8-1-2011

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UNM Technical Report: EECE-TR-2011-07-29

Report Date: July 29, 2011

Abstract

Fourier analysis techniques remain the staple tools for the processing of sinusoidal signals with stationary frequency content. These techniques however, are unsuitable for the analysis of signals with time-varying frequency content such as those with frequency and amplitude modulation unless approximations regarding stationarity are made over smaller signal time windows. In this paper, we study and extend a recently introduced Sturm-Liouville model for both continuous and discrete FM modulation for the purpose of studying orthogonal modes of continuous and discrete FM modulation. These FM modes are first shown to form an orthogonal system and further shown to have the same connection with respect to the FM Sturm-Liouville system operator, that complex exponentials or sinusoids have with LTI systems and the convolution operator. The notion of a finite Sturm-Liouville FM spectrum that measures the strength of the orthogonal FM modes present in a FM signal analogous to the Finite Fourier Transform, is formally introduced.

Keywords

Frequency modulation, Sturm-Liouville differential and difference equation, orthogonal FM modes,
Sturm-Liouville FM spectrum, instantaneous frequency response.

Sinusoidal signals and complex exponentials play a significant role in signal processing and spectral analysis of signals with stationary frequency content. However, they are unsuitable for analysis of signals whose frequency content such as speech signals, ECG waveforms, that are not stationary with respect to their frequency content. Frequency modulated (FM) signals in particular are a class of these signals, where the information resides in the instantaneous frequency of the signal. Traditional approaches for the analysis of these signals such as the spectrogram are based on assumptions of stationary frequency content over smaller time segments of the signal. Other time-frequency approaches such as the fractional Fourier transform approaches [10] are specific to just chirp signals.

Signal processing of these frequency modulated signals using the AM–FM signal model and the quasi-eigenfunction approximation was discussed in [8, 7]. The *energy separation algorithm* (ESA) and its discrete version DESA were studied in [8] as a methodolgy for the demodulation of AM–FM signals. In [7] it was shown that AM–FM signals can only be approximate eigenfunctions of LTI systems and consequently they will undergo harmonic distortion when they are subjected to LTI filtering. Constraints on the frequency response of a filter for minimizing the eigenfunction approximation and bounds on the demodulation error for AM–FM signals were developed. However, when these constraints are not met, the eigenfunction approximation incurs significant demodulation error.

Quasi-orthogonal signals comprised of up-chirps, i.e., with a linearly increasing IF and down-chirps, i.e., with a linearly decreasing IF are key components of a chirp-based system that has been proposed for spread spectrum communications [3]. Orthogonal FM transforms that are derived from simple permutations of the DFT phase were investigated in [9] for the purpose of concentrating the energy of an image in a few transform domain coefficients. Recently a Sturm-Liouville (S-L) model for the analysis of FM signals was introduced in [15]. Orthogonal modes of continuous and discrete frequency modulation were developed using the differential or difference equation satisfied by the FM signal. In this paper, we consolidate and extend the S-L model for FM, by first studying the orthogonal FM modes to develop a system theory framework for frequency modulated signals. We introduce the notion of a the finite S-L FM spectrum for FM signals that is analogous to the Fourier spectrum for sinusoidal signals.

1 Continuous Time FM

Let us first consider a sinusoidal signal of the form:

$$x(t) = \cos(\omega_o t + \theta_o).$$

This signal satisfies the constant coefficient, homogenous, second-order differential equation of the classical harmonic oscillator:

$$\ddot{x} + \omega_o^2 x = 0.$$

Sinusoids are *eigenfunctions* of a LTI system operator and form the basis for LTI system theory:

$$L(\exp(j\omega_o t)) = H(j\omega_o)\exp(j\omega_o t),$$

where $H(j\omega_o)$ represents the complex eigenvalue. Now consider a frequency modulated signal of the form:

$$x(t) = \cos(\phi(t)) = \cos\left(\int_{-\infty}^{t} \omega_i(\tau)d\tau\right),$$

where $\omega_i(t)$ is the instantaneous frequency and $\phi_i(t)$ is the instantaneous phase. This signal satisfies a second-order differential equation with time-varying coefficients of the form:

$$\ddot{x} - \frac{\hat{\omega}_i(t)}{\omega_i(t)}\dot{x} + \omega_i^2(t)x = \left(\mathcal{D}^2 - \frac{\mathcal{D}\omega_i}{\omega_i}\mathcal{D} + \omega_i^2\right)x = 0, \quad (1)$$

where $\mathcal{D}$ denotes the derivative operator. In general this system is a linear time-varying system. In the specific case of sinusoidal FM it becomes periodically time-varying. It is also known that even in the simple case, where the message waveform is sinusoidal, the bandwidth of the FM signal is infinite and requires truncation. In fact the Carson bandwidth of an FM signal retains just spectral components that have an amplitude of at least 10% of the maximum spectral amplitude [4].

1.1 Sturm-Liouville Differential Equation

The FM differential equation described in Eq. (1) does not correspond to a self-adjoint operator. The self-adjoint form of the FM differential equation is [1]:

$$\mathcal{D}\left(\frac{1}{\omega_i(t)}\mathcal{D}x(t)\right) + \omega_i(t)x(t) = 0$$

The self-adjoint form of the FM differential equation for the FM signal $x(t) = \cos(n\phi(t))$ is given by:

$$\left(\frac{1}{\omega_i}\mathcal{D}^2 - \frac{\mathcal{D}\omega_i}{\omega_i^2}\mathcal{D}\right)x = -n^2\omega_i x \longleftrightarrow \mathcal{H}(\omega_i)x = -n^2\omega_i x. \tag{2}$$

Comparing this to the differential form of the Sturm-Liouville differential equation:

$$\mathcal{D}\left(p(x)\mathcal{D}(y(x))\right) + q(x)y(x) = \lambda w(x)y(x),$$

where λ is the eigenvalue and $w(x)$ is the weight function, we can see that Eq. (2) is a specific case of the Sturm-Liouville problem with $\lambda_n = -n^2$, $p(t) = \frac{1}{\omega_i(t)}$, $q(t) = 0$ and weight function[1] $w(t) = w_i(t)$. Eq. (2) can in turn be formulated as a Sturm-Liouville system with periodicity by periodic extension or as a extended S-L system through extrapolation of the instantaneous frequency $\omega_i(t)$ without loss of generality.

