Plasma Expansion into a Vacuum with an Arbitrarily Oriented External Magnetic Field.

36th International Workshop on High Density Energy Physics with Intense Ion and Laser Beams.
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3rd February, 2016

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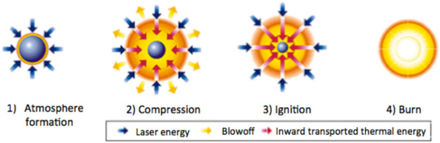
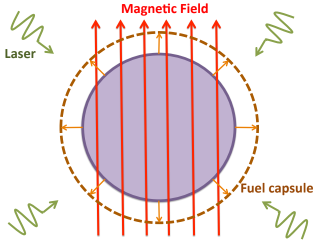
Motivation

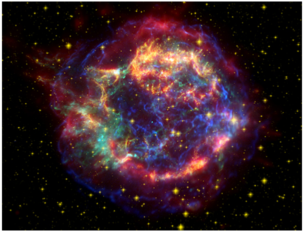
- **Inertial Confinement Fusion (ICF)**



- **Magneto Inertial Fusion (MIF)**



- **Astrophysics**[1]



[1] Picture source: *http://www.space.com/11425-photos-supernovas-star-explosions.html*

Previous Work

Magnetised plasma expansion

- **Ideal magnetohydrodynamics** framework.
- **Perpendicular** magnetic field.
- **Inconsistencies** when obtaining a complete self-similar solution.

✓ D. Anderson, M. Bonnnedal and M. Lisak, "*Effect of magnetic field on self-similar plasma expansion into vacuum*", Phys. Scr. **22**, 507-509, (1980).

✓ D. Bennaceur-Doumaz and M. Djebli, "*Effect of transverse magnetic field on laser produced plasma expansion into vacuum*", Phys. Plasmas **18**, 084507 (2011).

What is new in this work?

Novelties

- ✓ **Complete** self-similar **magnetised** plasma expansion against a **magnetised** vacuum.
- ✓ **Arbitrarily oriented** external magnetic field.

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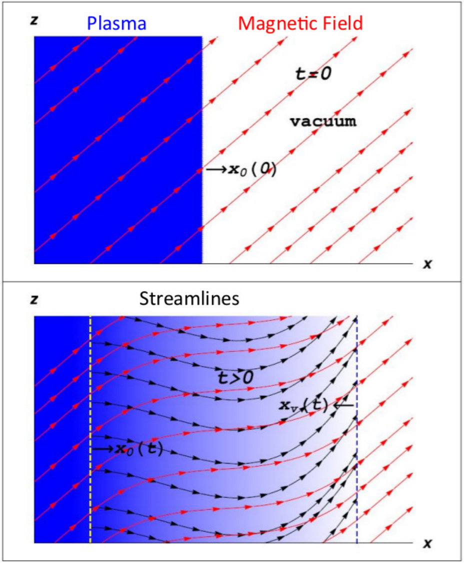
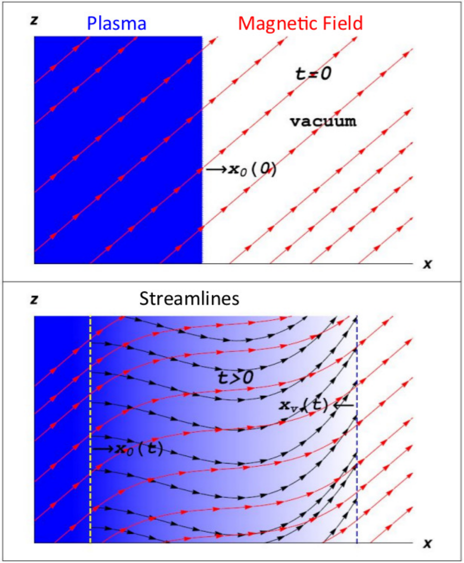
Governing Equations



Governing Equations



Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0,$$

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \vec{V} \cdot \nabla \vec{V} + \frac{1}{\mu_0} \vec{B} \times \left(\nabla \times \vec{B}\right) + \nabla p = 0,$$

