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# Finite-Time Stability of Discrete-Time Nonlinear Systems: Analysis and Design

S. Mastellone, P. Dorato, C. T. Abdallah

**Abstract**—Finite-time stability of nonlinear discrete-time systems is studied. Some new analysis results are developed and applied to controller design.

## I. INTRODUCTION

In this work we propose a new analysis result for finite time stability of deterministic and stochastic discrete-time nonlinear systems. We also extend existing results in finite-time stability to the design of discrete-time stochastic systems. In many practical problems it is of interest to investigate the stability of a system over a finite interval of time. Consider for example the problem of driving a car across a tunnel for which the distance between the boundaries is a known quantity  $2\beta$ , knowing that the mission lifetime is  $N$ , we can reformulate the problem in term of finite-time stability since we have specific constraints on state bounds and time. Classical control theory does not directly address this requirement because it focuses mainly on the asymptotic behavior of the system (over an infinite time interval), and does not usually specify bounds on the trajectories. On the other hand, finite-time stability (or short-time stability [7], [12]) plays an important role in the study of the transient behavior of systems.

It is important to underline that the two stability concepts are disconnected. In fact, a system may be finite-time stable, i.e. a state starting within a “specified” bound  $\alpha$  does not exceed a “specified” bound  $\beta$  in a specified time interval  $[0, N]$ , but may become unstable after the specified interval of time. On the other hand, the state trajectory might exceed the given bound over a certain time interval, but asymptotically go to zero. Asymptotic stability is specified with respect to arbitrary bounds, i.e. a trajectory starting within a bound  $\delta(\epsilon)$  stays in an “arbitrary”  $\epsilon$  and eventually converges to the origin, while finite-time stability is always defined with respect to pre-specified bounds  $\alpha$  and  $\beta$ .

At first the concept of finite-time stability emerged under the name of “practical stability” [22], in which specific bounds on the state were given. The finite-time stability analysis problem has been discussed for linear [3], [6], [7], [8], [9], [10] and nonlinear systems [15], [16], [17], [18], [26]. A stochastic version of finite-time stability has been developed in [13] for analysis and in [20], [21] for optimal control design. Deterministic finite-time stability theory has

been applied to several control problems in linear systems [4], [5], [11]. It is interesting to notice the time gap between 1972 and recent papers. After a brief discussion of the deterministic version in section (II), we mainly focus on stochastic finite-time stability (III). In particular, we introduce in section (III-A) some useful bounds, then in section (III-B) we use those bounds to state sufficient conditions for a stochastic system to be finite-time stable. Section (III-C) compares and discussed the results in the previous sections. We then proceed in section (III-D) to extend the analysis techniques to designing controllers. Finally in section (III-E), we propose an optimal feedback law for finite-time stability of a dynamical stochastic system.

## II. DETERMINISTIC FINITE-TIME STABILITY

We focus on discrete-time dynamical systems described by

$$x_{k+1} = f(x_k), x \in \mathbb{R}^n, x(0) = x_0 \quad (1)$$

Where  $x$  is the system state, and  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a vector function. For notational simplicity, we use  $x_k = x(k)$ . Also from now on we will denote  $\|\cdot\| \equiv \|\cdot\|_2^2$ . We are interested in studying the state trajectory of the system in a finite time interval.

**Definition 1: Finite-Time Stability** [1], [23] The system (1) is finite-time stable (FTS) with respect to the 4-tuple  $(\alpha, \beta, N, \|\cdot\|)$ ,  $\alpha \leq \beta$  if every trajectory  $x_k$  starting in  $\|x_0\| \leq \alpha$  satisfies the bound  $\|x_k\| \leq \beta$  for all  $k = 1, \dots, N$ .

Some extensions of the FTS concept are presented in [2],[15]. Next we present a new analysis result for FTS of nonlinear discrete-time systems. We consider three classes of systems described in Figure (1): a) systems for which the state trajectories always increase in the norm, b) systems for which states always decrease in the norm, and c) systems whose state trajectories behavior’s is mixed.

