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A Proposal for the Use of the Method of Characteristics as a Condition on the Numerical Solutions of Two-Dimensional Lagrangian Isentropic Flow

Christian D. Anderson

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CHARACTERISTICS AS A CONDITION ON THE NUMERICAL SOLUTIONS OF TWO-DIMENSIONAL LAGRANGIAN ISENTROPIC FLOW

ANDERSON

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A PROPOSAL FOR THE USE OF THE METHOD OF
CHARACTERISTICS AS A CONDITION ON THE
NUMERICAL SOLUTIONS OF TWO-DIMENSIONAL
LAGRANGIAN ISENTROPIC FLOW,

by

Christian D. Anderson

A Thesis

Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Physics

The University of New Mexico

1961



This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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INTRODUCTION

The method of characteristics has been widely used in two-dimensional problems (i.e., one spacial dimension and time or two spacial dimensions for the steady state), and it was felt that this method, in all or part, could be applied to time-dependent, two-space hydrodynamics.

In this paper we have done the following:

1. Derived the two-dimensional, time-dependent, nonviscous Lagrangian Equations;
2. Discussed, in general, some aspects of the method of characteristics;
3. Applied the method to our Lagrangian equations; and finally,
4. Proposed a possible application of our results to the numerical solution of the Lagrangian equations.

In general, Sections I and II are devoted to the restatement of the material in the references, while Sections III and IV contain the new results.

THE STATE

The State of New York, in and for the County of Albany, do hereby certify that the following is a true and correct copy of the original as the same appears on file in the office of the Secretary of State.

Witness my hand and the seal of the Secretary of State at Albany, this 1st day of January, 1900.

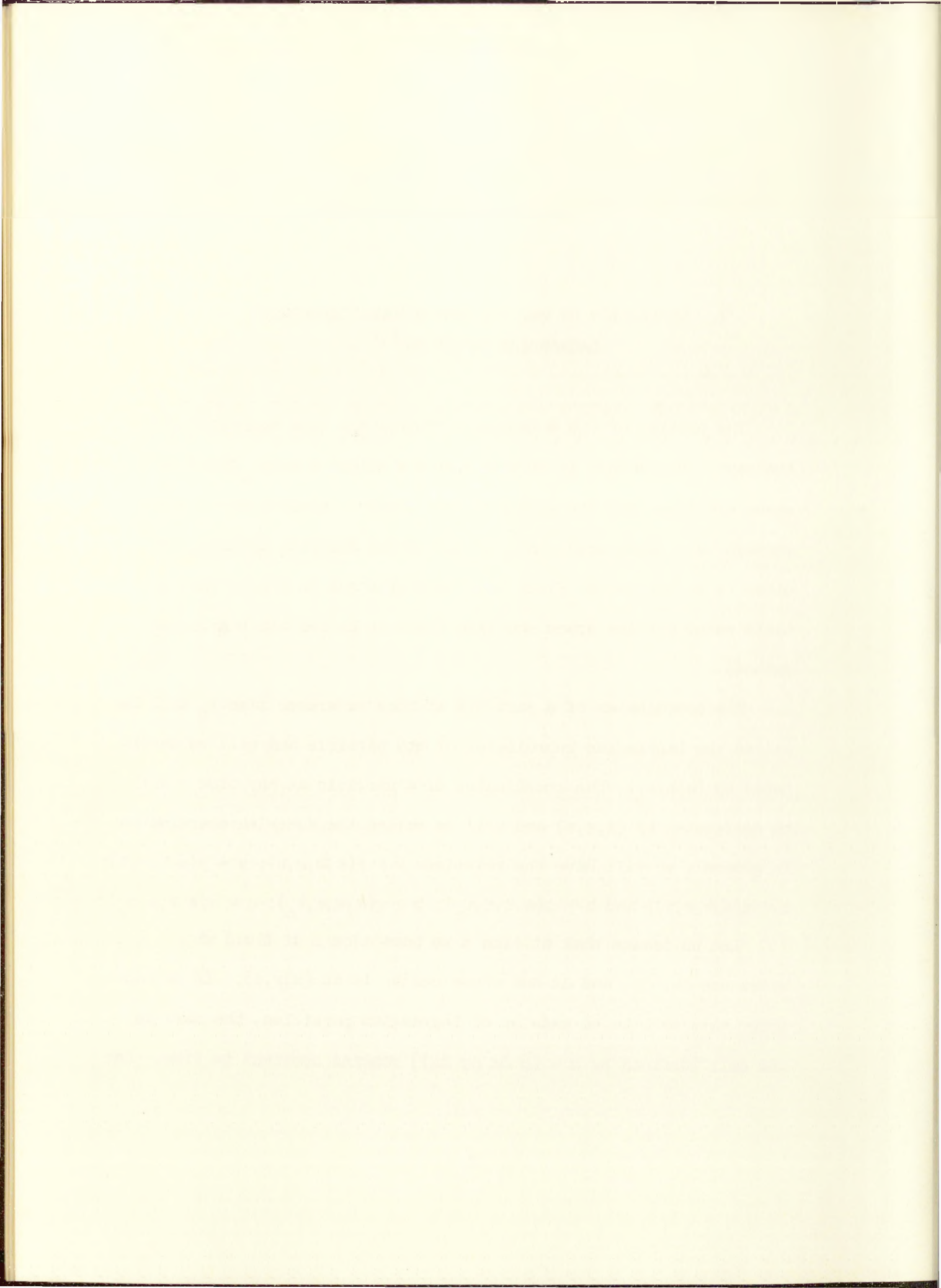
Secretary of State

I. DERIVATION OF THE TWO-DIMENSIONAL ISENTROPIC LAGRANGIAN EQUATIONS^{1,2}

The problem of the dynamics of fluids has been approached in two ways. One method is to consider a stationary mesh, fixed in space and time, and the motion of the fluid is studied as it moves through this fixed mesh (this is called the Eulerian method); the other is to divide the fluid into particles and to follow the particle paths through space and time (this is called the Lagrangian method).

The coordinates of a particle at some reference time t_0 will be called the Lagrangian coordinates of the particle and will be designated by (a,b,c) . The coordinates of a particle at any time t will be designated by (x,y,z) and will be called the Eulerian coordinates. In general, we will have the relations $x = x(a,b,c,t)$; $y = y(a,b,c,t)$; $z = z(a,b,c,t)$; and $a = x(a,b,c,t_0)$; $b = y(a,b,c,t_0)$; $c = z(a,b,c,t_0)$.

Let us assume that at time t we have a cell of fluid whose sides are Δx , Δy , and Δz and whose center is at (x,y,z) . If we consider this cell to be made up of Lagrangian particles, the mass in the cell [defined by $m = (\rho \Delta x \Delta y \Delta z)$] remains constant in time. Let



us further assume that the problem is two-dimensional in the (x,y) plane (i.e., all quantities are independent of z).

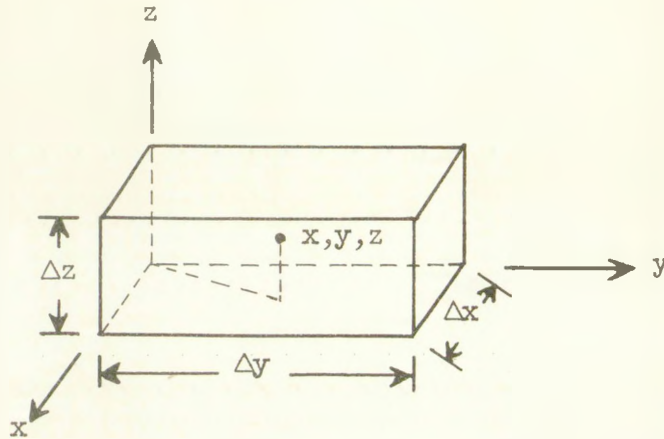


Figure 1.

Force is defined as the time rate of change of momentum (i.e., $(\partial/\partial t)m\vec{r}$), but since $\partial m/\partial t = 0$ for a Lagrangian cell, $\vec{F} = m(\partial\vec{r}/\partial t)$, or component wise,

$$F_x = m\ddot{x}, \quad F_y = m\ddot{y}, \quad \text{and} \quad F_z = m\ddot{z}.$$

Now let us look at the cell in the x,y plane. Since we are considering isentropic flow, the only force acting on the cell is the isentropic pressure P .