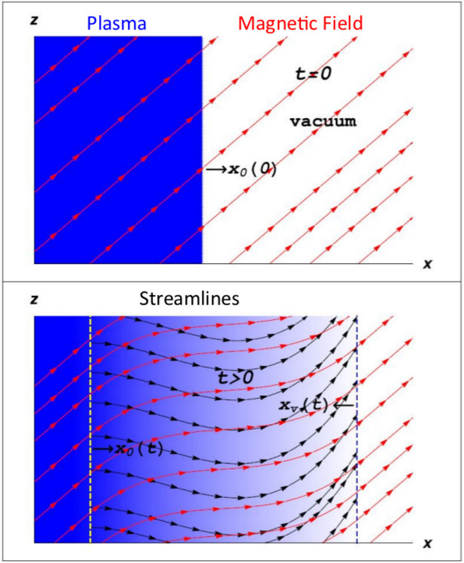
$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}.$$

Maxwell's equations

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times \left(\vec{V} \times \vec{B}\right) = 0,$$

$$\nabla \cdot \vec{B} = 0.$$

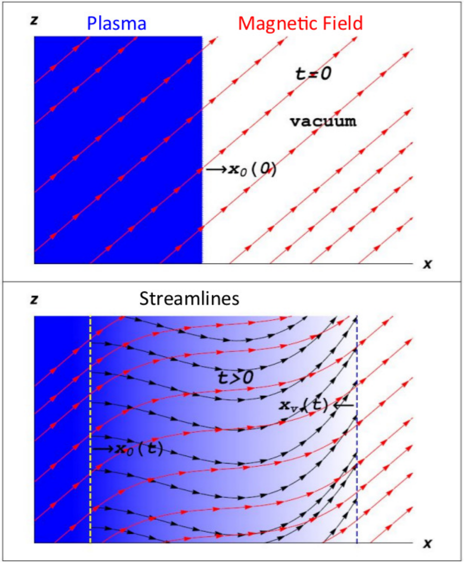
Boundary Conditions (I)



Boundary Condition at $x \to -\infty$

- ✓ Recovery of the **unperturbed plasma**.
- ✓ **Frontier** $x = x_0(t)$ travelling at constant velocity towards the unperturbed plasma.
- ✓ **None perturbation** for $x < x_0(t)$.

Boundary Conditions (II)



Border with the Vacuum

- ✓ **Frontier** $x = x_v(t)$ travelling at constant velocity towards $x > 0$.
- ✓ It is a **fluid surface**:

$$V_x(t, x = x_v) = \mathrm{d}x_v(t)/\mathrm{d}t.$$

- ✓ **Momentum conservation** yields **jump conditions**.

Self-Similarity and Dimensionless Equations

✓ **No characteristic length** → Dependence on $\xi = x/(c_0 t)$.

✓ **Dimensionless variables**:

$$n(\xi) = \rho/\rho_0, \quad b_z(\xi) = B_z/B_0, \quad u(\xi) = V_x/c_0, \quad w(\xi) = V_z/c_0.$$

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Dimensionless governing equations

$$\begin{gathered}
(u-\xi)\, n' + n u' = 0, \\
n\,(u-\xi)\, u' + n^{\gamma-1} n' + \frac{\left(b_z^2\right)'}{\beta} = 0, \\
n\,(u-\xi)\, w' - \frac{2\cos\alpha}{\beta} b_z' = 0, \\
(u-\xi)\, b_z' + b_z u' - w' \cos\alpha = 0.
\end{gathered}$$

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$$(u-\xi)\, b_z' + b_z u' - w' \cos\alpha = 0.$$

Type of solutions

✓ Trivial solution: **plateau**.

$$n, u, w, b_z = \text{const}.$$

✓ **Non-linear waves**: Dispersion relation.

$$f\,(n, u, w, b_z; \xi) = 0.$$

Self-Similarity and Dimensionless Equations

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Governing parameters

α = Initial magnetic field inclination

Type of solutions

✓ Trivial solution: **plateau**.

$$n, u, w, b_z = \text{const}.$$

✓ **Non-linear waves**: Dispersion relation.

