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A Method of Changing Certain Infinite Series To New But Equivalent Series

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A METHOD OF CHANGING CERTAIN INFINITE SERIES TO NEW BUT EQUIVALENT SERIES

By

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INTRODUCTION

The purpose of this investigation is to prove that the sum of a series of variable terms of a certain type is equal to a constant plus the sum of another series of variable terms. As a result of this proof it may be hoped that certain series, which so far have not been summed, may be shown equal to a constant plus another series for which the sum is known.

SECTION I

Thesis: If g(n) has certain particular properties, then g'(n) has them, also.

In this section we shall show that if the coefficient g(n) of the series $\sum_{n=0}^{\infty} g(n) z^n$ has the properties assigned to it in the following theorem, the coefficient g'(n) of the series $\sum_{n=0}^{\infty} g'(n) z^n$ has the same properties, g'(n) being the first derivative of g(n).

Theorem: Suppose that the coefficient g(n) occurring in the general term of the power series,

$$(1) \quad \sum_{n=0}^{\infty} g(n) z^n, \text{ radius of convergence} = \infty \quad ,$$

may be regarded as a function, g(w), of the complex variable w=x+iy and as such satisfies the following two conditions:

(a) is single valued and analytic throughout the

finite w-plane,

(b) is such that for all values of x and y one may write

$$\left| \frac{g(x+\frac{1}{2}+iy)}{g(x)} \right| < Ke^{\pm(k-1)\pi y},$$

where K is a constant independent of x and y, and k is any given positive integer ≥ 2, and where the upper or lower of the signs $\pm$ appearing therein is to be taken according as y is positive or negative.

Then the integral function f(z) defined by the series (1) when considered for all values of z satisfying the condition,

$$-\pi < \arg\left[(-1)^k z\right] < \pi,$$

may be expressed in the form,

$$f(z) = \int_{-L-\frac{1}{2}}^{\infty} \left\{ g(x) \left[(-1)^k z\right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$- \sum_{m=-L}^{-1} g(m) z^m - \zeta_k(L, z) \; ;$$

wherein L is an arbitrary positive integer and, whatever the value chosen for it, the expression $\zeta_k(L, z)$ is such that

$$\lim_{\mathrm{mod}\, z \to 0} z^{L} \zeta_k(L, z) = 0. \text{ [1]}$$

The coefficient $g'(n)$ satisfies the condition (a) of this theorem for it is well known that "if $f(z)$ is analytic throughout an open region T, then its derivatives of all orders exist at each point of T, and each of them is thus also analytic throughout T."[2]

If $g'(n)$ is to fulfill condition (b) of the theorem, it is necessary to show that

$$\left| \frac{g'(x+\frac{1}{2}+iy)}{g'(x)} \right| < K e^{\pm(k-1)\pi y},$$

wherein K and k have the conditions imposed by the theorem.

By definition

$$(2) \quad g'(x+\tfrac{1}{2}+iy) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy)}{(\Delta x+i\Delta y)}$$

and

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[1] Newsom, C. V., On the Behavior of Entire Functions in Distant Portions of the Plane, p. 10-11.

[2] Curtiss, David Raymond, Analytic Functions of a Complex Variable, p. 98.

$$\lim \; [illegible] \; (\zeta, \psi) = [illegible]$$
$$\zeta \to \zeta$$

The coefficient $g'(\zeta)$ [illegible] (a) of this theorem [illegible] it will be seen that $f(z)$ is analytic [illegible] the derivatives of all orders [illegible] of each [illegible] and each of them is thus also analytic [illegible].

If $g(w)$ is to fulfill condition (c) of the theorem, it is necessary to show that

$$\left. \frac{f\,[g(w)]}{g'(w)} \right| [illegible]$$

wherein f and [illegible] the proof [illegible] theorem.

By definition:

(2) $g'(x + iy)$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{g(x+\Delta x + i(y+\Delta y)) - g(x+iy)}{\Delta x + i\Delta y}$$

and

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[1] [illegible], Functions of a Complex Variable, p. [illegible].

[2] [illegible], Functions of a Complex Variable, p. [illegible].

