

7-9-2007

# Rate-Limited Stabilization for Network Control Systems

Chaouki T. Abdallah

I. Lopez

Follow this and additional works at: [https://digitalrepository.unm.edu/ece\\_fsp](https://digitalrepository.unm.edu/ece_fsp)

---

## Recommended Citation

Abdallah, Chaouki T. and I. Lopez. "Rate-Limited Stabilization for Network Control Systems." *2007 American Control Conference* (2007): 275-280. doi:10.1109/ACC.2007.4282830.

This Article is brought to you for free and open access by the Engineering Publications at UNM Digital Repository. It has been accepted for inclusion in Electrical & Computer Engineering Faculty Publications by an authorized administrator of UNM Digital Repository. For more information, please contact [disc@unm.edu](mailto:disc@unm.edu).

# Rate-Limited Stabilization for Network Control Systems

I. Lopez, C.T. Abdallah

Electrical & Computer Engineering Department  
MSC01 1100

1 University of New Mexico  
Albuquerque, NM 87131-0001

{ilopez, chaouki}@ece.unm.edu

**Abstract**—In this paper, we extend results from packet-based control theory, and present sufficient conditions on the rate of a packet network to guarantee asymptotic stability of unstable discrete LTI system, with less inputs than states. Two types of Network Control Systems are considered in the absence of communication delays, then for one of the two types, the case of a constant time delay is discussed. Examples and simulations are provided to demonstrate the results.

## I. INTRODUCTION

Feedback control systems whose control loops are closed through a real-time network are called Network Control Systems (NCS) [3], [6]. The primary advantage of a NCS is in reducing the number of dedicated control and measurement channels (wires) which simplifies maintenance and diagnosis, as well reducing the cost. When feeding back through a network however, the assumptions of classical control may need to be revisited. For example, the delays encountered by signals in the control loop may become time-varying or random. This particular issue has been analyzed in works [5] and [7]. The new problems arise because the sensed data and the control signals are no longer connected directly through a “wire”, but instead through a packet network which has finite data rate, a propagation delay, and may be shared by many other systems.

In recent years, much research and development have been expanded in this area and, because of the attractive benefits of remote control, several reliable protocols have been developed for robust real-time control purposes. For example, our own research on Internet-based protocols are such an approach [4]. With the decreasing cost of networking infrastructure, general networks are becoming even more suitable for control applications. However, in the absence of dedicated control protocols, new issues arise and a new theory is needed for networked control design. In particular, the communication channel between the plant and the controller may no longer remain unmodelled, since it can carry a finite number of bits/s and the conventional assumption of infinite capacity of the channel no longer holds. In addition to suffering from both delay and quantization effects, the finite data rate forces us to determine the usefulness of the

number of bits [8]. This is precisely the issue on which we focus in this work. The question we posed and attempt to answer is: how many bits are needed in the sensor-to-controller and controller-to-actuator networks to control an unstable system?

In 1999, Wong and Brockett [16] considered a feedback system communicating through a digital channel with finite capacity, and since asymptotic stability was deemed unrealistic, the concept of containability was introduced. Since then, several researchers have studied the problem. Mitter [9] and collaborators have contributed to the development of a new theory that matches classical control theory with traditional information theory, [2], [11], [12] and [10]. This research however, considered only a digital channel of communication instead of a packet-based network which can include time delays.

A theory for control over a packet-based network was recently proposed in [13], [14] and [15]. The authors considered a general, discrete, linear time invariant (LTI) system and found a sufficient rate for exponential stabilization of an unstable plant of order  $n$ , under the rather limiting assumption that the system has  $n$  inputs and an invertible input matrix  $B$ . The work included finite rate issues, packet dropping, as well as uncertainties in the plant model. We believe that the assumption of an invertible  $B$  matrix is conservative, since it fails to hold when the system has a single input, or more generally, when it has less inputs than the order of the system. Moreover, for a scalar system with constant time-delay (as may occur in a network control system), the idea of augmentation of the system no longer applies since the matrix  $B$  of the augmented system fails to be invertible.