$$f\,(n, u, w, b_z; \xi) = 0.$$

$$\beta = \frac{2\gamma\mu_0 p_0}{B_0^2} = 2\left(\frac{c_0}{v_{A0}}\right)^2$$

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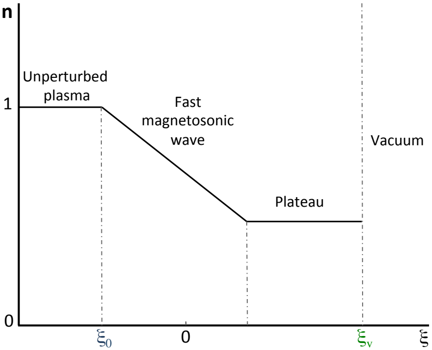
Dispersion Relation

$$\alpha = \frac{\pi}{2},\ B_x = 0,\ w = 0$$

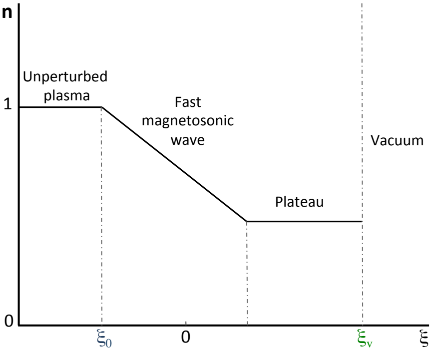


✓ **Dispersion relation**: Magnetosonic wave.

$$(u - \xi)^2 = n^{\gamma - 1} + 2\frac{b_z^2}{n\beta},$$

Dispersion Relation

$$\alpha = \frac{\pi}{2},\ B_x = 0,\ w = 0$$



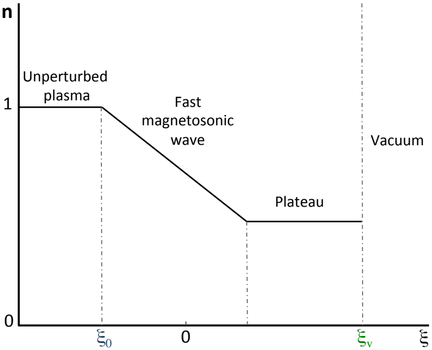
- ✓ **Dispersion relation**: Magnetosonic wave.

$$(u-\xi)^2 = n^{\gamma-1} + 2\frac{b_z^2}{n\beta},$$

$$(u-\xi)^2 = \left(\frac{c}{c_0}\right)^2 + \left(\frac{v_A}{c_0}\right)^2.$$

Dispersion Relation

$$\alpha = \frac{\pi}{2}, \; B_x = 0, \; w = 0$$



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- ✓ The **unperturbed plasma** can be retrieved by the magnetosonic wave.

$$\xi_0 = -\sqrt{1+\frac{2}{\beta}}.$$

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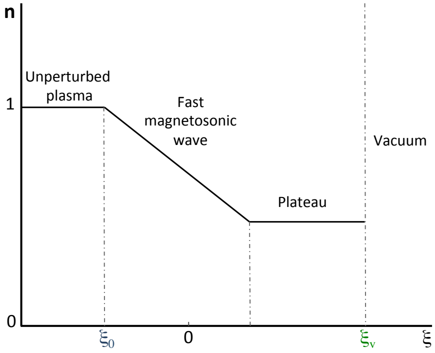