1.2 Orthogonal FM Modes

This in turn implies that the operator $\mathcal{H}$ has real and positive eigenvalues and a full set of orthogonal eigenfunctions $\psi_n(t)$ with respect to the weight function $\omega_i(t)$:

$$\int_{-\infty}^{\infty} w(t)\psi_m(t)\psi_n(t)dt = 0, m \neq n. \tag{3}$$

This result is consistent with earlier work on FAM-lets[2] [6], where the sequence of functions:

$$\begin{aligned} \gamma_n(t) &= \sqrt{\omega_i(t)}\cos\left(n\phi(t)\right), \\ \psi_n(t) &= \sqrt{\omega_i(t)}\sin\left(n\phi(t)\right). \end{aligned} \tag{4}$$

Consequently the complex exponential version of the FAM-lets given by:

$$\begin{aligned} \alpha_n(t) &= \sqrt{\omega_i(t)}\exp\left(jn\phi(t)\right) \\ &= \sqrt{\omega_i(t)}\exp\left(jn\omega_c t + jn\omega_m \int_{-\infty}^{t} \omega_i(\tau)d\tau\right) \end{aligned} \tag{5}$$

[1]For the Sturm-Liouville framework to hold the weight function $\omega_i(t)$ should be strictly positive. This is not restrictive and is assumed in most FM modulation systems.

[2]FAM-lets are constant Q basis functions because both the carrier frequencies and frequency deviations of the FM modes scale linearly. In the context of sinusoidally modulated FM signals and computer generated music this is called harmonic FM [4]. When the ratio of the carrier frequency to the frequency deviation is not rational it is called non-harmonic FM.

is also an eigenvector of the S-L system. This is a intuitively satisfying result in that it is analogous to the correspondence between complex exponentials and LTI systems. It is easily seen by a simple substitution of variables $u = \phi(t)$ that the two basis of eigenfunction defined in Eq. (4) indeed form an orthogonal sequence of functions [6]:

$$\begin{aligned} < \gamma_m(t), \gamma_n(t) > &= \int_{-\infty}^{\infty} \omega_i(t)\gamma_m(t)\gamma_n(t)dt, m \neq n \\ &= \int_0^{2\pi} \cos(mu)\cos(nu)du = 0. \end{aligned} \tag{6}$$

It is also well known that many of the special functions encountered in quantum mechanics such as Legendre or Hermite functions satisfy the Sturm-Liouville framework for specific discrete values of the eigenvalue λ and the weight function $w(x)$ [16].

1.3 System Theoretic Implications

There are three important consequences of expressing the FM differential equation in the Sturm-Liouville form. The first implication is that if the FM signal $x(t)$ is input to the system $\mathcal{H}(\omega_i)$ then the output is just a scalar multiple of the input signal. In other words, the S-L system $\mathcal{H}$ does not introduce any instantaneous frequency distortion and that instantaneous frequency of the input signal $x(t)$ remains invariant:

$$\begin{aligned} \mathcal{H}\left(\sum_{k=0}^{\infty} a[k]\cos(k\phi(t))\right) &= \sum_{k=0}^{\infty} a[k]\mathcal{H}(\cos(k\phi(t))) \\ &= \omega_i(t)\sum_{k=0}^{\infty} b[k]\cos(k\phi(t)) \\ b[k] &= -k^2 a[k]. \end{aligned} \tag{7}$$

The second implication is that results analogous to LTI systems and sinusoids such as a Fourier series and Fourier transforms can be developed for FM signals. With $\phi_k(t) = \cos(k\phi(t))$ [5]:

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} c[k]\phi_k(t) \\ c[k] &= \frac{\int_{-\infty}^{\infty} x(t)\phi_k(t)\omega_i(t)dt}{\int_{-\infty}^{\infty} |\phi_k(t)|^2\omega_i(t)dt}, \end{aligned} \tag{8}$$

where $c[k]$, the S-L coefficient, measures the strength of a particular FM mode in the signal. The third implication of the S-L framework is that sequence of S-L coefficients $c[k]$ is stationary in frequency content even though the underlying FM signal has non-stationary frequency content. This means traditional signal processing concepts such as convolution and filtering can be applied to the S-L coefficients:

$$\begin{aligned} b[k] &= a[k]\lambda_k, \lambda_k = -k^2 \\ \mathcal{F}^{-1}(b[k]) &= \mathcal{F}^{-1}(a[k]) * \mathcal{F}^{-1}(\lambda_k), \end{aligned} \tag{9}$$

where $\mathcal{F}^{-1}$ here denotes the inverse DTFT operator and $*$ denotes the convolution operator. In this sense, the S-L coefficients constitute the stationary portion of the FM signal. The ratio of the S-L coefficients of the output to the input:

$$\lambda_k = H[k] = \frac{b[k]}{a[k]}$$

can in essence be interpreted as the discrete *instantaneous frequency response* (IFR) of the S-L system analogous to the frequency response for LTI systems.

2 Discrete time FM

At this point, it is worthwhile pointing out that one could in theory substitute discrete versions of the derivative operator in the definition of the continuous S-L operator $\mathcal{H}$ to yield different discrete versions of the S-L operator. However, they would only serve as discrete approximations of the continuous counterpart. Instead, the approach used here for generating a S-L framework for discrete time FM is to work directly with the difference equation satisfied by the FM signal. First consider the sinusoidal sequence $s[n] = \cos(\Omega_o n)$ which satisfies the second-order difference equation:

$$s[n] - 2\cos(\Omega_o)\, s[n-1] + s[n-2] = 0.$$

Now consider the discrete time FM sequence $x[n]$ given by:

$$x[n] = \cos(\Theta[n]) = \cos\left(\int_o^n \Omega_i[m]dm + \theta_o\right),$$

where the instantaneous phase $\Theta[n]$ is modeled as a first difference:

$$\Theta[n] = \Theta[n-1] + \Omega_i[n].$$

It is easily seen that this satisfies a second-order generating difference equation of the form [14]:

$$x[n] - c_1[n]x[n-1] + c_2[n]x[n-2] = 0,$$

where the time-varying coefficients are given by:

$$\begin{aligned} c_1[n] &= \frac{\sin\left(\Omega_i[n] + \Omega_i[n-1]\right)}{\sin\left(\Omega_i[n-1]\right)} \\ c_2[n] &= \frac{\sin(\Omega_i[n])}{\sin(\Omega_i[n-1])}. \end{aligned} \tag{10}$$

