The Classification of Fuzzy Subgroups of some Finite Abelian $p-$Groups of rank 3

By

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Abstract

An important trend in fuzzy group theory in recent years has been the notion of classification of fuzzy subgroups using a suitable equivalence relation. In this dissertation, we have successfully used the natural equivalence relation defined by Murali and Makamba in [81] and a natural fuzzy isomorphism to classify fuzzy subgroups of some finite abelian p-groups of rank three of the form $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for any fixed prime integer p and any positive integer n. This was achieved through the usage of a suitable technique of enumerating distinct fuzzy subgroups and non-isomorphic fuzzy subgroups of G. We commence by giving a brief discussion on the theory of fuzzy sets and fuzzy subgroups from the perspective of group theory through to the theory of sets, leading us to establish a linkage among these theories. We have also shown in this dissertation that the converse of theorem 3.1 proposed by Das in [24] is incorrect by giving a counter example and restate the theorem. We have then reviewed and enriched the study conducted by Ngcibi in [94] by characterising the non-isomorphic fuzzy subgroups in that study. We have also developed a formula to compute the crisp subgroups of the under-studied group and provide its proof. Furthermore, we have compared the equivalence relation under which the classification problem is based with various versions of equivalence studied in the literature. We managed to use this counting technique to obtain explicit formulae for the number of maximal chains, distinct fuzzy subgroups, non-isomorphic maximal chains and non-isomorphic fuzzy subgroups of these groups and their proofs are provided.

KEYWORDS : Fuzzy subgroups, normal subgroups, level subgroups, alpha-cuts, equivalent fuzzy subgroups, isomorphic fuzzy subgroups.

Contents

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DEDICATION

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Chapter 1

INTRODUCTION

1.1 A BRIEF HISTORICAL ACCOUNT OF GROUP THEORY

In the 19th century, assortments of algebraic concepts were propounded by the heavy weights in mathematics. However, some of these concepts did not gain the needed pivotal recognition in mathematics as more emphases were placed on the development of the fundamental algebraic structures. The concept of a group is unquestionably one of the best tools forging algebraic homogeneity. Generally, contemporary algebra has the group-properties as the cohesive force which networks the various algebraic concepts. It is therefore conclusive that the group-theoretic techniques have enriched algebra in totality. The notion of a group commenced its activity in a solidified mode. However, its abstract characterization, which we will be highlighting later in the preliminaries, was not easily resolved until the early years of the 20th century.

Group theory, which can be described as the study of the algebraic structures referred to as a group in mathematics, specifically abstract algebra, has a long and rich history in its evolution. We briefly highlight three historical roots, namely: number theory, the theory of algebraic equations and geometry, which had earlier been researched into by Lagrange, Abel and Galois [66].

Originally, the concept of a group and its name is largely an invention of the 19th century. Its antecedents, however, can be traced back at least to Lagrange in the late 18th century and his study of roots of polynomials, though this depended heavily upon a pre-natal work from the mid-17th century, [36], [93] and [121].

However, it is Galois and Cauchy who are accredited with the major promotion of group theory. Even though Galois did not categorically define a "group" with adequate axioms, he was the first to use it. Furthermore, in his final notes on the papers he wrote the night prior to his untimely death, he did mention "closure axioms", a manifestation that he had a "closed system" or an "object". Analogous to Galois's work, Ruffini and Abel studied the problem of the solvability of the general quintic equation by radicals though their definition of a group was not straightforward. However, they really had a great notion of Galois' object. Cauchy, in 1844 attuned with Ruffini's work and part of Galois' previous work, which provided a characterization of an entity he termed a "conjugate system of substitutions" [121].

Thereafter, an extensive effort was initiated by Cayley in 1854 to give "group" an entirely abstract character. Even though Cayley is being acknowledged by few authors for the usage of the name "group" in its contemporary abstract taste for the inaugural stage, Boole and others in the 1840s, as asserted by Kline, cannot be overlooked [67]. In 1870, Jordan endorsed Galois' idea of a " group" which then became the accepted name [37]; [44]; [64]; [65]; [70]; [77]; [121].

Kronecker in 1870 came up with another almost-definition of a group. However, he did not link his notion to group theory [121]. Thereafter, Von Dyck and Weber in 1882 [18] both presented definitions of a group, again close to the contemporary one . Burnside in 1897 defined a group in terms of closure, associativity and inverse [66]. Simultaneously, Frobenius, Holder and Weber made headway in the direction of the contemporary definition independent of Burnside's, [52]; [76]; [107].

Hasse [51] in 1926, used the equivalence relation defined in algebra as a generalization of the notion of equality between elements of a set in the realm of group theory and defines a relation on the group G by letting aRb if $ab^{-1} \in H$. It is proved that this is an equivalence relation and the right cosets are the equivalence classes. The right cosets thus form a partition, (a decomposition of a set into disjoint subsets whose union is the set), of the group G. Delsarte [30] painstakingly studied all subgroups of a given group. Vogt [119] also studied the lattice of subgroups.

Here are some of the recent works in the realm of abstract group theory that will be useful in this dissertation.

Thomas Stehling [109] in 1992 studied how to compute the number of sub-

groups of a finite abelian group. A new method of proving some classical theorems of abelian groups was established by *Tărnăuceanu* [112] in 2007 and subsequently he presented an arithmetic method of counting the subgroups of a finite abelian group[113] in 2010. On the other hand, Grigore *Călugăreanu* [16] also worked on the total number of subgroups of a finite abelian group. Mario Hampejs and *László Tóth* [50] worked on the subgroups of finite abelian groups of rank three and proposed that the total number of subgroups of $(\mathbb{Z}_p)^3$ is determined by $s(p) = \sum_{k=0}^{3} \begin{bmatrix} 3 \\ k \end{bmatrix} = 2\left(p^2 + p + 2\right)$ where $\begin{bmatrix} 3 \\ k \end{bmatrix}$ is the Gaussian coefficient for $0 \leq k \leq 3$ and an arbitrary positive integer $1 \leq$ p ≤ 50.
Based on the above developments, group theory can be seen as a twentieth century phenomenon, since it is the century in which it received massive and rapid advancements. This paved the way for conventional set theory which in essence has enhanced the abstract nature of group theory.

1.2 A BRIEF HISTORICAL ACCOUNT OF SET THEORY

Unlike most fields in mathematics where their emergence involves more than one mathematician, conventional set theory which forms part of mathematical logic, being described as the study of sets, is an autonomous field and rather different. The whole transfinite landscape can be viewed as having been articulated by Cantor in significant part to solve the Continuum Problem.
Though initiated by Georg Cantor, his decisions and ability to forge on were influenced by Richard Dedekind [28] in the 1870s. However, it was Cantor's paper [17], published in 1874, that positioned set theory on a suitable mathematical footing. The notion of infinity was a great concern from the time of the Greeks, one of whom is Zeno of Elea who made a great contribution around 450BC through his investigation on the infinite, see [130]. But for the intervention of Ernst Zermelo [126] (1904), set theory was nearly rejected due to numerous paradoxes discovered by Cantor himself and many other mathematicians of his time.
In 1904 Zermelo [126] officially devised, tested and proved the axiom of choice when he established that every set can be well-ordered, which is Cantor's conjecture. He further developed the workable axiomatic set theory in 1908 [127], which was improved upon by Fraenkel [42] in 1922, (The axiomatic set theory

- ZF). *Gödel* [46] in 1940 tested the Zermelo-Fraenkel axiom of choice while proving the famous consistency of continuum hypodissertation which was the brain-child of Georg Cantor, leading to Gödel's prestigious inner theorem. Subsequently, Paul Cohen [23] in 1963 established that the axiom of choice is an exclusive of the alternative axioms of set theory.

1.3 A BRIEF HISTORICAL ACCOUNT OF FUZZY SET AND FUZZY GROUP THEORY

As characterized by Cantor [17] and axiomatised by Zermelo [127] and Fraenkel [42] in conventional set theory, an object is either an element of a set or not. In fuzzy set theory however, this status quo was liberalized by Lotfi A. Zadeh [123]. According to him, an object has a degree of membership in a set, a number between 0 and 1. He therefore described a fuzzy set as a collection of objects with imprecise boundaries in which the transition from membership to non-membership is gradual rather than abrupt. For instance, the grade of membership of an individual in the set of "tall people" is more relaxed than a simple yes or no answer and can be a real number such as 0.85. Although fuzzy set was the brain-child of Zadeh[123] in 1965, its root could be traced back to a multiplicity of notable thinkers such as Plato, Georg Wilhelm Friedrich Hegel, Karl Marx , Jan Lukasiewicz and others. Furthermore Buddha, the founder of Buddhism, played a momentous role towards the advancement of fuzzy logic around 500BC. His beliefs were based on the notion that the universe is packed with contradictions, that virtually everything comprises a number of its reverse, or that things can be A and not A at the same time, see [129]. In the early part of the 20th century, Brouwer [53] was even led to question the validity of the law of excluded middle, a basic law of logic which states that every statement is either true or false. Brouwer reasoned that there may be a third possibility, that is three-valued logic. This viewpoint was later shared by Bertrand Russell [102] in 1923 and subsequently by Charles Sanders Peirce [98] in 1931 that "logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought".
It is without any doubt that there is a correlation between Buddha's belief and the modern fuzzy logic which forms part of fuzzy mathematics. After the introduction of the concept of a fuzzy set by Zadeh [123], considerable

facets of fuzzy subsets were studied. In 1971, Rosenfeld [101] imported the notion of fuzzy sets to the domain of group theory and coined the notion of fuzzy subgroups of a group. As a matter of fact, he showed how manifolds of fundamental properties in group theory should be extended in a basic way to establish the theory of fuzzy groups. Though Anthony and Sherwood [6] in 1979 redefined fuzzy groups, Rosenfeld's definition appears to be the most essential and accepted one. Since then, the classification of fuzzy subgroups of a finite group has been one of the most considered problems in fuzzy theory. This topic has enjoyed a constant evolution since its inception.

Based on the Rosenfeld's definition of fuzzy group, several fuzzy algebraic concepts were investigated and vigorous attempts have been made to "fuzzify" a number of important classical mathematical structures such as topological spaces, algebras, categories and groups and also to consider fuzzy automata, fuzzy programmes, fuzzy graphs, fuzzy probability, and so on. P. Das [24] in 1981 characterized all fuzzy subgroups of finite cyclic groups using what he termed "level subgroups" of a fuzzy subgroup which is based on the concept of a "level subset" introduced earlier by Zadeh [123]. Das's definition of level subgroup was transformed by Ajmal [2] by restricting $t \in \text{Im } \mu$. This modified definition of level subgroups has been used by Jain and Ajmal [61] to define a new category of fuzzy subgroups. We give both definitions of level subgroups in this dissertation. Alkhamees [5] continued then with the investigation of fuzzy cyclic subgroups based on fuzzy cyclic p-groups. In a series of papers, Mukherjee and Bhattacharya [10], [11], [12], [13] have developed fuzzy analogs of a number of notions in classical group theory, and proved fuzzy generalization of a couple of valuable theorems such as Lagrange's and Cayley's theorems, thereby enriching the theory of fuzzy groups. Bhattacharya [13] in 1987 showed that two fuzzy subgroups of a finite group with identical level subgroups are equal if and only if their image sets are equal. Further, Bhattacharya characterized all fuzzy subgroups of a finite group, thereby generalizing the earlier results by Rosenfeld [101] and P. Das[24]. Extending the effect of group homomorphisms on fuzzy groups as earlier studied by Rosendeld [101], Anthony and Sherwood [6], Sidky and Mishref [108], Kumar [69] and Akgul [4], Sebastian and Babu Sunder [104] investigated group homomorphism on the chains of level subgroups of an arbitrary fuzzy group and obtained generalizations of the earlier results on finite level cardinality in [125]. B.B. Makamba [72] extended the notion of fuzzy normality introduced by Mukherjee and Bhattacharya to

the notion of normality of a fuzzy subgroup in another fuzzy group. Murali and Makamba [84] have determined the number of distinct fuzzy subgroups of Abelian groups of the order $p^n q^m$ under suitably defined equivalence relation on fuzzy subgroups of a group, where p and q are different primes. Zhang and Zou [128] considered a similar problem and obtained the number of fuzzy subgroups of cyclic groups of order p^n, where p is a prime number. Several other notions of equivalence of fuzzy subgroups were studied by the following researchers, Volf [120], Branimir and Tepavcevic [15], Degang et al [29], Tarnauceanu and Bentea [116], Ghafur and Sulaiman [110], Ajmal [3], Mordeson [78], Dixit et al [38], [39], Zhang and Zou [128], Jain [60], Tarnauceanu [115], Mashinchi and Mukaidonon [75] and Iranmanesh and Naraghi [55]. Makamba and Murali in [74] used a group theoretical-property to obtain fuzzy subgroups and continued to establish a congruence relation. The notion of external direct product of fuzzy subgroups was a brain-child of Sherwood [105], while the idea of the internal direct product of fuzzy subgroups was introduced by Makamba [73]. Rosenfeld [101] established that a homomorphic image of a fuzzy subgroup is a fuzzy subgroup if and only if the fuzzy subgroup has a sup-property. A homomorphic pre-image of a fuzzy subgroup is always a fuzzy subgroup. Subsequently, Anthony and Sherwood [6] showed that the homomorphic image of a fuzzy subgroup is a fuzzy subgroup irrespective of the sup-property. Some of the most recent investigations into homomorphic images and pre-images of fuzzy subgroups were studied by the likes of Makamba [73], Murali [80], Abou-Zaid [1], Sidky and Mishref [108] and Kumar [69]. Kaufmann [63] examined the fuzzy relation that was earlier defined on a set by Zadeh [123], [125] and subsequently dealt with by Rosenfeld [101]. M.K.Chakraborty and M.Das [19], [20], [21] also considered fuzzy relations in connection with fuzzy functions and equivalence relations.
In parallel to Das and Chakraborty's [19] work, Murali and Makamba in a series of papers [81], [82],[83], [84] studied fuzzy relations and then a natural equivalence relation on the class of all fuzzy sets of a set. The aforementioned was used to established the number of distinct equivalence classes of fuzzy subgroups of p-groups. Murali and Makamba [82] further used keychains to classify fuzzy subgroups of some finite groups. The concept of a pinned flag was also introduced by Murali and Makamba [83] to deal with the operations such as union, intersection and sum associated with the aforementioned natural equivalence. O. Ndiweni [89] used the natural equivalence relation defined

by Murali and Makamba [81], [82],[83], [84] to characterize fuzzy subgroups of some finite groups and further determined the number of equivalence classes of fuzzy subgroups, specifically for the symmetric group S_3, dihedral group D_4, the quaternion group Q_8, cyclic p-groups $G = \mathbb{Z}_{p^n}$; $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$; $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m} + \mathbb{Z}_r$; and $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m} + \mathbb{Z}_{r^s}$, where p,q,r are all distinct primes and n,m,s are all positive integers and its extension. Sulaiman and Ghafur [111] worked on a particular case of finite cyclic groups namely $G = \mathbb{Z}_p + \mathbb{Z}_q + \mathbb{Z}_r + \mathbb{Z}_s$ with p, q ,r , s all distinct prime numbers. Humera and Raza [54] determined the number of fuzzy subgroups of finite Abelian groups $G = \mathbb{Z}_{p^n} + Z_q$ and $G = \mathbb{Z}_{p_1} + \mathbb{Z}_{p_2} + \cdots + \mathbb{Z}_{p_n}$, where $p_1, p_2, ..., p_n$ are all distinct primes. They also determined the number of fuzzy subgroups of the Abelian groups. In both groups, the equivalence relation used on the fuzzy subgroups of any group G is the one by Sulaiman and Abdul Ghafur [111]. Ngcibi [94] classified the fuzzy subgroups of finite Abelian groups of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ for any prime p and $n = 1, 2, 3$. Further they developed the general formulae for the number of maximal chains, the number of crisp subgroups and, for the case $n = m$, the number of non-equivalent fuzzy subgroups. S. Ngcibi [95] used the equivalence relation characterized by Murali and Makamba [81], [82],[83], [84] to investigate the number of the distinct equivalence classes of fuzzy subgroups of a finite abelian p- group G of rank two and devised formulae for both the number of maximal chains and the number of distinct equivalence classes of fuzzy subgroups. M. *Tărnăuceanu* and L. Bentea [117] successfully used a recurrence relation indicated in [117] to count the number of distinct fuzzy subgroups for two classes of finite abelian groups, namely, finite cyclic groups and finite elementary abelian p-groups and have given an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ where p is a prime. This may be considered as one of the remarkable studies towards our study. The same authors, later in [114] extended their study of classifying fuzzy subgroups of abelian finite groups to a class of non-abelian groups, specifically hamiltonian groups. They obtained an explicit formula for the number of distinct fuzzy subgroups of a finite hamiltonian group in a particular case. Esengul Salturk and Irfan Siap [103] also used the equivalence relation defined by Murali and Makamba in [81] to examine the structure of equivalence classes of fuzzy subgroups of $\mathbb{Z}_p^n$ for any given prime p and positive integer n indicating the rank of the group. It has become an important problem in fuzzy group theory to classify fuzzy subgroups of finite groups. Without any equivalence relation on fuzzy subgroups of a group, the number of fuzzy subgroups is infinite even for the trivial group $\{e\}$. In this dissertation we will briefly compare the different notions of fuzzy equivalence relations studied by other researchers, however, we will focus and use the notion studied by Murali and Makamba [81], [82],[83]. These equivalence relations have been used to characterise fuzzy subgroups of finite groups, however different results have been obtained in some cases on the same groups. This is primarily because the classification and enumeration techniques used in each case depend on both the type of fuzzy equivalence relation and the counting techniques developed from that particular concept of fuzzy equivalence. In this dissertation we focus on fuzzy subgroups of an abelian group G of rank 3. In particular we use $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$, where p is any fixed prime and any arbitrary positive integer n. This is an extension of Ngcibi's work in [94].

Chapter 2

PRELIMINARIES

2.1 INTRODUCTION

As mentioned earlier, the conventional set theory is built on the primary notion of “set” of which an entity is either an element or not an element. A sharp, crisp, and obvious dissimilarity exists between an associate and a non-associate for any well-defined “set” of objects in this theory. Moreover, there is an extremely defined and clear limit to specify if an object belongs to the set. So given a set A, an object x either is in A or not in A. This is true for both the deterministic and the stochastic logic. Furthermore, one might pose a question such as “What is the likelihood that the object x is an element of A?” in the domain of statistics and probability. In this logic though, a response might be akin to “The likelihood for this object to be in set A is 95 percent”. The ultimate result (i.e. conclusion) is still either “it is” or “it is not” a part of the set. The probability of 95 percent does not imply that the object has 95 percent membership in the set or that it has 5 percent non-membership. Specifically, in the conventional set theory, it is unacceptable that an element is in a set and not in the set at the same time. As a result, several real-world application problems cannot be illustrated and handled by the conventional set theory, as well as all those elements having a partial membership to a set. On the contrary, fuzzy set theory does acknowledge incomplete memberships, and thus generalizes the conventional set theory to some extent.

In order to introduce the notion of fuzzy sets and fuzzy subgroups, we first appraise some fundamental concepts in the basic set theory and group theory of the conventional mathematics. It will be seen that the fuzzy set theory is a very natural extension of the conventional set theory, and is also a rigorous mathematical concept.

2.2 BASIC DEFINITIONS AND CONCEPTS IN THE CONVENTIONAL SET THEORY

Definition 2.2.1 A set is a collection of clearly defined objects referred to as elements or members of the set. Usually we refer to the whole under-study set as universe of discourse, and denoted by U and if an element x is in set U, we write $x \in U$, otherwise we use the notation $x \notin U$ and read it as x is not in U.

Normally we denote sets by using upper case letters.

Definition 2.2.2 Let U be the universe of discourse which includes all possible elements associated with the given problem. If we define a set S in the universe set U such that there exists an element $x \in U$ but $x \notin S$, then we say S is a proper subset of the set U and we write $S \subset U$.
On the other hand, S is referred to as improper subset of U and we denote it as $S \subseteq U$ if whenever $x \in S$ then $x \in U$.

However if S is not contained in U, we represent it as $S \nsubseteq U$ and refer to S as not a subset of U.
A set that has no element is called an empty set and we denote it by $\emptyset$ or $\{\}$.
An empty set is a subset of any set.
Basically, in the conventional set theory, a finite set can be represented by enumerating all its elements using $S = \{s_1, s_2, s_3..., s_n\}$ for some finite positive number n. Apart from this representation, sets can be defined by specifying the conditions of the elements. For instance, if the elements of set S should satisfy the conditions $b_1, b_2, b_3, ..., b_n$, then the set S is described as $S = \{a : a \text{ satisfies } b_1, b_2, ..., b_n\}$.
Additionally, if these elements $s_i(i = 1, 2, 3, ..., k)$ of S form a subset A of S, then for all elements $x \in S$ the set A can be represented by its indicator or characteristic function $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$
Similarly for S as a subset of U. The above indicator or characteristic function explicitly describes sets A and S. Mathematically, we express these S as

a mapping $S : U \to [0,1]$. So if $S(x) = 1$, then x is a member of S and if $S(x) = 0$ then x is not a member of S. Thus in the conventional set theory $\chi_S(x)$ has only the values 0 ('false') and 1 ('true'). We refer to such sets as crisp sets.
Clearly, one can see that crisp set theory is incapable of representing classifications and descriptions in several cases. In fact, crisp set theory provides inadequate representation for most cases.
If the under-studied set is an n-dimensional Euclidean space, then we represent its size by using the number of elements it contains. This number is called the cardinality. We denote the cardinality of the set S by $|S|$. If the cardinality $|S|$ is a finite number, then the set S is a finite set. On the other hand, if $|S|$ is infinite, then S is an infinite set.

2.3 OPERATIONS ON THE CONVENTIONAL (CRISP) SET

Definition 2.3.1 Let A and B be nonempty sets. Then
(i) The union of A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
Clearly $A \cup B = B \cup A$. (Commutativity)
(ii) The intersection of A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
Clearly $A \cap B = B \cap A$. (Commutativity)
(iii) A and B are equal and we write $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
(iv) The difference of A and B is the set $A - B$ or $A|B = \{x \in A : x \notin B\}|$.
It is also referred to as the relative complement. The complement of set A is denoted by A^c or $\neg A$.
(v) The Cartesian product of A and B is the set $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

2.4 SET-THEORETIC PROPERTIES OR AXIOMS

We review the operational characteristics of union, intersection and complement on the conventional set.

Note 2.1.0. Let A, B and C be sets on the universe of discourse U. Then the following axioms hold
(i) $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$ $(Associativity)$.

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $(Distributivity)$.

(iii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ $(Transitivity)$.

(iv) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ and $A \cap U = A$ and $A \cup U = U$ $(Identity)$.

(v) $A \cap A = A$ and $A \cup A = A$ $(Idempotency)$.

(vi) $(A^c)^c = A$ $(Involution)$.

(vii) $A \cap A^c = \emptyset$ $(Axiom\ of\ Contradiction)$.

(viii) $A \cup A^c = U$ $(Axiom\ of\ the\ Excluded\ Middle)$.

The De Morgan's law is satisfied with the union, intersection and complement operations:
(ix) $(A \cup B)^c = A^c \cap B^c$

(x) $(A \cap B)^c = A^c \cup B^c$
The indicator or characteristic function of a set S is defined by $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$
The above indicates the membership $\chi_S(x)$ in the set S for the element x in the universe of discourse. Each member x of S is assigned to one of the two elements of $\{0, 1\}$.

2.5 SET-THEORETIC OPERATIONS TO FUNCTION-THEORETIC OPERATIONS

Under this subtitle, we first define the set operations and then characterize the set-theoretic operations in terms of functions.
Let S and T be two sets on the universal set U. Then
(i) The union: $S \cup T = \{a \in U : a \in S \text{ or } a \in T\}$ is expressed in terms of membership functions as $\chi_{S\cup T}(a) = \chi_S(a) \vee \chi_T(a) = \max(\chi_S(a), \chi_T(a)) \forall\ a \in U$
(ii) The intersection: $S \cap T = \{a \in U : a \in S \text{ and } a \in T\}$ is defined on membership functions as $\chi_{S\cap T}(a) = \chi_S(a) \wedge \chi_T(a) = \min(\chi_S(a), \chi_T(a)) \forall\ a \in U$
(iii) The complement: S^c or $\neg S = \{a \in U : a \notin S\}$ is given in terms of membership functions as $\chi_{S^c}(a)$ or $\chi_{\neg}S = 1 - \chi_S(a) \forall\ a \in U$
(iv) The difference: $S - T$ or $S \setminus T = \{a \in S : a \notin T\}$ is expressed as $\chi_{S-T}(a)$ or $\chi_{S\setminus T}(a) = \chi_S(a) \wedge \chi_{T^c}(a) = \min(\chi_S(a), 1 - \chi_T(a)) \forall\ a \in U$
(v) The containment $S \subseteq T$ is defined as $\chi_S(a) \leq \chi_T(a) \forall\ a \in U$

2.6 SOME BASIC DEFINITIONS AND CONCEPTS IN GROUP THEORY

Definition 2.6.1 Let S be a nonempty set. A rule that combines pairs such as (x, y) of S to get another element of S (S is closed under the rule) is referred to as a Binary Operation. This operation defines a map $S \times S \to S$.
Employing the multiplicative notation, the operation is said to be
1. associative if $(xy)z = x(yz)$ for all $x, y, z \in S$
(i). If the operation is associative then the product of any n elements (ordered) is well-defined
2. commutative if $xy = yx$ for all $x, y \in S$
3. has an identity 1 if for all $x \in S$, $1x = x = x1$
(i). An element x is said to be invertible if there exists x^{-1} such that $xx^{-1} = 1 = x^{-1}x$.

Applying the additive notation, the identity is denoted by 0 and the inverse of x is denoted by $-x$. The exponents x^n become multiples nx. All results will still be true whichever notation is used.

Definition 2.6.2 A group is a pair $(G, \bullet)$ consisting of a nonempty set G and binary operation
$\bullet : G \times G \to G$ such that the following axioms are satisfied
(i) $g \bullet (h \bullet k) = (g \bullet h) \bullet k, \forall g, h, k \in G$ (associativity)
(ii) $\exists\, e \in G$ such that $e \bullet g = g \bullet e = g \ \ \forall g \in G$ (identity element)
(iii) $\forall g \in G, \exists\, g^{-1} \in G$ such that $g \bullet g^{-1} = g^{-1} \bullet g = e$ (inverse)
(iv) If g has a left and right inverse, they must be equal.
(v) An inverse is unique.
(vi) Inverses multiply in opposite order: $(xy)^{-1} = y^{-1}x^{-1}$.
The group G is said to be commutative or abelian group if the operation $\bullet$ on G satisfies the additional axiom
(vii) Commutativity: $xy = yx$ for all $x, y \in G$.

