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A Unification Between Nonlinear-Nonquadratic Optimal Control and Integrator Backstepping

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Abstract

In this paper we develop an optimality-based framework for backstepping controllers. Specifically, using a nonlinear-nonquadratic optimal control framework we develop a family of globally stabilizing backstepping controllers parameterized by the cost functional that is minimized. Furthermore, it is shown that the control Lyapunov function guaranteeing closed-loop stability is a solution to the steady-state Hamilton-Jacobi-Bellman equation for the controlled system and thus guarantees both optimality and stability. The results are specialized to the case of integrator backstepping.

1. Introduction

General nonlinear systems are notoriously hard to stabilize. Control system designers have usually resorted to Lyapunov methods in order to obtain stabilizing controllers for such systems. Unfortunately, however, there does not exist a unified procedure for finding a control Lyapunov function candidate that will stabilize the closed-loop for general nonlinear systems. Recent work involving differential geometric methods has made the design of controllers for certain classes of nonlinear systems more methodical. Such frameworks include the concepts of zero dynamics and feedback linearization. These techniques, however, usually rely on canceling out system nonlinearities using feedback and may therefore lead to inefficient designs since feedback linearizing controllers may generate unnecessarily large control effort to cancel beneficial system nonlinearities.

Backstepping control has recently received a great deal of attention in the nonlinear control literature [1]. The popularity of this control methodology can be explained in a large part due to the fact that it provides a framework for designing stabilizing nonlinear controllers for a large class of nonlinear dynamic cascade systems. This framework guarantees stability by providing a systematic procedure for finding a control Lyapunov function for the closed-loop system and choosing the control such that the time derivative of the control Lyapunov function along the trajectories of the closed-loop dynamic system is negative. Furthermore, the controller is obtained in such a way that the nonlinearities of the dynamic system which may be useful in reaching performance objectives need not be canceled as in state or output feedback linearization techniques. However, no analytical measure of performance or notions of optimality have been shown to exist for controllers derived via backstepping approaches.

In this paper we develop an optimality-based theory for backstepping controllers. The key motivation for developing an optimal nonlinear backstepping control theory is that it provides a family of candidate backstepping controllers parameterized by the cost functional that is minimized. In order to address the optimality-based backstepping nonlinear control problem we use the nonlinear-nonquadratic optimal control framework developed in [2]. The basic underlying ideas of the results in [2] rely on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a control Lyapunov function for the nonlinear controlled system thus guaranteeing both optimality and stability. The nonlinear feedback control law is chosen so that the Hamilton-Jacobi-Bellman optimality conditions are satisfied. In this paper we extend the framework developed in [2] to cascade systems for which the backstepping control design methodology is applicable. Specifically, we show that a particular controller derived via backstepping methods corresponds to the solution of an optimal control problem that minimizes a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the control Lyapunov derivative is negative along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing solutions to the Hamilton-Jacobi-Bellman equation. Thus, our results allow us to derive globally asymptotically stabilizing backstepping controllers for nonlinear systems that minimize a corresponding nonlinear-nonquadratic performance functional.

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2. Optimal Nonlinear-Nonquadratic Feedback Control

In this section we consider affine systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$, and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, with performance functional

$$J(x_0, u(\cdot)) = \int_0^\infty [L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u]dt, \quad (2)$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}, L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{P}^{m\times m}$. Furthermore define the set of asymptotically stabilizing controllers by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \in \mathcal{U} \text{ and } x(\cdot) \text{ given by (1) satisfies } x(t) \to 0 \text{ as } t \to \infty\}. \quad (3)$$