First note that the signal $y[n] = \sin\Theta[n]$ also satisfies the same difference equation. This again is an intuitively satisfying result in that the complex exponential version of the FM signal given by $x[n] = \exp\left(j\Theta[n]\right)$ will also be an eigenfunction of the S-L FM operator. It can also be verified that this difference equation will reduce to that of the sinusoid in the stationary case, i.e., $\Omega_i[n] = \Omega_o$. The corresponding self-adjoint difference equation obtained by the S-L difference equation framework described in [11] is given by:

$$\nabla_-\left(p[n]\Delta_+(x[n])\right) + w[n]C[n]x[n] = 0, \tag{11}$$

where the weight function $w[n]$, $p[n]$, and $C[n]$ are given by:

$$\begin{aligned} w[n] &= \prod_{r=0}^{n-1} \frac{\sin(\Omega_i[r])}{\sin(\Omega_i[r+2])} = \frac{\sin(\Omega_i[0])\sin(\Omega_i[1])}{\sin(\Omega_i[n])\sin(\Omega_i[n+1])} \\ p[n] &= \sin\left(\Omega_i[n]\right) w[n] = \frac{\sin(\Omega_i[0])\sin(\Omega_i[1])}{\sin(\Omega_i[n+1])} \\ C[n] &= \sin(\Omega_i[n]) + \sin(\Omega_i[n+1]) \\ &- \sin(\Omega_i[n+1] + \Omega_i[n]) \end{aligned} \tag{12}$$

and the symbols ∇_- and Δ_+ denote the one-sample backward and forward difference operators. As in the continuous case, the S-L operator is in general a linear time-varying system. It should be noted here that the form of the FM difference equation and as a result the self-adjoint S-L difference equation are sensitive to the form of discretization of the instantaneous phase $\Theta[n]$. As in the continuous case, the difference equation in Eq. (11) can be formulated as a extended/periodic S-L system by either: (a) periodic extension of the instantaneous frequency $\Omega_i[n]$ at the boundaries [2, 17], which would imply a discrete Fourier series representation for the IF or (b) extrapolation of the IF at the boundaries under the assumption that IF is varying slowly, where the boundary values can be repeated [8]. The solution to the discrete S-L difference equation is then formulated as the solution to a weighted, tridiagonal eigenvalue problem of the form

$$\mathcal{L}(\mathbf{x}) = \lambda \mathbf{W}\mathbf{x}, \tag{13}$$

where $\mathbf{W} = \mathrm{diag}(w[0], \ldots, w[N-1])$ is a diagonal matrix of the positive weights and λ is the eigenvalue[3].

The symmetric, tridiagonal, weighted eigenvalue problem is encountered in the context of the theory of orthogonal polynomials which satisfy an associated three term recursion. These polynomials are orthogonal with respect to a weighted inner product. In the limiting case of the S-L operator, where the S-L operator is a circulant, Toeplitz-tridiagonal, its eigenvectors are sinusoids, the IF's of the eigenvectors are constant, and the orthogonal polynomials associated with the S-L operator are the Chebyshev polynomials of the second kind [18]. As the modulation depth decreases, the eigenvector approaches a sinusoid. Expressions for the eigenvectors in terms of the associated orthogonal polynomials and its roots can also be found in [18].

2.1 Orthogonal FM Modes

As in the continuous case, the eigenvectors of the S-L operator:

$$\mathcal{L}(p[n]) = \nabla_{-} p[n]\Delta_{+} + w[n]C[n]$$

corresponding to distinct eigenvalues are orthogonal with respect to the positive weight function $w[n]$:

$$< v_p[n], v_q[n] > = \sum_{n=0}^{N-1} w[n] v_p[n] v_q[n] = 0, \;\; p \neq q. \tag{14}$$

The corresponding expansion of the discrete FM signal in terms of the eigenvectors $v_k[n]$ of the S-L operator is given by:

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} c[k] v_k[n], \\ c[k] &= \frac{\sum_{n=0}^{N-1} w[n] x[n] v_k[n]}{\sum_{n=0}^{N-1} w[n] |v_k[n]|^2} \end{aligned} \tag{15}$$

As in the continuous case, the weight function can be absorbed into the orthogonal basis of eigenvectors to produce an orthonormal basis:

$$\gamma_k[n] = \sqrt{w[n]} v_k[n]. \tag{16}$$

These eigenvectors contain both amplitude and frequency modulation and the IF of the eigenvectors of the matrix $\mathcal{L}$ furthermore have a form specified by the IF of the input signal, $\omega_i[n]$. For example if we use the tridiagonal formulation of the S-L operator with no corner correction we obtain:

$$\begin{aligned} v_k[n] &= \sqrt{\frac{2}{N+1}} a_k[n] \sin\left(\omega_c^{(k)}(n+1) + \phi_k[n]\right), \\ \phi_k[n] &= \omega_m^{(k)} \int_0^n q[m] dm, \\ \omega_c^{(k)} &= \frac{\pi}{N+1}(k+1), 0 \leq k \leq N-1, \end{aligned} \tag{17}$$

[3] The S-L eigenvalue problem can be solved in various senses: exactly using the MATLAB functions `eig(A,B)` or `qz(A,B)` or in the min-norm sense using `svd.m` or `gsvd.m`. For situations where the signal of interest and consequently the estimate of the IF, $\Omega_i[n]$, are noisy, a generalized SVD version of Eq. (13) is employed

where $q[n]$ is the normalized message signal. Note that in the limiting case, where the input signal is a sinusoid, the eigenvector basis $v_k[n]$ will reduce to the *discrete sine transform* (DST) basis, specifically the symmetric version of DST-I [12]:

$$\Upsilon_k[n] = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi}{N+1}(n+1)(k+1)\right). \tag{18}$$

Note that unlike the continuous case, the corresponding cosine version of the sequence or the DCT-I sequence do not constitute eigenvectors of the same operator.

If on the other hand, if we use the "Toeplitz plus near Hankel," framework described in [13] we obtain the other versions of the DCT. Specifically if we add/subtract and subtract/add 1 from the diagonal corners of the S-L operator we obtain the DCT-4/DST-4 pair:

$$\Upsilon_k[n] = \sqrt{\frac{2}{N}} a_k[n] \sin\left(\frac{\pi}{N}(n+0.5)(k+0.5) + \phi_k[n]\right). \tag{19}$$

However, they are obtained from two different operators.

We can now compute the inner-product of two distinct S-L DST-I based FM modes as:

$$\begin{aligned} & < v_p[n], v_q[n] >= T_1 + T_2 \\ T_1 &= -\sum_{n=0}^{N-1} \frac{w[n]a_p[n]a_q[n]}{N+1} \times \\ & \cos\left(\omega_c^{(n)}(p+q+2) + \phi_p[n] + \phi_q[n]\right) \\ T_2 &= \sum_{n=0}^{N-1} \frac{w[n]a_p[n]a_q[n]}{N+1} \times \\ & \cos\left(\omega^{(n)}(p-q) + \phi_p[n] - \phi_q[n]\right). \end{aligned} \tag{20}$$

Each of the terms in the expression above is the inner-product of a lowpass waveform $w[n]a_p[n]a_q[n]$, i.e., with spectral content around DC, with a bandpass waveform, i.e., with spectral content around a much higher carrier frequency. The lowest carrier frequency of the first term with the $(p+q+2)$ factor is $\frac{2\pi}{N+1}$. Consequently there is no spectral overlap and therefore by Parseval's theorem, this first term is approximately zero:

$$\begin{aligned} T_1 &= \sum_{n=0}^{N-1} w[n]a_p[n]a_q[n] \times \\ & \cos\left(\omega^{(n)}(p+q+2) + \phi_p[n] + \phi_q[n]\right) \\ &\approx 0 \end{aligned} \tag{21}$$