The first step consists of exploring the state trajectories using a discrete version of the continuous-time Bellman-Gronwall inequality [19]. If the state trajectory is always increasing (in the norm) during the time interval of interest, then it is enough to verify that the state at the last time of the interval does not exceed the bound. In the case where the trajectory is always decreasing and it starts inside the bound, the FTS is guaranteed. In the case of a mixed behavior, it is necessary to explore if the trajectory is bounded at each time step. In the next theorem we formulate the conditions for finite-time stability of the system (1).

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*Theorem 1:* The system (1) is finite-time stable with respect to  $(\alpha, \beta, N, \|\cdot\|)$ ,  $\alpha \leq \beta$ , if for a function  $V(x_k, k) = V_k \geq 0$  such that  $\delta_1 \|x_k\| \leq V_k \leq \delta_2 \|x_k\|$ , where  $\delta_1 > 0$ ,  $\delta_2 > 0$ ,  $\gamma = \delta_1 \beta$ ,  $\gamma_0 = \delta_2 \alpha$ ,  $V_0 \leq \gamma_0$  and  $S_\beta = \{x_k : \|x_k\| \leq \beta\}$  we have  $\forall k = 0, \dots, N, \forall x_k \in S_\beta$

$$\Delta V_k \leq \rho_k V_k \quad (2)$$

and one of the following three conditions occur:

- **Case 1:**  $\rho_k \geq 0$

$$\frac{\gamma}{\gamma_0} \geq \prod_{i=0}^{N-1} (1 + \rho_i) \quad (3)$$

The value of  $\rho_k \geq 0$  implies that the bounds on the increments of  $V_k$  are as a worse case always greater than one, which is the case of monotonically increasing functions.

- **Case 2:**  $0 \geq \rho_k > -1$

No additional conditions are required.

The condition  $0 \geq \rho_k > -1$  restricts the bounds on the increments of  $V_k$  to be always between zero and one, which constrains the function to be monotonically decreasing.

- **Case 3:**  $\rho_k > -1$

$$\frac{\gamma}{\gamma_0} \geq \sup_k \prod_{i=0}^{k-1} (1 + \rho_i) \quad (4)$$

The case  $\rho_k > -1$  contains the two previous cases, that is the function  $V_k$  may be increasing and decreasing.

*Proof:* The proof is available in [25] ■

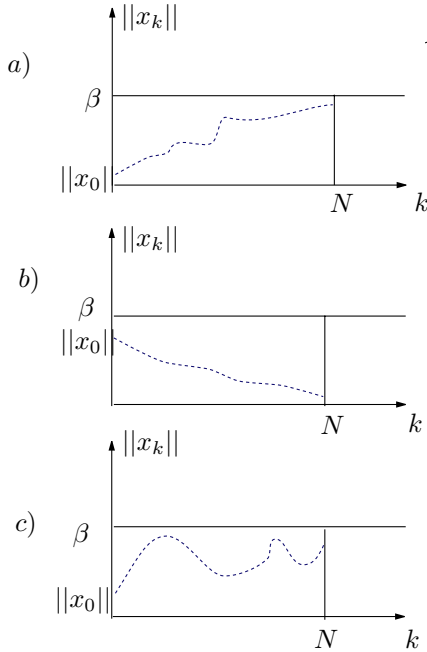


Fig. 1. a) Increasing dynamics. b) Decreasing dynamics. c) Mixed dynamics.

### III. STOCHASTIC FINITE-TIME STABILITY

Next, we describe how finite-time stability, which was originally defined for deterministic systems may be extended to stochastic systems. Consider a discrete time, stochastic dynamical system

$$x_{k+1} = f(x_k, \theta_k), x \in \mathbb{R}^n, x(0) = x_0 \quad (5)$$

Where  $x$  is the system state, and  $f : \mathbb{R}^n \times \mathcal{B} \rightarrow \mathbb{R}^n$  is a vector function,  $\mathcal{B}$  is the family of Borel subsets of points on  $\mathbb{R}$ ; also  $\{\theta_k\}$  is a stationary independent random sequence, with mean  $\mu_\theta = \mathbb{E}[\theta_k] = \mathbb{E}[\theta_k^2]$  and variance  $\sigma_\theta$ , which makes  $x_k$  a Markov process in  $\mathbb{R}^n$ . In stochastic dynamical systems it is meaningful to consider the probability for the trajectory not to exceed a given bound over a finite time interval. Therefore we consider the following definitions