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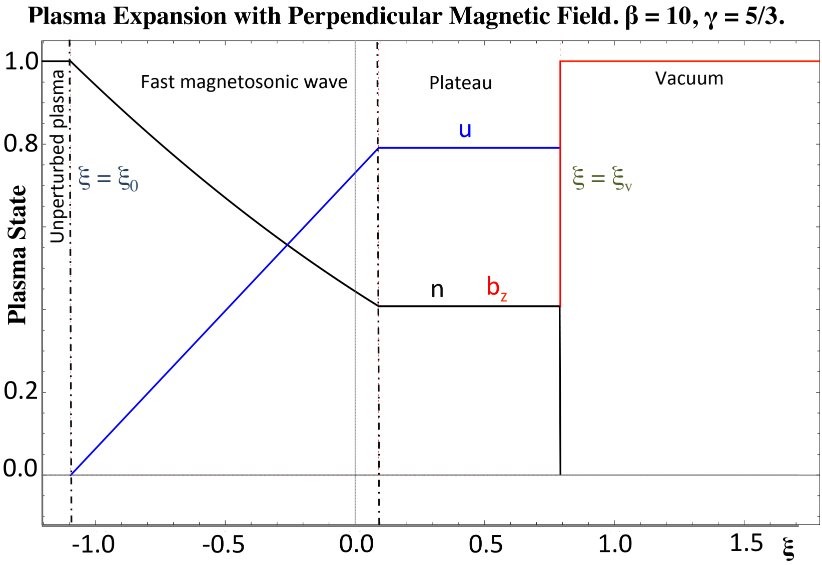
$$(u - \xi)^2 = \left(\frac{c}{c_0}\right)^2 + \left(\frac{v_A}{c_0}\right)^2.$$

- ✓ The **unperturbed plasma** can be retrieved by the magnetosonic wave.

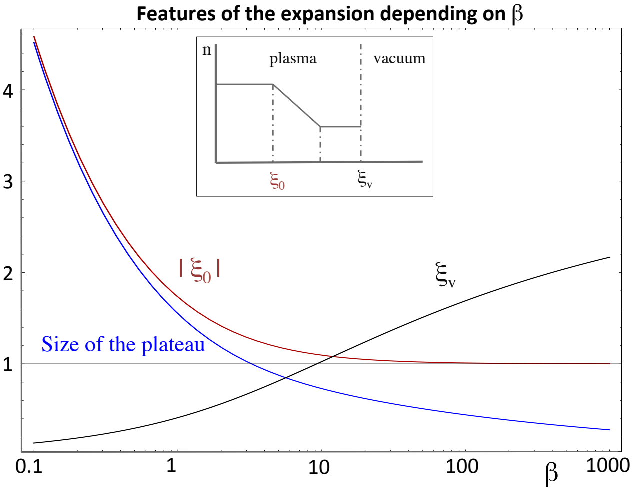
$$\xi_0 = -\sqrt{1 + \frac{2}{\beta}}.$$

- ✓ The **frontier with the vacuum** **cannot** be retrieved by the magnetosonic wave.

Solution for $\gamma = 5/3$



Effect of the Magnetic Field Intensity



$\beta \ll 1$, **Strong magnetic field**	$\beta \gg 1$, **Weak magnetic field**
High confinement	Standard fluid

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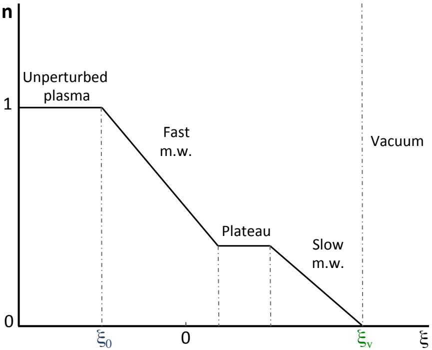
Dispersion Relation

$$\alpha < \frac{\pi}{2},\ B_x = \text{const} \neq 0,\ w \neq 0$$

$$(u-\xi)^2 = \frac{1}{2}\left(\frac{c}{c_0}\right)^2 + \frac{1}{2}\left(\frac{v_A}{c_0}\right)^2 \pm \frac{1}{2}\sqrt{\left[\left(\frac{c}{c_0}\right)^2 + \left(\frac{v_A}{c_0}\right)^2\right]^2 - 4\left(\frac{c}{c_0}\right)^2\left(\frac{v_A}{c_0}\right)^2\cos^2\theta}.$$



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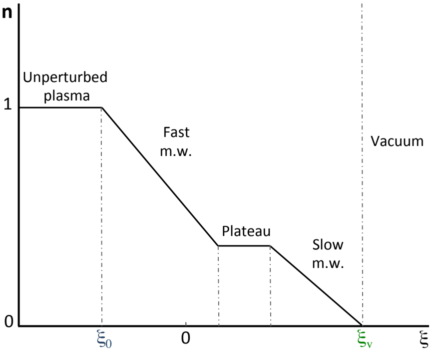


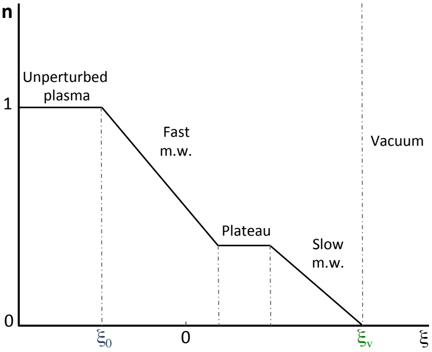
Fast Magnetosonic Wave.

