5-29-1959

Transient Response of a Mass Mounted on a Nonlinear, Strain-Rate Sensitive Element

Wiley T. Holmes

TRANSIENT RESPONSE OF A MASS - HOLMES

TRANSIENT RESPONSE OF A MASS MOUNTED ON A
NONLINEAR, STRAIN-RATE SENSITIVE ELEMENT

A Thesis

By

Wiley T. Holmes

Submitted in Partial Fulfillment of the
Requirements for the Degree of Master
of Science in Engineering

The University of New Mexico

1959

This thesis, directed and approved by the candidate's committee, has been accepted by the Graduate Committee of the University of New Mexico in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

E F Castetter
DEAN

May 29, 1959
DATE

Thesis committee

R. C. Dove
CHAIRMAN

C. T. Grace

Victor J. Skoglund

TABLE OF CONTENTS

	Page
NOMENCLATURE	iii
INTRODUCTION	1
RELATED INVESTIGATIONS	3
CONCISE STATEMENT OF THE PROBLEM	5
DEVELOPMENT OF SOLUTION	6
Phase-Plane-Delta Analysis	6
Analytical Treatment	14
APPLICATION TO AN ACTUAL PROBLEM	22
RESULTS	27
CONCLUSIONS	28
REFERENCES	29
ACKNOWLEDGEMENTS	31
APPENDIX	32

NOMENCLATURE

x Absolute displacement of mass

$\dot{x} = dx/dt$ Absolute velocity of mass

$\ddot{x} = d^2x/dt^2$ Absolute acceleration of mass

m Mass of system

k Stiffness of restoring element

t Time

ω Circular frequency - radians per second; quasi-natural frequency of system

y, $y(t)$ Ground or base displacement

$\ddot{y} = d^2y/dt^2$ Acceleration of base

z Relative displacement of mass with respect to system base

$\dot{z} = dz/dt$ Relative velocity of mass with respect to system base

$\ddot{z} = d^2z/dt^2$ Relative acceleration of mass with respect to system base

w Weight of mass of system

a_B Acceleration of base in accelerations of gravity

σ Stress

ϵ Strain

$\dot{\epsilon} = d\epsilon/dt$ Rate of strain

A Area

μ One millionth of a quantity

F Force

v Velocity

INTRODUCTION

Numerous treatises on the vibrations of systems with nonlinear spring characteristics, may now be found in the literature. The majority of these deal with either free or forced oscillations under steady state conditions. Only a few investigators however, have examined the response of a nonlinear system to a transient disturbance.

The object of the present transient motion study was not that of investigation of response of a nonlinear system alone, but was projected to include the effect of a strain-rate sensitive, nonlinear restoring element.

With the advent of the relatively recent introduction of the polyurethane and similar plastic materials, this problem has become increasingly important. For these materials, widely used for shock mitigating, purposes; exhibit strain-rate sensitivity to an important degree. Thus a comparatively new field of research has presented itself and thus far appears to have received little attention in the literature.

At present, designers are confronted with numerous problems involving transient motions. The response of the system to these transients may easily be catastrophic if satisfactory methods are not available for their prediction. These problems appear in a variety of physical applications. For instance, (a) the protection of electronic apparatus, (b) the prevention of failures in structural members, and (c) the attempt, by eliminating the shock hazard, to gain assurance of continued operation of mechanical devices. The damaging phenomenon is invariably excessive acceleration caused by improper selection of energy absorbing or energy

Numerous treatments [illegible] with [illegible] characteristics, [illegible] may be found in the [illegible] literature. [illegible] these deal with either free vibrations or [illegible] conditions. Only a few investigators [illegible] have [illegible] the response of a nonlinear system to a transient disturbance.

The object of the present investigation [illegible] investigation of response of a [illegible] to include the effect of [illegible]-rate sensitivity, nonlinear [illegible] element.

With the advent of the relatively recent introduction of [illegible] thene and similar plastic materials, this problem has become [illegible] important. For these materials [illegible] exhibit strain-rate sensitivity [illegible] tively new field of research [illegible] have received little attention in the literature.

At present, designers are confronted with numerous [illegible] transient motions. The response of [illegible] easily be catastrophic if [illegible] their prediction. These [illegible] tions. For instance, (a) [illegible] prevention of failures in [illegible] and (b) the [illegible] eliminating the shock hazard [illegible] of mechanical devices. [illegible] acceleration caused by [illegible] storing elements.

The crux of the problem is then, to develop a method that has the following merits:

(a) Generality - It should be sufficiently general to cover a large variety of cushioning elements with dissimilar stress-strain characteristics, and should not be limited as to the form and duration of the transient impulse.

(b) Simplicity - It should be relatively simple to apply, so that one may obtain solutions as quickly as possible.

(c) Reliability - It must be reliable so that it may be used with confidence.

This thesis develops a method that had as its primary objectives, these ideals. The factors of generality and reliability are present to a large degree. Considering the complexity of the problem, the method may be said to be reasonably simple.

one" (by the authors) and it was applied to several nonlinear systems of different characteristics in combination with various shapes of applied pulses. No significant work of similar nature seems to have been accomplished until 1952 when Jacobsen (6) improved the phase-plane method and extended its application to even a broader variety of problems.

The majority of the work on strain-rate sensitive systems, outside of material properties testing, seems to appear in the fields of creep and relaxation. Several investigators (7) have performed theoretical analyses of nonlinear strain-rate sensitive elements in these areas. The method usually considers the use of rheological models (such as those of Maxwell, Voight, and Kelvin) to describe the phenomena. No work, with one exception (8), on the response of nonlinear, strain-rate sensitive systems to a transient disturbance was discovered in the literature research of the writer.

ing" (by the author) [illegible]

of different [illegible]

applied pulses. [illegible]

been accomplished until [illegible]

method and extended [illegible]

The majority of the [illegible]

of material properties testing, [illegible]

and relaxation. [illegible]

analyses of nonlinear [illegible]

The method usually [illegible]

of Maxwell, Voight, and [illegible]

one exception (3), on the response [illegible]

systems to a transient [illegible]

research of the writer.

CONCISE STATEMENT OF THE PROBLEM

The general problem has been described in the fore-going text. More specifically, the problem is: (See Fig. 1)

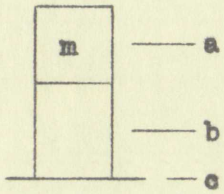


Fig. 1

Predict the response of a mass (a), mounted on a nonlinear rate-sensitive element (b), to an acceleration pulse applied at the base (c). The stress-strain relations (at various rates) of element (b), and the magnitude of the pulse as a function of time having been previously determined. However, the method to be described is general, as previously stated, and is not restricted to this particular case but may be applied to many other cases.

DEVELOPMENT OF SOLUTION

Phase-Plane-Delta Analysis

The phase-plane-delta method of graphical solution will not be discussed in great detail here. For a detailed study, one is referred to the references previously cited. Some review of the method is necessary however, in order to clearly define its application to the present problem.

In the phase-plane-delta method a point in the phase plane is determined by the usual coordinates $(\dot{x}/\omega, x)$, where $\dot{x}$ is velocity, $\omega = \sqrt{k/m}$, and x is displacement. The phase trajectories, a step-wise process, are described by a series of circular arcs with centers δ, located on the displacement axis only. It will be seen how δ is defined, as the principles are developed.

First write the differential equation describing the oscillatory motion as,

$$m\ddot{x} + f(\dot{x},x,t) + kx = 0 \; . \qquad (a)$$

After dividing by m, and setting $k/m = \omega^2$, one has,

$$\ddot{x} + f(\dot{x},x,t)/m + \omega^2 x = 0 \; . \qquad (b)$$

The above equation may be rewritten as,

$$\ddot{x} + \omega^2(x + \delta) = 0 \qquad (c)$$

where the second parameter in the displacement term is,

$$\delta = f(\dot{x},x,t)/k \; .$$

Introducing the phase plane coordinates

$$x = x; \quad \dot{x}/\omega = v$$

The phase-plane [illegible] will be [illegible]

discussed in [illegible] to the [illegible] [illegible]

necessary however, in order [illegible] the [illegible] of the

present problem.

In the phase-plane [illegible] in the phase plane [illegible]

determined by the usual coordinates [illegible] where [illegible] is the

$\omega = \sqrt{k/m}$, and x is displacement. [illegible] otherwise

process, are described by [illegible] with centers [illegible]

located on the displacement axis only. [illegible]

as the principles are developed.

First write the differential equation [illegible] the conditions [illegible]

motion as,

[illegible]

After dividing by m, and setting [illegible]

[illegible]

The above equation may be rewritten as,

[illegible]

where the second parameter in the displacement term is

[illegible]

Introducing the phase-plane [illegible]

[illegible]

and changing the independent variable,

$$\ddot{x} = \omega\, dv/dt = \omega(dv/dx)(dx/dt) = \omega^2 v\, dv/dx \;.$$

Equation (c) can be written as,

$$dx/dv = -v/(x + \delta) \;. \qquad \text{(d)}$$

Integrating Equation (d), (δ is treated as a constant throughout any particular step), gives,

$$(x + \delta)^2 + v^2 = C \;.$$

Now let $C = r^2$, and Equation (e) is seen to be the equation of a circle of radius, $r = \sqrt{(x + \delta)^2 + v^2}$, with its center located at $(x + \delta)$. Fig. 2 illustrates the geometrical concepts evolved from Equations (d) and (e).