(3) $$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x}.$$

The absolute value of the ratio of the limits given in (2) and (3) is

(4) $$\left|\frac{g'(x+\tfrac{1}{2}+iy)}{g'(x)}\right| = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left|\frac{\Delta x}{\Delta x + i\Delta y}\right| \cdot \left|\frac{g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy)}{g(x+\Delta x)-g(x)}\right|.$$

If in the fraction,

$$\left|\frac{g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy)}{g(x+\Delta x)-g(x)}\right| ,$$

the function $g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)$ is replaced by C, the function $g(x+\tfrac{1}{2}+iy)$ by D, the function $g(x+\Delta x)$ by A, and the function $g(x)$ by B, it follows that

$$\left|\frac{g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy)}{g(x+\Delta x)-g(x)}\right| = \left|\frac{C-D}{A-B}\right| = \left|\frac{C}{A-B} - \frac{D}{A-B}\right|.$$

By a simple algebraic manipulation it may be observed that

$$\left|\frac{C}{A-B} - \frac{D}{A-B}\right| = \left|\frac{C}{A} + \frac{C}{A}\cdot\frac{B}{A-B} - \frac{D}{A} - \frac{D}{A}\cdot\frac{B}{A-B}\right|.$$

Since the absolute value of the difference of two quantities is less than, or equal to, the sum of the

absolute value of the first quantity and the absolute value of the second, the expression

$$\left| \frac{C}{A} + \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} - \frac{D}{A} \cdot \frac{B}{A-B} \right|$$

will be less than, or equal to,

$$\left| \frac{C}{A} \right| + \left| \frac{D}{A} \right| + \left| \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} \cdot \frac{B}{A-B} \right| .$$

Now let us consider the first two parts of this sum separately.

If one recalls the values designated by C and A, it is seen that

$$\left| \frac{C}{A} \right| \text{ becomes } \left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \right| .$$

But $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \right|$

$= \left| \frac{g(x+\frac{1}{2}+iy)}{g(x)} \right|$ which is postulated less than $Ke^{\pm(k-1)\pi y}$

in condition (b) of the theorem.

Likewise if the values designated by D and A are employed,

$$\left| \frac{D}{A} \right| \text{ becomes } \left| \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \right| .$$

But $\lim_{\Delta x \to 0} \left| \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \right| = \left| \frac{g(x+\frac{1}{2}+iy)}{g(x)} \right|$ which again is less than $Ke^{\pm(k-1)\pi y}$.

Thus it has been shown that

(5) $$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy)}{g(x+\Delta x)-g(x)} \right|$$

$$< 2Ke^{\pm(k-1)\pi y} + \left| \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} \cdot \frac{B}{A-B} \right|.$$

If the original values for C, D and A are substituted in the fractions $\frac{C}{A}$ and $\frac{D}{A}$, then

(6) $$\left| \frac{C}{A} \cdot \frac{B}{A-B} - \frac{D}{A} \cdot \frac{B}{A-B} \right|$$

becomes equal to

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)} \left[\frac{B}{A-B}\right] - \frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)} \left[\frac{B}{A-B}\right] \right|.$$

Since the function, f(z), is analytic over any region of our interest as stated in the theorem, it is also continuous.[1]

By the definition of continuity,

$\left| g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\frac{1}{2}+iy) \right| < \epsilon$, where ϵ is any arbitrarily small positive quantity if $|\Delta x|$ and $|\Delta y|$ are taken sufficiently small.

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[1]Townsend, E. J., Functions of a Complex Variable, p. 79.

In any inequality involving positive members, a fraction numerically less than unity may be so chosen that when the larger of the two quantities is multiplied by that fraction, the result is equivalent to the small quantity. So there exists a θ, $|\theta| < 1$, such that the following statement is true,

$$(7) \qquad g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy)=\theta\epsilon \quad .$$

It is to be observed that

$$g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy)+\theta\epsilon \quad ,$$

and

$$\frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)}=\frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)}+\frac{\theta\epsilon}{g(x+\Delta x)}$$

if $g(x+\Delta x)$ is assumed to be not equal to zero for sufficiently small values of Δx.

If the value for $\dfrac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)}$ that has just been found is substituted for it in statement (6),

$$\left|\frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta y)}{g(x+\Delta x)}\left[\frac{B}{A-B}\right]-\frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)}\left[\frac{B}{A-B}\right]\right| \text{ becomes}$$

$$\left|\frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)}\left[\frac{B}{A-B}\right]+\frac{\theta\epsilon}{g(x+\Delta x)}\left|\frac{B}{A-B}\right|-\frac{g(x+\frac{1}{2}+iy)}{g(x+\Delta x)}\left[\frac{B}{A-B}\right]\right| \text{ which}$$

is equal to $\left|\dfrac{\theta\epsilon}{g(x+\Delta x)}\left[\dfrac{B}{A-B}\right]\right|$.

