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# A Method of Changing Certain Infinite Series To New But Equivalent Series

Moneta Gunilla Johnson

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A METHOD OF CHANGING CERTAIN INFINITE SERIES  
TO NEW BUT EQUIVALENT SERIES

By

Moneta Gunilla Johnson

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A Thesis Submitted for the Degree of  
Master of Arts in Mathematics

The University of New Mexico

1936

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#### ACKNOWLEDGEMENT

Acknowledgement is due Dr. Newsom for his suggestion of the subject of this thesis and for his encouragement during its preparation, to Marian Pierce for her co-operation and assistance in investigating certain phases of the subject, (to Professor C. A. Barnhart and to Dr. E. J. Workman for their critical reading of the manuscript.)

8/1/36 Miller 100



## INTRODUCTION

The purpose of this investigation is to prove that the sum of a series of variable terms of a certain type is equal to a constant plus the sum of another series of variable terms. As a result of this proof it may be hoped that certain series, which so far have not been summed, may be shown equal to a constant plus another series for which the sum is known.

The purpose of this investigation is to determine  
that the sum of the angles of a triangle is 180 degrees.  
This type is known as a direct proof. In a direct proof,  
another series of related facts are used to show that  
this proof is true. In this case, we will use the fact  
so far have not been proved. We will use the fact that  
constant with another series of related facts.

known.

11/15/15



## SECTION I

Thesis: If  $g(n)$  has certain particular properties, then  $g'(n)$  has them, also.

In this section we shall show that if the coefficient  $g(n)$  of the series  $\sum_{n=0}^{\infty} g(n) z^n$

has the properties assigned to it in the following theorem, the coefficient  $g'(n)$  of the series

$\sum_{n=0}^{\infty} g'(n) z^n$  has the same properties,  $g'(n)$  being the

first derivative of  $g(n)$ .

**Theorem:** Suppose that the coefficient  $g(n)$  occurring in the general term of the power series,

$$(1) \quad \sum_{n=0}^{\infty} g(n) z^n, \text{ radius of convergence} = \infty,$$

may be regarded as a function,  $g(w)$ , of the complex variable  $w=x+iy$  and as such satisfies the following two conditions:

(a) is single valued and analytic throughout the

Let  $f(x)$  be a function of  $x$  and let  $g(x)$  be a function of  $x$ .

Then  $f(x) + g(x)$  is a function of  $x$ .

In this section we shall show that if  $f(x)$  and  $g(x)$  are functions of  $x$  and if  $f(x)$  and  $g(x)$  are both continuous at  $x = a$ , then  $f(x) + g(x)$  is also continuous at  $x = a$ .

Let  $f(x)$  and  $g(x)$  be functions of  $x$  and let  $f(x)$  and  $g(x)$  be both continuous at  $x = a$ .

Then the properties of  $f(x)$  and  $g(x)$  at  $x = a$  are:

For  $f(x)$ , the condition (1) of the definition of continuity is satisfied.

$$\sum_{n=0}^{\infty} g(n) x^n$$

First derivative of  $f(x)$  and  $g(x)$  at  $x = a$  are:

Then:  $f'(x)$  and  $g'(x)$  are the first derivatives of  $f(x)$  and  $g(x)$  at  $x = a$ .

Using the general form of the Taylor series for  $f(x)$  and  $g(x)$  at  $x = a$ , we have:

$$(1) \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

may be regarded as a function  $F(x)$  of  $x$  and the series

variable  $x$  only and as such a function of  $x$ .

two conditions:

(a) is a single value and the series is convergent.

finite w-plane,

(b) is such that for all values of  $x$  and  $y$  one may write

$$\left| \frac{g(x+\frac{1}{2}+iy)}{g(x)} \right| < Ke^{\pm(k-1)\pi y},$$

where  $K$  is a constant independent of  $x$  and  $y$ , and  $k$  is any given positive integer  $\geq 2$ , and where the upper or lower of the signs  $\pm$  appearing therein is to be taken according as  $y$  is positive or negative.

Then the integral function  $f(z)$  defined by the series (1) when considered for all values of  $z$  satisfying the condition,

$$-\pi < \arg [(-1)^k z] < \pi,$$

may be expressed in the form,

$$f(z) = \int_{-L-\frac{1}{2}}^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$-\sum_{m=-L}^{-1} g(m)z^m - \zeta_k(L, z);$$

wherein  $L$  is an arbitrary positive integer and, whatever the value chosen for it, the expression  $\zeta_k(L, z)$  is such that

Finite groups

(b) In order to show that...

may write

$$\left| \frac{g(x) - f(x)}{g(x)} \right|$$

where  $N$  is a constant... is any given positive integer... or form of the group... taken according to... Then the later...

series (1) can be expressed... Using the condition...

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n x^n}{n!}$$

$$\sum_{n=1}^{\infty} a_n x^n$$

series (1) is an arbitrary... over the finite domain... is such that

$$\lim_{\text{mod } z \rightarrow 0} z^{L} \int_k^{\infty} (1, z) = 0. \quad 1$$

The coefficient  $g'(n)$  satisfies the condition (a) of this theorem for it is well known that "if  $f(z)$  is analytic throughout an open region  $T$ , then its derivatives of all orders exist at each point of  $T$ , and each of them is thus also analytic throughout  $T$ ."<sup>2</sup>

If  $g'(n)$  is to fulfill condition (b) of the theorem, it is necessary to show that

$$\left| \frac{g'(x + \frac{1}{2} + iy)}{g'(x)} \right| < K e^{\pm(k-1)\pi y},$$

wherein  $K$  and  $k$  have the conditions imposed by the theorem.

By definition

$$(2) \quad g'(x + \frac{1}{2} + iy) \\ = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{g(x + \frac{1}{2} + iy + \Delta x + i \Delta y) - g(x + \frac{1}{2} + iy)}{(x + \frac{1}{2} + iy + \Delta x + i \Delta y) - (x + \frac{1}{2} + iy)}$$

and

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<sup>1</sup>Newsom, C. V., On the Behavior of Entire Functions in Distant Portions of the Plane, p. 10-11.

<sup>2</sup>Curtiss, David Raymond, Analytic Functions of a Complex Variable, p. 98.