*Definition 2: Inclusion Probability* [20] Consider the dynamical stochastic system (5), the associated inclusion probability with respect to  $(\alpha, \beta, N, \|\cdot\|)$  is defined as follows:

$$P_{in}(x_k; \alpha, \beta, N) = P\{\|x_k\| \leq \beta : 0 \leq k \leq N; \|x_0\| \leq \alpha\}$$

*Definition 3: Exit Probability* Consider the dynamical stochastic system (5), the associated exit probability with respect to  $(\alpha, \beta, N, \|\cdot\|)$  is defined as follows:

$$P_{ex}(x_k; \alpha, \beta, N) = P\left\{ \sup_{N \geq k \geq 0} \|x_k\| > \beta; \|x_0\| \leq \alpha \right\}$$

Note that  $P_{ex}(x_k; \alpha, \beta, N) = 1 - P_{in}(x_k; \alpha, \beta, N)$ . Therefore, we define stochastic finite-time stability:

*Definition 4: Finite Time Stochastic Stability (FTSS)* The dynamical system (5) is FTSS with respect to  $(\alpha, \beta, N, \lambda, \|\cdot\|)$  if

$$P_{in}(x_k; \alpha, \beta, N) \geq (1 - \lambda), \text{ or} \quad (6)$$

$$P_{ex}(x_k; \alpha, \beta, N) < \lambda \quad (7)$$

We will show next how FTSS can be indirectly determined by studying the exit and inclusion probabilities associated with a function  $V(x_k, k)$  defined for the dynamical system.

#### A. Bounds on Exit Probability

In order to analyze and to eventually design for the finite-time stability of a process, we provide in this section upper bounds on the exit probability of the process (5) and on the associated function  $V_k$ . These upper bounds will allow us to indirectly study the FTSS of the system. The first theorem we present is from [13], [14].

*Theorem 2:* [13] Consider a discrete-time Markov process  $x_k, k = 0, 1, \dots$ . Also consider the function  $V(x_k, k) = V_k \geq 0$  and the open set  $S_\gamma = \{x_k : V_k \leq \gamma\}$ . If the following conditions are satisfied  $\forall x_k \in S_\gamma, \phi_k \geq 0$

$$\mathbb{E}_{x_k} [V(x_{k+1}, k+1)] \leq \infty \forall x_k \in S_\gamma,$$

$$\mathbb{E}_{x_k} [V(x_{k+1}, k+1) - V(x_k, k)] \leq \phi_{k+1}$$

Then for the initial condition  $x(0) = x_0$  we have

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \quad (8)$$

where  $\Phi_N = \sum_{i=1}^N \phi_i$

*Proof:* See [13] or [14] ■

The last theorem gives an upper bound for the exit probability of  $V_k$ . This upper bound depends on the initial conditions through  $V_0$ , on the desired bound through  $\gamma$ , and on the time interval and state dynamics indirectly through  $\Phi_N$ . Next, we bound the exit probability of the state dynamics of (5) directly.

*Theorem 3:* Consider the dynamical system (5) and its exit probability with respect to  $(\alpha, \beta, N, \|\cdot\|)$ ,  $P_{ex}(x_k; \alpha, \beta, N)$ , also consider the function  $V_k$  as described previously, we have the following upper bound

$$P_{ex}(x_k; \alpha, \beta, N) \leq \mathbb{E} \left[ \frac{\sup_{N \geq k \geq 0} \|x_k\|}{\beta}; \|x_0\| \leq \alpha \right]$$

*Proof:* The proof easily follows from Chebychev inequality [24]. In the following,  $I$  is the indicator function, for brevity  $I = I_{\{\sup_{N \geq j \geq 0} \|x_j\| > \beta\}}$ . Also recalling that  $P(x \leq t) = \mathbb{E}[I_{x \leq t}]$ , then

$$\begin{aligned} P_{ex}(x_k; \alpha, \beta, N) &= P\left\{ \sup_{N \geq k \geq 0} \|x_k\| > \beta; \|x_0\| \leq \alpha \right\} \\ &= \mathbb{E} \left[ I\left( \sup_{N \geq j \geq 0} \|x_j\| \right); \|x_0\| \leq \alpha \right] \\ &\leq \mathbb{E} \left[ I\left( \sup_{N \geq j \geq 0} \|x_j\| \right) \frac{\sup_{N \geq k \geq 0} \|x_k\|}{\beta}; \|x_0\| \leq \alpha \right] \\ &\leq \mathbb{E} \left[ \frac{\sup_{N \geq k \geq 0} \|x_k\|}{\beta}; \|x_0\| \leq \alpha \right] \end{aligned}$$

Again the bound on  $P_{ex}(x_k; \alpha, \beta, N)$  is directly related to the bounds on the state  $\alpha, \beta$ , to the state dynamics, and to the time interval.

