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Data Rates Conditions for Network Control System Stabilization

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Abstract— **In this paper we present sufficient conditions on the rate of a packet network to guarantee asymptotic stability of unstable discrete LTI system with linear state feedback control. Two types of Network Control Systems are considered in the absence of communication delays. For one type we consider the case where we have invertible B matrix and the case where this does not occurred. Examples and simulations are provided to demonstrate the results.**

I. Introduction

Feedback control systems whose control loops are closed through a real-time network are called Networked Control Systems (NCS) [2], [4]. Although these systems have the advantage of low cost and simplified maintenance and diagnosis, the assumptions of classical control may need to be revisited in order to design them. The new problems arise because the sensed data and

In particular, the communication channel between the plant and the controller may no longer remain unmodelled, since it can carry a finite number of bits/s and the conventional assumption of infinite capacity of the channel no longer holds. In addition to suffering from both delay and quantization effects, the finite data rate forces us to determine the usefulness of the number of bits [5]. This is precisely the issue we focus on this work. The question we pose and attempt to answer is: how many bits are needed in the sensor-to-controller and controller-to-actuator networks to control an unstable system when the controller structure is a state feedback controller?

Several researchers have studied the problem. Mitter [6] and collaborators have contributed to the development of a new theory that matches classical control theory with traditional information theory, [1], [8], [9], and [7]. The results on these works considered only a digital channel of communication instead of a packet-based network which can include time delays and packet dropouts. Also, all such works considered the encoded state estimation error as the message that is sent through the channel.

A theory for control over a packet-based network was recently proposed in [10] and [11], as well as in [3]. The authors considered state encoding instead of the estimation error coding. Some assumptions of these works were relaxed in [12]. In our present work we include the case where a linear feedback controller is used instead of the control sequence that was built in [12]. There it was shown that for a discrete-time unstable LTI under a state encoding/decoding scheme with equal bit allocation per component of the state, we have the following sufficient condition on the data rate to stabilize it: $\frac{R}{n} \geqslant \lfloor \log(\|A^n\|) + 1 \rfloor + 1$; where $\frac{R}{n}$ are the number of bits per sample that are allocated for each state. A logical question that arise is, is the same R sufficient when, we have a linear state feedback controller, $u(k) = -K\bar{x}(k)$? As we will see in the following example the answer is no. This is surprising since a previous paper [8] used a linear state feedback controller for stabilization with a minimum rate. The difference arises because of the inefficiency in encoding the state instead of the estimation error. However, the state encoding has the advantage of easier implementation.

Next, we present an example to show how the packet rate obtained in [12] is not sufficient when a controller of the form $u = -K\bar{x}$ is used.

A. Example

Consider the following system

$$x(k+1) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k) \qquad (1)$$

We assume the initial condition to be $x(0) = \begin{bmatrix} 1.33 & 3.768 & 8.44 \end{bmatrix}'$. If we choose $u(k) = -K\bar{x}(k)$, where $\bar{x}(k)$ is an estimation of the state that consists in the R/n most significant bits of the binary representation of each component of the state $x(k)$, we get $x(k+1) = (A - BK)x(k) + BK\varepsilon(k)$. Where $\varepsilon(k)$ is the error between the actual state and the estimation $\bar{x}(k)$. For $K = \begin{bmatrix} 10.912 & 39.711 & -37.023 \end{bmatrix}$ the matrix $(A - BK)$ will be stable with eigenvalues $\lambda = \{0.9, 0.8, 0.7\}$. Now, if we use the bit rate per state given in [12], i.e., $\frac{R}{n} \geqslant \lfloor \log(\|A^n\|) + 1 \rfloor + 1 = 10$ bits/time-step, we see that this is not enough to stabilize the system (Figure 1). Therefore, we need to find a new condition on the rate and that is partially the goal of this work.

We want to clarify that in all the simulations in this paper, although $x(k)$ is discrete and exists just in the instants $k = \{0, 1, 2, \ldots\}$, we plotted them like a continuous signal for visualization purposes.