In applications involving [illegible]

fraction multiplied [illegible]

that when the larger [illegible]

plied by that fraction, the result [illegible]

the small quantity. [illegible]

such that the following statement holds:

(7) g(x+ly [illegible]

It is to be observed that

g(x+ly+ [illegible]

and

[illegible]

if $g(x+\Delta x)$ is assumed [illegible]

sufficiently small values of Δx.

If the value [illegible]

just been found to [illegible]

$\frac{g(x+ly+ [illegible])}{g(x+\Delta x)}$ [illegible]

$\frac{g(x+ly)}{g(x+\Delta x)}$ [illegible]

is equal to [illegible]

Now if a final substitution is made for A and B, it follows that

$$\left| \frac{\theta\epsilon}{g(x+\Delta x)} \left[\frac{B}{A-B}\right] \right| = \left| \frac{\theta\epsilon}{g(x+\Delta x)} \cdot \frac{g(x)}{g(x+\Delta x)-g(x)} \right| .$$

Since the absolute value of a product is equal to the product of the absolute values of the factors,

$$\left| \frac{\theta\epsilon}{g(x+\Delta x)} \cdot \frac{g(x)}{g(x+\Delta x)-g(x)} \right| = \left| \frac{g(x)}{g(x+\Delta x)} \right| \cdot \left| \frac{\theta\epsilon}{g(x+\Delta x)-g(x)} \right| .$$

But $\lim\limits_{\Delta x \to 0} \left| \frac{g(x)}{g(x+\Delta x)} \right| = \left| \frac{g(x)}{g(x)} \right| = 1$.

Therefore

$$\lim_{\Delta x \to 0} \left| \frac{g(x)}{g(x+\Delta x)} \right| \cdot \left| \frac{\theta\epsilon}{g(x+\Delta x)-g(x)} \right| = \lim_{\Delta x \to 0} \left| \frac{\theta\epsilon}{g(x+\Delta x)-g(x)} \right| .$$

By an application of the Law of the Mean, $g(x+\Delta x)-g(x)$ is equal to $g'(x_1)\Delta x$ where $x < x_1 < x+\Delta x$; and since it may be assumed that $g'(x_1) \neq 0$, it follows readily that

$$\lim_{\Delta x \to 0} \left| \frac{\theta\epsilon}{g(x+\Delta x)-g(x)} \right| = \left| \frac{1}{g'(x_1)} \right| \cdot \lim_{\Delta x \to 0} \left| \frac{\theta\epsilon}{\Delta x} \right| .$$

Inasmuch as

$$g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy) = \theta\epsilon$$

is analytic and "the existence of the derivative of a function at a point requires that the difference

quotient have the same limit no matter how Δx and Δy approach zero,"[1] a mode of approach may be selected of the type $\Delta y = m \Delta x$.

Thus the expression

$$g(x+\tfrac{1}{2}+iy+\Delta x+i\Delta y)-g(x+\tfrac{1}{2}+iy)$$

becomes a function of Δx for any fixed value of x and y and the function will approach zero as Δx approaches zero.

Therefore $\theta\epsilon$ approaches zero in the limit as Δx approaches zero.

In (7), $\theta\epsilon$ is seen to be equal to an analytic expression and so it can be expressed by a Taylor's series in terms of Δx when x and y are fixed since an analytic function can be expressed in a Taylor's series convergent in some region.[2]

Such a series will have no constant term because $\theta\epsilon$ becomes zero as Δx approaches zero.

If $\theta\epsilon$ is expressed as the series,

$$a_1\Delta x + a_2(\Delta x)^2 + a_3(\Delta x)^3 + a_4(\Delta x)^4 + \text{-----},$$

which is the form of its Taylor series development, then the fraction $\left|\frac{\theta\epsilon}{\Delta x}\right|$ becomes the series,

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[1]Curtiss, David Raymond, Analytic Functions of a Complex Variable, p. 47.

[2]<u>Ibid</u>., p. 121.

$$\left| a_1 + a_2 \Delta x + a_3 (\Delta x)^2 + a_4 (\Delta x)^3 + \text{------} \right| .$$

This series converges and will equal zero as Δx approaches zero if a_1 is equal to zero. If a_1 is not equal to zero, the series will equal a constant in the limit.