- ✓ Sign $+$,
- ✓ Connects with $\xi = \xi_0$.

Dispersion Relation

$$\alpha < \frac{\pi}{2},\ B_x = \text{const} \neq 0,\ w \neq 0$$

$$(u-\xi)^2 = \frac{1}{2}\left(\frac{c}{c_0}\right)^2 + \frac{1}{2}\left(\frac{v_A}{c_0}\right)^2 \pm \frac{1}{2}\sqrt{\left[\left(\frac{c}{c_0}\right)^2 + \left(\frac{v_A}{c_0}\right)^2\right]^2 - 4\left(\frac{c}{c_0}\right)^2\left(\frac{v_A}{c_0}\right)^2\cos^2\theta}.$$



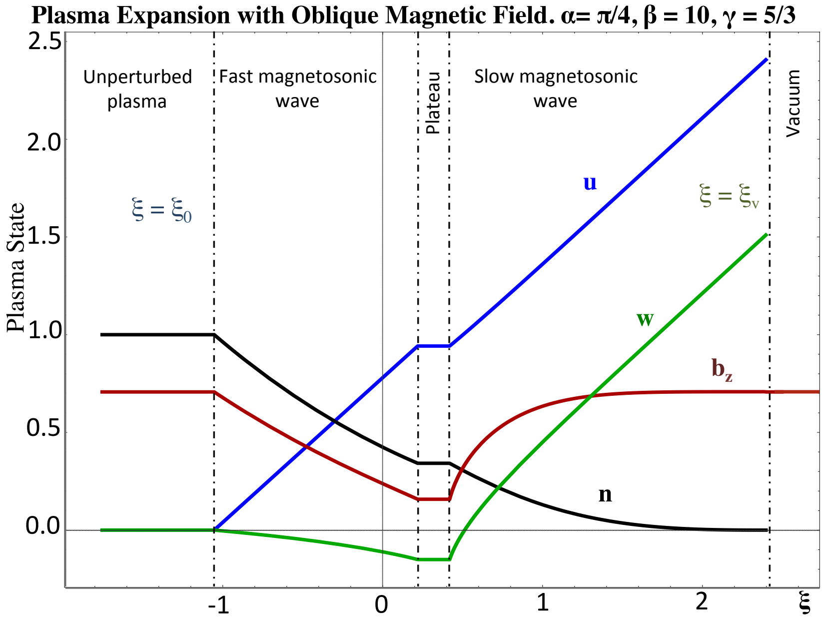
Fast Magnetosonic Wave.

- ✓ Sign $+$,
- ✓ Connects with $\xi = \xi_0$.

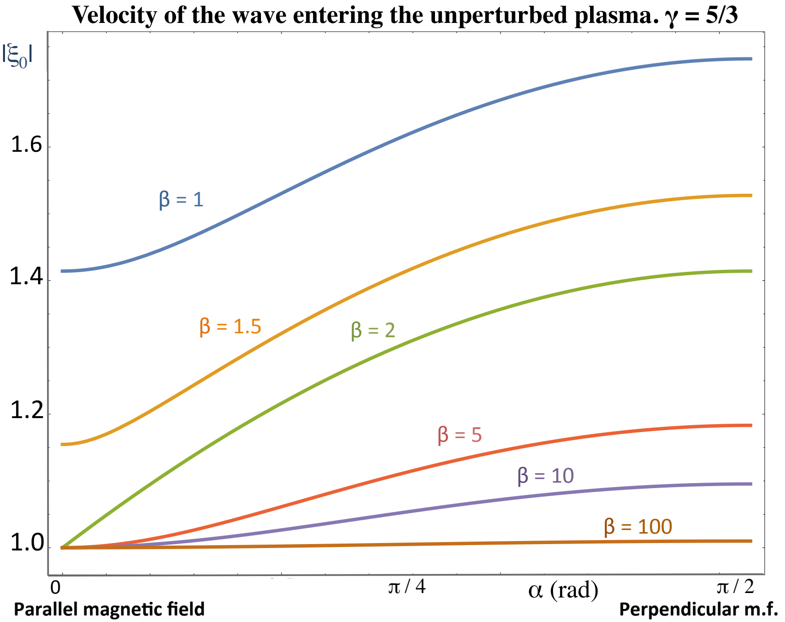
Slow Magnetosonic Wave.

