On the cohomology of the inverse semigroup $\mathcal{G}$ of the G-sets of a groupoid G

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***Abstract*—Renault has defined in [7] the cohomology of the inverse semigroup $\mathcal{G}$ of the G-sets of a given groupoid G as a functor from the category of $\mathcal{G}$-presheaves to that of abelian groups. We show in our paper that $\mathcal{G}$-presheaves is isomorphic to $\mathbf{Ab}^{D(\mathcal{G})}$ where $D(\mathcal{G})$ is the division category defined from Loganathan in [6] and used there to give another description of the Lausch cohomology of inverse semigroups. This isomorphism allows us in turn to prove that Renault and Lausch cohomology groups of $\mathcal{G}$ are isomorphic.**

Key words: Groupoid, cohomology, presheaves, inverse semigroup.

I. Introduction and Preliminaries

We give in this section a few basic notions from groupoids and inverse semigroups associated to them and show how cohomology groups of a groupoid are defined. All these can be found in [7]. By definition, a groupoid G is a set endowed with a product map $(x, y) \mapsto xy$: $G^2 \to G$ where G^2 is a subset of $G \times G$ called the set of composable pairs, and an inverse map $x \times x^{-1} : G \to G$ such that the following relations are satisfied:

(i) $(x^{-1})^{-1} = x$;
(ii) $(x, y), (y, z) \in G^2$, then $(xy, z), (x, yz) \in G^2$ and $(xy)z = x(yz)$;
(iii) $(x^{-1}, x) \in G^2$ and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$;
(iv) $(x, x^{-1}) \in G^2$ and if $(z, x) \in G^2$, then $(zx)x^{-1} = z$.

For every $x \in G$, we define $d(x) = x^{-1}x$ as the domain of x and $r(x) = xx^{-1}$ as the range of x. Note that a pair (x, y) is composable only if $r(y) = d(x)$. Also the relations $xd(x) = x = r(x)x$, suggest that we call the set $G^0 = r(G) = d(G)$ the unit space of G. Here is a non trivial example of a groupoid.

Example I.1 Let U be a set and S a group which acts on U on the right. The action of s on u is denoted by $u \cdot s$. We let G be $U \times S$ and define the following groupoid structure: (u, s) and (v, t) are composable only if $v = u \cdot s$; $(u, s)(u \cdot s, t) = (u, st)$, and $(u, s)^{-1} = (u \cdot s, s^{-1})$. Then, $r(u, s) = (u, e)$ and $d(u, s) = (u \cdot s, e)$. The map $(u, e) \mapsto u$ identifies G^0 with U.

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An important notion in the theory of groupoids is that of a G-set. Let G be a groupoid and S a subset of G. We call S a G-set if the restriction of r and d to it is one-to-one, or equivalently if $SS^{-1}, S^{-1}S \subseteq G^0$. The set $\mathcal{G}$ of all G-sets of G can be made into an inverse semigroup, for if S and T are G-sets, then their product ST is again a G-set, and for any $S \in \mathcal{G}$, $G^{-1} \in \mathcal{G}$.
To define the cohomology of $\mathcal{G}$ we need to define first the presheaves. For this, let $\mathcal{C}$ be any category and A_0 a set. The set 2^{A_0} of all subsets of A_0 when ordered by inclusion becomes a category: there is an arrow $U \to V$ if $V \subseteq U$. By definition a $\mathcal{C}$-presheaf $\mathcal{A}$ from 2^{A_0} to $\mathcal{C}$ is a contravariant functor whose object map is denoted by $U \to \mathcal{A}_U$ and its morphism map by $\mathcal{A}_U \to \mathcal{A}_V$ whenever $V \subseteq U$. A partial isomorphism, or a partial symmetry ϕ of $\mathcal{A}$ is a bijection $\phi : V \to U$ where V and U are subsets of A_0 together with isomorphisms $\phi : \mathcal{A}_{V'} \to \mathcal{A}_{\phi(V')}$ for any $V' \subseteq V$, which are compatible with the restriction morphism. The latter means that for every $V'' \subseteq V'$, the following diagram commutes

$$\begin{tikzcd} \mathcal{A}_{V'} \arrow[r] \arrow[d] & \mathcal{A}_{\phi(V')} \arrow[d] \\ \mathcal{A}_{V''} \arrow[r] & \mathcal{A}_{\phi(V'')} \end{tikzcd}$$