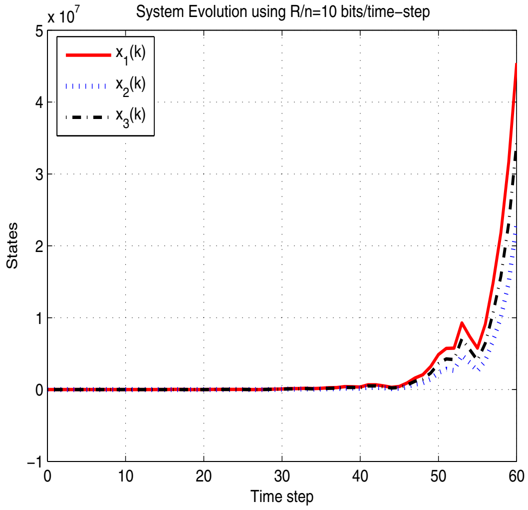

Fig. 1. Closed-loop network control system with state-feedback controller (Type I): Using $\frac{R}{n} = 10$ bits/time-step.

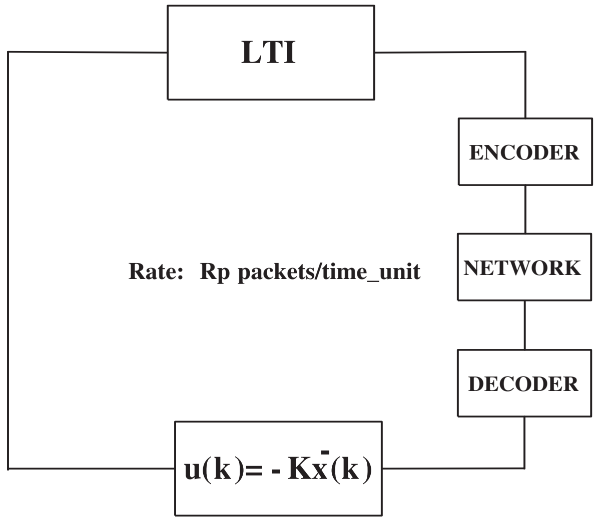

Fig. 2. Closed-loop network control system with state-feedback controller: Type I

II. Problem Setup

A. Network Control System: Type I

We are interested in improving the results on [12]. We thus consider the same two possible configurations for the packet-based network control system. The first system, referred to as *Network Control System Type I*, has a rate of R_{p1} packets/sample-time. The packet based network considers a packet size of D_{Max} bits used for data (although the protocol information needs extra bits in the packet, it is not needed for this analysis) and assume the closed loop shown in Figure 2 given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ u(k) &= -K\bar{x}(k) \end{aligned} \tag{2}$$

where A is $n \times n$ and we assume that it is diagonal $A = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$ and $|\lambda_j| \geq 1, \forall j \in \{1, \dots, n\}$, and $\lambda_i \neq \lambda_j$ if $j \neq i$, $x(k)$ is $n \times 1$, B is $n \times m$ and $u(k)$ is $m \times 1$.

Knowing that $\bar{x}(k) = x(k) - \varepsilon(k)$, system (2) can be rearranged as

$$x(k+1) = (A - BK)x(k) + BK\varepsilon(k) \tag{3}$$

We assume that the controller does not saturate, and that the packet-network does not drop packets nor is it subjected to disturbances (noise) or time delay. Basically with these assumptions we are just focusing on the implications of a limited network rate. We assume that the plant is able to send the complete states measurements through the link, i.e, that the states are measured. We also assume perfect synchronization of the encoder and decoder so that the decoder knows exactly the encoding scheme used by the encoder at all times.

B. Network Control System: Type II

The second type of packet-based network, to be referred to as *Network Control System Type II*, consists of the same discrete LTI system given by equation (2), but with the addition of a second network between the controller and the actuator with rate R_{p2} as shown in Figure 3. This consideration leads to the following system

$$\begin{aligned} x(k+1) &= Ax(k) + B\bar{u}(k) \\ \bar{u}(k) &= u(k) - \varepsilon_u(k) \\ u(k) &= -K\bar{x}(k) \end{aligned} \tag{4}$$

where A is $n \times n$ and we assume that it is diagonal $A = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$ and $|\lambda_j| \geq 1, \forall j \in \{1, \dots, n\}$, and $\lambda_i \neq \lambda_j$ if $j \neq i$, $x(k)$ is $n \times 1$, B is $n \times m$, $\bar{u}(k)$ is $m \times 1$, $u(k)$ is $m \times 1$ and $\varepsilon u(k)$ is $m \times 1$.