In this paper, we extend the results of [15] to the case of discrete-time, linear, time-invariant systems with  $m$  inputs such that  $m < n$ , where  $n$  is the order of the system. We ignored the packet-losses considered in [15], since our work is focused in issue of the invertibility of the  $B$  matrix and in the extension of the results to the case where we have a constant time-delay induced by the sensor-to-controller network.

As it was considered in [15], we discuss two types of network control systems: one that includes a network between the sensors and the controller, and another that

The research of I. Lopez is partially supported by Conacyt. The research of C.T. Abdallah is partially supported by NSF award CNS 0626380 under the FIND initiative.

models two networks in the loop, one between the sensors and controller, and another between the controller and the actuator.

## II. PROBLEM SETUP

As discussed above, we are interested in generalizing results shown in [15]. We thus consider the same two possible configurations for the packet-based network control system. The first system, referred to as *Network Control System Type I*, has a rate of  $R_{p1}$  packets/sample-time, the packet based network considers a packet size of  $D_{Max}$  bits used for data (although the protocol information need extra bits in the packet, it is useless for this analysis) and considers a discrete LTI system shown in Figure 1 and given by

$$x(k+1) = Ax(k) + Bu(k), \quad (1)$$

where  $A$  is  $n \times n$  and we assume that it is diagonal  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $|\lambda_j| \geq 1, \forall j \in \{1, \dots, n\}$ , and  $\lambda_i \neq \lambda_j$  if  $j \neq i$ ,  $x(k)$  is  $n \times 1$ ,  $B$  is  $n \times m$  and  $u(k)$  is  $m \times 1$ .

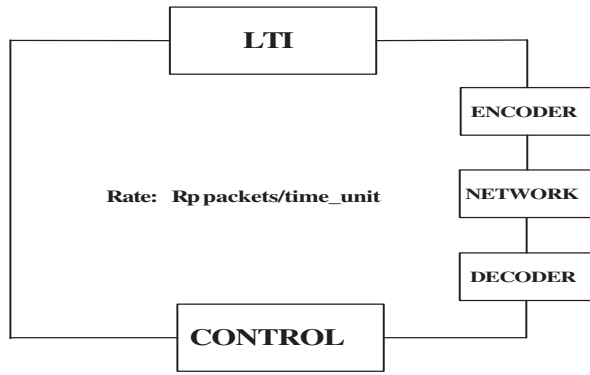


Fig. 1. Close-loop network control system: Type 1

The second type of packet-based network, to be referred to as *Network Control System Type II*, consists of the same discrete LTI system given by equation (1), but with the addition of a second network between the controller and the actuator with rate  $R_{p2}$  as shown in Figure 2.

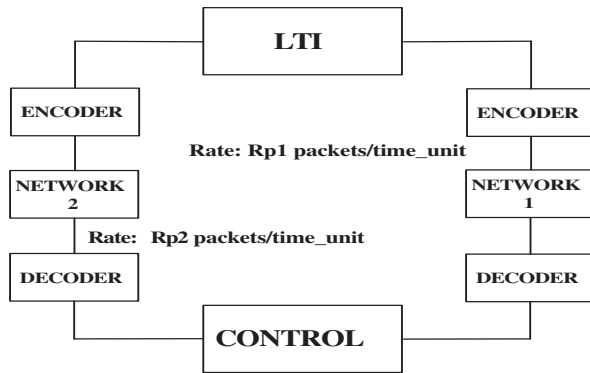


Fig. 2. Close-loop network control system: Type 2

We assume that the controller does not saturate, and that the packet-network does not drop packets nor is it subjected

to disturbances (noise). For both NCS types, we assume that the plant is able to send the complete states measurements through the link, i.e., that the states are measured. We also assume perfect synchronization of the encoder and decoder so that the decoder knows exactly the encoding scheme used by the encoder at all times.