- ✓ Sign -,
- ✓ Connects with $\xi = \xi_v$,
- ✓ Weak discontinuity.

Solution for $\gamma = 5/3$

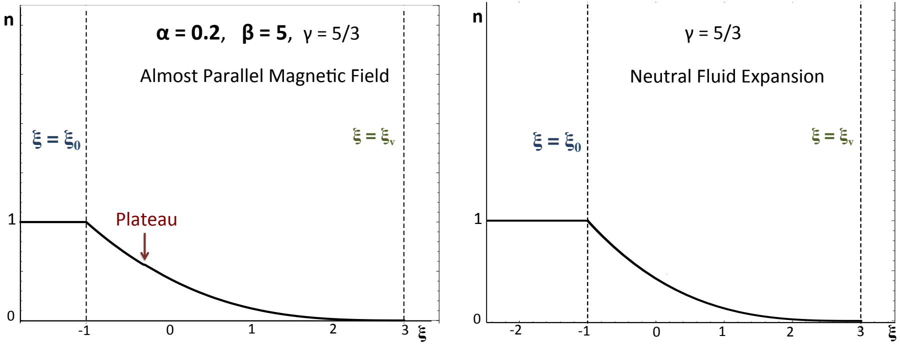


Effect of the Inclination of the Magnetic Field Lines



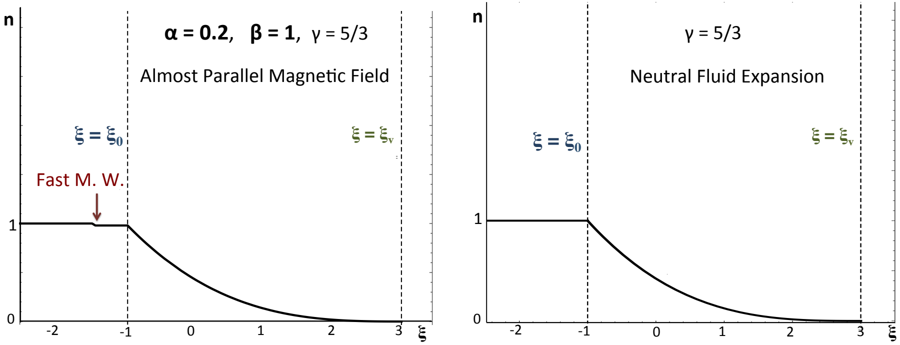
Recovery of the Standard Fluid Expansion, $\beta > 2$

