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EXCURSIONS OF A RANDOM WALK RELATED TO THE STRONG LAW OF LARGE NUMBERS

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1. Introduction. Let $\{X, i = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random variables, each normally distributed with mean μ and variance σ^2. For $n = 1, 2, 3, \dots$, define $S_n = \sum_{i=1}^n X_i = S_0 \equiv 0$. It follows from the Kolmogorov strong law of large numbers (see [**1**, p. 274]), that $\lim_{n\to\infty}(S_n - n\mu)/n^\alpha = 0$ a.s. for all $\alpha > 1/2$. Consequently, for each real number $c > 0$, the inequality

$$S_n - n\mu > cn^\alpha \tag{1.1}$$

is satisfied for only finitely many indices n.

We define an excursion of the random walk $\{S_n, n = 1, 2, 3, \dots\}$ to be a complete sequence of consecutive indices for which the inequality (1.1) holds. More precisely, we say that an excursion of length k, $k = 1, 2, 3, \dots$, begins at index n, $n = 1, 2, 3, \dots$, if

$$(S_{n-1} - (n-1)\mu \le c(n-1)^\alpha, S_{n+i-1} - (n+i-1)\mu > c(n+i-1)^\alpha \\ \text{for } i = 1, 2, 3, \dots, k, S_{n+k} - (n+k)\mu \le c(n+k)^\alpha).$$

For $n = 1, 2, 3, \dots$, define the event A_n by $A_n = (S_n - n\mu > cn^\alpha, S_{n+1} - (n+1)\mu \le c(n+1)^\alpha)$ and define the random variable $X(c)$ by

$$X(c) = \sum_{n=1}^\infty I(A_n). \tag{1.2}$$

($I(A)$ denotes the indicator function of the event A.) $X(c)$ represents the number of excursions. It follows from (1.1) that $X(c)$ is finite-valued. (We suppress, in the notation, the dependence of $X(c)$ on α.)

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Here we investigate the expected number of excursions. The main result of this paper, Theorem 2.1, establishes asymptotically close upper and lower bounds on the expected value of $X(c)$ for $1/2 < \alpha \leq 1$. (See Section 2.4) This provides a characterization of the relationship between the fluctuations of the random walk $\{S_n\}$ and the strong law of large numbers.

The results in this paper contrast with earlier investigations on the random variable $N(c)$, which represents the number of indices for which (1.1) occurs. Specifically, define the random variable $N(c)$ by

$$N(c) = \sum_{n=1}^{\infty} I(B_n) \tag{1.3}$$

where $B_n = (S_n - n\mu > cn^\alpha)$. The random variable $N(c)$ and characterizations of its moments have been studied in a variety of settings. (See, for example, Lai and Lan [**4**], Slivka [**6**], Stratton [**8**], Razanadrakoto and Severo [**5**] and Klebaner [**3**].) Results on $N(c)$ most closely related to Theorem 2.1 appear in Drucker, Fodor, and Fisher [**2**] and Slivka and Severo [**7**]. (We note that in the latter two references the variable $N(c)$ is defined by (1.3) with $B_n = (|S_n - n\mu| > cn^\alpha)$.)

In Section 2 we state Theorem 2.1 and discuss various implications of the theorem. We include tables of values for the upper and lower bounds established by the theorem for a variety of values of c and α and relate the expected value of $X(c)$ with the expected value of $N(c)$. The proof of Theorem 2.1 comprises Section 3 of the paper.

2. Statement and discussion of Theorem 2.1.

2.1. *Notation and statement of results.* We assume the notation and definitions as described in the introduction. For notational convenience, we make the following definitions. Let $\rho = c/\sigma$, let $\Phi(\cdot)$ represent the cumulative distribution function of the standard normal distribution, and define

$$A(\rho, \alpha) = \frac{1}{\pi(4\alpha - 2)} \left(\frac{2}{\rho^2}\right)^{1/(4\alpha-2)} \Gamma\left(\frac{1}{4\alpha - 2}\right). \tag{2.1}$$