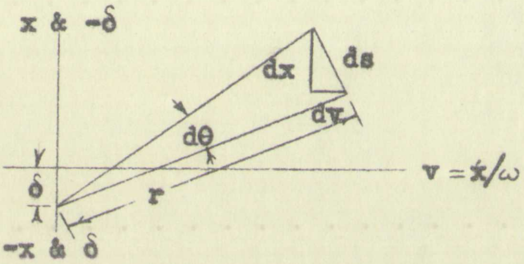


Fig. 2

Considering the geometry of Fig. 2,

$$d\theta = ds/r = \sqrt{dx^2 + dv^2} \Big/ \sqrt{v^2 + (x + \delta)^2} = \left[\sqrt{1 + (dv/dx)^2} \Big/ \sqrt{1 + (x + \delta/v)^2}\right](dx/v) = dx/v$$

which can be related to the time element by writing $\dot{x} = \omega v$ as,

$$dt = (1/\omega)\cdot(dx/v) = (1/\omega)d\theta \;. \qquad \text{(f)}$$

Integrating Equation (f) for the duration of a step, one has,

$$t_{n+1} - t_n = (1/\omega)(\theta_{n+1} - \theta_n) \;.$$

For the orientation of axes used, increasing time is proportional to the counterclockwise angular variation of the phase trajectory normal.

and changing the independent variable [illegible]

Equation (a) can be written as [illegible]

[illegible]

Integrating Equation (d) [illegible] gives a [illegible] particular [illegible]

[illegible]

Now let [illegible], and Equation (e) [illegible] of radius, [illegible] Fig. 2 illustrates the [illegible] (d) and (e).

Fig. 2

Considering the geometry of Fig. 2,

[illegible]

[illegible]

which can be related to [illegible]

[illegible]

Integrating Equation [illegible] for the [illegible]

[illegible]

For the orientation [illegible] counterclockwise [illegible]

All of the parameters necessary for the stepwise construction of a graphical solution of Equation (c) have been reviewed.[1] Application to the present problem is now considered.

Referring to the system in Fig. 3,

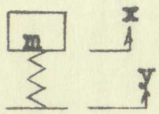


Fig. 3

Equation (c) may be written,

$$\ddot{x} + \omega^2(x - y(t) + \delta) = 0 \qquad (g)$$

where, x - is displacement of the mass

y(t) - is the base displacement acting on the mass through the restoring element

Introducing the relation,

$z = y - x$, where z is the displacement of the mass relative to the base

into Equation (7) one obtains,

$$(\ddot{y} - \ddot{z}) + \omega^2(y - z - y + \delta) = 0$$

$$-\ddot{z} + \omega^2(-z + \delta + \ddot{y}/\omega^2) = 0$$

$$\ddot{z} + \omega^2(z - [\delta + \ddot{y}/\omega^2]) = 0 \qquad (h)$$

so that $\delta_{eff} = \delta + \ddot{y}/\omega^2$.

Now if y is expressed in g's (accelerations of gravity) instead of units of length/sec.^{2}, one has,

---

[1]Equations (a) through (f) were taken from reference (6), where a discussion in even greater detail is given.

All of the [illegible]
practical [illegible]
the present problem [illegible]

Referring to the [illegible]

[illegible]

Equation (6) may be written

[illegible]

where, x - is displacement of the [illegible]

y(t) - is the base displacement [illegible] mass

(through the [illegible])

Introducing the relation

[illegible]

[illegible]

into Equation (7) one obtains

[illegible]

[illegible]

[illegible]

[illegible]

Now if y is expressed in [illegible]

units of length [illegible]

---

[1] Equations (4) through (7) [illegible]
discussion in [illegible]

$$y/\omega^2 = a_s g/\omega^2 = a_s g/(kg/w) = a_s w/k \qquad (j)$$

where, a_s - is acceleration of the base in g's

w - is weight of the mass in pounds

and Equation (8) becomes,

$$\ddot{z} + \omega^2(z - [\delta + a_s/k]) = 0 \qquad (k)$$

Equation (h) or Equation (k) is used to solve the system in Fig. 2 the choice of the equation to be used being governed by the acceleration units in which the driving term is expressed.

The phase-plane-delta method is especially suited to the present problem. It presents graphically the variation of velocity with displacement. This is precisely the principle made use of in the present method.

The method will be described with the aid of Fig. 4 and Fig. 5.

Consider the set of stress-strain curves, for a rate sensitive element, shown in Fig. 4.

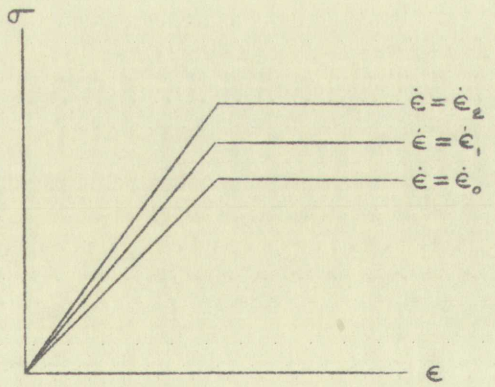


Fig. 4

These curves are first converted to load-displacement curves at various velocities. Fig. 5 shows the load-displacement curves and a phase-plane

[illegible]

where [illegible]

[illegible]

and Equation (8) becomes

[illegible]

Equation (b) or Equation (c) [illegible]

P the choice of the [illegible]

acceleration units in which [illegible]

The phase-plane-delta method is [illegible] to the present prob-

lem. It presents graphically the variation of velocity with displacement.

This is precisely the principle made use of in the present method.

The method will be described with the aid of [illegible]

Consider the [illegible]

element, shown in Fig. [illegible]

Fig. [illegible]

These curves are first constructed [illegible]

velocities. Fig. [illegible]

diagram of the response of a mass, mounted on this element, to an impulse.

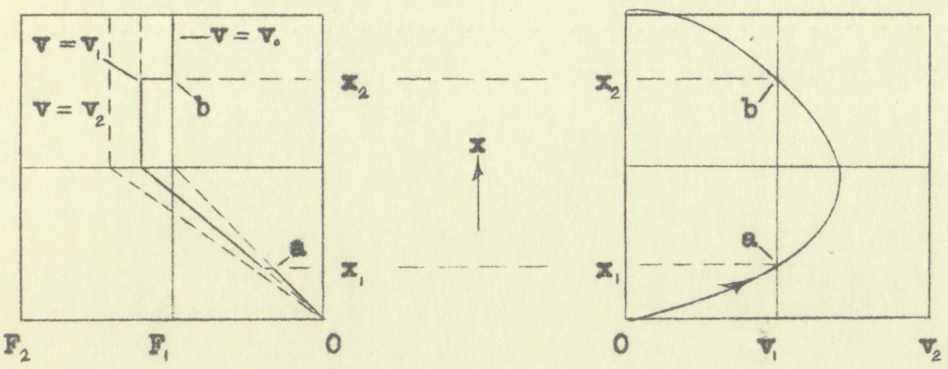


Fig. 5

The set of load-displacement curves have been rotated 90° so that the displacement will correspond to the displacement on the phase diagram. Referring to Fig. 5, the restoring element is assumed to follow the static straining curve ($\dot{e}=\dot{e}_0$ or $v = v_0$) until the phase trajectory indicates the value $v = v_1$. A jump (indicated by 'a' in Fig. 5) is then made to the load-displacement curve of rate $v = v_1$. This curve is followed to point 'b' where $v = v_1$, since the phase trajectory shows that the mass does not reach the velocity $v = v_2$. A jump is now made back to the static curve, which the restoring element follows until the mass reaches maximum displacement.

The above procedure is repeated as often as necessary and the jumps are made to curves of increasing or decreasing strain-rates accordingly, as to whether the response velocity is increasing or decreasing. If a judicious choice of jumps determined by a consideration of the geometry of the load-displacement curves and the phase-plane trajectory is made, the method will give good accuracy.

In the solution, the phase-plane-delta analysis proceeds in the usual manner. An initial load-displacement curve is selected and from it and the mass quantity, k and ω are determined. It should be pointed out that the initial curve may not necessarily be the static one, for if an applied pulse attains large acceleration magnitudes in a small time interval, the phase trajectory for some systems may indicate a sharp increase of velocity, initially.

Application of the method will be further illustrated by applying it to a hypothetical problem.

Consider the set of load-displacement curves shown in Fig. 6.

