Some Classes Of Semigroups That Have Medial Idempotent And Some Construction Theorem

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ABSTRACT

It's known that the set of idempotents of the semigroup, plays an important role for the structure of this semigroup. Specially, in the regular semigroups, an important role plays presence of the medial idempotent and normal medial idempotent. Blyth,T. S and R. B. McFadden have studied and constructed the regular semigroups which contain a normal medial idempotent in terms of idempotent-generated regular semigroups with a normal medial idempotent and inverse semigroups with an identity. M.Loganathan has described the construction of the regular semigroups which contain a medial idempotent. In this paper we will study further properties of medial idempotents on abundant semigroups. We will apply also the construction theory of abundant semigroups with a medial idempotent to quasi-adequate semigroups and will make a description of structure of those subsemigroups.

Keywords: *abundant semigroup, quasi-adequate semigroup, adequate semigroup, medial idempotent, normal medial idempotent.*

BASIC DEFINITIONS AND PROPERTIES

On a semigroup S, the elements *a* and *b* are related by * (*) if and only if they are related by Green's relation ( ) in some oversemigroup of S. El-Qallali in [1] has shown that:

$$a \quad {}^*b \Leftrightarrow (\forall(x,y) \in S^1 \times S^1,\ xa = ya \Leftrightarrow xb = yb) \qquad (1)$$

$$a \quad {}^*b \Leftrightarrow (\forall(x,y) \in S^1 \times S^1,\ ax = ay \Leftrightarrow bx = by) \qquad (2)$$

from which it follows: If $e \in E(S)$, then

$$a \quad {}^*e \Leftrightarrow ea = a \wedge (\forall(x,y) \in S^1 \times S^1,\ xa = ya \Rightarrow xe = ye) \qquad (3)$$

$$a \quad {}^*e \Leftrightarrow ae = a \wedge (\forall(x,y) \in S^1 \times S^1,\ ax = ay \Rightarrow ex = ey) \qquad (4)$$

***Definition* 1:** The semigroup S is called *abundant* if and only if we have :

$$\forall a \in S,\ \mathrm{R}_a^* \cap E(S) \neq \phi \wedge \mathrm{L}_a^* \cap E(S) \neq \phi$$

***Definition* 2:** The abundant semigroup S is called *quasi – adequate* if and only if $E(S)$ is subsemigroup of S.

***Definition* 3:** The quasi – adequate semigroup S is called *adequate* if and only if $E(S)$ is commutative.

We can see from the definition, that * and * are respectively left congruence and right congruence on the S. It is also that in every semigroup $\subset$ * and $\subset$ * but when the elements *a* and *b* are regular we will have: $a \quad b \Leftrightarrow a \quad {}^*b$ and $a \quad b \Leftrightarrow a \quad {}^*b$. Furthermore, if S is regular semigroup, then $=$ * and $=$ *.

Let be S an abundant semigroup and let $e \in R_a^* \cap E(S),\ f \in L_a^* \cap E(S)$. From (3) and (4) it easy to see that $a = ea = af = eaf$ (5). From [2] we have this Lemma:

***Lemma* 1:** A semigroup S is adequate if and only if each * – class and each *– class contain a unique idempotent and the subsemigroup < $E(S)$ > $= \overline{E(S)}$ is regular.

We shall denote by a^+ and a^* the unique idempotent respectively of the classes R^*_a and L^*_a for some $a \in S$, where S is adequate. So $\{a^+\} = R^*_a \cap E(S)$ and $\{a^*\} = L^*_a \cap E(S)$. From this definition for the elements a^+ and a^*, we have immediately $a = a^+a = aa^* = a^+aa^*$, furthermore for $a \in S$ and $e \in E(S)$ we have also $(ea)^+ = ea^+$ and $(ae)^* = a^*e$.

***Definition* 4:** An idempotent *u* of the regular or abundant semigroup S is called *medial* if $\forall x \in \overline{E(S)},\ xux = x$ and a medial idempotent *u* is called *normal* if the band $u\overline{E}u$ is commutative.