Two partial isomorphisms ϕ and ϕ' can be composed: if $\phi : V \to U$ and $\phi' : V' \to U'$, we let V'' be $\phi^{-1}(U' \cap V)$ and U'' be $\phi(U' \cap V)$; $\phi'' = \phi \circ \phi'$ is the bijection $V'' \to U''$ obtained by composing ϕ and ϕ'; and for $W \subseteq V''$ we define $\phi'' : \mathcal{A}_W \to \mathcal{A}_{\phi''(W)}$ by composing

$$\mathcal{A}_W \xrightarrow{\phi\text{''}} \mathcal{A}_{\phi'(W)} \xrightarrow{\phi} \mathcal{A}_{\phi\circ\phi'(W)}.$$

The inverse of a partial isomorphism is defined in the obvious way. In this way the set $\mathcal{T}(\mathcal{A})$ of partial isomorphisms of $\mathcal{A}$ becomes an inverse semigroup which we call the isomorphism inverse semigroup of the given $\mathcal{C}$-presheaf $\mathcal{A}$.
For a given inverse semigroup $\mathcal{G}$, we define a $\mathcal{G}$-presheaf $(\mathcal{A}, \mathcal{L})$ to be a $\mathcal{C}$-presheaf $\mathcal{A}$ together with a homomorphism $\mathcal{L} : \mathcal{G} \to \mathcal{T}(\mathcal{A})$ such that $\mathcal{L}^0 : \mathcal{G}^0 \to 2^{A_0}$ is an injection.
Given a $\mathcal{G}$-presheaf $(\mathcal{A}, \mathcal{L})$ of abelian groups one can form

the following cochain complex. A n-cochain is a function $f : \mathcal{G}^n \to \mathcal{A}$ which satisfies the following conditions:

(i) $f(s_0, s_1, ..., s_{n-1}) \in \mathcal{A}_{r(s_0 s_1 ... s_{n-1})}$;

(ii) f is compatible with the restriction map, that is if $U = r(s_0 s_1 ... s_{n-1})$ and $V = r(t_0 t_1 ... t_{n-1})$ where $t_i = e s_i$ for some idempotent e_i, then $f(t_0, t_1, ..., t_{n-1})$ is the restriction of $f(s_0, s_1, ..., s_{n-1}) \in \mathcal{A}_U$ to V; and

(iii) for $n > 0$, $f(s_0, ..., s_i, ... s_{n-1}) \in 2^{A_0}$ whenever s_i is an idempotent

The set $C^n(\mathcal{G}, \mathcal{A})$ of n-cochains is an abelian group under pointwise addition. The sequence

$$0 \longrightarrow C^0(\mathcal{G}, \mathcal{A}) \longrightarrow C^1(\mathcal{G}, \mathcal{A}) \longrightarrow \cdots$$

$$\longrightarrow C^n(\mathcal{G}, \mathcal{A}) \xrightarrow{\delta^n} C^{n+1}(\mathcal{G}, \mathcal{A}) \longrightarrow \cdots$$

where

$$\delta^0(f(s)) = \mathcal{L}(s) f \circ d(s) - f \circ r(s) \text{ and}$$

$$\begin{aligned}\delta^n f(s_0, ..., s_n) &= \mathcal{L}(s_0) f(s_1, ..., s_n) \\ &+ \sum_{i=1}^{n} (-1)^i f(s_0, ... s_{i-1} s_i, ..., s_n) \\ &+ (-1)^{n+1} f(s_0, ..., s_{n-1})\end{aligned}$$

is a cochain complex. We denote by $Z^n(\mathcal{G}, \mathcal{A})$ and $B^n(\mathcal{G}, \mathcal{A})$ the groups of n-cocycles and that of n-coboundaries. The n-th cohomology group $Z^n(\mathcal{G}, \mathcal{A})/B^n(\mathcal{G}, \mathcal{A})$ will be denoted by $H^n(\mathcal{G}, \mathcal{A})$.