When the series, and consequently $\left| \frac{\theta\epsilon}{\Delta x} \right|$ is equal to zero,

$$\lim_{\Delta x \to 0} \left| \frac{\theta\epsilon}{g(x+\Delta x)\ g(x)} \right| = \left| \frac{1}{g'(x)} \right| \cdot \lim_{\Delta x \to 0} \left| \frac{\theta\epsilon}{\Delta x} \right| = 0 .$$

And so

$$\left| \frac{g(x+\frac{1}{2}+iy+\Delta x+i\Delta x)-g(x+\frac{1}{2}+iy)}{g(x+\Delta x)-g(x)} \right|$$

$$< 2\,Ke^{\pm(k-1)\pi y} .$$

When a_1 is not equal to zero, and the series becomes equal to a constant when Δx approaches zero this constant can be made less than $2\,Ke^{\pm(k-1)\pi y}$, for no matter what values are chosen for y, K may be chosen large enough to make this condition true.

Therefore the left hand member of (4),

$$\left| \frac{g'(x+\frac{1}{2}+iy)}{g'(x)} \right| ,$$ is less than

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left| \frac{\Delta x}{\Delta x + i\,\Delta y} \right| \cdot 3\,Ke^{\pm(k-1)\pi y}.$$

But if, as before, Δy is chosen equal to m Δx,

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left| \frac{\Delta x}{\Delta x + i\,\Delta y} \right| = \lim_{\Delta x \to 0} \left| \frac{\Delta x}{\Delta x(1+im)} \right| = \left| \frac{1}{1+im} \right|,$$

or

$$\left| \frac{g'(x+\frac{1}{2}+iy)}{g'(x)} \right| < K_1 e^{\pm(k-1)\pi y}.$$

Thus condition (b) of the Theorem is satisfied for the coefficient $g'(n)$.

SECTION II

Thesis: A particularization of the integral expression in the theorem given in Section I.

We shall now consider briefly the function, f(z), defined by the integral expression,

$$(8)\quad f(z) = \int_{-L-\frac{1}{2}}^{\infty} \left\{ g(x) \left[(-1)^k z\right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$- \sum_{m=-L}^{-1} g(m) z^m - \varsigma_k(L, z),$$

wherein L is any arbitrary positive integer, and, whatever the value chosen for it, the expression $\varsigma_k(L, z)$ is such that

$$\lim_{\mathrm{mod}\ z \to \infty} z^L \varsigma_k(L, z) = 0$$

If L is fixed as unity, the integral,

$$\int_{-1-\frac{1}{2}}^{\infty} \left\{ g(x) \left[(-1)^k z\right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx,$$

$$\text{equals} \int_{-\frac{3}{2}}^{\infty} \left\{ g(x) \left[(-1)^k z \right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx.$$

This integral may be expressed as the sum of two integrals; namely,

$$\int_{o}^{\infty} \left\{ g(x) \left[(-1)^k z \right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$+ \int_{-\frac{3}{2}}^{o} \left\{ g(x) \left[(\ 1)^k z \right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx.$$

The second of these two integrals exists and may be denoted by a constant, C_1, for any particular value of k. This statement is apparent when one observes that the limits upon the integral are finite and that the integrand exists at all points of the closed interval represented by the range of integration.

Even at $x=o$ and at $x=-1$ where $\sin \pi x$ in the denominator of the integrand vanishes, the numerator vanishes likewise to an equal order; thus the integrand exists. Moreover, $\left[(-1)^k z \right]^x$ exists for all negative values of x in view of the fact that the absolute value of z is to be taken large. Also $g(w)$, $w=x+iy$, is postulated to be analytic in the finite part of the plane.

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This integral may be [illegible] as the [illegible]

integrals, namely

[illegible]

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of z is [illegible] large. [illegible]

postulated to [illegible]

plane.

The summation $\sum_{m=-l}^{-1} g(m)z^m$ becomes zero and disappears if l=1.

The third part, $\varrho_k(l,z)$, will become zero if z is allowed to approach infinity, so it may be ignored if z becomes sufficiently large. This, then, will be the assumption imposed upon z in the remainder of this discussion.

SECTION III

Thesis: <u>The sum of the series,</u> $\sum_{n=0}^{\infty} g(n)\, z^n$, <u>is equal to the sum of the series,</u> $\sum_{n=0}^{\infty} g'(n)\, z^n$, <u>plus some constant quantity.</u>

It has been proved that if k is a positive integer greater than, or equal to 2, and the restrictions on $g(n)\, z^n$ are the same as those postulated in Section I, that

$$\sum_{n=0}^{\infty} g(n)\, z^n = \int_{0}^{\infty} \left\{ g(x) \left[(-1)^k z\right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$+ \int_{-L-\frac{1}{2}}^{0} \left\{ g(x) \left[(-1)^k z\right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx$$

$$- \sum_{m=-L}^{-1} g(m) z^m - \rho_k(L, z).$$

of which the last three terms either disappear or become constants if L is equal to unity and if z is taken of large modulus.