The inner product can therefore be approximated as:

$$\begin{aligned} < v_p[n], v_q[n] > &\approx \sum_{n=0}^{N-1} \frac{w[n]a_p[n]a_q[n]}{N+1} \times \\ & \cos\left(\omega_c^{(n)}(p-q) + \phi_p[n] - \phi_q[n]\right) \\ &= 0, \quad p \neq q, \end{aligned} \tag{22}$$

where the last result follows from the same observation that there will be no spectral overlap between the low-pass term $w[n]a_p[n]a_q[n]$ and the term with carrier frequency $\frac{\pi(p-q)}{N+1}$ unless $p = q$. The lowpass approximation employed here is a common assumption in narrowband communications systems, where the carrier frequency is much larger than the message bandwidth [19]. The exact claim of orthogonality of the S-L FM modes follows from the fact that they are solutions to a S-L difference equation and standard results of S-L theory apply [11].

2.2 System Theoretic Implications

The response of the discrete S-L operator to this signal is given by:

$$y_k[n] = \mathcal{L}(z_k[n]) = \lambda_k w[n] z_k[n]. \tag{23}$$

Suppose the input to the system is a superposition of these FM modes, then the corresponding output is:

$$\begin{aligned} \mathcal{L}\left(\sum_{k=0}^{N-1} \alpha[k] z_k[n]\right) &= \sum_{k=0}^{N-1} \alpha[k] \mathcal{L}(z_k[n]) \\ &= \sum_{k=0}^{N-1} \lambda_k \alpha[k] w[n] z_k[n] \\ &= \sum_{k=0}^{N-1} \beta[k] w[n] z_k[n]. \end{aligned} \tag{24}$$

Specifically the IF modes that are present in the output of the S-L operator are the same IF modes present in the input to the S-L operator. The ratio of the S-L coefficients is analogous to the frequency response of LTI systems:

$$\lambda_k = H[k] = \frac{\beta[k]}{\alpha[k]} = \frac{\left(\mathbf{V}^T \mathbf{W} \mathbf{D} \mathbf{y}\right)_k}{\left(\mathbf{V}^T \mathbf{W} \mathbf{D} \mathbf{x}\right)_k}, \tag{25}$$

where **D** denotes the unit sample advance operator, **V** is the matrix of S-L eigenvectors, and **W** is a diagonal matrix of S-L weights. Furthermore, the generalized Fourier coefficient sequences $\beta[k]$ and $\alpha[k]$ are connected via convolution through:

$$\mathcal{F}^{-1}(\beta[k]) = \mathcal{F}^{-1}(\alpha[k]) * \mathcal{F}^{-1}(\lambda_k) = \mathcal{F}^{-1}(\alpha[k]) * h[k], \tag{26}$$

where $\mathcal{F}^{-1}$ here denotes the inverse DFT matrix. This relationship is significant in that conventional LTI system theory can be applied to the generalized Fourier coefficients $\beta[k]$ and $\alpha[k]$ even though the underlying signals $y(t)$ and $x(t)$ are frequency modulated. Specifically the quantity:

$$\alpha[k] = \left(\mathbf{V}^T \mathbf{W} \mathbf{D} \mathbf{x}\right)_k = \sum_{m=0}^{N-1} v_k[m] w[m] x[m+1] \tag{27}$$

is formally defined as the finite Sturm-Liouville FM spectrum[4] of the FM signal $x(t)$, analogous to the Fourier spectrum for sinusoidal signals, except in this case the spectrum indicates the strength of a particular FM mode in the signal. The operation of truncation, i.e., retaining just the S-L coefficients above a certain power threshold dependent on the SNR, implemented in the example in Fig. (4), is therefore equivalent to applying an ideal brick-wall bandpass filter on the noisy S-L coefficients.

Fig. (1), fig. (2), and fig. (4) describe the application of the discrete S-L approach to three different signals: (a) sinusoidally modulated FM signal, where the MATLAB function `eig(A,B)` is employed, (b) FM signal with a triangular IF, where the MATLAB function `qz.m` is employed, and (c) FM signal with a triangular IF in noise, where the generalized SVD function in MATLAB `gsvd.m` is employed. For the triangular IF example in (b), the carrier frequency of the input FM signal is intentionally chosen to be an integer multiple of $\frac{\pi}{N+1}$ so that it coincides with the carrier frequency of one of the normal FM modes. Fig. (2)(e,f) compare the S-L FM spectrum which is a one dimensional spectrum to the MA-CDFRFT spectrum that is a two dimensional spectrum [10]. The distinguishing characteristics of the S-L approach from discrete fractional Fourier transform based approaches is that the IF of the eigenvectors are of the same form as the IF of the FM signal being analyzed and that it is not specific to just chirps.

[4] The non-causal definition of the spectrum is a direct result of the non-causal formulation of the S-L operator that is defined with a one sample noncausal shift.

Fig. (3)(a,b) describes the center-frequencies and frequency deviations of the orthogonal FM modes for an input signal with sinusoidal FM modulation. Fig. (3)(c) describes the frequency modulation index for selected FM modes. S-L eigenvectors with more zero-crossings correspond to high-frequency FM modes, while the eigenvectors with fewer zero-crossings correspond to lowpass FM modes. While the carrier frequencies of the FM modes are linearly spaced apart as with the FAM-lets, the frequency deviations of the modes are not linearly spaced apart, but are rather symmetric about a central mode. The FM modes with modulation index larger than 1 are considered wideband, while the modes with index less than 1 are narrowband. The results with the S-L FM spectrum in all three examples, where the difference between the largest peak and its nearest neighbor is around 30 dB, are also indicative of the fact that the S-L orthogonal FM modes provide significant energy compaction in just a few FM modes. Fig. (5)(a) studies the ESA frequency demodulation error between the IF of the mode corresponding to the S-L spectral peak and the IF of the input FM signal versus the S-L system size. As the size increases, the error decreases indicating that one of the FM modes will eventually capture the input FM signal. Fig. (5)(b) depicts the frequency demodulation error when the input signal is one of the normal FM modes.