***Proposition* 1:** If S is an abundant semigroup with a medial idempotent *u*, then we have:

$$(\forall x \in S,\ e \in R^*_x \cap E,\ f \in L^*_x \cap E),\ x = eux = xuf = euxuf$$

It follows from (5): $x = ex = eue \cdot x = eu \cdot ex = eux$,
$x = xf = x \cdot fuf = xf \cdot uf = xuf$, so from $x = eux$ and $x = xuf$ we have $x = euxuf$. In [3] we have proved this proposition:

***Proposition* 2:** If *S* is regular semigroup that contains a medial idempotent *u*, then *uS*, *Su*, and *u*S*u* are orthodox semigroups such that:

$$E(uS) = u\overline{E} = uE, \quad E(Su) = \overline{E}u = Eu, \quad E(uSu) = u\overline{E}u = uEu$$

and for any $x \in \overline{E}$, we have $xux' = xx'$ and $x'ux = x'x$ and in similar way we can see also:

***Proposition* 3:** If S is an abundant semigroup with a medial idempotent *u* then uS, Su and

uSu are quasi-adequate semigroups. Therefore

$$E(uS) = u\overline{E} = uE,\ E(Su) = \overline{E}u = Eu,\ E(uSu) = u\overline{E}u = uEu$$

In [3] we have proved the following lemma:

***Lemma* 2: I.** Let $e, e' \in \overline{E}u$, $a \in u\overline{E}u$. If $e\mathcal{R}e'$ in $\overline{E}u$ and *ue*ě *a = a*, then *ue'*ě *a = a*,
$ea = e'a$

II. Let $f, f' \in u\overline{E}$, $b \in u\overline{E}u$. If $f\mathcal{L}f'$ in $u\overline{E}$ and *b*ě *fu = b*, then *b*ě *f'u = b*,
$bf = bf'$

THE REGULAR SEMIGROUPS WITH A MEDIAL IDEMPOTENT

In this section we will prove that every regular semigroup *S*, with a medial idempotent *u*, can be described in terms of the subsemigroup generated by idempotents of S, $\overline{E(S)}$, and of the subsemigroup uSu. We will see that the subsemigroup uSu is orthodox semigroup with identity.

Let $\overline{E}$ be an idempotent-generated regular semigroup with a medial idempotent *u*. Let S be an orthodox semigroup with identity whose band of idempotents

$$E(\mathrm{S}) \cong u\overline{E}u.$$

From this isomorphism we will identify the structures of E(S) and $u\overline{E}u$, so *u* becomes the identity of the semigroup S.

Really, if e is the identity of S, then $e^2 = e$, so $e \in E$(S). And since u is identity of $u\overline{E}u \cong E$(S), it follows that u becomes the identity of E(S).

So we have: $eu = ue = u$ (e is identity of S, i.e. e is also identity of E(S))

$eu = ue = \mathrm{e}$ (u is identity of $u\overline{E}u$, i.e. u is also identity of E(S))

Hence we have:

$u = e$. Let $\overline{E}u/\mathcal{R} = \{ \mathrm{R}_e : e \in \overline{E}u \}$ and $u\overline{E}/\mathcal{L} = \{\mathrm{L}_f : f \in u\overline{E} \}$ respectively the set of $\mathcal{R}$ - classes in $\overline{E}u$ and the set of $\mathcal{L}$ - classes in $u\overline{E}$.
In [3] we have proved:

***Proposition* 4:** For any $e \in \overline{E}u$, $uR_e = R_{ue} = \{ g \in u\overline{E}u : g\mathcal{R}ue$ in $u\overline{E}u \}$

and for any $f \in u\overline{E}$, $L_f u = L_{fu} = \{ h \in u\overline{E}u :$ $h\mathcal{L}fu$ in $u\overline{E}u \}$.