If the group G is abelian, it is customary to denote the operation additively, using a $(+)$ symbol, and to use the symbol 0 for the identity element.

Definition 2.6.3 The order of a group $(G, \bullet)$ is the cardinality or the number of elements of the set G, written $|G|$.

Definition 2.6.4 A subgroup of a group G is a group $H \subseteq G$ usually written as $H \leq G$, under binary operation of G restricted to H.

Definition 2.6.5 A subgroup $H < G$ is said to be proper if H is not equal to G. Otherwise, H is referred to as an improper subgroup. The subgroup $\langle K \rangle$ generated by a set $K \subseteq G$ is the minimal subgroup in G containing K, i.e. the subgroup of all combinations of the elements in K and their inverses.

Definition 2.6.6 If H and K are subgroups of group G, then
(i) The intersection: $H \cap K$ or $H \wedge K$ is the maximal subgroup contained in both H and K.
(ii) The join: $H \cup K$ or $H \vee K = \langle H, K \rangle$ is the minimal subgroup containing both H and K.

2.7 ON FUZZY SETS

As indicated earlier in section 2.1, the conventional set can best be described by enumerating or outlining its elements with the aid of an indicator or a characteristic function. This function only assigns a value 0 or 1 to each individual in the universe of discourse, thereby discriminating between elements and non-elements of the under-studied conventional set. The aforementioned function can be generalized to an extent that the values apportioned to the members of the universe of discourse lie within a precise range and specify the degree of membership of these elements in the under-considered set. Larger values symbolize higher grades of set membership. Such a function is referred to as a membership function and the set defined by it, a fuzzy set. We shall give an explicit definition of universe of discourse, fuzzy set, fuzzy subset, review the standard operations on fuzzy set with few examples and discuss some basic concepts in fuzzy set theory under this section. We shall also characterize fuzzy subgroups and give some basic concepts in fuzzy subgroups.

2.8 CHARACTERIZATION OF UNIVERSE OF DISCOURSE

The universe of discourse for fuzzy sets in fuzzy logic was primarily described only on the integers. Nowadays, the universe of discourse for fuzzy sets and fuzzy relations is characterized with three numbers. The first two numbers indicate the start and end of the universe of discourse, and the third argument indicates the increment between elements. This provides the user more suppleness in selecting the universe of discourse. For instance, the fuzzy set of numbers described in the universe of discourse $U = \{u_j\} = \{1, 2, 3, 4, \cdots, 15\}$ is presented as Universe of Discourse $\to \{1, 15, 1\}$. In this example, the first number 1 signifies the start and the second number 15 the end respectively while the third number 1, the increment between elements.

Definition 2.8.1 Let X be a nonempty set. A fuzzy set S in X is characterized by its membership function $\Psi_S : X \to [0, 1]$ and $\Psi_S(q)$ is interpreted as the degree of membership of element q in fuzzy set S for each $q \in X$. It is obvious that the set of tuples given by $S = \{(q), \Psi_S(q) \,| q \in X\}$ completely determines S. For convenience sake, we will write $S(q)$ instead of $\Psi_S(q)$.

Definition 2.8.2 Let S and T be two fuzzy sets of a conventional set X. Then we refer to S as a fuzzy subset of T if $S(q) \leq T(q)$, for all $q \in X$.

Note 2.1.1. Throughout this dissertation, if no confusion would arise, we will simply call a fuzzy subset a fuzzy set, keeping in mind that it has to be a subset of some universe set and has to have a pre-described membership function associated with it.
This membership function is a generalization of an indicator or a characteristic function of a subset defined for $\{0, 1\}$. More generally, one can use a complete lattice L instead of $[0, 1]$ in a definition of a fuzzy subset S. We usually denote the family of all fuzzy sets in X by $\mathbb{F}(X)$ or I^X where $I = [0, 1]$.
Note 2.1.2. In this dissertation, we used I^X to represent the family of all the fuzzy subsets of X.

Example 2.8.3 Let $X = \{1, 2, 3, 4\}$ be a given universe of discourse and $M_{\mathbb{F}}(x) = \{0.25, 0.50, 0.75, 1.00\}$ be the grades of membership assigned to all element in X in an orderly manner. Clearly, one can observe that the cardinality of $X = 4$ and that of $M_{\mathbb{F}}(x) = 4$, hence there are $256 = 4^4$ fuzzy sets as members in $\mathbb{F}(X)$ among which $16 = 2^4$ are crisp subsets of X.

2.9 REPRESENTATION OF FINITE AND INFINITE FUZZY SETS

We present here two ways of representing fuzzy sets
Suppose $X = \{a_1, a_2, a_3, ..., a_n\}$ is a finite set and S is a fuzzy set in X. Then we often use the notation $S = x_1/a_1 + x_2/a_2 + + x_n/a_n$ where the term x_i/a_i, for $i = 1, ..., n$ signifies that x_i is the degree of membership of $a_i \in S$ and the plus sign represents the union. On the other hand, we use the notation $S = \int \Psi_S(a)/x$ when the universe is continuous and infinite.

2.10 OPERATIONS ON FUZZY SETS

Since crisp sets and its associated indicator or characteristic functions may respectively be considered as unique cases of fuzzy sets and membership functions, we extend the conventional set theoretic operations from elementary set theory to fuzzy sets. We observe that all those operations which are extensions of non-fuzzy (crisp) notions reduce to their usual meaning whenever the fuzzy sets have membership grades that are drawn from $\{0, 1\}$. For this reason, when extending the operations to fuzzy sets, the same notation as in set theory is

used.
Let β, η and κ be fuzzy sets on the universal set X. For a given element $a \in X$, the following function-theoretic operations for the set-theoretic operations of union, intersections, complement, difference, containment and equality are defined for β, η and κ on X:
The union: $\Psi_\beta(a) \cup \Psi_\eta(a) = \Psi(a) \vee \Psi(a)$ for all $a \in X$
The intersection: $\Psi_\beta(a) \cap \Psi_\eta(a) = \Psi(a) \wedge \Psi(a)$ for all $a \in X$
Containment: The fuzzy set β is said to be contained in fuzzy set η if and only if for every $a \in X$ we have $\Psi_\beta(a) \leq \Psi_\eta(a)$

Theorem 2.10.1 Let β and κ be two fuzzy subsets of X. If $\mu(x) = (\beta \cup \kappa)(x)$ and $\nu(x) = (\beta \cap \kappa)(x)$ for all $x \in X$, then:
(a) $\nu \subset \mu$
(b) $\beta \subset \mu$ and $\kappa \subset \mu$.
(c) $\nu \subset \beta$ and $\nu \subset \kappa$.

Proof: See [122]
The complement: $\Psi_{\beta^c}(a) = 1 - \Psi_\beta(a)$ for all $a \in X$.
The difference: $(\beta - \eta)(a)$ or $(\beta \setminus \eta)(a) = \Psi_{\beta-\eta}(a)$ or $\Psi_{\beta\setminus\eta}(a) = \Psi_\beta(a) \wedge \Psi_{\eta^c}(a)$ $= \min(\Psi_\beta(a), 1 - \Psi_\eta(a))$ for all $a \in X$
Equality: Two fuzzy sets β and η on the same universe of discourse X are said to be equal if and only if for all $a \in X$, we have $\Psi_\beta(a) = \Psi_\eta(a)$
Apart from the above expression, union and intersection of two fuzzy sets can be defined through T-conorm (or S-norm) and T-norm operators respectively. These two operators are functions S,T : $[0,1] \times [0,1] \to [0,1]$ satisfying some convenient boundary, monotonicity, commutativity and associativity properties. As introduced by Zadeh [123], a more intuitive but equivalent definition of union of fuzzy set is the "smallest" fuzzy set containing both η and κ. Analogously, the intersection of η and κ is the "largest" fuzzy set which is contained in both η and κ.
Note 2.1.3. More generally, the intersection of a fuzzy set and its complement is not the empty fuzzy set (whose membership set contains only the number 0) as it is in conventional set theory unless the fuzzy set is a crisp subset, see [91].
Note 2.1.4. The union of a fuzzy set and its complement is not the universal

set X.
For completeness' sake, we shall utilize the examples below to further elaborate the operations on fuzzy set discussed above. Before proceeding with the examples, we first consider U as a finite universal set consisting of $1 \leq k$ elements and $|U| = r$. If we let each element $u \in U$ occupying the i^{th} position in the set be given by $u_{i-1} + (i-1)j$, for $1 \leq i \leq r$ and j the constant difference, then all fuzzy sets $\Psi \subseteq U$ can be expressed as $\Psi_u =$
$\big\{(a_{i-1} + (i-1)j, \frac{i-1}{r-i}), \cdots, (a_{r-1} + (r-1)j, \frac{r-1}{r-i}) = 1\big\}$, where the $\frac{i-1}{r-i}$ denote the grades of membership of all fuzzy sets contained in U within the closed unit interval $I = [0, 1]$.
Note 2.1.5. The membership grades are not all necessarily distinct. Besides, the enumeration of fuzzy sets with the special properties are unaffected by the uniformly chosen grades in Ψ_u and also conform with the preferential equality discussed in [86].
Note 2.1.6. Although fuzzy sets and probability both can take on similar values, it is essential to note that membership grades are not probabilities. This is because membership grades of fuzzy sets on a finite universal set do not satisfy any condition that their summation must be equal 1, unlike probabilities. Again, the specification of membership functions is subjective, which means that the membership functions specified for the same concept by different persons may vary considerably. This subjectivity comes from individual differences in perceiving or expressing abstract concepts and has little to do with randomness. Therefore, the subjectivity and non-randomness of fuzzy sets is one of the primary differences between the study of fuzzy sets and probability theory, which deals with objective treatment of random phenomena. See [68]chap. 4

Example 2.10.2 . Let U be a universal set consisting of positive integers less than or equal to 15, thus $U = \{1, 2, 3, \cdots, 15\}$. Let β and η be two fuzzy sets of U with $\beta = \{(1, 0.00), (2, 0.07), (3, 0.14), (4, 0.21), (5, 0.29), (6, 0.36), (7, 0.43),$
$(8, 0.50), (9, 0.57), (10, 0.64), (11, 0.71), (12, 0.79), (13, 0.86), (14, 0.93), (15, 1.00)\}$
and $\eta = \{(1, 0.14), (2, 0.07), (3, 0.21), (4, 0.29), (5, 0.36), (6, 0.43), (7, 0.50),$
$(8, 0.57), (9, 0.64), (10, 0.71), (11, 0.79), (12, 0.86), (13, 0.93), (14, 1.00), (15, 0.00)\}$
Then:
$\beta \cup \eta = \{(1, 0.14), (2, 0.07), (3, 0.21), (4, 0.29), (5, 0.36), (6, 0.43), (7, 0.50),$

$(8, 0.57), (9, 0.64), (10, 0.71), (11, 0.79), (12, 0.86), (13, 0.93), (14, 1.00), (15, 1.00)\}$

$\beta \cap \eta = \{(1, 0.00), (2, 0.07), (3, 0.14), (4, 0.21), (5, 0.29), (6, 0.36), (7, 0.43),$
$(8, 0.50), (9, 0.57), (10, 0.64), (11, 0.71), (12, 0.79), (13, 0.86), (14, 0.93), (15, 0.00)\}$

$\beta^c = \{(1, 1.00), (2, 0.93), (3, 0.86), (4, 0.79), (5, 0.71), (6, 0.64), (7, 0.57),$
$(8, 0.50), (9, 0.43), (10, 0.36), (11, 0.29), (12, 0.21), (13, 0.14), (14, 0.07), (15, 0.00)\}$
$\beta - \eta = \{(1, 0.00), (2, 0.07), (3, 0.14), (4, 0.21), (5, 0.29), (6, 0.36), (7, 0.43),$
$(8, 0.43), (9, 0.36), (10, 0.29), (11, 0.21), (12, 0.14), (13, 0.07), (14, 0.00), (15, 1.00)\}$

Example 2.10.3 . Consider a universal set U consisting of positive integers ≤ 15, thus $U = \{1, 2, 3, \cdots, 15\}$. Let $\eta = \{(1, 0.02), (2, 0.03), (3, 0.04), (4, 0.05), (5, 0.06), (6, 0.07), (7, 0.08), (8, 0.09), (9, 0.1)\}$ and $\beta = \{(1, 0.00), (2, 0.01), (3, 0.02), (4, 0.03), (5, 0.04), (6, 0.05), (7, 0.06), (8, 0.07), (9, 0.08)\}$

It can be easily seen that $\beta \subseteq \eta$

2.11 SOME BASIC CONCEPTS IN FUZZY SET THEORY

With the exception of the axiom of the excluded middle and the axiom of contradiction which are in variance with fuzzy sets, the remaining properties such as associativity, distributivity, idempotency, identity as well as the De Morgans's laws hold on fuzzy sets
We present some basic concepts in fuzzy set theory that are desirable:

2.11.1 An empty fuzzy set

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Definition 2.11.1 . A fuzzy set $\emptyset$ of X is said to be an empty fuzzy set if for each $x \in X$, we have $\emptyset(x) = 0$.

2.11.2 A Fuzzy Point

Definition 2.11.2 . A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except for one and only one element of X, say $x \in X$.

2.11.3 A Crossover Point of a Fuzzy Set

Definition 2.11.3 If $\Psi_\beta(x) = 0.5$, for a certain point $x \in X$, then such a point is called a crossover point of a fuzzy set β.

2.11.4 The Core of a Fuzzy Set

Definition 2.11.4 . The core of a fuzzy set β, denoted by core(β), is the set of all points $x \in X$ such that $\Psi_\beta(x) = 1$.

Example 2.11.5 . Consider $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $\beta = \{(a_1, 0), (a_2, 0.3), (a_3, 0.7), (a_4, 1), (a_5, 0.5), (a_6, 0.9)\}$. Then the core(β) $= \{(a_4, 1)\}$

2.11.5 The Support of a Fuzzy Set

Definition 2.11.6 Let β be a fuzzy set of X. The support of β, denoted supp(β), is the crisp subset of X whose elements all have nonzero membership grades in β. Thus supp(β) $= \{x \in X : \beta(x) > 0\}$.

Example 2.11.7 . Consider a space X consisting of ages of undergraduate students in the University of Fort Hare, say $X = \{15, 18, 21, 24, 27, 30, 33, 36\}$. Let us define a fuzzy set β on X such that $\beta = \{(15, 0.00), (18, 0.50), (21, 0.55), (24, 0.60), (27, 0.65), (30, 0.70), (33, 0.75), (36, 0.00)\}$.

Then supp(β) $= \{18, 21, 24, 27, 30, 33\}$.

2.11.6 A Co-support of a Fuzzy Set

Definition 2.11.8 . The co-support of β denoted by co-supp(β), is a non-fuzzy set which involves all elements that are completely found outside a given fuzzy set and is given by co-supp(β) $= \{x \in X : \beta(x) = 0\}$

Example 2.11.9 . From example 2.11.7, it is easy to see that the co-supp$(\beta) = \{15, 36\}$

2.11.7 Fuzzy Singleton

Definition 2.11.10 . A fuzzy set whose support is a single point in X with $\Psi(x) = 1$ is referred to as a fuzzy singleton

2.11.8 On Alpha-cut (α-cut) Set

Definition 2.11.11 A weak α-cut set or an α level-set of S denoted by S_α is a non-fuzzy set which is made up of members whose membership grades are not less than α and is given by $S_\alpha = \{x \in X : \Psi_S(x) \geq \alpha\}$, where $\alpha \in (0, 1]$.

Definition 2.11.12 . If $S^\alpha = \{x \in X : \Psi_S(x) > \alpha\}$, for $\alpha \in [0, 1]$, then S^α is referred to as a strong α-cut.

Proposition 2.11.13 . Let $\beta, \eta \in I^X$, then
1. $\beta = \eta$ if and only if $\beta^\alpha = \eta^\alpha$ for all $\alpha \in I$.
2. $\beta = \eta$ if and only if $\beta_\alpha = \eta_\alpha$ for all $0 < \alpha < 1$.

Proof:. See [71]

Corollary 2.11.14 . (1). $S_\alpha = X$ whenever $\alpha = 0$
(2). $S^\alpha = \emptyset$ whenever $\alpha = 1$

Example 2.11.15 Using example 2.11.7 and by setting $\alpha = 0.55$, we have $S_{0.55} = \{21, 24, 27, 30, 33\}$

Definition 2.11.16 . Let β be a fuzzy subset of the set X. Then β is said to be normal if sup$\beta(x) = 1$, for $x \in X$.

2.11.9 Characterization of Images and Pre-images of Fuzzy Sets

Let X and Y be two non-empty sets and f a mapping from X into Y. For fuzzy sets $\lambda, \lambda_k \subseteq X$ and $\mu, \mu_k \subseteq Y$, define the image $f(\lambda)$ of λ under f for $y \in Y$ as

$$f(\lambda)(y) = \begin{cases} sup\{\lambda(x) : x \in f^{-1}(x)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{, Otherwise} \end{cases}$$

The pre-image denoted by $f^{-1}(\mu)$ of μ under f is the fuzzy subset of X such that, for $x \in X$, $f^{-1}(\mu)(x) = \mu(f(x))$. Thus $f^{-1}(\mu)$ consists of precisely all the elements of λ that are mapped to elements of μ by f.

2.11.10 An f-invariant of a Fuzzy Set

Definition 2.11.17 [101]. A fuzzy set β in X is said to be f-invariant if $\beta(a_1) = \beta(a_2)$ whenever $f(a_1) = f(a_2)$ for all $a_1, a_2 \in X$.

2.11.11 On Sup Property

Definition 2.11.18 . Let X be any given set. A fuzzy subset $\beta : X \to [0, 1]$ of X has the sup property if for any subset S of the set X there exists $x_0 \in S$ such that $\beta(x_0) = sup\{\beta(x) : x \in S\}$.

2.12 ON FUZZY SUBGROUPS

First of all, we present some basic concepts and results on fuzzy subgroups.

Definition 2.12.1 . As defined by Rosenfeld in [101]. Let υ be a fuzzy set on a group G. Then υ is said to be a fuzzy subgroup of G if for all $a, b \in G$, we have

1. $\upsilon(ab) \geq \min\{\upsilon(a), \upsilon(b)\}$
2. $\upsilon(a^{-1}) = \upsilon(a)$.

For the identity element $e \in G$, we have $\upsilon(a) \leq \upsilon(e)$, for all $a \in G$

Definition 2.12.2 [11] . If υ is a fuzzy subgroup on a group G and ϑ is a map from G onto itself, we define a map $\upsilon^{\vartheta} : G \to [0,1]$ by $\upsilon^{\vartheta}(g) = \upsilon(g^{\vartheta})$, for all $g \in G$ where g^{ϑ} is the image of g under ϑ.

Definition 2.12.3 [80]. We define $\upsilon \circ \upsilon(g) = \sup_{g=g_1 g_2} (\upsilon(g_1) \wedge (\upsilon(g_2))$.

Proposition 2.12.4 A fuzzy subset υ of G is a fuzzy subgroup of G if and only if
(a) $\upsilon \circ \upsilon \leq \upsilon$ and
(b) $\upsilon^{-1} = \upsilon$ where υ^{-1} is defined as $\upsilon^{-1} : G \to I$, for all $g \in G, \upsilon^{-1}(g) = \upsilon(g^{-1})$.

Proof. See [89]

2.13 ON LEVEL SUBGROUPS

Aware of Zadeh's[123] idea of level subsets, Das in [24] applied that concept to characterize level subgroups of a fuzzy group. Since then, several properties of fuzzy groups have been defined by using Das' level subgroups, hence it has become one of the essential tools used in the study of fuzzy groups.
We first define level subset, state a theorem and then proceed with the Das's definition of level subgroup, state theorem, corollary and provide their proofs. We also give the definition of level subgroup revised by Ajmal[2] and Jain [60].
Recall from definition 2.11.11: Let υ be a fuzzy subset of S and $t \in [0,1]$. Then $\upsilon_t = \{s \in S : \upsilon(s) \geq t\}$ is called the level subset of υ at t.

Theorem 2.13.1 [24]. Let G be a group and υ be a fuzzy subgroup of G, then the level subset υ_t, for $t \in [0,1]$, $t \leq \upsilon(e)$, is a subgroup of G, where e is the identity of G

Proof. See [24]

Definition 2.13.2 [24]. Let υ be a fuzzy subgroup of the group G and $0 \leq t \leq \upsilon(e)$, then υ_t is referred to as the level subgroup of υ at t.

Theorem 2.13.3 [24] Theorem 3.1
Let G be a group and υ be a fuzzy subgroup of G. Two level subgroups υ_{t_1}, υ_{t_2}(with $t_1 < t_2$) of υ are equal if and only if there is no $x \in G$ such that $t_1 < \upsilon(x) < t_2$.

Note 2.1.5: Only one implication of the above theorem is true, its converse is false as the next example shows.

Example 2.13.4 Let $G = S_3 = \{e, a, a^2, b, ab, a^2b\}$ such that $a^3 = e = b^2$, be the symmetric group on 3 symbols. Define a fuzzy subset of S_3 as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{1}{2} & \text{if} \quad x = b \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

Now no x in G exists such that $\frac{1}{3} < \mu(x) < \frac{1}{2}$, but $\mu_{\frac{1}{2}} = \{e, b\} \neq \mu_{\frac{1}{3}} = S_3$. This is contrary to the above theorem. Hence the theorem should read

Theorem 2.13.5 Let G be a group and υ be a fuzzy subgroup of G. If two level subgroups υ_{t_1}, υ_{t_2}(with $t_1 < t_2$) of υ are equal, then there is no $x \in G$ such that $t_1 < \upsilon(x) < t_2$.

Corollary 2.13.6 [24]Corollary 3.1
Let G be a finite group of order n and υ be a fuzzy subgroup of G.
Let $\text{Im}(\upsilon) = \{t_i | \upsilon(x) = t_i \text{ for some } x \in G\}$. Then $\{\upsilon_{t_i}\}$ are the only level subgroups of υ.

Proof: Assume $\text{Im}(\upsilon)$ has the elements $t_0 < t_1 < t_2 \cdots < t_k$ since G is finite.
Claim : $\upsilon_{t_0} = G$.
Clearly $\upsilon_{t_0} \subseteq G$. Let $x \in G$, then $\upsilon(x) = t_j$ for some $j \in J = \{t_0, t_1, \cdots, t_k\} = \text{Im}(\upsilon)$. Therefore $x \in \upsilon_{t_j}$, $t_j \geq t_0$, implying $\upsilon_{t_j} \subseteq \upsilon_{t_0}$, thus $x \in \upsilon_{t_0}$. Therefore $G \subseteq \upsilon_{t_0} \Rightarrow G = \upsilon_{t_0}$.
Suppose $\exists\ t \in [0, 1] : t_i < t < t_{i+1}$ and $\upsilon_t \neq \upsilon_{t_m}\ \forall\ m \in J$. Then
$\upsilon_{t_{i+1}} \subsetneq \upsilon_t \subsetneq \upsilon_{t_i} \ \cdots (1)$
Let $x \in \upsilon_t$, then $\upsilon(x) > t$ since $\upsilon(x) \neq t$. Therefore $\upsilon(x) \geq t_{i+1}$ since $\upsilon(x) \in J$,

$\Rightarrow x \in \upsilon_{t_{i+1}}$, therefore $\upsilon_t \subseteq \upsilon_{t+1}$ contradicting (1) above. Thus no $t \in [0,1]$ exists such that $t_i < t < t_{i+1}$ and $\upsilon_t \neq \upsilon_{t_m}$ $\forall$ $m \in J$. Therefore any level subgroup of G is of the form υ_t, $t \in J$. This completes the proof.

Proposition 2.13.7 [104] Prop 2.1. Let υ be a fuzzy subgroup of G with $\text{Im}(\upsilon) = \{t_j : j \in J\}$ and $f = \left\{\upsilon_{t_j} : j \in J\right\}$ where J is an arbitrary index set. Then:
(a) there exists a unique $j_0 \in J$ such that $t_{j_0} \geq t_j$, for every $j \in J$,
(b) $\upsilon_{t_{j_0}} = \bigcap_{j \in J} \upsilon_{t_j}$,
(c) $G = \bigcup_{j \in J} \upsilon_{t_j}$,
(d) the members of f form a chain.

Proof. See [104].

Definition 2.13.8 [3] Let G be a group and μ be a fuzzy subgroup of G. The subgroups μ_α, $\alpha \in [0,1]$ and $\alpha \leq \mu(e)$ are called level subgroups of G.
Ajmal in [2] (see also Jain [60]) revised this definition of level subgroup by restricting $\alpha \in \text{Im } \mu$ and gave the following,

Definition 2.13.9 [2]. The level subgroup of G $\mu_t^{>} = \{\mu(x) > t : x \in G, t \in \text{Im } \mu\}$

Definition 2.13.10 [60]. Let $\mu \in L(G)$. Then $P(\mu) = \{x \in G : \mu(x) > Inf\mu\}$ is a subgroup of G, called the penultimate subgroup of μ.

2.14 ON HOMOMORPHIC IMAGES AND PRE-IMAGES OF FUZZY SUBGROUPS

We reproduce some propositions on homomorphic images and pre-images of fuzzy subgroups from the work studied earlier by Rosenfeld and that of S.Sebastian and S.B.Sundar.

Proposition 2.14.1 [101] Prop 5.8
A homomorphic image or pre-image of a fuzzy subgroup is a fuzzy subgroup.

Proof. See [101]

Proposition 2.14.2 [104] Prop 3.1
If F^* is a fuzzy subgroup of G^* and $\left\{{F^*}_{t_j} : j \in J\right\}$ is the collection of all level subgroups of F^*, then $\left\{f^{-1}(F^*_{t_j} : j \in J)\right\}$ is the collection of all level subgroups of $f^{-1}(F^*)$.

Proof. See [104]

Proposition 2.14.3 [104] Prop 4.1. If f is a surjection, F has sup-property and $\left\{F_{t_j} : j \in J\right\}$ is the collection of all level subgroups of F, then $\left\{f(F_{t_j}) : j \in J\right\}$ is the collection of all level subgroups of $f(F)$.