3 Orthogonal FM Modes and Angular Mathieu Functions

Fig. (6) describes the similarity between the orthogonal FM modes and *angular Mathieu functions* (AMF) described in [21, 20]. For instantaneous frequencies with negative frequency deviations, the orthogonal FM modes from the S-L framework exhibit the same symmetry or antisymmetry about their mid-point that the cosine and sine elliptic AMF's do:

$$\begin{aligned} \mathrm{ce}_n(z,-q) &= \pm\mathrm{ce}_n(\pi/2-z,q), \\ \mathrm{se}_n(z,-q) &= \pm\mathrm{se}_n(\pi/2-z,q). \end{aligned} \tag{28}$$

This is illustrated in Fig. (6)(a,b), where the orthogonal FM modes for negative frequency deviation values are specific modes are plotted. The orthogonal S-L FM modes also satisfy the same asymptotic behavior as the AMF's in that in the limit as the frequency deviation goes to zero we obtain sinusoids:

$$\begin{aligned} \lim_{q\to 0} ce_n(z,q) &= \cos(nz), \\ \lim_{q\to 0} se_n(z,q) &= \sin(nz). \end{aligned} \tag{29}$$

This property is illustrated in Fig. (6)(e), where the first orthogonal FM mode is plotted for different frequency deviation parameters, depicting the change from a FM modulated signal to a purely sinusoidal signal. Fig. (6)(c,d) depict the IF'S of selected AMF's obtained using the approach in [21] and the associated MATLAB functions. Notice that the AMF's are sinusoidally modulated, where a change in the parameter q results in a increase in the frequency deviation of the underlying IF's. This result is also very similar to the results seen with the orthogonal S-L FM modes, where the modes are also FM modulated with an IF of the same form as the input FM signal. Furthermore the AMF's obtained through the framework in [21] also have the same linear spacing of the FM mode center-frequencies of $\frac{\pi}{N}$ as depicted in the ESA IF estimates of selected AMF's in Fig. (6)(c,d), a result very similar to that seen in the orthogonal S-L FM modes. Additionally the orthogonal S-L FM modes exhibit both amplitude and frequency modulation, a property that is also seen in approximate solutions to the Mathieu differential equation [20].

Effectively these similarities imply that the carrier-spacing of the FM modes is related to the parameter z and the frequency deviation of the FM modes is related to the parameter q of the AMF's. Specifically the framework in [21] and Fig. (6)(f) imply that the parameter q of the AMF's is a odd function of frequency deviation of the modes. These striking similarities combined with the results from Fig. (3)(b) lead us to the conjecture that the S-L orthogonal FM modes are contained in the span of a finite dimensional subset of AMF's for specific discrete values of the parameters.

$$v_k[n] = \sum_{r=0}^{N-1} p_k[r]\mathrm{ce}_k(q_r,z_r) + q_k[r]\mathrm{se}_k(q_r,z_r). \tag{30}$$

In the case of the symmetric DST based orthogonal FM modes this becomes:

$$\begin{aligned} \omega_o &= \frac{\pi}{N+1},\ z_k = (k+1)\omega_o, 0 \leq k \leq N-1 \\ \omega_m^{(k)} &= C\csc(z_k),\ q_k = \sinh(\omega_m^{(k)}). \end{aligned} \tag{31}$$

The result that the AMF's are sinusoidally modulated, i.e., have a sinusoidal IF, and yet are able to represent a general FM waveform as depicted in Fig. (6)(c,d) is consistent with the ESA framework in [7, 8], that allows for any IF that can be represented through a finite Fourier series of cosines/sines:

$$\Omega_i[n] = \Omega_o + \sum_{k=1}^{N} \alpha[k]\cos(\Omega_k n + \Theta_k), \tag{32}$$

except at discontinuity points, where the estimated IF goes through the mid-point of the discontinuity.

4 Conclusions

We have extended and expanded on the S-L framework for continuous and frequency modulation introduced in [15]. Orthogonal FM modes arising from the eigenfunctions or eigenvectors of the S-L FM operator are shown to undergo no IF distortion when subjected to the S-L FM system. A generalized Fourier series representation of a modulated waveform in terms of the orthogonal FM modes was presented and the notion of the finite S-L FM spectrum that describes the strength of the FM modes prevalent in a modulated signal was presented. Simulation results presented indicate that the orthogonal FM modes provide significant energy compaction in terms of representing a modulated waveform with a few transform coefficients.

These orthogonal S-L FM modes furthermore, reduce to the standard Fourier basis or the symmetric sine basis, in the limit when the modulation strength becomes negligible. More significantly it was also shown that S-L coefficients of a FM signal with respect to the orthogonal FM modes are stationary eventhough the underlying signal has nonstationary frequency content. The implication is that standard system theory results such as convolution, filtering, and the DTFT can be applied to the S-L coefficients. In the continuous-time case, the S-L orthogonal FM modes reduce to the better known FAM-let basis, while in the discrete-time case, the striking similarities between the orthogonal S-L FM modes and angular Mathieu functions were examined and it was conjectured that the orthogonal FM modes are in the span of a finite dimensional subset of the angular Mathieu functions for specific discrete values of the underlying parameters.

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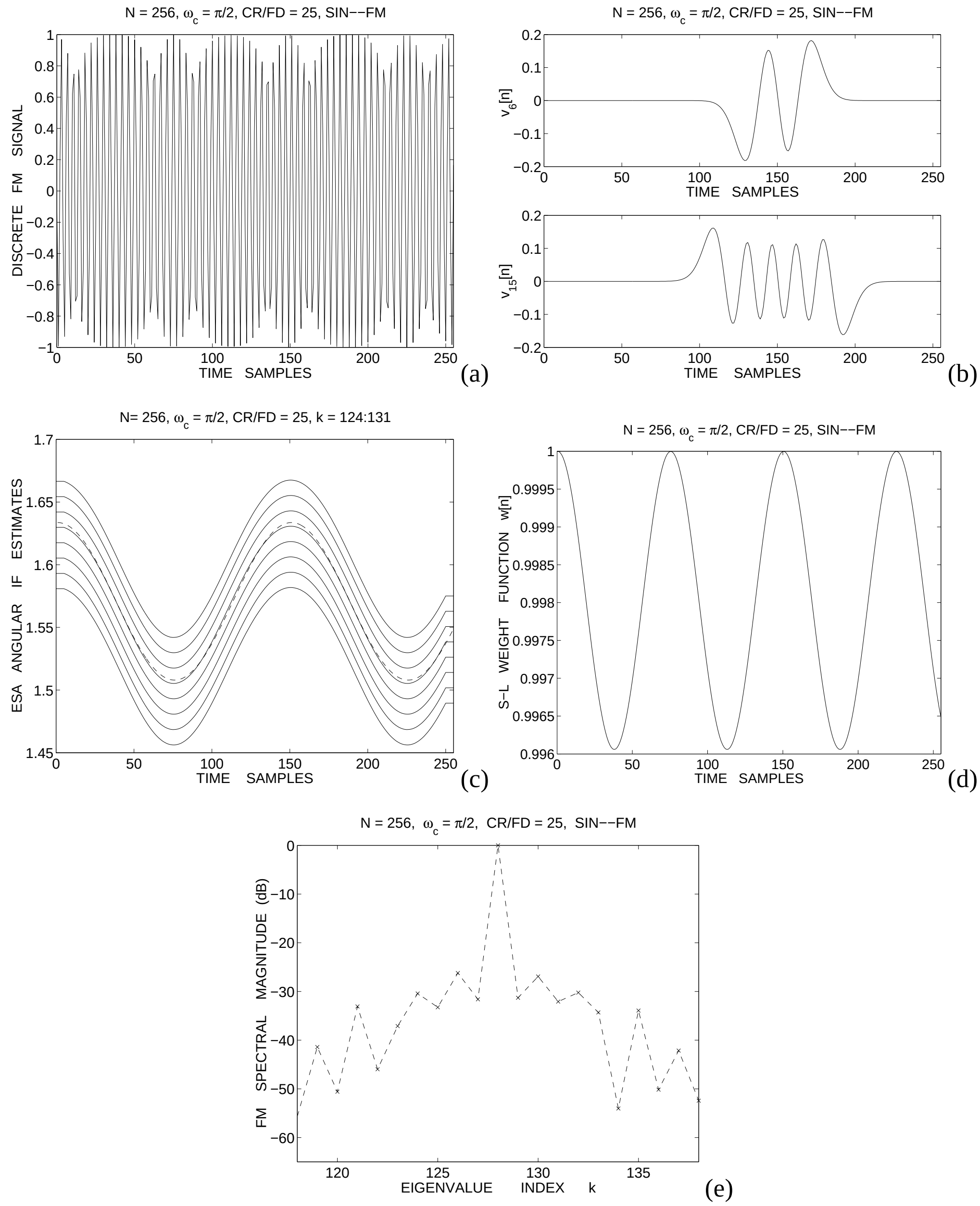