If, in the above integral, k is made equal to 2, the representation of the series simply becomes

$$(9) \qquad \sum_{n=0}^{\infty} g(n)z^n = \int_0^{\infty} g(x)z^x\,dx + C_1 .$$

When the formula for integration by parts; namely,

$$\int u\,dv = uv - \int v\,du,$$

is applied to the term in the second member of (9) involving the integral wherein u is made equal to g(x) and dv to $z^x dx$, it follows that

$$(10) \qquad \int_0^{\infty} g(x)z^x dx = \frac{g(x)z^x}{\log z}\Big]_0^{\infty} - \frac{1}{\log z} \int_0^{\infty} g'(x)z^x dx.$$

In the fraction, $\frac{g(x)z^x}{\log z}$, if x be permitted to approach infinity, the numerator will approach zero. This follows from the fact that the series

$$\sum_{n=0}^{\infty} g(n)z^n$$

has been assumed at the start to be convergent inside of a circle of infinite radius, and a necessary condition for convergence is that the general term will approach zero as the number of the terms approaches infinity.

If the lower limit, zero, is substituted in the same fraction, it may be seen that

(11) $\quad g(o)z^{o} = A,$

where A is merely the constant term of the series,

$$\sum_{n=o}^{\infty} g(n)z^{n} \; .$$

Since $g'(n)$ has the same properties as $g(n)$, the theorem stated in Section I may be applied to the series

$$\sum_{n=o}^{\infty} g'(n)z^{n} \; ,$$

thus giving

$$\sum_{n=o}^{\infty} g'(n)z^{n} = \int_{o}^{\infty}\left\{g'(x)\left[(-1)^{k}z\right]^{x}\frac{\sin(k-1)\pi x}{\sin\pi x}\right\}dx$$

$$+\int_{-1-\frac{1}{2}}^{o}\left\{g'(x)\left[(-1)^{k}z\right]^{x}\frac{\sin(k-1)\pi x}{\sin\pi x}\right\}dx$$

$$-\sum_{m=-1}^{-1} g(m)z^{m} - \varphi_{k}(1,z).$$

The same statements as those made in regard to expression (8) are applicable here.

Therefore the series,

$$\sum_{n=0}^{\infty} g'(n)z^n ,$$

is equivalent to

$$\int_0^{\infty}\left\{ g'(x) \left[(-1)^k z\right]^x \frac{\sin(k-1)\pi x}{\sin \pi x} dx + C_2 .\right.$$

If k is assumed to be equal to 2 as before, it is apparent that

$$(12) \qquad \sum_{n=0}^{\infty} g'(n)z^n = \int_0^{\infty} g'(x)z^x \, dx + C_2 .$$

When the values from statements (11) and (12) are substituted in (10), it is observed that

$$\int_0^{\infty} g(x)z^x \, dx = -\frac{1}{\log z}\left\{ A + \sum_{n=0}^{\infty} g'(n)z^n - C_2 \right\}.$$

When this value is substituted in equation (9), the original series,

$$\sum_{n=0}^{\infty} g(n)z^n ,$$

is seen to be equivalent to

$$-\frac{1}{\log z}\left\{ A + \sum_{n=0}^{\infty} g'(n)z^n - C_2 \right\} + C_1 \ .$$

In conclusion, we have

$$\sum_{n=0}^{\infty} g(n)z^n = \alpha - \frac{1}{\log z}\left\{ \beta + \sum_{n=0}^{\infty} g'(n)z^n \right\} ,$$

(α and β are constants) provided that the conditions postulated upon the series,

$$\sum_{n=0}^{\infty} g(n)z^n ,$$

are satisfied.

. . . .

SECTION IV

Thesis: Additional Observations.

It may now be observed that in the previous discussion, the limitation placed upon the value of k in the integral,

$$\int_{-1-\frac{1}{2}}^{\infty} \left\{ g(x) \left[(-1)^k z\right]^x \quad \frac{\sin(k-1)\pi x}{\sin \pi x} \right\} dx,$$

has been unduly restrictive. Since k may be any integer greater than or equal to 2, we may, for example, let it be 3.