### B. Stochastic Finite-Time Stability Analysis

In the previous section we showed how the exit probability relative to the state dynamics  $x_k$  and to the associated function  $V(x_k, k)$  can be bounded and how the bound depends on the parameters describing the finite-time stability objective. In this section we use the described bound to provide sufficient conditions for FTSS stability of system (5).

*Theorem 4:* Consider the dynamical system (5) and a function  $V_k$  such that for given  $\delta_1, \delta_2$  we have  $\delta_1 \|x_k\| \leq V(x_k, k) \leq \delta_2 \|x_k\|$ , and  $\gamma = \beta \delta_1, \gamma_0 = \alpha \delta_2, V_0 \leq \gamma_0, \delta_1 > 0, \delta_2 > 0$ . Then the system is finite-time stochastically stable with respect to  $(\alpha, \beta, N, \|\cdot\|, \lambda)$ , if any of the following three conditions is satisfied

(i)

$$\begin{aligned} \mathbb{E}_{x_k} [V_{k+1}] &\leq \infty \\ \mathbb{E}_{x_k} [\Delta V_k] &\leq \phi_{k+1} \\ \frac{[\alpha \delta_2 + \Phi_N]}{\beta \delta_1} &\leq \lambda \\ \Phi_N &= \sum_{k=1}^N \phi_k, \forall x_k \in S_\gamma, \phi_k \geq 0 \end{aligned} \quad (9)$$

(ii)

$$\mathbb{E} \left[ \frac{\sup_{N \geq k \geq 0} \|x_k\|}{\beta}; \|x_0\| \leq \alpha \right] \leq \lambda \quad (10)$$

(iii)

$$P\{\Delta V_k \leq \rho_k V_k\} \geq (1 - \lambda) \quad (11)$$

$$\frac{\gamma}{\gamma_0} \geq \sup_k \prod_{i=0}^{k-1} (1 + \rho_i) \quad (12)$$

$$\forall x_k \in S_\gamma, \rho_k > -1, \forall k = 0, \dots, N$$

*Proof:* In order to prove the above statements we verify that (i) – (iii) imply finite-time stability for the system. Finite-time stability easily follows from point (i) considering that for  $\delta_1 \|x_k\| \leq V(x_k, k) \leq \delta_2 \|x_k\|, \forall k = 0, \dots, N$  and  $\gamma_0 = \delta_2 \alpha, \gamma = \delta_1 \beta$  we have

$$P_{ex}(x_k; \alpha, \beta, N) \leq P_{ex}(V_k; \gamma_0, \gamma, N) \quad (13)$$

and therefore from theorem 2 and (i)

$$P_{ex}(x_k; \alpha, \beta, N) \leq \lambda \quad (14)$$

Now recalling that  $P_{ex}(x_k; \alpha, \beta, N) + P_{in}(x_k; \alpha, \beta, N) = 1$  we have that finite-time stability for the system (5) with respect to  $(\alpha, \beta, N, \|\cdot\|, \lambda)$  i.e.

$$P_{in}(x_k; \alpha, \beta, N) \geq (1 - \lambda) \quad (15)$$

For point (ii), from the upper bound on  $P_{ex}(x_k; \alpha, \beta, N)$  provided in theorem 3, with the same principle as before directly follows that

$$P_{ex}(x_k; \alpha, \beta, N) \leq \lambda \quad (16)$$

and therefore

$$P_{in}(x_k; \alpha, \beta, N) \geq (1 - \lambda) \quad (17)$$

Finally for the proof of point (iii) let us consider the following for  $\rho_k > -1$  and  $\forall k = 0, \dots, N$

$$P\{\Delta V_k \leq \rho_k V_k\} = P\{V_{k+1} - (1 + \rho_k)V_k \leq 0\} \quad (18)$$