Figure 1: Discrete S-L problem, sinusoidal-FM :(a) sinusoidal FM signal, (b) selected eigenvectors of the discrete S-L operator using the MATLAB function `eig(A,B)` depicting different number of zero crossings, (c) IF of selected eigenvectors extracted using the ESA [8], (d) weighting function of the discrete S-L problem, (e) S-L FM spectrum of the signal.

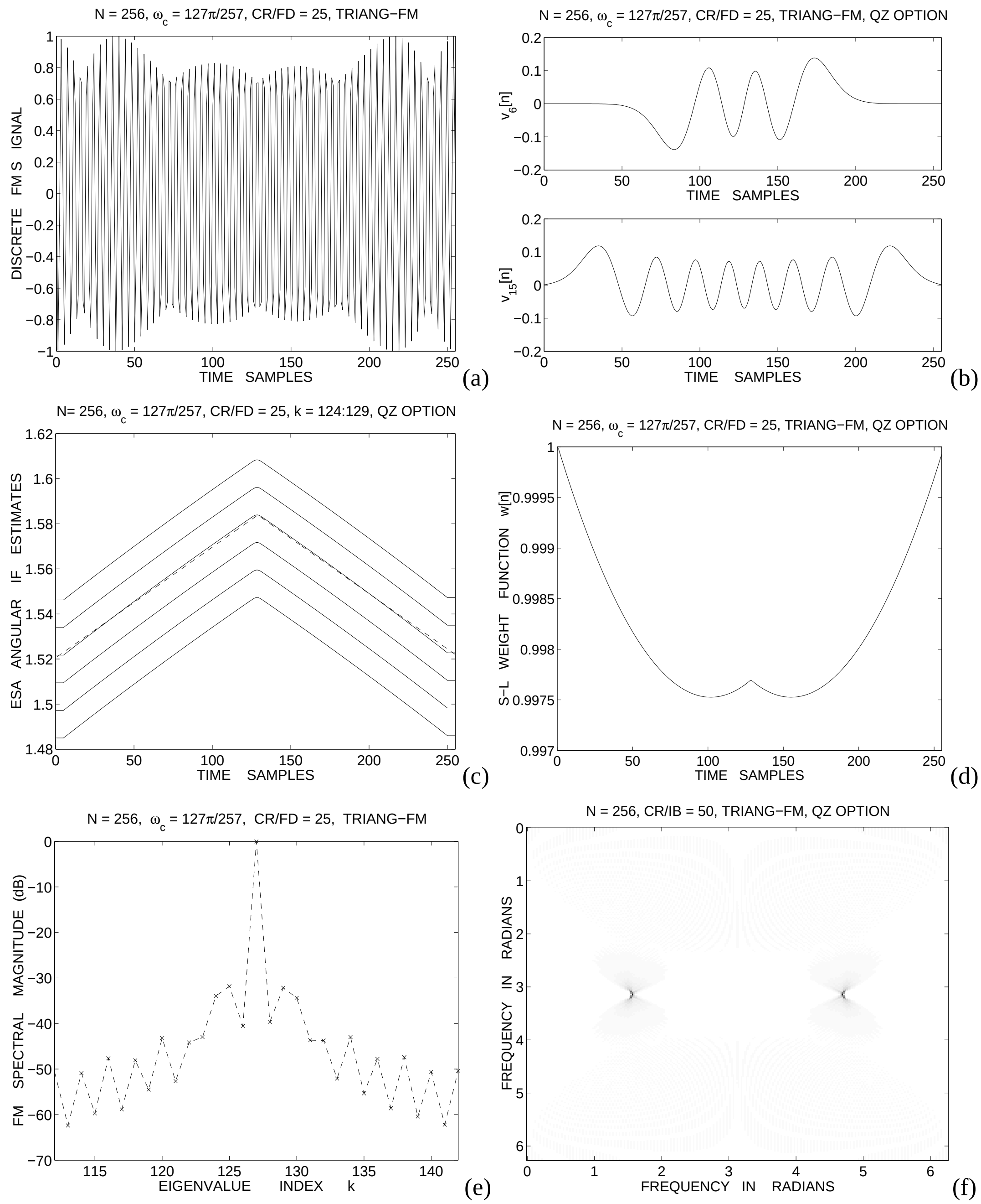

Figure 2: Discrete S-L problem, triangular frequency modulation : (a) FM signal, (b) selected eigenvectors of the discrete S-L operator using a QZ decomposition for the generalized eigenvalue problem, (c) ESA instantaneous frequency estimate of selected eigenvectors: $k = 124 : 129$ of the discrete S-L operator using the DESA, (d) weight function associated with the discrete S-L operator, and (e,f) S-L FM spectrum and discrete fractional Fourier spectrum of the FM signal. Note that the eigenvectors resemble discrete versions of Gauss-Hermite functions and that the IF of the eigenvectors is of the same form as the IF of the input FM signal.