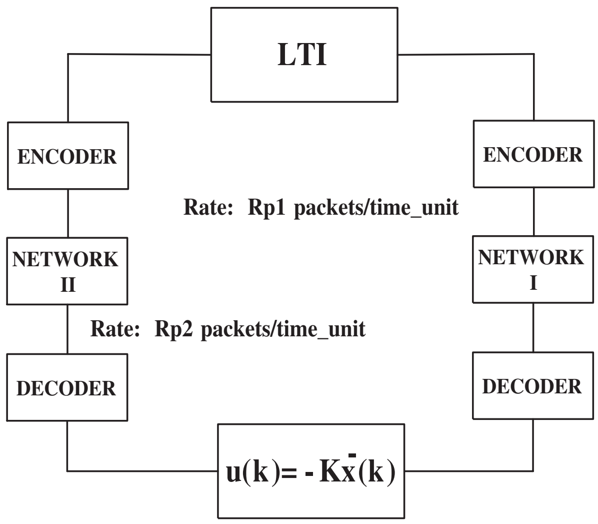

Fig. 3. Closed-loop network control system with state-feedback controller: Type II

Knowing that $\bar{x}(k) = x(k) - \varepsilon_x(k)$ and $\bar{u}(k) = u(k) - \varepsilon_u(k)$, system (4) can be rearranged as

$$x(k+1) = (A - BK)x(k) + BK\varepsilon_x(k) - B\varepsilon_u(k) \tag{5}$$

III. Results

A. Network Control System Type I with State-Feedback Controller

For the case where we have a NCS with State-Feedback Controller (Type I), we have the following result.

Theorem 3.1: Assuming an equal allocation of bits per state and (A,B) is a controllable pair, a sufficient condition for system (2) to be asymptotically stabilizable is

$$R_p \geqslant \left\lceil \frac{R}{D_{Max}} \right\rceil$$

where $R = n\left\lceil \log\left(\frac{\|BK\|}{\frac{1}{2}-\|A-BK\|}\right)\right\rceil$, $\lceil\rceil$ is the **ceil** function and every state is allocated in $\frac{R}{n}$ bits/sample.

Proof:

Let us assume that the binary expansion of the state $x(k)$ is given by:

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} = \begin{bmatrix} \sum_{i=-\infty}^{M_1} \alpha_{1i}2^i \\ \sum_{i=-\infty}^{M_2} \alpha_{2i}2^i \\ \vdots \\ \sum_{i=-\infty}^{M_n} \alpha_{ni}2^i \end{bmatrix} \tag{6}$$

Where $\alpha_{ij} \in \{0,1\}$ and $M_j \in \mathbb{N}$. For the sake of simplification we also assume that in the binary expansion $x_j(k) > 0, \forall j$. This is possible since the sign of each state mode could be considered by adding n extra bits in the rate, one bit per state sign. Also, we know that $x_j \leqslant 2^{M_j+1}$.

Now, let's assume that $M_{max} = \max(M_1, M_2, \ldots, M_n)$ if we take the norm of the state, we have:

$$\begin{aligned} \|x(k)\| &\leqslant \|x_1(k)\| + \ldots + \|x_n(k)\| \\ &\leqslant n2^{M_{max}+1} \end{aligned} \tag{7}$$