Before we proceed to the next section we want to clarify some notation. From here on, the **log** function is in base 2, the norm symbol ( $\|\cdot\|$ ) will denote the Euclidean norm and  $\lceil \cdot \rceil$  is the **ceil** function. Also, we will be using the variable  $\mu$  to denote the controllability index which in multivariable linear systems theory [1] is defined as the least integer  $k$  such that

$$\text{rank} [B \mid AB \mid \dots \mid A^{k-1}B] = n. \quad (2)$$

## III. RESULTS

### A. Network Control System: Type I

For the case where we have a NCS Type I, we have the following result.

**Theorem 3.1:** Assuming an equal allocation of bits per state component, a network rate,  $R_p$  of packets/bits, and  $(A, B)$  is a controllable pair with controllability index  $\mu$ , a sufficient condition for system (1) to be asymptotically stabilizable is

$$R_p \geq \left\lceil \frac{R}{D_{Max}} \right\rceil,$$

where  $R = n \lceil \log(\|A^\mu\|) + 1 \rceil$  and every state can allocate  $\frac{R}{n}$  bits/sample.

*Proof:* Let us assume that the binary expansion of the state  $x(k)$  is given by

$$\begin{aligned} x(k) &= [x_1(k) \ x_2(k) \ \dots \ x_n(k)]' \\ &= \left[ \sum_{i=-\infty}^{M_1} \alpha_{1i} 2^i \quad \sum_{i=-\infty}^{M_2} \alpha_{2i} 2^i \quad \dots \quad \sum_{i=-\infty}^{M_n} \alpha_{ni} 2^i \right]'. \end{aligned} \quad (3)$$

Where  $\alpha_{ij} \in \{0, 1\}$  and  $M_j \in \mathbb{N}$ . For simplicity sake, we also assume that in the binary expansion  $x_j(k) > 0, \forall j$ . This is possible, since the sign of each state component may later be considered, by adding  $n$  extra bits to the rate (one extra bit per state component). We then know that  $x_j \leq 2^{M_j+1}$ . Now, let us assume that  $M_{max} = \max\{M_1, M_2, \dots, M_n\}$ , and if we take the norm of the state, we have

$$\begin{aligned} \|x(k)\| &\leq \|x_1(k)\| + \dots + \|x_n(k)\| \\ &\leq n 2^{M_{max}+1}. \end{aligned} \quad (4)$$

We know that we can represent  $n 2^{M_{max}+1}$  by a minimum number of bits,  $\tilde{M} = M_{max} + \log_2(n) + 1$ , and therefore,  $2^{\tilde{M}-1} < \|x(k)\| \leq 2^{\tilde{M}}$ . Now, let us consider an equal allocation of bits per state component,  $\frac{R}{n}$ , so that the encoded version of  $x(k)$  is given by  $\bar{x}(k)$ , and

$$\bar{x}(k) = \left[ \sum_{i_1}^{M_1} \alpha_{1i_1} 2^{i_1} \quad \sum_{i_2}^{M_2} \alpha_{2i_2} 2^{i_2} \quad \dots \quad \sum_{i_n}^{M_n} \alpha_{ni} 2^{i_n} \right]', \quad (5)$$

where  $i_1 = M_1 - \frac{R}{n} + 1, i_2 = M_2 - \frac{R}{n} + 1, \dots, i_n = M_n - \frac{R}{n} + 1$ . The error between the actual state and the encoded version,

$\varepsilon(k) = x(k) - \bar{x}(k)$ , is given by

$$\varepsilon(k) = \begin{bmatrix} \sum_{i=-\infty}^{M_1 - \frac{R}{n}} \alpha_{1i} 2^i & \sum_{i=-\infty}^{M_2 - \frac{R}{n}} \alpha_{2i} 2^i & \dots & \sum_{i=-\infty}^{M_n - \frac{R}{n}} \alpha_{ni} 2^i \end{bmatrix}' \quad (6)$$