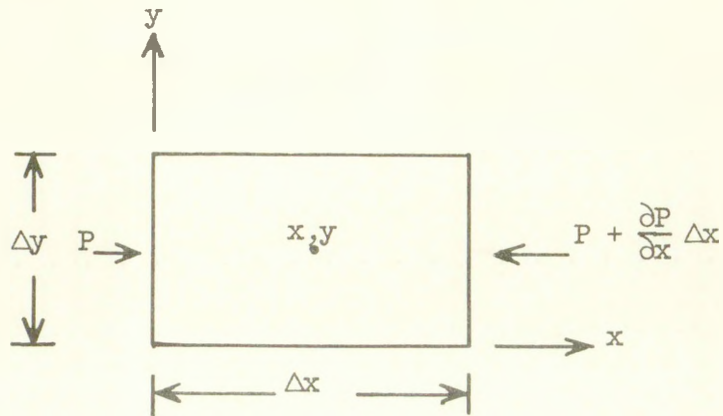


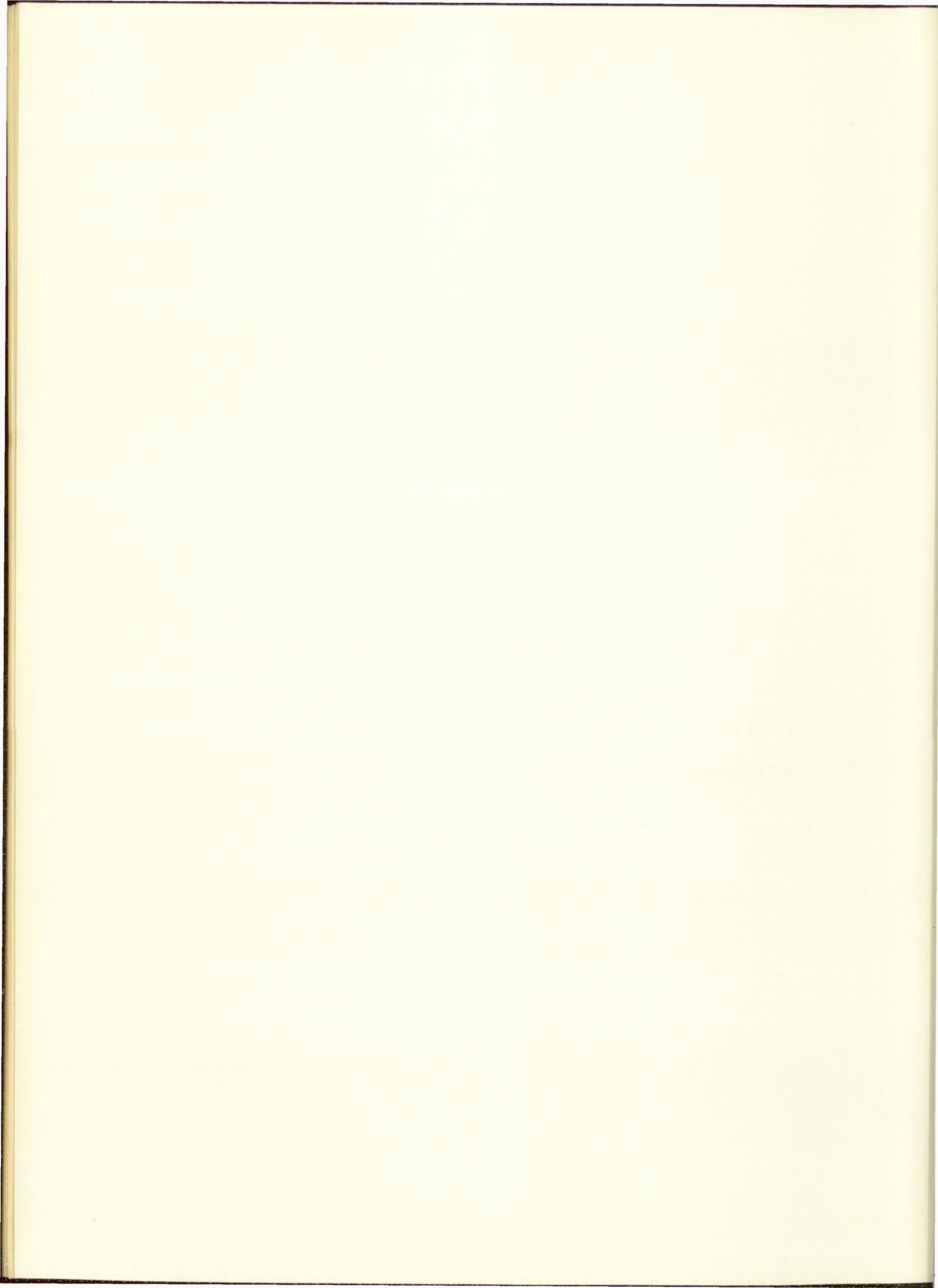
Figure 2.

The force in the x direction at $[x - (\Delta x/2)]$ is $F_x = P \Delta y \Delta z$, and at $[x + (\Delta x/2)]$ the force in the x direction is $F_x = [P + (\partial P/\partial x) \times \Delta x] \Delta y \Delta z$. The net force in the x direction on our cell is the difference between these two forces,

$$F_x = P \Delta y \Delta z - [P + (\partial P/\partial x) \Delta x] \Delta y \Delta z,$$

or

$$F_x = - \frac{\partial P}{\partial x} \Delta x \Delta y \Delta z.$$



But F_x also equals $m\ddot{x}$; therefore,

$$m\ddot{x} = (\rho \Delta x \Delta y \Delta z)\ddot{x} = - \frac{\partial P}{\partial x} \Delta x \Delta y \Delta z,$$

or

$$\frac{\partial^2 x}{\partial t^2} = - \frac{1}{\rho} \frac{\partial P}{\partial x}. \quad (\text{I.1})$$

Likewise in the y direction,

$$\frac{\partial^2 y}{\partial t^2} = - \frac{1}{\rho} \frac{\partial P}{\partial y}. \quad (\text{I.2})$$

Now $x = x(a,b,t)$ and $y = y(a,b,t)$; hence we can change variables so that

$$a = a(x,y,t) \text{ and } b = b(x,y,t),$$

provided that the Jacobian of the transformation

$$J = \frac{\partial(x,y)}{\partial(a,b)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} \end{vmatrix}$$

is not equal to zero. Expanding the determinant we get

$$J = \frac{\partial x}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial x}{\partial b} \frac{\partial y}{\partial a}.$$



Now let us do the following manipulation; since $x = x[a(x,y), b(x,y)]$ and $y = y[a(x,y), b(x,y)]$,

$$1 = \frac{\partial x}{\partial x} = \frac{\partial x}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial x}{\partial b} \frac{\partial b}{\partial x} ,$$

$$0 = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial y}{\partial b} \frac{\partial b}{\partial x} .$$

Solving these two equations for $\partial a/\partial x$ and $\partial b/\partial x$, we have

$$\frac{\partial a}{\partial x} = \frac{1}{J} \frac{\partial y}{\partial b} \text{ and } \frac{\partial b}{\partial x} = - \frac{1}{J} \frac{\partial y}{\partial a} .$$

By the same method

$$\frac{\partial a}{\partial y} = - \frac{1}{J} \frac{\partial x}{\partial b} \text{ and } \frac{\partial b}{\partial y} = \frac{1}{J} \frac{\partial x}{\partial a} .$$

Now returning to Equation I.1 we obtain the equations of motion entirely in terms of the Lagrangian independent variables as follows:

$$\frac{\partial^2 x}{\partial t^2} = - \frac{1}{\rho} \frac{\partial P}{\partial x} = - \frac{1}{\rho} \left(\frac{\partial P}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial P}{\partial b} \frac{\partial b}{\partial x} \right),$$

but

$$\frac{\partial a}{\partial x} = \frac{1}{J} \frac{\partial y}{\partial b} \text{ and } \frac{\partial b}{\partial x} = - \frac{1}{J} \frac{\partial y}{\partial a} ,$$



therefore

$$\frac{\partial^2 x}{\partial t^2} = - \frac{1}{\rho J} \left(\frac{\partial P}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial P}{\partial b} \frac{\partial y}{\partial a} \right) . \quad (I.3)$$

Likewise, for Equation I.2 we have,

$$\frac{\partial^2 y}{\partial t^2} = - \frac{1}{\rho J} \left(\frac{\partial P}{\partial b} \frac{\partial x}{\partial a} - \frac{\partial P}{\partial a} \frac{\partial x}{\partial b} \right) . \quad (I.4)$$

Now let us derive the equation of continuity.

The following equation is a relationship between the Lagrangian and Eulerian differential volume elements.

$$\Delta z \Delta x \Delta y = \Delta z J \left(\frac{xy}{ab} \right) \Delta a \Delta b .$$

The mass of the rectangle in Figure 2 is given by $\rho \Delta x \Delta y \Delta z$ and so we can write,

$$\rho \Delta z \Delta x \Delta y = \rho \Delta z J \Delta a \Delta b .$$

Since the mass in a Lagrangian cell is always constant,

$$(\rho \Delta z J \Delta a \Delta b)_{t=0} = (\rho \Delta z J \Delta a \Delta b)_{t=t' > 0} ;$$

but $J = 1$ and $\rho = \rho_0$ at $t = 0$, and therefore



$$\rho_0 \Delta z \Delta a \Delta b = \rho J \Delta z \Delta a \Delta b ,$$

or

$$\rho_0 = \rho J , \quad (I.5)$$

the equation of continuity.