then iterating the partial difference inequalities and considering the upper bound on  $V_0 \leq \gamma_0$  we get  $\forall k = 0, \dots, N$

$$P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma_0 \prod_{i=0}^{k-1} (1 + \rho_i)\} \quad (19)$$

then using the condition (12) from (iii) it follows that  $\forall k = 0, \dots, N$

$$P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma\} \quad (20)$$

and moreover  $\forall k = 0, \dots, N$

$$(1 - \lambda) \leq P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma\} \quad (21)$$

that implies finite time stability with respect to  $(\alpha, \beta, N, \|\cdot\|, \lambda)$  ■

### C. Relations of FTS Conditions

In this section we compare the above results for FTS analysis. First we study how the two upper bounds presented in section (III-A) are related. In particular let us consider (recall theorem 2) the following

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \quad (22)$$

where  $\Phi_N = \sum_{i=1}^N \phi_i$  and from theorem 3, and from the definition of function  $V_k$ , we have

$$P_{ex}(x_k; \alpha, \beta, N) \leq \mathbb{E} \left[ \frac{\sup_{N \geq k \geq 0} V_k}{\gamma}; V_0 < \gamma_0 \right]$$

then using the fact that  $\delta_1 \|x_k\| \leq V(x_k) \leq \delta_2 \|x_k\|$  and  $\gamma = \delta_1 \beta$  we have

$$P_{ex}(x_k; \alpha, \beta, N) \leq P_{ex}(V_k; \gamma_0, \gamma, N) \quad (23)$$

and moreover, by Chebychev inequality

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \mathbb{E} \left[ \frac{\sup_{N \geq k \geq 0} V_k}{\gamma}; V_0 < \gamma_0 \right] \quad (24)$$

from inequalities (23,24) we conclude that, to find a the least conservative upper bound on  $P_{ex}(x_k; \alpha, \beta, N)$ , we only need to compare the two bounds on  $P_{ex}(V_k; \gamma_0, \gamma, N)$ , in (22) and (24). In particular we observe that in (22), starting from  $V_0$ , pessimistic bounds are set on the trajectory of  $V_k$  at each step by  $\phi_k$ 's. In (24) we are actually considering the expected value of supremum over all  $V_k$  in the studied interval. In principle the bound in (24) is less conservative than the one in (22) and does not require evaluation of the increment at each step, but on the other hand it is not easy to directly calculate the value of the supremum of  $V_k$ .

Now let us consider part (iii) of theorem 4 from which we have for  $k = 0, \dots, N$

$$\begin{aligned} P\{\Delta V_k \leq \rho_k V_k\} &\leq P\{V_k \leq \gamma_0 \sup_k \prod_{i=0}^{k-1} (1 + \rho_i)\} \\ &\leq P\{V_k \leq \gamma\} \\ &= 1 - P\{\sup_{N \geq k \geq 0} V_k > \gamma\} \end{aligned}$$

we then observe how the last term (the inclusion probability), is the complement of the exit probability for  $V_k$ , and then a bound analogous to the one in (24) applies.

Since the three parts of theorem 4 are comparable, from now on we will just focus on the first part (i), since it is more general and does not directly require the knowledge of the state of the system.

### D. Finite-Time Stochastic Stability Design

The previous section focused on analysis but may be extended to designing controllers that stochastically stabilize a system over a finite time. Consider the discrete-time, stochastic dynamical system in which the state is a Markov process in  $\mathbb{R}^n$

$$x_{k+1} = f(x_k, \theta_k) + g(x_k)u_k, \quad x \in \mathbb{R}^n, \quad x(0) = x_0 \quad (25)$$

Where  $x$  is the system state,  $u_k$  is a one-dimensional control input,  $f$  and  $g$  are vector functions, and  $\{\theta_k\}$  is an independent stationary random sequence with mean  $\mu_\theta$ . In particular, we consider systems in which the random sequence  $\{\theta_k\}$  appears linearly in the system i.e.  $f(x_k, \theta_k) = f(x_k)\theta_k$ . In order to simplify notation, we will use the following forms  $g(x_k) = g_{x_k}$  and  $f(x_k) = f_{x_k}$ .

