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PHASE-CONVEX ARCS IN ROOT SPACE AND THEIR APPLICATION TO ROBUST SPR PROBLEMS

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ABSTRACT

This paper considers the problem of identifying regions in the complex-plane, such that polynomials having roots in those regions, have their phase bounded by that of a few extreme polynomials. We present sufficient and testable conditions for disfferent regions satisfying this property. Applications of the results to the robust SPR analysis and synthesis problems are illustrated.

1 Introduction

This paper considers the problem of identifying regions in the complex-plane, such that the phase of polynomials having roots in those regions, is bounded by that of a few extreme polynomials. More specifically, given a family of polynomials $P(z^{-1})$ with members $p(z^{-1})$, and with a nominal member $p_0(z^{-1})$. Assume that the family is prescribed as follows

$$P(z^{-1}) = \{p(z^{-1}); p(z_i^{-1}) = 0, z_i \in \Omega_i\} \tag{1}$$

where Ω_i is a set about the corresponding root of $p_0(z^{-1})$. We are interested in finding the sets Ω_i such that the phase of every member in $P(z^{-1})$ is determined by a few extreme members. This will allow us to study and to enforce the SPRness of a family of transfer functions, by considering a finite number of members of that family. The problem was considered by many authors, and our development parallels that of [?]. The motivation given there and elsewhere [1], [2], is that an SPR condition is frequently invoked to prove convergence of adaptive algorithms. In general, the family of polynomials $P(z^{-1})$ is a related to the plant description and as such is uncertain in an adaptive algorithm or filtering setting.

We differ from [?] and [?] in that will not assume any knowledge of the spectral density of the regressor that may lead to consider different positivity regions. In order to allow different convergence rates, we will consider the complement of the circle of radius $1/r_0$ as the positivity region. However, the results presented here can be extended to regions other than the complement of circular regions. As it is well-known in certain cases the SPR condition can be enhanced by means of filter design. In this case, a filter $f(z^{-1})$ is sought such that the transfer function $p(z^{-1})/f(z^{-1})$ is SPR for every $p(z^{-1}) \in P(z^{-1})$. As it has been pointed out in [?] and [?], the design procedure can be extraordinarily simplified when the phase of the family is bounded by that of two members. This advantage is more evident when considering even-order polynomials but also for odd-order polynomials due to the smoothness of the phase-bounding functions.

Of course, there is another benefit of an extreme-point result for the phase, namely when checking the SPRness of the family, since this can be accomplished by checking the phase of a small number of selected polynomials.

This paper is organized as follows. The simple case of roots lying in regions bounded by parametrized arcs is first discussed in section 2 where the concept of phase-convex arc is introduced. The more special case of straight-line and circular arcs are also studied in section 2. The general case of roots lying in Domains is given in 3. The applications of the results to the robust SPR analysis and synthesis problems are presented in section 4. Numerical examples are discussed in section 5 and our conclusions are given in section 6.

2 Phase-Convex arcs

Let $\lambda \to F(\lambda)$, $\lambda \in [\lambda_0, \lambda_1]$ be a continuously differentiable parameterization of an arc in the complex plane. It will be convenient to write $F(\lambda) = X(\lambda) + jY(\lambda)$, where X and Y are two real functions of the real variable λ. *****Show Figure**** We will consider first pairs of complex conjugate roots, i.e., polynomials with the following structure

$$p(z^{-1}, \lambda) = [1 - (X(\lambda) + jY(\lambda))z^{-1}][1 - (X(\lambda) - jY(\lambda))z^{-1}] \tag{2}$$

Note that this defines a family of second order polynomials with roots on $F(\lambda)$ and $F^*(\lambda)$. Throughout the text we will assume that $|F(\lambda)| < 1/r_0$, $\lambda \in [\lambda_0, \lambda_1]$ which will be necessary for stability (meaning here roots in the circle of radius $1/r_0$) of the polynomials considered. Let $\phi(\omega, \lambda) = \arg\{p(r_0^{-1}e^{-j\omega}, \lambda)\}$ we seek conditions that guarantee

$$\phi(\omega, \lambda_0) \le \phi(\omega, \lambda) \le \phi(\omega, \lambda_1) \tag{3}$$

or

$$\phi(\omega, \lambda_1) \le \phi(\omega, \lambda) \le \phi(\omega, \lambda_0) \tag{4}$$

for all $\omega \in [0, 2\pi)$. We will call an arc satisfying properties (3) or (4) a phase-convex arc or simply say that an extreme-point property holds for the phase. From now on we will drop the dependence with λ wherever there is no possible confusion. Noting that the phase function can be written in the following form,

$$\phi(\omega, \lambda) = -2\omega + \arg\{e^{j\omega} - (X + jY)r_0\} + \arg\{e^{j\omega} - (X - jY)r_0\} \tag{5}$$

we can state our first result.

