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Interconnected Hybrid Systems: A Framework for Multi-agent Systems with Hybrid Interacting Dynamics

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Abstract

We present a new framework for describing multi-agent systems with hybrid interacting dynamics where the interaction between agents occurs at both the continuous and discrete levels. We formally define these multi-agent systems as Interconnected Hybrid Systems and then recast fundamental hybrid concepts such as a hybrid metric, hybrid execution, and reachability in this new interconnected hybrid systems framework. We then prove a necessary and sufficient condition for the existence and uniqueness of the interconnected hybrid executions extending previous work on hybrid systems.

Keywords

Interconnected hybrid systems, reachability, metric, execution.

1 Introduction

In most of the work reported on cooperative systems, individual models for cooperating agents are described by purely continuous dynamics [2, 4, 6, 9, 12–14, 18]. There are few exceptions, where discrete event system theory is applied [3]. In exploring new communication network paradigms [7, 16] we sometimes find the use of purely continuous dynamics to be restrictive as explained below.

We envision a network in which functions (e.g. routing) are not fixed to physical nodes, but are instead implemented by software agents that are free to migrate from node to node, depending on resources that they may have to compete for [15]. This approach gives rise to a new type of multi-agent system where agent dynamics are composed by discrete states that represent the location of the agent in the network and its operating mode, and by continuous states that represent the amount of resources that the agent is receiving from the network. The node dynamics are also composed by discrete and continuous states. The discrete states represent changes in the agents hosted by the node, while continuous states represent the evolution of the resource availability due to the competition of agents for such resources. Agents start at initial locations in the network and with a given set of resources. Nodes start at discrete states that reflect the initial distribution of agents and at continuous states corresponding the initial availability of resources. The continuous states of the agents may then evolve according the agents requirements affecting the availability of resources in the nodes. Agents may also jump to different locations depending on the conditions in the nodes. These jumps will affect the continuous evolution of other agents and nodes, and will also cause discrete jumps in the nodes reflecting the new agent distribution. A pictorial example of this situation is depicted in Figure 1.

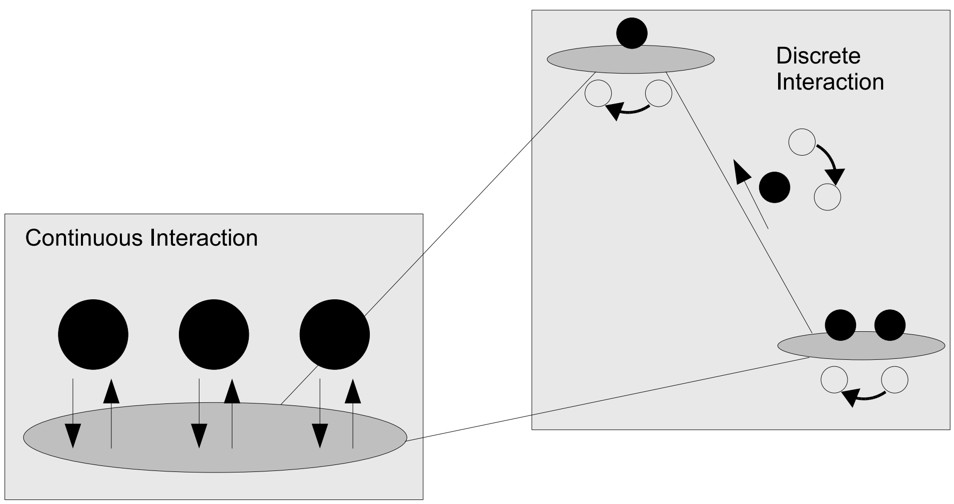

Figure 1: Example of dynamical behavior of agents and nodes. Agents are as hybrid automata. Each mode in an automaton corresponds to a possible location of an agent in the network (Agents on top). Each transition between modes represents a change of location made by an agent (agent at the bottom). The dynamics of the nodes are also modeled as hybrid systems. Each mode represents a number of agents residing at a node paired with the availability of resources that varies in discrete manner. The agents on top are located on a node, therefore have a discrete state fixed and the continuous dynamics of agents and the nodes that hosts them are interacting. The agent at the bottom is moving between nodes, so a discrete transition is happening.

It is not clear how to capture the operation of such a system with existing hybrid frameworks. The interactions between the hybrid systems that model agents and nodes happen at both the continuous and discrete levels. The continuous and discrete dynamics of the agents depend on both the continuous and discrete states of the nodes and viceversa. We attempt to capture this interaction with a new class of systems: the interconnected hybrid systems. Such systems are not mere parallel compositions, or products, of the component subsystems [19]. The existence and evolution of an individual subsystem can be meaningless if isolated. Moreover, interactions are not limited to common or uncommon events. In our case, the hybrid state in one of the systems modifies the execution in another one. Therefore we formally define the interconnected hybrid system such that the continuous evolution in one agent depends on the continuous states of agents that are connected to it, and similarly the discrete dynamics depend on continuous and discrete dynamics of neighboring agents. This definition also includes a description of the connectivity of the multi-agent system in each agent's hybrid state. We then extend our previous work [17] defining a metric for this new class of systems and explaining the properties of this metric. Finally we recast the reachability and the hybrid execution concepts from hybrid systems theory into the new framework, and provide a necessary and sufficient condition for the existence and uniqueness of the interconnected hybrid execution (the hybrid analog to the state's evolution in continuous dynamical systems), in terms of the components of each agent's hybrid model, extending some of the concepts in [11].

The reminder of this paper is organized as follows: In Section 2 we define the interconnected Hybrid System and explain the key features of this new concept. In Section 3 we introduce an Interconnected hybrid metric and provide its properties, while in Section 4 we define the interconnected hybrid execution and state the necessary and sufficient conditions for its existence and uniqueness. Section 5 provides the proofs to the results in the previous section and Section 6 outlines our conclusions.

2 Interconnected Hybrid Systems

A hybrid system is denoted $\mathbf{H}_i$, where $i \in I$ indexes the systems in a group. ν_i denotes dependence of ν on i. ν_{q_i} denotes dependence of ν on both q_i and i. ν^n, denotes the n^{th} element of a sequence in ν, and $\nu(t)$ denotes the value of ν at time t. Finally, with some abuse of notation, ν_0 marks an initial condition.

Let Q_i be the set of discrete states of $\mathbf{H}_i$, where $Q_i = O_i \times D_i$, where O_i is the set of operating states and D_i is the set of connectivity states. Each $o_i \in O_i$ represents a different operating condition of $\mathbf{H}_i$. Each $d_i \in D_i$, represents different connectivity conditions. $(o_i, d_i) \in Q_i$ is denoted as q_i. Each q_i has an associated set $V(q_i) \subseteq I$ $\forall q_i \in Q_i$ and $\forall i \in I$, which stores the indexes of the systems that are connected to $\mathbf{H}_i$, i.e., if $j \in V(q_i)$ then $\mathbf{H}_j$ is connected to $\mathbf{H}_i$. Note that $V(q) = V(q')$ for all $q = (o,d), q' = (o',d') \in Q_i$ that satisfy $d = d'$.

Let $\Sigma_i = \{\Sigma_{q_i}\}_{q_i \in Q_i}$ where $\Sigma_{q_i} = (X_{q_i}, f_{q_i}, U_{q_i}, \mathbb{R}^+)$ is a continuous dynamical system that corresponds to $q_i \in Q_i$ with X_{q_i} being the continuous state space, f_{q_i} the continuous dynamics, U_{q_i} the set of continuous controls, and $\mathbb{R}^+ = [0, \infty)$ the time set.

Let $\mathbf{S}_i = \{S_{q_i}\}_{q_i \in Q_i}$ be the set of discrete transition labels of $\mathbf{H}_i$. Symbol $s_{q_i} \in S_{q_i}$ determines the discrete state after a transition from $q_i \in Q_i$ in system $\mathbf{H}_i$. We consider two types of transitions: Transitions triggered by local external events and transitions that are functions of the states of the local system and the systems connected to it.

Let $\mathbf{G}_i = \{G_{q_i}\}_{q_i \in Q_i}$ be the set of guard conditions for $\mathbf{H}_i$. G_{q_i} is a map that determines when a transition is possible from $q_i \in Q_i$. The set G_{q_i} is defined below for each type of transition. Let $\mathbf{Z}_i = \{Z_{q_i}\}_{q_i \in Q_i}$ be the set of transition maps of $\mathbf{H}_i$, where $Z_{q_i} : G_{q_i} \times S_{q_i} \rightarrow \bigcup_{p_i \in Q_i} \{X_{p_i}\}$ determines the continuous state of $\mathbf{H}_i$ after a transition $s_{q_i} \in S_{q_i}$.

Definition 1 (Interconnected Hybrid System) *An* Interconnected Hybrid System *(IHS) is a set* $\mathbf{H}^* = \{\mathbf{H}_i\}_{i \in I}$ *of Controlled Hybrid Dynamical Systems [1]* $\mathbf{H}_i$ *indexed by the set I. For each* $i \in I$, $\mathbf{H}_i = [Q_i, \Sigma_i, \mathbf{G}_i, \mathbf{Z}_i, \mathbf{S}_i]$*, such that*

- *The continuous control inputs in* U_{q_i} *are the continuous states of the systems that are connected to* $\mathbf{H}_i$*. Therefore* $U_{q_i} = X_{q_i} \times \prod_{j \in V(q_i)} X_j$*. Note that that the dimension of* U_{q_i} *is lesser or equal to the dimension of* $\prod_{i \in I} X_i$ *for all* $q_i \in Q_i$ *for all* $i \in I$*.*

- *A guard condition for an event-triggered transition is denoted as* $G^{\mathrm{E}}_{q_i}$. *This guard must satisfy a condition on the state of the system(s) and on the existence of an event i.e,* $G^{\mathrm{E}}_{q_i} : S_{q_i} \to E_i \times X_{q_i} \times \prod_{j\in V(q_i)} H_j$ *where* E_i *is the set of possible events of* $\mathbf{H}_i \in \mathbf{H}^*$. *A guard condition for a state-based transition, denoted* $G^{\mathrm{S}}_{q_i}$ *needs to satisfy a condition on the state of the system(s) only i.e,* $G^{\mathrm{S}}_{q_i} : S_{q_i} \to X_{q_i} \times \prod_{j\in V(q_i)} H_j$.