In [3] we have seen also that the set:

$W = \{ (R_e, x, L_f) \in \overline{E}u/\mathcal{R} \text{ x } S \text{ x } u\overline{E}/\mathcal{L} :$

it exists $x` \in V(x)$, $ue = xx`$ and $fu = x`x \}$,

where $e \in \overline{E}u$ and $f \in u\overline{E}$, with the binary operation defined as follows:

$$(R_e, x, L_f)(R_h, y L_k) = (R_{ea}, xfhy, L_{bk}) \in W$$

where $a, b \in E(S)$ such that $a\, \mathcal{R}\, xfhy\, \mathcal{L}\, b$ in S, is regular semigroup with medial idempotent. Now, if S is an regular smigroup with a medial idempotent medial u and $E = E(S)$ is the set of idempotents of S, then we have demonstrate (in ***Proposition* 2**) that uSu is orthodox subsemigroup with identity u and $E(uSu) = u\overline{E}u = uEu$. So, we can consider the semigroup $W = W(\overline{E(S)}, uSu)$ described above. Finally, our goal is to show that this semigroup S, can described in terms of $\overline{E(S)}$ and uSu, so we will prove:

***Theorem* 1:** If S is an regular semigroup, containing an medial idempotent medial u, then $S \cong W = W(\overline{E(S)}, uSu)$, where the function: θ: $S \rightarrow W = W(\overline{E(S)}, uSu)$ such that $x\theta = (R_{xx'}, uxu, L_{x'x})$ is an isomorphism of S to W and $x' \in V(x) \cap uSu$

***Proof*:** Let be $x', x'' \in V(x) \cap uSu$ such that: $x\theta = (R_{xx'}, uxu, L_{x'x})$ and $x\theta = (R_{xx''}, uxu, L_{x''x})$. Then $xx' = (xx''x)x' = (xx'')(xx')$ and $xx'' = (xx'x)x'' = (xx')(xx'')$ where $xx', xx'' \in E(S)$. But, $x', x'' \in uSu$ i.e. $x'u = x'$ and $x''u = x''$, hence $xx' = xx'u \in \overline{E(S)}u$. So, $xx'\mathcal{R}\, xx''$ in $\overline{E(S)}u$ and equally we have $x'x\, \mathcal{L}\, x''x$ in $u\overline{E(S)}$, that means $R_{xx'} = R_{xx''}$ and $L_{x'x} = L_{x''x}$, and so we have $(R_{xx'}, uxu, L_{x'x}) = (R_{xx''}, uxu, L_{x''x})$. This shows that θ is well defined. We se also that, for any $x \in S$,

$V(x) \cap uSu \neq \Phi$. Indeed, since S is regular semigroup, each element $x \in S$, has at least an inverse $x' \in V(x)$. On the other hand, we see that: $x \cdot ux'u \cdot x = xx'x = x$ and

$ux'u \cdot x \cdot ux'u = ux'xx'u = ux'u$. So, the element $ux'u \in uSu$ is also an inverse of x, hence $ux'u \in V(x)$ and we have $ux'u \in V(x) \cap uSu$ i.e. $V(x) \cap uSu \neq \Phi$.

Lets show now, that $\forall x \in S$, $(R_{xx'}, uxu, L_{x'x}) \in W(\overline{E(S)}, uSu)$. Since $x' \in V(x) \bigcap uSu$, we have $x' = uyu$ for some $y \in S$. But, when uyu is an inverse of x, we have:

$$uxu \cdot uyu \cdot uxu = uxu \cdot x' \cdot uxu = uxx'xu = uxu \quad \text{and}$$

$$uyu \cdot uxu \cdot uyu = uyu \cdot x \cdot uyu = x'xx' = x' = uyu$$

so, $uyu \in V(uxu)$ or $uyu = (uxu)'$ therefore:

$uxx' = ux \cdot uyu = uxu \cdot uyu = (uxu)(uxu)'$ and $x'xu = uyu \cdot xu = uyu \cdot uxu = (uxu)'(uxu)$

showing that $(R_{xx'}, uxu, L_{x'x}) \in W(\overline{E(S)}, uSu) = \{ (R_e, uxu, L_f) \in \overline{E(S)}u/\mathcal{R} \text{ x } uSu \text{ x } u\overline{E(S)}/\mathcal{L}$ where $(uxu)(uxu)' = ue$ and $(uxu)'(uxu) = fu \}$, for every element $x \in S$.