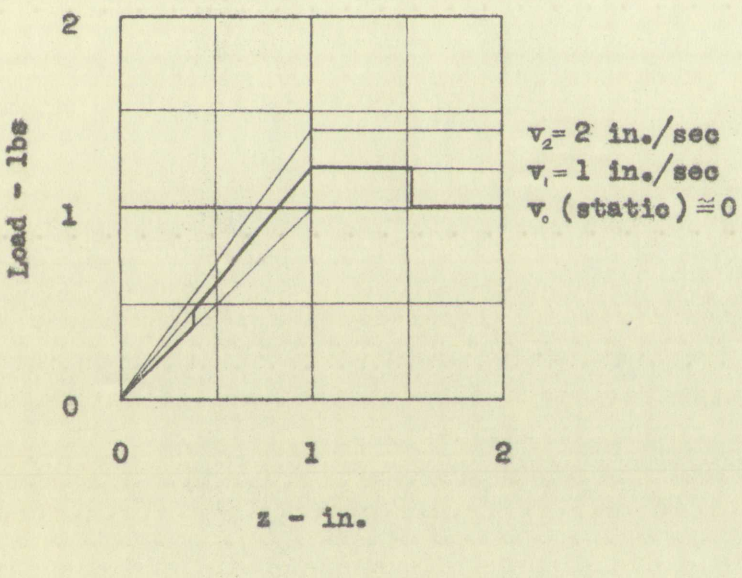


Fig. 6

and let these apply to the system shown in Fig. 3. Now determine the response of a mass, $m = 1$, when a semi-sinusoidal pulse, $\ddot{y} = 2$ sine $2t$ in./sec^2. is applied to the base.

Equation (h) is seen to be the appropriate one for this case, and

In the [illegible]

usual manner [illegible]

it and the [illegible]

out that the [illegible]

an applied pulse [illegible]

internal, the [illegible]

increase of velocity [illegible]

Application [illegible]

it to a hypothetical [illegible]

Consider the [illegible]

[illegible]

and let these apply to the [illegible]

response of a mass, [illegible]

in./sec², is applied to the [illegible]

Equation (6) is [illegible]

the phase-plane construction is shown in Fig. 7. The details of plotting will be omitted here.

Referring to Fig. 7, note that the load-displacement curve that the system will follow initially, will be the static one, which it continues to follow until the trace of the phase trajectory reaches a value $\dot{z}/\omega = 1$ in./ sec. A jump is now made to the load-displacement curve of rate $v_1 = 1$ in./sec. As the phase diagram proceeds using this curve, it is readily seen that insufficient velocity will be attained to cause a further jump to another rate curve before the yield point is reached. After the yield point, velocity gradually decreases to 1 in./sec. where a jump is made back to the static curve.

The absolute acceleration of the supported mass may be obtained by either of two methods. First by relating the time back to the load-displacement curve used in the construction of the phase-plane diagram. Using Newton's second principle, the force is then divided by the mass of the oscillatory system, and one obtains the absolute acceleration as a function of time. This method is illustrated in Fig. 8 below.

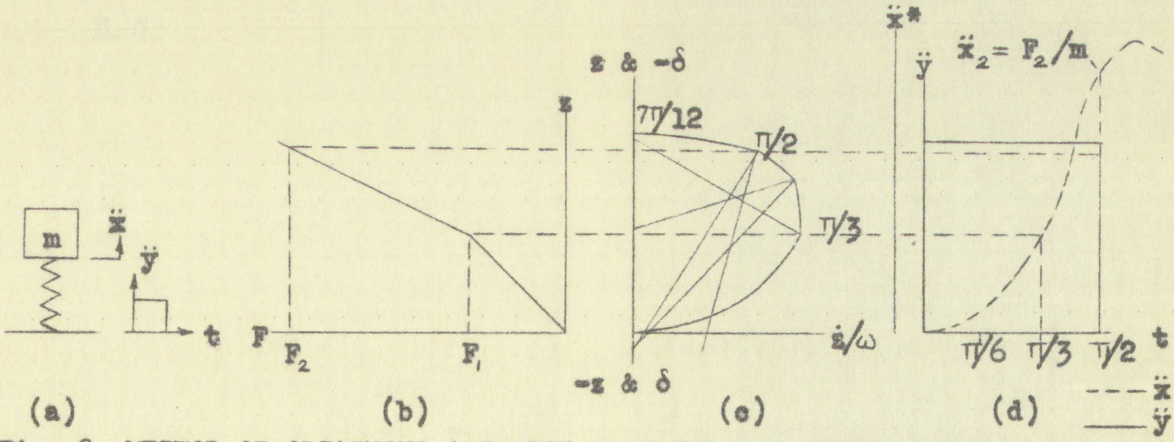
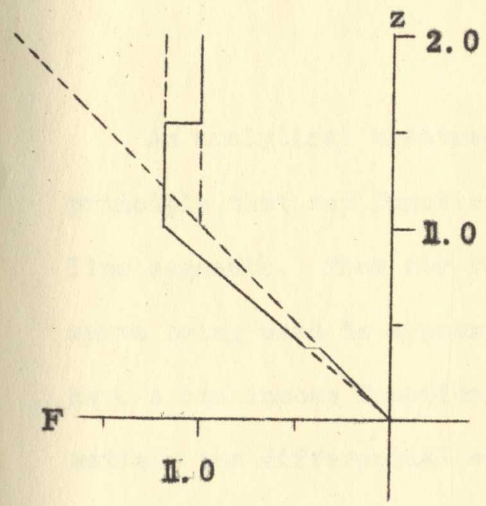


Fig. 8 METHOD OF OBTAINING ABSOLUTE ACCELERATION DIRECTLY FROM PHASE DIAGRAM

*Note - $\ddot{x}$ and $\ddot{y}$ will have different scales.


z
2.0
1.0
F
1.0

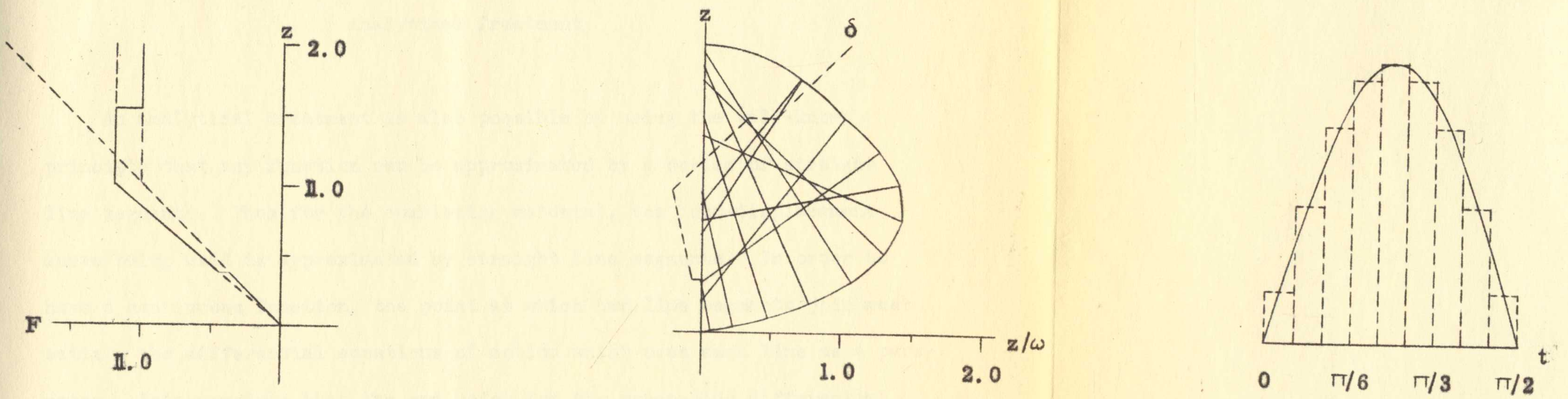


Fig. 7 PHASE-PLANE DIAGRAM OF HYPOTHETICAL PROBLEM

In this illustration a square wave is applied to the base of a system (Fig. 8a) with a bilinear spring (Fig. 8b). The resulting acceleration, $\ddot{x}$, of the mass is shown in Fig. 8d.

The second method of obtaining the same result is as follows: The phase trajectory will give directly, relative velocity as a function of time. This may be graphically differentiated to give relative acceleration. Then absolute acceleration may be obtained by the formula,

$$\ddot{x}=\ddot{y} - \ddot{z} .$$

Analytical Treatment

An analytical treatment is also possible by using the well-known principle that any function can be approximated by a series of straight line segments. Thus for the cushioning material, the load-displacement curve being used is approximated by straight line segments. In order to have a continuous function, the point at which two line segments join must satisfy the differential equations of motion which uses each line as a parameter. This requires that the end point for the preceeding differential equation determine the boundary conditions for the next differential equation.

This method is recommended only as a check for the phase-plane analysis or perhaps as a more accurate analysis to be used after the phase-plane method is accomplished. For one must plot displacement versus velocity in order to determine the steps for a rate sensitive system. Furthermore, the solution of a transcendental equation (sometimes tedious) is involved

In this illustration a step-curve is applied to the case of a system (Fig. 5a) with a nonlinear spring (Fig. 5b). The resulting acceleration, $\ddot{x}$, of the mass is shown in Fig. 5c.

The second method of obtaining the same result is as follows: The phase trajectory will give directly, relative velocity as a function of time. This may be graphically differentiated to give relative acceleration. Then absolute acceleration may be obtained by the formula.