Line 1 (1.1) - 1.1  
Page 2

# MEMORANDUM

The attached report (1) was prepared by the

(a) of this organization. It is a copy of the

(b) is a copy of the report prepared by the

The derivatives of all orders of the

and each of them is a copy of the

If (a) is to be used in the

thereon, it is necessary to use the

$$\frac{1}{2} \left( \frac{1}{2} \right) = \frac{1}{4}$$

whereas 1 and 2 are the only two factors of 2.

thereon

by definition

$$(2) \quad 2(x) = 2x$$

$$\frac{1}{2} \left( \frac{1}{2} \right) = \frac{1}{4}$$

and

Thereon, the only factors of 2 are 1 and 2.  
It is clear that 1 and 2 are the only two factors of 2.  
Thereon, the only factors of 2 are 1 and 2.  
Complex variables, p. 22

$$(3) \quad g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}.$$

The absolute value of the ratio of the limits given in (2) and (3) is

$$(4) \quad \left| \frac{g'(x+\frac{1}{2}+iy)}{g'(x)} \right| \\ = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left| \frac{\Delta x}{\Delta x + i \Delta y} \right| \cdot \left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y) - g(x+\frac{1}{2}+iy)}{g(x+\Delta x) - g(x)} \right|.$$

If in the fraction,

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y) - g(x+\frac{1}{2}+iy)}{g(x+\Delta x) - g(x)} \right|,$$

the function  $g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)$  is replaced by C, the function  $g(x+\frac{1}{2}+iy)$  by D, the function  $g(x+\Delta x)$  by A, and the function  $g(x)$  by B, it follows that

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y) - g(x+\frac{1}{2}+iy)}{g(x+\Delta x) - g(x)} \right| = \left| \frac{C-D}{A-B} \right| = \left| \frac{C}{A-B} - \frac{D}{A-B} \right|.$$

By a simple algebraic manipulation it may be observed that

$$\left| \frac{C}{A-B} - \frac{D}{A-B} \right| = \left| \frac{C}{A} + \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} - \frac{D}{A} \cdot \frac{B}{A-B} \right|.$$

Since the absolute value of the difference of two quantities is less than, or equal to, the sum of the

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$$(3) \quad \frac{d^2x}{dt^2} = -\frac{g}{L}x$$

The general solution of (3) is

$$(4) \quad x(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$\omega = \sqrt{\frac{g}{L}}$$

If in the fraction

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

the function  $f(t) = \cos(\omega t)$  and  $g(t) = \sin(\omega t)$  are solutions, then the function  $f(t) = \sin(\omega t)$  and  $g(t) = \cos(\omega t)$  are also solutions.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

By a simple change of phase, we can write the general solution as

$$x(t) = C \cos(\omega t - \phi)$$

Since the total energy is constant, we have



absolute value of the first quantity and the absolute value of the second, the expression

$$\left| \frac{G}{A} + \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} - \frac{D}{A} \cdot \frac{B}{A-B} \right|$$

will be less than, or equal to,

$$\left| \frac{C}{A} \right| + \left| \frac{D}{A} \right| + \left| \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} \cdot \frac{B}{A-B} \right|.$$

Now let us consider the first two parts of this sum separately.

If one recalls the values designated by C and A, it is seen that

$$\left| \frac{C}{A} \right| \text{ becomes } \left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \right|.$$

$$\text{But } \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \right|$$

$$= \left| \frac{g(x+\frac{1}{2}+iy)}{g(x)} \right| \text{ which is postulated less than } Ke^{\pm(k-1)\pi y}$$

in condition (b) of the theorem.

Likewise if the values designated by D and A are employed,

$$\left| \frac{D}{A} \right| \text{ becomes } \left| \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \right|.$$

absolute value of the first term is less than the

value of the second term.

$$\frac{1}{2} < \frac{1}{3} < \frac{1}{4} < \dots$$

will be less than the value of the

$$\frac{1}{2} < \frac{1}{3} < \frac{1}{4} < \dots$$

How far we travel in the direction of a point of the

axis depends on the value of the

It is clear that

it is seen that

$$\frac{1}{2} < \frac{1}{3} < \frac{1}{4} < \dots$$

But in the case of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  the terms are

$$\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \dots$$

is condition (b) of the theorem.

Therefore the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges.

explicitly,

$$\frac{1}{1^2} > \frac{1}{2^2} > \frac{1}{3^2} > \dots$$

But  $\lim_{\Delta x \rightarrow 0} \left| \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \right| = \left| \frac{g(x+\frac{1}{2}+iy)}{g(x)} \right|$  which again is

less than  $Ke^{\pm(k-1)\pi y}$ .

Thus it has been shown that

$$(5) \quad \left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy)}{g(x+\Delta x)-g(x)} \right| \\ < 2Ke^{\pm(k-1)\pi y} + \left| \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} \cdot \frac{B}{A-B} \right|.$$

If the original values for C, D and A are substituted in the fractions  $\frac{C}{A}$  and  $\frac{D}{A}$ , then

$$(6) \quad \left| \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} \cdot \frac{B}{A-B} \right|$$

becomes equal to

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] - \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] \right|.$$

Since the function,  $f(z)$ , is analytic over any region of our interest as stated in the theorem, it is also continuous.<sup>1</sup>

By the definition of continuity,

$\left| g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy) \right| < \epsilon$ , where  $\epsilon$  is any arbitrarily small positive quantity if  $|\Delta x|$  and  $|\Delta y|$  are taken sufficiently small.

---

<sup>1</sup>Townsend, E. J., Functions of a Complex Variable, p. 79.

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In any inequality involving positive members, a fraction numerically less than unity may be so chosen that when the larger of the two quantities is multiplied by that fraction, the result is equivalent to the small quantity. So there exists a  $\theta$ ,  $|\theta| < 1$ , such that the following statement is true,

$$(7) \quad g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy)=\theta\epsilon .$$

It is to be observed that

$$g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy)+\theta\epsilon ,$$

and

$$\frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} = \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} + \frac{\theta\epsilon}{g(x+\Delta x)}$$

if  $g(x+\Delta x)$  is assumed to be not equal to zero for sufficiently small values of  $\Delta x$ .

If the value for  $\frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)}$  that has

just been found is substituted for it in statement (6),

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] + \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] \right| \text{ becomes}$$

$$\left| \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] + \frac{\theta\epsilon}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] - \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] \right| \text{ which}$$

$$\text{is equal to } \left| \frac{\theta\epsilon}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] \right| .$$

In an irregularly shaped area, the fraction of the total area that is occupied by the larger of the two shapes is determined by the ratio of the area of the larger shape to the total area. The smaller the ratio, the smaller the fraction of the total area that is occupied by the larger shape.

$$(7) \quad \frac{A(x)}{A(x) + B(x)}$$

COMMITTEE

MEMORANDUM

It is noted that the ratio of the area of the larger shape to the total area is a function of the ratio of the side lengths of the two shapes. If the ratio of the side lengths is  $k$ , then the ratio of the areas is  $k^2$ .