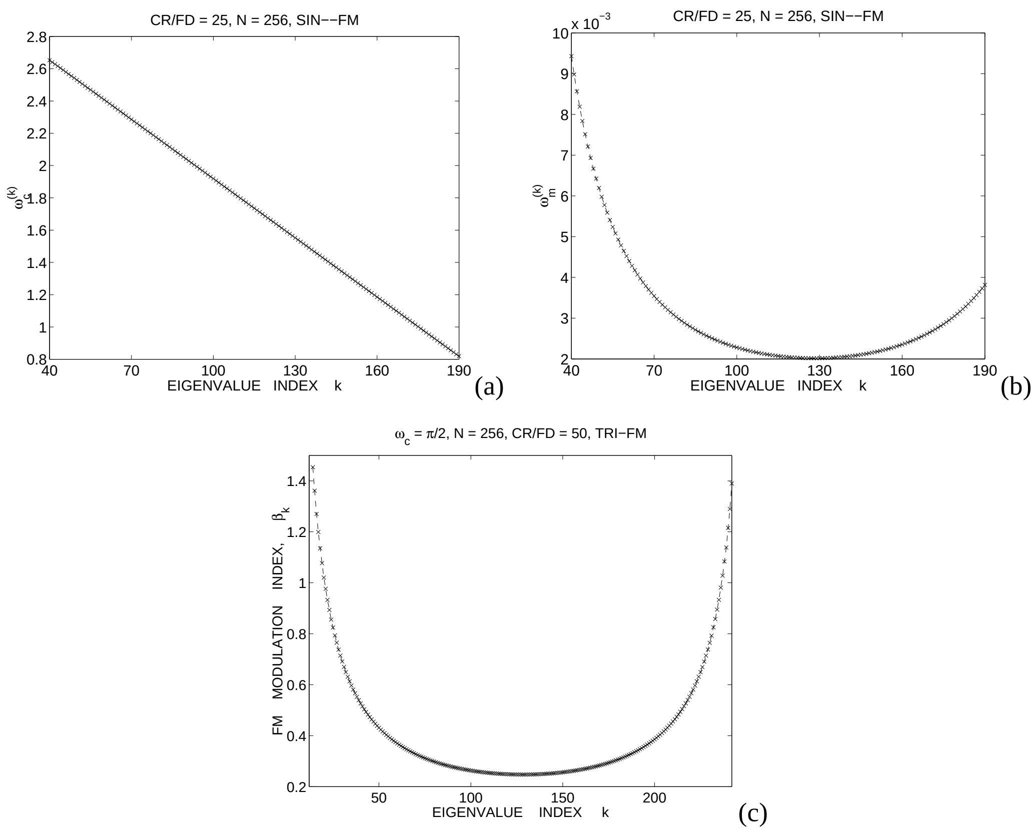

Figure 3: (a,b) center frequency and frequency deviation of selected FM modes of the discrete S-L operator for the first sinusoidally modulated example, and (c) frequency modulation indices for specific FM modes for a triangular IF.

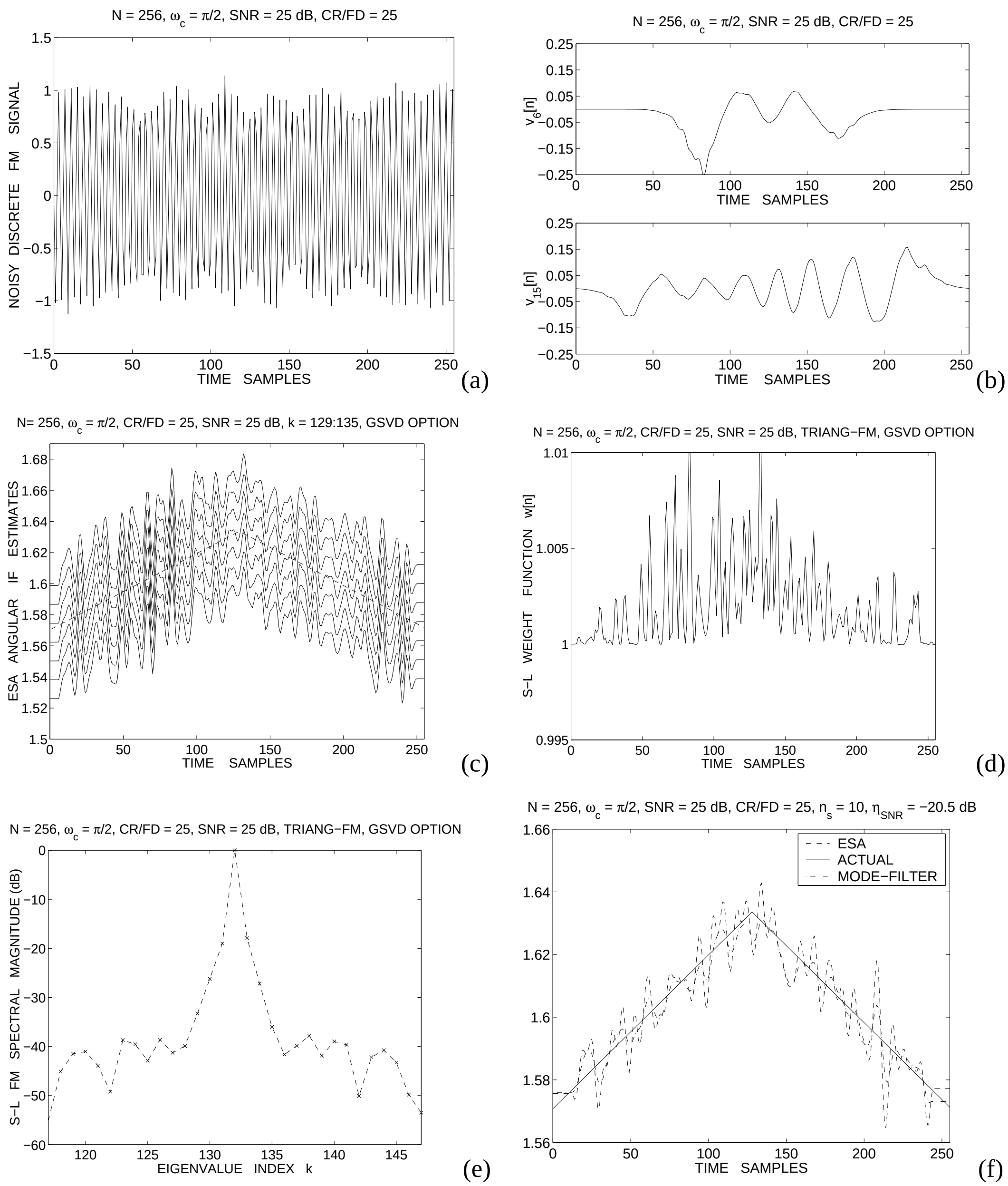

Figure 4: FM orthogonal mode decomposition in AWGN using the generalized SVD version of Eq. (13): (a) noisy FM signal with SNR = 25 dB, (b,c) selected eigenvectors of the generalized SVD solution and ESA IF estimates for selected eigenvectors, where the dashed line represents the ESA-IF estimate of the FM signal in part (a), (d) corresponding discrete S-L weight function, and (e,f) FM spectrum for noisy signal and ESA IF estimate after FM mode rejection below a threshold of η_{SNR} = -20.5 dB using 10 times simple binomial smoothing of the IF estimates.