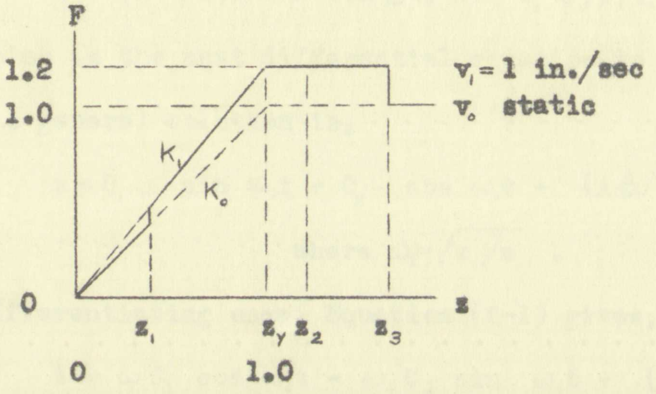
$$a_a = \ddot{x} .$$

Analytical Treatment

An analytical treatment is also possible by using the well-known principle that any function can be approximated by a series of straight line segments. Thus for the cushioning material, the load-displacement curve being used is approximated by straight line segments. In order to have a continuous function, the point at which two line segments join must satisfy the differential equations of motion which uses each line as a parameter. This requires that the end point for the preceeding differential equation determine the boundary conditions for the next differential equation.

This method is recommended only as a check for the phase-plane analysis or perhaps as a more accurate analysis to be used after the phase-plane method is accomplished. For one must plot displacement versus velocity in order to determine the slope for a rate sensitive system. Furthermore, the solution of a transcendental equation (sometimes tedious) is involved for determining boundary conditions for some of the equations.

The method will be applied to the same problem used in the preceeding analysis so that the results of the two methods may be compared. The parameters are repeated in Fig. 9.



$m = 1$

$A = 2$

$\beta = 2$

$\omega_0 = \sqrt{k_0/m} = 1$

$\omega_1 = \sqrt{k_1/m} = \sqrt{1.2} = 1.09$

Fig. 9

The first differential equation that governs the relative motion is,

$$\ddot{z} + (k_0/m)z = A \sin\beta t \text{ , for } |z| < z_1 \qquad \text{(a-1)}$$

which has as its solution

$$z = (A/(\omega^2 - \beta^2))(\sin \beta t - (\beta/\omega) \sin \omega t) \qquad \text{(b-1)}$$

when the boundary conditions are $z(0) = 0$, $\dot{z}(0) = 0$. Taking the first derivative with respect to t, of Equation (b-1) and substituting the problem parameters, one obtains,

$$\dot{z} = -(4/3)(\cos 2t - \cos t) \qquad \text{(c-1)}$$

which becomes

$$1 - (4/3) \cos t = -(4/3) \cos 2t$$

or

$$4 \cos t - 3 = 4 \cos 2t \qquad \text{(d-1)}$$

after z_1 1 is substituted. The value of the argument t, of immediate interest,[1] that satisfies Equation (d-1) is, $t_1^* = 46.9°$. Substituting this value and the problem parameters in Equation (b-1) one has,

$$z_1 = -(2/3)(\sin 2(46.9°) - 2 \sin 46.9°) = 0.309 \; .$$

$\dot{z}(t_1) = 1$, and $z(t_1) = 0.309$, become the boundary conditions for

$$\ddot{z} + (k_1/m)z = A \sin \beta t, \text{ for } z_1 < |z| < z_y = 1 \qquad \text{(e-1)}$$

which is the next differential equation to be considered.

Its general solution is,

$$z = C_1 \cdot \sin \omega_1 t + C_2 \cdot \cos \omega_1 t + (A/(\omega^2 - \beta^2)) \sin \beta t \qquad \text{(f-1)}$$

where $\omega_1 = \sqrt{k_1/m}$.

Differentiating once, Equation (f-1) gives,

$$\dot{z} = \omega_1 C_1 \cos \omega_1 t - \omega_1 C_2 \sin \omega_1 t + (A\beta/(\omega^2 - \beta^2)) \cos \beta t \; . \qquad \text{(g-1)}$$

From Equations (f-1) and (g-1) respectively, one gets the set,

$$0.779 \, C_1 + 0.627 \, C_2 = 1.007$$

$$0.684 \, C_1 - 0.850 \, C_2 = 0.693$$

after the problem parameters and boundary conditions are substituted.

From which,

$$C_1 = 1.18 \; ; \; C_2 = 0.126.$$

The solution of (f-1) is then,

$$z = 1.18 \sin 1.09t + 0.126 \cos 1.09t - 0.714 \sin 2t,$$

$$\text{for } 46.9° < t < t_y \; . \qquad \text{(h-1)}$$

---

[1]Other roots of Equation (d-1) are not of interest for they will violate the limits for z in Equation (e-1).

*The times t_1, t_y, t_2, and t_3 will correspond respectively to the displacements z_1, z_y, z_2, and z_3.

after x = 1 is substituted. The [illegible] [1]

interest, then substituting Equation (F-1) [illegible] [illegible]

this value and the problem parameters in Equation F-1, one gets

$z_p = (P/3)(\sin 0.1 [illegible]) - 2 \sin [illegible]$

$z(t_1) = 1$, and $\dot{z}(t_1) = 0.275$, because the nonhomogeneous equation

$\ddot{z} + (k_e/m)z = A \cos 2t$, for $t_1 \leq t \leq t_2$ [2]

which is the next differential equation to be considered.

Its general solution is,

$z = C_1 \sin \omega_1 t + C_2 \cos \omega_1 t + [illegible] \sin 2t$

where $\omega_1 = \sqrt{k_e/m}$

Differentiating once, Equation (F-1) gives,

$\dot{z} = \omega_1 C_1 \cos \omega_1 t - \omega_1 C_2 \sin \omega_1 t$ [illegible]

From Equations (F-1) and (F-1) respectively, one gets

$0.779\,C_1 + 0.627\,C_2 = 1.097$

$0.691\,C_1 - 0.590\,C_2 = [illegible]$

after the problem parameters and boundary conditions are substituted.

From which,

$C_1 = 1.15$, $C_2 = [illegible]$

The solution of (F-1) is then,

$z = 1.15 \sin 1.09t + [illegible] \cos 1.09t - 0.731 \sin 2t$

for $t_1 \leq t \leq t_2$

---

[1] Other roots of Equation (G-1) are not of interest, for they will violate the limits for x in Equation (F-1).

[2] The times t_1, t_2, t_3, and t_4 will correspond respectively to the displacements x_1, x_2, x_3, and x_4.

Equation (h-1) is now used to determine the boundary conditions for the succeeding differential equaltion. Substituting $z(t_y) = z_y = 1$, one gets,

$$1 - 1.18 \sin 1.09t = 0.126 \cos 1.09t - 0.714 \sin 2t \qquad (j-1)$$

from which $t_y = 82.6^\circ$. Differentiating Equation (h-1) and substituting the value t_y yields,

$$\dot{z}(t_y) = 1.285 \cos 90^\circ - 0.1375 \sin 90^\circ - (4/2.8) \cos 165.2^\circ = 1.243 \ .$$

The differential equation that now governs the motion is,

$$\ddot{z} + (k_1/m)z_y = A \sin \beta t \ , \text{ for } z_y < |z| < z_2 \qquad (k-1)$$

which has as its general solution,

$$z = -(A/\beta^2) \sin\beta t - \omega_1^2 z_y(t^2/2) + C_1 t + C_2,$$

$$\text{for } t_y < t < t_2 = {}^*90^\circ \qquad (1-1)$$

Differentiation of Equation (1-1) gives,

$$\dot{z} = -(A/\beta) \cos\beta t - \omega_1^2 z_y t + C_1, \text{ for } t_y < t < 90^\circ \, (m-1)$$

Substituting, $z(t_y) = 1$, $\dot{z}(t_y) = 1.243$, ($t_y = 82.6^\circ = 1.442$ radians) into Equations (1-1) and (m-1) respectively, one has,

$$1 = -0.500 \sin 165.2^\circ - 0.721(1.442)(1.2) + 1.442 C_1 + C_2$$

$$1.243 = -\cos 165.2^\circ - 1.2(1.442) + C_1$$

from which, $C_1 = 2.01$, $C_2 = -0.524$ and the solution becomes,

$$z = -0.500 \sin 2t - 0.60t^2 + 2.01t - 0.524 \ ,$$

$$\text{for } t_y < t < t_2 = 90^\circ \qquad (n-1)$$

The boundary conditions for the next differential equation describing the motion are obtained from Equation (n-1) and the equation resulting from taking its first Newtonian derivative, and substituting $t_2 = 90^\circ$ in

---

*Applied impulse vanishes at t_2.