Just as the ratio of the areas is a function of the ratio of the side lengths, so the ratio of the perimeters is a function of the ratio of the side lengths.

$$\frac{A(x)}{A(x) + B(x)} = \frac{k^2}{k^2 + 1}$$
$$\frac{P(x)}{P(x) + Q(x)} = \frac{k}{k + 1}$$

As usual, the ratio of the perimeters is a function of the ratio of the side lengths.

Now if a final substitution is made for A and B, it follows that

$$\left| \frac{\theta \epsilon}{g(x+\Delta x)} \left[ \frac{B}{A-B} \right] \right| = \left| \frac{\theta \epsilon}{g(x+\Delta x)} \cdot \frac{g(x)}{g(x+\Delta x)-g(x)} \right|.$$

Since the absolute value of a product is equal to the product of the absolute values of the factors,

$$\left| \frac{\theta \epsilon}{g(x+\Delta x)} \cdot \frac{g(x)}{g(x+\Delta x)-g(x)} \right| = \left| \frac{g(x)}{g(x+\Delta x)} \right| \cdot \left| \frac{\theta \epsilon}{g(x+\Delta x)-g(x)} \right|.$$

$$\text{But } \lim_{\Delta x \rightarrow 0} \left| \frac{g(x)}{g(x+\Delta x)} \right| = \left| \frac{g(x)}{g(x)} \right| = 1.$$

Therefore

$$\lim_{\Delta x \rightarrow 0} \left| \frac{g(x)}{g(x+\Delta x)} \right| \cdot \left| \frac{\theta \epsilon}{g(x+\Delta x)-g(x)} \right| = \lim_{\Delta x \rightarrow 0} \left| \frac{\theta \epsilon}{g(x+\Delta x)-g(x)} \right|.$$

By an application of the Law of the Mean,  $g(x+\Delta x)-g(x)$  is equal to  $g'(x_1)\Delta x$  where  $x < x_1 < x+\Delta x$ ; and since it may be assumed that  $g'(x_1) \neq 0$ , it follows readily that

$$\lim_{\Delta x \rightarrow 0} \left| \frac{\theta \epsilon}{g(x+\Delta x)-g(x)} \right| = \left| \frac{1}{g'(x_1)} \right| \cdot \lim_{\Delta x \rightarrow 0} \left| \frac{\theta \epsilon}{\Delta x} \right|.$$

Inasmuch as

$$g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy) = \theta \epsilon$$

is analytic and "the existence of the derivative of a function at a point requires that the difference

The following table shows the results of the experiment.

The results are as follows:

Time (min)	Temperature (°C)
0	20.0
10	21.5
20	23.0
30	24.5
40	26.0
50	27.5
60	29.0
70	30.5
80	32.0
90	33.5
100	35.0

The results show that the temperature increases over time.

The rate of increase is approximately 0.15°C per minute.

This is consistent with the theoretical prediction.

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80	32.0
90	33.5
100	35.0

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The rate of increase is approximately 0.15°C per minute.

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The following table shows the results of the experiment.

The results are as follows:



quotient have the same limit no matter how  $\Delta x$  and  $\Delta y$  approach zero,<sup>1</sup> a mode of approach may be selected of the type  $\Delta y = m \Delta x$ .

Thus the expression

$$g(x + \frac{1}{2} + iy + \Delta x + i \Delta y) - g(x + \frac{1}{2} + iy)$$

becomes a function of  $\Delta x$  for any fixed value of  $x$  and  $y$  and the function will approach zero as  $\Delta x$  approaches zero.

Therefore  $\theta_\epsilon$  approaches zero in the limit as  $\Delta x$  approaches zero.

In (7),  $\theta_\epsilon$  is seen to be equal to an analytic expression and so it can be expressed by a Taylor's series in terms of  $\Delta x$  when  $x$  and  $y$  are fixed since an analytic function can be expressed in a Taylor's series convergent in some region.<sup>2</sup>

Such a series will have no constant term because  $\theta_\epsilon$  becomes zero as  $\Delta x$  approaches zero.

If  $\theta_\epsilon$  is expressed as the series,

$$a_1 \Delta x + a_2 (\Delta x)^2 + a_3 (\Delta x)^3 + a_4 (\Delta x)^4 + \dots,$$

which is the form of its Taylor series development,

then the fraction  $\left| \frac{\theta_\epsilon}{\Delta x} \right|$  becomes the series,

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<sup>1</sup>Curtiss, David Raymond, *Analytic Functions of a Complex Variable*, p. 47.

<sup>2</sup>Ibid., p. 121.

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$$\left| a_1 + a_2 \Delta x + a_3 (\Delta x)^2 + a_4 (\Delta x)^3 + \dots \right|.$$

This series converges and will equal zero as  $\Delta x$  approaches zero if  $a_1$  is equal to zero. If  $a_1$  is not equal to zero, the series will equal a constant in the limit.

When the series, and consequently  $\left| \frac{\theta \epsilon}{\Delta x} \right|$  is equal to zero,

$$\lim_{\Delta x \rightarrow 0} \left| \frac{\theta \epsilon}{g(x+\Delta x)g(x)} \right| = \left| \frac{1}{g'(x)} \right| \lim_{\Delta x \rightarrow 0} \left| \frac{\theta \epsilon}{\Delta x} \right| = 0.$$

And so

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta x)-g(x+\frac{1}{2}+iy)}{g(x+\Delta x)-g(x)} \right| < 2Ke^{\pm(k-1)\pi y}.$$

When  $a_1$  is not equal to zero, and the series becomes equal to a constant when  $\Delta x$  approaches zero this constant can be made less than  $2Ke^{\pm(k-1)\pi y}$ , for no matter what values are chosen for  $y$ ,  $K$  may be chosen large enough to make this condition true.

Therefore the left hand member of (4),

$$\left| \frac{g'(x+\frac{1}{2}+iy)}{g'(x)} \right|, \text{ is less than}$$

1. The function  $f(x) = \frac{1}{x^2}$  is continuous at  $x = 1$ .

This series converges to the value of the function at  $x = 1$ . The series is equal to zero, the value of the function at  $x = 1$ .

When the series is evaluated at  $x = 1$ , it is equal to zero.

$$\frac{1}{1^2} = 1$$

$$\frac{1}{1^2} = 1$$

1/1^2 = 1

This is a geometric series with first term 1 and common ratio 1/2. The sum of the series is equal to 2. This constant can be used to find the value of the function at  $x = 1$ . The function is equal to 2.

$$\frac{1}{1^2} = 1$$

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left| \frac{\Delta x}{\Delta x + i \Delta y} \right| \leq K e^{\pm(k-1)\pi y}.$$

But if, as before,  $\Delta y$  is chosen equal to  $m \Delta x$ ,

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \left| \frac{\Delta x}{\Delta x + i \Delta y} \right| = \lim_{\Delta x \rightarrow 0} \left| \frac{\Delta x}{\Delta x(1+im)} \right| = \left| \frac{1}{1+im} \right|,$$

or

$$\left| \frac{g'(x + \frac{1}{2} + iy)}{g'(x)} \right| < K_1 e^{\pm(k-1)\pi y}.$$

Thus condition (b) of the Theorem is satisfied for the coefficient  $g'(n)$ .