Theorem 1 *An extreme-point result holds for $\phi(\omega, \lambda)$ if for any $\lambda \in (\lambda_0, \lambda_1)$ the functions*

$$g_1(\lambda) = (1 - \alpha^2)\frac{\partial X}{\partial \lambda} - 2\alpha\frac{\partial Y}{\partial \lambda} \tag{6}$$

$$g_{-1}(\lambda) = (1 - \beta^2)\frac{\partial X}{\partial \lambda} + 2\beta\frac{\partial Y}{\partial \lambda} \tag{7}$$

both have the same sign, where

$$\alpha = \frac{Yr_0}{(1 - Xr_0)} \tag{8}$$

$$\beta = \frac{Yr_0}{(1 + Xr_0)} \tag{9}$$

Proof: Clearly, (3) or (4) will hold if $\phi(\omega, \lambda)$ is a monotonic function of λ for every ω. For this, we require that for a fixed ω either $\partial\phi/\partial\lambda \ge 0$ or $\partial\phi/\partial\lambda \le 0$ for every $\lambda \in (\lambda_0, \lambda_1)$. Let $u = \cos\omega$ and $v = \sin\omega$. Differentiating ϕ with respect to X and Y, we obtain

$$\frac{\partial\phi}{\partial X} = \frac{2vr_0}{D}[1 + (Xr_0)^2 - 2uXr_0 - (Yr_0)^2] \tag{10}$$

$$\frac{\partial\phi}{\partial Y} = \frac{-4vr_0}{D}(Yr_0)(u - Xr_0) \tag{11}$$

where $D = [(u - Xr_0)^2 + (v - Yr_0)^2][(u - Xr_0)^2 + (v + Yr_0)^2]$. Then, it is possible to write

$$\frac{\partial \phi}{\partial \lambda} = \frac{2vr_0}{D} \left(\left[1 + (Xr_0)^2 - 2uXr_0 - (Yr_0)^2\right] \frac{\partial X}{\partial \lambda} - 2(u - Xr_0)Yr_0 \frac{\partial Y}{\partial \lambda} \right) \tag{12}$$

Note that due to the antisymmetry of the phase of a real polynomial, we only need to consider the interval $[0, \pi)$ for ω or, conversely, $v \geq 0$. Also, $\phi(0, \lambda) = 0$ for any $\lambda \in (\lambda_0, \lambda_1)$. From these two facts, we conclude that the factor v in (12) can be discarded for considerations of extremality. The same can be said about the term $2r_0/D$ since it is always positive.

Now, looking at the term in brackets, we see that it depends in an affine manner on $u = \cos\omega$. Therefore, one way of guaranteing monotonicity of ϕ with λ is that the term in brackets has the same sign for $u = 1$ and $u = -1$ and for every $\lambda \in (\lambda_0, \lambda_1)$. Note however that this is only a sufficient condition. For instance, it is still valid that the term in brackets have different signs at $u = 1$ and $u = -1$ if the change of sign of the derivative happens at some u_0 independently of λ. In this case, it is clear that the phase function is constant with λ at $u = u_0$ and hence condition (3) still holds with an equality.

Note also that the sign condition is maintained if we divide the term in brackets by a positive number. Thus, let $g_1(\lambda)$ denote this term for $u = 1$ divided by $(1 - Xr_0)^2$ and $g_{-1}(\lambda)$ denote the term for $u = -1$ divided by $(1 + Xr_0)^2$. Then, we arrive to the condition for phase extremality as

$$sgn\{g_{-1}(\lambda)\} = sgn\{g_1(\lambda)\} \tag{13}$$

This completes the proof.