The state space of the IHS $\mathbf{H}^*$ *is* $H^* = \prod_{i\in I} H_i$ *where* $H_i = Q_i \times \bigcup_{q_i\in Q_i} X_{q_i}$ *is the state space of hybrid system* $\mathbf{H}_i$, *and the state of the IHS is denoted as* $\vec{h} = (\vec{q}, \vec{x}_{\vec{q}})$ *where* $\vec{q} = (q_i)^T_{i\in I}$, *and* $\vec{x}_{\vec{q}} = (x^T_{q_i})^T_{i\in I}$, *where* $q_i \in Q_i$ *for all* $i \in I$, *and* $x_{q_i} \in X_{q_i}$ *for all* $q_i \in Q_i$ *and for all* $i \in I$.

Note in Definition 1 that the discrete states of the systems are divided into operating states, which are used to describe modes of operation of each individual agent in the system, and connectivity states, which describe the possible configurations for information exchange between agents in the system. If one thinks in the usual graph theoretic argument that describes the connectivity between agents in multi-agent systems literature [2, 4, 6, 9, 12–14, 18] different connectivity states in each agent correspond to its different possible neighborhoods. We however, do not limit the connectivity description of the IHS to the use of graph theory. Also note that no assumptions are made about symmetry on the connectivity, so this definition includes the possibility of agent $i \in I$ being connected to agent $j \in I : j \neq i$ without j being connected to i, which corresponds to a directed graph on the graph theoretic argument.

The interactions between the agents in the systems are achieved in the continuous dynamics through the continuous control inputs. The continuous control inputs of agent $i \in I$ in the IHS are functions of the continuous state of agent $i \in I$ and the continuous states of the agents that are directly connected to agent $i \in I$. Therefore the continuous evolution of each agent is influenced by the continuous dynamics of the agents that are connected to it.

The interactions between the discrete dynamics of the agents in the system are achieved through the transition guards. In both cases (the event-triggered, and the state-based transition) the transition guards of agent $i \in I$ set conditions on the continuous states of agent $i \in I$ and on the hybrid states of the agents that are connected to agent $i \in I$. So, for the case of state-based transitions, a discrete transition may occur when both the continuous state of agent i and the hybrid states the the agents connected to $i \in I$ reach a guard condition. In the event-triggered case, a discrete transition may occur on agent $i \in I$ when this agent experiences an external event if the condition on the states of agent i and the agents connected to it is satisfied. Therefore in both state-based and event-triggered transitions of agent $i \in I$, the discrete dynamics are influenced by the hybrid states of the agents that are connected to agent $i \in I$. Note that the events are assumed to be local, i.e an event in agent $i \in I$ has direct influence only on this agent's dynamics. However, since an event will generate an state change in agent $i \in I$, such state change will potentially affect the dynamics of the agents that are connected to agent $i \in I$. For this reason we believe that the assumption of the events being local should not represent a restriction.

To summarize, Definition 1 presents a hybrid analog to the standard multi-agent setting [2, 4, 6, 9, 12–14, 18] where each agent uses the states of its neighbors to update its own evolution. The following is an standing assumption for the rest of this paper.

Assumption 1 *The sets of discrete states* Q_i *are finite for all* $i \in I$. *There exist a vector space* X_i *such that* $X_{q_i} \subseteq X_i \subseteq \mathbb{R}^d$ *for all* $q_i \in Q_i$ *for all* $i \in I$ *where* d *is an integer. The vector fields* $f_{q_i}(x_{q_i}, u_{q_i}, t)$ *are globally Lipschitz continuous [8] on both* x_{q_i} *and* u_{q_i} *with Lipschitz constants* $L_{x_{q_i}}$ *and* $L_{u_{q_i}}$ *for all* $q_i \in Q_i$ *for all* $i \in I$.

3 A Metric for Interconnected Hybrid Systems

In [17] we introduce a new notion of hybrid metric. We extend this concept for interconnected hybrid systems. Let the directed graph that represents the hybrid system $\mathbf{H}$ [10] be denoted as $\mathcal{G}_H$.

Definition 2 (Discrete Distance [17]) *Let the distance between two discrete states of a hybrid system q and q' be the length of the shortest path*[1] *from node q to node q' in $\mathcal{G}_H$. This distance is denoted by $d_D(q,q')$.*

Definition 3 (Interconnected hybrid distance (IHD)) *Let the distance from $\vec{h} \in H^*$ to $\vec{h}' \in H^*$ be:*

$$d_H^*(\vec{h},\vec{h}') = \max_{i \in I}(d_D(q_i,q_i')) + \tanh(\|\vec{x}_{\vec{q}} - \vec{x}'_{\vec{q}'}\|)$$

where for each $i \in I$, q_i and q_i' are the components of $\vec{q} = (q_i)_{i\in I}^T$ and $\vec{q}' = (q_i')_{i\in I}^T$.

Note that $\vec{x}_{\vec{q}} = (x_{q_i}^T)_{i\in I}^T$ and $\vec{x}'_{\vec{q}'} = (x'^T_{q_i'})_{i\in I}^T$ where each $x_{q_i}^T$ and $x'^T_{q_i'}$ is a vector. Then $\vec{x}$ and $\vec{x}'$ are vectors formed by concatenating the vector states of each individual system in $\mathbf{H}^*$. Therefore the norm $\|\vec{x}_{\vec{q}} - \vec{x}'_{\vec{q}'}\|$ is well defined on $\prod_{i\in I} X_i$. In the reminder of this section we drop the subindex notation on $\vec{x}_{\vec{q}}$ for simplicity because the correspondence between $\vec{x}_{\vec{q}}$ and $\vec{q}$ is clear from the context.

The $\tanh(.)$ function of the norm in the interconnected hybrid distance provides a mechanism to distinguish the discrete and the continuous parts of the distance between two interconnected hybrid states: The interconnected hybrid distance is composed by an integer and a fractionary part. The integer part provides the exact number of discrete transitions that the system needs to experience to reach one discrete state from another, while the fractionary part results from the application of an invertible function to the standard notion of distance between two continuous states.

Theorem 1 (Properties of the IHD) *Given three interconnected hybrid states $\vec{h} = (\vec{q},\vec{x}), \vec{h}' = (\vec{q}',\vec{x}'), \vec{h}'' = (\vec{q}'',\vec{x}'') \in H^*$, the following properties hold:*

1. *$d_H^*(\vec{h},\vec{h}') \geq 0$ for all $h,h' \in H^*$.*
2. *$d_H^*(\vec{h},\vec{h}') = 0$ if and only if $\vec{h} = \vec{h}'$.*
3. *$d_H^*(\vec{h},\vec{h}'') \leq d_H^*(\vec{h},\vec{h}') + d_H^*(\vec{h}',\vec{h}'')$ for all $h,h',h'' \in H^*$.*

Proof:

1. From Def. 2 here, and Def. 11 and Prop. 2 in [17] $d_D(q_i,q_i') \geq 0\ \forall i \in I$, then $\max_{i\in I} d_D(q_i,q_i') \geq 0$. By properties of norm and of tanh, $\tanh\|\vec{x}-\vec{x}'\| \geq 0$ for all $\vec{x},\vec{x}' \in \prod_{i\in I} X_i$. Thus $d_H^*(\vec{h},\vec{h}') \geq 0$ for all $h,h' \in H^*$.
2. $(\Rightarrow)$ If $\vec{h} = \vec{h}'$, $\vec{q} = \vec{q}'$ and $\vec{x} = \vec{x}'$. $\vec{q} = \vec{q}'$ implies $q_i = q_i'\ \forall i \in I$. Then $d_D(q_i,q_i') = 0\ \forall i \in I$, which implies $\max_{i\in I}(d_D(q_i,q_i')) = 0$. $\vec{x} = \vec{x}'$ implies $\tanh(\|\vec{x}-\vec{x}'\|) = 0$. Thus $\vec{h} = \vec{h}' \Rightarrow d_H^*(\vec{h},\vec{h}') = 0$.

   $(\Leftarrow)$ Since $\max_{i\in I} d_D(q_i,q_i') \geq 0$ and $\tanh(\|\vec{x}-\vec{x}'\|) \geq 0\ \forall \vec{h},\vec{h}' \in H^*$, $d_H^*(\vec{h},\vec{h}') = 0$ implies that $\max_{i\in I} d_D(q_i,q_i') = 0$ and $\tanh(\|\vec{x}-\vec{x}'\|) = 0$. From Def. 2 $\max_{i\in I} d_D(q_i,q_i') = 0$ implies $d_D(q_i,q_i') = 0\ \forall i \in I$, which together with Prop. 1 in [17] implies $q_i = q_i'\ \forall i \in I$, which implies $\vec{q} = \vec{q}'$. $\tanh(\|\vec{x}-\vec{x}'\|) = 0$ implies $\|\vec{x}-\vec{x}'\| = 0$, which implies $\vec{x} = \vec{x}'$. Thus $d_H^*(\vec{h},\vec{h}') = 0 \Rightarrow \vec{h} = \vec{h}'$.

   $(\Rightarrow)$ and $(\Leftarrow)$ imply $d_H^*(\vec{h},\vec{h}') = 0 \Longleftrightarrow \vec{h} = \vec{h}'$.
3. **Discrete:** From Lemma 2 in [17] $d_D(q_i,q_i'') \leq d_D(q_i,q_i') + d_D(q_i',q_i'')$. Suppose $\exists \vec{q},\vec{q}',\vec{q}'' \in \prod_{i\in I} Q_i$ such that $\max_{i\in I} d_D(q_i,q_i'') > \max_{i\in I} d_D(q_i,q_i') + \max_{i\in I} d_D(q_i',q_i'')$. Then, $\exists i,j,k \in I$ such that $d_D(q_i,q_i'') > d_D(q_j,q_j') + d_D(q_k',q_k'')$. Note that this implies $d_D(q_j,q_j') \geq d_D(q_i,q_i')$ and $d_D(q_k',q_k'') \geq d_D(q_i',q_i'')$. This implies $d_D(q_i,q_i'') > d_D(q_i,q_i') + d_D(q_i',q_i'')$, which contradicts Lemma 2 in [17]. Therefore $\max_{i\in I} d_D(q_i,q_i'') \leq \max_{i\in I} d_D(q_i,q_i') + \max_{i\in I} d_D(q_i',q_i'')\ \forall \vec{q},\vec{q}',\vec{q}'' \in \prod_{i\in I} Q_i$.