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## SECTION II

Thesis: A particularization of the integral expression in the theorem given in Section I.

We shall now consider briefly the function,  $f(z)$ , defined by the integral expression,

$$(8) \quad f(z) = \int_{-L^{-\frac{1}{2}}}^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$- \sum_{m=-L}^{-1} g(m) z^m - \zeta_k(L, z),$$

wherein  $L$  is any arbitrary positive integer, and, whatever the value chosen for it, the expression  $\zeta_k(L, z)$  is such that

$$\lim_{\text{mod } z \rightarrow \infty} z^L \zeta_k(L, z) = 0$$

If  $L$  is fixed as unity, the integral,

$$\int_{-L^{-\frac{1}{2}}}^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx,$$

Theorem 1. Let  $f(x)$  be a function defined on the interval  $[a, b]$ .  
 Then the function  $F(x)$  defined by the expression  

$$F(x) = \int_a^x f(t) dt$$
 is a primitive of  $f(x)$  on the interval  $[a, b]$ .

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F(x)$  is any primitive of  $f(x)$  on the interval  $[a, b]$ .  
 This is the fundamental theorem of calculus.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(\xi_k) \Delta x_k$$



equals  $\int_{-\frac{3}{2}}^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx.$

This integral may be expressed as the sum of two integrals; namely,

$$\int_0^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$+ \int_{-\frac{3}{2}}^0 \left\{ g(x) [(1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx.$$

The second of these two integrals exists and may be denoted by a constant,  $C_1$ , for any particular value of  $k$ . This statement is apparent when one observes that the limits upon the integral are finite and that the integrand exists at all points of the closed interval represented by the range of integration.

Even at  $x=0$  and at  $x=-1$  where  $\sin \pi x$  in the denominator of the integrand vanishes, the numerator vanishes likewise to an equal order; thus the integrand exists. Moreover,  $[(-1)^k z]^x$  exists for all negative values of  $x$  in view of the fact that the absolute value of  $z$  is to be taken large. Also  $g(w)$ ,  $w=x+iy$ , is postulated to be analytic in the finite part of the plane.

$$\frac{1}{x^2} = x^{-2} \Rightarrow \frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}$$

This integral is the same as the integral of  $x^{-2}$ .

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

The integral of  $x^{-n}$  is  $\frac{x^{-n+1}}{-n+1} + C$  for  $n \neq 1$ .  
For  $n=1$ , the integral is  $\ln|x| + C$ .  
This is the general formula for the integral of a power function.