We aim to design a state-feedback control law  $u_k = u(x_k)$ , such that the closed-loop system is FTSS with respect to the parameters  $(\alpha, \beta, N, \|\cdot\|, \lambda)$ . The proposed design technique is based on part (i) of theorem 4. In particular the control law has to guarantee the finite-time stochastic stability condition (i) is satisfied. From now on, we also restrict our study to the choice of  $V_k = x_k^T x_k$ , which will lead to conservative results.

*Theorem 5:* Let us consider the Markov process defined in (25), and denote with  $\mu_\theta$  the mean of each random variable  $\theta_k$ ,  $k = 0, \dots, N$ . Consider the FTSS condition (9), and let us choose  $\phi_k = \frac{\gamma\lambda - \gamma_0}{N}$ ,  $\forall k = 0, \dots, N$ , and therefore  $\Phi_N = \gamma\lambda - \gamma_0$ . Then, the system is stabilizable over a finite time with respect to  $(\alpha, \beta, N, \|\cdot\|, \lambda)$  and  $V(x_k) = x_k^T x_k$ , if there exists an input law  $u(x_k)$  such that,  $\forall k = 0, \dots, N, \forall x_k \in S_\gamma = \{x_k : V(x_k) \leq \gamma\}$

$$\begin{aligned} \text{if} \quad &g_{x_k}^T g_{x_k} = (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k}) = 0 \\ \text{then} \quad &(f_{x_k}^T f_{x_k} - x_k^T x_k) < \phi_k \end{aligned} \quad (26)$$

otherwise

$$\mathbb{E}_{x_k} [V(x_{k+1}, k+1)] \leq \infty \quad (27)$$

$$\begin{aligned} \mu_\theta f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 \\ + \mu_\theta (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k}) u_k \leq \phi_k; \end{aligned} \quad (28)$$

$$\begin{aligned} \mu_\theta^2 (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k})^2 \geq \\ 4g_{x_k}^T g_{x_k} (\mu_\theta f_{x_k}^T f_{x_k} - x_k^T x_k - \phi_k) \end{aligned} \quad (29)$$

The set of possible control laws is given by

$$\begin{aligned} u_1 \leq u_k \leq u_2, \quad \text{for } g_{x_k}^T g_{x_k} \neq 0, \text{ and} \\ (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k}) \neq 0; \\ u_k = 0, \quad \text{for } g_{x_k}^T g_{x_k} = (g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k}) = 0 \end{aligned}$$

Next, let  $A_1 = (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k})$ ,  $B_1 = (\mu_\theta f_{x_k}^T f_{x_k} - x_k^T x_k - \phi_k)$ ,

$$u_{1,2} = \frac{-\mu_\theta A_1 \pm \sqrt{\mu_\theta^2 (A_1^2) - 4g_{x_k}^T g_{x_k} B_1}}{2g_{x_k}^T g_{x_k}}$$

*Proof:* Consider condition (28). Because of the choice of  $\phi_k$  we have

$$\mathbb{E}_{x_k} [V_{k+1} - V_k] \leq \frac{\gamma\lambda - \gamma_0}{N}, \quad \forall k = 0, \dots, N \quad (30)$$

and also

$$\Phi_N = \sum_{k=0}^N \frac{\gamma\lambda - \gamma_0}{N} = \gamma\lambda - \gamma_0, \quad (31)$$

from theorem 2 the above conditions imply

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \leq \frac{[\gamma_0 + \gamma\lambda - \gamma_0]}{\gamma} = \lambda \quad (32)$$

and therefore finite-time stability follows.  $\blacksquare$   
The proposed design technique guarantees closed-loop finite-time stability under the theorem's assumptions. However, we actually designed to meet the specified bound by fixing  $\phi_k$ . This is a constraint that makes the above conditions on the existence of the controller only sufficient.