Theorem 2.1 [2]. Consider the controlled system (1) with performance functional (2). Assume there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that

$$V(0) = 0, \quad L_2(0) = 0, \quad V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (4)$$

$$V'(x)[f(x) - \tfrac{1}{2}g(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \tfrac{1}{2}g(x)R_2^{-1}(x)g^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)] < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (5)$$

and $V(x) \to \infty$ as $\|x\| \to \infty$. Then the solution $x(t) = 0$, $t \geq 0$, of the closed-loop system

$$\dot{x}(t) = f(x(t)) + g(x(t))\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (6)$$

is globally asymptotically stable with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[L_2^{\mathrm{T}}(x) + g^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)], \quad (7)$$

and the performance functional (2), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x), \quad (8)$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \quad (9)$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \quad (10)$$

3. Optimal Integrator Backstepping Controllers

In this section we consider the cascade system

$$\dot{x}(t) = f(x(t)) + g(x(t))\hat{x}(t), \quad x(0) = x_0, \quad t \geq 0, \quad (11)$$

$$\dot{\hat{x}}(t) = u(t), \qquad \hat{x}(0) = \hat{x}_0, \quad (12)$$

where (1) has been augmented by an input subsystem consisting of m integrators. For the following result define

$$\tilde{L}(x, \hat{x}, u) \triangleq \tilde{L}_1(x, \hat{x}) + \tilde{L}_2(x, \hat{x})u + u^{\mathrm{T}}R_2(x, \hat{x})u. \quad (13)$$

Theorem 3.1. Consider the cascade system (11), (12) with performance functional

$$\tilde{J}(x_0, \hat{x}_0, u(\cdot)) \triangleq \int_0^\infty \tilde{L}(x(t), \hat{x}(t), u(t))\mathrm{d}t, \quad (14)$$

where $u(\cdot)$ is admissible, $(x(t), \hat{x}(t))$, $t \geq 0$, solves (11), (12), and $\tilde{L}(x, \hat{x}, u)$ is given by (13). Assume there exist C^1 functions $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and $V_{\mathrm{sub}} : \mathbb{R}^n \to \mathbb{R}$, a function $\tilde{L}_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{1\times m}$, and a positive-definite matrix $\hat{P} \in \mathbb{P}^{m\times m}$ such that

$$\alpha(0) = 0,\ \tilde{L}_2(0,0) = 0,\ V_{\mathrm{sub}}(0) = 0, \quad (15)$$

$$V_{\mathrm{sub}}(x) > 0,\ x \in \mathbb{R}^n,\ x \neq 0, \quad (16)$$

$$V'_{\mathrm{sub}}(x)[f(x) + g(x)\alpha(x)] < 0,\ x \in \mathbb{R}^n,\ x \neq 0, \quad (17)$$

$$(\hat{x} - \alpha(x))^{\mathrm{T}}\Big\{g^{\mathrm{T}}(x)V'^{\mathrm{T}}_{\mathrm{sub}}(x) - 2\hat{P}\Big[\alpha'(x)(f(x) + g(x)\hat{x})$$
$$+R_2^{-1}(x, \hat{x})[\hat{P}(\hat{x} - \alpha(x)) + \tfrac{1}{2}\tilde{L}_2^{\mathrm{T}}(x, \hat{x})]\Big]\Big\} < 0,$$
$$(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^m,\ \hat{x} \neq \alpha(x). \quad (18)$$

Then the solution $(x(t), \hat{x}(t)) = (0, 0)$, $t \geq 0$, of the cascade system (11), (12) is globally asymptotically stable with the feedback control law $u = \tilde{\phi}(x, \hat{x})$, where

$$\tilde{\phi}(x, \hat{x}) = -R_2^{-1}(x, \hat{x})\hat{P}[\hat{x} - \alpha(x)] - \tfrac{1}{2}R_2^{-1}(x, \hat{x})\tilde{L}_2^{\mathrm{T}}(x, \hat{x}). \quad (19)$$