Therefore, we have  $\varepsilon_j(k) < 2^{M_j - \frac{R}{n} + 1}$ , and

$$\begin{aligned} \|\varepsilon(k)\| &\leq \|\varepsilon_1(k)\| + \dots + \|\varepsilon_n(k)\| \\ &\leq n 2^{M_{\max} - \frac{R}{n} + 1} \\ &= 2^{\tilde{M} - \frac{R}{n}}. \end{aligned} \quad (7)$$

Let us then consider the evolution of the system starting at time  $k$

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ x(k+2) &= Ax(k+1) + Bu(k+1) \\ &= A^2x(k) + ABu(k) + Bu(k+1) \\ &\vdots \\ x(k+r) &= A^r x(k) + \sum_{i=1}^r A^{r-i} Bu(k+i-1); \quad \forall r \geq 3. \end{aligned}$$

Recalling that  $\mu$  represents the controllability index then, if we stop at  $k + \mu$  we have

$$\begin{aligned} x(k+\mu) &= A^\mu x(k) + A^{\mu-1} Bu(k) + A^{\mu-2} Bu(k-1) \\ &\quad + \dots + Bu(k+\mu-1). \end{aligned} \quad (8)$$

This equation may be re-arranged as  $x(k+\mu) = A^\mu x(k) + \zeta_\mu \mathbb{U}(k)$ , where

$$\begin{aligned} \zeta_\mu &= [B \mid AB \mid \dots \mid A^{\mu-1}B] \\ &= [\delta_1 \mid \delta_2 \mid \dots \mid \delta_j \mid \dots \mid \delta_{n\mu}] \end{aligned}$$

and

$$\begin{aligned} \mathbb{U}(k) &= \begin{bmatrix} u(k+\mu-1) & \dots \\ u(k) & \end{bmatrix}' \\ &= [u_1 \quad \dots \quad u_j \quad \dots \quad u_{m\mu}]', \end{aligned}$$

noting that  $\delta_j$  is the  $j$ th column in  $\zeta_\mu$  and  $u_j$  is the  $j$ th element in the vector  $\mathbb{U}(k)$ . Let us select the first  $n$  independent columns of  $\zeta_\mu$  and build a new matrix, called  $\zeta_n$ . Let us also select the elements of  $\mathbb{U}(k)$  corresponding to the columns chosen from  $\zeta_\mu$  and form a new vector, called  $\mathbb{U}_n(k)$ . Recalling that  $x(k) = \bar{x}(k) + \varepsilon(k)$  we have  $x(k+\mu) = A^\mu \bar{x}(k) + A^\mu \varepsilon(k) + \zeta_\mu \mathbb{U}(k)$ . If we choose the control law  $\mathbb{U}_n(k) = -\zeta_n^{-1} A^\mu \bar{x}(k)$ , we may reconstruct  $\mathbb{U}(k)$  replacing  $u_j$  with the corresponding values of  $\mathbb{U}_n(k)$  in the proper sequence order and letting  $u_j = 0$  for the remaining elements. After  $\mu$  steps, and by applying the control sequence  $\mathbb{U}(k)$  we obtain

$$x(k+\mu) = A^\mu \varepsilon(k). \quad (9)$$

Then, from equations (9) and (7) and the properties of matrix norms, we obtain

$$\begin{aligned} \|x(k+\mu)\| &= \|A^\mu \varepsilon(k)\| \\ &\leq \|A^\mu\| \|\varepsilon(k)\| \\ &\leq \|A^\mu\| 2^{\tilde{M} - \frac{R}{n}}. \end{aligned}$$