Theorem 2.1. *Let $\{X_i = 1, 2, 3, \ldots\}$ be a sequence of independent, normally distributed random variables with mean μ and variance σ^2.*

Let $c > 0$ and $1/2 < \alpha \leq 1$. Let $X(c)$ be as defined by (1.2)*. Then*

$$\max\{L(\rho,\alpha),0\} \leq EX(c) \leq U(\rho,\alpha)$$

where

$$L(\rho,\alpha) = A(\rho,\alpha) - \frac{1-\alpha}{4\alpha-2} - \frac{\rho(1-\alpha)^2 2^{3/2-\alpha}}{(2\alpha-1)\sqrt{\pi}}(1-\Phi(\rho)) - \frac{3}{\sqrt{2\pi}} \tag{2.2}$$

and

$$U(\rho,\alpha) = A(\rho,\alpha) + \frac{\alpha\Phi(\rho\alpha)}{2\alpha-1}. \tag{2.3}$$

Remark. We note that, as expected, the bounds on $EX(c)$ are based on c and σ through $\rho = c/\sigma$.

The upper and lower bounds simplify considerably in the linear case, which we record as the following corollary.

Corollary 2.2. *Under the same conditions as Theorem* 2.1*, suppose that $\alpha = 1$. Then*

$$\max\left\{\frac{1}{\rho\sqrt{2\pi}} - \frac{3}{\sqrt{2\pi}}, 0\right\} \leq EX(c) \leq \frac{1}{\rho\sqrt{2\pi}} + \Phi(\rho).$$

2.2. *The asymptotic behavior of $EX(c)$.* For fixed α satisfying $1/2 < \alpha \leq 1$, it is clear that $\lim_{c\to 0} EX(c) = \infty$. It follows easily from Theorem 2.1 that

$$EX(c) \sim \frac{1}{\pi(4\alpha-2)} 2^{1/(4\alpha-2)} \Gamma\left(\frac{1}{4\alpha-2}\right) \rho^{-1/(2\alpha-1)} \tag{2.4}$$

as $c \to 0$ and so

$$EX(c) = O(\rho^{-1/(2\alpha-1)}) \tag{2.5}$$

as $c \to 0$.

2.3. *The relationship between $X(c)$ and $N(c)$.* It is of interest to compare the asymptotic behavior as $c \to 0$ of the expected number of excursions and the expected number of steps of the normal random walk above the boundaries described in Theorem 2.1. It follows from Drucker, Fodor, and Fisher [**2**, Theorem 2.3 and proof of Corollary 3.1] and (2.4) that

$$\frac{EN(c)}{EX(c)} \sim (2\alpha - 1)\sqrt{\pi}\frac{\Gamma((2\alpha+1)/(4\alpha-2))}{\Gamma(1/(4\alpha-2))}(2/\rho^2)^{1/(4\alpha-2)}$$

as $c \to 0$ and hence that

$$\frac{EN(c)}{EX(c)} = O(1/\rho^{2\alpha-1})$$

as $c \to 0$.

2.4. *Remarks and examples related to the upper and lower bounds on $EX(c)$.* Define

$$d(\rho, \alpha) = U(\rho, \alpha) - L(\rho, \alpha).$$

In the linear case ($\alpha = 1$) of Corollary 2.2, it is easily seen that $d(\rho, 1) \leq \Phi(\rho) + 3/(\sqrt{2}\pi) \leq 1.68$ for all $\rho > 0$. For $\rho \leq \sqrt{\pi}/3$, it is clear that $d(p,1) = \Phi(\rho) + 3/(\sqrt{2}\pi)$ and that $\lim_{\rho\to 0} d(p,1) = 1/2 + 3/(\sqrt{2}\pi) \approx 1.2$. Hence, we note that the upper and lower bounds on $EX(c)$ are uniformly close in the case of a linear boundary.