   Continuous: $\tanh(\|x-x''\|) \leq \tanh(\|x-x'\|) + \tanh(\|x -' x''\|)$ follows from Lemma 3 in [17]. **Discrete** and **Continuous** parts imply the claim.

[1]For a definition of a path, see [5].

■

Note that the interconnected hybrid distance does not satisfy the symmetry property that metrics usually do because of the use of the discrete distance of Definition 2. However, we believe that the absence of this property is actually desirable because the number of transitions that are required to reach $\vec{q}$ from $\vec{q}'$ may be different from the number of transitions required to reach $\vec{q}'$ from $\vec{q}$.

It is possible to reformulate Definition 3 and Theorem 1 to prevent simultaneous discrete transitions among different individual systems. In such case a more meaningful notion of distance would be $d_H^*(\vec{h},\vec{h}') = \sum_{i\in I}(d_D(q_i,q_i')) + \tanh(\|\vec{x}-\vec{x}'\|)$.

4 Interconnected Hybrid Execution

In this section we introduce the Interconnected Hybrid Execution (IHE) based on the concept of hybrid execution in [11]. The IHE is the analog of the state evolution of a continuous multi-agent dynamical system, and captures the system's hybrid behavior with respect to both discrete and continuous interactions of the agents among themselves and with and with its environment. Then we provide conditions for the existence and uniqueness of an infinite IHE. These conditions are stated as a function of each agent in the system. Therefore the desired global behavior of the system (existence and uniqueness of its execution), can be guaranteed by the specification of local design variables inside each agents dynamics.

A *Time Trajectory* is a sequence $\bar{\tau} = \{\bar{\tau}^0, \bar{\tau}^1, \ldots, \bar{\tau}^n, \ldots, \bar{\tau}^{\bar{N}}\}$, where $\bar{\tau}^n \le \bar{\tau}^{n+1}$ for all $n = \{0,1,\ldots,\bar{N}-1\}$. $\bar{\tau}$ is infinite if $\bar{N} = \infty$ and is finite otherwise. τ is an *Interconnected Hybrid Time Trajectory* (IHTT) if τ is a time trajectory and if 1) $\tau^0 \in \tau$ is the time when $\mathbf{H}^*$ starts its evolution, 2) $\tau^n \in \tau$ is the time at which there is a system $\mathbf{H}_i \in \mathbf{H}^*$ that makes a discrete transition from q_i^n to q_i^{n+1} for $n = \{0,1,\ldots,N-1\}$, such that the Interconnected Hybrid System $\mathbf{H}^*$ makes a discrete transition from $\vec{q}^n$ to $\vec{q}^{n+1}$, and 3) $\tau^N \in \tau$ is the time when $\mathbf{H}^*$ ends its evolution. $\hat{\tau}$ is an *Event Time Trajectory* (ETT), if $\hat{\tau}$ is a time trajectory and $\hat{\tau}^n$ is the time when there is a system $\mathbf{H}_i \in \mathbf{H}^*$ that experiments a discrete event $e_i^n \in E_i$ for all $n \in \{0,1,\ldots,\hat{N}\}$ where $\hat{N}$ is the number of events that $\mathbf{H}^*$ experiments.

The IHTT and the ETT are used to encode timing information for the continuous and discrete dynamics of the IHS $\mathbf{H}^*$. The IHTT stores the times when a discrete transition takes place at least on one of the agents in the system. As a consequence the IHTT also specifies time intervals between two consecutive elements in the sequence where uninterrupted continuous evolution takes place. On the other hand the ETT stores information about the specific times that events happen somewhere in the system. Note that this two sequences are considered completely independent. This is useful because the occurrence of an event does not necessarily imply that a discrete transition takes place.

The IHTT and the ETT as defined above allow to have more than one hybrid system in the overall IHS taking a discrete transition or experimenting an event at the same time. These definitions may be reformulated to exclude this possibility. Any time trajectory $\bar{\tau}$ is linearly ordered by the relation $\prec$ defined by $t_1 \prec t_2$ for $t_1 \in [\bar{\tau}_i, \bar{\tau}_{i+1}]$ and $t_2 \in [\bar{\tau}_j, \bar{\tau}_{j+1}]$ if $t_1 < t_2$ or $i < j$. We say $\bar{\tau} = \{\bar{\tau}^0, \bar{\tau}^1, \ldots, \bar{\tau}^{\bar{N}}\}$ is a prefix of $\tilde{\tau} = \{\tilde{\tau}^0, \tilde{\tau}^1, \ldots, \tilde{\tau}^{\tilde{N}}\}$ (written $\bar{\tau} \sqsubseteq \tilde{\tau}$) if either they are identical, or $\bar{\tau}$ is finite, $\bar{N} \le \tilde{N}$, $\bar{\tau}_n = \tilde{\tau}_n$ for all $n \in \{0,1,\ldots,\bar{N}-1\}$, and $[\bar{\tau}_{N-1}, \bar{\tau}_N[\subseteq [\tilde{\tau}_{N-1}, \tilde{\tau}_N[$, where $[$ is either $]$ or $)$.

A *Group Event Sequence* of $\mathbf{H}^*$ is a collection $\mathcal{E}^* = (\hat{\tau}, \mathrm{Es})$ where $\hat{\tau}$ is an ETT and $\mathrm{Es} = (e^0_{\alpha^0}, e^1_{\alpha^1}, \ldots, e^{\hat{N}}_{\alpha^{\hat{N}}})$ is the sequence of events that $\mathbf{H}^*$ experiments, where $\alpha^n \in I$ for all $n \in \{0,1,\ldots\hat{N}\}$, such that $e^n_{\alpha^n} \in E_{\alpha^n}$ specifies the event that occurs at $\hat{\tau}^n$, and the individual system $\mathbf{H}_{\alpha^n}$ that experiments such event for all $n \in \{0,1,\ldots\hat{N}\}$. Therefore the group event sequence contains ordered pairs composed by the time when an event occur at any agent in the system and an event label that indicates what event occurred, and identifies the agent that experimented such event.

In the following, in order to simplify the description of our results we divide the transition guards into a local part, a remote part, and an event part when needed. The local part verifies that the state of the agent experimenting a discrete transition satisfies the transition guard. The remote part verifies that the states of the agents connected to the one that is experimenting the transition satisfy the transition guard, and finally the event part (in the case of an event-triggered transition) verifies that the agent experimenting the transition has also experimented a discrete event that enables such transition. Let $G_{q_i}^{\text{S/Local}}(s) \subseteq X_{q_i}$ denote the first element in the cartesian product of the state-based transition guard $G_{q_i}^{\text{S}}(s)$. Let $G_{q_i}^{\text{S/Remote}}(s) \subseteq \prod_{j\in V(q_i)} H_j$ denote the remainder of the elements of the cartesian product of the state-based transition guard $G_{q_i}^{\text{S}}(s)$. Let $G_{q_i}^{\text{E/E}}(s) \subseteq E_i$ denote the first element in the cartesian product of the event-triggered transition guard $G_{q_i}^{\text{E}}(s)$. Let $G_{q_i}^{\text{E/Local}}(s) \subseteq X_{q_i}$ denote the second element in the cartesian product of the event-triggered transition guard $G_{q_i}^{\text{E}}(s)$. Finally let $G_{q_i}^{\text{E/Remote}}(s) \subseteq \prod_{j\in V(q_i)} H_j$ denote the remainder of the elements of the cartesian product of the event-triggered transition guard $G_{q_i}^{\text{E}}(s)$. We also use the following notation: $q_i \bar{\in} \vec{q}$ if q_i is a component of the vector $\vec{q}$. $x_{q_i} \bar{\in} \vec{x}_{\vec{q}}$ if x_{q_i} is a component of $\vec{x}_{\vec{q}}$ where $q_i \bar{\in} \vec{q}$ (Similarly for $s_{q_i} \bar{\in} \vec{s}$ and $u_{q_i} \bar{\in} \vec{u}$). $h_i \bar{\in} \vec{h}$ if h_i is a component of $\vec{h}$. Finally since $\vec{h} = (\vec{q}, \vec{x}_{\vec{q}})$ we also say $q_i \bar{\in} \vec{h}$ if $q_i \bar{\in} \vec{q}$ and $x_{q_i} \bar{\in} \vec{h}$ if $x_{q_i} \bar{\in} \vec{x}_{\vec{q}}$.

We say that $\vec{h}(t)$ satisfies the state-based transition guard $G_{q_i}^{\text{S}}(s)$ if $x_{q_i}(t) \in G_{q_i}^{\text{S/Local}}(s)$, and $(h_j)_{j\in V(q_i)}(t) \in G_{q_i}^{\text{S/Remote}}(s)$, where $q_i \bar{\in} \vec{h}$, $x_{q_i} \bar{\in} \vec{h}$, and $h_i \bar{\in} \vec{h}$ for all $i \in I$. Similarly $\vec{h}(t)$ satisfies the event-triggered transition guard $G_{q_i}^{\text{E}}(s)$ if $x_{q_i}(t) \in G_{q_i}^{\text{E/Local}}(s)$, and $(h_j)_{j\in V(q_i)}(t) \in G_{q_i}^{\text{E/Remote}}(s)$, where where $q_i \bar{\in} \vec{h}$, $x_{q_i} \bar{\in} \vec{h}$, and $h_i \bar{\in} \vec{h}$ for all $i \in I$. Finally $(\hat{\tau}^k, e_{\alpha^k}^k)$ satisfies the event-triggered transition guard $G_{q_i}^{\text{E}}(s)$ at time t if $\alpha^k = i$, $\hat{\tau}^k = t$, and $e_{\alpha^k}^k \in G_{q_i}^{\text{E/E}}(s)$,

Definition 4 (Interconnected Hybrid Execution) *An* Interconnected Hybrid Execution *(IHE)* $\chi(\vec{h}_0, \mathcal{E}^*)$ *with initial conditions* $\vec{h}_0$ *and group event sequence* $\mathcal{E}^*$ *is a collection* $(\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$*, where:*