Let proved now that $\overline{}$ is injective. If $x, y \in S$ such that $\overline{x} = \overline{y}$. Then:

$$\overline{x} = \overline{y} \Rightarrow (R_{xx'}, uxu, L_{x'x}) = (R_{yy'}, uyu, L_{y'y}) \Rightarrow R_{xx'} = R_{yy'},\ uxu = uyu,\ L_{x'x} = L_{y'y}$$

and, from $x', y' \in uSu$ it follows $ux'u = ux' = x'u = x'$ and $uy'u = uy' = y'u = y'$,

then: $y'xy' = (y'u)x(uy') = y'(uxu)y' = y'(uyu)y' = y'yy' = y'$. So,

$R_{xx'} = R_{yy'} \Rightarrow xx'\mathcal{R}yy' \Rightarrow xx' = yy't$ for some $t \in \overline{E(S)}u$, hence

$$xx'x = yy'tx \Rightarrow x = yy'tx \Rightarrow y'x = y'yy'tx \Rightarrow y'x = y'tx.$$

Now: $x = yy'tx = yy'x \Rightarrow xy' = yy'xy' \Rightarrow xy' = yy' \Rightarrow xy'y = yy'y = y$ so $xy'y = y$ (5)

On the other hand, from

$L_{x'x} = L_{y'y}$ it follows $x'x \mathcal{L} y'y$ in $u\overline{E(S)}$ so $y'y = lx'x$ (6)

for some $l \in u\overline{E(S)}$ from which it follows $y'yx' = lx'xx'$ or $lx' = y'yx'$ (7)

Now, we see: $uyy'yu = uyu = uxu$

or $(uyu)y'(uyu) = uxu \Rightarrow (uxu)y'(uxu) = uxu \quad / \cdot x'$

$$\Rightarrow x'uxy'xux' = x'uxux' \Rightarrow x'xy'xx' = x'xx' \Rightarrow x'xy'xx' = x' \quad / \cdot x$$

$$\Rightarrow xx'xy'xx'x = xx'x \Rightarrow xy'x = x \quad (8)$$

Thus, from (5), (6), (7) and (8) we have:

$$y = xy'y = x(y'y) = x(lx'x) = x(lx')x = x(y'yx')x = x(y'u)y(ux')x = xy'(uyu)x'x$$
$$= xy'uxux'x = xy'xx'x = xy'x = x$$

that shows that $\overline{}$ is injective.

Let be now $(R_e, uxu, L_f) \in W(\overline{E(S)}, uSu)$ and $y = exf \in S$. We can prove that

$y^{-} = (R_e, uxu, L_f)$. Since $y^{-} = (exf)^{-} = (R_{(exf)(exf)'}, uexfu, L_{(exf)'(exf)})$, we must prove that:

$$uexfu = uxu, \quad R_{(exf)(exf)'} = R_e, \quad L_{(exf)'(exf)} = L_f$$

Since $fu = (uxu)'(uxu) \Rightarrow uxu \cdot fu = uxu \Rightarrow uxfu = uxu$ and

$ue = (uxu)(uxu)' \Rightarrow ue \cdot uxu = uxu \Rightarrow uexu = uxu$ [where $(uxu)' \in V(uxu)$] we have: $uexfu = (ueu)xfu = ue(uxfu) = ue(uxu) = uexu = uxu$.

So, $uexfu = uxu$ that means the middle component is the same.

Now, $(R_e, uxu, L_f) \in W$ and $(R_{(exf)(exf)'}, uxu, L_{(exf)'(exf)}) \in W$, means that there are inverses $(uxu)', (uxu)'' \in V(uxu)$ such that $ue = (uxu)(uxu)'$ and $u(exf)(exf)' = (uxu)(uxu)''$ moreover $(uxu)(uxu)' \mathcal{R} (uxu)(uxu)''$ in $\overline{E(S)}\, u$ or $ue \,\mathcal{R}\, u(exf)(exf)'$ or $e \mathcal{R} (exf)(exf)'$ in $\overline{E(S)}\, u$ and therefore $R_{(exf)(exf)'} = R_e$ in

$\overline{E(S)}\,u$ (dually we can show that $L_{(exf)'(exf)} = L_f$ in $u\,\overline{E(S)}$). We have shown: $(exf)^{-} = (R_{(exf)(exf)'}, uexfu, L_{(exf)'(exf)})$ = (R_e,*uxu*,L_f), that means $^{-}$ is surjective.