Proof. See [104]

Theorem 2.14.4 . Let $f : G \to G$ be a homomorphism of G into G. If υ is a fuzzy subgroup of G, then the image $f(\upsilon)$ of υ under f is a fuzzy subgroup of G.

Proof. See [104]

Proposition 2.14.5 . Let $f : G \to G'$ be a homomorphism and υ a fuzzy subgroup of a group G'. Then the pre-image $f^{-1}(\upsilon)$ of υ under f is a fuzzy subgroup of G.

Proof. See [104]

2.15 ON NORMAL FUZZY SUBGROUPS

Definition 2.15.1 [104]. Let υ be a fuzzy subgroup of a group G. Then a fuzzy subgroup υ of the group G is said to be a fuzzy normal subgroup if for all $x, y \in G$, we have $\upsilon(xy) = \upsilon(yx)$.

Equivalently υ is fuzzy normal if and only if $\upsilon(xyx^{-1}) = \upsilon(y)$, for all $x, y \in G$.

Proposition 2.15.2 [104]. Let υ be a fuzzy normal subgroup of G. Let $f : G \to G'$ be a homomorphism where G' is a group. Then the image $f(\upsilon)$ of υ under f is fuzzy normal in $f(G)$.

Proof. See [104]

2.15.1 On Fuzzy Cosets

Definition 2.15.3 [79]. If υ is a fuzzy subgroup of a group G, then for any $x \in G$, we define a left fuzzy coset of υ, denoted $x\upsilon$, as the fuzzy subset G defined by $(x\upsilon)(y) = \upsilon(x^{-1}y)$ for all $y \in G$. A right fuzzy coset of υ is also defined by $(\upsilon x)(y) = \upsilon(yx^{-1})$.

If υ is fuzzy normal, then the set $G_\upsilon = \{x\upsilon : x \in G\}$ is a group under the binary operation defined by $(x\upsilon)(y\upsilon) = (xy)\upsilon, \forall\ x, y \in G$. Besides, we also have $x\upsilon = \upsilon x$ for all $x \in G$.. (See [79], Proposition 4.3 and Theorem 4.5).

Proposition 2.15.4 [79]. Let υ be a fuzzy subgroup of G. Then υ is fuzzy normal if and only if $x\upsilon = \upsilon x$ for all $x \in G$.

Proof. See [79]

2.15.2 On Fuzzy Conjugate

Definition 2.15.5 [11]. Let G be a group. If υ is a fuzzy subgroup of G and $x \in G$, then the fuzzy subset $Ax(\upsilon)$ of G defined by $Ax(\upsilon)(g) = \upsilon(x^{-1}gx)$, for all $g \in G$ is called the fuzzy conjugate of υ determined by x.

Theorem 2.15.6 [11]. A fuzzy subgroup υ of a group G is a fuzzy normal subgroup if and only if υ is constant on the conjugate classes of G.

Proof. See [11]

2.15.3 On Fuzzy Abelian

Definition 2.15.7 [12]. Let G be a group and let υ be a fuzzy subgroup of G, then υ is said to be fuzzy abelian if G_υ is an abelian subgroup of G.

Proposition 2.15.8 [12]. A non-empty subset A of G is an abelian subgroup of G if and only if X_A is a fuzzy abelian subgroup of G.

Proof. See [12]

Chapter 3

FUZZY EQUIVALENCE AND FUZZY ISOMORPHISM

3.1 INTRODUCTION

The most excellent technique to examine similarities of like objects in a compilation of objects is to subdivide them into disjoint subsets called cells such that all the similar ones are in one cell. Such cells are called a partition of the set of all the objects. This is possible through the utilization of the concept of an equivalence relation. This concept of an equivalence relation oversimplifies equality in that objects perceived identical can effortlessly be associated by means of class belonging. An indispensable verity, connecting equivalence relations and partitions is that every equivalence relation on a set determines a specific partition of the set and every partition of a set determines a specific equivalence relation on the set. These operations are inverse to each other. Equivalence relations which permeate mathematics with several salient applications have considerably been investigated in different contexts by different authors. Consequently, different names have been associated with them which heavily depend on the author and the context where they have been investigated. For instance, Zadeh [123], [125] initially described them as similarity relations, Valverde [118], Demirci [31], [32], Boixader and Jacas [14], Jacas [56], Jacas and Recasens [57],[58],[59] refer to them as indistinguishability operators. $\Im$-equivalence was how De Baets et al [25],[26], [27] termed them, and many-valued equivalence relations by [33], [34] just to name a few. The effects of an equivalence relation on fuzzy subsets of a set were considered by a number of authors, amongst them Murali in [80] who characterized and studied properties of fuzzy equivalence relations and certain lattice theoretical properties of fuzzy equivalence relations. Chakraborty and Das [19],[20], Dubois and Prade [40], Ounalli and Jaoua [96] and Nemitz [92] studied fuzzy functions as fuzzy relations and fuzzy partitions. Fuzzy equivalence relations studied in a fuzzy framework, oversimplify the crisp equivalence relation and equality. In this chapter we give a definition of a partition, relation between two sets, binary relation on a set, equivalence relation, equivalence classes, and further define fuzzy relation and fuzzy partition. We also reproduce the definitions of fuzzy equivalence relation introduced by Murali and Makamba [81] that we use in this dissertation and briefly examine different forms of fuzzy equivalence relations that include fuzzy reflexive relations and fuzzy transitive relations. We again extend our discussion to cover fuzzy isomorphism introduced by Murali and Makamba [81]. In addition we study and compare the Murali and Makamba [81] definition of fuzzy equivalence to other notions of equivalence of fuzzy subgroups that are found in literature. We conclude the chapter by introducing the two counting techniques that are derived from this fuzzy equivalence relation.

3.2 GENERAL EQUIVALENCE RELATION

Definition 3.2.1 [43]. Let $\mathcal{S}$ be a collection of nonempty subsets of a set A. Then $\mathcal{S}$ is said to be a *partition* of A if
(1) $S \cap S^{'} = \emptyset$, for any distinct S and $S^{'}$ in $\mathcal{S}$, and
(2) $A = \bigcup \{S | S \in \mathcal{S}\}$.

Definition 3.2.2 . Given two nonempty sets X and Y, a relation between X and Y is a subset $\Re \subseteq X \times Y$. For a relation $\Re \subseteq X \times Y$ and $x \in X$, $y \in Y$, if $(x, y) \in \Re$, we write $x\Re y$ (we say x is $\Re$-related to y).

Definition 3.2.3 . A binary relation on a nonempty set X is a relation $\Re$ between X and X, that is, a subset $\Re \subseteq X \times X$.

Definition 3.2.4 [43]. A relation $\Re$ on a nonempty set A is an equivalence relation if and only if $\Re$ is
(i) reflexive, i.e $(a, a) \in \Re$,for all $a \in A$
(ii) symmetric, $(a, b) \in \Re \Rightarrow (b, a) \in \Re$ and

(iii) transitive, $(a, b) \in \Re$ and $(b, c) \in \Re$, then $(a, c) \in \Re$.

Definition 3.2.5 [45]. (i) If $\Re$ is an equivalence relation on a nonempty set A, then for an element $a \in A$, the set $[a] = \{x | a\Re x\}$ is called the equivalence class of a.
(ii) The element a in the bracket above is called a representative of the equivalence class.

3.3 FUZZY RELATION

The concepts of a fuzzy relation and fuzzy similarity relation were introduced and studied by Zadeh [123] in 1965 and further followed by [124] in 1966 and [125] in 1971. He therefore introduced the concept of fuzzy equivalence class as a natural generalisation of the notion of a crisp equivalence class. Murali in [80] defined a fuzzy equivalence relation on a set and observed that there is a correspondence between fuzzy equivalence relations and certain classes of fuzzy sets. De Baets et al in [25] and Ovchinnikov et al in [97] studied fuzzy equivalence relations in terms of fuzzy partitions. We give the following definition by Murali in [80]

Definition 3.3.1 Let X and Y be two universes of discourse. A fuzzy relation on $X \times Y$ denoted by $\Re(a, b)$ or $\Re$ is defined as the set $\Re$ characterized by the membership function $\mu_{\Re}(a, b)$ where $\Re = \{((a, b), \mu_{\Re}(a, b)) | (a, b) \in X \times Y, \mu_{\Re}(a, b) \in [0, 1]\}$.

Note 3.1.0:
(1) Since the relation $\Re$ defined above is a binary relation, it is said to be:
(i) Reflexive if $\mu_{\Re}(a, a) = 1,$ for all $a \in X$
(ii) Symmetric if $\mu_{\Re}(a, b) = \mu_{\Re}(b, a),$ for all $a, b \in X$
(iii) Transitive if $\mu_{\Re} \circ \mu_{\Re} \leq \mu_{\Re}$ where $\mu_{\Re} \circ \mu_{\Re}$ is defined by $\mu_{\Re} \circ \mu_{\Re}(a, b) = sup_{c \in X}(\mu_{\Re}(a, c) \wedge \mu_{\Re}(c, b))$
If the conditions (i), (ii) and (iii) hold in a fuzzy relation on a set X, then such a relation is called a fuzzy equivalence relation on X.
(2) Because fuzzy relations are fuzzy sets in product space, set theoretic operations such as union, intersection and complement can be defined for them.

Though quite a number of researchers normally prefer working with the standard reflexive, symmetric and transitive conditions of fuzzy relations, not all fuzzy equivalence relations satisfy conditions $(i)-(iii)$ in definition 3.2.4, hence a number of versions of fuzzy equivalence relations have been introduced based on this definition. For instance in [125], Zadeh introduced fuzzy counterparts of classical properties of reflexivity, symmetry, transitivity and antisymmetry of binary relations. He exploited these properties to examine types of fuzzy relations that include fuzzy equivalence, compatibility and ordering relations. Reservations and objections to this definition based on the standard conditions (i)-(iii) can be found in Behounek et al [8],[9], Boixader and Recasens [14], Gottwald [47] and Jacas and Recasens [57]. Researchers have redefined new types of fuzzy relations based on the three conditions (i)-(iii). Gupta and Gupta in [48] pointed out that condition $\forall x \in X$, $\mu(x,x) = 1$, for instance is too strong for defining a fuzzy reflexive relation (see also Dutta et al in [41]). They then proposed positive values for all $\mu(x,x)$ and that $\mu(y,z) \leq \mu(x,x)$ for all $y \neq z$ and $x \in X$. With the aid of this definition of a fuzzy reflexive relation, they were able to redefine fuzzy equivalence relations that either supercede or improve on most of the theorems of Murali [80]. Other concepts studied on reflexive equivalence relations include G-reflexive fuzzy relation by Gupta and Gupta [48] as a generalisation of Zadeh's[125] reflexive fuzzy relation. For more on reflexive fuzzy relations see Gupta and Singh [49] and Chakraborty and Das [19]. We use the standard definition of a reflexive fuzzy relation $\mu(x,x) = 1 \; \forall x \in X$ that has been used by Murali in [80] and Nemitz in [92].

Generally most researchers are at ease with the use of the symmetric law $\forall x, y \in X, \mu(x,y) = \mu(y,x)$, see also Zadeh [125], Chakraborty and Das [19], [20] and Cho [22]. Just like the reflexive part of the definition, different approaches to a fuzzy transitive relation have been studied. The first type is the one introduced by Zadeh [123], the second is the one studied by Demirci and Recanesens [35] and also by Jayaram and Mesiar [62]. This is defined with the help of t-norms and is called the T-transitivity of fuzzy relations. Other notions of a transitive relation include among others $\in$-fuzzy transitivity studied by Beg and Samina in [7] that lead to the notion of $\in$-fuzzy dissimilarity relation.

In the following section we give the Murali and Makamba [81] definition of fuzzy equivalence relation on fuzzy subgroups and compare this definition to some of its variants and duals.

3.3.1 The Fuzzy Partition Of A Fuzzy Subset

Definition 3.3.2 . Let X be a nonempty set and let μ be a fuzzy subset of a set X. Then
$\sum = \{\kappa : \kappa \text{ is a fuzzy subset of a set } X \text{ and } \kappa \subseteq \mu\}$ is said to be a fuzzy partition of μ if
(a). $\bigcup_{\kappa\in\sum} \kappa = \mu$ and
(b). any two members of $\sum$ are either identical or disjoint

3.4 EQUIVALENCE RELATION ON SUBGROUPS

Definition 3.4.1 . A fuzzy relation μ on a group G is said to be a fuzzy equivalence relation on G if:
(i) $\mu(x, x) = 1, \forall x \in G$
(ii) $\mu(x, y) = \mu(y, x)$, $\forall x, y \in G$
(iii) $\mu \circ \mu \leq \mu$

Equivalence relations of fuzzy sets have been used to study equivalence of fuzzy subgroups. It is clear from our discussion in the previous section that not all fuzzy equivalence relations satisfy the fuzzy equivalence relation defined by Zadeh [123], hence the existence of several versions of fuzzy equivalence relations. In this dissertation we use the definition of Murali and Makamba [81]. Other notions of fuzzy equivalence were defined by Volf [120], Branimir and Tepavcevic [15], Degang et al [29], Tarnauceanu and Bentea [116], Ghafur and Sulaiman [111], Dixit et al [39], Dixit et al [38], Zhang and Zou [128], Ajmal [3], Mordeson [78], Jain [60], Mashinchi and Mukaidonon [75] and Iranmanesh and Naraghi [55].
The Murali and Makamba definition is as follows:

Definition 3.4.2 [81]. Let μ and ν be any fuzzy sets in I^X where $I = [0, 1]$ and a nonempty set X. We define an equivalence relation on I^X as follows:

$\mu \sim \nu$ if and only if for all $a, b \in X$, $\mu(a) > \mu(b)$ if and only if $\nu(a) > \nu(b)$ and $\mu(a) = 0$ if and only if $\nu(a) = 0$.

Note 3.1.1: The condition $\mu(a) = 0$ if and only if $\nu(a) = 0$ simply means that the supports of μ and ν are equal.
In the condition of equivalence relation $\mu \sim \nu$, the strict inequality can be replaced by $\geq$. Thus $\mu \sim \nu$ if and only if $\forall\, a, b \in X$, $\mu(a) \geq \mu(b)$ if and only if $\nu(a) \geq \nu(b)$ and $\mu(a) = 0$ if and only if $\nu(a) = 0$. It is easily seen that either of the inequalities in the definition determines the same equivalence class of fuzzy sets.
The condition $\mu(a) = 0$ if and only if $\nu(a) = 0$ is an important part of the equivalence relation as the following example illustrates.

Example 3.4.3 . Consider $D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = b^2 = (ab)^2 = e$ for e an identity element of the group. Define the fuzzy sets μ and ν on D_4 as follows:

$$\mu(x) = \begin{cases} 1 & if \quad x = e \\ \frac{1}{5} & if \quad x = a, a^2, a^3 \\ \frac{1}{7} & otherwise \end{cases}$$

and

$$\nu(x) = \begin{cases} 1 & if \quad x = e \\ \frac{1}{5} & if \quad x = a, a^2, a^3 \\ 0 & otherwise \end{cases}$$

It can be seen that the
$\text{Supp}(\mu) \neq \text{Supp}(\nu)$ even though $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$ $\forall\, x, y \in D_4$.
Note 3.1.2. If $\mu \sim \nu$, then $|\text{Im}(\mu)| = |\text{Im}(\nu)|$.
The converse is not true, thus if $|\text{Im}(\mu)| = |\text{Im}(\nu)|$ or even if $\text{Im}(\mu) = \text{Im}(\nu)$ and $\text{Supp}(\mu) = \text{Supp}(\nu)$, it is not necessary to have $\mu \sim \nu$, as verified by the example below.

Example 3.4.4 . Let $S_3 = \{e, a, a^2, b, ab, a^2b\}$ generated by a and b where $a^3 = e = b^2$ and e the identity element.
Define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} 1 & if \quad x = e \\ \frac{1}{5} & if \quad x = b \\ \frac{1}{7} & otherwise \end{cases}.$$

and

$$\nu(x) = \begin{cases} 1 & if \quad x = e \\ \frac{1}{5} & if \quad x = ab \\ \frac{1}{7} & otherwise \end{cases}.$$

Observe that in the above example,
$\mathrm{Im}\,(\mu) = \mathrm{Im}\,(\nu)$ and $\mathrm{Supp}\,(\mu) = \mathrm{Supp}\,(\nu) = S_3$
However, $\mu\,(b) > \mu\,(ab)$ but $\nu\,(b) \not> \nu\,(ab)$. Therefore μ is not equivalent to ν.

Proposition 3.4.5 . Let μ and ν be two fuzzy subsets of X. Suppose for each $t > 0$ there exists an $s > 0$ such that $\mu^t = \nu^s$. Then $\mu \sim \nu$.

Proof. See [81]

Proposition 3.4.6 . Let μ be a fuzzy subgroup of a finite group G. If t_i, t_j are elements of the image set of μ such that $\mu_{t_i} = \mu_{t_j}$, then $t_i = t_j$.

Proof. See [12]

3.4.1 Comparison of Equivalence Relations

Before we compare the above definition with other notions of fuzzy equivalence relations, we give the definitions of distinct fuzzy subgroups and right limited point of Im μ.

Definition 3.4.7 [89] Two fuzzy subgroups μ and ν of a group G are said to be distinct iff $[\mu] \neq [\nu]$, where $[\mu]$ and $[\nu]$ are equivalence classes containing μ

and ν respectively.

Definition 3.4.8 [29] Let μ be a fuzzy set of X and $a \in X$. For any $\epsilon > 0$, if there exist $b \in X$ such that $\mu(a) + \epsilon > \mu(b) > \mu(a)$, then $\mu(a)$ is called the right limited point of Im μ.

The notion of equivalence of fuzzy subgroups introduced by Volf in [120] and Tarnauceanu and Bentea in [116] differs from that of Murali and Makamba [81]. They define an equivalence of two fuzzy subgroups μ and ν of G as follows,
$\mu \approx \nu \Longleftrightarrow \forall x, y \in G,\ \mu(x) > \mu(y) \Longleftrightarrow \nu(x) > \nu(y)$.
This definition is closely connected to the concept of level subgroups. Thus according to Volf [120], two fuzzy subgroups μ and ν of G are equivalent if they have the same set of level subgroups. So the necessary and sufficient condition for equivalence of two fuzzy subgroups is the equality of level subgroups. This is a generalization of the definition given by Murali and Makamba in [81] since the condition that their supports are equal has been discarded. The second definition equivalent to the Tarnauceanu and Bentea [116], Dixit et al [39], Zhang and Zou [128] and Mordeson [78] is given by Ghafur and Sulaiman [111]. It defines the equivalence relation on fuzzy subgroups as follows:
Let $\mu, \nu \in F(G)$ of the form:

$$\mu(x) = \begin{cases} \rho_1 & x \in A_1 \\ \rho_2 & x\ A_1 \setminus A_2 \\ . \\ . \\ . \\ \rho_n & x \in A_n \setminus A_{n-1} \end{cases}$$

$$\nu(x) = \begin{cases} \rho_1 & x \in B_1 \\ \rho_2 & x\ B_1 \setminus B_2 \\ . \\ . \\ . \\ \rho_n & x \in B_m \setminus B_{m-1} \end{cases}$$

then we say that μ and ν are equivalent if (1) m=n (2) $A_i(\mu) = B_i(\nu)$, $\forall i \in \{1, 2, ..., n\}$
It is a generalisation of the Murali and Makamba [80] definition. It is set theoretical, in that the number of fuzzy subgroups is influenced by the structure

and pattern of diagrams of lattices of subgroups of the group. Perhaps the only difference between all these definitions that use level subgroups is that they use the definition of level subgroup given by Das [24] while others use the definition of level subgroup proposed by Ajmal [3].
For example Mordeson's [78] definition is given as follows: Let G be a finite group and $\mu, \nu \in L_1(G)$. Then $\forall x, y \in G$, $\mu \equiv \nu$ if $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$, (where $L_1(G)$ is the set of all fuzzy subgroups μ of G such that $\mu(e) = 1$) . It is clear that this definition is similar to the one proposed by Volf [120], Tarnauceanu and Bentea [116]. We also notice that the definition of equivalence given by Volf [120], Tarnauceanu and Bentea [116], Dixit et al [39], Zhang and Zou [128] and Mordeson [78] uses the notion of level subgroups in the sense of Das [24] hence differ from the Jain [60] and Ajmal [3] definition because for their level subgroups μ_t and ν_t, $t \in [0, 1]$ while Ajmal [3] requires that $t \in$ Im μ.
Branimir and Tepavcevic [15] are the proponents of the other version, given as follows:
Let μ, ν : $X \to L$. (L is a complete lattice, which in particular can be $[0, 1]$). Then μ is equivalent to ν iff μ and ν have equal families of cuts.
In this definition the use of cuts has been justified because of the following reasons: firstly there are normally uncountably many distinct fuzzy sets on the same domain, be it finite or infinite, so is the case with fuzzy subgroups of a group. Since it can be shown that not all these fuzzy subgroups are "different", they argue that the equality of cut-sets could then be an essential classification criterion. They generalize the notion of equality of fuzzy sets given by Murali and Makamba [81]. This equivalence relation is generally defined for lattice-valued fuzzy sets but can be restricted to fuzzy sets with finite values in $[0, 1]$. This notion of equivalence is set theoretical and algebraic because of its reliance on isomorphism between two lattices. It is clear that if this equivalence relation defined by Branimir and Tepavcevic [15] holds, then the one defined by Murali and Makamba in [81] holds but not conversely.
The analysis method was introduced by Degang C et al in [29]. This notion defines the equivalence of two fuzzy subsets as follows:
Let μ and ν be fuzzy subsets of X, then μ and ν are strong equivalent if $\mu^R = \nu^R$, where μ^R denotes the collection of all $a \in X$ such that $\mu(a)$ is a right limited point of $Im(\mu)$.
In addition to strong equivalence, they define two fuzzy subgroups μ and

ν of a group X to be S^*-equivalent, denoted $\mu \cong \nu$ if $Im(\mu) = Im(\nu)$, $sup(\mu) \cong sup(\mu)$ and for any $t \in [0,1]$, $\mu^t \neq \emptyset$ implies that there exists an $s \in [0,1]$ such that $\mu^t \cong \nu^s$ and for any $s \in [0,1]$, $\nu^s \neq \emptyset$ implies there exists a $t \in [0,1]$ such that $\mu^s \cong \nu^t$. We note that a replacement of $sup(\mu) \cong sup(\nu)$ by $sup(\mu) = sup(\nu)$ and $\mu^t \cong \nu^s$ by $\mu^t = \nu^s$ transforms S^*-equivalence relation to strong equivalence relation. For a finite group, if two fuzzy subgroups μ and ν of a group G are equivalent using the definition of Murali and Makamba then they are S^*-equivalent, but the converse may not always be true mainly because two isomorphic groups may not be equal. So the equivalence relation used in this dissertation defined in [81],[82],[83] is a special case of S^*-equivalence. In computing the equivalence classes of fuzzy subgroups of finite groups, it is clear that the number of distinct fuzzy subgroups will be reduced if S^*-equivalence is used instead of the definition given by Murali and Makamba [81]. Another difference between these two notions of equivalence is that in the infinite case when dealing with the definition of equivalence given by Murali and Makamba [81],[82],[83], the number of fuzzy subgroups of an infinite cyclic group with an infinite number of membership values is infinite, but with S^*-equivalence it is finite. So in [29] Degang et al emphasise the use of S^*-equivalence to classify distinct fuzzy subgroups of infinite groups.

Mashinchi and Mukaidonon [75] proposed the following definition : μ is equivalent to ν denoted $\mu \cong_t \nu$ if there exist an isomorphism f from Supp μ to Supp ν such that $\forall x, y \in$ Supp μ we have $\mu(x) > \nu(y) \Leftrightarrow \nu(f(x)) > \nu(f(y))$ and Iranmanesh and Naraghi [55] gave the following definition: let G be a group and μ, $\nu \in F(G)$, μ is equivalent to ν denoted $\mu \sim_k \nu$ if there exist a one-to-one and onto function $f : F_\mu \to F_\nu$ such that $\forall \mu_t \in F_\mu$ $\mu_t \cong f(\mu_t)$. We observe that for any finite group G and for any $\mu, \nu \in F(G)$ if μ and ν are equivalent in the sense of Mashinchi and Mukaidonon [75] then they are equivalent in the sense of the definition of Iranmanesh and Naraghi [55].

On the other hand Naraghi [88] defined the equivalence of fuzzy subgroups as follows, let $\mu, \nu \in F(G)$, then μ is equivalent ν denoted $\mu \sim_t \nu$ if and only if $F_\mu = F_\nu$ and Suppμ=Suppν. This definition is equivalent to the Murali and Makamba's [81].

Jain in [60] and Ajmal in [3] gave the following definition, two fuzzy subgroups μ and ν are equivalent denoted $\mu \approx \nu$ if μ and ν have the same chain of level subgroups, that is $\{\mu_t\}_{t \in Im\ \mu} = \{\nu_s\}_{s \in Im\ \nu}$. Firstly this definition is a variant to the Murali and Makamba definition because it does not require finiteness of

the group. Secondly unlike the Murali and Makamba definition that requires that the supports be equal, Jain's [60] in definition 2.13.10 uses the equality of penultimate subgroups in place of supports. This definition basically says that two fuzzy subgroups in an arbitrary group G that are equivalent necessarily have the same penultimate subgroup but their supports may or may not be different. It is also instructive to note that since sup $\mu \in$ Im μ, Inf $\mu \in$ Im μ and sup $\mu >$ Inf $\mu \geq 0$. It is clear that if Sup $\mu =$ Inf μ then the definition of Jain [60] and Ajmal [2] is the same as that of Murali and Makamba [81] since both penultimate subgroups of μ and ν coincide with their supports. If Sup μ $>$ Inf μ, the definition of equivalence in the sense of Murali and Makamba [81] implies equivalence in the sense of Ajmal[2] and Jain [60]. Lastly we observe also that the two definitions coincide when the least element 0 in the evaluation lattice $[0, 1]$, $0 \in$ Im μ. This implies that Inf $\mu = 0$ hence the supports of μ and ν coincide with their penultimate subgroups. It is clear that the definitions of Mordeson [78] and Ajmal [3] are a generalisation of the notion of equality of sets, as is the definition of Mashinchi and Mukaidono in [75]. We note also that the definitions of Mordeson [78] and Ajmal [3] are equivalent. The Murali and Makamba [80] definition was developed for the case of finite group whereas Jain's [60]definition is for any arbitrary group.
The following proposition by Murali and Makamba [81] characterises a fuzzy equivalence relation. It was further verified and improved by Degang et al in [29].