For  $n=2$ , the integral is  $-\frac{1}{x} + C$ .  
For  $n=3$ , the integral is  $-\frac{1}{2x^2} + C$ .  
For  $n=4$ , the integral is  $-\frac{1}{3x^3} + C$ .  
For  $n=5$ , the integral is  $-\frac{1}{4x^4} + C$ .  
For  $n=6$ , the integral is  $-\frac{1}{5x^5} + C$ .  
For  $n=7$ , the integral is  $-\frac{1}{6x^6} + C$ .  
For  $n=8$ , the integral is  $-\frac{1}{7x^7} + C$ .  
For  $n=9$ , the integral is  $-\frac{1}{8x^8} + C$ .  
For  $n=10$ , the integral is  $-\frac{1}{9x^9} + C$ .  
For  $n=11$ , the integral is  $-\frac{1}{10x^{10}} + C$ .  
For  $n=12$ , the integral is  $-\frac{1}{11x^{11}} + C$ .  
For  $n=13$ , the integral is  $-\frac{1}{12x^{12}} + C$ .  
For  $n=14$ , the integral is  $-\frac{1}{13x^{13}} + C$ .  
For  $n=15$ , the integral is  $-\frac{1}{14x^{14}} + C$ .  
For  $n=16$ , the integral is  $-\frac{1}{15x^{15}} + C$ .  
For  $n=17$ , the integral is  $-\frac{1}{16x^{16}} + C$ .  
For  $n=18$ , the integral is  $-\frac{1}{17x^{17}} + C$ .  
For  $n=19$ , the integral is  $-\frac{1}{18x^{18}} + C$ .  
For  $n=20$ , the integral is  $-\frac{1}{19x^{19}} + C$ .  
For  $n=21$ , the integral is  $-\frac{1}{20x^{20}} + C$ .  
For  $n=22$ , the integral is  $-\frac{1}{21x^{21}} + C$ .  
For  $n=23$ , the integral is  $-\frac{1}{22x^{22}} + C$ .  
For  $n=24$ , the integral is  $-\frac{1}{23x^{23}} + C$ .  
For  $n=25$ , the integral is  $-\frac{1}{24x^{24}} + C$ .  
For  $n=26$ , the integral is  $-\frac{1}{25x^{25}} + C$ .  
For  $n=27$ , the integral is  $-\frac{1}{26x^{26}} + C$ .  
For  $n=28$ , the integral is  $-\frac{1}{27x^{27}} + C$ .  
For  $n=29$ , the integral is  $-\frac{1}{28x^{28}} + C$ .  
For  $n=30$ , the integral is  $-\frac{1}{29x^{29}} + C$ .  
For  $n=31$ , the integral is  $-\frac{1}{30x^{30}} + C$ .  
For  $n=32$ , the integral is  $-\frac{1}{31x^{31}} + C$ .  
For  $n=33$ , the integral is  $-\frac{1}{32x^{32}} + C$ .  
For  $n=34$ , the integral is  $-\frac{1}{33x^{33}} + C$ .  
For  $n=35$ , the integral is  $-\frac{1}{34x^{34}} + C$ .  
For  $n=36$ , the integral is  $-\frac{1}{35x^{35}} + C$ .  
For  $n=37$ , the integral is  $-\frac{1}{36x^{36}} + C$ .  
For  $n=38$ , the integral is  $-\frac{1}{37x^{37}} + C$ .  
For  $n=39$ , the integral is  $-\frac{1}{38x^{38}} + C$ .  
For  $n=40$ , the integral is  $-\frac{1}{39x^{39}} + C$ .  
For  $n=41$ , the integral is  $-\frac{1}{40x^{40}} + C$ .  
For  $n=42$ , the integral is  $-\frac{1}{41x^{41}} + C$ .  
For  $n=43$ , the integral is  $-\frac{1}{42x^{42}} + C$ .  
For  $n=44$ , the integral is  $-\frac{1}{43x^{43}} + C$ .  
For  $n=45$ , the integral is  $-\frac{1}{44x^{44}} + C$ .  
For  $n=46$ , the integral is  $-\frac{1}{45x^{45}} + C$ .  
For  $n=47$ , the integral is  $-\frac{1}{46x^{46}} + C$ .  
For  $n=48$ , the integral is  $-\frac{1}{47x^{47}} + C$ .  
For  $n=49$ , the integral is  $-\frac{1}{48x^{48}} + C$ .  
For  $n=50$ , the integral is  $-\frac{1}{49x^{49}} + C$ .  
For  $n=51$ , the integral is  $-\frac{1}{50x^{50}} + C$ .  
For  $n=52$ , the integral is  $-\frac{1}{51x^{51}} + C$ .  
For  $n=53$ , the integral is  $-\frac{1}{52x^{52}} + C$ .  
For  $n=54$ , the integral is  $-\frac{1}{53x^{53}} + C$ .  
For  $n=55$ , the integral is  $-\frac{1}{54x^{54}} + C$ .  
For  $n=56$ , the integral is  $-\frac{1}{55x^{55}} + C$ .  
For  $n=57$ , the integral is  $-\frac{1}{56x^{56}} + C$ .  
For  $n=58$ , the integral is  $-\frac{1}{57x^{57}} + C$ .  
For  $n=59$ , the integral is  $-\frac{1}{58x^{58}} + C$ .  
For  $n=60$ , the integral is  $-\frac{1}{59x^{59}} + C$ .  
For  $n=61$ , the integral is  $-\frac{1}{60x^{60}} + C$ .  
For  $n=62$ , the integral is  $-\frac{1}{61x^{61}} + C$ .  
For  $n=63$ , the integral is  $-\frac{1}{62x^{62}} + C$ .  
For  $n=64$ , the integral is  $-\frac{1}{63x^{63}} + C$ .  
For  $n=65$ , the integral is  $-\frac{1}{64x^{64}} + C$ .  
For  $n=66$ , the integral is  $-\frac{1}{65x^{65}} + C$ .  
For  $n=67$ , the integral is  $-\frac{1}{66x^{66}} + C$ .  
For  $n=68$ , the integral is  $-\frac{1}{67x^{67}} + C$ .  
For  $n=69$ , the integral is  $-\frac{1}{68x^{68}} + C$ .  
For  $n=70$ , the integral is  $-\frac{1}{69x^{69}} + C$ .  
For  $n=71$ , the integral is  $-\frac{1}{70x^{70}} + C$ .  
For  $n=72$ , the integral is  $-\frac{1}{71x^{71}} + C$ .  
For  $n=73$ , the integral is  $-\frac{1}{72x^{72}} + C$ .  
For  $n=74$ , the integral is  $-\frac{1}{73x^{73}} + C$ .  
For  $n=75$ , the integral is  $-\frac{1}{74x^{74}} + C$ .  
For  $n=76$ , the integral is  $-\frac{1}{75x^{75}} + C$ .  
For  $n=77$ , the integral is  $-\frac{1}{76x^{76}} + C$ .  
For  $n=78$ , the integral is  $-\frac{1}{77x^{77}} + C$ .  
For  $n=79$ , the integral is  $-\frac{1}{78x^{78}} + C$ .  
For  $n=80$ , the integral is  $-\frac{1}{79x^{79}} + C$ .  
For  $n=81$ , the integral is  $-\frac{1}{80x^{80}} + C$ .  
For  $n=82$ , the integral is  $-\frac{1}{81x^{81}} + C$ .  
For  $n=83$ , the integral is  $-\frac{1}{82x^{82}} + C$ .  
For  $n=84$ , the integral is  $-\frac{1}{83x^{83}} + C$ .  
For  $n=85$ , the integral is  $-\frac{1}{84x^{84}} + C$ .  
For  $n=86$ , the integral is  $-\frac{1}{85x^{85}} + C$ .  
For  $n=87$ , the integral is  $-\frac{1}{86x^{86}} + C$ .  
For  $n=88$ , the integral is  $-\frac{1}{87x^{87}} + C$ .  
For  $n=89$ , the integral is  $-\frac{1}{88x^{88}} + C$ .  
For  $n=90$ , the integral is  $-\frac{1}{89x^{89}} + C$ .  
For  $n=91$ , the integral is  $-\frac{1}{90x^{90}} + C$ .  
For  $n=92$ , the integral is  $-\frac{1}{91x^{91}} + C$ .  
For  $n=93$ , the integral is  $-\frac{1}{92x^{92}} + C$ .  
For  $n=94$ , the integral is  $-\frac{1}{93x^{93}} + C$ .  
For  $n=95$ , the integral is  $-\frac{1}{94x^{94}} + C$ .  
For  $n=96$ , the integral is  $-\frac{1}{95x^{95}} + C$ .  
For  $n=97$ , the integral is  $-\frac{1}{96x^{96}} + C$ .  
For  $n=98$ , the integral is  $-\frac{1}{97x^{97}} + C$ .  
For  $n=99$ , the integral is  $-\frac{1}{98x^{98}} + C$ .  
For  $n=100$ , the integral is  $-\frac{1}{99x^{99}} + C$ .

The summation  $\sum_{m=-l}^{-1} g(m)z^m$  becomes zero and dis-

appears if  $l=1$ .

The third part,  $\mathcal{G}_k(l, z)$ , will become zero if  $z$  is allowed to approach infinity, so it may be ignored if  $z$  becomes sufficiently large. This, then, will be the assumption imposed upon  $z$  in the remainder of this discussion.

# WORLD WAR II

The war was fought between the Axis powers and the Allied powers.

1941-1945

The war was fought between the Axis powers and the Allied powers. The Axis powers included Germany, Italy, and Japan. The Allied powers included the United States, Great Britain, and the Soviet Union. The war was fought from 1941 to 1945. The Axis powers were defeated in 1945. The war was the deadliest conflict in human history.

### SECTION III

Thesis: The sum of the series,  $\sum_{n=0}^{\infty} \underline{g(n) z^n}$ , is

equal to the sum of the series,  $\sum_{n=0}^{\infty} \underline{g'(n) z^n}$ , plus some

constant quantity.