Substituting Equation I.5 into Equations I.3 and I.4, we have

$$\frac{\partial^2 x}{\partial t^2} = - \frac{1}{\rho_0} \left(\frac{\partial P}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial P}{\partial b} \frac{\partial y}{\partial a} \right) \quad (I.6)$$

and

$$\frac{\partial^2 y}{\partial t^2} = - \frac{1}{\rho_0} \left(\frac{\partial x}{\partial a} \frac{\partial P}{\partial b} - \frac{\partial x}{\partial b} \frac{\partial P}{\partial a} \right). \quad (I.7)$$

At constant entropy, $\partial P/\partial a = (\partial P/\partial \rho)(\partial \rho/\partial a)$ and $\partial P/\partial b = (\partial P/\partial \rho)(\partial \rho/\partial b)$, and furthermore it has been shown² that for isentropic flow the sound speed, C , satisfies the following equation,

$$C^2 = \left. \frac{\partial P}{\partial \rho} \right|_s .$$

(Note: The sound speed is the speed of a signal with respect to Eulerian coordinates.)

We can now write $\partial P/\partial a = C^2(\partial \rho/\partial a)$ and $\partial P/\partial b = C^2(\partial \rho/\partial b)$ and substituting these results in Equations I.6 and I.7 we have, after dividing through by C_0^2 and noting that $(1/C_0^2)(\partial^2 x/\partial t^2)$ can be written $\partial^2 x/\partial(C_0 t)^2$,



$$\frac{\partial^2 x}{\partial (c_0 t)^2} = -\frac{c^2}{\rho_0 c_0^2} \left(\frac{\partial \rho}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial \rho}{\partial b} \frac{\partial y}{\partial a} \right),$$

and

$$\frac{\partial^2 y}{\partial (c_0 t)^2} = -\frac{c^2}{\rho_0 c_0^2} \left(\frac{\partial x}{\partial a} \frac{\partial \rho}{\partial b} - \frac{\partial x}{\partial b} \frac{\partial \rho}{\partial a} \right).$$

When the following definitions are made,

$$\frac{\partial x}{\partial (c_0 t)} = u, \quad \frac{\partial x}{\partial a} = r, \quad \frac{\partial x}{\partial b} = s, \quad \frac{\partial y}{\partial (c_0 t)} = v, \quad \frac{\partial y}{\partial a} = w, \quad \frac{\partial y}{\partial b} = z, \quad (I.8)$$

our equations become

$$\frac{\partial u}{\partial (c_0 t)} = \frac{c^2}{\rho_0 c_0^2} \left(-z \frac{\partial \rho}{\partial a} + w \frac{\partial \rho}{\partial b} \right), \quad (I.9)$$

and

$$\frac{\partial v}{\partial (c_0 t)} = \frac{c^2}{\rho_0 c_0^2} \left(s \frac{\partial \rho}{\partial a} - r \frac{\partial \rho}{\partial b} \right). \quad (I.10)$$



II. METHOD OF CHARACTERISTICS^{3,4}

In this section we will discuss the method of characteristics as applied to the determination of characteristic curves and surfaces for a family of quasi-linear differential equations. The domain of dependence and region of influence as related to these curves and surfaces will also be defined.

If we are given the following,

$$A \frac{\partial f}{\partial x} + B \frac{\partial f}{\partial y} + C = 0,$$

where the function f and the coefficients A , B , and C are all functions of, and defined for, all (x,y) .

This equation defines a derivative of f at any point (x_0, y_0) in the direction defined by the equation

$$\frac{dx}{dy} = \frac{B(x_0, y_0)}{A(x_0, y_0)} .$$

Again, if one is given the pair of equations



$$\left. \begin{aligned}
 (1) \quad A_1 \frac{\partial u}{\partial x} + B_1 \frac{\partial u}{\partial y} + C_1 \frac{\partial v}{\partial x} + D_1 \frac{\partial v}{\partial y} + E_1 &= 0, \\
 (2) \quad A_2 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y} + C_2 \frac{\partial v}{\partial x} + D_2 \frac{\partial v}{\partial y} + E_2 &= 0,
 \end{aligned} \right\} \begin{array}{l} \text{the coefficients} \\ \text{can all be} \\ \text{functions of } x, y, u, v. \end{array}$$

then one can attempt to find multipliers λ_1 and λ_2 (for 1 and 2, respectively) such that u and v are both differentiated in the same direction at any point (x_0, y_0) .

If there are two such directions the equations are called hyperbolic, if there is only one such direction the equations are called parabolic, and if no such direction exists then the equations are called elliptic.

The equations for these directions define a set of curves in the x, y space called characteristic curves, i.e.,

$$\frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2} = \frac{\lambda_1 C_1 + \lambda_2 C_2}{\lambda_1 D_1 + \lambda_2 D_2} = \frac{dx}{dy}.$$

This method can be continued for k equations in k functions with n independent variables.

In particular we wish to develop the theory for nine equations, nine functions, and three independent variables (a, b, t) or symbolically,



$$L_{\mu}(u) = \sum_{\nu, \kappa} A_{\mu\kappa}^{\nu} u_{x_{\nu}}^{\kappa} + f_{\mu} = 0; \quad \mu = 1, 2, \dots, 9, \\ \kappa = 1, 2, \dots, 9, \quad (\text{II.1}) \\ \nu = 1, 2, 3,$$

with

$$A_{\mu\kappa}^{\nu} = A_{\mu\kappa}^{\nu}(u^{\kappa}, x_{\nu}); \quad f_{\mu} = f_{\mu}(u^{\kappa}, x_{\nu}).$$

Through any point (a_1, b_1, t_1) we place a plane characterized by a normal vector $\vec{\xi}$ whose components are (ξ_a, ξ_b, ξ_t) .

We call such planes "exceptional" if a linear combination $L = \lambda_{\mu} L_{\mu}$ of the differential equations L_{μ} can be found such that L involves derivatives of the u^{κ} only in directions lying in these planes. The direction of the derivative of u^{κ} is parallel to the vector $\vec{\phi}$ whose components are $\lambda_{\mu} A_{\mu\kappa}^{\nu}$.

The condition for an exceptional plane can be written as

$$\vec{\phi} \cdot \vec{\xi} = \lambda_{\mu} A_{\mu\kappa}^{\nu} \xi^{\nu} = 0; \quad \mu = 1, 2, \dots, 9, \\ \nu = 1, 2, 3.$$

We have nine homogeneous equations in nine unknowns λ_{μ} , and the condition for the existence of nontrivial multipliers λ_{μ} is that the determinant

$$\left| A_{\mu\kappa}^{\nu} \xi^{\nu} \right| = 0. \quad (\text{II.2})$$



This equation will give us a relationship between the components of the normal vector \vec{E} , and these vectors will form a "cone"⁵ with vertex at (a_1, b_1, t_1) .

If we are given a surface where (a_1, b_1, t_1) is any arbitrary point on this surface, then we call the surface "characteristic" if

(1) The surface is normal at (a_1, b_1, t_1) to a vector determined by Equation II.2 originating at this point, and

(2) This is true for all points on the surface without exception.

If we are using finite difference schemes, the characteristic surface at (a_1, b_1, c_1) is approximated by the cone which is the envelope of the exceptional planes at (a_1, b_1, c_1) . Such a cone will be called a characteristic cone.

Domain of Dependence⁶

In two dimensions let us assume we have a characteristic network C^+ and C^- for a set of two hyperbolic equations and a curve S that is nowhere a characteristic line.

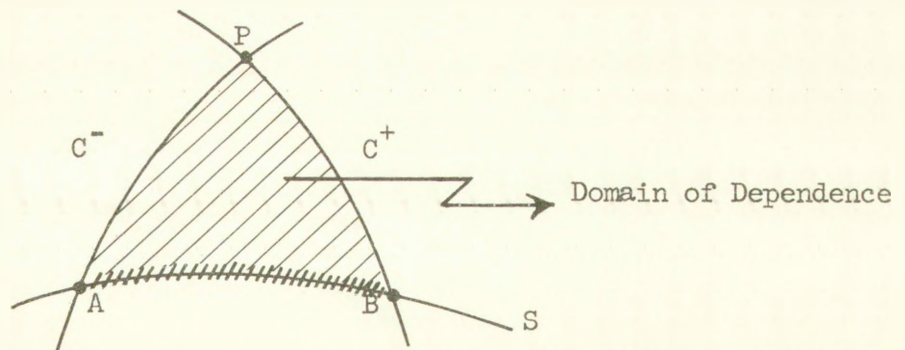


Figure 3.



Let C^- and C^+ be the characteristic equations through point P . These characteristic curves intersect S at points A and B , respectively. It can be shown by the method of iterations that the solutions at P of our equations are dependent only upon the solutions given on the line segment AB of S .

This proof⁷ can be extended to three dimensions where a characteristic cone intersects a plane.