We know that we can represent $n2^{M_{max}+1}$ by a minimum number of bits, $\tilde{M} = M_{max} + \log_2(n) + 1$, and therefore, $2^{\tilde{M}-1} < \|x(k)\| \leqslant 2^{\tilde{M}}$. Now, let us consider an equal allocation of bits per state component, $\frac{R}{n}$, so that the encoded version of $x(k)$ is given by $\bar{x}(k)$, and:

$$\bar{x}(k) = \begin{bmatrix} \sum_{i=M_1-\frac{R}{n}+1}^{M_1} \alpha_{1i}2^i \\ \sum_{i=M_2-\frac{R}{n}+1}^{M_2} \alpha_{2i}2^i \\ \vdots \\ \sum_{i=M_n-\frac{R}{n}+1}^{M_n} \alpha_{ni}2^i \end{bmatrix} \tag{8}$$

The error between the actual state and the encoded version, $\varepsilon(k) = x(k) - \bar{x}(k)$, is given by:

$$\varepsilon(k) = \begin{bmatrix} \sum_{i=-\infty}^{M_1-\frac{R}{n}} \alpha_{1i}2^i \\ \sum_{i=-\infty}^{M_2-\frac{R}{n}} \alpha_{2i}2^i \\ \vdots \\ \sum_{i=-\infty}^{M_n-\frac{R}{n}} \alpha_{ni}2^i \end{bmatrix} \tag{9}$$

Therefore, we have $\varepsilon_j(k) < 2^{M_j-\frac{R}{n}+1}$, and

$$\begin{aligned} \|\varepsilon(k)\| &\leqslant \|\varepsilon_1(k)\| + \ldots + \|\varepsilon_n(k)\| \\ &\leqslant n2^{M_{max}-\frac{R}{n}+1} \\ &= 2^{\tilde{M}-\frac{R}{n}}. \end{aligned} \tag{10}$$

If we analyze the evolution of the system starting at time k given by $x(k+1) = Ax(k) + Bu(k)$. If $u(k) = -K\bar{x}(k) = -K(x(k) - \varepsilon(k))$ then

$$x(k+1) = (A-BK)x(k) + BK\varepsilon(k) \tag{11}$$

We have then:

$$\begin{aligned} \|x(k+1)\| &\leqslant \|A-BK\|\|x(k)\| + \|BK\|\|\varepsilon(k)\| \\ &\leqslant \|A-BK\|2^{\tilde{M}} + \|BK\|2^{\tilde{M}-\frac{R}{n}} \end{aligned}$$

To shrink the state we need:

$$\begin{aligned} \|A-BK\|2^{\tilde{M}} + \|BK\|2^{\tilde{M}-\frac{R}{n}} &< 2^{\tilde{M}-1} \\ \|A-BK\| + \|BK\|2^{-\frac{R}{n}} &< 2^{-1} \end{aligned}$$

Solving for $\frac{R}{n}$ we get:

$$\frac{R}{n} > \log_2\left(\frac{\|BK\|}{\frac{1}{2} - \|A-BK\|}\right) \tag{12}$$

The $\lceil . \rceil$ function was introduced since $\frac{R}{n}$ must be an integer number of bits for each state component. Now, R is the sufficient number effective bits that we need to transmit of the whole state for stabilization. But, knowing that a packet has a maximum length of D_{Max}, then if, $R \leq D_{Max}$, we will need a packet rate of $R_p = 1$ packet/sample-time. However, if we have $R > D_{Max}$ then, we will need a minimum of $\left\lceil \frac{R}{D_{Max}} \right\rceil$ packets/time-step. Actually, this last expression covers both cases, since $\frac{R}{D_{Max}} < 1$ gives a 1 packet/sample-time when the ceil function is applied.

■

However, to get a physically realizable R we need that $\frac{1}{2} - \|A-BK\| > 0$. Unless B is invertible, the appropriate K to accomplish this condition is difficult or impossible to get as shown in the following example.