$$\beta = 2\left(\frac{c_0}{v_{A0}}\right)^2$$



Recovery of the Standard Fluid Expansion, $\beta < 2$

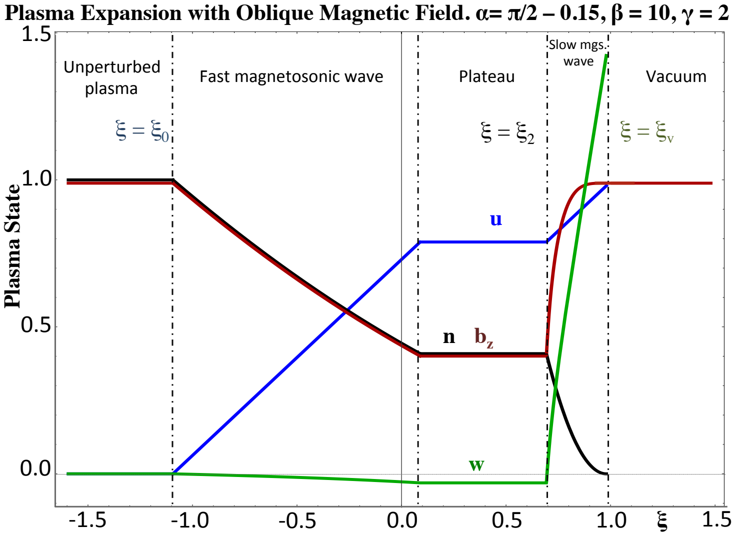
$$\beta = 2\left(\frac{c_0}{v_{A0}}\right)^2$$



Recovery of the Perpendicular Magnetic Field Expansion

$|\alpha - \pi/2| = \eta \ll 1$

$\xi_v - \xi_2 = \mathcal{O}(\eta^2)$



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Conclusions

- ✓ Plasma expansion into vacuum with an external magnetic field is a **self-similar** process.
- ✓ Solution: **piecewise matching non-linear waves** and **plateaus**.

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- ✓ Solution: **piecewise matching non-linear waves** and **plateaus**.
- ✓ **Perpendicular** magnetic field → **Density discontinuity** at the edge of the expansion.

Conclusions

- ✓ Plasma expansion into vacuum with an external magnetic field is a **self-similar** process.
- ✓ Solution: **piecewise matching non-linear waves** and **plateaus**.
- ✓ **Perpendicular** magnetic field → **Density discontinuity** at the edge of the expansion.
- ✓ **Oblique** magnetic field → **Weak discontinuity** at the edge of the expansion.

Conclusions

- ✓ Plasma expansion into vacuum with an external magnetic field is a **self-similar** process.
- ✓ Solution: **piecewise matching non-linear waves** and **plateaus**.
- ✓ **Perpendicular** magnetic field → **Density discontinuity** at the edge of the expansion.
- ✓ **Oblique** magnetic field → **Weak discontinuity** at the edge of the expansion.
- ✓ The **net effect** of the magnetic field is to **confine** the expansion.

Conclusions

- ✓ Plasma expansion into vacuum with an external magnetic field is a **self-similar** process.
- ✓ Solution: **piecewise matching non-linear waves** and **plateaus**.
- ✓ **Perpendicular** magnetic field $\rightarrow$ **Density discontinuity** at the edge of the expansion.
- ✓ **Oblique** magnetic field $\rightarrow$ **Weak discontinuity** at the edge of the expansion.
- ✓ The **net effect** of the magnetic field is to **confine** the expansion.
- ✓ Serve as **test case** for M.H.D. code validation.

F. García Rubio, A. Ruocco, J. Sanz, *"Plasma expansion into a vacuum with an arbitrarily oriented external magnetic field"*. Phys, Plasmas **23**, 012103 (2016).

THANK YOU FOR YOUR ATTENTION

Any questions?

Boundary Conditions

Recovery of the **unperturbed plasma** at $x \to -\infty$:

- ✓ Frontier $x = x_0(t)$ travelling at constant velocity towards the unperturbed plasma.
- ✓ None perturbation for $x < x_0(t)$.

Border with the vacuum

- ✓ Frontier $x = x_v(t)$ travelling at constant velocity towards $x > 0$.
- ✓ It is a fluid surface:

$$V_x(t, x = x_v) = \mathrm{d}x_v(t)/\mathrm{d}t.$$

- ✓ Momentum conservation impose:

$$p_0 \left(\frac{\rho(t, x_v)}{\rho_0} \right)^{\gamma} + \frac{B_z^2(t, x_v)}{2\mu_0} = \frac{B_0^2 \sin^2 \alpha}{2\mu_0},$$

$$\cos \alpha (B_z(t, x_v) - B_0 \sin \alpha) = 0.$$

- ✓ **Perpendicular magnetic field** $(\alpha = \pi/2) \to$ **Density jump**.
- ✓ **Oblique magnetic field** $(\alpha < \pi/2) \to$ **Weak discontinuity**, $\rho(t, \xi_v) = 0$.