- τ *is an interconnected hybrid time trajectory.*
- $\mathbf{q} = \{\vec{q}^0, \vec{q}^1, \ldots, \vec{q}^n, \ldots, \vec{q}^N\}$ *is a sequence of vectors of discrete locations* $\vec{q}^n = (q_i^n)_{i\in I}^T$ *where* q_i^n *is the discrete mode of system* $\mathbf{H}_i$ *at the n step on the sequence.*
- $\mathbf{s} = \{\vec{s}^0, \vec{s}^1, \ldots, \vec{s}^n, \ldots, \vec{s}^N\}$ *is a sequence of vectors of switching labels* $\vec{s}^n = (s_{q_i}^n)_{i\in I}^T$ *where* $s_{q_i}^n$ *is the switching label of system* $\mathbf{H}_i$ *at n step in the execution.*
- $\mathbf{x} = \{\vec{x}^0, \vec{x}^1, \ldots, \vec{x}^n, \ldots, \vec{x}^N\}$ *is a sequence of continuous evolution* $\vec{x}^n = (x_{q_i^n}^T)_{i\in I}^T$ *where* $x_{q_i^n}$ *is a differentiable map* $x_{q_i^n} : [\tau^{n-1}, \tau^n[\to X_{q_i^n}$ *of system* $\mathbf{H}_i$ *at the n step on the sequence for all* $i \in I$.
- $\mathbf{u} = \{\vec{u}^0, \vec{u}^1, \ldots, \vec{u}^n, \ldots, \vec{u}^N\}$ *is a sequence of continuous control inputs* $\vec{u}^n = \left(u_{q_i^n}\right)_{i\in I}^T$ *where* $u_{q_i^n}$ *is a differentiable map* $u_{q_i^n} : [\tau^{n-1}, \tau^n[\to U_{q_i^n}$ *at the n step on the sequence for all* $i \in I$.

The interconnected hybrid execution $\chi(\vec{h}_0, \mathcal{E}^*)$ *satisfies the following conditions:*

- *Initial Condition:* $\vec{h}_0 = (\vec{q}^0, \vec{x}^0(0))$ *is an initial condition of* $\mathbf{H}^*$.
- *Continuous Dynamics: for all* $t \in [\tau^{n-1}, \tau^n[$*, for all* $n \in \{1, 2, \ldots, N\}$*, and for all* $i \in I$, $\dot{x}_{q_i^n}(t) = f_{q_i^n}(x_{q_i^n}, u_{q_i^n}, t)$, $x_{q_i^n} \in X_{q_i^n}$ *and* $u_{q_i^n} \in U_{q_i^n}$ *where* $q_i^n \bar{\in} \vec{q}^n$, $x_{q_i^n} \bar{\in} \vec{x}^n$*, and* $u_{q_i^n} \bar{\in} \vec{u}^n$.
- *Discrete Dynamics: Either the* ***event-triggered transition conditions*** *or the* ***state-based transition conditions*** *hold for each* $n \in \{0, 1, 2, \ldots, N-1\}$ *and for all* $i \in I$. *The* ***event-triggered transition conditions*** *are:*
  - $q_i^{n+1} = s_{q_i^n} \in S_{q_i^n}$*, where* $q_i^{n+1} \bar{\in} \vec{q}^{n+1}$ *and* $s_{q_i^n} \bar{\in} \vec{s}^n$,
  - *There exists a* $(\hat{\tau}^k, e_{\alpha^k}^k) \in \mathcal{E}^*$ *that satisfies* $G_{q_i^n}^{\text{E}}(s_{q_i^n})$ *at time* τ^n.
  - $\vec{h}^n(\tau^n)$ *satisfies* $G_{q_i^n}^{\text{E}}(s_{q_i^n})$.

- $x_{q_i^{n+1}}(\tau^n) \in Z_{q_i^n}\big(e^k_{\alpha^k}, x_{q_i^n}, (h_j^n)_{j\in V(q_i^n)}, s_{q_i^n}\big)$.

*The **state-based transition conditions** are:*

- $q_i^{n+1} = s_{q_i^n} \in S_{q_i^n}$, *where* $q_i^{n+1} \bar{\in} \vec{q}^{n+1}$ *and* $s_{q_i^n} \bar{\in} \vec{s}$,
- $\vec{h}^n(\tau^n)$ *satisfies* $G^{\mathrm{S}}_{q_i^n}(s_{q_i^n})$.
- $x_{q_i^{n+1}}(\tau^{n+1}) \in Z_{q_i^n}(x_{q_i^n}, (h_j^n)_{j\in V(q_i^n)}, s_{q_i^n})$.

The IHE provides the information about the continuous and discrete states and inputs of the system at each instant of its evolution. It is the analog of the state-input trajectory in continuous time systems. The conditions imposed in the second part of Definition 4 are required for it to be valid to $\mathbf{H}^*$. Therefore an IHE should start at a valid initial condition. The continuous evolution between two times in the interconnected hybrid time trajectory should satisfy the continuous dynamics of each agent, and the discrete transitions should have valid transition guards and transition maps. Note that we used $\chi(\vec{h}_0, \mathcal{E}^*)$ to denote an IHE with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$.

In order to prove the existence and uniqueness we need to define additional concepts. We say that an IHE $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ of $\mathbf{H}^*$ with $N+1$ elements is a prefix of another IHE $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*) = (\tilde{\tau}, \tilde{\mathbf{q}}, \tilde{\mathbf{s}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}})$ of $\mathbf{H}^*$ with $\tilde{N}+1$ elements (written $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$) if $\tau \sqsubseteq \tilde{\tau}$, and for all $n \in \{0, 1, \ldots, N\}$ and for all $t \in [\tau^{n-1}, \tau^n[$ $(\vec{q}^n, \vec{s}^n, \vec{x}^n(t), \vec{u}^n(t)) = (\vec{\tilde{q}}^n, \vec{\tilde{s}}^n, \vec{\tilde{x}}^n(t), \vec{\tilde{u}}^n(t))$. We say that $\chi(\vec{h}_0, \mathcal{E}^*)$ is a strict prefix of $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ (written $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubset \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$) if $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$, and $\chi(\vec{h}_0, \mathcal{E}^*) \neq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$.

An IHE $\chi(\vec{h}_0, \mathcal{E}^*)$ is called maximal if it is not a strict prefix of any other execution. An IHE $\chi(\vec{h}_0, \mathcal{E}^*)$ is finite if τ is a finite sequence and the last elements of $\mathbf{u}$ and $\mathbf{x}$ are defined over compact intervals of time, i.e. $\vec{u}^N : [\tau^{N-1}, \tau^N] \to \prod_{i\in I} U_{q_i^n}$, and $\vec{x}^N : [\tau^{N-1}, \tau^N] \to \prod_{i\in I} X_{q_i^n}$. $\chi(\vec{h}_0, \mathcal{E}^*)$ is infinite if τ is an infinite sequence or if $\tau^N = \infty$.

$\chi^S(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all IHEs with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$, and similarly $\chi^F(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all finite IHEs, $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all infinite IHEs, and $\chi^M(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all maximal IHEs with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$. Init denotes the set of all initial conditions, and ESS denotes the set of all possible group event sequences.

We say that $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$ if $\tau = \{\tau^0, \tau^1, \ldots, \tau^N\}$ and $\vec{h} = (\vec{q}^N, \vec{x}^N(\tau^N))$. An interconnected hybrid state $\vec{h} \in Reach(\vec{h}_0, \mathcal{E}^*)$ if there exists a finite IHE $\chi(\vec{h}_0, \mathcal{E}^*) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$. The set of states $\vec{h}$ that can be reached from any initial condition and with any group event sequence $Reach_{\mathbf{H}^*} = \bigcup_{(\vec{h}_0, \mathcal{E}^*)\in \mathrm{Init}\times\mathrm{ESS}} Reach(\vec{h}_0, \mathcal{E}^*)$ is called *Interconnected Reachable Set*.

Let $\psi(q_i, x_{q_i}, u_{q_i}, t)$ denote the continuous flow of $f_{q_i}(x_{q_i}, u_{q_i}, t)$ for all $i \in I$. We define the set for which continuous evolution is impossible as $Out_{\mathbf{H}^*} = \{\vec{h} \in \prod_{i\in I} X_i \times \prod_{i\in I} Q_i; \forall \varepsilon > 0, \exists t \in [0, \varepsilon)$ and $\exists i \in I$, such that $\psi(q_i, x_{q_i}, u_{q_i}, t) \notin X_{q_i}$, where $q_i \bar{\in} \vec{h}, x_{q_i} \bar{\in} \vec{h}\}$. This set specifies what states in the system require a discrete transition for the system to continue its evolution.

We say that $\mathbf{H}^*$ is deterministic if given $\vec{h}_0$ and $\mathcal{E}^*$, $\chi^M(\vec{h}_0, \mathcal{E}^*)$ contains at most one element. The following result provides the necessary and sufficient conditions for existence of an infinite execution given that the system is deterministic. These conditions combined with the condition for a IHS to be deterministic yield the existence and uniqueness of an infinite IHE. The proof of Lemma 1 is provided in Section 5.

Lemma 1 (Existence if deterministic) *Suppose $\mathbf{H}^*$ is deterministic. Then given an initial condition $\vec{h}_0$ and a group event sequence, $\mathcal{E}^*$, $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is nonempty if and only if for all $\vec{h} \in Reach_{\mathbf{H}^*} \bigcap Out_{\mathbf{H}^*}$ either one of the following conditions holds:*

1. *There exist a $s \in S_{q_i}$ for some $q_i \bar{\in} \vec{h}$ such that $\vec{h}$ satisfies $G^{\mathrm{S}}_{q_i}(s)$.*

2. *There exist a $s \in S_{q_i}$ for some $q_i \bar{\in} \vec{h}$ such that $\vec{h}$ satisfies $G^{\mathrm{E}}_{q_i}(s)$, and there exists $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$ that satisfies $G^{\mathrm{E}}_{q_i}(s)$ at τ^N where τ^N is the time of the last element of the finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ that maps the system $\mathbf{H}^*$ from $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$.*

Note that the conditions in this lemma essentially require that whenever the system gets into an state where continuous evolution is impossible, it is guaranteed that a discrete transition from that state exists. In the following we state the necessary and sufficient conditions for an IHS to be deterministic. The proof of Lemma 2 is provided in Section 5.