We illustrate Corollary 2.2 with the following table of results (rounded off to the third decimal place) calculated using *Mathematica* (see [**9**]). We take $\sigma = 1$. For selected values of c, we display the interval containing $EX(c)$ obtained through Corollary 2.2 and, for purposes of comparison, we display the interval containing $EN(c)$ obtained through the application of the aforementioned result in [**2**, Theorem 2.3].

c	$EN(c)$	$EX(c)$
1	[0,.5]	[0,1.240]
.1	[49.5,50]	[3.314,4.529]
.01	[4999.5, 5000]	[39.219, 40.398]
.001	[499,999.5, 500,000]	[398.267,399.443]

For the case $1/2 < \alpha < 1$, we observe that

$$\lim_{\rho \to 0} d(\rho, \alpha) = \frac{3}{\sqrt{2\pi}} + \frac{1}{4\alpha - 2}$$

so that the relative error in approximating $EX(c)$ through Theorem 2.1 approaches zero as c approaches zero. As we did earlier for the case $\alpha = 1$, we illustrate Theorem 2.1 for $\sigma = 1$ with selected values of c and α. Here we also display an approximation of $EN(c)$ showing its order of magnitude for comparison with $EX(c)$.

α	c	$EN(c)$	$EX(c)$
.75	.1	15,000	[62.731,64.457]
.6	.5	483,340	[189.687,193.346]
.55	.8	2.8×10^{10}	[11380.3,11387.3]

3. Proof of Theorem 2.1. We divide the proof of Theorem 2.1 into two sections. In Section 3.1 we derive the upper bound (2.3), and in Section 3.2 we derive the lower bound (2.2).

Define $X_n^* = (X_n - \mu)/\sigma$ and $S_n^* = \sum_{k=1}^n X_k^*$ for $n = 1, 2, 3, \ldots$. It follows from (1.2) that

$$EX(c) = \sum_{n=1}^{\infty} P(S_n^* > \rho n^\alpha, S_n^* + X_{n+1}^* \le \rho(n+1)^\alpha)$$

so that

(3.1)

$$\begin{aligned} EX(c) &= \sum_{n=1}^{\infty} \int_{-\infty}^{\rho((n+1)^\alpha - n^\alpha)} P(\rho n^\alpha < S_n^* < \rho(n+1)^\alpha - y)\phi(y)\, dy \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\rho((n-1)^\alpha - n^\alpha)} \left\{ \Phi\left(\frac{\rho(n+1)^\alpha - y}{\sqrt{n}}\right) - \Phi\left(\frac{\rho n^\alpha}{\sqrt{n}}\right) \right\} \phi(y)\, dy \end{aligned}$$

where $\phi(\cdot)$ represents the standard normal density function.

3.1. *Derivation of the upper bound.* We apply the mean value theorem to the first factor of the latter integrand in (3.1) and use the fact that

$\phi'(y) = -y\phi(y)$ to obtain

$$\begin{aligned} EX(c) &\leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} \exp(-\rho^2 n^{2\alpha-1}/2) \\ &\qquad \cdot \int_{-\infty}^{\rho(n+1)^\alpha - \rho n^\alpha} (\rho(n+1)^\alpha - \rho n^\alpha - y)\phi(y)\, dy \\ &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} (\rho(n+1)^\alpha - \rho n^\alpha) \\ &\qquad \cdot \Phi(\rho(n+1)^\alpha - \rho n^\alpha) \exp(-\rho^2 n^{2\alpha-1}/2) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} \phi(\rho(n+1)^\alpha - \rho n^\alpha) \exp(-\rho^2 n^{2\alpha-1}/2). \end{aligned} \tag{3.2}$$

It follows from the monotonicity of $\Phi(\cdot)$ and the inequality $(n+1)^\alpha - n^\alpha \leq \alpha n^{\alpha-1}$ for $\alpha \leq 1$, that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} (\rho(n+1)^\alpha - \rho n^\alpha) \Phi(\rho(n+1)^\alpha - \rho n^\alpha) \exp(-\rho^2 n^{2\alpha-1}/2) \\ \leq \sum_{n=1}^{\infty} \frac{\rho\alpha\Phi(\rho\alpha)}{\sqrt{2\pi} n^{3/2-\alpha}} \exp(-\rho^2 n^{2\alpha-1}/2) \\ \leq \frac{\rho\alpha\Phi(\rho\alpha)}{\sqrt{2\pi}} \int_0^\infty x^{\alpha-3/2} \exp(-\rho^2 x^{2\alpha-1}/2)\, dx. \end{aligned}$$