Now remains to show that $^{-}$ is a homomorphism. Let *x*, *y* ∈ *S* and we will have:

$(xy)^{-} = (R_{(xy)(xy)'}, uxyu, L_{(xy)'(xy)})$ and

$x^{-}\,y^{-} = (R_{xx'}, uxu, L_{x'x})\,(R_{yy'}, uyu, L_{y'y}) = (R_{xx'a}, uxux'xyy'uyu, L_{by'y}) = (R_{xx'a}, uxyu, L_{by'y})$

where $a\mathcal{R}\,uxyu\,\mathcal{L}\,b$ in *uSu* and *a,b* are idempotents in *uSu*. So, the middle component for $(xy)^{-}$ and $x^{-}\,y^{-}$ is the same. But, $(R_{(xy)(xy)'}, uxyu, L_{(xy)'(xy)}) \in W$ and $(R_{xx'a}, uxyu, L_{by'y}) \in W$, means there are (*uxyu*)', (*uxyu*)''∈V(*uxyu*) such that *u*(*xy*)(*xy*)' = (*uxyu*)(*uxyu*)' and

uxx'a = (*uxyu*)(*uxyu*)'' moreover (*uxyu*)(*uxyu*)'$\mathcal{R}$(*uxyu*)(*uxyu*)'' or *u*(*xy*)(*xy*)'$\mathcal{R}$ *uxx'a* from which we have: *xx'*· *u*(*xy*)(*xy*)'$\mathcal{R}$ *xx'*· *uxx'a* or (*xy*)(*xy*)'$\mathcal{R}$ *xx'a* because $\mathcal{R}$ is left congruence and *xx'* ∈ $\overline{E(S)}\,u$, therefore $R_{(xy)(xy)'} = R_{xx'a}$ and (dually) $L_{(xy)'(xy)} = L_{by'y}$.

So we have $(xy)^{-} = x^{-}\,y^{-}$.

Finally we have prove: S ≅ *W* = *W*($\overline{E(S)}, uSu$) ■

THE CONSTRUCTION OF QUASI-ADEQUATE SEMIGROUPS WITH NORMAL MEDIAL IDEMPOTENTS

In this section we will describe the construction of quasi-adequate semigroups with normal medial idempotents in terms of the band E with a normal medial idempotent *u* and of adequate semigroup S with the semilattice of idempotents $E^0 = uEu$. Let be $\mathcal{R}$ and $\mathcal{L}$ the Green's relations on E . Then we have:

***Lemma* 3:** If S is an abundant semigroup with a normal medial idempotent *u*, then

(1) S is a quasi-adequate semigroup if and only if u is a middle unit element

(2) S is an adequate semigroup if and only if u is a unit element

Proof:

(1) If u is a middle unit of S, then for all *x* in E we have $x^2 = xx = xux = x$ that means S is quasi-adequate semigroup. Now suppose that S is quasi-adequate semigroup. Then E is a band and from [4], $\mathfrak{D}$ is congruence on E, moreover the $\mathfrak{D}$ – classes on E are rectangular bands, that means $e\,\mathfrak{D}\,eue\,\mathfrak{D}\,ueu$ for all e in E or $uefu\,\mathfrak{D}\,ef\,\mathfrak{D}\,ueu\,ufu$ for all e, f in E. Now, since *u* is a normal medial idempotent in E, from [5] we have $\mathfrak{D} = \mathcal{F}$ on E, where $e\,\mathcal{F}\,f \Leftrightarrow ueu = ufu$. So, $uefu = ueu\,ufu$ from which it follows $ef = euf$. Hence *u* is a middle unit for E. But *u* is also a middle unit for S, because for any $x, y \in S$, there exists $e, f \in E$ such that $x\,\mathcal{L}^*\,e$ and $y\,\mathcal{R}^*f$ so that $xy = xe\,fy = xe\,u\,fy = xuy$.