In the next section we will show that for any inverse semigroup S, S-presheaves form a category and that this category is isomorphic to the functor category $\mathbf{Ab}^{D(S)}$ where $D(S)$ has objects all the idempotents of S and morphisms $e \to f$ are triples (e, x, x') where x' is the inverse of x and $e \geq xx'$, $x'x = f$. The main result of [2] states that there is only one cohomology functor from a given category to $\mathbf{Ab}$, therefore the Lausch cohomology defined on $\mathbf{Ab}^{D(S)}$ has to coincide to that of renault define on S-presheaves.

II. S-PRESHEAVES AS FUNCTORS

Let S be an inverse semigroup, X a presheaf of abelian groups over $E(S)$ and $\alpha : S \to \mathcal{T}(X)$ be a representation of S by partial symmetries of X.

Lemma II.1 *Representation α gives rise to an S-module in the sense of Lausch.*

Proof. Theorem 5.8 ($(i) \Leftrightarrow (ii)$) of [5] states that α can be regarded as an action of S on the right of the presheaf X with values in $\mathbf{Ab}$. Then as shown in p. 33 of [5] one can construct a clifford semigroup $(\mathbf{X}, \otimes)$ with semilattice of idempotents $E(S)$ and with a right action of S on X given by

$$a \circ s = \alpha(es) \rho^e_{ess^{-1}}(a).$$

which satisfies all the properties of an S-module. ■

Let S be a fixed inverse semigroup, we form the category of S-presheaves with objects representations of S by partial symmetries of presheaves of abelian groups over $E(S)$ and morphisms between two representations $\alpha : S \to \mathcal{T}(X)$ and $\beta : S \to \mathcal{T}(Y)$ are S-module morphisms $\tau : \mathbf{X} \to \mathbf{Y}$ between the corresponding S-modules of Lemma II.1 such that $\forall s \in S$,

$$\tau(\alpha(s)(x)) = \beta(s)(\tau(x)). \tag{1}$$

Here $\alpha(s)$ is meant to be be one of the components of the corresponding family and $x \in X(e)$ where $X(e)$ is the domain of that component of $\alpha(s)$. We have to show that S-presheaves is indeed a category. The only thing we have to check is that if $\alpha : S \to \mathcal{T}(X)$, $\beta : S \to \mathcal{T}(Y)$ and $\gamma : S \to \mathcal{T}(Z)$ are objects from S-presheaves and $\tau_1 : \alpha \to \beta$, $\tau_2 : \beta \to \gamma$ are morphisms, then for every $s \in S$ and x from some domain of some component of $\alpha(s)$ we have

$$\tau_2 \tau_1(\alpha(s)(x)) = \gamma(s)(\tau_2 \tau_1(x)). \tag{2}$$

From the definitions of τ_1 and τ_2 we have

$$\tau_1(\alpha(s)(x)) = \beta(s)(\tau_1(x)) \tag{3}$$

and

$$\tau_2(\beta(s)(y)) = \gamma(s)(\tau_2(y)). \tag{4}$$

Then replacing in (4) y by $\tau_1(x)$ we get

$$\tau_2(\beta(s)(\tau_1(x))) = \gamma(s)(\tau_2 \tau_1(x)). \tag{5}$$

Now (3) and (5) imply (2).