Proposition 3.4.9 [81] Suppose μ and ν are two fuzzy subsets of X such that μ is equivalent to ν. Then for each $t \in [0, 1]$ there is an $s \in [0, 1]$ such that $\mu^t = \nu^s$ or $\mu^t = \nu^{-1}(s, 1]$.

3.5 FUZZY ISOMORPHISM

In [100] Ray introduced his definition of isomorphic fuzzy groups, however it has been shown that this definition of fuzzy isomorphism is weaker than the Murali and Makamba [81] definition used in this dissertation. The same authors in [87] classified the relationship between various notions of isomorphism. They compared this notion of isomorphism to that of an equivalence relation of fuzzy subgroups of finite groups. They discovered that this notion of equivalence is finer than the notion of isomorphism. So this definition of fuzzy isomorphism is a generalisation of the definition in [81] of fuzzy equivalence. For a panorama on isomorphic fuzzy groups and isomorphism between finite chains of subgroups, see Pruszyriska and Dudzicz in [99] and Ray [100], Murali and Makamba [87]. For completeness, we begin by defining a group homomorphism.

Definition 3.5.1 [47]. Let (G, θ) and $(G^{'}, \circ)$ be groups. A homomorphism is a mapping $f : G \to G^{'}$ such that $f(a\theta b) = f(a) \circ f(b)$, $\forall a, b \in G$.

Note 3.1.2: If $f : G \to G^{'}$ is a homomorphism, by $f(\mu)$ we mean the image of a fuzzy subset μ of G and is a fuzzy subset of $G^{'}$ defined by $(f(\mu))(g^{'})$=sup$\{\mu(g) : g \in G, f(g) = g^{'}\}$ if $f^{-1}(g') \neq \emptyset$ and $f(\mu)(g^{'}) = 0$ if $f^{-1}(g^{'}) = \emptyset$ for $g^{'} \in G^{'}$. Similarly if ν is a fuzzy subset of $G^{'}$, the pre-image of ν, $f^{-1}(\nu)$ is a fuzzy subset of G defined by $(f^{-1}(\nu))(g) = \nu(f(g))$.
Exploiting the above definition, we now define a mapping that preserves both structure and group operation.

Definition 3.5.2 [89]. An isomorphism is a homomorphism that is bijective.

Murali and Makamba in [81] gave the following definition of isomorphic fuzzy subgroups:

Definition 3.5.3 . Let $\mu \in F(G)$ and $\nu \in F(G^{'})$. We say μ is fuzzy isomorphic to ν, denoted by $\mu \cong \nu$, if and only if there exists an isomorphism $f : G \to G^{'}$ such that $\mu(x) > \mu(y) \Leftrightarrow \nu(f(x)) > \nu(f(y))$ and $\mu(x) = 0 \Leftrightarrow \nu(f(x)) = 0$.

Proposition 3.5.4 If $\mu \cong \nu$ then $f(\mu) \cong f(\nu)$.

Proposition 3.5.5 If $\mu \cong \nu$ in $G^{'}$ then $f^{-1}(\mu) \cong f^{-1}(\nu)$ in G.

Proof. Straightforward.

Chapter 4

ON EQUIVALENCE OF FUZZY SUBGROUPS OF FINITE ABELIAN p-GROUPS OF RANK TWO

4.1 INTRODUCTION

In the last few years, there has been a growing interest in the notion of a characterisation of a fuzzy subgroup of a finite group. This has led to the study of equivalence relations conducted by several researchers. It is a fact that without any equivalence relation on a fuzzy subgroups of a finite group G, the number of fuzzy subgroup is uncountable, even for the trivial group $\{e\}$. One of the most important aspects that have been considered in this direction is the use of an equivalence relation to classify fuzzy subgroups of a finite p-group. Thus in [82], [84], [85] and [120], the number of distinct fuzzy subgroups for cyclic groups of order $p^n q^m$ (p, q primes) is determined, while [82] deals with this number on a finite cyclic group of square-free order. Also, in [116] a recurrence relation is obtained which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: (arbitrary) finite cyclic groups and finite elementary abelian p-groups of the form $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ where p is a prime number. An explicit formula is given for the first class in [114]. For the case of finite hamiltonian groups, an interesting application has been presented in [116]. Employing the equivalence relation studied in [81], Saltürk and Şiap [103]managed to determine the nature and the number of the fuzzy subgroups of the group of the form $\mathbb{Z}_p + \mathbb{Z}_p + \cdots + \mathbb{Z}_p$ for the same prime number p. Remarkable results were obtained by S. Ngcibi in [94] and [95] with an equivalence relation introduced in [81] to determine the number

of fuzzy subgroups of a finite abelian p-group of rank two, specifically for the forms $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ and $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$ for a fixed prime p and positive integers n and m. This and others may be considered as a special case of our study.
The purpose of this chapter is first to review the studies conducted by Ngcibi in [94] and [95] and further proceed with a brief discussion of the study performed by Saltürk and Şiap [103]. This laid the foundation for our study. We shall employ the natural equivalence relation introduced in [81] to determine the number and nature of fuzzy subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for any fixed prime number p and $n \in \mathbb{N}$. As a principle guide in determining the number of these classes, we first find the number of maximal chains of G. We also employ the definition of isomorphism given in chapter two to determine the number of equivalence and non-isomorphic classes of fuzzy subgroups of this group. We then compare the number of equivalence and isomorphic classes for the group.

4.2 ON FUZZY SUBGROUPS OF p-GROUPS OF THE FORM $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$

4.2.1 Introduction

Before proceeding with the classification of the fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$, we have to give a definition of a p-group and the notion of a cyclic p-group. We also give two theorems on cyclic p-groups.

Definition 4.2.1 [43]. Let G be a group and p a prime. We say G is a p-group if the order of every element of G is a power of p. That is, for all $g \in G$, $|g| = p^k$ for some k depending on g.

Definition 4.2.2 [43]. Let G be a group and r be an element in G such that $G = \{nr : n \in \mathbb{Z}\}$, then G is referred to as a cyclic group. Such an element $r \in G$ is referred to as a generator of G and we denote such by $G =< r >$ which signifies that G is a cyclic group generated by r.

Definition 4.2.3 [43]. A cyclic p-group is a group $G \cong \mathbb{Z}_{p^n}$ for some positive integer n.

4.2.2 On Maximal Chains and Fuzzy Subgroups of $\mathbb{Z}_{p^n}$

The only maximal chain for a cyclic group $\mathbb{Z}_{p^n}$ is $\mathbb{Z}_{p^n} \supset \mathbb{Z}_{p^{n-1}} \supset \mathbb{Z}_{p^{n-2}} \supset \cdots \supset \{0\}$.
Here we will use the above maximal chain to show how non-equivalent fuzzy subgroups are determined in this group. Now let us consider the maximal chain $\mathbb{Z}_{p^n} \supset \mathbb{Z}_{p^{n-1}} \supset \mathbb{Z}_{p^{n-2}} \supset \cdots \supset \{0\}$. For $1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0$, we define a fuzzy subgroup μ on $\mathbb{Z}_{p^n}$ such that: μ assumes λ_n on $\mathbb{Z}_{p^n}$, λ_{n-1} on $\mathbb{Z}_{p^n-1}$,λ_{n-2} on $\mathbb{Z}_{p^n-2}$,$\cdots \lambda_1$ on $\mathbb{Z}_p$ and 1 on 0. The above fuzzy subgroup is simply represented by $1\lambda_1\lambda_2\lambda_3\cdots\lambda_{n-1}\lambda_n$. This group becomes $\mathbb{Z}_p$ when $n = 1$ and any fuzzy subgroup of $\mathbb{Z}_p$ is equivalent to one of the following three: 11,1λ,10 by [81]. The trivial crisp subgroup $\mathbb{Z}_p$ is denoted by 11 and the trivial subgroup 0 by 10 whereas 1λ denotes the fuzzy subgroup $\mu(x) = 1$ if $x = 0$ and $\mu(x) = \lambda$ if $x \neq 0$.

Proposition 4.2.4 [81]. For any given natural number n, there are $2^{n+1} - 1$ distinct fuzzy subgroup of $\mathbb{Z}_{p^n}$.

Theorem 4.2.5 [43]. If a is a generator of a finite cyclic group G of order n , then the other generators of G are elements of the form ra, where gcd $(r, n) = 1$.

Theorem 4.2.6 [43]. Every subgroup of a cyclic group is cyclic. Moreover , if $| < a > | = n$, then the order of any subgroup of $< a >$ is a divisor of n, and for each divisor k of n , the group $< a >$ has exactly one subgroup of order k which is $< a^{\frac{n}{k}} >$.

4.2.3 On Crisp Subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$

As a principle guide in classifying the number of distinct fuzzy subgroups of the above group G, we first find the number of the crisp subgroups and the maximal chains of G. We shall also give the number of non-isomorphic classes of G.
We now use the examples below with specific prime numbers p= $2, 3, 5$ and positive integers $n = 1, \cdots , 4$ as the starting point for our discussion.
For p= $2, 3, 5$ and $n = 1$.

Example 4.2.7 . (1) The group $G = \mathbb{Z}_2 + \mathbb{Z}_2$ contains the following elements $\{(0,0),(0,1),(1,0),(1,1)\}$ and the following crisp subgroups: $\mathbb{Z}_2+\mathbb{Z}_2$, $\mathbb{Z}_2+\{0\}$, $\{0\}+\mathbb{Z}_2$, $<(1,1)>$ and $\{(0,0)\}$
(2) For $G = \mathbb{Z}_3 + \mathbb{Z}_3$, we have the following elements $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$ and the following crisp subgroups: $\mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_2 + \{0\}$, $\{0\} + \mathbb{Z}_3$, $<(1,1)>$, $<(1,2)>$ and $\{(0,0)\}$
(3) The group $G = \mathbb{Z}_5 + \mathbb{Z}_5$, has 25 elements and 8 crisp subgroups: $\mathbb{Z}_5+\mathbb{Z}_5$, $\mathbb{Z}_5+\{0\}$, $\{0\}+\mathbb{Z}_5$, $<(1,1)>$, $<(1,2)>$, $<(1,3)>$, $<(1,4)>$ and $\{(0,0)\}$

From the above examples, we observed that the total number of crisp subgroups in the group is being influenced by the prime p, n and the rank of the group. Hence we have the following proposition.

Proposition 4.2.8 . The group $G = \mathbb{Z}_p + \mathbb{Z}_p$ has $np + (n + 2)$ crisp subgroups.

Proof. See [94]

4.2.4 The Number of Crisp Subgroups of $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p$

Now let us enumerate the total number of crisp subgroups of the group above where $n = 2$. We begin with $p = 2$, 3 and 5
For $p = 2$ and $n = 2$, the group $G = \mathbb{Z}_{2^2} + \mathbb{Z}_2$ has 8 elements and the following subgroups: $\mathbb{Z}_{2^2} + \mathbb{Z}_2$, $\mathbb{Z}_{2^2} + \{0\}$, $\{0\} + \mathbb{Z}_2$, $<(1,1)>$, $<(2,0)>$, $<(2,1)>$, $<(2,0),(0,1)>$ and $\{(0,0)\}$
For $p = 3$ and $n = 2$, $G = \mathbb{Z}_{3^2} + \mathbb{Z}_3$, there are 27 elements and a total number of 10 crisp subgroups, listed below:
$\mathbb{Z}_{3^2} + \mathbb{Z}_3$, $\mathbb{Z}_{3^2} + \{0\}$, $\{0\} + \mathbb{Z}_3$, $<(1,1)>$, $<(1,2)>$, $<(3,0)>$, $<(3,1)>$, $<(3,2)>$, $<(3,0),(0,1)>$ and $\{(0,0)\}$
For $p = 5$ and $n = 2$, we have:
$\mathbb{Z}_{5^2} + \mathbb{Z}_5$, $\mathbb{Z}_{5^2} + \{0\}$, $\{0\} + \mathbb{Z}_5$, $<(1,1)>$, $<(1,2)>$, $<(1,3)>$, $<(1,4)>$, $<(5,0)>$, $<(5,1)>$, $<(5,2)>$, $<(5,3)>$, $<(5,4)>$, $<(5,0),(0,1)>$ and $\{(0,0)\}$
Next, we look at the crisp subgroups of $G = \mathbb{Z}_{p^3} + \mathbb{Z}_p$
For $G = \mathbb{Z}_{2^3} + \mathbb{Z}_2$, there are 16 elements and 11 crisp subgroups: $\mathbb{Z}_{2^3} + \mathbb{Z}_2$, $\mathbb{Z}_{2^3} + \{0\}$, $\{0\} + \mathbb{Z}_2$, $<(1,1)>$, $<(2,0)>$, $<(2,1)>$, $<(4,0)>$, $<(4,1)>$,

$<(2,0),(0,1)>$, $<(4,0),(0,1)>$ and $\{(0,0)\}$.
For $G=\mathbb{Z}_{3^3}+\mathbb{Z}_3$, there are 81 elements and 14 crisp subgroups: $\mathbb{Z}_{3^3}+\mathbb{Z}_3$, $\mathbb{Z}_{3^3}+\{0\}$, $\{0\}+\mathbb{Z}_3$, $<(1,1)>$, $<(1,2)>$, $<(3,0)>$, $<(3,1)>$, $<(3,2)>$, $<(9,0)>$, $<(9,1)>$, $<(9,2)>$, $<(3,0),(0,1)>$, $<(9,0),(0,1)>$ and $\{(0,0)\}$
For $p=5$, we have the following crisp subgroups:
$\mathbb{Z}_{5^3}+\mathbb{Z}_5$, $\mathbb{Z}_{5^3}+\{0\}$, $\{0\}+\mathbb{Z}_5$, $<(1,1)>$, $<(1,2)>$, $<(1,3)>$, $<(1,4)>$, $<(5,0)>$, $<(5,1)>$, $<(5,2)>$, $<(5,3)>$, $<(5,4)>$, $<(25,0)>$, $<(25,1)>$, $<(25,2)>$ $<(25,3)>$ $<(25,4)>$, $<(5,0),(0,1)>$, $<(25,0),(0,1)>$ and $\{(0,0)\}$
Now let's consider $G=\mathbb{Z}_{p^4}+\mathbb{Z}_p$ and $p=2,3$ and 5. We obtain the following crisp subgroups:
For $p=2$: $\mathbb{Z}_{2^4}+\mathbb{Z}_2$, $\mathbb{Z}_{2^4}+\{0\}$, $\{0\}+\mathbb{Z}_2$, $<(1,1)>$, $<(2,0)>$, $<(2,1)>$, $<(4,0)>$, $<(4,1)>$, $<(8,0)>$, $<(8,1)>$, $<(2,0),(0,1)>$, $<(4,0),(0,1)>$, $<(8,0),(0,1)>$ and $\{(0,0)\}$.
For $p=3$: $\mathbb{Z}_{3^4}+\mathbb{Z}_3$, $\mathbb{Z}_{3^4}+\{0\}$, $\{0\}+\mathbb{Z}_3$, $<(1,1)>$, $<(1,2)>$, $<(3,0)>$, $<(3,1)>$, $<(3,2)>$, $<(9,0)>$, $<(9,1)>$, $<(9,2)>$, $<(27,0)>$, $<(27,1)>$, $<(27,2)>$, $<(3,0),(0,1)>$, $<(9,0),(0,1)>$ $<(27,0),(0,1)>$ and $\{(0,0)\}$
For $p=5$: $\mathbb{Z}_{5^4}+\mathbb{Z}_5$, $\mathbb{Z}_{5^4}+\{0\}$, $\{0\}+\mathbb{Z}_5$, $<(1,1)>$, $<(1,2)>$, $<(1,3)>$, $<(1,4)>$, $<(5,0)>$, $<(5,1)>$, $<(5,2)>$, $<(5,3)>$, $<(5,4)>$, $<(25,0)>$, $<(25,1)>$, $<(25,2)>$ $<(25,3)>$ $<(25,4)>$, $<(125,0)>$, $<(125,1)>$, $<(125,2)>$, $<(125,3)>$, $<(125,4)>$, $<(5,0),(0,1)>$, $<(25,0),(0,1)>$, $<(125,0),(0,1)>$ and $\{(0,0)\}$
Based on the above examples, we conjecture a generalisation for the total number of crisp subgroups of the group $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p$ in the form of a theorem

Theorem 4.2.9 . The group $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p$ has $n(p+1)+2$ subgroups.

Proof. By induction on n. Let $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p$. Then G has the following maximal subgroups: $\mathbb{Z}_{p^{n-1}}+\mathbb{Z}_p$; $\mathbb{Z}_{p^n}+0$; $<(k,1)>$ for $k=1,2,3\cdots,p-1$. This gives $p+1$ maximal subgroups. Clearly the theorem is true for $n=1$ since $\mathbb{Z}_p+\mathbb{Z}_p$ has $p+3$ subgroups. Assume the theorem is true for any $k<n$. Let $G=\mathbb{Z}_{p^{k+1}}+\mathbb{Z}_p$. So G has the maximal subgroups $\mathbb{Z}_{p^k}+\mathbb{Z}_p$; $\mathbb{Z}_{p^n}+0$; $<(s,1)>$ for $s=1,2,3\cdots,p-1$. By induction, $\mathbb{Z}_{p^k}+\mathbb{Z}_p$ has $k(p+1)+2$ subgroups. Each cyclic maximal subgroup contributes only itself. Adding all the subgroups of G

gives the total number of subgroups to be $k(p+1)+2+p+1 = (k+1)(p+1)+2$. This completes the proof. □

4.2.5 On Maximal Chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$

Without maximal chains, it will be tedious if not impossible to determine the total number of fuzzy subgroups in a group. We begin our discussion by first defining a maximal chain and then proceed with specific examples based on the above group. We also provide some few results in the form of lemmas.

Definition 4.2.10 . A chain of subgroups of the group G is said to be maximal if it cannot be refined any more or no more subgroups can be inserted in the chain.

Definition 4.2.11 [89]. In a maximal chain δ on G, a subgroup that distinguishes the maximal chain from others is called a distinguishing factor. If there is more that one such subgroups, they are called distinguishing factors.

EXAMPLES OF MAXIMAL CHAINS For $p = 3$ and $n = 3$, we obtain the following maximal chains:

$G \supseteq \mathbb{Z}_{3^3} + \{0\} \supseteq < (3,0) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,1) > \supseteq < (3,0) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,2) > \supseteq < (3,0) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (3,0) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (3,1) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (3,2) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (0,1),(9,0) > \supseteq \{0\} + \mathbb{Z}_3 \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (0,1),(9,0) > \supseteq < (9,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (0,1),(9,0) > \supseteq < (9,1) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(3,0) > \supseteq < (0,1),(9,0) > \supseteq < (9,2) > \supseteq \{(0,0)\}$.

For $p = 5$ and $n = 3$, we obtain the following maximal chains:

$G \supseteq \mathbb{Z}_{5^3} + \{0\} \supseteq < (5,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,1) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,2) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,3) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,4) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,1) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,2) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,3) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,4) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq \{0\} + \mathbb{Z}_5 \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,1) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,2) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,3) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,4) > \supseteq \{(0,0)\}$.

For $p = 3$ and $n = 4$, we obtain the following maximal chains:
$G \supseteq \mathbb{Z}_{3^4} + \{0\} \supseteq < (3,0) > \supseteq < (9,0) > \supseteq \langle (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (1,1) > \supseteq < (3,0) > \supseteq < (9,0) > \supseteq < (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (1,2) > \supseteq < (3,0) > \supseteq < (9,0) > \supseteq < (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (3,0) > \supseteq < \supseteq < (27,0) > (9,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (3,1) > \supseteq < \supseteq < (27,0) > (9,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (3,2) > \supseteq < \supseteq < (27,0) > (9,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (9,0) > \supseteq < (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (9,1) > \supseteq < (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (9,2) > \supseteq < (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (0,1), (27,0) > \supseteq < (27,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (0,1), (27,0) > \supseteq < (27,1) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (0,1), (27,0) > \supseteq < (27,2) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (3,0) > \supseteq < (0,1), (9,0) > \supseteq < (0,1), (27,0) > \supseteq \{0\} + \mathbb{Z}_3 \supseteq \{(0,0)\}$.

Note: 4.1.1. Looking at the maximal chains above, we observe that the length of each maximal chain is given by $n + 2$, where $n \in \mathbb{N}$ is the exponent and 2 is the rank of the group.
For any natural number n, we have the following numbers of maximal chains:

Lemma 4.2.12 $G = \mathbb{Z}_p + \mathbb{Z}_p$ has $p + 1$ maximal chains.

Lemma 4.2.13 $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p$ has $2p + 1$ maximal chains. (For $n = 2$)

Lemma 4.2.14 $G = \mathbb{Z}_{p^3} + \mathbb{Z}_p$ has $3p + 1$ maximal chains. (For $n = 3$)

In general, for $n \in N$ and any prime number p, we have the following:

Proposition 4.2.15 [94]. $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ has $(n-1)(p-1) + (p+1) + (n-1)$ maximal chains.

Proof. See [94].

4.2.6 On Keychains

In this subsection we examine what is called a keychain which inherently resulted in the study of fuzzy subsets of a finite set X. These membership grades of elements of X are considered in the unit interval. We will first recall the basic definitions and some pertinent known results on keychains and then discuss usefulness of keychains in the counting of distinct fuzzy subgroups of a group in this subsection.

Definition 4.2.16 By [82]. A collection of real numbers on $[0, 1]$ of the form $1 > \lambda_1 > \lambda_2 \cdots > \lambda_{n-1} > \lambda_n$, where the last entry may or may not be zero is called a finite n-chain. This is usually expressed in the descending order as $1\lambda_1\lambda_2 \cdots \lambda_n$.

Definition 4.2.17 [82]. The numbers $1, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n$ are referred to as pins. Observe that 1 occupies the first position whilst λ_i occupies the (i + 1)th position for $i = 1, 2, 3 \cdots, n$. Thus the length of an n-chain is $n + 1$, and hence the n-chain contains $n + 1$ available positions.

These positions will be indispensable in our further discussions.

Definition 4.2.18 . An n-chain is called a keychain if $1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_{n-1} \geq \lambda_n$.

Note: 4.1.2. The pins in a keychain may either be distinct or not. The keychain with only pins 1 and 0 can be considered as a crisp set.

Definition 4.2.19 [83]. A flag on a finite group G is an increasing maximal chain of $n+1$ subgroups of G starting with the trivial subgroup $\{0\}$.

Definition 4.2.20 [83]. Let ξ be a flag on G and ι be a keychain. The pair $\{\xi, \iota\}$ is called a pinned-flag and we represent it as follows: $0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \subset G_3^{\lambda_3} \subset \cdots \subset G_n^{\lambda_n}$.

4.3 ENUMERATING NON-EQUIVALENT FUZZY SUBGROUPS

There are various techniques that are used in the counting of distinct fuzzy subgroups of a finite group. These counting techniques are derived from the interpretation of the definition of fuzzy equivalence relations used. In this section we give a brief explanation of the two counting techniques used in the enumeration of distinct equivalence classes of fuzzy subgroups. Maximal subgroups, maximal chains (flags) of the group and distinguishing factors of a flag play a pivotal role in the enumeration process. Basically the procedure is to list down the maximal subgroups of the group first and then form maximal chains of the group.

For instance, if we consider the following maximal chains

$G \supseteq \mathbb{Z}_{2^2} + \{0\} \supseteq < (2,0) > \supseteq \{(0,0)\}$

$G \supseteq < (1,1) > \supseteq < (2,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(2,0) > \supseteq < (2,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(2,0) > \supseteq < (2,1) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1),(2,0) > \supseteq \{0\} + \mathbb{Z}_2 \supseteq \{(0,0)\}$

of the finite group $G = \mathbb{Z}_{2^2} + \mathbb{Z}_2$, we notice the following:

(i). With $n = 2$, there are $2^{n+1} - 1$ distinct equivalence classes of fuzzy subgroups contributed by the first chain [81].

(ii) The next maximal chain contributes a number excluding those counted in chain (i) and it follows in that order till we exhaust all the maximal chains. We assert this idea in the following propositions:

Proposition 4.3.1 . Suppose G has the following maximal chains $G = A_n \supset A_{n-1} \supset A_{n-2} \supset \cdots \supset A_0$ (i), $G = B_n \supset B_{n-1} \supset B_{n-2} \supset \cdots \supset B_0$ (ii) and a third maximal chain $G = C_n \supset C_{n-1} \supset C_{n-2} \supset \cdots \supset C_0$(iii) distinct from (i) and (ii), and suppose $\exists j \in \mathbb{N}$ such that $B_j \neq A_j$,$C_j \neq B_j$ and $C_j \neq A_j$, then the number of distinct fuzzy subgroups contributed by (ii) is equal to the number contributed by (iii) for $n \geq 2$

Proof. See [89]

Proposition 4.3.2 [89]. In the course of enumerating distinct fuzzy subgroups, let the first maximal chain have $2^{n+1} - 1$ fuzzy subgroups. Suppose chain (i) has a distinguishing factor, then the number of fuzzy subgroups of maximal chain (i), $i \neq 1$ is equal to $\frac{2^{n+1}}{2}$for $n \geq 3$.

Proposition 4.3.3 [89]. In the process of counting fuzzy subgroups, let (k) be a maximal chain $G = K_n \supset K_{n-1} \supset K_{n-2} \supset \cdots \supset K_0$ such that all the K_i's have appeared in some previous maximal chain (i) for $i = 1, 2, 3, ..., k$ and have been used as distinguishing factors. If a pair of subgroups in the chain has not appeared in any previous chain, then the number of fuzzy subgroups of (k) is equal to $\frac{2^{n+1}}{2^2}$ for $n \geq 3$.

Note 4.1.4: If there is no distinguishing factor (new subgroup) in a maximal chain (i) but there is a new pair or a distinguishing pair (not used in the $i-1$ chains) then the number of fuzzy subgroups of the maximal chain (i) is equal to $\frac{2^{n+1}}{2^2}$. Inductively, if there is no distinguishing pair of subgroups but there is a distinguishing triple of subgroups in (i), then the number of fuzzy subgroups contributed by the maximal chain (i),is equal to $\frac{2^{n+1}}{2^3}$. Thus this argument continues inductively.