Definition 5 (Forced Transition Condition) *Given an initial condition $\vec{h}_0$ and a group event sequence $\mathcal{E}^*$, we say that $\vec{h} \in Reach_{\mathbf{H}^*}$ satisfies the Forced Transition (FT) condition if one of the following holds:*

- *If there exists a $s \in S_{q_i}$ for some $q_i \bar{\in} \vec{h}$ such that $\vec{h}$ satisfies $G^{\mathrm{S}}_{q_i}(s)$, then $\vec{h} \in Out_{\mathbf{H}^*}$.*
- *If there exists a $s \in S_{q_i}$ for some $q_i \bar{\in} \vec{h}$ such that $\vec{h}$ satisfies $G^{\mathrm{E}}_{q_i}(s)$, and there exists $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$ that satisfies $G^{\mathrm{E}}_{q_i}(s)$ at τ^N where τ^N is the last element of τ in the finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$, then $\vec{h} \in Out_{\mathbf{H}^*}$*

Definition 6 (Disjoint Transition Guard Condition) *Given an initial condition $\vec{h}_0$ and a group event sequence $\mathcal{E}^*$, we say that $\vec{h} \in Reach_{\mathbf{H}^*}$ satisfies the Disjoint Transition Guard (DTG) condition if one of the following holds:*

- *If there exist $s, s' \in S_{q_i}$ for some $\mathbf{H}_i \in \mathbf{H}^*$, where $s \neq s'$ and at least one of s, s' is a state-based transition, then either $x_{q_i} \notin G^{T/\mathrm{Local}}_{q_i}(s) \bigcap G^{T'/\mathrm{Local}}_{q_i}(s')$, or $(h_j)_{j \in V(q_i)} \notin G^{T/\mathrm{Remote}}_{q_i}(s) \bigcap G^{T'/\mathrm{Remote}}_{q_i}(s')$, where T, T' denote* S *or* E *depending on the type of transition that s and s' may be, $x_{q_i} \bar{\in} \vec{h}$, $q_i \bar{\in} \vec{h}$, and $h_i \bar{\in} \vec{h}$ for all $i \in I$.*
- *If there exists $s, s' \in S_{q_i}$ for some $\mathbf{H}_i \in \mathbf{H}^*$, where $s \neq s'$ and both s, s' are event-triggered transitions, then either $x_{q_i} \notin G^{\mathrm{E/Local}}_{q_i}(s) \bigcap G^{\mathrm{E/Local}}_{q_i}(s')$, or $(h_j)_{j \in V(q_i)} \notin G^{\mathrm{E/Remote}}_{q_i}(s) \bigcap G^{\mathrm{E/Remote}}_{q_i}(s')$ or $e^k_{\alpha^k} \notin G^{\mathrm{E/E}}_{q_i}(s) \bigcap G^{\mathrm{E/E}}_{q_i}(s')$, where $x_{q_i} \bar{\in} \vec{h}$, $q_i \bar{\in} \vec{h}$, and $h_i \bar{\in} \vec{h}$ for all $i \in I$, and $e^k_{\alpha^k}$ belongs to $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$, with $\hat{\tau}^k = \tau^N$ and $\alpha^k = i$ where τ^N is the last element of τ in the finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$.*

Definition 7 (Singleton Transition Map Condition) *Given an initial condition $\vec{h}_0$ and a group event sequence $\mathcal{E}^*$, we say that $\vec{h} \in Reach_{\mathbf{H}^*}$ satisfies the Singleton Transition Map (STM) condition if on of the following holds:*

- *If there exists a $s \in S_{q_i}$ for some $\mathbf{H}_i \in \mathbf{H}^*$ such that $\vec{h}$ satisfies $G^{\mathrm{S}}_{q_i}(s)$, then $Z_{q_i}(x_{q_i}, (h_j)_{j \in V(q_i)}, s)$ contains at most one element, where $x_{q_i} \bar{\in} \vec{h}$, $q_i \bar{\in} \vec{h}$, and $h_i \bar{\in} \vec{h}$ for all $i \in I$.*
- *If there exists a $s \in S_{q_i}$ for some $\mathbf{H}_i \in \mathbf{H}^*$ such that $\vec{h}$ satisfies $G^{\mathrm{E}}_{q_i}(s)$, and there exists $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$ that satisfies $G^{\mathrm{E}}_{q_i}(s)$ at τ^N where τ^N is the last element of τ in the finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$, then $Z_{q_i}(e^k_{\alpha^k}, x_{q_i}, (h_j)_{j \in V(q_i)}, s)$ contains at most one element, where $x_{q_i} \bar{\in} \vec{h}$, $q_i \bar{\in} \vec{h}$, and $h_i \bar{\in} \vec{h}$ for all $i \in I$.*

Lemma 2 (Determinism) *Given an initial condition $\vec{h}_0$ and a group event sequence $\mathcal{E}^*$, $\chi^M(\vec{h}_0, \mathcal{E}^*)$ contains at most one element if and only if for all $\vec{h} \in Reach_{\mathbf{H}^*}$ the Forced Transition, the Disjoint Transition Guard, and the Singleton Transition Map conditions are satisfied.*

The conditions in this lemma rule out any possibility where the system may take more than one path at the same time: If a discrete transition is possible then continuous evolution is impossible and viceversa (FT condition). If there exist two possible transitions then only one of their transition guards may be completely satisfied (DTG condition). And for every state that may originate a discrete transition there is only one possible destination point after such transition takes place (STM condition)

Combining Lemmas 1 and 2 we obtain the following result. This holds because an infinite IHE is maximal by definition.

Theorem 2 (Existence and Uniqueness) *Given an initial condition $\vec{h}_0$ and a group event sequence $\mathcal{E}^*$, $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ contains exactly one element if and only if the conditions in Lemmas 1 and 2 hold.*

Note that Theorem 2 states the necessary and sufficient conditions for the existence and uniqueness of an infinite IHE in terms of each agent's model. These conditions may be used to design the dynamics of each agent in local form such that the existence and uniqueness of the multi-agent system's execution is guaranteed. Some of the conditions may seem difficult to satisfy. However some design guidelines may be followed in order to obtain a well behaved system:

- Discrete events should be regarded as control inputs, forcing the discrete transition if the guard is satisfied. Otherwise it becomes practically impossible to guarantee that the FT condition holds for event-triggered transitions.
- The state based transition guards on each agents should overlap with the set where continuous evolution is impossible. In this form the conditions for existence and and the FT condition for state-based transitions are satisfied.
- The event-triggered transition guards should not coincide with the set where continuous evolution is impossible because there is no guarantee for the occurrence of an event that enables a discrete transition, which would violate the existence of the IHE.

5 Proofs

The proofs of Lemmas 1 and 2 require the following result.

Lemma 3 *Let $\vec{f}_{\vec{q}}(\vec{x}_{\vec{q}},t) = (f_{q_i}(x_{q_i},u_{q_i},t))_{q_i\in Q_i, i\in I: q_i \bar{\in} \vec{q}}$ where $x_{q_i} \bar{\in} \vec{x}_{\vec{q}}$. $\vec{f}_{\vec{q}}(\vec{x}_{\vec{q}},t)$ is globally Lipschitz on $\vec{x}_{\vec{q}}$ for all $\vec{q} \in \prod_{i\in I} Q_i$.*

Proof: Note that from Definition 1 u_{q_i} is a vector formed by the continuous states x_{q_j} of the agents $j \in V(q_i)$, then $f_{q_i}(x_{q_i},u_{q_i},t) = f_{q_i}(x_{q_i},(x_{q_j})_{q_j\in Q_j, j\in V(q_i)},t)$, which can be rewritten as $f_{q_i}(\vec{x}_{\vec{q}},t)$ where the components of $\vec{x}_{\vec{q}}$ that are different from x_{q_i} or x_{q_j} for all $q_j \in Q_j$ for all $j \in V(q_i)$ are absent. Therefore $\vec{f}_{\vec{q}}(\vec{x}_{\vec{q}},t) = (f_{q_i}(x_{q_i},u_{q_i},t))_{q_i\in Q_i, i\in I: q_i \bar{\in} \vec{q}}$ is well defined.

Consider $\left\|\vec{f}_{\vec{q}}(\vec{x}_{\vec{q}},t) - \vec{f}_{\vec{q}}(\vec{\tilde{x}}_{\vec{q}},t)\right\| := LHS$ where $\vec{x}_{\vec{q}}, \vec{\tilde{x}}_{\vec{q}} \in \prod_{i\in I} X_i$. By triangle inequality

$$LHS \leq \sum_{i\in I} \left\| f_{q_i}(\vec{x}_{\vec{q}},t) - f_{q_i}(\vec{\tilde{x}}_{q_i},t) \right\|$$

where $q_i \bar{\in} \vec{q}$ for all $i \in I$. By the previous argument, $f_{q_i}(\vec{x}_{\vec{q}},t) = f_{q_i}(x_{q_i},u_{q_i},t)$ and $f_{q_i}(\vec{\tilde{x}}_{q_i},t) = f_{q_i}(\tilde{x}_{q_i},\tilde{u}_{q_i},t)$, so

$$LHS \leq \sum_{i\in I} \left\| f_{q_i}(x_{q_i},u_{q_i},t) - f_{q_i}(\tilde{x}_{q_i},\tilde{u}_{q_i},t) \right\|$$

adding $f_{q_i}(\tilde{x}_{q_i},u_{q_i},t) - f_{q_i}(\tilde{x}_{q_i},u_{q_i},t) = 0$ to the term inside the norm in the right hand side above, and applying triangle inequality we obtain:

$$\begin{aligned} LHS \leq \quad & \textstyle\sum_{i\in I} \left\| f_{q_i}(x_{q_i},u_{q_i},t) - f_{q_i}(\tilde{x}_{q_i},u_{q_i},t) \right\| + \ldots \\ & \textstyle\sum_{i\in I} \left\| f_{q_i}(\tilde{x}_{q_i},u_{q_i},t) - f_{q_i}(\tilde{x}_{q_i},\tilde{u}_{q_i},t) \right\| \end{aligned}$$

which by Assumption 1 implies

$$LHS \leq \sum_{i \in I} L_{x_{q_i}} \left\| x_{q_i} - \tilde{x}_{q_i} \right\| + L_{u_{q_i}} \left\| u_{q_i} - \tilde{u}_{q_i} \right\|$$