■

As we will later see, for a geometrical interpretation of (13) it is convenient to redefine g_{-1} and g_1 in the following way, provided that $\partial X/\partial\lambda \neq 0$,

$$g_1(\lambda) = (\alpha^2 - 1) + 2\alpha\gamma \tag{14}$$
$$g_{-1}(\lambda) = (\beta^2 - 1) - 2\beta\gamma \tag{15}$$

so that condition (13) remains the same. Here, $\gamma = \partial Y/\partial X$. It can be shown that the sign condition can be restated as

$$sgn\{h_{-1}(\lambda)\} = sgn\{h_1(\lambda)\} \tag{16}$$

where

$$h_1(\lambda) = |\alpha + \gamma| - \sqrt{\gamma^2 + 1} \tag{17}$$
$$h_{-1}(\lambda) = |\beta - \gamma| - \sqrt{\gamma^2 + 1} \tag{18}$$

In this expressions, γ can be regarded as the direction of the tangent to the arc, giving the following interpretation: for a given direction of the tangent, we can easily find those points $X + jY$ in the complex plane for which the sign condition holds, and consequently have a monotonic phase for every $\omega \in [0, 2\pi)$, thus building an arc for which there is a phase extremal result.

Note that $h_1 = 0$ and $h_{-1} = 0$ define the boundaries of the feasible region (for a given γ). In terms of the (X, Y), these two equations can be easily seen as 4 straight lines. Moreover, the following properties can be derived

1. The segment $[-1/r_0, 1/r_0]$ of the real axis always belongs to the feasible region for any γ.
2. The 4 straight lines intersect at points on the circle of radius $1/r_0$ forming a rectangle inscribed in this circle.

The feasible region for $\gamma = 1$ and $r_0 = 1$ is depicted in figure ??????.

2.1 Straight Line Segments

In the case of straight line segments, *****Show Figure**** with some slight modifications of the original form for $F(\lambda)$, it is possible to write $X(\lambda) = x_0 + \lambda a$, $Y(\lambda) = y_0 + \lambda b$ where $\lambda \in [0, 1]$, and a and b are real numbers. Now, whenever $a \neq 0$ we can write $\gamma = b/a$ and therefore an extreme-point result will hold when (16) is satisfied for (x_0, y_0) and (x_0+a, y_0+b) or, put another way, when these two points belong to the feasible rectangle.
The case $a = 0$ (vertical lines) is somewhat special since it is not possible to use (16). As a matter of fact, it can be shown that (13) does not hold. Here (12) becomes

$$\frac{\partial \phi}{\partial \lambda} = -4vr_0^2(u - Xr_0)Yb \tag{19}$$

Then, there are two possibilities that may lead to a change of sign in (19). First, there is a change of sign when $u = Xr_0$. This situation was already discussed in the proof of theorem 1 and does not affect the extreme point property of the arc since it turns out that for ω such that $u = Xr_0$ the phase is a constant with λ. The second situation where $\partial\phi/\partial\lambda$ could change sign is when $Y = 0$ for some λ_0. It is clear in this case that the phase will have an extreme for $\lambda = \lambda_0$. However, a close look at this fact allows us to restate the problem so an extreme-point result holds. Clearly, if $Y(\lambda_0) = y_0 + \lambda_0 b = 0$, for some $\lambda_0 \in (0, 1)$ then we have a segment that crosses the real axis. Due to the symmetry of the roots on $F(\lambda)$ it is clear that there are pairs of roots that are considered twice. Since this is not necessary, it is possible to redefine the arc as $Y(\lambda) = y_0 - \lambda y_0$, $\lambda \in [0, 1]$, such that $Y(1) = 0$ and such that an extreme-point result holds. We conclude from this discussion that vertical segments always admit the extreme-point property.

The case of real roots can be treated as a special case of the horizontal segments. Note that here $F(\lambda) \subset \mathbf{R}$ so first-order factors can be dealt with. However, since $F^*(\lambda) = F(\lambda)$, it is clear that the argument of $[1 - F(\lambda)z^{-1}]$ is half of that of $[1 - F(\lambda)z^{-1}][1 - F^*(\lambda)z^{-1}]$. Since for this second-order polynomial we have proven that an extreme-point result holds as long as $F(\lambda) \subset (-1/r_0, 1/r_0)$ the same follows for the first order polynomial with real roots. ****Show Figures*****