If k is allowed to be 3, the integral above becomes

$$\int_{-1-\frac{1}{2}}^{\infty} \left\{ g(x) \ (-z)^x \quad \frac{\sin 2\pi x}{\sin \pi x} \right\} dx.$$

Since

$$\sin 2\pi x = 2 \sin \pi x \cos \pi x,$$

this substitution changes the integral under consideration to

$$2 \int_{-1-\frac{1}{2}}^{\infty} g(x) \ (-z)^x \cos \pi x \, dx.$$

[illegible]

[illegible]

cases, the [illegible]

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[illegible]

are been unduly [illegible]

greater than [illegible]

be 0.

[illegible]

[illegible]

Hence

[illegible]

that [illegible]

then [illegible]

[illegible]

When this integral is integrated by parts, the following equation results.

$$2\int_{-l-\frac{1}{2}}^{\infty} g(x)\,(-z)^{x}\ \cos\pi x\,dx$$

$$=\frac{2}{\pi}\Big[g(x)\,(-z)^{x}\ \sin\ \pi x\Big]_{-l-\frac{1}{2}}^{\infty}$$

$$-\frac{2}{\pi}\ \log(-z)\int_{-l-\frac{1}{2}}^{\infty} g(x)\,(-z)^{x}\ \sin\pi x\,dx$$

$$-\frac{2}{\pi}\int_{-l-\frac{1}{2}}^{\infty} g'(x)\,(-z)^{x}\ \sin\pi x\,dx.$$

Another integration by parts will be necessary (if it is desired that k be consistently taken equal to 3) in order that the expressions under the integral signs shall be the same as that in the original integral except for the coefficient, g(x).

By such an integration, it develops that

$$2\int_{-l-\frac{1}{2}}^{\infty} g(x)\,(-z)^{x}\ \cos\pi x\,dx$$

$$=\frac{2}{\pi}\Big[g(x)\,(-z)\ \sin\pi x\Big]_{-l-\frac{1}{2}}^{\infty}$$

$$+\frac{2}{\pi^2}\log(-z)\left[g(x)(-z)^x\cos\pi x\right]_{-1-\frac{1}{2}}^{\infty}$$

$$+\frac{2}{\pi^2}\left[g'(x)(-z)^x\cos\pi x\right]_{-1-\frac{1}{2}}^{\infty}$$

$$-\frac{2}{\pi^2}\log^2(-z)\int_{-1-\frac{1}{2}}^{\infty} g(x)(-z)^x\cos\pi x\,dx$$

$$-\frac{4}{\pi^2}\log(-z)\int_{-1-\frac{1}{2}}^{\infty} g'(x)(-z)^x\cos\pi x\,dx$$

$$-\frac{2}{\pi^2}\int_{-1-\frac{1}{2}}^{\infty} g''(x)(-z)^x\cos\pi x\,dx.$$

By an algebraic manipulation one will observe that

$$2\int_{-1-\frac{1}{2}}^{\infty} g(x)(-z)^x\cos\pi x\,dx$$

$$=\frac{2\pi}{\pi^2+\log^2(-z)}\left[g(x)(-z)^x\sin\pi x\right]_{-1-\frac{1}{2}}^{\infty}$$

$$+\frac{2\log(-z)}{\pi^2+\log^2(-z)}\left[g(x)(-z)^x\cos\pi x\right]_{-1-\frac{1}{2}}^{\infty}$$

$$+ \frac{2}{\pi^2 + \log^2(-z)} \Big[g'(x) \, (-z)^x \, \cos \pi x \Big]_{-1-\frac{1}{2}}^{\infty}$$

$$- \frac{4 \log(-z)}{\pi^2 + \log^2(-z)} \int_{-1-\frac{1}{2}}^{\infty} g'(x) \, (-z)^x \cos \pi x \, dx$$

$$- \frac{2}{\pi^2 + \log^2(-z)} \int_{-1-\frac{1}{2}}^{\infty} g''(x) \, (-z)^x \cos \pi x \, dx.$$

By the argument of Section I, it can be shown that $g''(x)$, the second derivative of $g(x)$, has the same properties as $g'(x)$, and therefore of $g(x)$. Moreover, the entire analysis of the first three sections readily follows, thus permitting the change of the original series under consideration into another series.

Likewise k may be given the value 4, 5, or any other positive integer and essentially the same analysis follows. Of course, that which is theoretically possible may not be practically expedient. The case where k is 2 appears to be very desireable from the practical standpoint and thus has been stressed.

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