(2) If S is an adequate and $e \in E$, then $e = eue = e^2u = eu = ue$. So, $xu = xx^*u = xx^* = x$ and $ux = ux^+x = x^+x = x$ for all x in S, where $\{x^*\} = L^*_x \cap E$ and $\{x^+\} = R^*_x \cap E$. Hence u is a unit element of S.
Conversely, if u is a unit element of S, then, by **(1)**, u is a middle unit of S and E is band, moreover, for all e, f in E we have: $ef = ueu \cdot ufu = ufu \cdot ueu = fe$ that means S is adequate semigroup. ■

***Lemma* 4:** If E is band with a normal medial idempotent *u*, then E is a normal band.

Proof: From **(1)** it follows that *u* is a middle unit of E and since uEu is commutative we have: $exye = e \cdot uxu \cdot uyu \cdot e = e \cdot uyu \cdot uxu \cdot e = eyxe$, for all *e, x, y* in E. So, E is a normal band.

Now let E be a band with a normal medial idempotent u and S an adequate semigroup with the semilattice of idempotents $E^0 = uEu$. Let be $\mathcal{R}$ and $\mathcal{L}$ the

Green's relations on E. ■ From the last lemma it follows that E is a normal band and further we have:

***Lemma* 5:** If $x, y \in S,\ e, f \in E$ and $e\,\mathcal{R}\,x^*$, $f\,\mathcal{L}\,y^+$, then

(1) $xefy = xy$ **(2)** $(xef)^+ = (xy)^+,\ (efy)^* = (xy)^*$

Proof:

(1) Since $e\,\mathcal{R}\,x^*$ and $f\,\mathcal{L}\,y^+$ we have $x^*e = e$, $ex^* = x^*$, $fy^+ = f$ and $y^+f = y^+$. So $e \in uE$, $f \in Eu$, moreover

$$xefy = xx^*efy^+y = xx^* \cdot ueu \cdot ufu \cdot y^+y = x \cdot ueu \cdot x^*y^+ \cdot ufu \cdot y = x(ex^*) \cdot (y^+f)y = xx^*y^+y = xy$$

(2) $(x \in S,\ ef \in uEu = E^0) \Rightarrow xef \in S$. Now we have $y^+\mathcal{R}^*\,y \Rightarrow (xef)y^+\mathcal{R}^*\,(xef)y \Rightarrow$

$(xefy^+)^+\,\mathcal{R}^*(xefy)^+ \Rightarrow (xefy^+)^+ = (xefy)^+ \Rightarrow (xef)^+ = (xy)^+$ and similarly $(efy)^* = (xy)^*$ ■

Let be

$$Q = Q(E, S) = \{\, (e, x, f) \in Eu \times S \times uE \ :\ e\,\mathcal{L}\,x^+,\ f\,\mathcal{R}\,x^* \,\}$$

and we define a binary operation on Q: $(e, x, f)\,(g, y, h) = (e(xy)^+, xy, (xy)^*h)$

This binary operation is well-defined because

$e\,\mathcal{L}\,x^+ \Rightarrow e(xy)^+\,\mathcal{L}\,x^+(xy)^+ \Rightarrow e(xy)^+\,\mathcal{L}\,(xy)^+$ and dually $(xy)^*h\,\mathcal{R}\,(xy)^+$

that means $(e(xy)^+, xy, (xy)^*h) \in Eu \times S \times uE$ = Q. Now we have this:

Theorem 2: 1) Q with the binary operation defined above is semigroup

2) $E(Q) = \{\, (e, x, f) \in Q : x \in E^0 \}$
3) Q is a quasi-adequate semigroup
4) $\bar{u} = (u,u,u)$ is normal medial idempotent of Q

5) $E(Q) \cong E$ and $\bar{u} Q \bar{u} \cong S$

Proof:

1) Let be $(e, x, f), (g, y, h), (s, z, t) \in Q$. We see that:

$[(e, x, f)(g, y, h)](s, z, t) = [e(xy)^+, xy, (xy)^*h](s, z, t) = (e(xy)^+(xyz)^+, xyz, (xyz)^*t) =$

$(e((xy)^+(xy)z)^+, xyz, (xyz)^*t) = (e(xyz)^+, xyz, (xyz)^*t)$. Similarly we find that

$(e, x, f)[(g, y, h)(s, z, t)] = (e(xyz)^+, xyz, (xyz)^*t)$. So Q is semigroup.