2.2 Circular Arcs

The form of these arcs is $F(\psi) = (x_0 + jy_0) + re^{j\psi}$, $\psi \in [\psi_0, \psi_1]$. *****Show Figure**** Therefore, g_1 and g_{-1} in (6,7) become

$$g_1(\psi) = \left[\frac{(y_0 + r\sin\psi)r_0}{1 - (x_0 + r\cos\psi)r_0}\right]^2 - 1 - 2\frac{(y_0 + r\sin\psi)r_0\cos\psi}{[1 - (x_0 + r\cos\psi)r_0]\sin\psi} \tag{20}$$

$$g_{-1}(\psi) = \left[\frac{(y_0 + r\sin\psi)r_0}{1 + (x_0 + r\cos\psi)r_0}\right]^2 - 1 + 2\frac{(y_0 + r\sin\psi)r_0\cos\psi}{[1 + (x_0 + r\cos\psi)r_0]\sin\psi} \tag{21}$$

After some straightforward algebraic manipulations, it can be shown that g_1 and g_{-1} have the same signs as respectively h_1 and h_{-1} defined below

$$h_1(\psi) = r^2 r_0^2 \sin\psi + 2ry_0 r_0^2 + y_0^2 r_0^2 - (1 - x_0 r_0)^2 \sin\psi - 2y_0 r_0 (1 - x_0 r_0) \cos\psi \quad (22)$$

$$h_{-1}(\psi) = r^2 r_0^2 \sin\psi + 2ry_0 r_0^2 + y_0^2 r_0^2 - (1 + x_0 r_0)^2 \sin\psi - 2y_0 r_0 (1 + x_0 r_0) \cos\psi \quad (23)$$

Now, given x_0, y_0 and r it is possible to compute the values of ψ for which the sign of h_1 and h_{-1} is the same. This is done by calculating the roots of $h_1 = 0$ and $h_{-1} = 0$. Both equations have the form $A \sin\psi + B \cos\psi + C = 0$ with A, B, C real numbers. It is immediate to show that each equation can have at most 4 solutions in the interval $[0, 2\pi)$. These solutions divide this interval in subintervals where the sign condition holds or fails. Obviously, if $[\psi_0, \psi_1]$ is a subset of one of the subintervals where the sign condition is satisfied, then the arc $F(\psi)$ is phase-convex. The case $y_0 = 0$ is especially simple, since

$$h_1(\psi) = [r^2 r_0^2 - (1 - X_0 r_0)^2] \sin\psi \quad (24)$$

$$h_{-1}(\psi) = [r^2 r_0^2 - (1 + X_0 r_0)^2] \sin\psi \quad (25)$$

so it is clear that the two expressions have the same sign if and only if $x_0 \in (-1/r_0, 1/r_0)$. Then, the extremes of the phase are obtained for ψ_0, ψ_1, unless the arc intersects the real axis. If this were true, one (or both) of the extremes would be the intersection point (points). Actually, this matches the result of [?] where these type of solutions were sought. However, it is important to stress that the use of (22) and (23) allows for a much broader class of extreme-point results.

3 Generalization to Domains

In this section we extend the previous results roots lying in arbitrary domains. *****Show Figure**** Let Ω be an open region, simply connected and symmetric with respect to the real axis, $P(z^{-1})$ be the family of second-order polynomials $p(z^{-1})$ with roots in Ω and assume that the boundary of Ω can be written as

$$\partial\Omega = \bigcup_{i=1}^{N} F_i(\lambda_i), \quad \lambda_i \in [\lambda_i^0, \lambda_i^1] \subset \mathbf{R} \quad (26)$$

where $F_i(\lambda_i)$ is a continuously differentiable phase-convex arc, according to the discussion in previous sections. Obviously, $F_i(\lambda_i^1) = F_{i+1}(\lambda_{i+1}^0)$, $i = 1, \cdots, N+1$, and $F_N(\lambda_N^1) = F_1(\lambda_1^0)$. Let $\phi_p(z) = \arg\{p(z^{-1})\}$ and define

$$\bar{\phi}(z) = \sup_{p \in P} \phi_p(z) \quad (27)$$

$$\underline{\phi}(z) = \inf_{p \in P} \phi_p(z) \quad (28)$$

These functions are respectively termed lead and lag functions of the family $P(z^{-1})$ [?]. Also define the extreme polynomials as

$$p_i(z^{-1}) = [1 - F_i(\lambda_i^0) z^{-1}][1 - F_i^*(\lambda_i^0) z^{-1}], \; i = 1, \cdots, N \quad (29)$$

and let the corresponding extreme phase functions $\phi_i(z) = \arg\{p_i(z^{-1})\}$, $i = 1, \cdots, N$. Then, the following result can be stated,