It has been proved that if  $k$  is a positive integer greater than, or equal to 2, and the restrictions on  $g(n) z^n$  are the same as those postulated in Section I, that

$$\begin{aligned} \sum_{n=0}^{\infty} g(n) z^n &= \int_0^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx \\ &+ \int_{-L-\frac{1}{2}}^0 \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx \\ &- \sum_{m=-L}^{-1} g(m) z^m - \int_k^{\infty} (L, z). \end{aligned}$$

of which the last three terms either disappear or become constants if  $L$  is equal to unity and if  $z$  is taken of large modulus.

Section III

Topic: The sum of the series

Let  $f(x) = \frac{1}{1-x}$ , then

If we have  $f(x) = \frac{1}{1-x}$ , then  $f(x) = 1 + x + x^2 + x^3 + \dots$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (|x| < 1)$$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad (|x| < 1)$$

of which the first term is 1, and the rest are some other terms.

If, in the above integral,  $k$  is made equal to 2, the representation of the series simply becomes

$$(9) \quad \sum_{n=0}^{\infty} g(n)z^n = \int_0^{\infty} g(x)z^x dx + C_1.$$

When the formula for integration by parts; namely,

$$\int u dv = uv - \int v du,$$

is applied to the term in the second member of (9) involving the integral wherein  $u$  is made equal to  $g(x)$  and  $dv$  to  $z^x dx$ , it follows that

$$(10) \quad \int_0^{\infty} g(x)z^x dx = \left[ \frac{g(x)z^x}{\log z} \right]_0^{\infty} - \frac{1}{\log z} \int_0^{\infty} g'(x)z^x dx.$$

In the fraction,  $\frac{g(x)z^x}{\log z}$ , if  $x$  be permitted to approach infinity, the numerator will approach zero. This follows from the fact that the series

$$\sum_{n=0}^{\infty} g(n)z^n$$

has been assumed at the start to be convergent inside of a circle of infinite radius, and a necessary condition for convergence is that the general term will approach zero as the number of the terms approaches infinity.

It is known that the function  $f(x)$  is continuous on the interval  $[a, b]$  and that the function  $F(x)$  is defined by the equation

$$F(x) = \int_a^x f(t) dt \quad (8)$$

When the function  $f(x)$  is continuous on the interval  $[a, b]$ , the function  $F(x)$  is also continuous on the interval  $[a, b]$ .

It is known that the function  $f(x)$  is continuous on the interval  $[a, b]$  and that the function  $F(x)$  is defined by the equation

$$F(x) = \int_a^x f(t) dt \quad (9)$$

When the function  $f(x)$  is continuous on the interval  $[a, b]$ , the function  $F(x)$  is also continuous on the interval  $[a, b]$ .

It is known that the function  $f(x)$  is continuous on the interval  $[a, b]$  and that the function  $F(x)$  is defined by the equation

$$F(x) = \int_a^x f(t) dt \quad (10)$$

When the function  $f(x)$  is continuous on the interval  $[a, b]$ , the function  $F(x)$  is also continuous on the interval  $[a, b]$ .

It has been shown that the function  $f(x)$  is continuous on the interval  $[a, b]$  and that the function  $F(x)$  is defined by the equation

$$F(x) = \int_a^x f(t) dt \quad (11)$$

for continuous functions  $f(x)$  on the interval  $[a, b]$ .

It is known that the function  $f(x)$  is continuous on the interval  $[a, b]$  and that the function  $F(x)$  is defined by the equation



If the lower limit, zero, is substituted in the same fraction, it may be seen that

$$(11) \quad g(0)z^0 = A,$$

where A is merely the constant term of the series,

$$\sum_{n=0}^{\infty} g(n)z^n.$$

Since  $g'(n)$  has the same properties as  $g(n)$ , the theorem stated in Section I may be applied to the series

$$\sum_{n=0}^{\infty} g'(n)z^n,$$

thus giving

$$\begin{aligned} \sum_{n=0}^{\infty} g'(n)z^n &= \int_0^{\infty} \left\{ g'(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx \\ &+ \int_{-1-\frac{1}{2}}^0 \left\{ g'(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx \\ &- \sum_{m=-1}^{-1} g(m)z^m - \mathcal{J}_k(L, z). \end{aligned}$$

If the lower limit is zero, the integral is

some fraction of the whole.

(11)  $\int_0^1 x^2 dx = \frac{1}{3}$

where  $x$  is the variable.

# INTEGRATION

Since  $y = f(x)$  is the curve...

The area under the curve...

is

$$\int_a^b f(x) dx$$

Thus

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$\int_a^b f(x) dx = F(b) - F(a)$$

The same statements as those made in regard to expression (8) are applicable here.

Therefore the series,

$$\sum_{n=0}^{\infty} g'(n)z^n,$$

is equivalent to

$$\int_0^{\infty} \left\{ g'(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} dx + C_2 \right\}.$$

If  $k$  is assumed to be equal to 2 as before, it is apparent that

$$(12) \quad \sum_{n=0}^{\infty} g'(n)z^n = \int_0^{\infty} g'(x)z^x dx + C_2.$$

When the values from statements (11) and (12) are substituted in (10), it is observed that

$$\int_0^{\infty} g(x)z^x dx = -\frac{1}{\log z} \left\{ A + \sum_{n=0}^{\infty} g'(n)z^n - C_2 \right\}.$$

When this value is substituted in equation (9), the original series,

$$\sum_{n=0}^{\infty} g(n)z^n,$$

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is equivalent to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

It is a theorem that if  $f(x)$  is analytic at  $a$ , then

converges to

$$(1) \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

that the value of the series (1) at  $x=a$  is

substituted in (1), it is always true

$$\int_a^b f(x) dx = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \int_a^b (x-a)^n dx$$

When this series is substituted in (1),

the original series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

is seen to be equivalent to

$$-\frac{1}{\log z} \left\{ A + \sum_{n=0}^{\infty} g(n)z^{n-C_2} \right\} + C_1 .$$

In conclusion, we have

$$\sum_{n=0}^{\infty} g(n)z^n = a - \frac{1}{\log z} \left\{ \beta + \sum_{n=0}^{\infty} g'(n)z^n \right\} ,$$

( $a$  and  $\beta$  are constants) provided that the conditions postulated upon the series,

$$\sum_{n=0}^{\infty} g(n)z^n ,$$

are satisfied.

ROCKY MOUNT BOND

← PRE-CONTENTS →

As seen in the diagram...



The diagram illustrates...



(D) The force exerted by the...

is equal to the weight...



is equal to the weight...

PROBLEM 10

STATEMENT OF THE PROBLEM

## SECTION IV

### Thesis: Additional Observations.

It may now be observed that in the previous discussion, the limitation placed upon the value of  $k$  in the integral,

$$\int_{-1-\frac{1}{2}}^{\infty} \left\{ g(x) [(-1)^k z]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx,$$

has been unduly restrictive. Since  $k$  may be any integer greater than or equal to 2, we may, for example, let it be 3.