In order to force the state to decrease in the norm (after  $\mu$  steps), we shrink the upper bound of the state  $x(k+\mu)$  by forcing it to be less than the lower bound of the state  $x(k)$ , i.e.,  $\|A^\mu\| 2^{\tilde{M} - \frac{R}{n}} < 2^{\tilde{M}-1}$ . Finally, solving for the rate  $R$ , we get  $\frac{R}{n} \geq \lceil \log(\|A^\mu\|) + 1 \rceil$ . The  $\lceil \cdot \rceil$  function was introduced since  $\frac{R}{n}$  must be an integer number of bits for each state component. For the next  $n$  steps, we consider  $x(k+n)$  as the initial condition, and using the same algorithm to generate the control sequence and the same rate  $R$ , the state  $x(k+2n)$  will be a shrunken version of  $x(k+n)$ . Proceeding in the same fashion,  $x(k+rn)$  will tend to zero as  $r \in \mathbb{N}$  grows and, therefore,  $x$  will tend to zero and asymptotic stabilizability will be achieved. Now,  $R$  is the sufficient number effective bits that we need to transmit of the whole state for stabilization. But, knowing that a packet has a maximum length of  $D_{Max}$ , then if,  $R \leq D_{Max}$ , we will need a packet rate of  $R_p = 1$  packet/sample-time. However, if we have  $R > D_{Max}$  then, we will need a minimum of  $\lceil \frac{R}{D_{Max}} \rceil$  packets/time-step. Actually, this last expression covers both cases, since  $\frac{R}{D_{Max}} < 1$  gives a 1 packet/sample-time when the ceil function is applied. ■

It is important to note that if  $B$  were invertible, i.e.,  $\mu = 1$ , then our result gives  $\frac{R}{n} \geq \lceil \log(\|A\| + 1) \rceil$  which is more conservative than the result  $(\frac{R}{n} > \log(\|A\|))$  in [15]. The discrepancy is due to the fact that in the last step of our proof, we forced  $\|x(k+n)\|$  to be strictly less than  $2^{\tilde{M}-1}$ , i.e., we shrank the norm of the state every  $n$  steps by a factor of 2 while [15] assumes that  $\|x(k+1)\|$  is only less than  $2^{\tilde{M}}$ . Since both proofs provide only sufficient conditions, and for the case of a large  $\|A\|$ , the discrepancy is not large.

An immediate consequence of Theorem 3.1 in the specific case of a single input case is described in the following corollary.

**Corollary 3.1:** Assuming an equal allocation of bits per state component and  $(A, B)$  is a controllable pair, where  $B$  is  $n \times 1$  and the control law,  $u(k)$ , is  $1 \times 1$ , a sufficient condition for system (1) to be asymptotically stabilizable is

$$R_p \geq \left\lceil \frac{R}{D_{Max}} \right\rceil,$$

where  $R = n \lceil \log(\|A^n\|) + 1 \rceil$  and every state allocates  $\frac{R}{n}$  bits/sample.

*Proof:* The proof is the same as that of Theorem 3.1. If  $B$  is  $n \times 1$  and  $u(k)$  is  $1 \times 1$ , then  $\mu = n$ . Substituting  $\mu$  in  $R$  in the proof of Theorem 3.1, we obtain the rate given by the corollary. ■

### B. Network Control System Type I with Time Delay

One motivation for extending the results of [15] was the desire to include time delays that may be present in the network. As mentioned earlier, even for the scalar case, the invertibility requirement of  $B$  would not allow the augmentation used in [15]. Using Theorem 3.1 however, let us

consider the network control System type 1 and the discrete LTI system given by the following equation

$$x(k+1) = Ax(k) + Bu(k-p), \quad (10)$$

where  $A = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$  and  $|\lambda_j| \geq 1, \forall j \in \{1, \dots, n\}$ , and  $\lambda_i \neq \lambda_j$  if  $j \neq i$ ,  $x(k)$  is  $n \times 1$ ,  $B$  is  $n \times 1$  and  $u(k)$  is  $1 \times 1$  and  $p \in \mathbb{N}$  is the time delay. We assume here that the delay is a constant equal to  $p$  time-steps even though that the network probably imparts a time-varying and random delay. Under such conditions, we obtain

**Theorem 3.2:** Assuming an equal allocation of bits per state component, a network rate of  $R_p = \lceil \frac{R}{D_{Max}} \rceil$  packets/time-step, and  $(\mathbb{A}, \mathbb{B})$  is a controllable pair. A sufficient condition for system (10) to be asymptotically stabilizable is

$$R \geq (n+p) \lceil \log(\|\mathbb{A}^{n+p}\|) + 1 \rceil$$

$$\text{where } \mathbb{A} = \begin{bmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & & 1 \\ 0 & 0 & \vdots & \dots & 0 \end{bmatrix} \text{ and } \mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \text{ and every}$$

state can allocate  $\frac{R}{n+p}$  bits/sample..