4.4 COMPUTATION OF DISTINCT FUZZY SUBGROUPS USING KEYCHAINS APPROACH

INTRODUCTION. Here we will start by looking at how keychains are used in determining the number of distinct fuzzy subgroups of a finite group. This approach makes use of the crisp properties of a finite group G, amongst are the maximal subgroups that are used to generate the maximal chains (flags), followed by the lengths of the maximal chains of the group. This length allows us to discover the number of pins or the levels in every keychain and lastly the number of maximal chains of the group. Through the levels, we are then able to list down all the potential keychains that correspond to the fuzzy subgroups. We know for a fact that every one of the keychains corresponds to fuzzy subgroups, and the number of distinct fuzzy subgroups exclusively depends on the number of flags of the finite group G as well as the nature of the keychain itself. Moreover, we know that a flag with n levels can be characterised by $2^n - 1$ keychains. This enumerating method requires that, once the entire flags of the group G are cataloged , we choose a keychain and "run" it on every maximal chain of the group G to obtain the number of distinct fuzzy subgroups it contributes. Subsequently, we select a second keychain and repeat the technique until the entire keychains are exhausted. This procedure needs to be executed diligently to prevent over-counting. To obtain the total number of the distinct fuzzy subgroups for the entire group, we sum all distinct fuzzy subgroups obtained from every keychain. We use the example below to explicitly illustrate this technique.

Example 4.4.1 . Consider $G = \mathbb{Z}_3 + \mathbb{Z}_3 = \{(0,0),(0,1),(0,1),(1,0),(1,1),$ $(1,2),(2,0),(2,1),(2,2)\}$ and let $\mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_2 + \{0\}$, $\{0\} + \mathbb{Z}_3$, $< (1,1) >$, $< (1,2) >$ and $\{(0,0)\}$ be the subgroups of G where $(0,0)$ is the identity element. There are four maximal chains or flags for this group. These are:
$\mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \{0\} \supseteq \{(0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \mathbb{Z}_3 \supseteq \{(0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1) > \supseteq \{(0,0)\}$
$\mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,2) > \supseteq \{(0,0)\}$
We notice that each maximal chain or flag is of length three, hence we can characterise each fuzzy subgroup by a keychain with three pins and in fact we have $2^3 - 1 = 7$ such keychains, viz.
111 11λ 110 $1\lambda\lambda$ $1\lambda\beta$ $1\lambda 0$ 100
Now to get the total number of distinct equivalence classes of fuzzy subgroups we use all the seven keychains on the four flags as follows:
We define fuzzy subgroups μ,ν,γ, and ϕ on the 4 flags respectively to be represented by the keychain 11λ. Thus $\mu(x) = \begin{cases} 1 & if \quad x = (0,0) \\ 1 & if \quad x \in \mathbb{Z}_3 + \{0\} \setminus \{(0,0)\} \\ \lambda & if \quad x \in \mathbb{Z}_3 + \mathbb{Z}_3 \setminus \mathbb{Z}_3 + \{0\} \end{cases}$.

$$\nu(x) = \begin{cases} 1 & if \quad x = (0,0) \\ 1 & if \quad x \in \{0\} + \mathbb{Z}_3 \setminus \{(0,0)\} \\ \lambda & if \quad x \in \mathbb{Z}_3 + \mathbb{Z}_3 \setminus \{0\} + \mathbb{Z}_3 \end{cases}.$$

$$\gamma(x) = \begin{cases} 1 & if \quad x = (0,0) \\ 1 & if \quad x \in < (1,1) > \setminus \{(0,0)\} \\ \lambda & if \quad x \in \mathbb{Z}_3 + \mathbb{Z}_3 \setminus < (1,1) > \end{cases}.$$

$$\phi(x) = \begin{cases} 1 & if \quad x = (0,0) \\ 1 & if \quad x \in < (1,2) > \setminus \{(0,0)\} \\ \lambda & if \quad x \in \mathbb{Z}_3 + \mathbb{Z}_3 \setminus < (1,2) > \end{cases}.$$

It is clear that all 4 fuzzy subgroups are distinct because they are defined on distinct maximal chains although each is represented by the keychain 11λ. Thus we say the keychain 11λ yields 4 distinct fuzzy subgroups. Similarly each of the keychains $1\lambda\beta$, $1\lambda 0$ and 110 yields 4 distinct fuzzy subgroups. Each of the remaining 3 keychains 111, $1\lambda\lambda$ and 100 yields only one fuzzy subgroup μ_i as follows:

$\mu_1(x) = 1$ for all $x \in G$

$$\mu_2(x) = \begin{cases} 1 & \text{if} \quad x = (0,0) \\ \lambda & \text{otherwise} \end{cases}$$

$$\mu_3(x) = \begin{cases} 1 & \text{if} \quad x = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Hence the total number of distinct fuzzy subgroups of G is equal to $4{+}4{+}1{+}3 = 19$.

This counting technique used in the above example can be very cumbersome. Thus for the remainder of the dissertation we use the earlier counting technique.

4.5 COUNTING THE TOTAL NUMBER OF NON-EQUIVALENT FUZZY SUBGROUPS OF $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ FOR $n \geq 1$

In this section we will look at how to generalise the counting of non-equivalent fuzzy subgroups. First we list some useful results for a rank 2 abelian group.

Proposition 4.5.1 [94]. $G = \mathbb{Z}_p + \mathbb{Z}_p$ has $4p + 7$ non-equivalent fuzzy subgroups.

Proposition 4.5.2 [94]. $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p$ has $16p + 15$ distinct fuzzy subgroups.

Proposition 4.5.3 [94]. $G = \mathbb{Z}_{p^3} + \mathbb{Z}_p$ has $48p + 31$ distinct fuzzy subgroups.

More generally, we have the following

Theorem 4.5.4 By [95]. $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has $2^{n+1}C(n, 1)p + 2^{n+2} - 1$ distinct fuzzy subgroups.

4.6 ISOMORPHIC CLASSES OF FUZZY SUBGROUPS OF $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$

Before we start dealing with the isomorphic classes of fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_p$, let us recall that two or more flags or maximal chains are said to be isomorphic if their lengths are equal and their corresponding components are isomorphic subgroups.

Example 4.6.1 . Here we give an example for the case $p = 5$ and $n = 4$ which will facilitate our discussion on how to determine the number of non-isomorphic classes of fuzzy subgroups in the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$. Consider all the maximal chains of G as listed below:
$G \supseteq \mathbb{Z}_{5^4} + \{0\} \supseteq < (5,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (1,1) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (1,2) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (1,3) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (1,4) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,1) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,2) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,3) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (5,4) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,1) > \supseteq < (125,0) > \supseteq \{(0,0)\}$
$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,2) > \supseteq < (125,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,3) > \supseteq < (125,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (25,4) > \supseteq < (125,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (0,1), (125,0) > \supseteq < (125,0) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (0,1), (125,0) > \supseteq < (125,1) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (0,1), (125,0) > \supseteq < (125,2) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (0,1), (125,0) > \supseteq < (125,3) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (0,1), (125,0) > \supseteq < (125,4) > \supseteq \{(0,0)\}$

$G \supseteq < (0,1), (5,0) > \supseteq < (0,1), (25,0) > \supseteq < (0,1), (125,0) > \supseteq \{0\} + \mathbb{Z}_5 \supseteq \{(0,0)\}$

Looking closely at the above maximal chains, we can see that the first five chains are isomorphic, so they can be collapsed and considered as one, and consequently they yield $2^{4+2} - 1 = 2^6 - 1$ distinct fuzzy subgroups. The next five maximal chains are distinguished from the first five by the subgroup $< (0,1), (5,0) >$ and they are isomorphic. Thus they can be collapsed into a single maximal chain, yielding 2^5 distinct fuzzy subgroups by Proposition 4.3.2 and the discussion thereafter. The next five maximal chains are distinguished from the first and the second five by the subgroup $< (0,1), (25,0) >$ and they too are isomorphic. Thus they can be collapsed into a single maximal chain, yielding 2^5 distinct fuzzy subgroups by Proposition 4.3.2 and the discussion immediately following the proposition. The remaining five maximal chains are distinguished from the first fifteen by the subgroup $< (0,1), (125,0) >$ and are also isomorphic. So they too can be collapsed into a single maximal chain yielding 2^5 distinct fuzzy subgroups as in the other cases. Hence the total number of non-isomorphic fuzzy subgroups of G is $2^6 - 1 + 2^5 + 3$.

Note: 4.1.5. In the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$, we notice that there is only one maximal chain up to isomorphism for $n = 1$, while for $n \geq 2$, the group $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has n maximal chains up to isomorphism. We state this fact as a theorem.

Theorem 4.6.2 . The group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ has n non-isomorphic maximal chains.

Proof. Let $n = 1$. Then $G = \mathbb{Z}_p + \mathbb{Z}_p$ can only have one maximal chain up to isomorphism. Suppose $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ has k maximal chains. Let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p$. Then the maximal subgroups of G are $\mathbb{Z}_{p^{k+1}} + 0$ and $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ up to isomorphism. By induction, $\mathbb{Z}_{p^k} + \mathbb{Z}_p$ has k maximal chains while $\mathbb{Z}_{p^{k+1}} + 0$ has only 1 maximal chain up to isomorphism. Thus G has $k + 1$ non-isomorphic maximal chains. This completes the proof.

Theorem 4.6.3 . The total number of non-isomorphic fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ is equal to $2^{n+2} - 1 + (n - 1)2^{n+1}$.

Proof. Each maximal chain (up to isomorphism) of G has a single distinguishing factor (subgroup). Thus the number of non-isomorphic fuzzy subgroups is equal to $2^{n+2} - 1 + (n - 1)2^{n+1}$ since there are n maximal chains (up to isomorphism) in total. This completes the proof.
Observe that for $p = 5$ and $n = 4$, the formula gives the number of distinct fuzzy subgroups as $2^6 - 1 + 2^5 + 3$ which agrees with the previous example 4.6.1.

Chapter 5

CRISP SUBGROUPS AND MAXIMAL CHAINS OF $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$

5.1 INTRODUCTION

The counting of distinct (non-equivalent) fuzzy subgroups, as indicated earlier in the previous chapters, uses maximal chains of crisp subgroups. In this regard, there are more non-equivalent fuzzy subgroups in any group of order at least 2, than crisp subgroups. Delsarte [30] in 1948 studied all subgroups of a given group, Vogt [119] also studied the lattice of subgroups.
A group of the form $(\mathbb{Z}_p)^n$ is called an elementary abelian p-group for a prime number p. In our study of subgroups of $\mathbb{Z}_{p^n} + (\mathbb{Z}_p)^2$, we will make use of the well-known formulae
1. $1 + p + p^2 + p^3 + \cdots + p^{d_G - 1}$ for the number of maximal subgroups of a p-group G where d_G is the rank of G. The rank of a group G is the minimal number of generators of G.
2. $\prod_{k=0}^{n-1} \frac{p^{m-k}-1}{p^{n-k}-1}$ for the number of subgroups of order p^n in a p-group G of order p^m. [106]
Thomas Stehling [109] in 1992 considered how to compute the number of subgroups of a finite abelian group. A new method of proving some classical theorems of abelian groups was established by *Tărnăuceanu* [112] in 2007 and subsequently he presented an arithmetic method of counting the subgroups of a finite abelian group[113] in 2010. On the other hand, Grigore *Călugăreanu* [16] also worked on the total number of subgroups of a finite abelian group. Mario Hampejs and *László Tóth* [50] worked on the subgroups of finite abelian

groups of rank three and proposed that the total number of subgroups of $(\mathbb{Z}_p)^3 = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ is given by $s(p) = \sum_{k=0}^{3} \begin{bmatrix} 3 \\ k \end{bmatrix} = 2\,(p^2 + p + 2)$ where $\begin{bmatrix} 3 \\ k \end{bmatrix}$ is the Gaussian coefficient, for $0 \leq k \leq 3$, and an arbitrary positive integer $1 \leq$ p ≤ 50. M. *Tărnăuceanu* and L. Bentea [116] successfully used a recurrence relation studied in [116] to count the number of distinct fuzzy subgroups for two classes of finite abelian groups, namely finite cyclic groups and finite elementary abelian p-groups and have given an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ where p is a prime. Esengul Salturk and Irfan Siap [103] used the equivalence relation defined by Murali and Makamba in [81] to study the structure of equivalence classes of fuzzy subgroups of $\mathbb{Z}_p^n$ for any given prime p and positive integer n indicating the rank of the group. These studies however, centered only on the cases where the prime p is of an exponent one (1). In this dissertation we use the equivalence relation studied in [81] to study the structure of equivalence classes of fuzzy subgroups of a finite abelian group of rank three. Our focus is on the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for $n \geq 1$ and a fixed prime number p. In what follows, we manually list all the crisp subgroups in this particular group using the prime number $p = 2, 3, 5$ and 7 and $n = 1, 2, 3, 4$ as representative, leading to a generalisation of the crisp subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for any natural number n. We also construct their respective maximal chains and give a general formula for the number of the maximal chains of G.

5.2 ON CRISP SUBGROUPS OF $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$

Proposition 5.2.1 The group $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ has $2p^2 + 2p + 4$ subgroups.

Proof. By [94], the group $\mathbb{Z}_p + \mathbb{Z}_p$ has $p + 3$ subgroups. Now any maximal subgroup of G is isomorphic to $H = \mathbb{Z}_p + \mathbb{Z}_p + 0$ of order p^2. The number of such maximal subgroups is $\sigma_1 = 1 + p + p^2 + p^3 + \cdots + p^{d_G - 1}$ with $d_G = 3$. Thus $\sigma_1 = 1 + p + p^2$.
Next the number of subgroups of order p is $\sigma_1 = \prod_{k=0}^{n-1} \frac{p^{m-k} - 1}{p^{n-k} - 1}$. Now $m = 3$ and $n = 1$, thus $\sigma_1 = \frac{p^3 - 1}{p - 1} = p^2 + p + 1$. Adding the two trivial subgroups 0 and G, the total number of subgroups of G is $2p^2 + 2p + 4$. □

Now we aim to find a simple formula for the number of subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$. So next we consider the case when $p = 2$ and $n = 2$. We manually computed the number of subgroups to be $27 = 2(2.2.2 + 2 + 1) + 3 + 2$.
For $p = 3$ and $n = 2$, we manually computed the number of subgroups to be $50 = 3(2.3.2 + 2 + 1) + 3 + 2$.
Proceeding inductively, the number of subgroups of $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$ is equal to $p(2pn + n + 1) + 3 + n$.
In order to observe a clear pattern, we start by manually listing all the crisp subgroups contained in the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for $p = 2, 3, 5$ and 7 when n is 2.
Now for $p = 2$ and $n = 2$, the group $G = \mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2$ has 27 crisp subgroups, as listed below:
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2$, $\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \{0\}$, $\mathbb{Z}_{2^2} + \{0\} + \mathbb{Z}_2$, $\{0\} + \mathbb{Z}_2 + \mathbb{Z}_2$, $\mathbb{Z}_{2^2} + \{0\} + \{0\}$, $\{0\} + \mathbb{Z}_2 + \{0\}$, $\{0\} + \{0\} + \mathbb{Z}_2$, $< (0, 1, 1) >$, $< (1, 1, 1) >$, $< (1, 1, 0) >$, $< (1, 0, 1) >$, $< (2, 0, 0) >$, $< (2, 0, 1) >$, $< (2, 1, 0) >$, $< (2, 1, 1) >$, $< (1, 1, 1), (1, 1, 0) >$, $< (1, 1, 1), (1, 0, 1) >$, $< (1, 1, 1), (0, 1, 1) >$, $< (1, 1, 0), (1, 0, 1) >$, $< (2, 0, 0), (0, 1, 0), (0, 0, 1) >$, $< (2, 0, 0), (2, 1, 0) >$, $< (2, 0, 0), (2, 0, 1) >$, $< (2, 1, 0), (0, 1, 1) >$, $< (2, 1, 0), (2, 1, 1) >$, $< (2, 1, 1), (0, 1, 1) >$, $< (2, 1, 1), (2, 0, 1) >$ and $\{(0, 0, 0)\}$
Now for $p = 3$ and $n = 2$, the group $G = \mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3$ has 50 crisp subgroups, as listed below:
$\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \{0\}$, $\mathbb{Z}_{3^2} + \{0\} + \mathbb{Z}_3$, $\{0\} + \mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_{3^2} + \{0\} + \{0\}$, $\{0\} + \mathbb{Z}_3 + \{0\}$, $\{0\} + \{0\} + \mathbb{Z}_3$, $< (0, 1, 1) >$, $< (0, 1, 2) >$, $< (1, 0, 1) >$, $< (1, 0, 2) >$, $< (1, 1, 0) >$, $< (1, 1, 1) >$, $< (1, 1, 2) >$, $< (1, 2, 0) >$, $< (1, 2, 1) >$, $< (1, 2, 2) >$, $< (1, 1, 0), (1, 1, 1) >$, $< (1, 1, 0), (1, 2, 1) >$, $< (1, 1, 0), (1, 2, 2) >$, $< (1, 1, 1), (1, 2, 0) >$, $< (1, 1, 1), (1, 2, 1) >$, $< (1, 1, 1), (1, 2, 2) >$, $< (1, 1, 2), (1, 2, 0) >$, $< (1, 1, 2), (1, 2, 1) >$, $< (1, 1, 2), (1, 2, 2) >$, $< (1, 2, 0), (1, 2, 1) >$, $< (3, 0, 0), (0, 1, 0), (0, 0, 1) >$, $< (3, 0, 0) >$, $< (3, 0, 1) >$, $< (3, 0, 2) >$, $< (3, 1, 0) >$, $< (3, 1, 1) >$, $< (3, 1, 2) >$, $< (3, 2, 0) >$, $< (3, 2, 1) >$, $< (3, 2, 2) >$, $< (3, 0, 0), (3, 0, 1) >$, $< (3, 0, 0), (3, 1, 0) >$, $< (3, 0, 0), (3, 1, 1) >$, $< (3, 0, 0), (3, 1, 2) >$, $< (3, 1, 0), (3, 1, 1) >$, $< (3, 1, 0), (3, 2, 1) >$, $< (3, 1, 0), (3, 2, 2) >$, $< (3, 1, 1), (3, 2, 0) >$, $< (3, 1, 1), (3, 2, 1) >$,

$< (3,1,2),(3,2,0) >$, $< (3,1,2),(3,2,2) >$, $< (3,2,0),(3,2,1) >$, and $\{(0,0,0)\}$.
For $p = 5$ and $n = 2$, the group $G = \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5$ contains 120 crisp subgroups, as listed below:
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\}$, $\mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5$, $\{0\} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^2} + \{0\} + \{0\}$, $\{0\} + \mathbb{Z}_5 + \{0\}$, $\{0\} + \{0\} + \mathbb{Z}_5$, $< (0,1,1) >$, $< (0,1,2) >$, $< (0,1,3) >$, $< (0,1,4) >$,
$< (1,0,1) >$, $< (1,0,2) >$, $< (1,0,3) >$, $< (1,0,4) >$, $< (1,1,0) >$,
$< (1,1,1) >$, $< (1,1,2) >$, $< (1,1,3) >$, $< (1,1,4) >$, $< (1,2,0) >$,
$< (1,2,1) >$, $< (1,2,2) >$, $< (1,2,3) >$, $< (1,2,4) >$, $< (1,3,0) >$,
$< (1,3,1) >$, $< (1,3,2) >$, $< (1,3,3) >$, $< (1,3,4) >$, $< (1,4,0) >$,
$< (1,4,1) >$, $< (1,4,2) >$, $< (1,4,3) >$, $< (1,4,4) >$,
$< (1,1,0),(1,1,1) >$, $< (1,1,0),(1,2,1) >$, $< (1,1,0),(1,2,2) >$,
$< (1,1,0),(1,2,3) >$, $< (1,1,0),(1,2,4) >$, $< (1,1,1),(1,2,0) >$,
$< (1,1,1),(1,2,1) >$, $< (1,1,1),(1,2,2) >$, $< (1,1,1),(1,2,3) >$,
$< (1,1,1),(1,2,4) >$, $< (1,1,2),(1,2,0) >$, $< (1,1,2),(1,2,1) >$,
$< (1,1,2),(1,2,2) >$, $< (1,1,2),(1,2,3) >$, $< (1,1,2),(1,2,4) >$,
$< (1,1,3),(1,2,0) >$, $< (1,1,3),(1,2,1) >$, $< (1,1,3),(1,2,2) >$,
$< (1,1,3),(1,2,3) >$, $< (1,1,3),(1,2,4) >$, $< (1,1,4),(1,2,0) >$,
$< (1,1,4),(1,2,1) >$, $< (1,1,4),(1,2,2) >$, $< (1,1,4),(1,2,3) >$,
$< (1,1,4),(1,2,4) >$, $< (1,2,0),(1,2,1) >$, $< (1,3,0),(1,3,1) >$,
$< (1,4,0),(1,4,1) >$, $< (5,0,0),(0,1,0),(0,0,1) >$,
$< (5,0,0) >$, $< (5,0,1) >$, $< (5,0,2) >$, $< (5,0,3) >$, $< (5,0,4) >$,
$< (5,1,0) >$, $< (5,1,1) >$, $< (5,1,2) >$, $< (5,1,3) >$, $< (5,1,4) >$,
$< (5,2,0) >$, $< (5,2,1) >$, $< (5,2,2) >$, $< (5,2,3) >$, $< (5,2,4) >$,
$< (5,3,0) >$, $< (5,3,1) >$, $< (5,3,2) >$, $< (5,3,3) >$, $< (5,3,4) >$,
$< (5,4,0) >$, $< (5,4,1) >$, $< (5,4,2) >$, $< (5,4,3) >$, $< (5,4,4) >$,
$< (5,0,0),(5,0,1) >$, $< (5,0,0),(5,1,0) >$, $< (5,0,0),(5,1,1) >$,
$< (5,0,0),(5,1,2) >$, $< (5,0,0),(5,1,3) >$, $< (5,0,0),(5,1,4) >$,
$< (5,1,0),(5,1,1) >$, $< (5,1,0),(5,2,1) >$, $< (5,1,0),(5,2,2) >$,
$< (5,1,0),(5,2,3) >$, $< (5,1,0),(5,2,4) >$, $< (5,1,1),(5,2,0) >$,
$< (5,1,1),(5,2,1) >$, $< (5,1,1),(5,2,3) >$, $< (5,1,1),(5,2,4) >$,
$< (5,1,2),(5,2,0) >$, $< (5,1,2),(5,2,1) >$, $< (5,1,2),(5,2,2) >$,
$< (5,1,2),(5,2,3) >$, $< (5,1,3),(5,2,0) >$, $< (5,1,3),(5,2,2) >$,
$< (5,1,3),(5,2,3) >$, $< (5,1,3),(5,2,4) >$, $< (5,1,4),(5,2,0) >$,
$< (5,1,4),(5,2,1) >$, $< (5,1,4),(5,2,2) >$, $< (5,1,4),(5,2,4) >$,
$< (5,2,0),(5,2,1) >$, $< (5,3,0),(5,3,1) >$, $< (5,4,0),(5,4,1) >$,

and $\{(0,0,0)\}$.
For $p=7$ and $n=2$, the group $G=\mathbb{Z}_{7^2}+\mathbb{Z}_7+\mathbb{Z}_7$ contains 222 crisp subgroups. However, we do not list them here as they are too numerous.
Next we consider all the subgroups of the group $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$ for $p=2,3,5,7$ and $n=3$.
Now for $p=2$ and $n=3$, the group $G=\mathbb{Z}_{2^3}+\mathbb{Z}_2+\mathbb{Z}_2$ has 38 crisp subgroups, as listed below:
$\mathbb{Z}_{2^3}+\mathbb{Z}_2+\mathbb{Z}_2$, $\mathbb{Z}_{2^3}+\mathbb{Z}_2+\{0\}$, $\mathbb{Z}_{2^3}+\{0\}+\mathbb{Z}_2$, $\{0\}+\mathbb{Z}_2+\mathbb{Z}_2$, $\mathbb{Z}_{2^3}+\{0\}+\{0\}$, $\{0\}+\mathbb{Z}_2+\{0\}$, $\{0\}+\{0\}+\mathbb{Z}_2$, $<(0,1,1)>$, $<(1,1,1)>$, $<(1,1,0)>$,
$<(1,0,1)>$, $<(2,0,0)>$, $<(2,0,1)>$, $<(2,1,0)>$, $<(2,1,1)>$,
$<(1,1,1),(1,1,0)>$, $<(1,1,1),(1,0,1)>$, $<(1,1,1),(0,1,1)>$,
$<(1,1,0),(1,0,1)>$, $<(2,0,0),(0,1,0),(0,0,1)>$,
$<(2,0,0),(2,1,0)>$, $<(2,0,0),(2,0,1)>$, $<(2,1,0),(0,1,1)>$,
$<(2,1,0),(2,1,1)>$, $<(2,1,1),(0,1,1)>$, $<(2,1,1),(2,0,1)>$
$<(4,0,0),(0,1,0),(0,0,1)>$,
$<(4,0,0)>$, $<(4,0,1)>$, $<(4,1,0)>$, $<(4,1,1)>$,
$<(4,0,0),(4,1,0)>$, $<(4,0,0),(4,0,1)>$, $<(4,1,0),(0,1,1)>$,
$<(4,1,0),(4,1,1)>$, $<(4,1,1),(0,1,1)>$, $<(4,1,1),(4,0,1)>$
and $\{(0,0,0)\}$.
For $p=3$ and $n=3$, the group $G=\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3$ has 72 crisp subgroups, as listed below:
$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3$, $\mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\}$, $\mathbb{Z}_{3^3}+\{0\}+\mathbb{Z}_3$, $\{0\}+\mathbb{Z}_3+\mathbb{Z}_3$, $\mathbb{Z}_{3^3}+\{0\}+\{0\}$, $\{0\}+\mathbb{Z}_3+\{0\}$,
$\{0\}+\{0\}+\mathbb{Z}_3$, $<(0,1,1)>$, $<(0,1,2)>$, $<(1,0,1)>$, $<(1,0,2)>$,
$<(1,1,0)>$, $<(1,1,1)>$, $<(1,1,2)>$, $<(1,2,0)>$, $<(1,2,1)>$, $<(1,2,2)>$,
$<(1,1,0),(1,1,1)>$, $<(1,1,0),(1,2,1)>$, $<(1,1,0),(1,2,2)>$,
$<(1,1,1),(1,2,0)>$, $<(1,1,1),(1,2,1)>$, $<(1,1,1),(1,2,2)>$,
$<(1,1,2),(1,2,0)>$, $<(1,1,2),(1,2,1)>$, $<(1,1,2),(1,2,2)>$,
$<(1,2,0),(1,2,1)>$, $<(3,0,0),(0,1,0),(0,0,1)>$,
$<(3,0,0)>$, $<(3,0,1)>$, $<(3,0,2)>$, $<(3,1,0)>$, $<(3,1,1)>$,
$<(3,1,2)>$, $<(3,2,0)>$, $<(3,2,1)>$, $<(3,2,2)>$,
$<(3,0,0),(3,0,1)>$, $<(3,0,0),(3,1,0)>$, $<(3,0,0),(3,1,1)>$,
$<(3,0,0),(3,1,2)>$, $<(3,1,0),(3,1,1)>$, $<(3,1,0),(3,2,1)>$,
$<(3,1,0),(3,2,2)>$, $<(3,1,1),(3,2,0)>$, $<(3,1,1),(3,2,1)>$,