Given an inverse semigroup S with semilattice of idempotents E we define a category $\mathcal{P}(S)$ with objects the idempotents E of S and morphisms $e \to f$ are pairs $(e, s) \in E \times S$ such that $f = s^{-1} e s$. Composition is given by $(s^{-1} e s, t)(e, s) = (e, st)$. Let $P(S)$ the quotient of $\mathcal{P}(S)$ by the congruence on the hom-sets of $\mathcal{P}(S)$ generated by the pairs

$$(e, s) \sim (e, es) \text{ and } (e, e) \sim id_e.$$

We will write morphisms of $P(S)$ by the same symbols as their representatives in $\mathcal{P}(S)$. Note that the semilattice $E(S)$ is a subcategory of $P(S)$.

The next two lemmas show two properties of functors from $\mathbf{Ab}^{P(S)}$.

Lemma II.2 *Every $X \in \mathbf{Ab}^{P(S)}$ gives rise to a right action of S on the **Ab**-bundle $\mathbf{X} = \cup_{e \in E} X(e)$.*

Proof. Define a function $\circ : \mathbf{X} \times S \to \mathbf{X}$ by

$$a \circ s = X(e, s)(a) \text{ whenever } a \in X(e) \text{ .}$$

Let us check the three properties for the right action of S on $\mathbf{X}$.

(Act 3) If $a \in X(e)$, then from the definition $a \circ s \in X(s^{-1} e s)$ and the map $a \mapsto a \circ s$ is a morphism in $\mathbf{Ab}$ since $X(e, s)$ is such.

(Act 1) If $a \in X(e)$, then $a \circ e = X(e, e)(a) = id_{X(e)}(a) = a$.

(Act 2) $(a \circ s) \circ t = X(s^{-1} e s, t) X(e, s)(a) = X(e, st)(a) = a \circ (st)$. ■

Lemma II.3 *Every $X \in \mathbf{Ab}^{P(S)}$ gives rise to an S-module $\mathbf{X} = \cup_{e\in E} X|_E(e)$ where $X|_E$ is the restriction of X in $E(S)$.*

Proof. We will show that the clifford semigroup $\mathbf{X}$ has the structure of an S-module. From $(iii) \Rightarrow (ii)$ of Theorem 5.8 of [5] we have that the $\mathbf{Ab}$-bundle $\mathbf{X}$ of Lemma II.2 can be regarded as a representation of S by partial symmetries of a presheaf with values in $\mathbf{Ab}$ in the following way. First, as in the proof of Theorem 5.6 of [5] we form a semilattice of groups $X(e)$ (though we have one already) by defining for $e \geq f$, $\rho^e_f : X(e) \to X(f)$ by $\rho^e_f(a) = a \circ f$. But $a \circ f = X(e,f)(a) = a + f$. This shows that the clifford semigroup arising by restricting X in $E(S)$ is the same as the one described in Theorem 5.6 of [5]. Then define a partial function

$$a \cdot s = \begin{cases} a \circ s & \text{if} \quad a \in X(e) \text{ and } ss^{-1} = e \\ \text{undefined} & \text{else} \end{cases}$$

This is a right action of S on the presheaf $X|_E$ which satisfies (Rep 1)-(Rep 5) of Proposition 5.7 of [5] therefore from Example 3 of [5] $\mathbf{X}$ becomes an S-module with the S action defined by

$$a \star s = \rho^e_{ess^{-1}} \cdot (es) = \rho^e_{ess^{-1}}(a) \circ (es). \tag{6}$$

On the other hand we see that

$$\begin{aligned} \rho^e_{ess^{-1}}(a) \circ (es) &= X(ess^{-1}, es)X(e, ss^{-1})(a) \\ &= X(e, es)(a) \\ &= X(e, s)(a) \\ &= a \circ s. \end{aligned}$$

Comparing with (6) we see that actions $\star$ and $\circ$ are equal, therefore $\mathbf{X}$ is an S-module. ■

Define $G : S$-presheaves $\to \mathbf{Ab}^{P(S)}$ on objects by sending each representation $\alpha : S \to \mathcal{T}(X)$ to $G(\alpha) : P(S) \to \mathbf{Ab}$ which sends each idempotent e to $X(e)$ and each morphism $(e, s) : e \to s^{-1}es$ to the composite

$$G(\alpha)((e, s)) = \alpha(es)\rho^e_{ess^{-1}}. \tag{7}$$

The functorial properties of $G(\alpha)$ are easy to proof if we recall that (7) defines a right action on the presheaf X and that for $a \in X(e)$, $G(\alpha)((e, s))(a)$ is the same as $a \circ s$ of Example 3 of [5].