Asymptotic Expressions

Strong magnetic field $\beta \ll 1$

$$n_p = 1 - \frac{\beta}{\gamma} + \mathcal{O}(\beta^2),$$

$$\xi_0 = -\sqrt{\frac{2}{\beta}} + \mathcal{O}(\beta^{1/2}),$$

$$\xi_v = \frac{1}{\gamma}\sqrt{\frac{\beta}{2}} + \mathcal{O}(\beta^{3/2}).$$

Weak magnetic field $\beta \gg 1$

$$n_p = \left(\frac{\gamma}{\beta}\right)^{1/\gamma} + \mathcal{O}(\beta^{-3/\gamma}),$$

$$\xi_0 = -1 - \frac{1}{\beta} + \mathcal{O}(\beta^{-2}),$$

$$\xi_v = \frac{2}{\gamma - 1}\left[1 - \left(\frac{\gamma}{\beta}\right)^{(\gamma-1)/(2\gamma)}\right] + \mathcal{O}(\beta^{-1}).$$

Analytical Solution for $\gamma = 2$

- ✓ When $\gamma = 2$, the system of governing equations is **analytically integrable**.

- ✓ **Magnetosonic wave structure**:

$$u = 2\sqrt{1 + \frac{2}{\beta}}\left(1 - \sqrt{n}\right).$$

$$b_z = n \text{ (also true for a general } \gamma\text{)},$$

- ✓ Plasma state at the **plateau**, its **foot** and **end**:

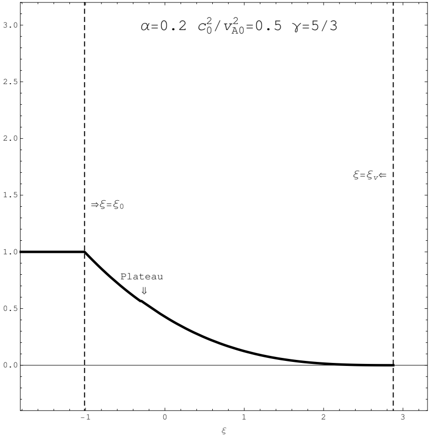
$$n_p = b_{zp} = \sqrt{\frac{2/\beta}{1 + 2/\beta}},$$

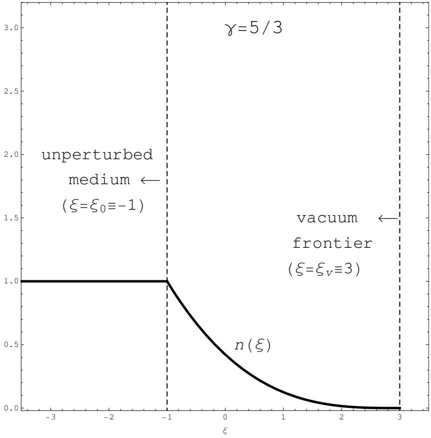
$$u_p = \xi_v = 2\sqrt{1 + 2/\beta}\left(1 - \sqrt[4]{\frac{2/\beta}{1 + 2/\beta}}\right),$$

$$\xi_1 = 2\sqrt{1 + 2/\beta}\left(1 - \frac{3}{2}\sqrt[4]{\frac{2/\beta}{1 + 2/\beta}}\right).$$

Recovery of the Standard Fluid Expansion

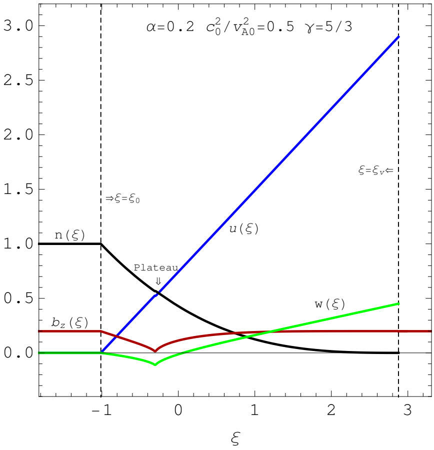
$\beta > 2$





Recovery of the Standard Fluid Expansion

$\beta = 5$



$\beta = 1$