Theorem 2 *The lead and lag functions of the family* $P(z^{-1})$ *for a given* $z = r_0 e^{j\omega}$, $\omega \in [0, 2\pi)$, *are obtained by maximizing and minimizing among the extreme phase functions, that is,*

$$\bar{\phi}(r_0 e^{j\omega}) = \max_{i=1,\cdots,N} \phi_i(r_0 e^{j\omega}) \tag{30}$$

$$\underline{\phi}(r_0 e^{j\omega}) = \min_{i=1,\cdots,N} \phi_i(r_0 e^{j\omega}) \tag{31}$$

Proof: We will present here the proof for the lead function, since the corresponding proof for the lag function follows along the same lines. From the fact that all the $F_i(\lambda_i)$ are phase-convex arcs, we can conclude that the maximum along $\partial\Omega$ can be calculated from maximization among the extremes of each segment. In order to prove that the supremum for all the points in Ω is on the boundary, we can employ the Maximum Modulus Principle [3] and the fact that every vertical segment is phase convex. In order to show this, consider that for any polynomial $q_0(z^{-1}) = [1-(x_0+jy_0)z^{-1}][1-(x_0-jy_0)z^{-1}]$, $x_0+jy_0 \in \Omega$, and for a given $z = r_0 e^{j\omega}$, $\omega \in [0, 2\pi)$, there must exist some $q_1(z^{-1}) = [1-(x_1+jy_1)z^{-1}][1-(x_1-jy_1)z^{-1}]$, $x_1+jy_1 \in \partial\Omega$ such that $arg\{q_0(r_0^{-1}e^{-j\omega})\} \leq arg\{q_1(r_0^{-1}e^{-j\omega})\}$. This x_1+jy_1 is simply one of the points at which the vertical line passing through x_0+jy_0 intersects $\partial\Omega$. This then completes the proof.

■

Obviously, it is possible that only a subset of the N extremes is necessary to bound the phase. A very important special case is summarized by the following corollary

Corollary 1 *Suppose that there exist unique* $k, l \in \{1, \cdots, N\}$ *such that the argument of the polynomials with roots in* $F_i(\lambda_i)$ *and* $F_i^*(\lambda_i)$ *for* $z = r_0 e^{j\omega}$, $\omega \in (0, \pi)$ *changes from strictly increasing to strictly decreasing with* $\lambda_i \in (\lambda_i^0, \lambda_i^1)$ *at* $i = k$ *and viceversa for* $i = l$. *Then, the argument is bounded by that of only two extremes, that is,*

$$\bar{\phi}(r_0 e^{j\omega}) = \phi_k(r_0 e^{j\omega}) \tag{32}$$

$$\underline{\phi}(r_0 e^{j\omega}) = \phi_l(r_0 e^{j\omega}) \tag{33}$$

for some $i, k \in \{1, \cdots, N\}$.

■

It is worth remarking that most of the arcs that have been considered so far (except the vertical line segment) satisfy the conditions in Corollary 1.

Theorem 2 can be also used to produce conservative (but simple) bounds. In fact, consider the family

$$Q(z^{-1}) = \{q(z^{-1}); q(z_0^{-1}) = 0 \Rightarrow z_0 \in \Gamma \quad or \quad z_0 \in \Gamma^*\} \tag{34}$$

for which $\Gamma \subset \Omega$ and define $\phi_q(z) = arg\{q(z^{-1})\}$. Then

$$\inf_{p \in P} \phi_p(z) \leq \inf_{q \in Q} \phi_q(z) \leq \sup_{q \in Q} \phi_q(z) \leq \sup_{p \in P} \phi_p(z) \tag{35}$$

for $z = r_0 e^{j\omega}$, $\omega \in [0, 2\pi)$.

A necessary and sufficient special case is the following. Let $G \in \partial\Omega$ be the set of points $F_i(\lambda_i^0)$, $F_i(\lambda_i^1)$ for which the extremes in (30) or (31) are attained for some $\omega \in [0, 2\pi)$.

Then, if $G \subset \partial\Omega \cap \partial\Gamma$, equation (35) holds with equalities in the leftmost and righmost expressions.