Let $\tau : \alpha \to \beta$ is a morphism in S-presheaves where $\alpha : S \to \mathcal{T}(X)$ and $\beta : S \to \mathcal{T}(Y)$. Define

$$G(\tau) : G(\alpha) \to G(\beta)$$

as the family

$$\{\tau_e : X(e) \to Y(e) | e \in E\}.$$

To show that $G(\tau)$ is natural we have to show that for each $e \in E$, every morphism $(e, s) : e \to s^{-1}es$ and every $a \in X(e)$, we have

$$\tau_{s^{-1}es} G(\alpha)(e, s)(a) = G(\beta)(e, s)\tau_e(a),$$

which from (7) is equivalent to

$$\tau(a \circ s) = \tau(a) \circ s.$$

This is true since from Lemma II.1 X and Y are S-modules with action $\circ$ and $\tau : X \to Y$ is an S-module morphism.

Define $G' : \mathbf{Ab}^{P(S)} \to S$-presheaves on objects X in the following way. From Lemma II.2 X gives rise to a right action of S on the $\mathbf{Ab}$-bundle $\mathbf{X} = \cup_{e\in E} X(e)$ and then as in the proof of $(iii) \Rightarrow (ii)$ of Theorem 5.8 of [5] one can define a representation $G'(X)$ of S by partial symmetries of the presheaf $X|_E$. It turns out that $G'(X) : S \to \mathcal{T}(X|_E)$ is defined by $s \mapsto X(ss^{-1}, s)$ where $X(ss^{-1}, s) : X(ss^{-1}) \to X(s^{-1}s)$ is the map $a \mapsto a \circ s$.

Lemma II.4 *The module of Lemma II.1 arising from the representation $G'(X)$ is the same as the module of Lemma II.3 arising from X.*

Proof. Theorem 5.8 ($(ii) \Rightarrow (i)$) and Example 3 of [5] show that the module of Lemma II.1 arising from the representation $G'(X)$ is the clifford semigroup $\mathbf{X}$ of Lemma II.3 consisting of groups $X(e)$ together with structure morphisms $\rho^e_f = X(e, f)$, and the action of S on $\mathbf{X}$ is given by

$$\begin{aligned} a \star s &= \rho^e_{ess^{-1}} \cdot (es) \\ &= X(ess^{-1}, es)X(e, ss^{-1})(a) \\ &= X(e, es)(a) \\ &= X(e, s)(a) \\ &= a \circ s. \end{aligned}$$

This proves the lemma. ■

Define G' on morphisms. If $\tau : X \to Y$ is a natural transformation of functors in $\mathbf{Ab}^{P(S)}$ then τ induces an S-module morphism $\tau^* : \mathbf{X} \to \mathbf{Y}$ of the corresponding S-modules $\mathbf{X}$ and $\mathbf{Y}$ of Lemma II.3. But Lemma II.4 claims that $\mathbf{X}$ matches to the module arising from $G'(X)$ and so does $\mathbf{Y}$ to $G'(Y)$. Also the fact that τ^* is a module morphism implies

$$\tau^* X(ss^{-1}, s) = Y(ss^{-1}, s)\tau^*,$$

which shows that $\tau^* : \mathbf{X} \to \mathbf{Y}$ can be regarded as a morphism between the respective representations $G'(X)$ and $G'(Y)$. We define

$$G'(\tau) = \tau^*.$$

The functorial properties are now clear.