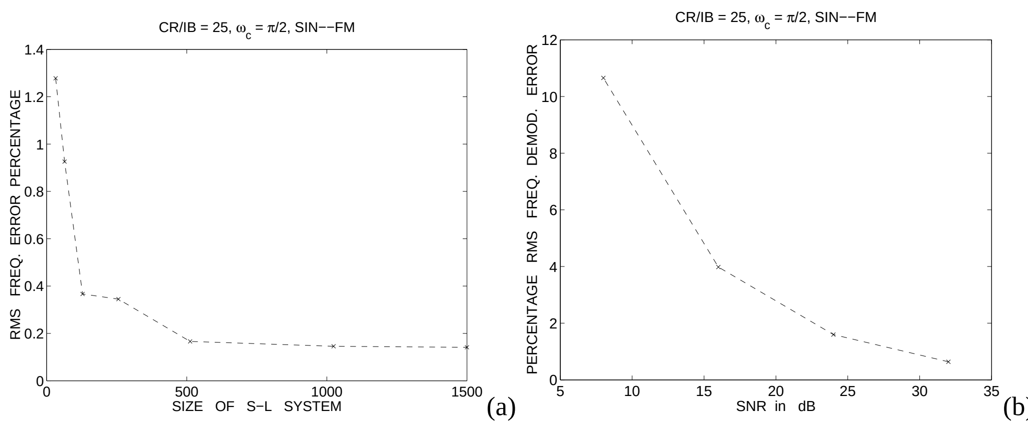

Figure 5: Effect of parameters: (a) RMS ESA frequency demodulation error percentage for different S-L operator sizes, (b) RMS frequency demodulation error percentage for different SNR's in AWGN averaged over 100 experiments

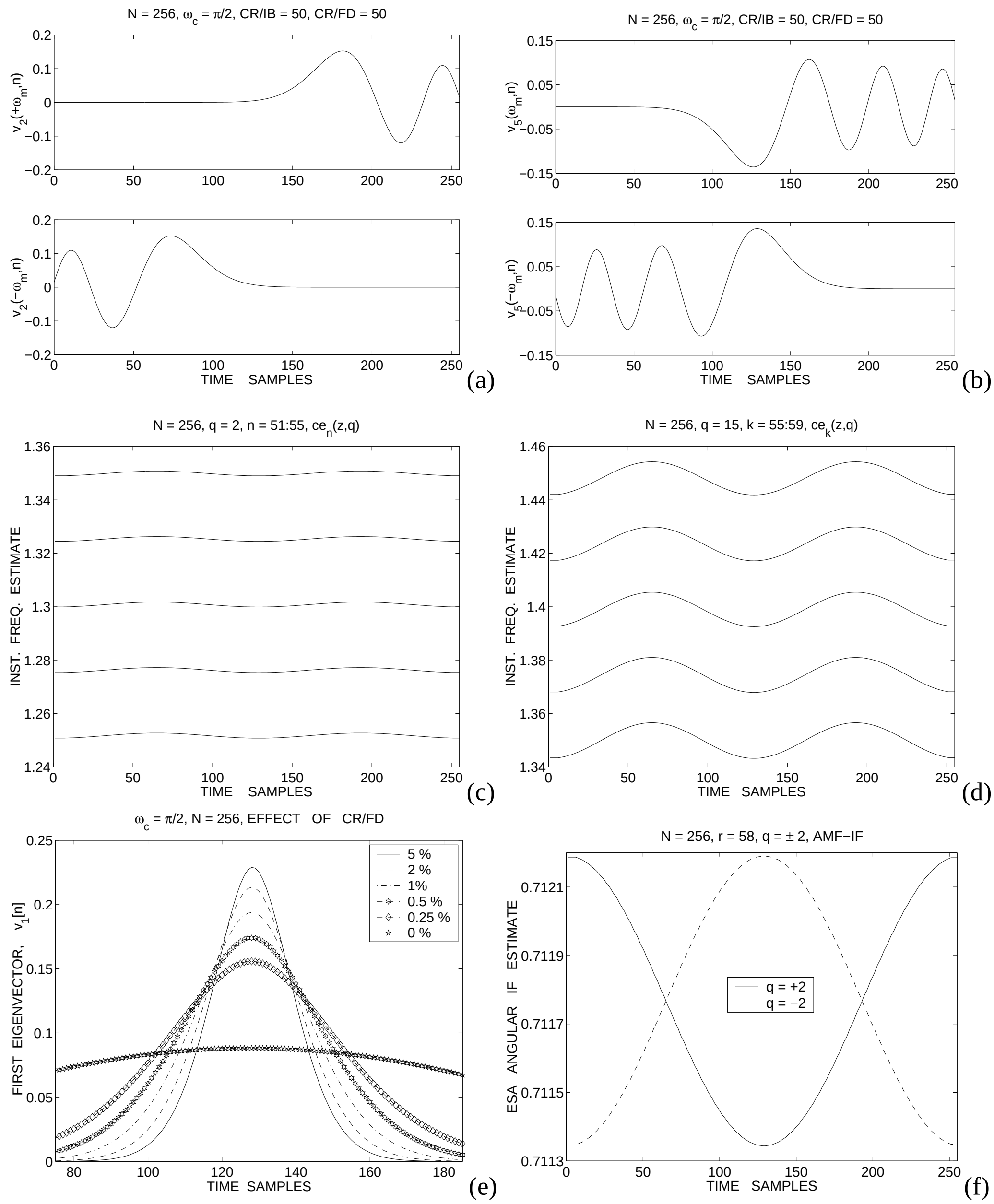

Figure 6: Orthogonal FM modes and angular Mathieu functions: (a,b) orthogonal FM modes from the S-L system for positive and negative frequency deviation depicting properties identical with angular Mathieu functions $ce_n(z,q)$ or $se_n(z,q)$ for negative q parameters, (c,d) ESA IF estimates of Mathieu functions evaluated using the framework and MATLAB functions in [21] for different q parameters depicting sinusoidal FM, (e) first orthogonal FM mode for different frequency deviation depicting that asymptotically as the frequency deviation goes to zero the FM modes become purely sinusoidal as is the case with Mathieu functions, (f) ESA IF estimates of a specific mode for both positive and negative values of q.