Consider the system given by

$$x(k+1) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

Let us assume that the norm used is $\|A-BK\|_\infty$. If we impose the condition of $\frac{1}{2}-\|A-BK\|_\infty > 0$ we need

$$\|A-BK\|_\infty = \left\|\begin{matrix} 2-k_1 & -k_2 \\ -k_1 & 3-k_2 \end{matrix}\right\|_\infty > \frac{1}{2}$$

Since the infinite norm is defined as: $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$. Therefore, we need to satisfy the two following inequalities

$$|2-k_1|+|-k_2| < \frac{1}{2} \tag{13}$$

$$|-k_1|+|3-k_2| < \frac{1}{2} \tag{14}$$

To satisfy inequality (13) we see that, at least, we need to satisfy $|2-k_1| < \frac{1}{2}$. This imply that $\frac{3}{2} < k_1 < \frac{5}{2}$, but these limits will make it impossible to satisfy the inequality (14). Therefore, there is no k_1 and k_2 such that $|A-BK|_\infty < \frac{1}{2}$. If B were invertible, then we can always choose $K = B^{-1}A$ and, therefore, $\frac{R}{n} > \lceil \log_2(\|A\|)+1 \rceil$, will be the sufficient rate to stabilize the unstable system with a linear state feedback controller. But this is very conservative and will help us only when designing from the beginning and under the assumption of multiple inputs. It does not provide a sufficient rate if we already have a specific stable structure $(A-BK)$. The following section will deal with this issue.

B. Network Control System Type I with State-Feedback Controller without invertibility property on B matrix.

We need to introduce some results of perturbed systems that will provide some tools to prove theorem 3.2.

1) Perturbed System: Consider the system

$$x(k+1) = Mx(k)+g(x) \tag{15}$$

where M is a stable matrix and $g(x)$ is a perturbation in the system, like a modeling error or a disturbance. Let's assume that $\|g(x)\|_2 \leqslant \gamma \|x(k)\|_2$ for all k and $x \in \mathbb{R}^n$ and $\gamma > 0$.

Let $Q = Q^T > 0$ and solve the discrete-time Lyapunov equation

$$M^T PM - P + Q = 0 \tag{16}$$

for P. We know that there is a unique solution $P = P^T > 0$.

If we propose a candidate Lyapunov function $V(x) = x^T Px$ we know that:

$$\lambda_{min}(P)\,\|x(k)\|_2^2 \leqslant V(x) \leqslant \lambda_{max}(P)\,\|x(k)\|_2^2$$

and

$$-x^T(k)Qx(k) \leqslant -\lambda_{min}(Q)\,\|x(k)\|_2^2$$

where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the smallest and greatest eigenvalues of P, respectively; and $\lambda_{min}(Q)$ is the smallest eigenvalue of Q.

Now, by the second method of Lyapunov we have

$$\begin{aligned}
\triangle V(x) &= V(x(k+1)) - V(x(k)) \\
&= x^T(k+1)Px(k+1) - x^T Px(k) \\
&= (Mx(k)+g(x))^T P(Mx(k)+g(x)) - x^T(k)Px(k) \\
&= x^T(k)(M^T PM - P)x(k) + g^T(x)PMx(k) + \\
&\quad +g^T(x)Pg(x) \\
&= -x^T(k)Qx(k) + g^T(x)PMx(k) + g^T(x)Pg(x) \\
&\leqslant -\lambda_{min}(Q)\,\|x(k)\|_2^2 + g^T(x)PMx(k) + g^T Pg(x) \\
&\leqslant -\lambda_{min}(Q)\,\|x(k)\|_2^2 + \gamma\|PM\|_2\,\|x(k)\|_2^2 + \\
&\quad +\gamma^2\,\|P\|_2\,\|x(k)\|_2^2
\end{aligned}$$

Then, for asymptotic stability we need that

$$-\lambda_{min}(Q) + \gamma\|PM\|_2 + \gamma^2\,\|P\|_2 < 0 \tag{17}$$

We also know that:

$$\gamma\|PM\|_2 + \gamma^2\,\|P\|_2 \quad < \quad \gamma\|P\|_2\,(\|M\|_2+\gamma)$$

and

$$\gamma\|P\|_2\,(\|M\|_2+\gamma) \quad < \quad \gamma\lambda_{max}(P)(\|M\|_2+\gamma)$$

Finally, to satisfy inequality (17), we need the condition $\gamma\lambda_{max}(P)(\|M\|_2+\gamma) < \lambda_{min}(Q)$. That can be rearranged as

$$\gamma(\|M\|_2+\gamma) < \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} \tag{18}$$

According to [13] the ratio given in the right side of equation (18) is maximized when $Q = \mathbb{I}$. Then

$$\gamma(\|M\|_2+\gamma) < \frac{1}{\lambda_{max}(P)} \tag{19}$$

Now, since $\|M\|_2$ and $\gamma > 0$ we can calculate the region for γ that satisfies inequality (19). First, we use the auxiliary variables $a = \|M\|_2$ and $b = \frac{1}{\lambda_{max}(P)}$. We can plot the function $f_1(\gamma) = \gamma^2 + a\gamma$ and $f_2(\gamma) = b$, as in Figure 4.