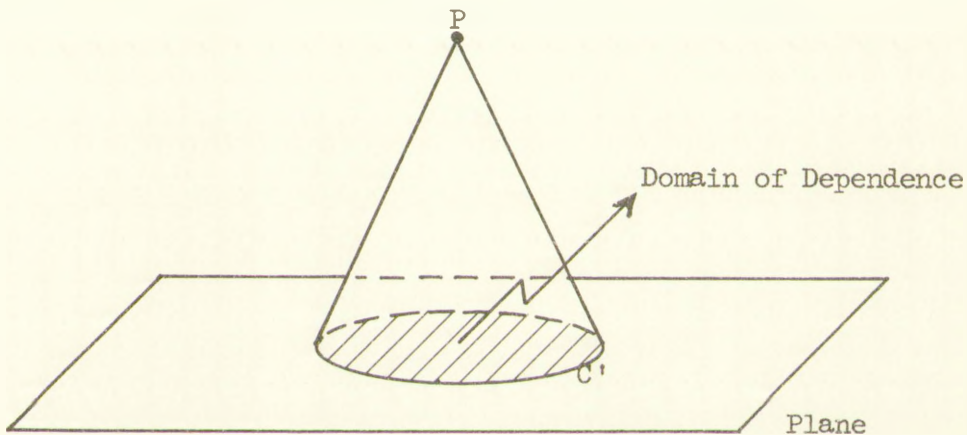


Figure 4.

The domain of dependence of the point P is the area of the plane P enclosed by the curve C' .

This means that only the values of our unknown functions given on C' and inside of C' contribute to the solution at point P .



Range of Influence⁸

Again in two dimensions let us assume we have a characteristic net C^+ and C^- and a point P where the lines C^+ and C^- intersect.

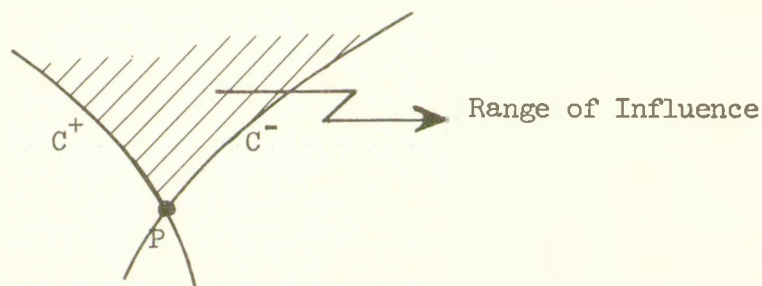


Figure 5.

The area lying between C^+ and C^- is called the range of influence of the point P . This means that the values of the solutions of our equations in this area are dependent upon the solutions at point P .

Again expanding into three dimensions, the characteristic cone emanating from the point P encloses a volume that is the range of influence of the point P .



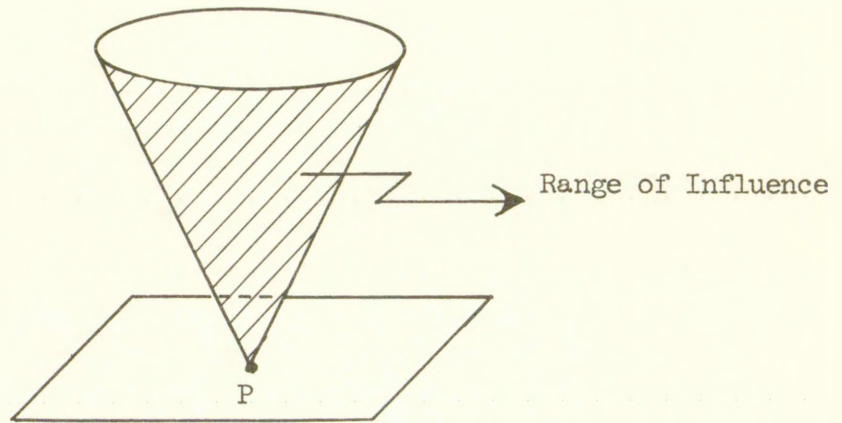


Figure 6.

The concepts of the domain of dependence will be used in the section on applications (i.e., Section IV).



III. APPLICATION OF THE METHOD OF CHARACTERISTICS
TO THE LAGRANGIAN EQUATIONS

We are now ready to apply the method of characteristics to the Lagrangian equations to obtain an equation relating the components of the vectors $\vec{\xi}$ which are normal to the exceptional planes at any point P, and then, from geometric considerations, show how to obtain points on the characteristic cone from this equation.

From Section I, Equations I.5, I.8, I.9, and I.10, we have the following:

$$\frac{\partial u}{\partial(C_o t)} + \left(\frac{c^2 z}{c^2_{o\rho_o}}\right) \frac{\partial \rho}{\partial a} + \left(\frac{-c^2 w}{c^2_{o\rho_o}}\right) \frac{\partial \rho}{\partial b} = 0 , \quad (\text{III.1})$$

$$\frac{\partial v}{\partial(C_o t)} + \left(\frac{-c^2 s}{c^2_{o\rho_o}}\right) \frac{\partial \rho}{\partial a} + \left(\frac{c^2 r}{c^2_{o\rho_o}}\right) \frac{\partial \rho}{\partial b} = 0 , \quad (\text{III.2})$$

$$\frac{\partial x}{\partial(C_o t)} - u = 0 , \quad (\text{III.3})$$

$$\frac{\partial y}{\partial(C_o t)} - v = 0 . \quad (\text{III.4})$$



The equations $(\partial x/\partial a) - r = 0$ and $(\partial x/\partial b) - s = 0$ in Equation I.8 are not independent of Equation III.3, and in order to prevent the occurrence of a trivial characteristic determinant we either must take the partial derivative of both of the equations with respect to time, or one of them with respect to time and the other with respect to a space variable such that you have a cross derivative.

Example

$$\frac{\partial x}{\partial a} - r = 0 \rightarrow \frac{\partial^2 x}{\partial a \partial b} - \frac{\partial r}{\partial b} = 0 .$$

If we assume x is continuous in its second-order partial derivatives, then we can interchange the order of differentiations, and we have

$$\frac{\partial s}{\partial a} - \frac{\partial r}{\partial b} = 0 , \tag{III.5}$$

and also

$$\frac{\partial x}{\partial b} - s = 0 \rightarrow \frac{\partial^2 x}{\partial b \partial (c_0 t)} - \frac{\partial s}{\partial (c_0 t)} = 0 \rightarrow \frac{\partial u}{\partial b} - \frac{\partial s}{\partial (c_0 t)} = 0 .$$

$$\frac{\partial u}{\partial b} - \frac{\partial s}{\partial (c_0 t)} = 0 . \tag{III.6}$$

Similarly, $(\partial y/\partial a) - w = 0$ and $(\partial y/\partial b) - z = 0$ in Equation I.8 are not independent of Equation III.4, and so we must use the above procedure for these equations.



Example

$$\frac{\partial y}{\partial a} - w = 0 \rightarrow \frac{\partial^2 y}{\partial a \partial b} - \frac{\partial w}{\partial b} = 0 .$$

If we assume y has continuous second order partial derivatives, then we can interchange the order of differentiations, and we have

$$\frac{\partial z}{\partial a} - \frac{\partial w}{\partial b} = 0 , \quad (\text{III.7})$$

and also

$$\frac{\partial y}{\partial b} - z = 0 \rightarrow \frac{\partial^2 y}{\partial b \partial (c_0 t)} - \frac{\partial z}{\partial (c_0 t)} = 0 \rightarrow \frac{\partial v}{\partial b} - \frac{\partial z}{\partial (c_0 t)} = 0 .$$

$$\frac{\partial v}{\partial b} - \frac{\partial z}{\partial (c_0 t)} = 0 . \quad (\text{III.8})$$

Now our continuity equation, $\rho_0 = \rho J = \rho(rz - sw)$, contains no derivatives of the new variables and, rather than solving for ρ and inserting this in our other equations, it is convenient to take the partial derivative of this equation with respect to any one of our independent variables.

Example

$$\rho(rz - sw) = \rho_0 \rightarrow (rz - sw) \frac{\partial \rho}{\partial a} + \rho r \frac{\partial z}{\partial a} + \rho z \frac{\partial r}{\partial a} - \rho s \frac{\partial w}{\partial a} - \rho w \frac{\partial s}{\partial a} = 0 .$$

$$(rz - sw) \frac{\partial \rho}{\partial a} + \rho r \frac{\partial z}{\partial a} + \rho z \frac{\partial r}{\partial a} - \rho s \frac{\partial w}{\partial a} - \rho w \frac{\partial s}{\partial a} = 0 . \quad (\text{III.9})$$



If we now insert the coefficients of Equations III.1 through III.9 into Equation II.2, noting that ξ_y has become $\xi_{C_0 t}$, we get the following determinant equation:

| | x | y | u | v | r | s | w | z | p |
|-------|---------------|---------------|---------------|---------------|------------------|-------------------|-------------------|------------------|--|
| III.1 | 0 | 0 | $\xi_{C_0 t}$ | 0 | 0 | 0 | 0 | 0 | $\frac{C^2}{C_0^2 \rho_0} (z\xi_a - w\xi_b)$ |
| III.2 | 0 | 0 | 0 | $\xi_{C_0 t}$ | 0 | 0 | 0 | 0 | $\frac{C^2}{C_0^2 \rho_0} (r\xi_b - s\xi_a)$ |
| III.3 | $\xi_{C_0 t}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| III.4 | 0 | $\xi_{C_0 t}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| III.5 | 0 | 0 | 0 | 0 | $-\xi_b$ | ξ_a | 0 | 0 | 0 |
| III.6 | 0 | 0 | ξ_b | 0 | 0 | $-\xi_{C_0 t}$ | 0 | 0 | 0 |
| III.7 | 0 | 0 | 0 | 0 | 0 | 0 | $-\xi_b$ | ξ_a | 0 |
| III.8 | 0 | 0 | 0 | ξ_b | 0 | 0 | 0 | $-\xi_{C_0 t}$ | 0 |
| III.9 | 0 | 0 | 0 | 0 | $(\rho z \xi_a)$ | $(-\rho w \xi_a)$ | $(-\rho s \xi_a)$ | $(\rho r \xi_a)$ | $(rz - sw)\xi_a$ |