For future reference, we note that, with the elementary change of variables $y = \rho x^{\alpha-1/2}$, one obtains the result

$$\int_0^\infty x^{\alpha-3/2} \exp(-(1/2)\rho^2 x^{2\alpha-1})\, dx = \frac{\sqrt{2\pi}}{\rho(2\alpha-1)}. \tag{3.3}$$

It follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} (\rho(n+1)^\alpha - \rho n^\alpha) \Phi(\rho(n+1)^\alpha - \rho n^\alpha) \\ \cdot \exp(-\rho^2 n^{2\alpha-1}/2) \leq \frac{\alpha\Phi(\rho\alpha)}{2\alpha-1}. \end{aligned} \tag{3.4}$$

We consider the latter sum in (3.2) and note that $\phi(\cdot) \le 1/\sqrt{2\pi}$ to obtain

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} \phi(\rho(n+1)^\alpha - \rho n^\alpha) \exp(-\rho^2 n^{2\alpha-1}/2)$$
$$\le \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} \exp\left(-\frac{\rho^2}{2} n^{2\alpha-1}\right)$$
$$\le \frac{1}{2\pi} \int_0^\infty x^{-1/2} \exp\left(-\frac{\rho^2}{2} x^{2\alpha-1}\right) dx.$$

The change of variables $y = \rho^2 x^{2\alpha-1}/2$ applied to the latter integral results in the inequality

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}} \phi(\rho(n+1)^\alpha - \rho n^\alpha) \exp(-\rho^2 n^{2\alpha-1}/2) \le \frac{1}{\pi(4\alpha-2)} \left(\frac{2}{\rho^2}\right)^{1/(4\alpha-2)} \Gamma\left(\frac{1}{4\alpha-2}\right). \tag{3.5}$$

Together the results (3.2)–(3.5) establish the upper bound of Theorem 2.1 described by (2.3).

3.2. *Derivation of the lower bound.* It follows from (3.1) and an application of the mean value theorem that

$$EX(c) \ge \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{-\infty}^{\rho(n+1)^\alpha - \rho n^\alpha} (\rho(n+1)^\alpha - \rho n^\alpha - y) \phi\left(\frac{\rho(n+1)^\alpha - y}{\sqrt{n}}\right) \phi(y)\, dy. \tag{3.6}$$

The change of variables $u = \rho(n+1)^\alpha - y$ applied to the integral in (3.6) yields

$$EX(c) \ge \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} \exp\left(-\frac{1}{2}\rho^2 (n+1)^{2\alpha}\right) \cdot \int_{\rho n^\alpha}^{\infty} (u - \rho n^\alpha) \cdot \exp\left\{-\frac{1}{2}\left(\frac{n+1}{n}\right)(u^2 - 2\rho n (n+1)^{\alpha-1} u)\right\} du. \tag{3.7}$$

Completing the square in the latter exponential function results in

$$EX(c) \geq \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} \exp\left(-\frac{1}{2}\rho^2 (n+1)^{2\alpha-1}\right) \cdot \int_{\rho n^\alpha}^{\infty} (u - \rho n^\alpha) \cdot \exp\left\{ -\frac{1}{2}\left(\frac{n+1}{n}\right)(u-\rho n(n+1)^{\alpha-1})^2\right\} du. \tag{3.8}$$