Equation (4-1) is now expressed in terms of [illegible]

[illegible] differential equation [illegible]

I = 1.15 sin (2.00t + 0.12) cos 3.54t [illegible]

From which [illegible] Differentiating [illegible] substituting

the values [illegible] yields,

[illegible]

The differential equation thereby [illegible]

[illegible]

which has as its general solution,

[illegible]

[illegible]

Differentiation of equation [illegible] yields,

[illegible]

Substituting [illegible] into

equations (4-1) and [illegible] respectively, one has*

[illegible]

[illegible]

from which, [illegible] and the solution becomes,

[illegible] = 0.500 sin 2t [illegible]

[illegible]

The boundary conditions for the [illegible]

the motion are obtained from Equation (4-1) and the [illegible]

from [illegible]

---

*Applied [illegible]

each. The resulting equations are,

$$z_2 = -0.60(1.57)^2 + 2.01(1.57) - 0.524 = 1.16$$

$$\dot{z}_2 = 1 - 1.20(1.57) + 2.01 = 1.13$$

Now the differential equation to be used is,

$$\ddot{z} + (k_1/m)z_y = 0 \text{ , for } z_2 < |z| < z_3 \qquad \text{(o-1)}$$

which has as its solution,

$$z = -\omega_1^2 z_y (t^2/2) + C_1 t + C_2 \qquad \text{(p-1)}$$

and the velocity equation becomes

$$\dot{z} = -\omega_1^2 z_y \; t + C_1 \qquad \text{(q-1)}$$

from which after substituting $\dot{z}(t_2) = 1.13$, $C_1 = 3.01$,

and from Equation (p-1), using $z(t_2) = 1.16$, one has,

$$1.16 = -0.60(1.57)^2 + 3.01(1.57) + C_2$$

from which $C_2 = -2.08$, and the particular solution of Equation (o-1) is,

$$z = -0.60t^2 + 3.01t - 2.08 \text{ , when } 90^\circ < t < t_3 . \qquad \text{(r-1)}$$

Using $\dot{z}(t_3) = 1$ in Equation (q-1) and substituting the value for C_1 along with the problem parameters, one has,

$$1.00 = -1.2t_3 + 3.01$$

from which $t_3 = 1.67$ radians $= 96^\circ$.

Likewise from Equation (p-1),

$$z = -0.60(1.67)^2 + 3.01(1.67) - 2.08 = 1.27$$

after substituting t_3, the value for C_2, and the problem parameters. $z(t_3) = 1.27$ and $\dot{z}(t_3) = 1$ are boundary conditions for the differential equation,

$$\ddot{z} + (k_0/m)z_y = 0 \text{ , when } z_3 < |z| \qquad \text{(s-1)}$$

which has the general solution

$$z = -\omega_0^2 z\ (t^2/2) + C_1 t + C_2 \qquad \text{(t-1)}$$

and the expression for the velocity is,

$$\dot{z} = -\omega_0^2 z_y t + C_1 \qquad \text{(u-1)}$$

from which, after substituting $\dot{z}(t_3) = 1$ and $\omega_0 = 1$, $C_1 = 2.67$.

Using Equation (t-1) and $z(t_3) = 1.27$, one obtains,

$$1.27 = -0.50(1.67)^2 + 2.67(1.67) + C_2$$

from which $C_2 = -2.80$ and the particular solution of Equation (s-1) is,

$$z = -0.50t^2 + 2.67t - 2.80,$$

$$\text{for } t > 96^\circ = 1.67 \text{ radians.} \qquad \text{(v-1)}$$

Sufficient equations for the solution of the response of the mass in terms of displacement, velocity, or acceleration, have been developed. The solution of this problem will consider only, the absolute acceleration of the mass.

The equations for relative acceleration are:

$$\ddot{z} = (4/3)\ (2 \sin 2t - \sin t), \text{ for } 0 < t < 46.9^\circ \qquad \text{(a-2)}$$

$$\ddot{z} = 4(0.714) \sin 2t - (1.09)^2(1.18 \sin 1.09t$$

$$+ 0.126 \cos 1.09t), \text{ for } 46.9^\circ < t < 82.6^\circ \qquad \text{(b-2)}$$

$$\ddot{z} = 2 \sin 2t - 1.2, \text{ for } 82.6^\circ = 1.44 < t < 90^\circ = 1.57 \qquad \text{(c-2)}$$

$$\ddot{z} = -1.2, \text{ which holds for } 90^\circ = 1.57 < t < 96^\circ = 1.67 \qquad \text{(d-2)}$$

although t does not appear in the equation.

$$\ddot{z} = -1, \text{ when } 96^\circ = 1.67 < t \qquad \text{(e-2)}$$

Equation (a-2) is obtained by differentiating once, Equation (c-1). Equations (b-2), (c-2), (d-2), and (e-2) are obtained by which has the general solution.

$$x = -\omega_2^2 (t^2/2) + C_1 t + C_2 \qquad \text{(f-1)}$$

and the expression for the velocity is,

$$\dot{x} = -\omega_2^2 t + C_1 \qquad \text{(t-1)}$$

from which, after substituting $\dot{x}(t_1) = 1$ and $\omega_2 = 1$, $C_1 = 2.67$.

Using Equation (f-1) and $x(t_1) = 1.27$, one obtains,

$$1.27 = -0.50(1.67)^2 + 2.67(1.67) + C_2$$

from which $C_2 = -2.80$ and the particular solution of Equation (e-1) is,

$$x = -0.50t^2 + 2.67t - 2.80,$$

$$\text{for } t > 96° = 1.67 \text{ radians.} \qquad \text{(e-1)}$$

Sufficient equations for the solution of the response of the mass in terms of displacement, velocity, or acceleration, have been developed. The solution of this problem will consider only, the absolute acceleration of the mass.

The equations for relative acceleration are:

$\ddot{x} = (4/3)$ (2 sin 2t − sin t), for $0 < t < 46.9°$ (a-2)

$\ddot{x} = 4(0.714)$ sin 2t − $(1.09)^2$(1.18 sin 1.09t

+ 0.126 cos 1.09t), for $46.9° < t < 82.5°$ (b-2)

$\ddot{x} = 2$ sin 2t − 1.2, for $82.5° = 1.44 < t < 90° = 1.57$ (c-2)

$\ddot{x} = -1.2$, which holds for $90° = 1.57 < t < 96° = 1.67$ (d-2)

although $\ddot{x}$ does not appear in the equation.

$\ddot{x} = -1$, when $96° = 1.67 < t$ (e-2)

Equation (a-2) is obtained by differentiating once, Equation (c-1). Equations (b-2), (c-2), (d-2), and (e-2) are obtained by differentiating twice, Equations (h-1), (o-1), (r-1), and (v-1) respectively.

The absolute acceleration is calculated from the formula,

$$\ddot{x} = \ddot{y} - \ddot{z} .$$

Fig. 10 shows two curves of absolute acceleration of the supported mass versus time, one obtained from the phase-plane analysis of Fig. 7 and the other by the analytical approach. The two curves are in good general agreement. The slight disagreement is caused by the more pronounced effect of the jumps in the load-displacement curve of the phase-plane solution, since a piecewise continuous smooth load-displacement curve without jump discontinuities was assumed in the analytical treatment.

differentiating twice, equations [illegible] and [illegible] res-
pectively.

The absolute [illegible] is calculated from the [illegible]

$x = $ [illegible]

Fig. 10 shows the [illegible] as function of the [illegible] mean versus time, [illegible] calculated from the theory [illegible] and the other by the [illegible] general agreement. The [illegible] discrepancy [illegible] more pronounced effect of the jump in the [illegible] of the [illegible] plane solution, [illegible] curve without jump discontinuities was assumed in [illegible] treatment.