*Proof:* We start out by augmenting the state vector, considering as new states the last  $p$  previous inputs. We then obtain

$$\begin{aligned} \mathbb{X}(k+1) &= \begin{bmatrix} \mathbf{x}(k+1) \\ x_{n+1}(k+1) \\ x_{n+2}(k+1) \\ \vdots \\ x_{n+p}(k+1) \end{bmatrix} \\ &= \begin{bmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & & 1 \\ 0 & 0 & \vdots & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_{n+1}(k) \\ x_{n+2}(k) \\ \vdots \\ x_{n+p}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k). \end{aligned}$$

This may be written as

$$\mathbb{X}(k+1) = \mathbb{A}\mathbb{X}(k) + \mathbb{B}u(k). \quad (11)$$

We then follow the same analysis as that of Theorem 3.1. The augmented system starting at time  $k$  evolves as

$$\begin{aligned} \mathbb{X}(k+1) &= \mathbb{A}\mathbb{X}(k) + \mathbb{B}u(k) \\ \mathbb{X}(k+2) &= \mathbb{A}\mathbb{X}(k+1) + \mathbb{B}u(k+1) \\ &= \mathbb{A}^2\mathbb{X}(k) + \mathbb{A}\mathbb{B}u(k) + \mathbb{B}u(k+1) \\ &\vdots \end{aligned}$$

If we stop at  $k+n+p$  we have

$$\mathbb{X}(k+n+p) = \mathbb{A}^{n+p}\mathbb{X}(k) + \zeta_2 \hat{u}_2(k), \quad (12)$$

where  $\zeta_2 = [\mathbb{B} \quad \mathbb{A}\mathbb{B} \quad \dots \quad \mathbb{A}^{n+p-1}\mathbb{B}]$  and  $\hat{u}_2(k) = [u(k+n+p-1) \quad \dots \quad u(k)]^T$ . Now, considering that each encoded state will be allocated  $\frac{R}{n+p}$  bits, we know that  $\mathbb{X}_j(k) = \sum_{i=-\infty}^{M_j} \alpha_{ji} 2^i$  and, therefore,  $\bar{\mathbb{X}}_j(k) =$

$\sum_{i=M_j-\frac{R}{n+p}+1}^{M_j} \alpha_{ji} 2^i$ . With this, we know then that the error in the state  $j$  will be given by  $\varepsilon_j(k) < 2^{M_j-\frac{R}{n+p}+1}$  and, consequently,  $\|\varepsilon(k)\|$  will be bounded by  $2^{\tilde{M}-\frac{R}{n+p}}$ , since  $\|\mathbb{X}(k)\| \leq 2^{\tilde{M}}$ , where  $\tilde{M} = \max(M_1, M_2, \dots, M_{n+p}) + \log_2(n+p) + 1$ .

From equation (12) and letting  $\hat{u}_2(k) = -\zeta_2^{-1} \mathbb{A}^{n+p} \bar{\mathbb{X}}(k)$  we obtain

$$\mathbb{X}(k+n+p) = \mathbb{A}^{n+p} \varepsilon(k). \quad (13)$$

Then, from equations (13), the new bound on  $\varepsilon(k)$  and properties of matrix norms, we have

$$\begin{aligned} \|\mathbb{X}(k+n+p)\| &= \|\mathbb{A}^{n+p} \varepsilon(k)\| \\ &\leq \|\mathbb{A}^{n+p}\| \|\varepsilon(k)\|. \end{aligned}$$