We write the sum in (3.8) as

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} \exp\left(-\frac{1}{2}\rho^2 (n+1)^{2\alpha-1}\right) \\ &\quad \cdot \int_{\rho n^\alpha}^{\infty} (u - \rho n^\alpha) \exp\left\{ -\frac{1}{2}\left(\frac{n+1}{n}\right)(u-\rho n(n+1)^{\alpha-1})^2\right\} du \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} \exp\left(-\frac{1}{2}\rho^2 (n+1)^{2\alpha-1}\right) \\ &\quad \cdot \int_{\rho n^\alpha}^{\infty} (u - \rho n(n+1)^{\alpha-1}) \\ &\qquad \cdot \exp\left\{ -\frac{1}{2}\left(\frac{n+1}{n}\right)(u-\rho n(n+1)^{\alpha-1})^2\right\} du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} (\rho n(n+1)^{\alpha-1} - \rho n^\alpha) \\ &\qquad \cdot \exp\left(-\frac{1}{2}\rho^2 (n+1)^{2\alpha-1}\right) \\ &\qquad \cdot \int_{\rho n^\alpha}^{\infty} \exp\left\{ -\frac{1}{2}\left(\frac{n+1}{n}\right)(u-\rho n(n+1)^{\alpha-1})^2\right\} du. \end{aligned} \tag{3.9}$$

We apply the change of variables $y = -((n+1)/(2n))(u - \rho n(n+1)^{\alpha-1})^2$ and $y = \sqrt{(n+1)/n}(u - \rho n(n+1)^{\alpha-1})$ to the second and third integrals that appear in (3.9), respectively. This, with (3.8), yields the inequality

$$EX(c) \geq T_1 - T_2 \tag{3.10}$$

where

$$T_1 = \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n}} \exp\left(-\frac{1}{2}\rho^2(n+1)^{2\alpha-1}\right) \cdot \left\{ \frac{n}{n+1} \exp\left(-\frac{1}{2}\left\{\frac{n+1}{n}\right\}\{\rho^2 n^2(n^{\alpha-1}-(n+1)^{\alpha-1})^2\}\right)\right\}$$

and

$$T_2 = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi(n+1)}}(\rho n^\alpha - \rho n(n+1)^{\alpha-1}) \cdot \Phi\left(-\sqrt{\frac{n+1}{n}}(\rho n^\alpha - \rho n(n+1)^{\alpha-1})\right) \cdot \exp\left(-\frac{1}{2}\rho^2(n+1)^{2\alpha-1}\right).$$

We consider T_2 and note that $0 \leq n^{\alpha-1} - (n+1)^{\alpha-1} \leq (1-\alpha)n^{\alpha-2}$ for $1/2 < \alpha \leq 1$. It follows from this and (3.3) that

$$\begin{aligned} T_2 &\leq \frac{\rho(1-\alpha)}{2\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2-\alpha}} \exp\left(-\frac{1}{2}\rho^2 n^{2\alpha-1}\right) \\ &\leq \frac{\rho(1-\alpha)}{2\sqrt{2\pi}} \left\{ \int_0^\infty x^{\alpha-3/2} \exp\left(-\frac{1}{2}\rho^2 x^{2\alpha-1}\right) dx\right\} \\ &= \frac{1-\alpha}{4\alpha-2}. \end{aligned} \tag{3.11}$$

The same remark leading to (3.11) leads to

$$T_1 \geq \sum_{n=1}^{\infty} \frac{1}{2\pi\sqrt{n+1}} \exp(-\rho^2(n+1)^{2\alpha-1}/2) \cdot \left\{ \sqrt{\frac{n}{n+1}} \exp\left(-\frac{\rho^2(1-\alpha)^2}{n^{2-2\alpha}}\right)\right\}. \tag{3.12}$$

Applying the inequalities $\exp(-x) \geq 1-x$ and $\sqrt{n/(n+1)} \geq 1 - 1/(2\sqrt{n(n+1)})$, we obtain

(3.13)
$$\sqrt{\frac{n}{n+1}}\exp\bigg(-\frac{\rho^2(1-\alpha)^2}{n^{2-2\alpha}}\bigg) \geq 1 - \frac{1}{2\sqrt{n(n+1)}} - \rho^2(1-\alpha)^2 n^{2\alpha-2}\bigg(1-\frac{1}{2\sqrt{n(n+1)}}\bigg).$$