= 0 ,

which is a condition on the components of the vector $\vec{\xi}$. Expanding this determinant equation and simplifying, we get

$$\xi_{C_0 t}^4 \xi_b^2 \xi_a \left\{ \left[\left(\frac{\partial x}{\partial b} \right)^2 + \left(\frac{\partial y}{\partial b} \right)^2 \right] \xi_a^2 + \left[\left(\frac{\partial x}{\partial a} \right)^2 + \left(\frac{\partial y}{\partial a} \right)^2 \right] \xi_b^2 - 2 \left(\frac{\partial x}{\partial a} \frac{\partial x}{\partial b} + \frac{\partial y}{\partial a} \frac{\partial y}{\partial b} \right) \xi_a \xi_b - \left(\frac{C_0^2 \rho_0}{C \rho} \right)^2 \xi_{C_0 t}^2 \right\} = 0 .$$

Now let us examine the first term (i.e., $\xi_{C_0 t}^4 \xi_b^2 \xi_a$). If this is equal to zero, we get an uninteresting solution (this will either



be a one-dimensional case or a nonphysical case); hence, the second term must equal zero. These extraneous solutions arise from the introduction of new variables that we did earlier, and if we try all the possible cases other than the example given, we will again get an uninteresting first term (e.g., $\xi_{C_0 t}^7$, $\xi_{C_0 t}^4 \xi_b \xi_a^2$, etc), but the second term will be the same. Therefore, the equation we want is

$$\left(\frac{C_0 \rho_0}{C \rho}\right)^2 \xi_{C_0 t}^2 = \left[\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2\right] \xi_a^2 + \left[\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2\right] \xi_b^2 - 2\left(\frac{\partial x}{\partial a} \frac{\partial x}{\partial b} + \frac{\partial y}{\partial a} \frac{\partial y}{\partial b}\right) \xi_a \xi_b, \quad (\text{III.10})$$

where $\xi_{C_0 t}$, ξ_a and ξ_b are components of a normal vector $\vec{\xi}$. (Note: The length of $\vec{\xi}$ has not been defined; hence, there are no conditions on the ξ_i components except Equation III.10.)

This equation defines our "cone" of normals $\vec{\xi}$. Let us make a change of variables such that the point we are on becomes the origin. For example, if we are on point $(a_0, b_0, C_0 t_0)$ and $(\bar{a}, \bar{b}, C_0 \bar{t})$ is any other point, then

$$\begin{aligned} t &= C_0 \bar{t} - C_0 t_0, \\ b &= \bar{b} - b_0, \\ a &= \bar{a} - a_0 \end{aligned} \quad (\text{III.11})$$



are our new variables. Note that the variables are all of the same dimension. Equation III.10 becomes

$$\left(\frac{c_{0\rho_0}}{c\rho}\right)^2 t^2 = \left[\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2\right]a^2 + \left[\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2\right]b^2 - 2\left[\frac{\partial x}{\partial a}\frac{\partial x}{\partial b} + \frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\right]ab, \quad (\text{III.12})$$

and for $t = \text{constant} \neq 0$ this equation is of the form

$$\alpha x^2 + \beta xy + \gamma y^2 = k^2.$$

If $\beta^2 - 4\alpha\gamma < 0$, this is the equation of an ellipse.

Testing Equation III.12 we have

$$\begin{aligned} \left(\frac{\partial x}{\partial a}\frac{\partial x}{\partial b} + \frac{\partial y}{\partial a}\frac{\partial y}{\partial b}\right)^2 - \left[\left(\frac{\partial x}{\partial b}\right)^2 + \left(\frac{\partial y}{\partial b}\right)^2\right]\left[\left(\frac{\partial x}{\partial a}\right)^2 + \left(\frac{\partial y}{\partial a}\right)^2\right] \\ = -\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}\right)^2; \end{aligned}$$

which is indeed less than zero and so the curve of intersection of the vectors \vec{E} with a plane ($t = \text{constant} \neq 0$) is an ellipse (call it C).

From the definition of a conical surface,⁹ we can now state that Equation III.12 is the equation of an elliptical conical surface or an elliptical cone.

For every vector $\vec{\xi}$ there is a vector $\vec{\phi}$ normal to $\vec{\xi}$ and lying in the plane formed by the t axis and the vector $\vec{\xi}$; the family of such vectors $\vec{\phi}$ form the characteristic cone. Now let us do the following: choose any t plane and call it the $t = k$ plane.

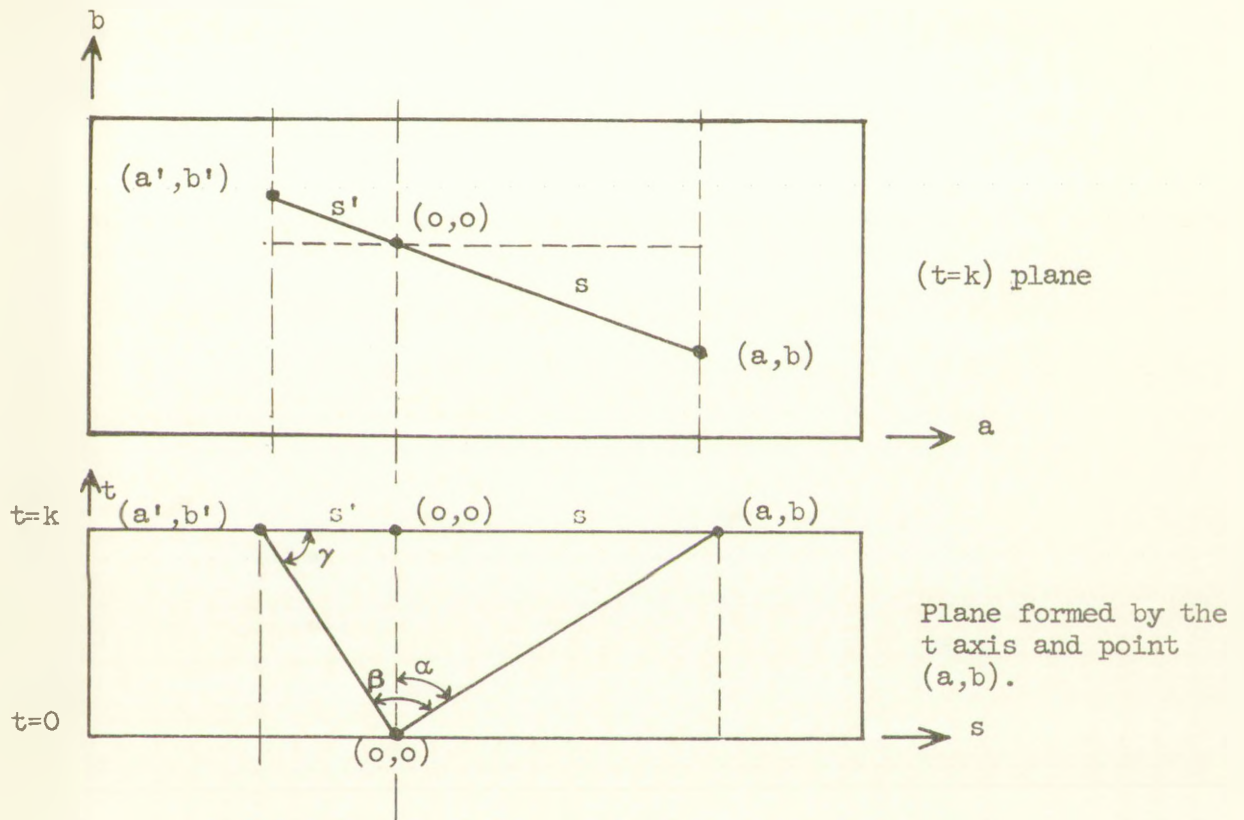


Figure 7.

$$\text{Given } \beta = 90^\circ; s = \sqrt{a^2 + b^2}; s' = \sqrt{a'^2 + b'^2}; \tan^2 \phi = \frac{b^2}{a^2} = \frac{b'^2}{a'^2};$$

then we have the following relations true,



$$\tan \gamma = \frac{k}{s'}; \tan \alpha = \frac{s}{k}; \text{ but } \alpha = \gamma \text{ if } \beta = 90^\circ;$$

therefore,

$$s' = \frac{k^2}{s} .$$

From Figure 7, we see that

$$\frac{a}{s} = \frac{-a'}{s'} \text{ but } s' = \frac{k^2}{s} ;$$

hence

$$a' = \frac{-k^2 a}{a^2 + b^2}$$

and likewise

$$b' = \frac{-k^2 b}{a^2 + b^2} . \tag{III.13}$$

Now if (a,b) is a point of the curve C , then we see that (a',b') is a point on the curve of intersection between $t = k$ and the characteristic cone (call this curve C'). For a visual example of these relations see Figure 8.