$<(3,1,2),(3,2,0)>$, $<(3,1,2),(3,2,2)>$, $<(3,2,0),(3,2,1)>$,
$<(9,0,0),(0,1,0),(0,0,1)>$,
$<(9,0,0)>$, $<(9,0,1)>$, $<(9,0,2)>$, $<(9,1,0)>$, $<(9,1,1)>$,
$<(9,1,2)>$, $<(9,2,0)>$, $<(9,2,1)>$, $<(9,2,2)>$,
$<(9,0,0),(9,0,1)>$, $<(9,0,0),(9,1,0)>$, $<(9,0,0),(9,1,1)>$,
$<(9,0,0),(9,1,2)>$, $<(9,1,0),(9,1,1)>$, $<(9,1,0),(9,2,1)>$,
$<(9,1,0),(9,2,2)>$, $<(9,1,1),(9,2,0)>$, $<(9,1,1),(9,2,1)>$,
$<(9,1,2),(9,2,0)>$, $<(9,1,2),(9,2,2)>$, $<(9,2,0),(9,2,1)>$,
and $\{(0,0,0)\}$
For $p = 5$ and $n = 3$, the group $G = \mathbb{Z}_{5^3} + \mathbb{Z}_5 + \mathbb{Z}_5$ contains 176 crisp subgroups, as listed below:
$\mathbb{Z}_{5^3} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^3} + \mathbb{Z}_5 + \{0\}$, $\mathbb{Z}_{5^3} + \{0\} + \mathbb{Z}_5$, $\{0\} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^3} + \{0\} + \{0\}$, $\{0\} + \mathbb{Z}_5 + \{0\}$, $\{0\} + \{0\} + \mathbb{Z}_5$, $<(0,1,1)>$, $<(0,1,2)>$, $<(0,1,3)>$, $<(0,1,4)>$,
$<(1,0,1)>$, $<(1,0,2)>$, $<(1,0,3)>$, $<(1,0,4)>$, $<(1,1,0)>$,
$<(1,1,1)>$, $<(1,1,2)>$, $<(1,1,3)>$, $<(1,1,4)>$, $<(1,2,0)>$,
$<(1,2,1)>$, $<(1,2,2)>$, $<(1,2,3)>$, $<(1,2,4)>$, $<(1,3,0)>$,
$<(1,3,1)>$, $<(1,3,2)>$, $<(1,3,3)>$, $<(1,3,4)>$, $<(1,4,0)>$,
$<(1,4,1)>$, $<(1,4,2)>$, $<(1,4,3)>$, $<(1,4,4)>$,
$<(1,1,0),(1,1,1)>$, $<(1,1,0),(1,2,1)>$, $<(1,1,0),(1,2,2)>$,
$<(1,1,0),(1,2,3)>$, $<(1,1,0),(1,2,4)>$, $<(1,1,1),(1,2,0)>$,
$<(1,1,1),(1,2,1)>$, $<(1,1,1),(1,2,2)>$, $<(1,1,1),(1,2,3)>$,
$<(1,1,1),(1,2,4)>$, $<(1,1,2),(1,2,0)>$, $<(1,1,2),(1,2,1)>$,
$<(1,1,2),(1,2,2)>$, $<(1,1,2),(1,2,3)>$, $<(1,1,2),(1,2,4)>$,
$<(1,1,3),(1,2,0)>$, $<(1,1,3),(1,2,1)>$, $<(1,1,3),(1,2,2)>$,
$<(1,1,3),(1,2,3)>$, $<(1,1,3),(1,2,4)>$, $<(1,1,4),(1,2,0)>$,
$<(1,1,4),(1,2,1)>$, $<(1,1,4),(1,2,2)>$, $<(1,1,4),(1,2,3)>$,
$<(1,1,4),(1,2,4)>$, $<(1,2,0),(1,2,1)>$, $<(1,3,0),(1,3,1)>$,
$<(1,4,0),(1,4,1)>$, $<(5,0,0),(0,1,0),(0,0,1)>$,
$<(5,0,0)>$, $<(5,0,1)>$, $<(5,0,2)>$, $<(5,0,3)>$, $<(5,0,4)>$,
$<(5,1,0)>$, $<(5,1,1)>$, $<(5,1,2)>$, $<(5,1,3)>$, $<(5,1,4)>$,
$<(5,2,0)>$, $<(5,2,1)>$, $<(5,2,2)>$, $<(5,2,3)>$, $<(5,2,4)>$,
$<(5,3,0)>$, $<(5,3,1)>$, $<(5,3,2)>$, $<(5,3,3)>$, $<(5,3,4)>$,
$<(5,4,0)>$, $<(5,4,1)>$, $<(5,4,2)>$, $<(5,4,3)>$, $<(5,4,4)>$,
$<(5,0,0),(5,0,1)>$, $<(5,0,0),(5,1,0)>$, $<(5,0,0),(5,1,1)>$,
$<(5,0,0),(5,1,2)>$, $<(5,0,0),(5,1,3)>$, $<(5,0,0),(5,1,4)>$,

$<(5,1,0),(5,1,1)>$, $<(5,1,0),(5,2,1)>$, $<(5,1,0),(5,2,2)>$,
$<(5,1,0),(5,2,3)>$, $<(5,1,0),(5,2,4)>$, $<(5,1,1),(5,2,0)>$,
$<(5,1,1),(5,2,1)>$, $<(5,1,1),(5,2,3)>$, $<(5,1,1),(5,2,4)>$,
$<(5,1,2),(5,2,0)>$, $<(5,1,2),(5,2,1)>$, $<(5,1,2),(5,2,2)>$,
$<(5,1,2),(5,2,3)>$, $<(5,1,3),(5,2,0)>$, $<(5,1,3),(5,2,2)>$,
$<(5,1,3),(5,2,3)>$, $<(5,1,3),(5,2,4)>$, $<(5,1,4),(5,2,0)>$,
$<(5,1,4),(5,2,1)>$, $<(5,1,4),(5,2,2)>$, $<(5,1,4),(5,2,4)>$,
$<(5,2,0),(5,2,1)>$, $<(5,3,0),(5,3,1)>$, $<(5,4,0),(5,4,1)>$,
$<(25,0,0),(0,1,0),(0,0,1)>$,
$<(25,0,0)>$, $<(25,0,1)>$, $<(25,0,2)>$, $<(25,0,3)>$, $<(25,0,4)>$,
$<(25,1,0)>$, $<(25,1,1)>$, $<(25,1,2)>$, $<(25,1,3)>$, $<(25,1,4)>$,
$<(25,2,0)>$, $<(25,2,1)>$, $<(25,2,2)>$, $<(25,2,3)>$, $<(25,2,4)>$,
$<(25,3,0)>$, $<(25,3,1)>$, $<(25,3,2)>$, $<(25,3,3)>$, $<(25,3,4)>$,
$<(25,4,0)>$, $<(25,4,1)>$, $<(25,4,2)>$, $<(25,4,3)>$, $<(25,4,4)>$,
$<(25,0,0),(25,0,1)>$, $<(25,0,0),(25,1,0)>$, $<(25,0,0),(25,1,1)>$,
$<(25,0,0),(25,1,2)>$, $<(25,0,0),(25,1,3)>$, $<(25,0,0),(25,1,4)>$,
$<(25,1,0),(25,1,1)>$, $<(25,1,0),(25,2,1)>$, $<(25,1,0),(25,2,2)>$,
$<(25,1,0),(25,2,3)>$, $<(25,1,0),(25,2,4)>$, $<(25,1,1),(25,2,0)>$,
$<(25,1,1),(25,2,1)>$, $<(25,1,1),(25,2,3)>$, $<(25,1,1),(25,2,4)>$,
$<(25,1,2),(25,2,0)>$, $<(25,1,2),(25,2,1)>$, $<(25,1,2),(25,2,2)>$,
$<(25,1,2),(25,2,3)>$, $<(25,1,3),(25,2,0)>$, $<(25,1,3),(25,2,2)>$,
$<(25,1,3),(25,2,3)>$, $<(25,1,3),(25,2,4)>$, $<(25,1,4),(25,2,0)>$,
$<(25,1,4),(25,2,1)>$, $<(25,1,4),(25,2,2)>$, $<(25,1,4),(25,2,4)>$,
$<(25,2,0),(25,2,1)>$, $<(25,3,0),(25,3,1)>$, $<(25,4,0),(25,4,1)>$,
and $\{(0,0,0)\}$.
For $p = 7$ and $n = 3$, the group $G = \mathbb{Z}_{7^3} + \mathbb{Z}_7 + \mathbb{Z}_7$ contains 328 crisp subgroups. However, the subgroups are too many to be listed here.
In what follows, we list all the subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for a fixed $n = 4$ and $p = 2, 3, 5, 7$.
Now for $p = 2$ and $n = 4$, the group $G = \mathbb{Z}_{2^4} + \mathbb{Z}_2 + \mathbb{Z}_2$ has 49 crisp subgroups. We do not list them here.
For $p = 3$ and $n = 4$, the group $G = \mathbb{Z}_{3^4} + \mathbb{Z}_3 + \mathbb{Z}_3$ has 94 crisp subgroups, as listed below:
$\mathbb{Z}_{3^4} + \mathbb{Z}_3 + \mathbb{Z}_3$, $\mathbb{Z}_{3^4} + \mathbb{Z}_3 + \{0\}$, $\mathbb{Z}_{3^4} + \{0\} + \mathbb{Z}_3$, $\{0\} + \mathbb{Z}_3 + \mathbb{Z}_3$,
$\mathbb{Z}_{3^4} + \{0\} + \{0\}$, $\{0\} + \mathbb{Z}_3 + \{0\}$, $\{0\} + \{0\} + \mathbb{Z}_3$, $<(0,1,1)>$,
$<(0,1,2)>$, $<(1,0,1)>$, $<(1,0,2)>$, $<(1,1,0)>$, $<(1,1,1)>$,

$< (1,1,2) >$, $< (1,2,0) >$, $< (1,2,1) >$, $< (1,2,2) >$,
$< (1,1,0),(1,1,1) >$, $< (1,1,0),(1,2,1) >$, $< (1,1,0),(1,2,2) >$,
$< (1,1,1),(1,2,0) >$, $< (1,1,1),(1,2,1) >$, $< (1,1,1),(1,2,2) >$,
$< (1,1,2),(1,2,0) >$, $< (1,1,2),(1,2,1) >$, $< (1,1,2),(1,2,2) >$,
$< (1,2,0),(1,2,1) >$, $< (3,0,0),(0,1,0),(0,0,1) >$,
$< (3,0,0) >$, $< (3,0,1) >$, $< (3,0,2) >$, $< (3,1,0) >$, $< (3,1,1) >$,
$< (3,1,2) >$, $< (3,2,0) >$, $< (3,2,1) >$, $< (3,2,2) >$,
$< (3,0,0),(3,0,1) >$, $< (3,0,0),(3,1,0) >$, $< (3,0,0),(3,1,1) >$,
$< (3,0,0),(3,1,2) >$, $< (3,1,0),(3,1,1) >$, $< (3,1,0),(3,2,1) >$,
$< (3,1,0),(3,2,2) >$, $< (3,1,1),(3,2,0) >$, $< (3,1,1),(3,2,1) >$,
$< (3,1,2),(3,2,0) >$, $< (3,1,2),(3,2,2) >$, $< (3,2,0),(3,2,1) >$,
$< (9,0,0),(0,1,0),(0,0,1) >$,
$< (9,0,0) >$, $< (9,0,1) >$, $< (9,0,2) >$, $< (9,1,0) >$, $< (9,1,1) >$,
$< (9,1,2) >$, $< (9,2,0) >$, $< (9,2,1) >$, $< (9,2,2) >$,
$< (9,0,0),(9,0,1) >$, $< (9,0,0),(9,1,0) >$, $< (9,0,0),(9,1,1) >$,
$< (9,0,0),(9,1,2) >$, $< (9,1,0),(9,1,1) >$, $< (9,1,0),(9,2,1) >$,
$< (9,1,0),(9,2,2) >$, $< (9,1,1),(9,2,0) >$, $< (9,1,1),(9,2,1) >$,
$< (9,1,2),(9,2,0) >$, $< (9,1,2),(9,2,2) >$, $< (9,2,0),(9,2,1) >$,
$< (27,0,0),(0,1,0),(0,0,1) >$,
$< (27,0,0) >$, $< (27,0,1) >$, $< (27,0,2) >$, $< (27,1,0) >$, $< (27,1,1) >$,
$< (27,1,2) >$, $< (27,2,0) >$, $< (27,2,1) >$, $< (27,2,2) >$,
$< (27,0,0),(27,0,1) >$, $< (27,0,0),(27,1,0) >$, $< (27,0,0),(27,1,1) >$,
$< (27,0,0),(27,1,2) >$, $< (27,1,0),(27,1,1) >$, $< (27,1,0),(27,2,1) >$,
$< (27,1,0),(27,2,2) >$, $< (27,1,1),(27,2,0) >$, $< (27,1,1),(27,2,1) >$,
$< (27,1,2),(27,2,0) >$, $< (27,1,2),(27,2,2) >$, $< (27,2,0),(27,2,1) >$,
and $\{(0,0,0)\}$.

For $p = 5$ and $n = 4$, the group $G = \mathbb{Z}_{5^4} + \mathbb{Z}_5 + \mathbb{Z}_5$ contains 232 crisp subgroups, as listed below:

$\mathbb{Z}_{5^4} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^4} + \mathbb{Z}_5 + \{0\}$, $\mathbb{Z}_{5^4} + \{0\} + \mathbb{Z}_5$, $\{0\} + \mathbb{Z}_5 + \mathbb{Z}_5$, $\mathbb{Z}_{5^4} + \{0\} + \{0\}$,
$\{0\} + \mathbb{Z}_5 + \{0\}$, $\{0\} + \{0\} + \mathbb{Z}_5$, $< (0,1,1) >$, $< (0,1,2) >$, $< (0,1,3) >$,
$< (0,1,4) >$, $< (1,0,1) >$, $< (1,0,2) >$, $< (1,0,3) >$, $< (1,0,4) >$,
$< (1,1,0) >$, $< (1,1,1) >$, $< (1,1,2) >$, $< (1,1,3) >$, $< (1,1,4) >$,
$< (1,2,0) >$, $< (1,2,1) >$, $< (1,2,2) >$, $< (1,2,3) >$, $< (1,2,4) >$,
$< (1,3,0) >$, $< (1,3,1) >$, $< (1,3,2) >$, $< (1,3,3) >$, $< (1,3,4) >$,
$< (1,4,0) >$, $< (1,4,1) >$, $< (1,4,2) >$, $< (1,4,3) >$, $< (1,4,4) >$,
$< (1,1,0),(1,1,1) >$, $< (1,1,0),(1,2,1) >$, $< (1,1,0),(1,2,2) >$,

$< (1,1,0),(1,2,3) >,\ < (1,1,0),(1,2,4) >,\ < (1,1,1),(1,2,0) >,$
$< (1,1,1),(1,2,1) >,\ < (1,1,1),(1,2,2) >,\ < (1,1,1),(1,2,3) >,$
$< (1,1,1),(1,2,4) >,\ < (1,1,2),(1,2,0) >,\ < (1,1,2),(1,2,1) >,$
$< (1,1,2),(1,2,2) >,\ < (1,1,2),(1,2,3) >,\ < (1,1,2),(1,2,4) >,$
$< (1,1,3),(1,2,0) >,\ < (1,1,3),(1,2,1) >,\ < (1,1,3),(1,2,2) >,$
$< (1,1,3),(1,2,3) >,\ < (1,1,3),(1,2,4) >,\ < (1,1,4),(1,2,0) >,$
$< (1,1,4),(1,2,1) >,\ < (1,1,4),(1,2,2) >,\ < (1,1,4),(1,2,3) >,$
$< (1,1,4),(1,2,4) >,\ < (1,2,0),(1,2,1) >,\ < (1,3,0),(1,3,1) >,$
$< (1,4,0),(1,4,1) >,\ < (5,0,0),(0,1,0),(0,0,1) >,$
$< (5,0,0) >,\ < (5,0,1) >,\ < (5,0,2) >,\ < (5,0,3) >,\ < (5,0,4) >,$
$< (5,1,0) >,\ < (5,1,1) >,\ < (5,1,2) >,\ < (5,1,3) >,\ < (5,1,4) >,$
$< (5,2,0) >,\ < (5,2,1) >,\ < (5,2,2) >,\ < (5,2,3) >,\ < (5,2,4) >,$
$< (5,3,0) >,\ < (5,3,1) >,\ < (5,3,2) >,\ < (5,3,3) >,\ < (5,3,4) >,$
$< (5,4,0) >,\ < (5,4,1) >,\ < (5,4,2) >,\ < (5,4,3) >,\ < (5,4,4) >,$
$< (5,0,0),(5,0,1) >,\ < (5,0,0),(5,1,0) >,\ < (5,0,0),(5,1,1) >,$
$< (5,0,0),(5,1,2) >,\ < (5,0,0),(5,1,3) >,\ < (5,0,0),(5,1,4) >,$
$< (5,1,0),(5,1,1) >,\ < (5,1,0),(5,2,1) >,\ < (5,1,0),(5,2,2) >,$
$< (5,1,0),(5,2,3) >,\ < (5,1,0),(5,2,4) >,\ < (5,1,1),(5,2,0) >,$
$< (5,1,1),(5,2,1) >,\ < (5,1,1),(5,2,3) >,\ < (5,1,1),(5,2,4) >,$
$< (5,1,2),(5,2,0) >,\ < (5,1,2),(5,2,1) >,\ < (5,1,2),(5,2,2) >,$
$< (5,1,2),(5,2,3) >,\ < (5,1,3),(5,2,0) >,\ < (5,1,3),(5,2,2) >,$
$< (5,1,3),(5,2,3) >,\ < (5,1,3),(5,2,4) >,\ < (5,1,4),(5,2,0) >,$
$< (5,1,4),(5,2,1) >,\ < (5,1,4),(5,2,2) >,\ < (5,1,4),(5,2,4) >,$
$< (5,2,0),(5,2,1) >,\ < (5,3,0),(5,3,1) >,\ < (5,4,0),(5,4,1) >,$
$< (25,0,0),(0,1,0),(0,0,1) >,$
$< (25,0,0) >,\ < (25,0,1) >,\ < (25,0,2) >,\ < (25,0,3) >,\ < (25,0,4) >,$
$< (25,1,0) >,\ < (25,1,1) >,\ < (25,1,2) >,\ < (25,1,3) >,\ < (25,1,4) >,$
$< (25,2,0) >,\ < (25,2,1) >,\ < (25,2,2) >,\ < (25,2,3) >,\ < (25,2,4) >,$
$< (25,3,0) >,\ < (25,3,1) >,\ < (25,3,2) >,\ < (25,3,3) >,\ < (25,3,4) >,$
$< (25,4,0) >,\ < (25,4,1) >,\ < (25,4,2) >,\ < (25,4,3) >,\ < (25,4,4) >,$
$< (25,0,0),(25,0,1) >,\ < (25,0,0),(25,1,0) >,\ < (25,0,0),(25,1,1) >,$
$< (25,0,0),(25,1,2) >,\ < (25,0,0),(25,1,3) >,\ < (25,0,0),(25,1,4) >,$
$< (25,1,0),(25,1,1) >,\ < (25,1,0),(25,2,1) >,\ < (25,1,0),(25,2,2) >,$
$< (25,1,0),(25,2,3) >,\ < (25,1,0),(25,2,4) >,\ < (25,1,1),(25,2,0) >,$
$< (25,1,1),(25,2,1) >,\ < (25,1,1),(25,2,3) >,\ < (25,1,1),(25,2,4) >,$
$< (25,1,2),(25,2,0) >,\ < (25,1,2),(25,2,1) >,\ < (25,1,2),(25,2,2) >,$

$< (25,1,2),(25,2,3) >$, $< (25,1,3),(25,2,0) >$, $< (25,1,3),(25,2,2) >$,
$< (25,1,3),(25,2,3) >$, $< (25,1,3),(25,2,4) >$, $< (25,1,4),(25,2,0) >$,
$< (25,1,4),(25,2,1) >$, $< (25,1,4),(25,2,2) >$, $< (25,1,4),(25,2,4) >$,
$< (25,2,0),(25,2,1) >$, $< (25,3,0),(25,3,1) >$, $< (25,4,0),(25,4,1) >$,
$< (125,0,0),(0,1,0),(0,0,1) >$,
$< (125,0,0) >$, $< (125,0,1) >$, $< (125,0,2) >$, $< (125,0,3) >$, $< (125,0,4) >$,
$< (125,1,0) >$, $< (125,1,1) >$, $< (125,1,2) >$, $< (125,1,3) >$, $< (125,1,4) >$,
$< (125,2,0) >$, $< (125,2,1) >$, $< (125,2,2) >$, $< (125,2,3) >$, $< (125,2,4) >$,
$< (125,3,0) >$, $< (125,3,1) >$, $< (125,3,2) >$, $< (125,3,3) >$, $< (125,3,4) >$,
$< (125,4,0) >$, $< (125,4,1) >$, $< (125,4,2) >$, $< (125,4,3) >$, $< (125,4,4) >$,
$< (125,0,0),(125,0,1) >$, $< (125,0,0),(125,1,0) >$, $< (125,0,0),(125,1,1) >$,
$< (125,0,0),(125,1,2) >$, $< (125,0,0),(125,1,3) >$, $< (125,0,0),(125,1,4) >$,
$< (125,1,0),(125,1,1) >$, $< (125,1,0),(125,2,1) >$, $< (125,1,0),(125,2,2) >$,
$< (125,1,0),(125,2,3) >$, $< (125,1,0),(125,2,4) >$, $< (125,1,1),(125,2,0) >$,
$< (125,1,1),(125,2,1) >$, $< (125,1,1),(125,2,3) >$, $< (125,1,1),(125,2,4) >$,
$< (125,1,2),(125,2,0) >$, $< (125,1,2),(125,2,1) >$, $< (125,1,2),(125,2,2) >$,
$< (125,1,2),(125,2,3) >$, $< (125,1,3),(125,2,0) >$, $< (125,1,3),(125,2,2) >$,
$< (125,1,3),(125,2,3) >$, $< (125,1,3),(125,2,4) >$, $< (125,1,4),(125,2,0) >$,
$< (125,1,4),(125,2,1) >$, $< (125,1,4),(125,2,2) >$, $< (125,1,4),(125,2,4) >$,
$< (125,2,0),(125,2,1) >$, $< (125,3,0),(125,3,1) >$, $< (125,4,0),(125,4,1) >$,
and $\{(0,0,0)\}$.
For $p = 7$ and $n = 4$, the group $G = \mathbb{Z}_{7^4} + \mathbb{Z}_7 + \mathbb{Z}_7$ contains 434 crisp subgroups. The above patterns of subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ lead us to the following theorem:

Theorem 5.2.2 . The number of subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for $n \geq 1$ is $p(2pn + n + 1) + 3 + n$ for a prime $p > 2$.

Proof. By induction on n. For $n = 1$, there are $1 + p + p^2$ maximal subgroups of G since G has rank 3. Since $n = 1$, all these subgroups must be isomorphic to $\mathbb{Z}_p + \mathbb{Z}_p + 0 =< (1,0,0);(0,1,0) >$. Each of the maximal subgroups has only cyclic nontrivial proper subgroups of order p. Thus the total number of subgroups of G is obtained by adding all the subgroups of the orders $1, p, p^2$ and p^3. The subgroup $H_1 = \mathbb{Z}_p + \mathbb{Z}_p + 0$ has $1 + p$ proper nontrivial subgroups. Next we consider $H_2 = \mathbb{Z}_p + 0 + \mathbb{Z}_p$. The subgroup $\mathbb{Z}_p + 0 + 0$ of H_2 has been

counted in H_1. So H_2 has p proper nontrivial subgroups not counted earlier. In $H_3 = 0 + \mathbb{Z}_p + \mathbb{Z}_p$, the subgroups $0 + \mathbb{Z}_p + 0$ and $0 + 0 + \mathbb{Z}_p$ have already been counted in H_1 and H_2. Thus H_3 has $p - 1$ proper nontrivial subgroups not counted above, viz. $< (0, r, 1) >$, $r = 1, 2, \cdots, p - 1$.
Next consider $H_4 =< (1, 0, 0), (0, 1, 1) >$. The maximal subgroups $< (1, 0, 0) >$ and $< (0, 1, 1) >$ have already been counted in H_1 and H_3. So the only new subgroups are: $< (1, 1, 1) >$; $< (1, 2, 2) >$; $< (1, 3, 3) >$, ... , $< (1, p-1, p-1) >$, $p - 1$ in total. Thus H_4 has $p - 1$ proper nontrivial subgroups not counted earlier.
In any of the remaining maximal subgroups of G, the only new subgroups (not counted above) will be of the form $< (1, r, s) >$ for $r, s = 1, 2, \cdots, p - 1$ with $r \neq s$. Thus the number of such subgroups is the permutation of $p-1$ distinct symbols taken 2 at a time, i.e. $\frac{(p-1)!}{(p-3)!} = (p - 1)(p - 2)$.
Thus adding all the subgroups, 1 of order 1; 1 of order p^3; $1 + p + p^2$ of order p^2 and $1 + p + p + 2(p - 1) + (p - 1)(p - 2)$ of order p, we have $2+1+p+p^2+1+p+p+2(p-1)+(p-1)(p-2) = (3+4p+p^2)+p^2-2p+1 = 2p^2 + 2p + 4 = p(2p.1 + 1 + 1) + 1 + 3$ is the total number of subgroups of G. Hence the theorem is true for $n = 1$.
Now we assume that the theorem is true for all $k < n$. So the group $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ has $p(2pk + k + 1) + 3 + k$ subgroups. The group $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$ has $1 + p + p^2$ maximal subgroups, and one of them $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ has $p(2pk + k + 1) + 3 + k$ subgroups.
Subgroups of order p^{k+2} other than $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ are all isomorphic. One of them is $H_1 = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$ which has $1 + p$ maximal subgroups. These subgroups are $< (1, 0, 0) >$; $< (0, 1, 0) >$; $< (1, 1, 0) >$; $< (1, 2, 0) >$; ... ; $< (1, p-1, 0) >$. One, viz $< (0, 1, 0) >$, has already been counted in $\mathbb{Z}_p+\mathbb{Z}_p+\mathbb{Z}_p$. Thus H_1 gives p maximal subgroups not counted before.
Now consider $H_2 = \mathbb{Z}_{p^{k+1}} + 0 + \mathbb{Z}_p$. One subgroup viz $< (0, 0, 1) >$, has already been counted in $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. Thus as above, H_2 yields p maximal subgroups not counted before.
The rest of the maximal subgroups not counted above are $< (1, r, s) >$, $r, s = 1, 2, \cdots, p-1$. So the number of such subgroups is $(p-1)^2$ as in the case $n = 1$ above. Summing the numbers of all the subgroups with the zero subgroup already counted in $\mathbb{Z}_{p^k}+\mathbb{Z}_p+\mathbb{Z}_p$, we have $2p+(p-1)^2+(1+p+p^2-1)+ [p(2pk+k+1)+3+k]+1 = 2p+p^2-2p+1+1+p+p^2-1+2P^2k+pk+p+3+k = p[2p(k + 1) + (k + 1) + 1] + (k + 1) + 3$ distinct subgroups. Thus the theorem

is true for $n = k + 1$. □

5.3 ON MAXIMAL CHAINS OF $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$

Here we construct the maximal chains for the $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ using the above crisp subgroups and give its generalisation.