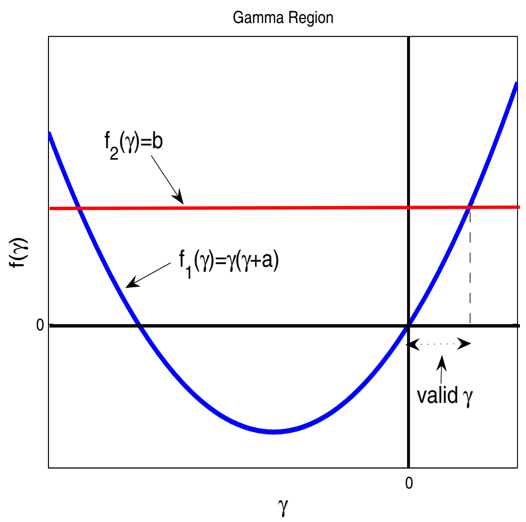

Fig. 4. Valid γ Region

Solving the inequality $\gamma(a+\gamma) < b$ for $\gamma > 0$ we get

$$\gamma \leqslant \frac{-a+\sqrt{a^2+4b}}{2} \tag{20}$$

Substituting the original variables we get

$$\gamma \leqslant \frac{-\|M\|_2 + \sqrt{\|M\|_2^2 + \frac{4}{\lambda_{max}(P)}}}{2} \tag{21}$$

2) Generalized Result for Network Type I: With the previous tools we can now state the following theorem that provides sufficient conditions for the bit rate when we have a discrete-time LTI system and a linear state feedback controller $u(k) = -K\bar{x}(k)$.

***Theorem** 3.2:* Assuming an equal allocation of bits per state and (A,B) is a controllable pair, a sufficient condition for system (2) to be asymptotically stabilizable is

$$R_p \geqslant \left\lceil \frac{R}{D_{Max}} \right\rceil$$

where $R = n\left\lceil \log_2\left(\frac{2\|BK\|_2}{-\|A-BK\|_2+\sqrt{\|A-BK\|_2^2+4/\lambda_{max}(P)}}\right)\right\rceil$, P is the solution of the discrete-time Lyapunov equation given by

$$(A-BK)^T P(A-BK) - P = -\mathbb{I}.$$

and every state can allocate $\frac{R}{n}$ bits/sample.

Proof: According to the previous subsection we see that system (3) is the same perturbed system that we just analyzed, with $M = A - BK$ and $g(x) = BK(x(k) - \bar{x}(k)) = BK\varepsilon(k)$, and $\|g(x)\|_2 \leqslant 2^{-\frac{R}{n}}\|BK\|_2\|x(k)\|_2$. We clearly see that for this case $\gamma = 2^{-\frac{R}{n}}\|BK\|_2$.

Substituting in inequality (21) we get

$$\|BK\|_2 2^{-\frac{R}{n}} \leqslant \frac{-\|A-BK\|_2+\sqrt{\|A-BK\|_2^2+4/\lambda_{max}(P)}}{2} \tag{22}$$

where P is the solution of the discrete-time Lyapunov equation

$$(A-BK)^T P(A-BK) - P = -\mathbb{I}.$$

If we solve for $\frac{R}{n}$ we get

$$\frac{R}{n} \geqslant \log_2\left(\frac{2\|BK\|_2}{-\|A-BK\|_2+\sqrt{\|A-BK\|_2^2+4/\lambda_{max}(P)}}\right)$$