2) If $(e, x, f) \in E(Q)$ then $(e, x, f)^2 = (e(xx)^+, xx, (xx)^*f) = (e(xx)^+, xx, (xx)^*f) = (e, x, f)$. So $x^2 = xx = x \in E^0$. Conversely, $(e, x, f) \in Q$ and $x^2 = x \in E^0$, we have

$(e, x, f)^2 = (e(xx)^+, xx, (xx)^*f) = (ex^+, x, x^*f) = (e, x, f)$, hence

$$E(Q) = \{ (e, x, f) \in Q : x \in E^0 \}$$

3) If $(e, x, f) \in Q$ we can prove that $(e, x, f) \mathcal{L}^* (e, x^*, f)$ in Q where $(e, x^*, f) \in E(Q)$.

<u>First</u> $(e, x, f)(e, x^*, f) = (e(xx^*)^+, xx^*, (xx^*)^*f) = (ex^+, x, x^*f) = (e, x, f)$

<u>Second</u> if $(g, y, h), (s, z, t) \in Q$ such that $(e, x, f)(g, y, h) = (e, x, f)(s, z, t)$ we must prove that $(e, x^*, f)(g, y, h) = (e, x^*, f)(s, z, t)$. But,

$(e, x, f)(g, y, h) = (e, x, f)(s, z, t) \Rightarrow (e(xy)^+, xy, (xy)^*h) = (e(xz)^+, xz, (xz)^*t) \Rightarrow$

$\Rightarrow e(xy)^+ = e(xz)^+ \wedge xy = xz \wedge (xy)^*h = (xz)^*t$ **(i)** and

$(e, x^*, f)(g, y, h) = (e(x^*y)^+, x^*y, (x^*y)^*h) \wedge (e, x^*, f)(s, z, t) = (e(x^*z)^+, x^*z, (x^*z)^*t)$

We notice that $x^* \in E$ and $x^* \mathcal{L}^* x$ which means $xy = xz \Rightarrow x^*y = x^*z \wedge e$ $(x^*y)^+ = e(x^*z)^+$

For the third component we have: $x^* \mathcal{L}^* x \Rightarrow x^*y \mathcal{L}^* xy$ and also $(x^*y)^* \mathcal{L}^*$ $(xy)^*$. So, $(x^*y)^* = (xy)^*$ and similarly $(x^*z)^* = (xz)^*$. Now, from (i) it follows:

$(xy)^*h = (xz)^*t \Rightarrow (x^*y)^*h = (x^*z)^*t$. That means $(e, x^*, f)(g, y, h) = (e, x^*, f)(s, z, t)$ Thus, $(e, x, f)(g, y, h) = (e, x, f)(s, z, t) \Rightarrow (e, x^*, f)(g, y, h) = (e, x^*, f)(s, z, t)$ and from (4)

follows that $(e, x, f) \mathcal{L}^* (e, x^*, f)$ in Q where $(e, x^*, f) \in E(Q)$. So each $\mathcal{L}^*$- class in Q has an idempotent. Dually we can prove that each $\mathcal{R}^*$- class in Q has an idempotent, hence Q is abundant semigroup. To show that Q is quasi-adequate we must prove that $E(Q)$ is subsemigroup of Q. So let $(e, x, f), (g, y, h) \in E(Q)$. Then $x, y \in E^0$ which means $xy \in E^0$ hence $(e, x, f)(g, y, h) = (e(xy)^+, xy, (xy)^*h) \in E(Q)$ and Q is quasi-adequate.

4) $\bar{u} = (u,u,u)$ is medial idempotent in Q because if $(e, x, f) \in \overline{E(Q)} = E(Q)$ we have:

$(e, x, f)(u,u,u)(e, x, f) = (ex^+, x, x^*u)(e, x, f) = (e, x, x^*)(e, x, f) = (e, x, f)$ and if $(u,u,u)(e, x, f)(u,u,u), (u,u,u)(g, y, h)(u,u,u) \in E(Q)$ then

$(u,u,u)(e, x, f)(u,u,u) \cdot (u,u,u)(g, y, h)(u,u,u) = (x, x, x)(y, y, y) = (xy, xy, xy) = \quad = (yx, yx, yx) = (u,u,u)(g, y, h)(u,u,u) \cdot (u,u,u)(e, x, f)(u,u,u)$.