Example 5.3.1 Let us construct the maximal chains for the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ with $n = 1$ and $p = 2, 3, 5, 7$ in that order.
For $p = 2$, the group $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ has the following 21 maximal chains:
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,1,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1), (0,1,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1), (0,1,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1), (0,1,1) > \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1), (1,0,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1), (1,0,1) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1), (1,0,1) > \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$

For $p = 3$ and $n = 1$, the group $G = \mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3$ has the following 52 maximal chains:

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \mathbb{Z}_3 + \{0\} \supseteq \mathbb{Z}_3 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \mathbb{Z}_3 + \{0\} \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \mathbb{Z}_3 + \{0\} \supseteq < (1,2,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \mathbb{Z}_3 + \{0\} \supseteq \{0\} + \mathbb{Z}_3 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \{0\} + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \{0\} + \mathbb{Z}_3 \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \{0\} + \mathbb{Z}_3 \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_3 + \{0\} + \mathbb{Z}_3 \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \mathbb{Z}_3 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \{0\} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,1,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0), (1,2,2) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,0) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,0) > \supseteq < (1,2,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,0) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,0) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,1) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,1) > \supseteq \{0\} + \mathbb{Z}_3 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,2) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,1), (1,2,2) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,2)> \supseteq \mathbb{Z}_3+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,1),(1,2,2)> \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,0)> \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,1)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,1)> \supseteq <(1,2,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,1)> \supseteq \mathbb{Z}_3+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,1)> \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,0,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,1,2),(1,2,2)> \supseteq \{0\}+\mathbb{Z}_3+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq <(1,2,0),(1,2,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_3 \supseteq \{(0,0,0)\}$

For $p=5$ and $n=1$ the group $G=\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5$ has the following 186 maximal chains:

$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\mathbb{Z}_5+\{0\} \supseteq \mathbb{Z}_5+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\mathbb{Z}_5+\{0\} \supseteq <(1,1,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\mathbb{Z}_5+\{0\} \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\mathbb{Z}_5+\{0\} \supseteq <(1,3,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\mathbb{Z}_5+\{0\} \supseteq <(1,4,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\mathbb{Z}_5+\{0\} \supseteq \{0\}+\mathbb{Z}_5+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\mathbb{Z}_5 \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\mathbb{Z}_5 \supseteq <(1,0,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\mathbb{Z}_5 \supseteq <(1,0,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\mathbb{Z}_5 \supseteq <(1,0,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \mathbb{Z}_5+\{0\}+\mathbb{Z}_5 \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \{0\}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \{0\}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq \{0\}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(0,1,3)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (0,1,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \mathbb{Z}_5 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,1,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,3,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,4,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,1) > \supseteq < (1,0,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,3,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,4,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (1,0,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,2,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,3,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,4,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (0,1,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,2,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,3,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,4,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (0,1,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1), (1,2,0) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1), (1,2,0) > \supseteq < (1,2,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1), (1,2,0) > \supseteq < (1,3,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,4,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (0,1,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,3,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,4,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,1) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,1) > \supseteq \{0\} + \mathbb{Z}_5 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,2) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,2) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,2) > \supseteq < (1,3,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,2) > \supseteq < (1,4,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,2) > \supseteq \mathbb{Z}_5 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,2) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,3) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,3) > \supseteq < (1,2,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,3) > \supseteq < (1,3,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,3) > \supseteq < (1,4,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,3) > \supseteq < (1,0,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,3) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,4) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,4) > \supseteq < (1,2,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,4) > \supseteq < (1,3,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,4) > \supseteq < (1,4,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,4) > \supseteq < (1,0,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,4) > \supseteq < (0,1,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,2,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,3,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,4,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_3 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,0) > \supseteq < (1,0,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,0) > \supseteq < (0,1,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,1) > \supseteq < (1,1,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,2),(1,2,1) > \supseteq < (1,3,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,1)> \supseteq <(1,4,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,1)> \supseteq <(1,0,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,1)> \supseteq <(0,1,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,3,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,4,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,2)> \supseteq <(1,0,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,2)> \supseteq \{0\}+\mathbb{Z}_5+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,3)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,3)> \supseteq <(1,2,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,3)> \supseteq <(1,3,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,3)> \supseteq <(1,4,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,3)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,3)> \supseteq <(0,1,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,4)> \supseteq <(1,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,4)> \supseteq <(1,2,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,4)> \supseteq <(1,3,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,4)> \supseteq <(1,4,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,4)> \supseteq \mathbb{Z}_5+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,2),(1,2,4)> \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,1,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,3,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,4,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,0,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(0,1,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(1,1,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(1,2,1)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(1,3,4)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(1,4,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq \mathbb{Z}_5+\{0\}+\{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(0,1,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(1,1,3)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$
$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(1,3,1)> \supseteq \{(0,0,0)\}$

$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,2) > \supseteq < (1,4,0) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,2) > \supseteq < (1,0,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,2) > \supseteq < (0,1,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,3) > \supseteq < (1,1,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,3) > \supseteq < (1,2,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,3) > \supseteq < (1,3,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,3) > \supseteq < (1,4,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,3) > \supseteq < (1,0,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,3) > \supseteq \{0\} + \mathbb{Z}_5 + \{0\} \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,4) > \supseteq < (1,1,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,4) > \supseteq < (1,2,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,4) > \supseteq < (1,3,0) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,4) > \supseteq < (1,4,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,4) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3), (1,2,4) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,0) > \supseteq < (1,1,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,0) > \supseteq < (1,2,0) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,0) > \supseteq < (1,3,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,0) > \supseteq < (1,4,2) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,0) > \supseteq < (1,0,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,0) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,1) > \supseteq < (1,1,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,1) > \supseteq < (1,2,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,1) > \supseteq < (1,3,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,1) > \supseteq < (1,4,0) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,1) > \supseteq < (1,0,2) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,1) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,2) > \supseteq < (1,1,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,2) > \supseteq < (1,2,2) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,2) > \supseteq < (1,3,0) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,2) > \supseteq < (1,4,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,2) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,2) > \supseteq < (0,1,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,3) > \supseteq < (1,1,4) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,3) > \supseteq < (1,2,3) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4), (1,2,3) > \supseteq < (1,3,2) > \supseteq \{(0,0,0)\}$$

$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,3)> \supseteq <(1,4,1)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,3)> \supseteq \mathbb{Z}_5+\{0\}+\{0\} \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,3)> \supseteq <(0,1,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,4)> \supseteq <(1,1,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,4)> \supseteq <(1,2,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,4)> \supseteq <(1,3,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,4)> \supseteq <(1,4,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,4)> \supseteq <(1,0,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,4),(1,2,4)> \supseteq \{0\}+\mathbb{Z}_5+\{0\} \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,0)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,1)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,2)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,3)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,2,0),(1,2,1)> \supseteq <(1,2,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,2,0),(1,2,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$$

$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,3,0),(1,3,1)> \supseteq <(1,3,0)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,3,0),(1,3,1)> \supseteq <(1,3,1)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,3,0),(1,3,1)> \supseteq <(1,3,2)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,3,0),(1,3,1)> \supseteq <(1,3,3)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,3,0),(1,3,1)> \supseteq <(1,3,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,3,0),(1,3,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$$

$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(1,4,0)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(1,4,1)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(1,4,2)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(1,4,3)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(1,4,4)> \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_5+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$$

For $p = 7$ and $n = 1$ the group $G = \mathbb{Z}_7 + \mathbb{Z}_7 + \mathbb{Z}_7$ has 456 maximal chains, but it is too cumbersome to list them here.
Our observation of the above examples shows that:

Proposition 5.3.2 For any given prime number p the group $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ has $(p+1)[p^2+p+1]$ maximal chains.

Proof. The group $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ has $1 + p + p^2$ maximal subgroups and each such subgroup is isomorphic to $\mathbb{Z}_p + \mathbb{Z}_p$, which has $p+1$ proper nontrivial subgroups by the rank 2 theory that was discussed in the previous chapter. Hence the proposition follows. □

Example 5.3.3 . Next, we construct the maximal chains of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$, where $n = 2$, beginning with prime 2,3,5 and 7 respectively. □
For $p = 2$ and $n = 2$, the group $\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2$ has the following 51 maximal chains:

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \mathbb{Z}_2 + \{0\} \supseteq \mathbb{Z}_{2^2} + \{0\} + \{0\} \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \mathbb{Z}_2 + \{0\} \supseteq < (1,1,0) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \mathbb{Z}_2 + \{0\} \supseteq < (2,0,0)\,, (2,1,0) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \mathbb{Z}_2 + \{0\} \supseteq < (2,0,0)\,, (2,1,0) > \supseteq < (2,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \mathbb{Z}_2 + \{0\} \supseteq < (2,0,0)\,, (2,1,0) > \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \{0\} + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \{0\} + \{0\} \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \{0\} + \mathbb{Z}_2 \supseteq < (1,0,1) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \{0\} + \mathbb{Z}_2 \supseteq < (2,0,0), (2,0,1) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \{0\} + \mathbb{Z}_2 \supseteq < (2,0,0), (2,0,1) > \supseteq < (2,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_{2^2} + \{0\} + \mathbb{Z}_2 \supseteq < (2,0,0), (2,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (1,1,0) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (1,0,1) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (2,0,0), (2,1,1) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (2,0,0), (2,1,1) > \supseteq < (2,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,0,1) > \supseteq < (2,0,0), (2,1,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,0) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0), (1,1,1) > \supseteq < (1,1,1) > \supseteq < (2,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,0),(1,1,1) >\supseteq< (2,0,0),(2,0,1) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,0),(1,1,1) >\supseteq< (2,0,0),(2,0,1) >\supseteq< (2,0,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,0),(1,1,1) >\supseteq< (2,0,0),(2,0,1) >\supseteq \{0\}+\{0\}+\mathbb{Z}_2 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(1,0,1) >\supseteq< (1,1,1) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(1,0,1) >\supseteq< (1,0,1) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(1,0,1) >\supseteq< (2,0,0),(2,1,0) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(1,0,1) >\supseteq< (2,0,0),(2,1,0) >\supseteq< (2,1,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(1,0,1) >\supseteq< (2,0,0),(2,1,0) >\supseteq \{0\}+\mathbb{Z}_2+\{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(0,1,1) >\supseteq< (1,1,1) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(0,1,1) >\supseteq \mathbb{Z}_{2^2}+\{0\}+\{0\} \supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(0,1,1) >\supseteq< (2,0,0),(2,1,1) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(0,1,1) >\supseteq< (2,0,0),(2,1,1) >\supseteq< (2,1,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (1,1,1),(0,1,1) >\supseteq< (2,0,0),(2,1,1) >\supseteq< (0,1,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,0,1) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,0,1) >\supseteq< (2,0,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,0,1) >\supseteq \{0\}+\{0\}+\mathbb{Z}_2 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,1,0) >\supseteq< (2,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,1,0) >\supseteq< (2,1,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,1,0) >\supseteq \{0\}+\mathbb{Z}_2+\{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{2^2}+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,1,1) >\supseteq<$

$(2,0,0) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,1,1) >\supseteq< (2,1,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,0,0),(2,1,1) >\supseteq< (0,1,1) >\supseteq \{(0,0,0)\}$
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$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,0),(2,0,1) >\supseteq< (0,1,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,0),(2,1,1) >\supseteq< (2,1,0) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,0),(2,1,1) >\supseteq< (2,1,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,0),(2,1,1) >\supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,1),(2,0,1) >\supseteq< (2,1,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,1),(2,0,1) >\supseteq< (2,0,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq< (2,1,1),(2,0,1) >\supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (0,1,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{2^2} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq< (2,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$.
For $p = 3$ and $n = 2$, the group $\mathbb{Z}_{3^2} + \mathbb{Z}_3 + \mathbb{Z}_3$ has 136 maximal chains. We do not list the chains here.
For $p = 5$ and $n = 2$, the group $\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5$ has the following 516 maximal chains:
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq \mathbb{Z}_{5^2} + \{0\} + \{0\} \supseteq< (5,0,0) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq< (1,1,0) >\supseteq< (5,0,0) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq< (1,2,0) >\supseteq< (5,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (1,3,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (1,4,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,2,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,3,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,4,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \mathbb{Z}_5 + \{0\} \supseteq < (5,0,0),(5,1,0) > \supseteq \supseteq \{0\} + \mathbb{Z}_5 + \{0\} \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \{0\} \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (1,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (1,0,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (1,0,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (1,0,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,2) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,3) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,4) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \mathbb{Z}_{5^2} + \{0\} + \mathbb{Z}_5 \supseteq < (5,0,0),(5,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,1,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,2,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,3,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,4,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,0,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,2,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,3,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,4,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,2) > \supseteq < (1,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,2) > \supseteq < (1,2,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,2) > \supseteq < (1,3,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,2) > \supseteq < (1,4,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,2) > \supseteq < (1,0,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,2) > \supseteq < (5,0,0),(5,1,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (5,0,0), (5,1,2) > \supseteq < (5,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (5,0,0), (5,1,2) > \supseteq < (5,2,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (5,0,0), (5,1,2) > \supseteq < (5,3,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (5,0,0), (5,1,2) > \supseteq < (5,4,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,2) > \supseteq < (5,0,0), (5,1,2) > \supseteq < (0,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,2,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,3,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,4,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (1,0,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (5,0,0), (5,1,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (5,0,0), (5,1,3) > \supseteq < (5,1,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (5,0,0), (5,1,3) > \supseteq < (5,2,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (5,0,0), (5,1,3) > \supseteq < (5,3,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (5,0,0), (5,1,3) > \supseteq < (5,4,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,3) > \supseteq < (5,0,0), (5,1,3) > \supseteq < (0,1,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,2,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,3,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,4,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (1,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (5,0,0), (5,1,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0), (1,2,4) > \supseteq < (5,0,0), (5,1,4) > \supseteq < (5,1,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,4) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,2,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,4) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,3,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,4) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,4,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,0),(1,2,4) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (0,1,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,1,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,2,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,3,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,4,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (1,0,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,1,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,1),(1,2,0) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,2,3) > \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,1),(1,2,2)> \supseteq <(1,1,1)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,1),(1,2,2)> \supseteq \mathbb{Z}_{5^2}+\{0\}+\{0\} \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,1,3)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,2,0)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,3,1)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,4,4)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(1,0,1)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(5,0,0),(5,1,2)> \supseteq <(5,2,4)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,0)> \supseteq <(5,0,0),(5,1,2)> \supseteq <(5,3,1)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(1,3,4)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(1,4,2)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq \mathbb{Z}_{5^2}+\{0\}+\{0\} \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,1)> \supseteq <(5,0,0),(5,1,3)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(1,2,2)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(1,3,1)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(1,0,4)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(5,0,0),(5,1,4)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,2)> \supseteq <(5,0,0),(5,1,4)> \supseteq <(5,1,4)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,3)> \supseteq <(1,2,3)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,3)> \supseteq <(5,0,0),(5,1,0)> \supseteq \{0\}+\mathbb{Z}_5+\{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,4)> \supseteq <(1,1,3)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,4)> \supseteq <(1,2,4)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,1,3),(1,2,4)> \supseteq <(1,3,0)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq < (1,1,3),(1,2,4) > \supseteq < (1,4,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq < (1,1,3),(1,2,4) > \supseteq < (1,0,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,3),(1,2,4) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq < (1,1,4),(1,2,0) > \supseteq < (1,1,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq < (1,1,4),(1,2,0) > \supseteq < (1,2,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq < (1,1,4),(1,2,0) > \supseteq < (1,3,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,0) > \supseteq < (5,0,0),(5,1,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,3) > \supseteq \mathbb{Z}_{5^2} + \{0\} + \{0\} \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,3) > \supseteq < (5,0,0),(5,1,4) > \supseteq < (5,0,0) > \supseteq$

$\{(0,0,0)\}$

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$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (1,1,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (1,2,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (1,3,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (1,4,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (1,0,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,2,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,3,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (5,0,0),(5,1,0) > \supseteq < (5,4,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,1,4),(1,2,4) > \supseteq < (5,0,0),(5,1,0) > \supseteq \{0\} + \mathbb{Z}_5 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (1,2,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,1) > \supseteq$

$\{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,2,0),(1,2,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (1,3,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (1,3,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (1,3,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (1,3,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (1,3,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,1) > \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,3,0),(1,3,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (1,4,0) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (1,4,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (1,4,2) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (1,4,3) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (1,4,4) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (1,4,0),(1,4,1) > \supseteq < (5,0,0),(5,0,1) > \supseteq < (5,0,2) > \supseteq$

$\{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,3)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,4)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(1,4,0),(1,4,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,1)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,2)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,3)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq <(5,0,4)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,0,1)> \supseteq \{0\}+\{0\}+\mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,0)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,0)> \supseteq <(5,1,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,0)> \supseteq <(5,2,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,0)> \supseteq <(5,3,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,0)> \supseteq <(5,4,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,0)> \supseteq \{0\}+\mathbb{Z}_5+\{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,1)> \supseteq <(5,0,0)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,1)> \supseteq <(5,1,1)> \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq <(5,0,0),(0,1,0),(0,0,1)> \supseteq <(5,0,0),(5,1,1)> \supseteq <$

$(5,2,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq <$
$(5,3,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq <$
$(5,4,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,1) > \supseteq <$
$(0,1,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,2) > \supseteq <$
$(5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,2) > \supseteq <$
$(5,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,2) > \supseteq <$
$(5,2,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,2) > \supseteq <$
$(5,3,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,2) > \supseteq <$
$(5,4,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,2) > \supseteq <$
$(0,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,3) > \supseteq <$
$(5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,3) > \supseteq <$
$(5,1,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,3) > \supseteq <$
$(5,2,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,3) > \supseteq <$
$(5,3,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,3) > \supseteq <$
$(5,4,2) > \supseteq \{(0,0,0)\}$

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$(0,1,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,4) > \supseteq <$
$(5,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,4) > \supseteq <$
$(5,1,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,0,0),(5,1,4) > \supseteq <$

$(5,2,3) > \supseteq \{(0,0,0)\}$

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$(5,3,2) > \supseteq \{(0,0,0)\}$

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$(5,4,1) > \supseteq \{(0,0,0)\}$

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$(0,1,4) > \supseteq \{(0,0,0)\}$

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$(5,1,0) > \supseteq \{(0,0,0)\}$

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$(5,1,1) > \supseteq \{(0,0,0)\}$

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$(5,1,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,1,0),(5,1,1) > \supseteq <$
$(5,1,3) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,1,0),(5,1,1) > \supseteq <$
$(5,1,4) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,1,0),(5,1,1) > \supseteq \{0\} +$
$\{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,1,0),(5,2,1) > \supseteq <$
$(5,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,1,0),(5,2,1) > \supseteq <$
$(5,2,1) > \supseteq \{(0,0,0)\}$

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$(5,3,2) > \supseteq \{(0,0,0)\}$

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$(5,4,3) > \supseteq \{(0,0,0)\}$

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$(5,0,4) > \supseteq \{(0,0,0)\}$

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$(0,1,1) > \supseteq \{(0,0,0)\}$

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$(5,1,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq < (5,0,0),(0,1,0),(0,0,1) > \supseteq < (5,1,0),(5,2,2) > \supseteq <$
$(5,2,2) > \supseteq \{(0,0,0)\}$

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$(5,3,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,0),(5,2,2) >\supseteq< (5,4,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,0),(5,2,2) >\supseteq< (5,0,3) >\supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,0) >\supseteq< (5,1,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,0) >\supseteq< (5,2,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,0) >\supseteq<$

$(5,3,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,0) >\supseteq<$
$(5,4,2) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,0) >\supseteq<$
$(5,0,3) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,0) >\supseteq<$
$(0,1,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,1) >\supseteq<$
$(5,1,4) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,1) >\supseteq<$
$(5,2,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,1) >\supseteq<$
$(5,3,3) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,1) >\supseteq<$
$(5,4,0) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,1) >\supseteq<$
$(5,0,2) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,1) >\supseteq<$
$(0,1,2) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,2) >\supseteq<$
$(5,1,4) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,2) >\supseteq<$
$(5,2,2) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,2) >\supseteq<$
$(5,3,0) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,2) >\supseteq<$
$(5,4,3) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,2) >\supseteq<$
$(5,0,1) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,2) >\supseteq<$
$(0,1,3) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,4) >\supseteq<$
$(5,1,4) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,4) >\supseteq<$
$(5,2,4) >\supseteq \{(0,0,0)\}$
$\mathbb{Z}_{5^2}+\mathbb{Z}_5+\mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,4) >\supseteq<$

$(5,3,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,4) >\supseteq< (5,4,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,4) >\supseteq< (5,0,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,1,4),(5,2,4) >\supseteq \{0\} + \mathbb{Z}_5 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,2,0),(5,2,1) >\supseteq< (5,2,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,2,0),(5,2,1) >\supseteq< (5,2,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,2,0),(5,2,1) >\supseteq< (5,2,2) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,2,0),(5,2,1) >\supseteq< (5,2,3) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,2,0),(5,2,1) >\supseteq< (5,2,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,2,0),(5,2,1) >\supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,3,0),(5,3,1) >\supseteq< (5,3,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,3,0),(5,3,1) >\supseteq< (5,3,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,3,0),(5,3,1) >\supseteq< (5,3,2) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,3,0),(5,3,1) >\supseteq< (5,3,3) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,3,0),(5,3,1) >\supseteq< (5,3,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,3,0),(5,3,1) >\supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,4,0),(5,4,1) >\supseteq< (5,4,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,4,0),(5,4,1) >\supseteq< (5,4,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,4,0),(5,4,1) >\supseteq<$

$(5,4,2) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,4,0),(5,4,1) >\supseteq< (5,4,3) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,4,0),(5,4,1) >\supseteq< (5,4,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq< (5,4,0),(5,4,1) >\supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (0,1,1) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (0,1,2) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (0,1,3) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (0,1,4) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \mathbb{Z}_5 + \{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{5^2} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq< (5,0,0),(0,1,0),(0,0,1) >\supseteq \{0\} + \mathbb{Z}_5 + \mathbb{Z}_5 \supseteq \{0\} + \{0\} + \mathbb{Z}_5 \supseteq \{(0,0,0)\}$.

For $p = 7$ and $n = 2$ the group $G = \mathbb{Z}_{7^2} + \mathbb{Z}_7 + \mathbb{Z}_7$ has 1296 maximal chains, but it is too bulky to list them here.
The above observation leads to:

Proposition 5.3.4 For $n = 2$, the group $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ has $(p+1) + (3p+2)[p^2+p]$ maximal chains for any given prime number p.

Proof. The group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ has $1 + p + p^2$ maximal subgroups. The subgroup $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ has $1 + p + p^2$ maximal subgroups, giving rise to $(1+p)(1+p+p^2)$ maximal chains. Each of the subgroups isomorphic to $\mathbb{Z}_{p^2} + \mathbb{Z}_p$ gives rise to $(n-1)(p-1) + p + 1 + n - 1 = p - 1 + p + 1 + 1 = 2p + 1$ maximal chains. Thus the total number of maximal chains is equal to $(1+p)(1+p+p^2) + (p+p^2)(2p+1) = (p+1) + (3p+2)[p^2+p]$. □
Next we construct the maximal chains for the group $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for $n = 3$. To avoid bulkiness, we focus mainly on the prime numbers $p = 2, 3$, since

prime numbers $p \geq 5$ yield bulky chains.