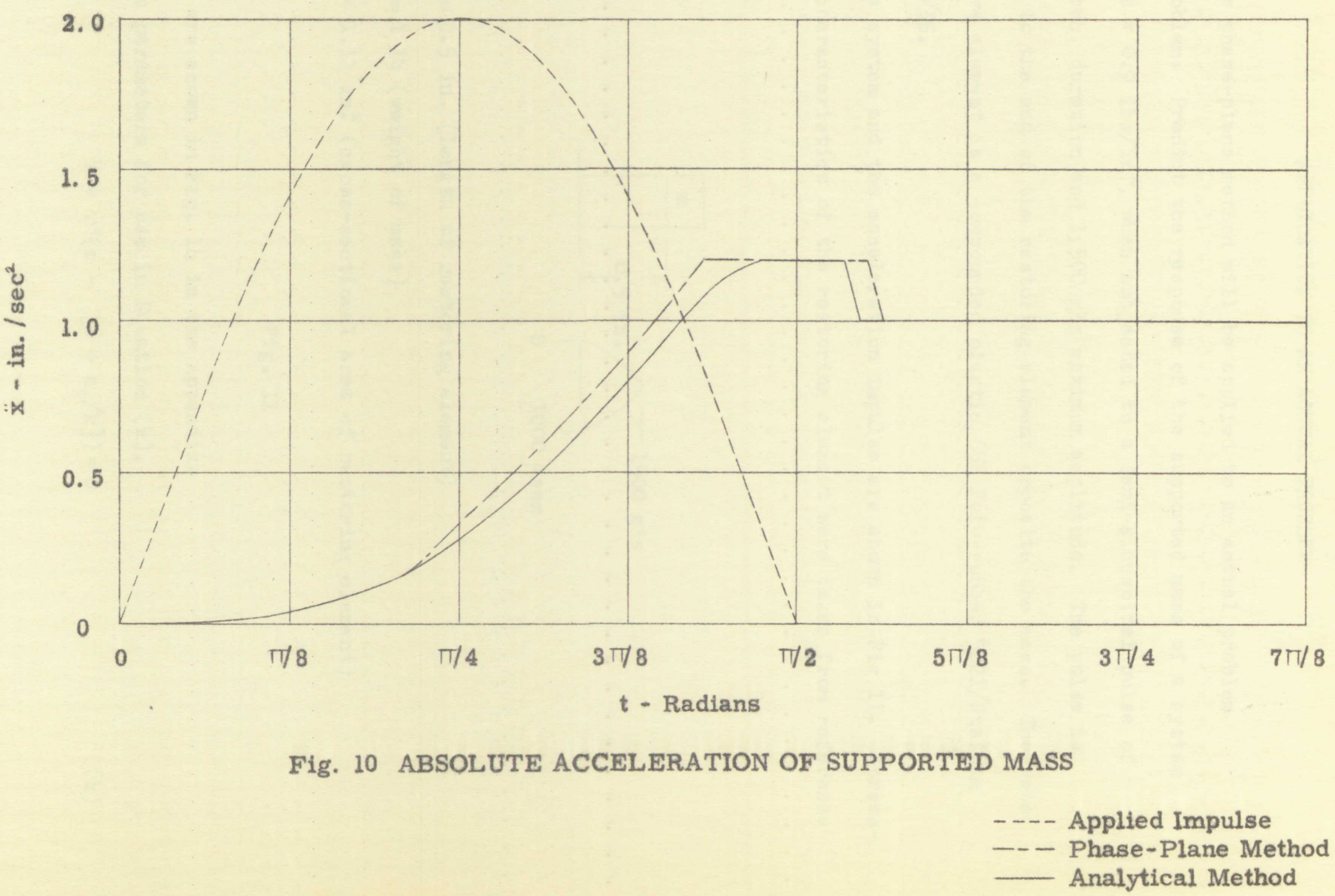


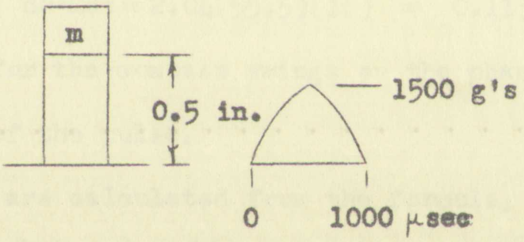
Fig. 10 ABSOLUTE ACCELERATION OF SUPPORTED MASS

APPLICATION TO AN ACTUAL PROBLEM

The phase-plane method will be applied to an actual problem.

Problem: Predict the response of the supported mass of a system with $w/A = 0.9$ lbs/in^2, when subjected to a semi-sinusoidal pulse of 1,000 μ sec. duration and 1,500 g's maximum amplitude. The pulse is applied to the end of the restoring element opposite the mass. The rate sensitive element is a compounded plastic, 65% Poly-rubber 5021/Stafoam 760, 65/35.

The system and the acceleration impulse are shown in Fig 11. Stress-strain characteristics of the restoring element were taken from reference



$h = 0.5$ in. (length of restoring element)

$w = 1$ lb (weight of mass)

$A = 1.11$ in^2 (cross-sectional area of restoring element)

Fig. 11

(8) and are shown in Fig. 14 in the Appendix.

The parameters for use in Equation (k),

$$\ddot{z} + \omega^2(z - [\delta + a_s/k]) = 0 \qquad (k)$$

[illegible] ROCKET [illegible]

The [illegible] system will be [illegible]

Problem: Predict the response of the [illegible]

with w/A = [illegible] lbs/in², were subjected to [illegible] of

1,000 μsec. [illegible] is

applied to the end of the [illegible]

sensitive element to [illegible]

760, 65/35 [illegible]

The system and the accelerometer [illegible] are shown in Fig. 10. [illegible]

strain characteristics of the [illegible] which [illegible]

h = 0.5 in. [illegible]

w = 1 lb (weight of mass)

A = 1.11 in² (cross-sectional area of [illegible])

(8) and are shown in Fig. 11 in the Appendix.

The parameters for [illegible]

the applicable one for this problem, are to be calculated.

Using the stress-strain curve for $\dot{\epsilon} = 5{,}000$ %/sec.,

$$k = \sigma A/z = 120(1.11)/0.0125 = 10{,}650 \text{ lbs/in.}$$

$$\text{where } z = \%\epsilon(h)\cdot(10)^{-2}$$

and from Equation (j),

$$\ddot{y}_{MAX}/\omega^2 = a_{B\,MAX}\, w/k = 1{,}500(1)/10{,}650 = 0.14 \text{ in.}$$

and

$$\omega = \sqrt{k/m} = (\sqrt{10.65(0.386)})(10)^3 = 2.04(10)^3 \text{ radians/sec.}$$

One more parameter, $d\theta$, is needed for phase-plane plotting. The sine pulse is divided into 18 parts so that $dt = 55.5\,\mu$sec. The average acceleration over this time interval is used in the plotting.

From Equation (f),

$$d\theta = \omega dt = 2.04(55.5)(10)^{-3} = 0.113 \text{ radians} = 6.50^\circ.$$

This is the basis for the compass swings on the phase-plane diagram during the period of the pulse.

Values of $\dot{z}/\omega$ are calculated from the formula,

$$\dot{z}/\omega = h(10)^{-2}\dot{\epsilon}/\omega = (0.50)(10)^{-2}\dot{\epsilon}/2.04(10)^3$$
$$= 2.44(10)^{-6}\dot{\epsilon}$$

and are given in Table 1.

The amplification [illegible] for [illegible] is to be calculated.

Using the [illegible] equation [illegible]

$$[illegible]$$

and from Equation (4.11)

$$[illegible]$$

and

$$[illegible]$$

One more parameter, [illegible], is needed for [illegible] the [illegible] sine pulse [illegible] [illegible] acceleration over this time interval is used in the [illegible]

From Equation (4.[illegible]) [illegible]

$$[illegible]$$

This is the basis for the [illegible] on the [illegible] during the period of the pulse.

Values of [illegible] are calculated from the formula

$$[illegible]$$

and are given in Table 4.

Table 1. Values of $\dot{z}/\omega$ for Various Strain-rates

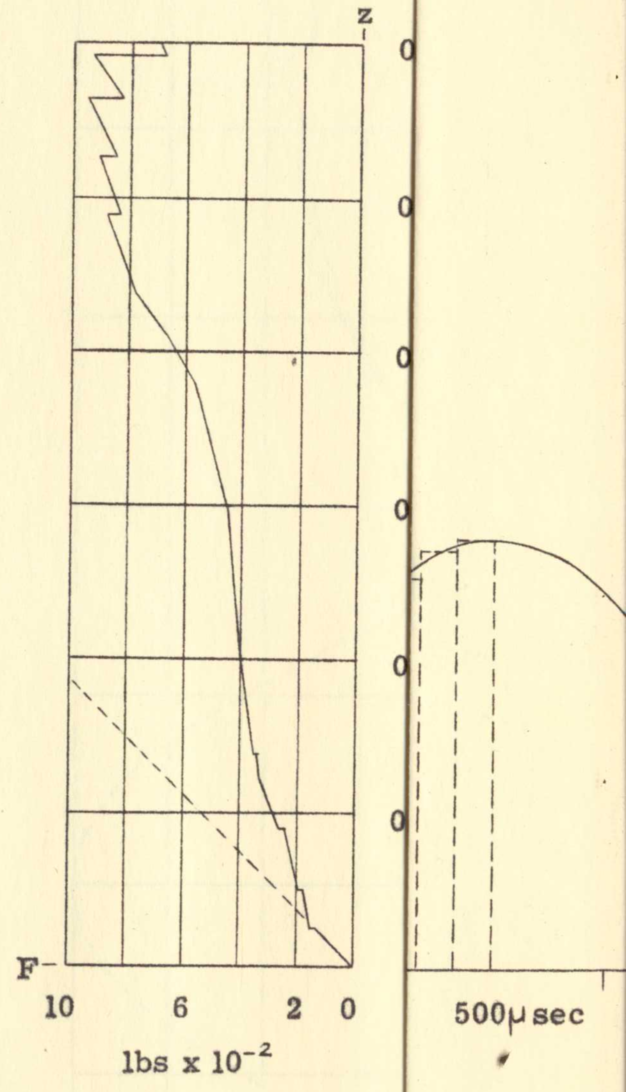
$\dot{\epsilon}$ %/sec	$\dot{z}$ in./sec	$\dot{z}/\omega$ in./sec
1,000	5	0.002
5,000	25	0.012
10,000	50	0.024
20,000	100	0.049
30,000	150	0.073
40,000	200	0.098
50,000	250	0.122
60,000	300	0.147

The phase-plane diagram is shown in Fig. 12 and the results obtained from it are shown in Fig. 13, where the applied pulse and absolute acceleration of the mass are shown as functions of time. The developed load-displacement curve is shown in Fig. 15 in the Appendix.

Table 1. Values of $\dot{z}/\omega$ for Various Strain-rates

$\dot{\varepsilon}$ 1/sec	$\dot{z}$ in./sec	$\dot{z}/\omega$ in./sec
1,000	5	0.002
5,000	25	0.012
10,000	50	0.024
20,000	100	0.049
30,000	150	0.073
40,000	200	0.098
50,000	250	0.122
60,000	300	0.147

The phase-plane diagram is shown in Fig. 12 and the results obtained from it are shown in Fig. 13, where the applied pulse and absolute acceleration of the mass are shown as functions of time. The developed load-displacement curve is shown in Fig. 15 in the Appendix.