Note that x_{q_i}, u_{q_i} are vectors formed by some but not necessarily all the components of $\vec{x}_{\vec{q}}$. Similarly $\tilde{x}_{q_i}, \tilde{u}_{q_i}$ are formed by components of $\vec{\tilde{x}}_{\vec{q}}$, which implies that $\left\| x_{q_i} - \tilde{x}_{q_i} \right\| \leq \left\| \vec{x}_{\vec{q}} - \vec{\tilde{x}}_{\vec{q}} \right\|$ as well as $\left\| u_{q_i} - \tilde{u}_{q_i} \right\| \leq \left\| \vec{x}_{\vec{q}} - \vec{\tilde{x}}_{\vec{q}} \right\|$. Therefore if we let $L = \sum_{i \in I} L_{x_{q_i}} + L_{u_{q_i}}$ we conclude that

$$\left\| \vec{f}_{\vec{q}}(\vec{x}_{\vec{q}}, t) - \vec{f}_{\vec{q}}(\vec{\tilde{x}}_{\vec{q}}, t) \right\| \leq L \left\| \vec{x}_{\vec{q}} - \vec{\tilde{x}}_{\vec{q}} \right\|$$

which proves the claim. ■

5.1 Proof of Lemma 1

($\Rightarrow$) Suppose for the sake of contradiction that $\mathbf{H}^*$ is deterministic, and for any $\vec{h}_0$ and $\mathcal{E}^*$ $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is nonempty, but there is a $\vec{h} \in Reach_{\mathbf{H}^*} \bigcap Out_{\mathbf{H}^*}$ for which none of 1) or 2) hold. Since $\vec{h} \in Reach_{\mathbf{H}^*}$ there is a finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ such that $\tau = \{\tau^0, \tau^1, \ldots, \tau^N\}$ and $\vec{h} = (\vec{q}^N, \vec{x}^N(\tau^N))$.

a) Suppose there exists another execution $\check{\chi}(\vec{h}_0, \mathcal{E}^*) = (\check{\tau}, \check{\mathbf{q}}, \check{\mathbf{s}}, \check{\mathbf{x}}, \check{\mathbf{u}})$ that extends $\chi(\vec{h}_0, \mathcal{E}^*)$ such that $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \check{\chi}(\vec{h}_0, \mathcal{E}^*)$ and $\check{\tau} = \{\tau^0, \tau^1, \ldots, \tau^{N-1}, \tau^N + \varepsilon\}$ for some $\varepsilon > 0$ (Lemma 3 makes this possible). Then there exists $t \in [0, \varepsilon)$ such that $\psi(q_i, x_{q_i}, u_{q_i}, t) \in X_{q_i}$ for all $i \in I$, which violates $\vec{h} \in Out_{\mathbf{H}^*}$.

b) Suppose there exists $\check{\chi}(\vec{h}_0, \mathcal{E}^*) = (\check{\tau}, \check{\mathbf{q}}, \check{\mathbf{s}}, \check{\mathbf{x}}, \check{\mathbf{u}})$ such that $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \check{\chi}(\vec{h}_0, \mathcal{E}^*)$ and $\check{\tau} = \{\tau^0, \tau^1, \ldots, \tau^N, \check{\tau}^{N+1}\}$, then there exists $\mathbf{H}_i \in \mathbf{H}^*$ that executes either a state-based transition or a an event-triggered transition at τ^N, therefore one of the following holds:

- If $\mathbf{H}_i$ executes a state-based transition, Definition 4 implies there exists a $s \in S_{q_i^{N-1}}$ such that $x_{q_i^{N-1}}(\tau^N) \in G^{\text{S/Local}}_{q_i^{N-1}}(s)$, $(h_j^{N-1})_{j \in V(q_i^{N-1})}(\tau^N) \in G^{\text{S/Remote}}_{q_i^{N-1}}(s)$, and $x_{q_i^N}(\tau^N) \in Z_{q_i^{N-1}}(G^{\text{S}}_{q_i^{N-1}}, s)$ where $q_i^n \bar{\in} \vec{h}^n$, $x_{q_i^n} \bar{\in} \vec{h}^n$, $h_j^n \bar{\in} \vec{h}^n$ for all $i, j \in I$ and for all $n \in \{N, N-1\}$. Note that this violates assumption that 1) does not hold.
- If $\mathbf{H}_i$ executes an event-triggered transition, Definition 4 implies there exists a $s \in S_{q_i^{N-1}}$ and a $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$, such that $\alpha^k = i$, $\hat{\tau}^k = \tau^N$, $e^k_{\alpha^k} \in G^{\text{E/E}}_{q_i^{N-1}}(s)$, $x_{q_i^{N-1}(\tau^N)} \in G^{\text{E/Local}}_{q_i^{N-1}}(s)$, $(h_j^{N-1})_{j \in V(q_i^{N-1})}(\tau^N) \in G^{\text{E/Remote}}_{q_i^{N-1}}(s)$, and $x_{q_i^N}(\tau^N) \in Z_{q_i^{N-1}}(G^{\text{E}}_{q_i^{N-1}}, s)$, where $q_i^n \bar{\in} \vec{h}^n$, $x_{q_i^n} \bar{\in} \vec{h}^n$, $h_j^n \bar{\in} \vec{h}^n$ for all $i, j \in I$ and for all $n \in \{N, N-1\}$. Note that this violates assumption that 2) does not hold.

a) and *b*) imply that $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ is maximal. However by assumption $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is nonempty, therefore there exists an infinite execution $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*) \in \chi^\infty(\vec{h}_0, \mathcal{E}^*)$. This execution is also maximal and different from $\chi(\vec{h}_0, \mathcal{E}^*)$, which implies that $\chi^M(\vec{h}_0, \mathcal{E}^*)$ has at least two different elements violating the assumption that $\mathbf{H}^*$ is deterministic, which proves the ($\Rightarrow$) part of our claim.

($\Leftarrow$) Suppose for the sake of contradiction that there is a $\vec{h}_0$ and a $\mathcal{E}^*$ for which $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is empty, but for all $\vec{h} \in Reach_{\mathbf{H}^*} \bigcap Out_{\mathbf{H}^*}$ either 1) or 2) hold.

Since $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is empty, we can find a finite, maximal execution $\chi(\vec{h}_0, \mathcal{E}^*) \in \chi^F(\vec{h}_0, \mathcal{E}^*) \bigcap \chi^M(\vec{h}_0, \mathcal{E}^*)$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$. $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ implies that the last elements of $\mathbf{x}$ and $\mathbf{u}$ are defined over the compact interval $[\tau^{N-1}, \tau^N]$, i.e., $\vec{x}^N : [\tau^{N-1}, \tau^N] \to \prod_{i \in I} X_{q_i^N}$ and $\vec{u}^N : [\tau^{N-1}, \tau^N] \to \prod_{i \in I} U_{q_i^N}$.

By assumption $\vec{h}^N \in Reach_{\mathbf{H}^*}$. If $\vec{h}^N \notin Out_{\mathbf{H}^*}$, then there exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon)$ and for all $i \in I$ $\psi(q_i^N, x_{q_i^N}, u_{q_i^N}, t) \in X_{q_i}$, where $q_i^N \bar{\in} \vec{h}$, $x_{q_i^N} \bar{\in} \vec{h}^N$. This implies that $\chi(\vec{h}_0, \mathcal{E}^*)$ can be extended to $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ such that $\chi(\vec{h}_0,\mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$, with $\tilde{\tau} = \{\tau^0,\tau^1,...,\tau^N+\varepsilon\}$, and $\vec{x}^N$ and $\vec{u}^N$ defined over the interval $[\tau^{N-1},\tau^N+\varepsilon[$. Therefore $\chi(\vec{h}_0,\mathcal{E}^*)$ is not maximal, contradicting our previous argument.

If $\vec{h}^N \in Out_{\mathbf{H}^*}$, then by assumption either 1 or 2) hold. If 1) holds, there exists a $\mathbf{H}_i \in \mathbf{H}^*$ such that there exists $s \in S_{q_i^N}$ with $x_{q_i^N} \in G^{\mathrm{S/Local}}_{q_i^N}(s)$, and $(h_j^N)_{j\in V(q_i^N)} \in G^{\mathrm{S/Remote}}_{q_i^N}(s)$ where $q_i^N \bar{\in} \vec{h}^N$, $x_{q_i^N} \bar{\in} \vec{h}^N$, and $h_i^N \bar{\in} \vec{h}^N$ for all $i \in I$. Then from Definition 4 $\chi(\vec{h}_0,\mathcal{E}^*)$ can be extended to $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ such that $\chi(\vec{h}_0,\mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ where $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*) = (\tilde{\tau},\tilde{\mathbf{q}},\tilde{\mathbf{s}},\tilde{\mathbf{x}},\tilde{\mathbf{u}}) = (\tau,\mathbf{q},\mathbf{s},\mathbf{x},\mathbf{u}) : (\tilde{\tau}^{N+1},\vec{\tilde{q}}^{N+1},\vec{\tilde{s}}^{N+1},\vec{\tilde{x}}^{N+1},\vec{\tilde{u}}^{N+1})$ where $\tilde{q}_i^{N+1} = s \in S_{q_i^N}$ and $\tilde{x}_{q_i^{N+1}}(\tilde{\tau}^{N+1}) \in Z_{q_i^N}(G^{\mathrm{S}}_{q_i^N},s)$, where $\tilde{q}_i^{N+1} \bar{\in} \vec{\tilde{q}}^{N+1}$ and $\tilde{x}_{q_i^{N+1}} \bar{\in} \vec{\tilde{x}}^{N+1}$. Therefore $\chi(\vec{h}_0,\mathcal{E}^*)$ is not maximal, contradicting our previous argument.