Now suppose we had a point whose coordinates are $(\bar{a}, \bar{b}, C_0 \bar{t})$ and we wish to find out if this point lies inside or outside of the characteristic cone. We would proceed as follows. (Note: This method will be used in Section IV.)

First, the point would be transformed using Equation III.11, the values of a, b, t would be inserted into Equation III.13 (where $k = t$) and a', b' would be obtained.



Now assuming the values of the coefficients in Equation III.12 are known, insert $a' = a$ into this equation and solve for b . We will obtain two roots ($b_1 \geq b_2$). If b_1 and b_2 are real and if $b_1 > b' > b_2$, then the point $(\bar{a}, \bar{b}, C_0 \bar{t})$ lies inside of the \vec{E} cone, and hence from geometric considerations (see Figures 7 and 8), it lies outside of our characteristic cone. If the above conditions are not satisfied, then the point lies inside of our characteristic cone.

It is also possible to calculate the radial distance that the point is inside or outside of the characteristic cone.



Graphical Representation of a Particular $\vec{\xi}$ Cone
and the Related Characteristic Cone

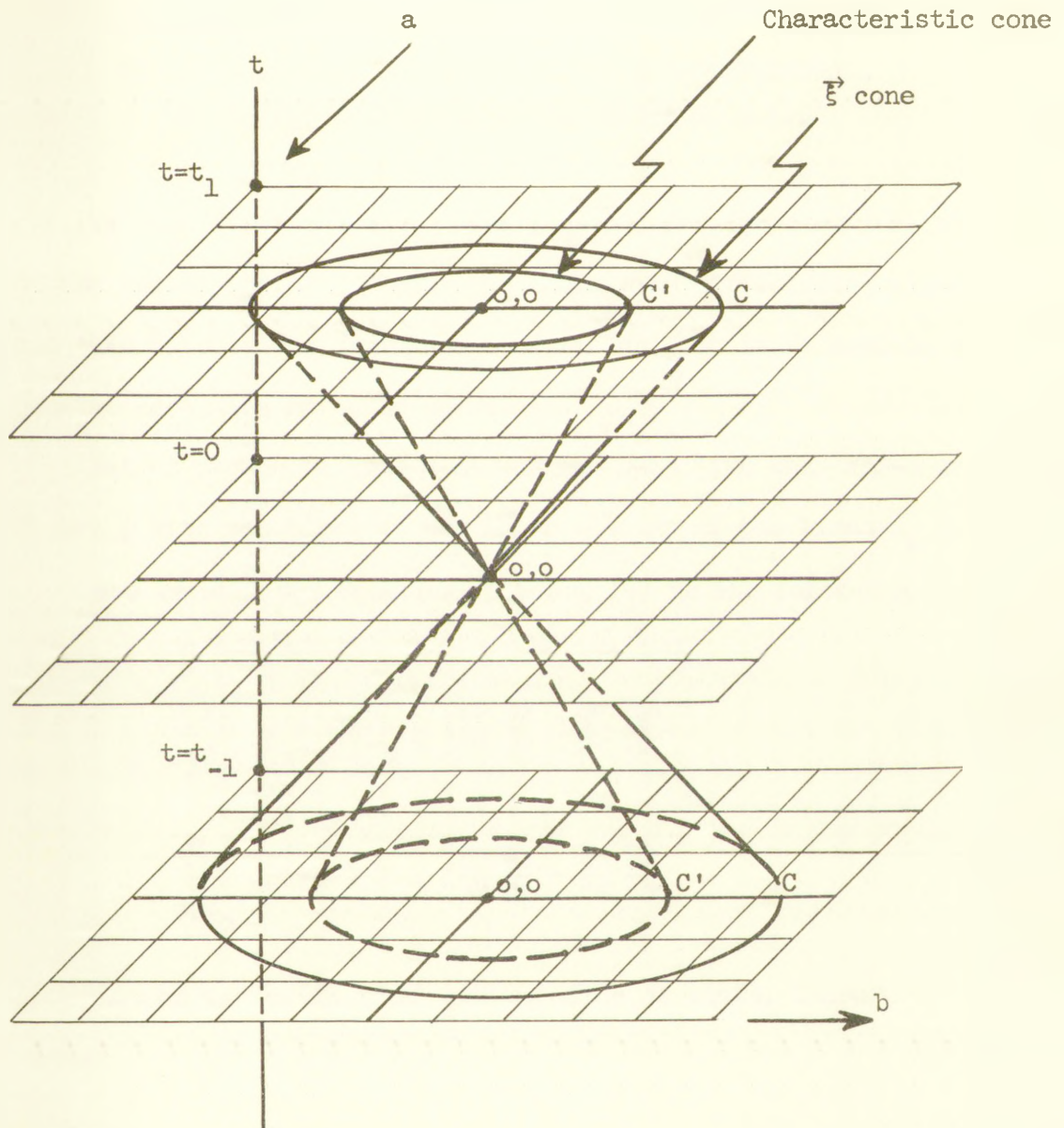


Figure 8.



IV. APPLICATION OF THE CHARACTERISTIC CONE AS A CONDITION
ON THE NUMERICAL SOLUTIONS OF THE TWO-DIMENSIONAL,
ISENTROPIC LAGRANGIAN DIFFERENTIAL EQUATIONS

We will first present a simple method for the numerical solution of the two-dimensional isentropic Lagrangian differential equations and then discuss how the characteristic cone could be used as a passive condition on these solutions.

Let us assume we have a difference mesh like that shown in Figure 9 with solutions at the intersections of the lines a_i and b_j .

Now we will use Equations I.6 and I.7 in the following form:

$$\frac{\partial^2 x}{\partial (c_o t)^2} = - \frac{1}{\rho_o c_o^2} \left(\frac{\partial P}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial P}{\partial b} \frac{\partial y}{\partial a} \right) , \quad (\text{IV.1})$$

$$\frac{\partial^2 y}{\partial (c_o t)^2} = - \frac{1}{\rho_o c_o^2} \left(\frac{\partial P}{\partial b} \frac{\partial x}{\partial a} - \frac{\partial P}{\partial a} \frac{\partial x}{\partial b} \right) . \quad (\text{IV.2})$$