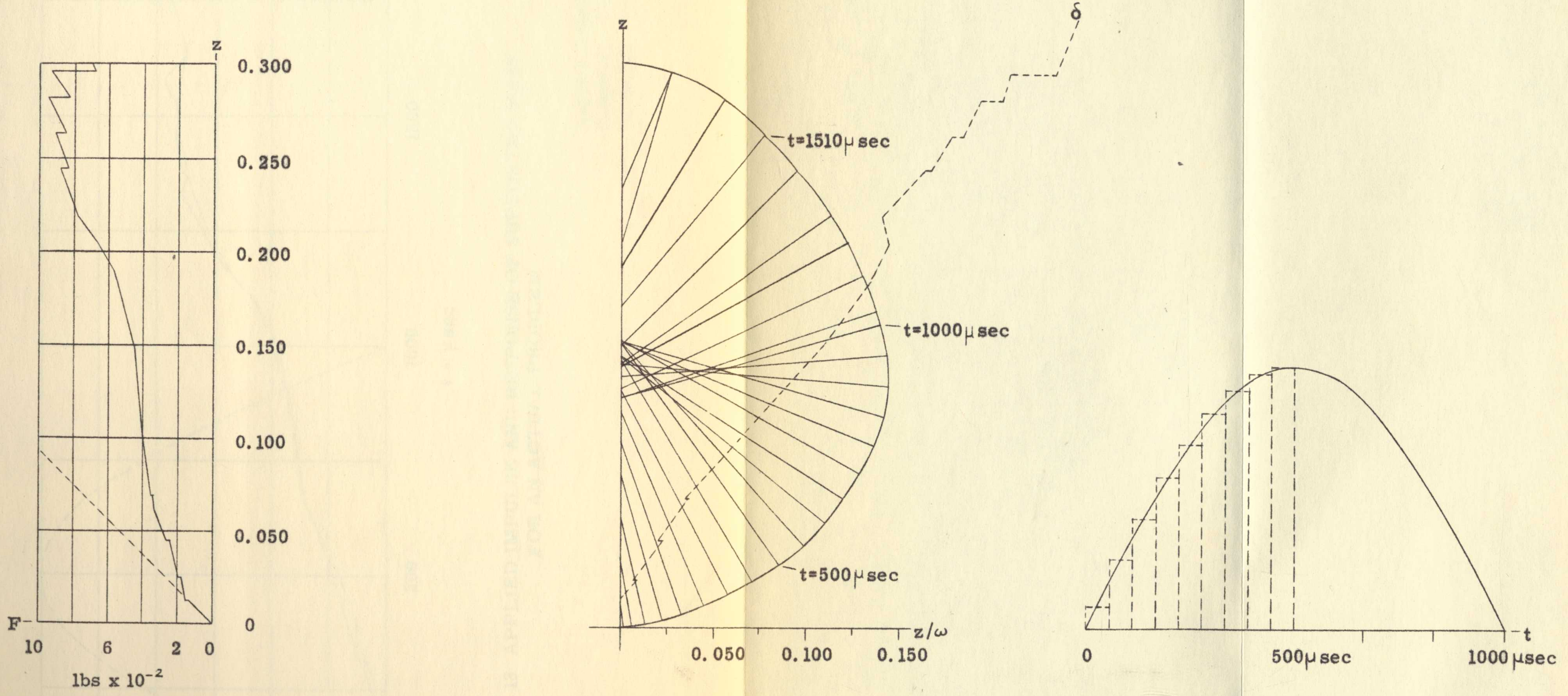


Fig. 12 PHASE-PLANE DIAGRAM FOR SOLUTION TO AN ACTUAL PROBLEM

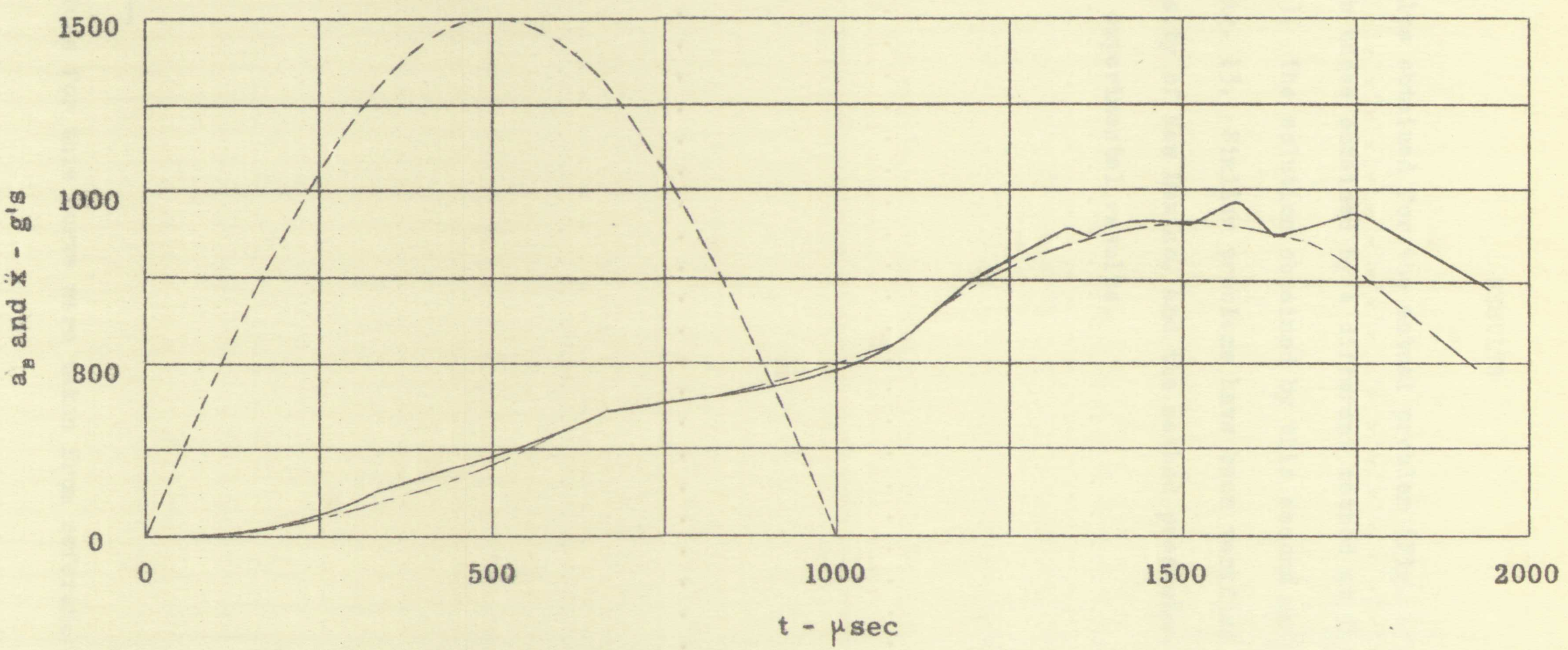


Fig. 13 APPLIED IMPULSE AND RESPONSE OF SUPPORTED MASS FOR AN ACTUAL PROBLEM

- - - - Applied Impulse
——— Response, Present Method
—- - - Response, Reference (9)

RESULTS

The results obtained for the actual problem (Fig. 13) are in close agreement with those obtained by a different method at The University of New Mexico (8). The solution obtained by this second method is also plotted[1] in Fig. 13. Similar problems have been verified experimentally at The University of New Mexico, and the method presented in (8) agrees well with the experimental results.

---

[1]Numerical values for this curve were taken from reference (9).

CONCLUSIONS

The modified phase-plane-delta method accomplishes, to a large degree, the objectives set forth in the Introduction. It has few limitations and for many problems it yields rapid solutions. The method is simple to apply and only requires a few ordinary drafting tools.

The analytical method has been shown to yield results comparable to those obtained by the above method. Its usefulness is impaired by the fact that it may become too lengthy. This will especially be true when numerous differential equations are necessary to describe the motion of a system.

CONCLUSION

The modified quasi-closed-point method fulfills, to a large degree, the objectives set forth in the introduction. It is [illegible] for many problems it yields zero solutions. The method is simple to apply and only requires a few ordinary drafting tools.

The analytical method has been shown to yield results comparable to those obtained by the above method. Its drawback lies in the fact that it may become too lengthy. This will especially be true when numerous differential equations are necessary to describe the motion of a system.

REFERENCES

(1) Stoker, J. J. Nonlinear Vibrations in Mechanical and Electrical Systems. Interscience Publishers, Inc., New York, 1950.

(2) Popoulis, A. Strongly Non-linear Oscillations. J. Math. and Physics, Vol. 37, No. 2, pp. 147-156, 1958.

(3) Rauscher, M. Steady Oscillations of Systems with Nonlinear and Unsymmetrical Elasticity. Trans. A.S.M.C., J. Appl. Mech., Vol. 5, No. 4, pp. A169-A177, 1938.

(4) Evaldson, R. L., Ayre, R. S., and Jacobsen, L. S. Response of an Elastically Non-linear System to Transient Disturbances. J. Franklin Inst., Vol. 248, No. 6, pp. 473-494, 1949.

(5) Lamoën, J. Etude Graphique des Vibrations de Systemes a un Seul Degre de Liberte. Rev. Universelle Mines, Ser. 8, Vol. 11, No. 7, 1935.

(6) Jacobsen, L. S. On a General Method of Solving Second-Order Differential Equations by Phase-Plane Displacements. Trans. A.S.M.E., J. Appl. Mech., Vol. 19, No. 4, pp. 543-553, 1952.