If  $k$  is allowed to be 3, the integral above becomes

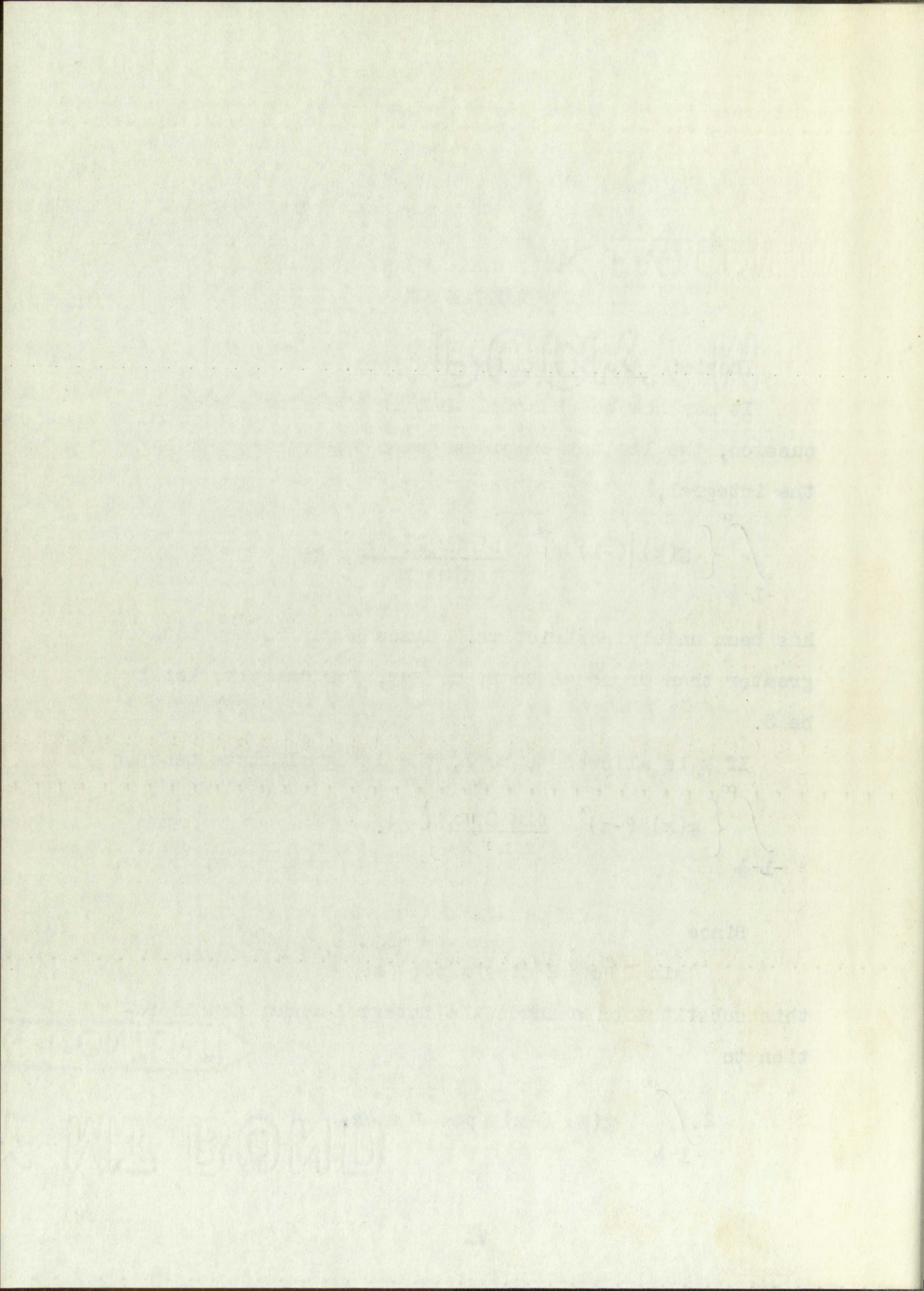
$$\int_{-1-\frac{1}{2}}^{\infty} \left\{ g(x) (-z)^x \frac{\sin 2\pi x}{\sin \pi x} \right\} dx.$$

Since

$$\sin 2\pi x = 2 \sin \pi x \cos \pi x,$$

this substitution changes the integral under consideration to

$$2 \int_{-1-\frac{1}{2}}^{\infty} g(x) (-z)^x \cos \pi x dx.$$





When this integral is integrated by parts, the following equation results.

$$\begin{aligned}
 & 2 \int_{-1-\frac{1}{2}}^{\infty} g(x) (-z)^x \cos \pi x \, dx \\
 &= \frac{2}{\pi} \left[ g(x) (-z)^x \sin \pi x \right]_{-1-\frac{1}{2}}^{\infty} \\
 &\quad - \frac{2}{\pi} \log(-z) \int_{-1-\frac{1}{2}}^{\infty} g(x) (-z)^x \sin \pi x \, dx \\
 &\quad - \frac{2}{\pi} \int_{-1-\frac{1}{2}}^{\infty} g'(x) (-z)^x \sin \pi x \, dx.
 \end{aligned}$$

Another integration by parts will be necessary (if it is desired that  $k$  be consistently taken equal to 3) in order that the expressions under the integral signs shall be the same as that in the original integral except for the coefficient,  $g(x)$ .

By such an integration, it develops that

$$\begin{aligned}
 & 2 \int_{-1-\frac{1}{2}}^{\infty} g(x) (-z)^x \cos \pi x \, dx \\
 &= \frac{2}{\pi} \left[ g(x) (-z) \sin \pi x \right]_{-1-\frac{1}{2}}^{\infty}
 \end{aligned}$$

# BOOK REVIEW

When the integral is improper, the following results apply:

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_{-\infty}^b f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_{-\infty}^b f(x) dx$$

Another important result will be necessary (it is stated that the conditions under the integral sign shall be the same as in the original integral except for the coefficient,  $g(x)$ ).  
By each an integration, it follows that

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \rightarrow -\infty} \int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_{-\infty}^b f(x) dx$$

$$\begin{aligned}
& + \frac{2}{\pi^2} \log(-z) \left[ g(x) (-z)^x \cos \pi x \right]_{-1-\frac{1}{2}}^{\infty} \\
& + \frac{2}{\pi^2} \left[ g'(x) (-z)^x \cos \pi x \right]_{-1-\frac{1}{2}}^{\infty} \\
& - \frac{2}{\pi^2} \log^2(-z) \int_{-1-\frac{1}{2}}^{\infty} g(x) (-z)^x \cos \pi x \, dx \\
& - \frac{4}{\pi^2} \log(-z) \int_{-1-\frac{1}{2}}^{\infty} g'(x) (-z)^x \cos \pi x \, dx \\
& - \frac{2}{\pi^2} \int_{-1-\frac{1}{2}}^{\infty} g''(x) (-z)^x \cos \pi x \, dx.
\end{aligned}$$