Further, since $n^2 \geq (n+1)/2$, it follows from (3.13) that

$$\sqrt{\frac{n}{n+1}}\exp\bigg(-\frac{\rho^2(1-\alpha)^2}{n^{2-2\alpha}}\bigg) \geq 1 - \frac{1}{\sqrt{2}(n+1)} - \rho^2(1-\alpha)^2 2^{1-\alpha}(n+1)^{\alpha-1}. \tag{3.14}$$

From (3.12) and (3.14), we observe that

$$T_1 \geq T_{11} - T_{12} - T_{13} \tag{3.15}$$

where T_{11}, T_{12} and T_{13} are defined by
(3.16)
$$\begin{cases} T_{11} = \dfrac{1}{2\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}\exp\bigg(-\frac{1}{2}\rho^2(n+1)^{2\alpha-1}\bigg) \\ T_{12} = \dfrac{1}{2\sqrt{2}\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)^{3/2}}\exp\bigg(-\frac{1}{2}\rho^2(n+1)^{2\alpha-1}\bigg) \\ T_{13} = \dfrac{\rho^2(1-\alpha)^2 2^{1-\alpha}}{2\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)^{3/2-\alpha}}\exp\bigg(-\frac{1}{2}\rho^2(n+1)^{2\alpha-1}\bigg). \end{cases}$$

We derive a lower bound on T_{11}, beginning with the relation

$$T_{11} \geq \frac{1}{2\pi}\int_1^{\infty}\frac{1}{\sqrt{x+1}}\exp\bigg(-\frac{1}{2}\rho^2(x+1)^{2\alpha-1}\bigg)\,dx. \tag{3.17}$$

The change of variables $y = \rho^2(x+1)^{2\alpha-1}/2$ leads to

$$\begin{aligned}\frac{1}{2\pi}\int_1^\infty \frac{1}{\sqrt{x+1}}\exp\bigg(-\frac{1}{2}\rho^2(x+1)^{2\alpha-1}\bigg)\,dx \\ \geq \frac{1}{\pi(4\alpha-2)}(2/\rho^2)^{1/(4\alpha-2)}\bigg\{\int_0^\infty y^{1/(4\alpha-2)-1}\exp(-y)\,dy \\ - \int_0^{\rho^2 2^{2\alpha-2}} y^{1/(4\alpha-2)-1}\,dy\bigg\} \\ = \frac{1}{\pi(4\alpha-2)}(2/\rho^2)^{1/(4\alpha-2)}\Gamma\bigg(\frac{1}{4\alpha-2}\bigg) - \frac{\sqrt{2}}{\pi},\end{aligned}$$

and we conclude that

$$T_{11} \geq \frac{1}{\pi(4\alpha-2)}(2/\rho^2)^{1/(4\alpha-2)}\Gamma\bigg(\frac{1}{4\alpha-2}\bigg) - \frac{\sqrt{2}}{\pi}. \tag{3.18}$$

It is elementary that

$$T_{12} \leq \frac{1}{2\sqrt{2}\pi}\int_0^\infty \frac{1}{(x+1)^{3/2}}\,dx = \frac{1}{\sqrt{2}\pi}. \tag{3.19}$$

Finally, we consider T_{13} and observe that

$$T_{13} \leq \frac{\rho^2(1-\alpha)^2}{2^\alpha\pi}\int_1^\infty x^{\alpha-3/2}\exp\bigg(-\frac{1}{2}\rho^2x^{2\alpha-1}\bigg)\,dx.$$

The same change of variables employed in deriving (3.3) leads to

$$\int_1^\infty x^{\alpha-3/2}\exp\bigg(-\frac{1}{2}\rho^2x^{2\alpha-1}\bigg)\,dx = \frac{2\sqrt{2\pi}(1-\Phi(\rho))}{\rho(2\alpha-1)}. \tag{3.20}$$

We apply (3.20) to obtain

$$T_{13} \leq \frac{2^{3/2-\alpha}\rho(1-\alpha)^2}{\sqrt{\pi}(2\alpha-1)}(1-\Phi(\rho)). \tag{3.21}$$

The results (3.18), (3.19), (3.21) and (3.15) along with (3.10) and (3.11) establish the lower bound described in (2.2). This completes the proof of Theorem 2.1. □

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