The results given so far can be extended to families of polynomials of any degree n. We will consider first the case of n even and complex conjugate roots and then include the remaining cases. Since it is possible to place the roots in different domains, we consider families of the form

$$P(z^{-1}) = \{p(z^{-1}) = \prod_{l=1}^{m} p_l(z^{-1}), \quad p_l(z^{-1}) \in P_l(z^{-1}), l = 1, \cdots, m\} \tag{36}$$

where

$$P_l(z^{-1}) = \{p_l(z^{-1}) = \prod_{i=1}^{M_l} [1 - z_i z^{-1}][1 - z_i^* z^{-1}], \quad z_i \in \Omega_l\} \tag{37}$$

and each region $\Omega_l \subset \mathbf{C}$, $l = 1, \cdots, m$ is such that its boundary is described by

$$\partial\Omega_l = \bigcup_{i=1}^{N_l} F_{i,l}(\lambda_{i,l}), \quad \lambda_{i,l} \in [\lambda_{i,l}^0, \lambda_{i,l}^1] \subset \mathbf{R} \tag{38}$$

where the $F_{i,l}(\lambda_{i,l})$ are continuously differentiable phase-convex arcs. Defining the extreme polynomials $p_{l,i}(z^{-1})$ as

$$p_{i,l}(z^{-1}) = \prod_{i=1}^{M_l} [1 - F_{i,l}(\lambda_{i,l}^0) z^{-1}][1 - F_{i,l}^*(\lambda_{i,l}^0) z^{-1}], \; i = 1, \cdots, N_l \tag{39}$$

$$\tag{40}$$

and the extreme phase functions as $\phi_{i,l}(z) = \arg\{p_{i,l}(z^{-1})\}$, $i = 1, \cdots, N_l$, $l = 1, \cdots, n/2$ we can state without proof the following result

Theorem 3

$$\bar{\phi}(r_0 e^{j\omega}) = \max_{\substack{l=1,\cdots,m \\ i=1,\cdots,N_l}} \sum_l \phi_{i,l}(r_0 e^{j\omega}) \tag{41}$$

$$\underline{\phi}(r_0 e^{j\omega}) = \min_{\substack{l=1,\cdots,m \\ i=1,\cdots,N_l}} \sum_l \phi_{i,l}(r_0 e^{j\omega}) \tag{42}$$

■

Note that the above maximization and minimization processes suffer from a "combinatorial-explosion" as n grows. However, frequency-dependent reduction techniques can be used, by identifying subintervals of $[0, 2\pi)$ where the functional expression of the lead and lag functions remains the same. Different important special cases can be identified. One such case is when the domains Ω_l are the same, that is, $m = 1$, since then the number of extremes required is that of the family of second-order polynomials with roots in Ω_l. Another important case is that for which every section $P_l(z^{-1})$ satisfies the conditions of Corollary 1. Then, only two extremes are required, no matter how large n is.

The case of n odd and real roots correspond tofactors of the form

$$P_l(z^{-1}) = \{\prod_{i=1}^{M_l}(1 - z_i z^{-1}), \quad z_i \in \Omega_l \subset (1/r_0, 1/r_0)\} \tag{43}$$

From Section 2.1, it is clear that these regions are phase-convex, so Theorem 3 extends in a very straightforward way to this case.

4 Application to SPR problems

The first application of our results has to do with SPR checking of families of polynomials. Suppose that the family $P(z^{-1})$ described in (36) is stable (note that this implies that every Ω_l is contained in the circle of radius $1/r_0$ which was one starting assumption). Then, the condition for SPRness on the complement of the circle of radius $1/r_0$ is that

$$Re\{p(r_0^{-1}e^{-j\omega})\} > 0, \quad \forall p \in P, \quad \forall \omega \in [0, 2\pi) \tag{44}$$

It is immediate to see that this condition can be transformed into one involving the phase, namely

$$\pi/2 < \underline{\phi}(r_0 e^{j\omega}) \leq \bar{\phi}(r_0 e^{j\omega}) < \pi/2; \quad \forall \omega \in [0, 2\pi) \tag{45}$$

From this condition, it is clear that if the root domains are bounded by closed curves that are piecewise phase-convex, the family will be SPR if and only if some adequately selected extreme polynomials are SPR. The importance of this result resides in the fact that the SPR checking of the family of polynomials can be simplified to a finite test. Moreover, recall that the possible combinatorial explosion can be generally alleviated or even eliminated (cf. previous section).