Similarly to the proof of theorem 3.2, the ceil function is finally added to get an integer number of bits and R is the sufficient number effective bits that we need to transmit of the whole state for stabilization. Also, knowing that a packet has a maximum length of D_{Max}, we need a minimum of $\left\lceil \frac{R}{D_{Max}} \right\rceil$ packets/time-step as was explained before. ∎

C. Generalized Result for Network Type II

With the previous approach we can now state the following theorem that provides sufficient conditions for the bit rates, $R_{p1} = \left\lceil \frac{R_1}{D_{Max}} \right\rceil$ and $R_{p2}\left\lceil \frac{R_2}{D_{Max}} \right\rceil$ when we have a Network Control System Type II.

***Theorem** 3.3:* Assuming an equal allocation of bits per state and (A,B) is a controllable pair, a sufficient condition for system (4) to be asymptotically stabilizable is

$$\|BK\|_2 2^{-\frac{R_1}{n}} + \|B\|_2\|K\|_2 2^{-R_2-\frac{R_1}{n}} + 2^{-R_2}\|B\|_2\|K\|_2 \leqslant \Omega$$

where Ω is given by

$$\Omega = \frac{-\|A-BK\|_2+\sqrt{\|A-BK\|_2^2+4/\lambda_{max}(P)}}{2}$$

and P is the solution of the discrete-time Lyapunov equation

$$(A-BK)^T P(A-BK) - P = -\mathbb{I}.$$

Proof: Similarly to the proof of theorem 3.3, we see that system (5) is the new perturbed system, with $M = A - BK$ and $g(x) = BK(x(k) - \bar{x}(k)) - B(u(k) - \bar{u}(k)) = BK\varepsilon_x(k) - B\varepsilon_u(k)$. Now

$$\begin{aligned}
\|g(x)\|_2 &= \|BK\varepsilon_x(k) - B\varepsilon_u(k)\|_2 \\
&\leqslant \|BK\varepsilon_x(k)\|_2 + \|B\varepsilon_u(k)\|_2 \\
&\leqslant 2^{-\frac{R_1}{n}}\|BK\|_2\|x(k)\|_2 + 2^{-R_2}\|B\|_2\|u(k)\|_2 \\
&\leqslant 2^{-\frac{R_1}{n}}\|BK\|_2\|x(k)\|_2 + \\
&\quad +2^{-R_2}\|B\|_2\left[2^{-\frac{R_1}{n}}\|K\|_2\|x(k)\|_2 + \|K\|_2\|x\|_2\right] \\
&= \gamma\|x\|_2
\end{aligned}$$

with

$$\gamma = \left[\|BK\|_2 2^{-\frac{R_1}{n}} + \|B\|_2\|K\|_2 2^{-R_2-\frac{R_1}{n}} + \|B\|_2\|K\|_2 2^{-R_2}\right].$$

Substituting in inequality (21) we get

$$\|BK\|_2 2^{-\frac{R_1}{n}} + \|B\|_2\|K\|_2 2^{-R_2-\frac{R_1}{n}} + \|B\|_2\|K\|_2 2^{-R_2} \leqslant \Omega \tag{23}$$

where Ω is given by

$$\Omega = \frac{-\|A-BK\|_2+\sqrt{\|A-BK\|_2^2+4/\lambda_{max}(P)}}{2} \tag{24}$$

and P is the solution of the discrete-time Lyapunov equation

$$(A-BK)^T P(A-BK) - P = -\mathbb{I}$$

Here again we need a minimum of $R_{p1} = \left\lceil \frac{R_1}{D_{Max}} \right\rceil$ packets/time-step for the sensor-controller network and a minimum of $R_{p2} = \left\lceil \frac{R_2}{D_{Max}} \right\rceil$ packets/time-step in the controller-actuator network.

IV. Simulations

To verify some of the results derived previously, we present several numerical examples and simulate them in Matlab®. We want to clarify that in the following plots, although $x(k)$ is discrete and exists just in the instants $k = \{0,1,2,\ldots\}$, we use continuous signals for visualization purposes.