### E. Minimization of the Exit Bound

In the previous section we designed a controller in order to meet given bounds on the inclusion probability  $P_{in}$  of the stochastic system (25). Here we proceed to develop design techniques to maximize the inclusion probability of the system. Instead of directly designing for the objective  $P_{in}$ , we base our design on the minimization of some upper bound on the objective  $P_{ex}$ . Consider the following optimization problem

$$\max_u P_{in}(x_k; \alpha, \beta, N) = \max_u P\{||x(k)|| \leq \beta : k \in [0, N]; ||x_0|| \leq \alpha\} \quad (33)$$

given the system (25). This objective can be achieved also by considering the equivalent problem

$$\min_u P_{ex}(x_k; \alpha, \beta, N) = \min_{u_k} P\{ \sup_{0 \leq k \leq N} ||x_k|| > \beta; ||x_0|| \leq \alpha \} \quad (34)$$

We can indirectly solve this problem by minimizing an upper bound on the function i.e.

$$\min_u L(x_k, u_k) \quad (35)$$

where  $L(x_k, u_k)$  is a cost function such that,  $\forall k = 0, \dots, N, \forall x_k \in S_\gamma$

$$P_{ex}(x_k; \alpha, \beta, N) \leq L(x_k, u_k) \quad (36)$$

In section (III-A) we provided some bounds on  $P_{ex}(V_k; \gamma_0, \gamma, N)$  and consequently on  $P_{ex}(x_k; \alpha, \beta, N)$ . Here we use those bounds in order to design for finite-time stability for the system (25) with respect to  $(\alpha, \beta, N, ||\cdot||, \lambda)$ , with  $\lambda$  as small as possible.

*Theorem 6:* Consider the system (25), and a function  $V(x_k) = x_k^T x_k$ . Then there exists a control law  $u_{opt}(x_k)$  that minimizes  $P_{ex}(V_k; \gamma_0, \gamma, N)$  i.e. stabilizes the system over a finite time with respect to  $(\alpha, \beta, N, ||\cdot||)$  if for  $g_{x_k}^T g_{x_k} \neq 0$ ,  $u_k$  minimizes the cost function  $L(x_k, u_k)$ , i.e.

$$L(x_k, u_{k,opt}) \leq L(x_k, u_k), \forall u, \forall k = 0, \dots, N \quad (37)$$

where  $\forall k = 0, \dots, N$

$$L(x_k, u_k) = \mu_\theta f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 + \mu_\theta (A_1) u_k$$

Moreover, the optimal control law is given by

$$u_k = \begin{cases} 0 & g_{x_k}^T g_{x_k} = 0 \\ -\frac{\mu_\theta (g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k})}{(2g_{x_k}^T g_{x_k})} & g_{x_k}^T g_{x_k} \neq 0 \end{cases} \quad (38)$$

for all  $k = 0, \dots, N$ .

*Proof:* The control law that minimizes  $\lambda$  can be found by considering once again the upper bound on the exit probability presented in theorem 2. The following sufficient conditions are given for the existence of such upper bound

$$P_{ex}(x_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma}$$

$$\mathbb{E}_{x_k} [V_{k+1} - V_k] \leq \phi_{k+1}, \forall x_k \in S_\gamma, \phi_k \geq 0 \quad (39)$$

where  $\Phi_N = \sum_{k=0}^N \phi_k$ . Since our objective is to maximize the inclusion probability or, equivalently, minimize the exit probability, we may minimize the upper bound on the exit probability since  $\gamma, \gamma_0, N$  are independent of the input  $u_k$ . We can then meet this requirement from the inequality (39) by minimizing each of the terms  $\mathbb{E}_{x_k} [V_{k+1} - V_k]$  for  $x_k \in S_\gamma$  or equivalently for  $V_k = x_k^T x_k$

$$L(x_k, u_k) = \mathbb{E}[(\theta_k^2 f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 + \theta_k (g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k}) u_k)]$$

that is an upper bound on  $\mathbb{E}_{x_k} [\Delta(V(x_k))]$ . Since  $\gamma, \gamma_0$  and  $\mathbb{E}[\theta_k^2] = \mathbb{E}[\theta_k] = \mu_\theta$  are fixed positive values we have

$$L(x_k, u_k) = [(\mu_\theta f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 + \mu_\theta (g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k}) u_k)] \quad \forall k = 0, \dots, N \quad (40)$$

We then obtain  $u_k$  in (38) that minimize  $L(x_k, u_k)$  by finding the solution to  $\frac{\partial}{\partial u_k} L(x_k, u_k) = 0$   $\blacksquare$

## IV. FINITE-TIME STABILITY DESIGN EXAMPLE

In this section we present an example to illustrate our design techniques.