Furthermore,

$$\tilde{J}(x_0, \hat{x}_0, \tilde{\phi}(x(\cdot), \hat{x}(\cdot))) = V(x_0, \hat{x}_0), \quad (x_0, \hat{x}_0) \in \mathbb{R}^n \times \mathbb{R}^m, \quad (20)$$

where

$$V(x, \hat{x}) = V_{\mathrm{sub}}(x) + [\hat{x} - \alpha(x)]^{\mathrm{T}}\hat{P}[\hat{x} - \alpha(x)], \quad (21)$$

and the performance functional (14), with

$$\tilde{L}_1(x, \hat{x}) = \tilde{\phi}^{\mathrm{T}}(x, \hat{x})R_2(x, \hat{x})\tilde{\phi}(x, \hat{x}) - V'_{\mathrm{sub}}(x)[f(x) + g(x)\hat{x}]$$
$$+2[\hat{x} - \alpha(x)]^{\mathrm{T}}\hat{P}\alpha'(x)[f(x) + g(x)\hat{x}], \quad (22)$$

is minimized in the sense that

$$\tilde{J}(x_0, \hat{x}_0, \tilde{\phi}(x(\cdot), \hat{x}(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} \tilde{J}(x_0, \hat{x}_0, u(\cdot)). \quad (23)$$

Remark 3.1. A particular choice of $\tilde{L}_2(x, \hat{x})$ satisfying condition (18) is given by

$$\tilde{L}_2(x, \hat{x}) = \left[V'_{\mathrm{sub}}(x)g(x)\hat{P}^{-1} - 2\left(f(x) + g(x)\hat{x}\right)^{\mathrm{T}}\alpha'^{\mathrm{T}}(x)\right] \cdot R_2(x, \hat{x}). \quad (24)$$

In this case with $u(t) \in \mathbb{R}$, $t \geq 0$, the feedback control law given by (19) specializes to the integrator backstepping controller given by Lemma 2.8 of [1] by setting $\hat{P} = \frac{1}{2}I_m$ and $R_2(x, \hat{x}) = 1/c$.

Remark 3.2. If $\tilde{L}_1(x, \hat{x}) \geq 0$, $R_2(x, \hat{x}) > 0$, and $\tilde{L}_2(x, \hat{x}) \equiv 0$, $(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^m$, then the feedback control law $\tilde{\phi}(x, \hat{x})$ given by (19) can be used to provide guaranteed gain margins to sector bounded input nonlinearities. Specifically, if $R_2(x, \hat{x}) = r_2^{-1}(x, \hat{x})I_m$, where

$$r_2(x, \hat{x}) = \begin{cases} \frac{\sqrt{(\beta^{\mathrm{T}}\gamma)^2 + (\beta^{\mathrm{T}}\beta)^2} - \beta^{\mathrm{T}}\gamma}{\beta^{\mathrm{T}}\beta}, & \hat{x} \neq \alpha(x), \\ 0, & \hat{x} = \alpha(x), \end{cases}$$

and $\beta \triangleq \hat{P}(\hat{x} - \alpha(x))$, $\gamma \triangleq 2\alpha'(x)[f(x) + g(x)\hat{x}] - \hat{P}^{-1}g^{\mathrm{T}}(x)V'^{\mathrm{T}}_{\mathrm{sub}}(x)$, then the control law (19) yields

$$\tilde{\phi}(x, \hat{x}) = \begin{cases} -\left[\frac{\sqrt{(\beta^{\mathrm{T}}\gamma)^2 + (\beta^{\mathrm{T}}\beta)^2} - \beta^{\mathrm{T}}\gamma}{\beta^{\mathrm{T}}\beta}\right]\beta, & \hat{x} \neq \alpha(x), \\ 0, & \hat{x} = \alpha(x). \end{cases}$$

In this case it can be shown that $\tilde{\phi}(x, \hat{x})$ guarantees closed-loop stability for nonlinear systems with component decoupled input nonlinearities in the conic sector $(\frac{1}{2}, \infty)$.

For a complete exposition of the results of this paper the interested reader is referred to [3].

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