If instead 2) holds, there exist a $\mathbf{H}_i \in \mathbf{H}^*$, and a $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$ such that $\alpha^k = i$, and such that there exists $s \in S_{q_i^N}$ with $x_{q_i^N} \in G^{\mathrm{E/Local}}_{q_i^N}(s)$, $(h_j^N)_{j\in V(q_i^N)} \in G^{\mathrm{E/Remote}}_{q_i^N}(s)$, and $e^k_{\alpha^k} \in G^{\mathrm{E/E}}_{q_i^N}(s)$, with $\hat{\tau}^k = \tau^N$. Then from Definition 4 $\chi(\vec{h}_0,\mathcal{E}^*)$ can be extended to $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ such that $\chi(\vec{h}_0,\mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ where $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*) = (\tilde{\tau},\tilde{\mathbf{q}},\tilde{\mathbf{s}},\tilde{\mathbf{x}},\tilde{\mathbf{u}}) = (\tau,\mathbf{q},\mathbf{s},\mathbf{x},\mathbf{u}) : (\tilde{\tau}^{N+1},\vec{\tilde{q}}^{N+1},\vec{\tilde{s}}^{N+1},\vec{\tilde{x}}^{N+1},\vec{\tilde{u}}^{N+1})$ where $\tilde{q}_i^{N+1} = s \in S_{q_i^N}$ and $\tilde{x}_{q_i^{N+1}}(\tilde{\tau}^{N+1}) \in Z_{q_i^N}(G^{\mathrm{S}}_{q_i^N},s)$, where $\tilde{q}_i^{N+1} \bar{\in} \vec{\tilde{q}}^{N+1}$ and $\tilde{x}_{q_i^{N+1}} \bar{\in} \vec{\tilde{x}}^{N+1}$. Therefore $\chi(\vec{h}_0,\mathcal{E}^*)$ is not maximal, contradicting our previous argument, and thus proving the $(\Leftarrow)$ part of the claim.

5.2 Proof of Lemma 2

$(\Leftarrow)$ Suppose for the sake of contradiction that $\chi^M(\vec{h}_0,\mathcal{E}^*)$ contains at least two elements but all FT, DTG, and SMT conditions hold. Then there exist $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ and $\check{\chi}(\vec{h}_0,\mathcal{E}^*)$ such that $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*) \neq \check{\chi}(\vec{h}_0,\mathcal{E}^*)$ and $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*),\check{\chi}(\vec{h}_0,\mathcal{E}^*) \in \chi^M(\vec{h}_0,\mathcal{E}^*)$.

Since both executions start at the same initial condition $\vec{h}_0$ and use the same group event sequence $\mathcal{E}^*$, there exists an IHE $\chi(\vec{h}_0,\mathcal{E}^*)$ that is a maximal prefix of both $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ and $\check{\chi}(\vec{h}_0,\mathcal{E}^*)$. Moreover $\chi(\vec{h}_0,\mathcal{E}^*)$ is finite because $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*) \neq \check{\chi}(\vec{h}_0,\mathcal{E}^*)$.

Let $\vec{h}^N$ be the state of $\mathbf{H}^*$ that is obtained from $\chi(\vec{h}_0,\mathcal{E}^*)$ with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$. Since $\chi(\vec{h}_0,\mathcal{E}^*)$ is finite, $\vec{x}^N \bar{\in} \vec{h}^N$ and $\vec{u}^N$ are defined over the compact interval $[\tau^{N-1},\tau^N]$. At this point the following cases are possible:

1. $\tau^N \notin \tilde{\tau}$ and $\tau^N \notin \check{\tau}$, therefore both $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ and $\check{\chi}(\vec{h}_0,\mathcal{E}^*)$ evolve from $\vec{h}^N$ on the system's continuous dynamics (This case establishes the sufficiency of Assumption 1 and Lemma 3). By definition of IHE (Definition 4, Lemma 3, and standard existence and uniqueness argument for continuous dynamical systems there exists $\varepsilon > 0$ such that for all $t \in [0,\varepsilon)$ and for all $i \in I$ $\psi(q_i^N, x^N_{q_i}, u^N_{q_i}, t) \in X_{q_i}$, where $q_i^N \bar{\in} \vec{h}^N$, $x^N_{q_i} \bar{\in} \vec{h}^N$, and $u^N_{q_i} \bar{\in} \vec{u}^N$ for all $i \in I$. Therefore there exists $\bar{\chi}(\vec{h}_0,\mathcal{E}^*) = (\bar{\tau},\bar{\mathbf{q}},\bar{\mathbf{s}},\bar{\mathbf{x}},\bar{\mathbf{u}})$ where $\bar{\tau} = \{\tau^0,\tau^1,\dots,\tau^{N-1},\tau^N+\varepsilon\}$, $\vec{x}^N$ and $\vec{u}^N$ are defined over $[\tau^{N-1},\tau^N+\varepsilon)$, $\chi(\vec{h}_0,\mathcal{E}^*) \sqsubset \bar{\chi}(\vec{h}_0,\mathcal{E}^*)$, and $\bar{\chi}(\vec{h}_0,\mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ and $\bar{\chi}(\vec{h}_0,\mathcal{E}^*) \sqsubseteq \check{\chi}(\vec{h}_0,\mathcal{E}^*)$ which contradicts discussion about $\chi(\vec{h}_0,\mathcal{E}^*)$ being the maximal prefix of $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ and $\check{\chi}(\vec{h}_0,\mathcal{E}^*)$.

2. $\tau^N \notin \tilde{\tau}$ and $\tau^N \in \check{\tau}$, therefore $\tilde{\chi}(\vec{h}_0,\mathcal{E}^*)$ evolves from $\vec{h}^N$ on the system's continuous dynamics, while $\check{\chi}(\vec{h}_0,\mathcal{E}^*)$ executes a discrete transition from $\vec{h}^N$ (This establishes sufficiency of the FT condition). The discrete transition that $\check{\chi}(\vec{h}_0,\mathcal{E}^*)$ executes from $\vec{h}^N$ may be and event-triggered transition or a state-based transition.

   If the transition is event-triggered, there exists a $\mathbf{H}_i \in \mathbf{H}^*$ such that there exists $s \in S_{q_i^N}$ with $s \neq nt$ such that $q_i^{N+1} = s$, $x_{q_i^N}(\tau^N) \in G^{\mathrm{E/Local}}_{q_i^N}(s)$, and $(h_j^N)_{j\in V(q_i^N)}(\tau^N) \in G^{\mathrm{E/Remote}}_{q_i^N}(s)$, and there exist $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$ such that $\tau = \hat{\tau}^k$, $\alpha^k = i$, and $e^k_{\alpha^k} \in G^{\mathrm{E/E}}_{q_i^N}(s)$. Then by the FT condition (second item) $\vec{h}^N \in Out_{\mathbf{H}^*}$. On the other hand, since $\tau^N \notin \tilde{\tau}$, definition of IHE (Definition 4, Lemma 3, and standard existence and uniqueness argument for continuous dynamical systems there exists $\varepsilon > 0$ such that for all $t \in [0,\varepsilon)$ and for all $i \in I$ $\psi(q_i^N, x_{q_i}^N, u_{q_i}^N, t) \in X_{q_i}$, where $q_i^N \bar{\in} \vec{h}^N$, $x_{q_i}^N \bar{\in} \vec{h}^N$, and $u_{q_i}^N \bar{\in} \vec{u}^N$ for all $i \in I$, which implies that $\vec{h}^N \notin Out_{\mathbf{H}^*}$ contradicting the previous conclusion.

If the transition is state-based, a similar argument leads to the same conclusion using the first item in the FT condition.

3. $\tau^N \in \tilde{\tau}$ and $\tau^N \notin \check{\tau}$. Symmetric to case 2.

4. $\tau^N \in \tilde{\tau}$ and $\tau^N \in \check{\tau}$, therefore both $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ and $\check{\chi}(\vec{h}_0, \mathcal{E}^*)$ execute a discrete transition from $\vec{h}^N$. (This case establishes the sufficiency of DTG and STM conditions). Each one of the IHE may take an event-triggered transition or a state-based transition, leading to four possibilities under this case.

   If $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ executes an event-triggered transition, then from definition of IHE, there exists a $\mathbf{H}_i \in \mathbf{H}^*$, such that there is $\tilde{s} \in S_{q_i^N}$ with $\tilde{s} \neq nt$ such that $x_{q_i^N}(\tau^N) \in G_{q_i^N}^{\mathrm{E/Local}}(\tilde{s})$ and $(h_j^N)_{j \in V(q_i^N)}(\tau^N) \in G_{q_i^N}^{\mathrm{E/Remote}}(\tilde{s})$, and there exists $(\hat{\tau}^k, e_{\alpha^k}^k) \in \mathcal{E}^*$ such that $\hat{\tau}^k = \tau^N$, $\alpha^k = i$, and $e_{\alpha^k}^k \in G_{q_i^N}^{\mathrm{E/E}}(\tilde{s})$. By assumption, there also exists an $\mathbf{H}_i \in \mathbf{H}^*$, such that there is $\check{s} \in S_{q_i^N}$, $\check{s} \neq nt$, $x_{q_i^N}(\tau^N) \in G_{q_i^N}^{\mathrm{T/Local}}(\check{s})$ and $(h_j^N)_{j \in V(q_i^N)}(\tau^N) \in G_{q_i^N}^{\mathrm{T/Remote}}(\check{s})$ where T denotes S or E depending on whether $\check{s}$ is event-triggered or state-based, and there exists (only in case $\check{s}$ is event-triggered) $(\hat{\tau}^k, e_{\alpha^k}^k) \in \mathcal{E}^*$ such that $\hat{\tau}^k = \tau^N$, $\alpha^k = i$, and $e_{\alpha^k}^k \in G_{q_i^N}^{\mathrm{E/E}}(\check{s})$.