The second application deals with filter design to enforce SPRness of a family of polynomials. In this case, we want to find a stable polynomial $f(z^{-1})$ such that given the stable family $P(z^{-1})$ as defined in (36), the following condition holds

$$Re\left\{\frac{p(r_0^{-1}e^{-j\omega})}{f(r_0^{-1}e^{-j\omega})}\right\} > 0, \quad \forall p \in P, \quad \forall \omega \in [0, 2\pi) \tag{46}$$

In [2] it is shown that a necessary and sufficient condition for the existence of such $f(z^{-1})$ is that for all $\omega \in [0, 2\pi)$,

$$\bar{\phi}(r_0 e^{j\omega}) - \underline{\phi}(r_0 e^{j\omega}) < \pi \tag{47}$$

Again, if the root domains for the family of polynomials satisfy the requirements for phase-convexity, it is clear that the previous condition can be transformed into a finite set of conditions involving the extreme polynomials. As before, it can be easily shown that this amounts a finite test to determine whether the family can be made SPR.

As was pointed out in [2], any stable polynomial with phase $[\bar{\phi}(r_0 e^{j\omega}) - \underline{\phi}(r_0 e^{j\omega})]/2$ satisfies (47). The problem is however to find the coefficients of such a polynomial. An especially appealing solution is obtained when a polynomial of degree n having the desired phase [?] can be found. This can be achieved when in (37) and (43) (depending on whether Ω_l is complex or real) M_l is even for every $l = 1, \cdots, m$ and every $P_l(z^{-1})$ satisfies the conditions of corollary 1 (i.e., phase bounded by only two extreme phase functions). Suppose, without loss of generality, that for every $l \in \{1, \cdots, m\}$, $\bar{\phi}$ and $\underline{\phi}$ are attained for the extremes $F_{1,l}(\lambda^0_{1,l})$ and $F_{N_l,l}(\lambda^0_{N_l,l})$. Then, the choice of $f(z^{-1})$ that will make the family SPR is

$$\begin{aligned} f(z^{-1}) \quad = \quad & \prod_{l=1}^{m} [1 - F_{1,l}(\lambda^0_{1,l})z^{-1}]^{\frac{N_l}{2}} [1 - F^*_{1,l}(\lambda^0_{1,l})z^{-1}]^{\frac{N_l}{2}} \\ & [1 - F_{N_l,l}(\lambda^0_{N_l,l})z^{-1}]^{\frac{N_l}{2}} [1 - F^*_{1,l}(\lambda^0_{N_l,l})z^{-1}]^{\frac{N_l}{2}} \end{aligned} \tag{48}$$

When $\Omega_l \subset \mathbf{R}$, (48) admits an obvious extension with first order sections in the product. If any of the N_l is odd, then the optimal solution has to be approximated, thus not guaranteeing that (46) is satisfied. In [**?**] some ideas are presented that can be useful in this approximation.

5 Numerical Examples

*******Show Examples********

6 Conclusions

Given a family of polynomials where the poles are uncertain but known to lie in a domain, we have provided a systematic characterization of domains for which the phase of the family can be bounded by that of a finite set of extremes, thus paving the way for finite tests to check if a family is SPR or can be made SPR by output error filtering. In addition, we have shown how in certain cases this characterization of the uncertainty can be succesfully exploited to find the structure of the optimal filter.

We have specialized our results for straight line segments and circular arcs, since we feel it is of value to use these type of simple sets in the process of identification of the unknown roots or to embed the uncertain set in a minimal domain with this simple structure. When the design of the filter $f(z^{-1})$ is not critical (i.e., there is a certain amount of freedom in the phase condition), this overbounding by a set that admits a two-extreme result will give a way of finding suitable $f(z^{-1})$ using the solution given in the paper.

Even though we have considered here the complement of a circle as a positivity region and structured uncertainties, our results can be adapted to consider other positivity regions, as in [**?**], [**?**], and to include unstructured uncertainties [**?**].

Research is in progress in the design of suboptimal filters $f(z^{-1})$ that can be used in cases where more than two extremes are necessary, such as that of rectangular domains for the roots, where more than four extremes are required.

References

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