Example 5.3.5 . For $p = 2$ and $n = 3$, the group $\mathbb{Z}_{2^3} + \mathbb{Z}_2 + \mathbb{Z}_2$ has 93 maximal chains which are not listed here.
For $p = 3$ and $n = 3$, the group $\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3$ has the following 256 maximal chains:
$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq \mathbb{Z}_{3^3}+\{0\}+\{0\} \supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (1,1,0) >\supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (1,2,0) >\supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (3,1,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (3,2,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (9,0,0),(9,1,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (9,0,0),(9,1,0) >\supseteq< (9,1,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (9,0,0),(9,1,0) >\supseteq< (9,2,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\mathbb{Z}_3+\{0\} \supseteq< (3,0,0),(3,1,0) >\supseteq< (9,0,0),(9,1,0) >\supseteq \{0\}+\mathbb{Z}_3+\{0\} \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\{0\}+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\{0\}+\{0\} \supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\{0\}+\mathbb{Z}_3 \supseteq< (1,0,1) >\supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\{0\}+\mathbb{Z}_3 \supseteq< (1,0,2) >\supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\{0\}+\mathbb{Z}_3 \supseteq< (3,0,0),(3,0,1) >\supseteq< (3,0,0) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3}+\mathbb{Z}_3+\mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3}+\{0\}+\mathbb{Z}_3 \supseteq< (3,0,0),(3,0,1) >\supseteq< (3,0,1) >\supseteq< (9,0,0) >\supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,2) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) > \supseteq < (9,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) > \supseteq < (9,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq \mathbb{Z}_{3^3} + \{0\} + \mathbb{Z}_3 \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,0) > \supseteq < (3,0,0) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,1) > \supseteq < (3,0,0) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,2) > \supseteq < (3,0,0) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,0) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,1) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (3,0,2) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) >$

$\supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) >$

$\supseteq < (9,0,1) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) >$

$\supseteq < (9,0,2) > \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,1,1) > \supseteq < (3,0,0),(3,0,1) > \supseteq < (9,0,0),(9,0,1) >$

$\supseteq \{0\} + \{0\} + \mathbb{Z}_3 \supseteq \{(0,0,0)\}$

$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,1,0) > \supseteq < (3,0,0) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

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$\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3 \supseteq < (1,1,0),(1,2,1) > \supseteq < (1,0,2) > \supseteq < (3,0,0) > \supseteq < (9,0,0) > \supseteq \{(0,0,0)\}$

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This exercise leads us to the following proposition:

Proposition 5.3.6 The group $\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$ has a total number of $(p+1)+(6p+3)[p^2+p]$ maximal chains, for $n=3$ and any given prime number p.

Proof. The group $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$ has $1+p+p^2$ maximal subgroups. The subgroup $\mathbb{Z}_{p^{n-1}}+\mathbb{Z}_p+\mathbb{Z}_p$ has $p+1+(3p+2)(p^2+p)$ maximal chains while the subgroup $\mathbb{Z}_{p^n}+\mathbb{Z}_p$ has $(n-1)(p-1)+p+1+n-1$ maximal chains. Thus the total number of maximal chains is equal to $p+1+(3p+2)(p^2+p)+(p^2+p)(2(p-1)+p+3=(p+1)+(6p+3)[p^2+p]$. □
We have also constructed the maximal chains for $\mathbb{Z}_{p^4}+\mathbb{Z}_p+\mathbb{Z}_p$ for $p=2$, and we obtained 147 maximal chains. We do not list the maximal chains here to avoid bulkiness. We obtained the following result:

Proposition 5.3.7 The group $\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$ has $(p+1)+(10p+4)[p^2+p]$ total number of maximal chains, for $n=4$ and for any prime number p.

Proof. Similar to the the case $n = 3$.
Now we characterise all the maximal chains of the group $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ in the theorem that follows.

Theorem 5.3.8 . The number of maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ is $(p+1)+(p^2+p)[\frac{n(n+1)}{2}p+n]$ for a natural number $n \geq 1$ and any prime number p.

Proof. We use induction on n. For $n = 1$, $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$.
One maximal chain of G is $G \supseteq \mathbb{Z}_p + \mathbb{Z}_p + \{0\} \supseteq \mathbb{Z}_p + \{0\} + \{0\} \supseteq \{(0, 0, 0)\}$. $\mathbb{Z}_p + \mathbb{Z}_p + \{0\}$ is a maximal subgroup of G of order p^2 and rank 2. All maximal subgroups of G must be of order p^2 and rank 2. There are $1 + p + \cdots + p^{rank(G)} - 1$ such subgroups, see [106]. Thus there are $1 + p + p^2$ maximal subgroups, since rank(G) = 3. Each maximal subgroup has maximal subgroups of order p and rank 1, so there are only $1 + p$ such subgroups for each maximal subgroup. Thus there are $(1 + p + p^2)(1 + p)$ maximal chains of G.
$(1+p+p^2)(1+p) = 1+p+p^2+p+p^2+p^3 = 1+2p+2p^2+p^3$. In the formula, if $n = 1$, then we have $p + 1 + (p^2 + p)(p + 1) = p + 1 + p^3 + p^2 + p^2 + p = 1 + 2p + 2p^2 + p^3$. Therefore the formula is true for $n = 1$.
Suppose now that the formula is true for $n = k$, i.e. $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ has $p+1+(p^2+p)[\frac{k(k+1)}{2}p+k]$ maximal chains. We show that if $G = \mathbb{Z}_{p^{k+1}}+\mathbb{Z}_p+\mathbb{Z}_p$, then G has $p + 1 + (p^2 + p)[\frac{(k+1)(k+2)}{2}p + k + 1]$ maximal chains. Now maximal subgroups of G are isomorphic to $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ or $\mathbb{Z}_{p^k+1} + \mathbb{Z}_p + \{0\}$. Thus they are of rank 3 or rank 2. Total number of maximal subgroups is $1 + p + p^2 + \cdots + p^{rank(G)-1} = 1 + p + p^2$. Therefore there are $p + p^2$ of rank 2 and 1 of rank 3 maximal subgroups.
By induction, the subgroup $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ of rank 3 yields $p + 1 + (p^2 + p)[\frac{k(k+1)}{2}p + k]$ maximal chains. The $p + p^2$ maximal subgroups of rank 2 of $G = \mathbb{Z}_{p^k+1} + \mathbb{Z}_p + \mathbb{Z}_p$ yield $(p+p^2)[(k+1)p+1]$ maximal chains, since a group $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has $np + 1$ maximal chains by [94]. Therefore the total number of maximal chains of $\mathbb{Z}_{p^k+1} + \mathbb{Z}_p + \mathbb{Z}_p$ is equal to $p + 1 + (p^2 + p)[\frac{k(k+1)}{2}p + k] + (p^2 + p)[(k + 1)p + 1] = (p + 1) + (p^2 + p)[\frac{(k+1)(k+2)}{2}p + k + 1]$. Thus the result is true for $n = k + 1$ and this completes the proof. □

Chapter 6

DISTINCT AND NON-ISOMORPHIC FUZZY SUBGROUPS

In this chapter, we use one of the two counting techniques discussed in chapter 3 to classify all non-equivalent fuzzy subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for a fixed prime integer p and $n \in \mathbb{N}$. We present a formula for the number of distinct fuzzy subgroups of G in the form of a theorem with a detailed proof. We begin this chapter with the classification of both the isomorphic and non-isomorphic maximal chains of this group and characterise them in the form of theorems. The chapter is organised as follows: In Section 1 we use only the maximal chains to enumerate all non-equivalent fuzzy subgroups of G. Section 2 deals with isomorphic and non-isomorphic maximal chains.

6.1 COMPUTATION OF NON-EQUIVALENT FUZZY SUBGROUPS USING MAXIMAL CHAINS

As stated in the preceding chapter, fuzzy subgroups are in fact pinned-flags. In a series of papers, Murali and Makamba [81],[82],[83] have defined an equivalence on the set of fuzzy subsets of a given group and meticulously classified the number of fuzzy subgroups of some finite abelian p-groups. The same authors have also studied the fuzzy subgroups of a group of the form $\mathbb{Z}_{p_1} + \mathbb{Z}_{p_2} + \cdots + \mathbb{Z}_{p_n}$ for some positive integer n and distinct prime numbers $p_1, p_2, \cdots, p_n$.
Ngcibi [94] classified the fuzzy subgroups of finite Abelian groups of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p$ for any prime number p and $n = 1, 2, 3$. *Tărnăuceanu* and Bentea [116] have developed an explicit formula to compute the number of fuzzy subgroups of $\mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$ where p is a prime integer. Later, Esengul Salturk and Irfan Siap [103] used the equivalence relation defined by Murali

and Makamba in [81] to study the structure of equivalence classes of fuzzy subgroups of $\mathbb{Z}_p^n$ for a fixed prime integer p and a positive integer n indicating the rank of the group. Employing the equivalence relation studied in [81], one calculates the total number of all fuzzy subgroups of any given finite abelian p-group. The study in [103] only treated the case where the prime integer p assumes an exponent of one (1) throughout. This study is however different from the studies discussed above since we have treated the case where the first fixed prime integer p in $\mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ is of an exponent $n \geq 1$ and it is a remarkable achievement of this dissertation.
Now let us use the maximal chains that follow to illustrate how to compute non-equivalent fuzzy subgroups of G. Our examples are only for $p = 2$ and $n = 1, 2$. Thereafter we give a characterisation in the form of a theorem for any fixed prime integer p and any natural number n.

Example 6.1.1 Let $G = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. In order to compute the number of fuzzy subgroups of the finite group G, firstly we construct and write down all the maximal chains or flags of the finite group G as follows:
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \mathbb{Z}_2 + \{0\} \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \mathbb{Z}_2 + \{0\} + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0),(1,0,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0),(1,0,1) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0),(1,0,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,0) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0),(1,1,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,0),(1,1,1) > \supseteq \{0\} + \{0\} + \mathbb{Z}_2 \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1),(0,1,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1),(0,1,1) > \supseteq < (0,1,1) > \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1),(0,1,1) > \supseteq \mathbb{Z}_2 + \{0\} + \{0\} \supseteq \{(0,0,0)\}$
$\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \supseteq < (1,1,1),(1,0,1) > \supseteq < (1,1,1) > \supseteq \{(0,0,0)\}$

$$\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq < (1,1,1),(1,0,1) > \supseteq < (1,0,1) > \supseteq \{(0,0,0)\}$$
$$\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2 \supseteq < (1,1,1),(1,0,1) > \supseteq \{0\}+\mathbb{Z}_2+\{0\} \supseteq \{(0,0,0)\}\,.$$

The chains are of length 4, thus the first chain contributes 2^4-1 distinct fuzzy subgroups. The second chain contributes a further $\frac{2^4}{2}=2^3$ distinct fuzzy subgroups since it has at least one subgroup not appearing in the first chain. Similarly, a total of 12 chains each has a unique (not used a a distinguishing factor elsewhere) subgroup. Thus these give 12×2^3 distinct fuzzy subgroups, while the remaining 8 chains each has a unique pair of subgroups, thus they contribute 8×2^2 distinct fuzzy subgroups. Thus our group G has $2^4-1+12\times 2^3+8\times 2^2=143$ distinct fuzzy subgroups.

Proposition 6.1.2 The number of distinct fuzzy subgroups of $G=\mathbb{Z}_p+\mathbb{Z}_p+\mathbb{Z}_p$ is equal to $2^4-1+2^3(2p^2+2p)+2^2p^3$

Proof. First we consider maximal subgroups of G. One of them is $H_1=\mathbb{Z}_p+\mathbb{Z}_p+0$. From the proof of the formula for the number of subgroups of G in chapter 4, we know that H_1 has $1+p$ maximal subgroups and the length of each maximal chain is 4. Thus all the maximal chains that contain H_1 yield 2^4-1+2^3p distinct fuzzy subgroups. The maximal subgroup $H_2=\mathbb{Z}_p+0+\mathbb{Z}_p$ has a maximal subgroup $\mathbb{Z}_p+0+0$ already appearing in H_1. However, each maximal chain involving H_2 has a distinguishing factor (new subgroup) not appearing in other maximal chains since H_2 itself does not appear in the maximal chains of H_1. Thus H_2 yields $2^3(1+p)$ distinct fuzzy subgroups.
The maximal subgroup $H_3=0+\mathbb{Z}_p+\mathbb{Z}_p$ has maximal subgroups $0+\mathbb{Z}_p+0$ and $0+0+\mathbb{Z}_p$ already appearing in H_1 and H_2 respectively. Thus there is a maximal chain containing H_3 with no distinguishing factor, but only a pair of distinguishing factors. Therefore H_3 yields 2^3p+2^2 distinct fuzzy subgroups.
The rest of the maximal subgroups contribute only the $(p-1)^2$ subgroups $< (1,r,s) >$, $r,s\neq 0$, not appearing earlier, as in the proof of the theorem on the number of subgroups of $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$, and this yields $2^3(p-1)^2$ distinct fuzzy subgroups. The number of the remaining distinguishing factors not yet used is now $(1+p+p^2-3)$, yielding a further $2^3(1+p+p^2-3)$ distinct fuzzy subgroups.
We have now exhausted all the distinguishing factors. So the rest of the maximal chains can only have pairs of distinguishing factors. The number of the remaining maximal chains is $[p+1+(p^2+p)(\frac{(1+1)}{2}p+1)-3(p+$

$1) - (p-1)^2 - (1+p+p^2-3)]$, where $p+1+(p^2+p)(\frac{(1+1)}{2}p+1)$ is the number of all the maximal chains as given in chapter 4. This yields $2^2[p+1+(p^2+p)(\frac{(1+1)}{2}p+1) - 3(p+1) - (p-1)^2 - (1+p+p^2-3)]$ distinct fuzzy subgroups.
Hence the total number of fuzzy subgroups of G is equal to $2^4 - 1 + 2^3p + 2^3(1+p) + 2^3p + 2^2 + 2^3(p-1)^2 + 2^3(1+p+p^2-3) + 2^2[1+p+1+(p^2+p)(\frac{(1+1)}{2}p+1) - 3(p+1) - (p-1)^2 - (1+p+p^2-3)] = 2^4 - 1 + 2^3[p+1+p+p+(p-1)^2+1+p+p^2-3] + 2^2[1+p+1+(p^2+p)(p+1) - 3p - 3 - (p^2-2p+1)^2 - 1 - p - p^2 + 3] = 2^4 - 1 + 2^3[p+1+p+p+p^2-2p+1+1+p+p^2-3] + 2^2[p+2+p^3+2p^2+p-3p-3-p^2+2p-1-1-p-p^2+3] = 2^4 - 1 + 2^3(2p^2+2p) + 2^2p^3$. □

Proposition 6.1.3 The number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p^2} + \mathbb{Z}_p + \mathbb{Z}_p$ is equal to $2^5 - 1 + 2^4(4p^2+3p) + 2^3(3p^3+p^2)$.

Proof. First we consider maximal subgroups of G. One of them is $H_1 = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$. All the maximal chains of G are of length 5. Thus by the above proposition, H_1 yields $2^5 - 1 + 2^4(2p^2+2p) + 2^3p^3$ distinct fuzzy subgroups.
Consider $H_2 = \mathbb{Z}_{p^2} + \mathbb{Z}_p + 0$ which has $1+p$ maximal subgroups: $\mathbb{Z}_p + \mathbb{Z}_p + 0$; $\mathbb{Z}_{p^2}$; $< (1,r,0) >$, $r = 1, 2, \cdots, p-1 >$. The proper subgroups of all these subgroups $< (1,r,0) >$ of H_2 have all been counted in H_1 or in $\mathbb{Z}_{p^2} + 0 + 0$. So H_2 yields $2^4(p+1)$ distinct fuzzy subgroups, corresponding to the subgroups H_2 and $< (1,r,0) >$, $r = 0, 1, 2, \cdots, p-1 >$.
Next consider $H_3 = \mathbb{Z}_{p^2} + 0 + \mathbb{Z}_p$ which has $1+p$ maximal subgroups, similar to H_2. The proper subgroups of all the subgroups of H_3 have all been counted in H_1 or H_2. Thus $< (1,0,r) >$, $r = 1, 2, \cdots, p-1 >$, are the only uncounted proper subgroups of H_3. So H_3 yields $2^4p + 2^3$ distinct fuzzy subgroups.
The rest of the maximal chains yield $(p-1)^2$ and $(1+p+p^2-3)$ new subgroups to be used as distinguishing factors, see above proposition. This yields a further $2^4[(p-1)^2 + (1+p+p^2-3)]$ distinct fuzzy subgroups.
Now the total number of maximal chains is $p+1+(p^2+p)(3p+2)$, see chapter 4. Thus maximal chains not used above yield a further $2^3[p+1+(p^2+p)(3p+2) - p - 1 - (p^2+p)(p+1) - 2(p+1) - (p-1)^2 - (1+p+p^2-3)]$ distinct fuzzy subgroups. Hence the total number of distinct fuzzy subgroups is equal to $2^5 - 1 + 2^4(2p^2+2p) + 2^3p^3 + 2^4(p+1) + 2^4p + 2^3 + 2^4[(p-1)^2 + (1+p+p^2-3)] + 2^3[p+1+(p^2+p)(3p+2) - p - 1 - (p^2+p)(p+1) - 2(p+1) - (p-1)^2 - (1+p+p^2-3)] = 2^5 - 1 + 2^4(4p^2+3p) + 2^3(3p^3+p^2)$. □

Theorem 6.1.4 . The number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ is equal to
$2^{n+3} - 1 + 2^{n+2}[p(2pn + n + 1)] + 2^{n+1}[p + 1 + (p^2 + p)(\frac{n(n+1)}{2}p + n) - p(2pn + n + 1) - 1]$.

Proof. By induction on n. The above two propositions prove the cases $n = 1$ and $n = 2$. Assume the theorem is true for $k < n$ and let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$. The maximal subgroup $H_1 = \mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ of G satisfies the formula of the theorem with $n = k$ by assumption. Now proceed as in the above proposition to work with maximal subgroups of $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$ and use them and the above propositions to show that the formula of the theorem is true for $n = k + 1$. □

6.1.1 On Isomorphic and Non-isomorphic Fuzzy Subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$

Recall: 1. Two fuzzy subgroups μ and ν of a group G are isomorphic if there is an isomorphism $f : G \to G$ such that for $x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\nu(f(x)) > \nu(f(y))$ and and $\mu(x) = \mu(y)$ if and only if $\nu(f(x)) = \nu(f(y))$.
If the two fuzzy subgroups are not isomorphic, then they are non-isomorphic.
2. Two maximal chains of a group G are isomorphic if they have the same length and corresponding subgroups in the chains are isomorphic.
For example if $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$, then all the maximal chains of G are isomorphic. When computing the number of non-isomorphic classes of fuzzy subgroups, it surfices to collapse all isomorphic maximal chains into 1 and then use the techniques of distinct fuzzy subgroups to calculate the number of non-isomorphic classes of fuzzy subgroups. Thus in $G = \mathbb{Z}_p + \mathbb{Z}_p + \mathbb{Z}_p$, the number of non-isomorphic classes of fuzzy subgroups is $2^4 - 1 = 15$.

Example 6.1.5 Using the maximal chains in Example 5.2.5 for $p = 3$ and $n = 3$, we discuss how to determine the number of isomorphic and non-isomorphic classes of fuzzy subgroups in the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$. The group $\mathbb{Z}_{3^3} + \mathbb{Z}_3 + \mathbb{Z}_3$ has 256 maximal chains.
Looking closely at the maximal chains, we can observe that the first three chains are isomorphic, so they can be collapsed and considered as one and

hence they yield $2^{n+3}-1$ i.e 2^6-1 non-isomorphic fuzzy subgroups. The next three maximal chains are distinguished from the first three by the subgroups generated by $<(3,0,0),(3,1,0)>$, hence they are isomorphic and can be collapsed into a single maximal chain yielding 2^5 non-isomorphic fuzzy subgroups. The next four maximal chains that follow are distinguished from the first and the second three by the subgroups generated by $<(9,0,0),(9,1,0)>$, hence they are isomorphic and can be collapsed into a single maximal chain yielding 2^5 non-isomorphic fuzzy subgroups. The remaining blocks that have the same number of maximal chains as the first 10 are either isomorphic to the first three chains that count 2^6-1 or the next three chains that count 2^5 or the next four chains that count 2^5 non-isomorphic fuzzy subgroups, hence they are all regarded as part of the ten chains that have already been counted. Next, we examine the maximal chains produced by the maximal subgroup of the form $<(3,0,0),(0,1,0),(0,0,1)>$. This subgroup is isomorphic to the group $\mathbb{Z}_{3^2}+\mathbb{Z}_3+\mathbb{Z}_3$ and it is distinguished from those considered, hence the first three chains on the block are collapsed into one chain and yield 2^5 non-isomorphic fuzzy subgroups. The next four chains have been distinguished from the first three by the subgroups generated by $<(9,0,0),(9,0,1)>$, hence they are isomorphic and can be collapsed into a single maximal chain, yielding 2^4 non-isomorphic fuzzy subgroups. Finally, we look at the maximal chains produced by the group generated $<(9,0,0),(0,1,0),(0,0,1)>$ which is isomorphic to the group $\mathbb{Z}_3+\mathbb{Z}_3+\mathbb{Z}_3$. This group is different from those considered above, all the chains under this group are regarded as a single chain and give a count of 2^5 non-isomorphic fuzzy subgroups. Hence in total we have $[2^6-1+2^5+2^5]+[2^5+2^4+2^5]=2^6-1+4*2^5+2^4=63+128+16=207$ non-isomorphic classes of fuzzy subgroups. This agrees with the theorems that follow.

Theorem 6.1.6 . The number of non-isomorphic maximal chains of subgroups of the group $G=\mathbb{Z}_{p^n}+\mathbb{Z}_p+\mathbb{Z}_p$ is equal to $\frac{n(n+1)}{2}$.

Proof. By induction on n. For $n=1$ all the maximal chains are isomorphic and each chain is of length 4. Thus there is only one maximal chain up to isomorphism. The formula $\frac{n(n+1)}{2}$ with $n=1$ also gives 1 maximal chain up to isomorphism.

Now assume the result is true for all $k<n$. Let $G=\mathbb{Z}_{p^{k+1}}+\mathbb{Z}_p+\mathbb{Z}_p$. Then G has 2 non-isomorphic maximal subgroups viz. $\mathbb{Z}_{p^k}+\mathbb{Z}_p+\mathbb{Z}_p$ and

$\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$. By induction, $\mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ has $\frac{k(k+1)}{2}$ non-isomorphic maximal chains. In chapter 3 we showed that $\mathbb{Z}_{p^n} + \mathbb{Z}_p$ has n maximal chains. Thus G has $\frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$ non-isomorphic maximal chains. Hence the theorem is true for $n = k + 1$. □

Theorem 6.1.7 . The number of non-isomorphic classes of fuzzy subgroups of the group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ is equal to $2^{n+3} - 1 + 2(n-1)2^{n+2} + \frac{(n-1)(n-2)}{2}2^{n+1}$.

Proof. By induction on n. For $n = 1$ we have only one maximal chain up to isomorphism, giving $2^4 - 1$ non-isomorphic fuzzy subgroups. Now assume the theorem is true for all $k < n$. Let $G = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + \mathbb{Z}_p$, then G has 2 non-isomorphic maximal subgroups viz. $H_1 = \mathbb{Z}_{p^k} + \mathbb{Z}_p + \mathbb{Z}_p$ and $H_2 = \mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$. By induction, H_1 yields $2^{k+4} - 1 + 2(k-1)2^{k+3} + \frac{(k-1)(k-2)}{2}2^{k+2}$ non-isomorphic fuzzy subgroups and H_2 yields 2 new subgroups up to isomorphism viz $\mathbb{Z}_{p^{k+1}} + \mathbb{Z}_p + 0$ and $\mathbb{Z}_{p^{k+1}} + 0 + 0$. Hence the total number of non-isomorphic fuzzy subgroups is equal to $2^{k+4} - 1 + 2(k-1)2^{k+3} + \frac{(k-1)(k-2)}{2}2^{k+2} + 2.2^{k+3} + [\frac{(k+1)(k+2)}{2} - 1 - 2k - \frac{(k-1)(k-2)}{2}]2^{k+2} = 2^{k+1+3} - 1 + 2(k+1-1)2^{k+1+2} + \frac{(k+1-1)(k+1-2)}{2}2^{n+1}$. Hence the theorem is true for $n = k + 1$. □

Chapter 7

CONCLUSSION

In this dissertation we have successfully used an enumeration technique introduced in [84] and an equivalence relation introduced by Murali and Makamba in [81] to investigate and classify distinct fuzzy subgroups of a finite abelian p-group of rank 3 of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ for any prime integer p and any positive integer n, a milestone and a core of this study. We started by reviewing formulae for the number of subgroups, maximal chains of subgroups and distinct fuzzy subgroups of the rank-2 group $\mathbb{Z}_{p^2} + \mathbb{Z}_p$, see chapter 4. In the same chapter, we computed the number of non-isomorphic classes of fuzzy subgroups.
Chapter 4 thus prepared the ground for the classification of fuzzy subgroups of the rank-3 group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$. We began this study by determining user-friendly formulae for the number of subgroups of this group G. We had the general formula $p(2pn + n + 1) + 3 + n$ for the number of subgroups of G. This then enabled us to count the maximal chains of subgroups of G. So we obtained the formula $(p+1) + (p^2+p)[\frac{n(n+1)}{2}p + n]$ for the number of maximal chains of subgroups of G.
The counting of distinct fuzzy subgroups requires maximal chains and how we distinguish them. We proceeded inductively to count the number of distinct fuzzy subgroups and finally obtained the formula
$2^{n+3} - 1 + 2^{n+2}[p(2pn + n + 1)] + 2^{n+1}[p + 1 + (p^2 + p)(\frac{n(n+1)}{2}p + n)$
- $p(2pn + n + 1) - 1]$ for this number.
By collapsing all isomorphic maximal chains into one, we obtained the formula $\frac{n(n+1)}{2}$ for the number of non-isomorphic maximal chains and $2^{n+3} - 1 + 2(n-1)2^{n+2} + \frac{(n-1)(n-2)}{2}2^{n+1}$ for the number of non-isomorphic fuzzy subgroups of

G. Our research would not have been successful without specific examples involving specific primes and exponents. So we looked laboriously at all subgroups of a specific G, maximal chains of subgroups and how to distinguish them. Painstakingly, we counted all distinct fuzzy subgroups. After many such examples, we started looking for patterns. There seems to be no way of guessing the results without looking first at specific cases.
Further research: The group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ can still be extended to $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{p^m} + \mathbb{Z}_{p^k}$ for any positive integers n, m and k. This should be a further research for a doctoral degree. We have done a lot of good work for a Masters degree. One could also extend $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ to $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p + \cdots + \mathbb{Z}_p$, i.e. we can attach any number of the summands $\mathbb{Z}_p$ to $\mathbb{Z}_{p^n}$. There is also the possibility of varying the primes, for example using $G = \mathbb{Z}_{p^n} + \mathbb{Z}_q + \mathbb{Z}_q$, where p and q are distinct primes. In fact there are many different possible permutations of the primes.
Finally we used the technique of starting with maximal chains of subgroups and then associating each maximal chain with all possible classes of keychains. This technique seems friendlier to handle than other techniques. The equivalence relation used in our classification of fuzzy subgroups seems more complex to handle compared to others available in literature, hence we are confident that our results on the maximal chains of subgroups, distinct fuzzy subgroups and non-isomorphic classes of fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_p + \mathbb{Z}_p$ have not been obtained before.

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