   Since $\vec{h}^N$ satisfies the guard conditions for both $\tilde{s}$ and $\check{s}$, $x_{q_i^N}(\tau^N) \in G_{q_i^N}^{\mathrm{E/Local}}(\tilde{s}) \bigcap G_{q_i^N}^{\mathrm{T/Local}}(\check{s})$, $(h_j^N)_{j \in V(q_i^N)}(\tau^N) \in G_{q_i^N}^{\mathrm{E/Remote}}(\tilde{s}) \bigcap G_{q_i^N}^{\mathrm{T/Remote}}(\check{s})$, and (only in case $\check{s}$ is event-triggered) $e_{\alpha^k}^k \in G_{q_i^N}^{\mathrm{E/E}}(\tilde{s}) \bigcap G_{q_i^N}^{\mathrm{E/E}}(\check{s})$. This implies by DTG condition that $\tilde{s} = \check{s}$ (which also discards the possibility of $\check{s}$ being state-based). Then by IHE definition $\tilde{q}_i^{N+1} = \check{q}_i^{N+1}$, which by the STM condition implies that $\tilde{x}_{\tilde{q}_i^{N+1}}(\tau^N) = \check{x}_{\check{q}_i^{N+1}}(\tau^N)$. Therefore $\chi(\vec{h}_0, \mathcal{E}^*)$ can be extended to $\bar{\chi}(\vec{h}_0, \mathcal{E}^*) = \chi(\vec{h}_0, \mathcal{E}^*) : (\bar{\tau}^{N+1}, \bar{\vec{q}}^{N+1}, \bar{\vec{s}}^{N+1}, \bar{\vec{x}}^{N+1}, \bar{\vec{s}}^{N+1})$ where $\bar{\chi}(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$, $\bar{\chi}(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \check{\chi}(\vec{h}_0, \mathcal{E}^*)$, and $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubset \bar{\chi}(\vec{h}_0, \mathcal{E}^*)$ which contradicts discussion about $\chi(\vec{h}_0, \mathcal{E}^*)$ being the maximal prefix of both $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$, and $\check{\chi}(\vec{h}_0, \mathcal{E}^*)$.

   If $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ executes a state-based transition, a similar discussion leads to the same conclusion proving this case

From all the previous cases the $(\Leftarrow)$ part of the claim is proved.

$(\Rightarrow)$ Suppose for the sake of contradiction that $\chi^M(\vec{h}_0, \mathcal{E}^*)$ contains at most one element, but that at least one of the FT, DTG, or STM conditions is not satisfied for $\vec{h}$. Since $\vec{h} \in Reach_{\mathbf{H}^*}$ there exists a finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ such that $\vec{h} = \vec{h}^N(\tau^N) = (\vec{q}^N, \vec{x}^N(\tau^N))$ where $\vec{q}^N$ and $\vec{x}^N$ are the last elements of $\mathbf{q}$ and $\mathbf{x}$ respectively, and $\vec{x}^N$ and $\vec{u}^N$ are defined over the compact interval $[\tau^{N-1}, \tau^N]$.

If the FT condition is violated, there exists $s \in S_{q_i^N}$ for some $\mathbf{H}_i \in \mathbf{H}^*$ such that $\vec{h}^N(\tau^N)$ satisfies the event-triggered transition guard (assuming s is event-triggered), and there exists $(\hat{\tau}^k, e_{\alpha^k}^k) \in \mathcal{E}^*$ that satisfies the event-triggered transition guard, but $\vec{h}^N(\tau^N) \notin Out_{\mathbf{H}^*}$. Therefore $\chi(\vec{h}_0, \mathcal{E}^*)$ can be extended with either a discrete transition or continuous evolution: In case of the discrete transition consider $\check{q}_i^{N+1} = s$ then there exists $\check{\chi}(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) : (\check{\tau}^{N+1}, \check{\vec{q}}^{N+1}, \check{\vec{s}}^{N+1}, \check{\vec{x}}^{N+1}, \check{\vec{u}}^{N+1})$, where $s \bar{\in} \check{\vec{s}}^{N+1}$, $\check{q}_i^{N+1} \bar{\in} \check{\vec{q}}^{N+1}$. In case of the continuous evolution consider $\varepsilon > 0$ such that $\psi(q_i^N, x_{q_i^N}, u_{q_i^N}, t) \in X_{q_i^N}$ for all $i \in I$ and for all $t \in [0, \varepsilon)$. Then there exists $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*) = (\tau^0, \vec{q}^0, \vec{s}^0, \vec{x}^0, \vec{u}^0), \ldots, (\tau^{N-1}, \vec{q}^{N-1}, \vec{s}^{N-1}, \vec{x}^{N-1}, \vec{u}^{N-1}), (\tau^N + \varepsilon, \vec{q}^N, \vec{s}^N, \vec{x}^N, \vec{u}^N)$ where $\vec{x}^N$ and $\vec{u}^N$ are defined over $[\tau^{N-1}, \tau^N + \varepsilon)$. Thus $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubset \check{\chi}(\vec{h}_0, \mathcal{E}^*)$ and $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubset \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$, and $\check{\chi}(\vec{h}_0, \mathcal{E}^*) \neq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ which implies that there is at least two maximal executions in $\chi^M(\vec{h}_0, \mathcal{E}^*)$ which contradicts assumption, therefore FT must hold. A similar conclusion is achieved in the case of a state-based transition.

If the DTG condition does not hold, there exists $s, s' \in S_{q_i^N}$ with $s \neq s'$ such that $x_{q_i^N} \in G_{q_l^n}^{\mathrm{T}_1/\mathrm{Local}}(s) \bigcap G_{q_i^N}^{\mathrm{T}_2/\mathrm{Local}}(s')$, $(h_j)_{j\in V(q_i^N)} \in G_{q_i^N}^{\mathrm{T}_1/\mathrm{Remote}}(s) \bigcap G_{q_i^N}^{\mathrm{T}_2/\mathrm{Local}}(s')$, and in case both s and s' are event-triggered, $e_{\alpha^k}^k \in G_{q_i^N}^{\mathrm{E/E}}(s) \bigcap G_{q_i^N}^{\mathrm{E/E}}(s')$. Then $\vec{h}^N(\tau^N)$ satisfies the transition guards of both s and s' simultaneously. Therefore $\chi(\vec{h}_0^N, \mathcal{E}^*)$ can be extended on two different discrete transitions $\check{q}_i^{N+1} = s$ and $\tilde{q}_i^{N+1} = s'$, where $\check{q}_i^{N+1} \neq \tilde{q}_i^{N+1}$. Then there exist two IHE $\check{\chi}(\vec{h}_0^N, \mathcal{E}^*) \neq \tilde{\chi}(\vec{h}_0^N, \mathcal{E}^*)$ where $\check{\chi}(\vec{h}_0^N, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) : (\check{\tau}^{N+1}, \vec{\check{q}}^{N+1}, \vec{\check{s}}^{N+1}, \vec{\check{x}}^{N+1}, \vec{\check{u}}^{N+1})$ and $\tilde{\chi}(\vec{h}_0^N, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) : (\tilde{\tau}^{N+1}, \vec{\tilde{q}}^{N+1}, \vec{\tilde{s}}^{N+1}, \vec{\tilde{x}}^{N+1}, \vec{\tilde{u}}^{N+1})$, where $\check{q}_i^{N+1} \bar{\in} \vec{\check{q}}^{N+1}$ and $\tilde{q}_i^{N+1} \bar{\in} \vec{\tilde{q}}^{N+1}$. Note that $\chi(\vec{h}_0^N, \mathcal{E}^*) \sqsubset \check{\chi}(\vec{h}_0^N, \mathcal{E}^*)$ and $\chi(\vec{h}_0^N, \mathcal{E}^*) \sqsubset \tilde{\chi}(\vec{h}_0^N, \mathcal{E}^*)$ so there exist at least two maximal execution in $\chi^M(\vec{h}_0^N, \mathcal{E}^*)$ contradicting our assumption. Thus DTG condition must hold. A similar conclusion is achieved in the case where both s and s' are state-based transitions.

If the STM condition does not hold for $\vec{h} = \vec{h}^N(\tau^N)$, there exists $s \in S_{q_i^N}$ for some $\mathbf{H}_i \in \mathbf{H}^*$ such that $\vec{h}$ satisfies event-triggered transition transition guard (assuming s is an event-triggered), and there also exists $(\hat{\tau}^k, e_{\alpha^k}^k) \in \mathcal{E}^*$ that satisfies the event-triggered transition guard. Since $Z_{q_i^N}(e_{\alpha^k}^k, x_{q_i^N}, (h_j)_{j\in V(q_i^N)}, s)$ contains at least two elements, $\chi(\vec{h}_0^N, \mathcal{E}^*)$ may be extended to $\check{\chi}(\vec{h}_0^N, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) : (\check{\tau}^{N+1}, \vec{\check{q}}^{N+1}, \vec{\check{s}}^{N+1}, \vec{\check{x}}^{N+1}, \vec{\check{u}}^{N+1})$ as well as to $\tilde{\chi}(\vec{h}_0^N, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) : (\tilde{\tau}^{N+1}, \vec{\tilde{q}}^{N+1}, \vec{\tilde{s}}^{N+1}, \vec{\tilde{x}}^{N+1}, \vec{\tilde{u}}^{N+1})$ where $\vec{\check{q}}^{N+1} = \vec{\tilde{q}}^{N+1}$ but $\vec{\check{x}}^{N+1} \neq \vec{\tilde{x}}^{N+1} \in Z_{q_i^N}(e_{\alpha^k}^k, x_{q_i^N}, (h_j)_{j\in V(q_i^N)}, s)$. This implies $\chi(\vec{h}_0^N, \mathcal{E}^*) \sqsubset \check{\chi}(\vec{h}_0^N, \mathcal{E}^*)$ and $\chi(\vec{h}_0^N, \mathcal{E}^*) \sqsubset \tilde{\chi}(\vec{h}_0^N, \mathcal{E}^*)$. Since $\check{\chi}(\vec{h}_0^N, \mathcal{E}^*) \neq \tilde{\chi}(\vec{h}_0^N, \mathcal{E}^*)$, $\chi^M(\vec{h}_0^N, \mathcal{E}^*)$ contains at least two elements contradicting our assumption. Therefore STM must hold. A similar conclusion is achieved in case s is state-based. This completes the proof of $(\Rightarrow)$, and of our claim.

6 Conclusion

We present an interconnected hybrid systems framework: a set of hybrid systems with interweaved continuous and discrete dynamics that form a multi-agent system with hybrid interacting dynamics. We extend the work in [11, 17] defining a metric, reachable sets, and executions for interconnected hybrid systems. We comment on the properties of the new metric and prove a necessary and sufficient conditions for the existence and uniqueness of interconnected hybrid executions that are written in terms of the local model of each hybrid agent.

We are currently working on the application of this conditions to the problem of designing future communication networks as explained in the introduction. We also expect that this new theoretical framework will enable us to analyze, control and perform abstractions on multi-agent systems with hybrid interacting dynamics.

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