Theorem II.1 *Categories $\mathbf{Ab}^{P(S)}$ and S-presheaves are isomorphic.*

Proof. Let us first show that for every $\alpha \in S$-presheaves we have $G'G\alpha = \alpha$. From the definition of G' we have that $G'G\alpha$ is the homomorphism

$$G'G\alpha : S \to \mathcal{T}(X)$$

defined by

$$s \mapsto G\alpha(ss^{-1}, s)$$

where from (7), $G(\alpha)(ss^{-1}, s)$ is the morphism

$$\begin{aligned} &G\alpha(ss^{-1}, s) : \\ &\quad X(ss^{-1}) \to X(s^{-1}s) = X(s^{-1}(ss^{-1})s). \end{aligned}$$

defined by

$$G\alpha(ss^{-1}, s) = \alpha((ss^{-1})s)\rho^{ss^{-1}}_{(ss^{-1})ss^{-1}} = \alpha(s),$$

therefore $G'G\alpha = \alpha$. Secondly we show that for every $X \in \mathbf{Ab}^{P(S)}$, $GG'X = X$. For this we have to show that $GG'X$ sends every morphism $(e, s) : e \to s^{-1}es$ of $P(S)$ to $X(e, s)$. From (7) we have

$$GG'X(e, s) = G'X(es)\rho^{e}_{ess^{-1}} \tag{8}$$

and from the definition of G' we have

$$G'X(es) = X((es)(es)^{-1}, es) = X(ess^{-1}, es). \tag{9}$$

But $\rho^{e}_{ess^{-1}} = X(e, ss^{-1})$ and then from (8) and (9) we have

$$\begin{aligned} GG'X(e, s) &= X(ess^{-1}, es)X(e, ss^{-1}) \\ &= X(e, (ss^{-1})(es)) \\ &= X(e, es) = X(e, s) \end{aligned}$$

as desired. ■

Proposition II.1 *For an inverse semigroup S, categories $P(S)$ and $D(S)$ of [6] coincide.*

Proof. First notice that $\mathcal{P}(S)$ coincides with $C(S)$ of [6]. Let $(e, x) : x \to x^{-1}ex$ be a morphism in $\mathcal{P}(S)$. We can write $x^{-1}ex$ as $(ex)^{-1}(ex)$ and observe that $e \geq (ex)(ex)^{-1}$, therefore (e, x) coincides with $(e, (ex), (ex)^{-1}) : e \to f = (ex)^{-1}(ex)$ of $C(S)$. Conversely, let $(e, x, x^{-1}) : e \to f$ be a morphism in $C(S)$. Since $e \geq xx^{-1}$, we have $e(xx^{-1}) = xx^{-1}$ and then $x^{-1}e(xx^{-1})x = x^{-1}xx^{-1}x$ which is equivalent to $x^{-1}ex = x^{-1}x$. But $f = x^{-1}x$, then $x^{-1}ex = f$ and as a consequence (e, x, x^{-1}) matches with $(e, x) : e \to x^{-1}ex$ of $\mathcal{P}(S)$. Lastly observe that our $\sim$ is the same as $\sim$ of p. 379 of [6], hence $P(S) = D(S)$. ■

Corollary II.1 *Cohomology groups of an inverse semigroup defined by Lausch are isomorphic to those defined by Renault.*

Proof. The cohomology of an inverse semigroup S after Renault is defined in S-presheaves which from Theorem II.1 and Proposition II.1 is isomorphic to $\mathbf{Ab}^{D(S)}$. But $\mathbf{Ab}^{D(S)}$ is an abelian category with enough injectives, therefore from the uniqueness theorem for cohomology functors [2] (see also [3]), there is only one cohomology functor on $\mathbf{Ab}^{D(S)}$. Since the cohomology after Lausch is defined in $\mathbf{Ab}^{D(S)}$, we have that both cohomologies coincide. ■

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