By an algebraic manipulation one will observe that

$$\begin{aligned}
& 2 \int_{-1-\frac{1}{2}}^{\infty} g(x) (-z)^x \cos \pi x \, dx \\
& = \frac{2\pi}{\pi^2 + \log^2(-z)} \left[ g(x) (-z)^x \sin \pi x \right]_{-1-\frac{1}{2}}^{\infty} \\
& + \frac{2\log(-z)}{\pi^2 + \log^2(-z)} \left[ g(x) (-z)^x \cos \pi x \right]_{-1-\frac{1}{2}}^{\infty}
\end{aligned}$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$f''(x) = \frac{6}{x^4}$$

$$f'''(x) = -\frac{24}{x^5}$$

$$f^{(4)}(x) = \frac{120}{x^6}$$

$$f^{(5)}(x) = -\frac{720}{x^7}$$

$$f^{(6)}(x) = \frac{5040}{x^8}$$

$$f^{(7)}(x) = -\frac{35280}{x^9}$$

$$\begin{aligned}
& + \frac{2}{\pi^2 + \log^2(-z)} \left[ g'(x) (-z)^x \cos \pi x \right]_{-1-\frac{1}{2}}^{\infty} \\
& - \frac{4 \log(-z)}{\pi^2 + \log^2(-z)} \int_{-1-\frac{1}{2}}^{\infty} g'(x) (-z)^x \cos \pi x \, dx \\
& - \frac{2}{\pi^2 + \log^2(-z)} \int_{-1-\frac{1}{2}}^{\infty} g''(x) (-z)^x \cos \pi x \, dx.
\end{aligned}$$

By the argument of Section I, it can be shown that  $g''(x)$ , the second derivative of  $g(x)$ , has the same properties as  $g'(x)$ , and therefore of  $g(x)$ . Moreover, the entire analysis of the first three sections readily follows, thus permitting the change of the original series under consideration into another series.

Likewise  $k$  may be given the value 4, 5, or any other positive integer and essentially the same analysis follows. Of course, that which is theoretically possible may not be practically expedient. The case where  $k$  is 2 appears to be very desirable from the practical standpoint and thus has been stressed.

# MEMORANDUM

TO : [Illegible]

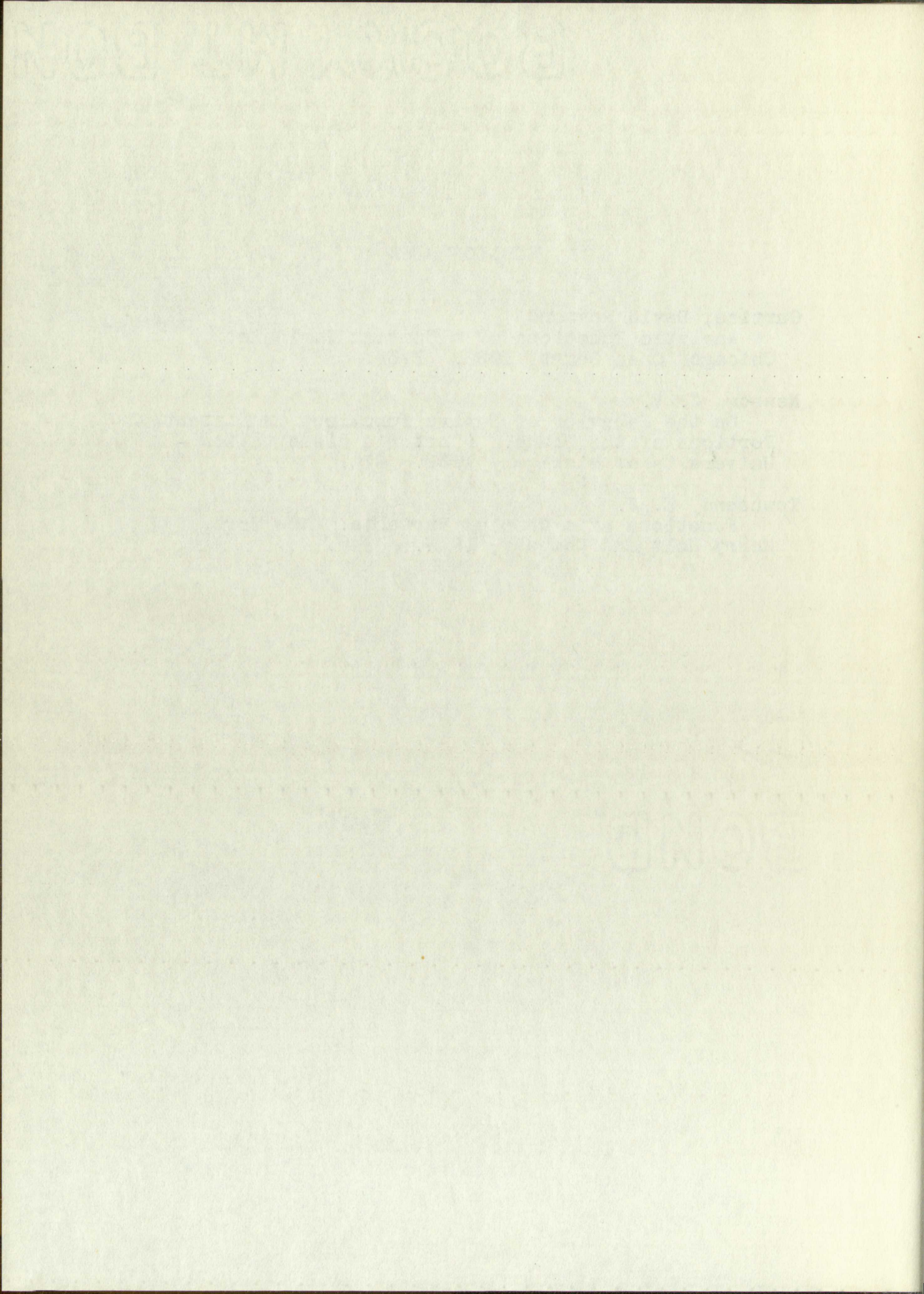
FROM : [Illegible]

SUBJECT : [Illegible]

[Illegible text block containing the main body of the memorandum]

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E. J. Workman.