A. Example for NCS Type I

Now, using the same system of section I-A. If we want to use theorem 3.2, we need to solve the corresponding Lyapunov equation. We get $\lambda_{max}(P) = 1.36 \times 10^8$. According to this, the bit rate per state is $\frac{R}{n} = 41$ bits/time-step. The simulation is given in Figure 5.

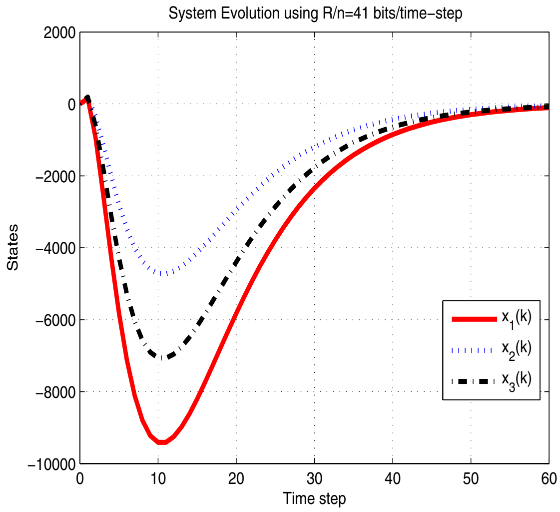

Fig. 5. Closed-loop network control system (Type I) using $\frac{R}{n} = 41$ bits/time-step

B. Example for NCS Type II

If we consider the same example that we have been working with and if we want to use theorem 3.3, we can use the same solution P that we calculated before in section IV-A, where $\lambda_{max}(P) = 1.36 \times 10^8$. According to theorem 3.3, we have to pick two rates that satisfy the inequality given in (23). In other words,

$$\|BK\|_2 2^{-\frac{R_1}{n}} + \|B\|_2 \|K\|_2 2^{-R_2-\frac{R_1}{n}} + 2^{-R_2} \|B\|_2 \|K\|_2 \leqslant \Omega$$

where $\Omega = 7.754 \times 10^{-11}$. Let us suppose that for the sensor-controller network we choose the same rate $\frac{R_1}{n} = 41$ bits/time-step as in the example of section IV-A. Then solving for R_2, the bit rate in controller-actuator network, we get $R_2 = 42$ bits/time-step. The simulation is given in Figure 6.

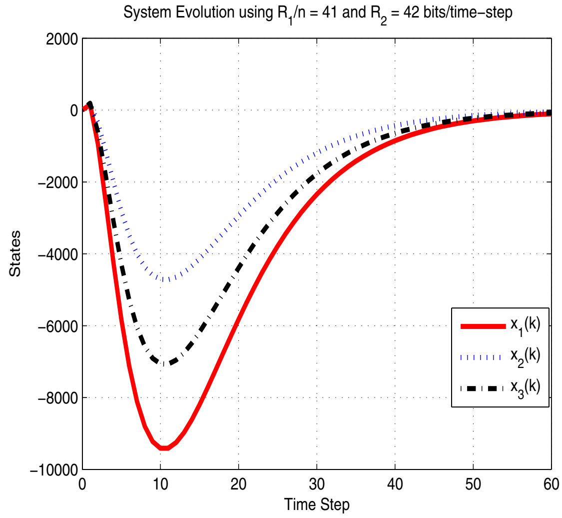

Fig. 6. Closed-loop network control system (Type II) using $\frac{R_1}{n} = 41$ and $R_2 = 42$ bits/time-step

V. Conclusions and Future Work

This paper has provided extensions of previous results on determining the sufficient rate of a packet-based networked control system. Here we relaxed the condition of using a specific control structure and replace it by the well known linear state feedback controller. The rates for Network Type I are much higher that the limits shown in previous works since we encoded the state itself and not the error between the state and its encoded version. We also obtained rates for a Network Type II where we included sensor-controller channel as well as controller-actuator channel.

Future work will include the inclusion of time delays and packet dropouts in the channels. Other ideas for future work include dealing with noise in the loop, the compensation in the networks rates for the extra information required by the decoder. We also know that the rates that we obtained via Lyapunov analysis are conservative, so it will be interesting to find sufficient conditions which are less conservative.

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