Let us assume an ideal gas equation of state; therefore, we can write,

$$P v = R T = \epsilon (\gamma - 1) \quad (\text{IV.3})$$



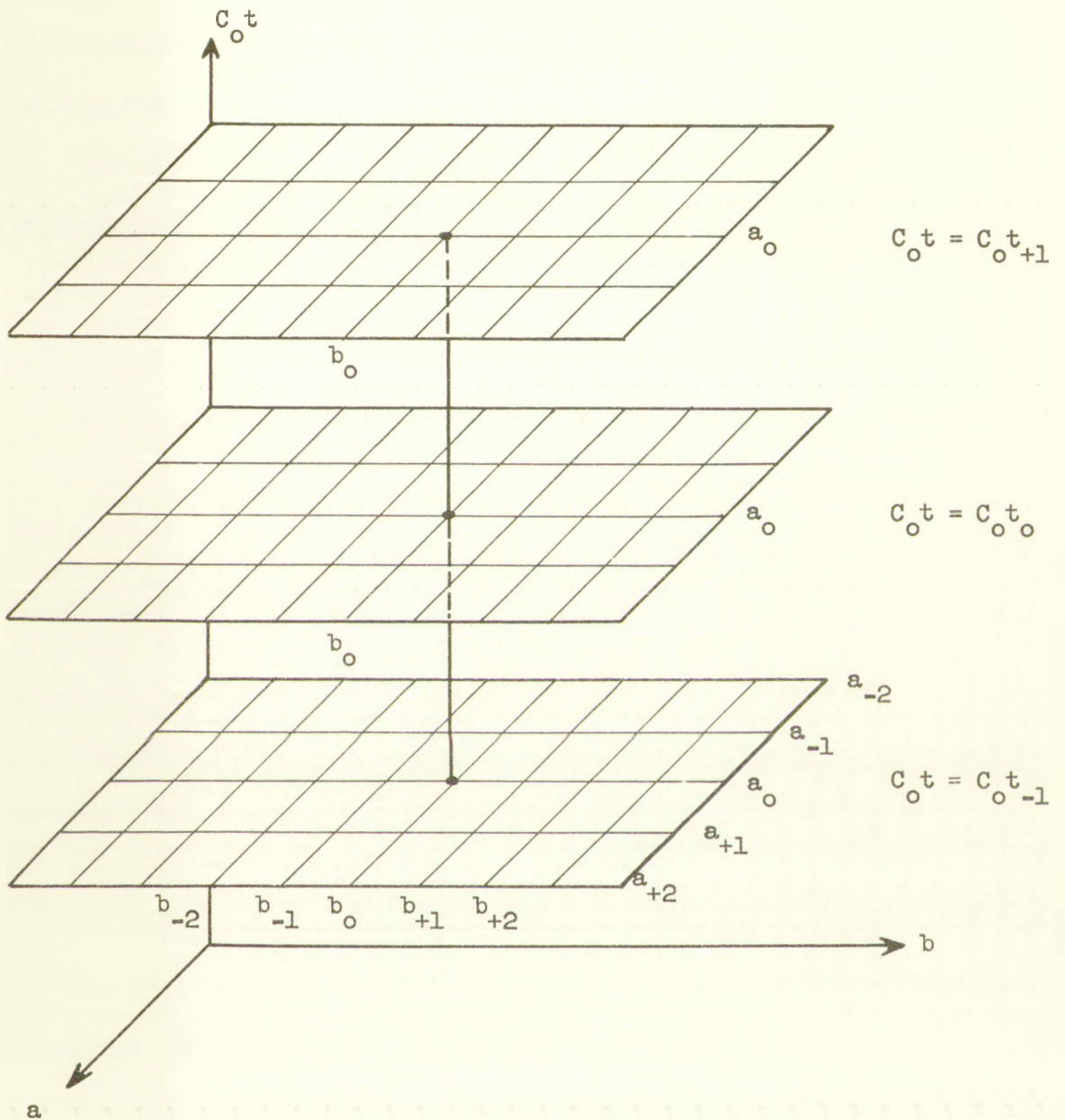


Figure 9.



at the point $(a_i, b_j, C_o t)$ where P , v , and ϵ are the pressure, specific volume and internal energy, respectively, and γ is a known constant of the material under study. Also assume ρ_o and C_o (the initial density and sound speed) are known.

The change in internal energy of an ideal gas at the point (a_i, b_j) under isentropic conditions is given by

$$\epsilon(C_o t_o) = \epsilon(C_o t_{-1}) - \left[\frac{P(C_o t_{-1}) + P(C_o t_o)}{2} \right] [v(C_o t_{-1}) - v(C_o t_o)], \quad (\text{IV.4})$$

where the last term is the $p dv$ work done on our cell from t_{-1} to t_o .

Now suppose we are at the point $(a_o, b_o, C_o t_o)$ and that we know $P(C_o t_{-1})$, $v(C_o t_{-1})$, $x(C_o t_o)$, $y(C_o t_o)$, and $\epsilon(C_o t_{-1})$ at all of our points (a_i, b_j) .

The volume of the element in Figure 1 was $\Delta x \Delta y \Delta z$, but in Section I it was shown that

$$V = \Delta z \Delta x \Delta y = \Delta z J \Delta a \Delta b.$$

Therefore, the specific volume of our Lagrangian cell is given by

$$v = V/V_o = J = \left(\frac{\partial x}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial x}{\partial b} \frac{\partial y}{\partial a} \right). \quad (\text{IV.5})$$



At C_{00} we have

$$\frac{\partial x}{\partial a}(a_0, b_0) = \frac{1}{2} \left[\frac{x(a_1, b_0) - x(a_0, b_0)}{a_1 - a_0} + \frac{x(a_0, b_0) - x(a_{-1}, b_0)}{a_0 - a_{-1}} \right], \quad (\text{IV.6})$$

$$\frac{\partial x}{\partial b}(a_0, b_0) = \frac{1}{2} \left[\frac{x(a_0, b_1) - x(a_0, b_0)}{b_1 - b_0} + \frac{x(a_0, b_0) - x(a_0, b_{-1})}{b_0 - b_{-1}} \right], \quad (\text{IV.7})$$

$$\frac{\partial y}{\partial a}(a_0, b_0) = \frac{1}{2} \left[\frac{y(a_1, b_0) - y(a_0, b_0)}{a_1 - a_0} + \frac{y(a_0, b_0) - y(a_{-1}, b_0)}{a_0 - a_{-1}} \right], \quad (\text{IV.8})$$

$$\frac{\partial y}{\partial b}(a_0, b_0) = \frac{1}{2} \left[\frac{y(a_0, b_1) - y(a_0, b_0)}{b_1 - b_0} + \frac{y(a_0, b_0) - y(a_0, b_{-1})}{b_0 - b_{-1}} \right]. \quad (\text{IV.9})$$

$$\Delta a = \frac{a_{+1} - a_{-1}}{2}; \quad \Delta b = \frac{b_{+1} - b_{-1}}{2}; \quad \Delta z = 1. \quad (\text{IV.10})$$

[Note: This will center P , v , and ϵ at the points (a_i, b_j) .]



Now insert Equations IV.6 to IV.10 into Equation IV.5 and we have determined the volume $v(a_o, b_o, C_o t_o)$.

Inserting this value in Equation IV.4 and eliminating $\epsilon(C_o t_o)$ between this equation and Equation IV.3, we have an equation that determines $P(a_o, b_o, C_o t_o)$.

Now at $C_o t_o$,

$$\frac{\partial P}{\partial a} = \frac{1}{2} \left[\frac{P(a_1, b_o) - P(a_o, b_o)}{a_1 - a_o} + \frac{P(a_o, b_o) - P(a_{-1}, b_o)}{a_o - a_{-1}} \right], \quad (IV.11)$$

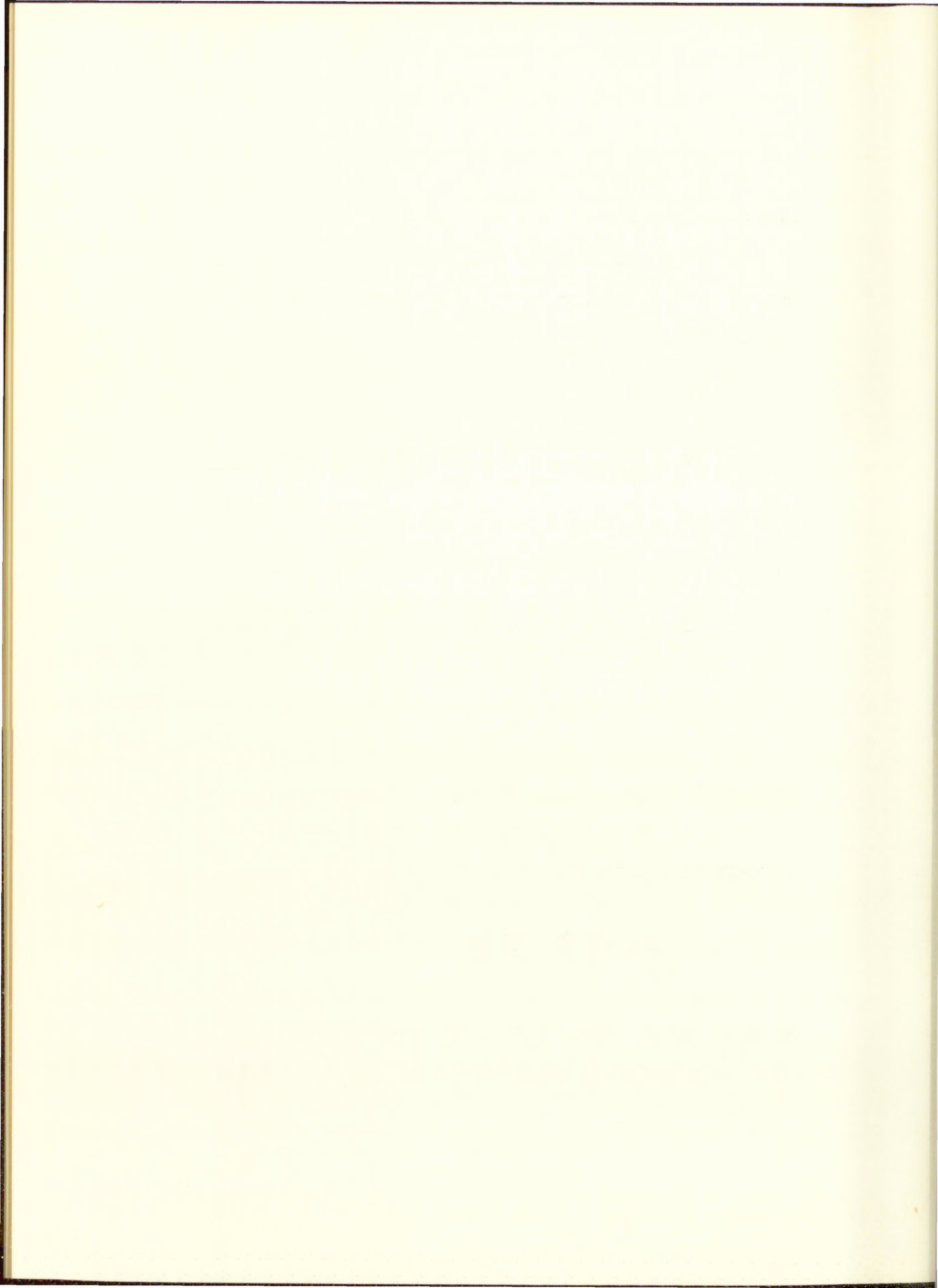
$$\frac{\partial P}{\partial b} = \frac{1}{2} \left[\frac{P(a_o, b_1) - P(a_o, b_o)}{b_1 - b_o} + \frac{P(a_o, b_o) - P(a_o, b_{-1})}{b_o - b_{-1}} \right]. \quad (IV.12)$$

Let $(\Delta C_o t)^+ = C_o t_1 - C_o t_o$ and $(\Delta C_o t)^- = C_o t_o - C_o t_{-1}$.