*Example 1:* Consider the system

$$x_{k+1} = 0.5e^{(x_k)} \theta_k + \sin(2\pi \frac{x_k}{5} - 7) u_k$$

where  $\theta_k \in \{0, 1\}$  is a process of i.i.d. random variables, with mean  $\mu_\theta = 0.5$ . We would like to choose  $u_k$  in such a way that the closed-loop system is finite-time stable with respect to  $(\alpha = 0.25, \beta = 1, N = 10, ||\cdot||, \lambda = 0.3)$ . We also want to minimize a bound on the exit probability  $P_{ex}$ .

By applying theorem 5 with  $\delta_1 = 1, \delta_2 = 1$  and therefore  $\phi_k = 0.005$  and choosing in the admissible range of controller  $u_k = -1.3$ , for  $\sin(2\pi \frac{x_k}{5} - 7) \neq 0$ , and  $u_k = 0$ , for  $\sin(2\pi \frac{x_k}{5} - 7) = 0$ , we obtain the closed-loop system

$$x_{k+1} = 0.5e^{(x_k)} \theta_k + \sin(2\pi \frac{x_k}{5} - 7) (-1.3 \text{sign}(|\sin(2\pi \frac{x_k}{5} - 7)|))$$

Also applying the input  $u_{opt}$  that minimizes  $\lambda$  we obtain the closed-loop dynamics,

$$\begin{aligned} x_{k+1} &= 0.5e^{(x_k)} \theta_k + \sin(2\pi \frac{x_k}{5} - 7) u_{opt}(k) \\ u_{opt}(k) &= \text{sign}(|\sin(2\pi \frac{x_k}{5} - 7)|) M(k) \\ M(k) &= \frac{-0.5e^{(x_k)} \sin(2\pi \frac{x_k}{5} - 7)}{2(\sin(2\pi \frac{x_k}{5} - 7))^2} \end{aligned}$$

In Figure (2) we compare a simulation of the closed-loop system, with the first controller  $u_k$  designed for FTSS with respect to  $(\alpha = 0.25, \beta = 1, N = 10, \|\cdot\|, \lambda = 0.3)$ , with the open-loop controller, and finally the closed-loop system with the second controller  $u_{opt}$ . Notice how in the open-loop case the bound  $\beta = 1$  is exceeded for more than three times over the first 10 seconds of simulation, while in the second case the bound is exceeded 3 times over the 10 seconds (i.e.  $P_{ex} = 0.3$ ) and in the third case is never exceeded, that shows how the design goals have been satisfied.

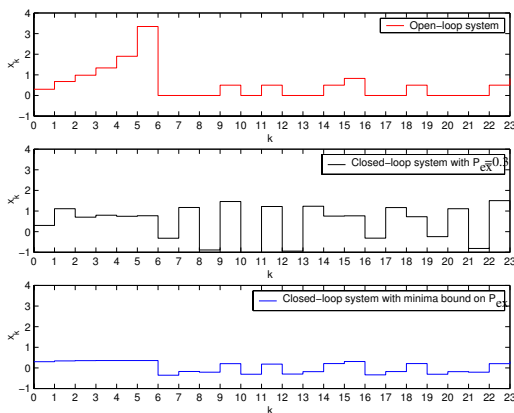


Fig. 2. Open loop system versus closed loop systems with exit probability  $P_{ex} \leq 0.3$  and minimal exit probability.

## V. CONCLUSION

We presented in this paper new results on finite-time stability for stochastic discrete-time nonlinear systems. Moreover, we explored how finite-time stability analysis techniques can be extended to control design.

After discussing deterministic FTS, and a new approach to its analysis, we considered a stochastic system and explored its finite-time stability. In particular, we described the concepts of “inclusion probability” and “exit probability”. We also showed how these quantities can be bounded by bounds that depend on the required finite-time stability parameters and that may be used to analyze FTSS, and to design for closed-loop FTSS. We finally described how an upper bound on the exit bound can be minimized, that is design for minimizing the probability of exceeding a bound over a finite time.

The most difficult aspect of applying our results is the checking the inequalities in the various theorems. It might of future value to study specific structures such as polynomial systems to alleviate such problems. It is also of interest to apply the FTSS results presented here to packet-dropping problems in networked control systems, and to study the effects of time delay.

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