(7) Whitman, R. V., Cheatham, J. F., Wang, W. S., Jacobsen, J. A., Landale, T. D., and Solteriades, M. C. The Behavior of Soils Under Dynamic Loading. 3. Final Report on Laboratory Studies. Mass. Inst. of Techn., Dept. of Civil and Sanitary Engr. Armed Forces Special Weapons Proj., Rpt. No. 118, Aug. 1954.

(8) Engineering Experiment Station. Dept. of Mech. Engr., Univ. of New Mexico. Effect of Loading Rate on Mechanical Properties. Proj. 58-1 ME. Quart. Prog. Rpt. No. 2, Jan. 1, 1959.

(9) Dove, R. C., Albuquerque, N. Mex. Computations for the Response of Cushioned Masses. Private Communication. Mar. 1959.

ACKNOWLEDGMENTS

The writer wishes to express his sincere appreciation:

For advice, to Dr. R. C. Dove whose untiring efforts and relentless energy are always an inspiration and added incentive to the student.

To Dr. F. D. Ju for his helpful suggestions leading to the problem solution.

APPENDIX

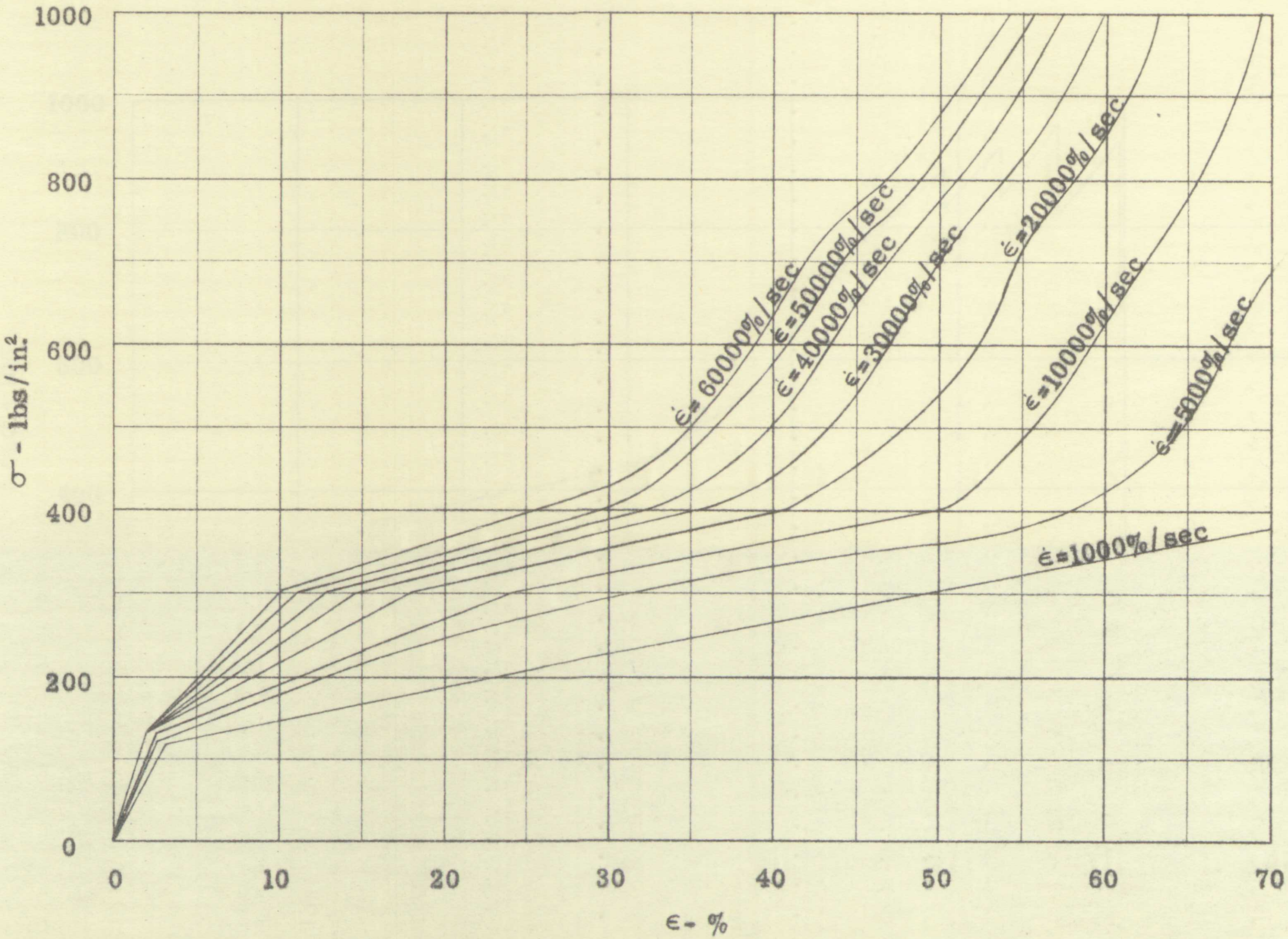


Fig. 14 STRESS-STRAIN CURVES AT VARIOUS STRAIN-RATES
Polyrubber 5021/Stafoam 760, 60/35 - Reference (8), Fig. (8)

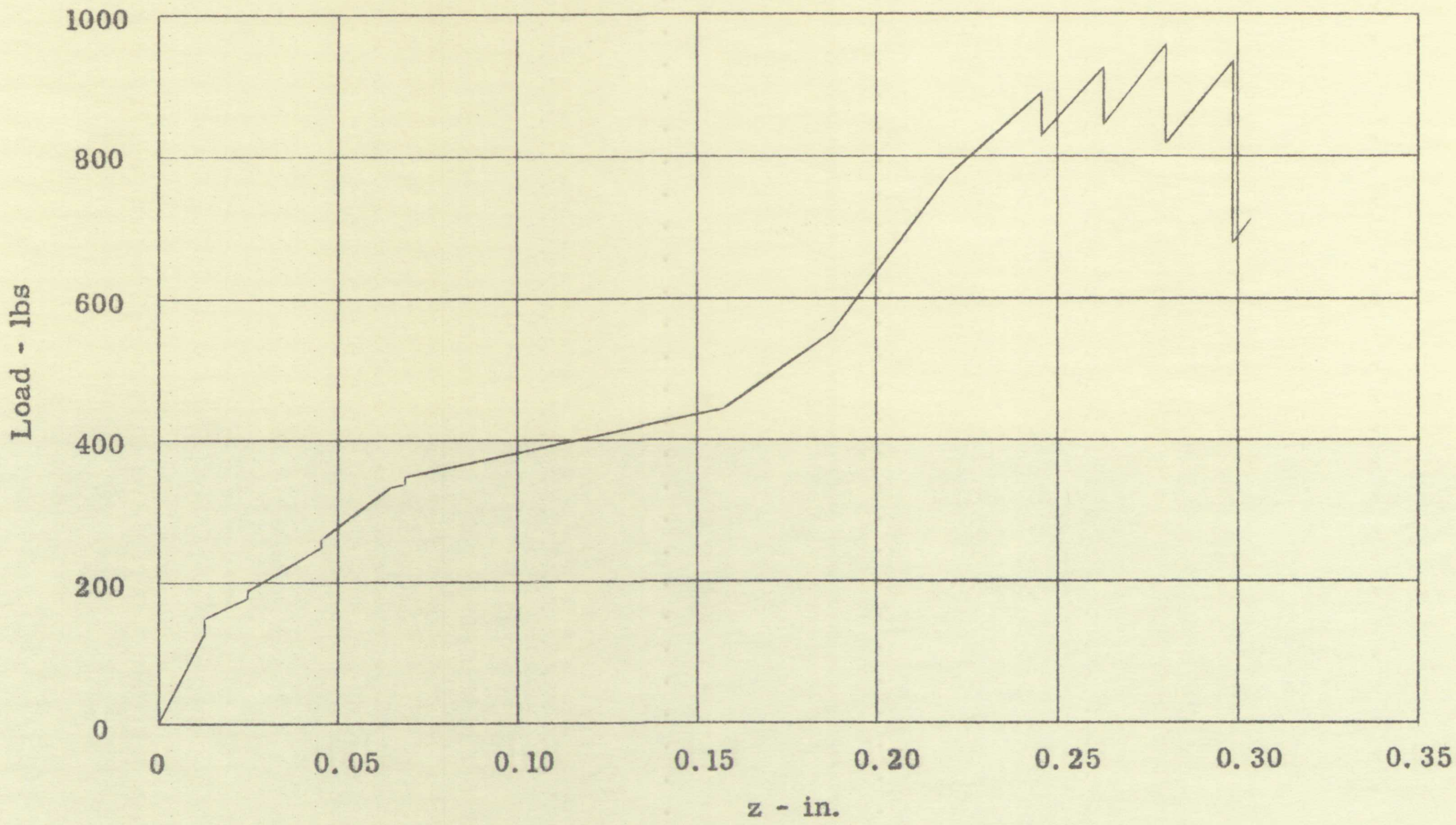


Fig. 15 DEVELOPED LOAD-DISPLACEMENT CURVE
FOR SOLUTION TO AN ACTUAL PROBLEM