In Equation IV.1 the value of the right hand side of the equation is a constant we can determine, and let us call it η .

$$-\frac{1}{\rho_o C_o^2} \left(\frac{\partial P}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial P}{\partial b} \frac{\partial y}{\partial a} \right) = \eta$$

at $(a_o, b_o, C_o t_o)$. Equation IV.1 can then be written,



$$\left[\frac{\frac{x(a_0, b_0, C_0 t_1) - x(a_0, b_0, C_0 t_0)}{\Delta(C_0 t)^+} - \frac{x(a_0, b_0, C_0 t_0) - x(a_0, b_0, C_0 t_{-1})}{\Delta(C_0 t)^-}}{\frac{1}{2}[\Delta(C_0 t)^+ + \Delta(C_0 t)^-]} \right] = \eta$$

The only unknown in this equation is $x(a_0, b_0, C_0 t_1)$, and hence we can solve to get its value.

By a similar process we can obtain $y(a_0, b_0, C_0 t_1)$ from Equation IV.2. Also, we can obtain the density ρ at $\rho = \rho(a_0, b_0, C_0 t_1)$, because

$$\rho = \frac{\rho_0}{\frac{\partial x}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial x}{\partial b} \frac{\partial y}{\partial a}}$$

from Equation I.5 and we can evaluate the right hand term. We now have outlined a difference method for solving the Lagrangian differential equations.

It is appropriate to insert a brief discussion of the stability of difference methods at this point.

If we are given a partial differential equation in x and t , and we devise a difference method for solving the equation numerically, it is not true to say that, in general, the smaller Δx and Δt , the more accurate the solution. It has been shown¹⁰ that for particular relative values of Δx and Δt , errors in the solution result that grow rapidly with time. This is called instability. Conditions on the relative values of Δx and Δt have been devised such that, if



these conditions are met, the difference solution approaches the exact solution as Δx and Δt approach zero.

For our equations the conditions would be $\Delta x/\Delta t \geq c$; $\Delta y/\Delta t \geq c$.¹¹ These are sometimes called the Courant conditions.

Suppose we have advanced the unknown x , y , and ρ to the $C_0 t_1$ plane and we have done this for any point (a_0, b_0) using the information at the points (a_0, b_0) ; (a_1, b_0) ; (a_{-1}, b_0) ; (a_0, b_1) ; and (a_0, b_{-1}) at time $C_0 t_0$. From Section II, these points should lie in the domain of dependence of the point $(a_0, b_0, C_0 t_1)$, but the stability condition is such that it requires the points to lie outside of the domain of dependence. It is possible to calculate the distance from the boundary of the domain of dependence of the point $(a_0, b_0, C_0 t_1)$ to the points (a_0, b_1) ; (a_0, b_{-1}) ; (a_1, b_0) ; (a_{-1}, b_0) from the discussion in Section III. If a point lies outside of the domain or on the boundary of the domain, then the distance will be labeled with a plus sign; if it lies inside the domain (in which case the problem is unstable), the distance will be labeled with a minus sign. This information can be tabulated for each point and used as a criterion for evaluating the dependability of the solutions at each point. In general, the farther outside the points lie, the less dependable the solution.



V. CONCLUSIONS

In this paper we have been concerned with characteristic surfaces and their usefulness as a condition on ordinary difference schemes as applied to the Lagrangian equations of Section I. We have entirely neglected to try and put the differential equations into characteristic form to see if they simplify to any degree, and the reason for this is that the coefficients of the differential Equations III.1 through III.9 are functions of the solutions of the equations themselves, and it is not clear that it could be done in any reasonable fashion.

The application given in Section IV may be tried on existing computer codes that solve the Lagrangian differential equations by difference methods.



REFERENCES

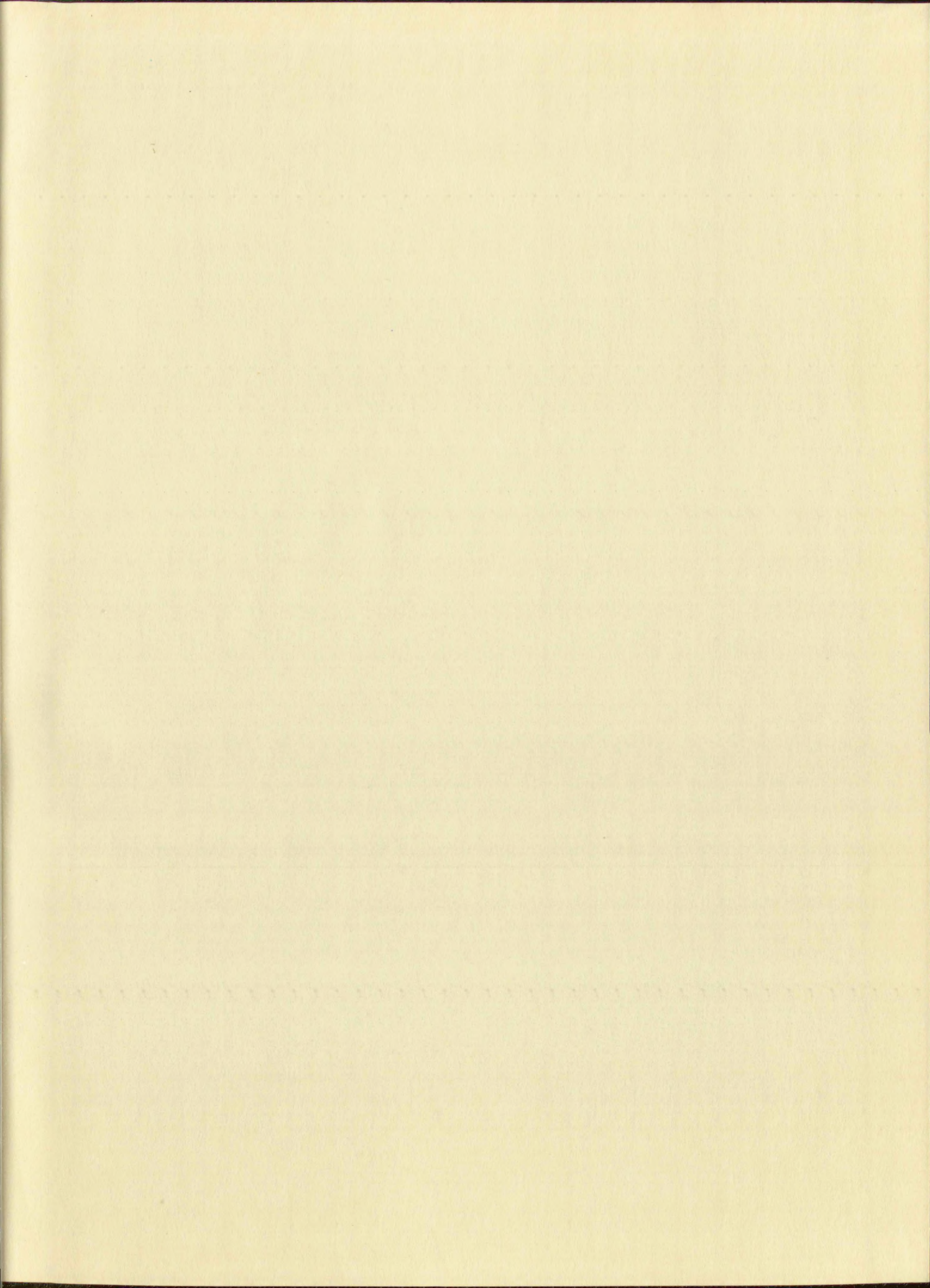
1. Sir Horace Lamb, Hydrodynamics (Dover Publications, New York, 1945), 6th ed., Chapter I.
2. Robert Lelevier, Lectures on Hydrodynamics and Shock Waves (University of California Radiation Laboratory Report UCRL-4333, Livermore, 1954).
3. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves (Interscience Publishers, Inc., New York, 1948), 1st ed., Vol. I, Section 32.
4. Richard von Mises, Mathematic Theory of Compressible Fluid Flow (Academic Press Inc., New York, 1958), pages 103-107.
5. Ibid., page 106.
6. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves (Interscience Publishers, Inc., New York, 1948), 1st ed., Vol. I, Section 24.
7. Ibid., page 49.
8. Ibid., Section 24.
9. G. James and R. C. James, Mathematics Dictionary (Van Nostrand Company, Inc., New York, 1959), page 72.
10. R. D. Richtmyer, Difference Methods for Initial Value Problems (Interscience Publishers, Inc., New York, 1957), pages 9-16.
11. Ibid., page 229.



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