So $\bar{u}E(Q)\bar{u}$ is commutative that means $\bar{u} = (u,u,u)$ is normal medial idempotent in Q.

5) Let be $\ddot{} : E(Q) \to E$ such that $\ddot{}[(e,x,f)] = ef$ for each elements $(e,x,f) \in E(Q)$

¨ - is surjective, because $\forall e\in E$ there exists $(eu,ueu,ue)\in E(\mathrm{Q})$ where ¨$[(eu,ueu,ue)]=euue=eue=e$. Now let be (e, x, f), $(g, y, h)\in E(\mathrm{Q})$ such that ¨$[(e,x,f)]=ef=gh=$¨$[(g,y,h)]$ then we have: $e\,\mathcal{L}\,x$, $f\,\mathcal{R}\,x$, $g\,\mathcal{L}\,y$, $h\,\mathcal{R}\,y$ from which follows

$ex=e,\ xe=x,\ xf=f,\ fx=x,\ gy=g,\ yg=y,\ yh=h,\ hy=y$. So,

$$ef=gh\Rightarrow exf=gyh\Rightarrow uexf=ugyh\Rightarrow xuef=yugh\Rightarrow xef=ygh\Rightarrow xf=yh\Rightarrow f=h,$$

$$ef=gh\Rightarrow exf=gyh\Rightarrow exfu=gyhu\Rightarrow efux=ghuy\Rightarrow efx=ghy\Rightarrow ex=gy\Rightarrow e=g,$$

$$ef=gh\Rightarrow xf=yh\Rightarrow xfu=yhu\Rightarrow fux=huy\Rightarrow fx=hy\Rightarrow x=y,$$ hence $(e,x,f)=(g,y,h)$ that means ¨ - is injective, consequently ¨ - is bijective, moreover we have:

¨$[(e,x,f)\cdot(g,y,h)]=$¨$[(exy,xy,xyh)]=exyh=efxygh=exfgyh=ef\cdot gh=$¨$[(e,x,f)]\cdot$¨$[(g,y,h)]$
.

So, we have proved that ¨ - is isomorphism and $E(\mathrm{Q})\cong E$.

Now let be $\cdot : \bar{u}\,\mathrm{Q}\,\bar{u}\to S$ such that $\cdot\,[(u,u,u)(e,x,f)(u,u,u)]=x$.

$\cdot$ - is surjective, because $\forall x\in S$ there exists $(u,u,u)(x^{+},x,x^{*})(u,u,u)\in\bar{u}\,\mathrm{Q}\,\bar{u}$ such that $\cdot\,[(u,u,u)(x^{+},x,x^{*})(u,u,u)]=x$. Also, $\cdot$ - is injective because:

$$x=y\Rightarrow(x^{+},x,x^{*})=(y^{+},y,y^{*})\Rightarrow(u,u,u)(x^{+},x,x^{*})(u,u,u)=(u,u,u)(y^{+},y,y^{*})(u,u,u)$$

Therefore $\cdot\,[(u,u,u)(e,x,\mathrm{f})(u,u,u)\cdot(u,u,u)(g,y,h)(u,u,u)]=$

$$=\cdot\,[(u,u,u)(ex^{+},x,x^{*})\cdot(g,y,h)(u,u,u)]=$$

$$=\cdot\,[(u,u,u)(e(xy)^{+},xy,(xy)^{*})(u,u,u)]=xy=$$

$$=\cdot\,[(u,u,u)(e,x,\mathrm{f})(u,u,u)]\cdot\ \cdot\,[(u,u,u)(g,y,h)(u,u,u)].$$ So,

$\bar{u}\,\mathrm{Q}\,\bar{u}\cong S$. ■

From this construction we have the following theorem:

***Theorem* 3:** Any quasi-adequate semigroup S with a normal medial idempotent u can be constructed as above, in terms of the band $E(S) = E$ and of the adequate subsemigroup uSu.

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