In order to shrink the upper bound of the state  $\mathbb{X}(k+n+p)$  we need that  $\|\mathbb{A}^{n+p}\| 2^{\tilde{M}-\frac{R}{n+p}} < 2^{\tilde{M}-1}$ . Finally, solving for the rate  $R$ , we obtain

$$\frac{R}{n+p} \geq \lceil \log(\|\mathbb{A}^{n+p}\|) + 1 \rceil.$$

Similarly to previous proofs, we will need a minimum of  $R_p = \lceil \frac{R}{D_{Max}} \rceil$  packets/time-step in the network rate. ■

### C. Network Control System: Type II

The last case considered in this work is the NCS Type II. We are able to prove the following result

**Theorem 3.3:** Assuming an equal allocation of bits per state component, network rates of  $R_{p1} = \lceil \frac{R_1}{D_{Max}} \rceil$  and  $R_{p2} = \lceil \frac{R_2}{D_{Max}} \rceil$  for network 1 and 2, respectively. Assuming also that  $(A, B)$  is a controllable pair, where  $B$  is  $n \times 1$ , the controllability matrix is given by  $\zeta = [B \quad AB \quad \dots \quad A^{n-1}B]$  and the control law,  $u(k)$ , is  $1 \times 1$ , a sufficient condition for system (1) to be asymptotically stabilizable is

$$\|A^n\| 2^{-\frac{R_1}{n}+1} + \|\zeta\| \|\zeta^{-1}A\| 2^{-R_2+1} < 1.$$

*Proof:* Since we now have a communication constraint from the controller to the plant actuators, we can no longer apply the calculated control signal  $u(k)$  directly to the plant. Instead, only the bits encoding  $u(k)$  according to the available rate,  $R_2$  may be used. This encoded control signal  $\tilde{u}(k)$  is the one that is received by the plant. We then have

$$x(k+1) = Ax(k) + B\tilde{u}(k). \quad (14)$$

Let us assume that we have exactly the same encoding and decoding schemes used in Theorem 3.1. If we start from  $x(k)$ , then the evolution of the system state into  $x(k+n)$  is given by  $x(k+n) = A^n x(k) + \zeta \tilde{\mathbb{U}}(k)$ , where  $\tilde{\mathbb{U}}(k) = [\tilde{u}(k+n-1) \quad \dots \quad \tilde{u}(k)]^T$ . Now, if we choose the control signal as  $\mathbb{U}(k) = -\zeta^{-1} A^n \bar{x}(k)$ , then  $\|\mathbb{U}(k)\| \leq \|\zeta^{-1} A^n\| 2^{\tilde{M}}$ . But, since  $\tilde{u}(k)$  represents the  $R_2$  most significant bits of  $u(k)$  we know that

$$\|\mathbb{U}(k) - \tilde{\mathbb{U}}(k)\| \leq \|\zeta^{-1} A^n\| 2^{\tilde{M}-R_2}. \quad (15)$$

Notice that in this case, a single  $u_j(k)$  is sent at each time, and since it is a scalar, we can allocate  $R_2$  bits to it and not  $\frac{R_2}{n}$  as done for the states. From equation (15) and recalling that  $x(k) = \bar{x}(k) + \varepsilon(k)$  and, similarly to previous proofs,  $\|\varepsilon(k)\| < 2^{\bar{M} - \frac{R_1}{n}}$ , we have

$$\begin{aligned} \|x(k+n)\| &= \left\| A^n \bar{x}(k) + A^n \varepsilon(k) + \zeta \tilde{U}(k) \right\| \\ &= \left\| \zeta \left( \zeta^{-1} A^n \bar{x}(k) + \tilde{U}(k) \right) + A^n \varepsilon(k) \right\| \\ &\leq \|\zeta\| \left\| \tilde{U}(k) - \tilde{U}(k) \right\| + \|A^n \varepsilon(k)\| \\ &\leq \|\zeta\| \left\| \zeta^{-1} A \right\| 2^{\bar{M} - R_2} + \|A^n\| 2^{\bar{M} - \frac{R_1}{n}}. \end{aligned}$$

If we want to guarantee the shrinking of  $x(k+n)$ , we enforce that  $\|\zeta\| \|\zeta^{-1} A\| 2^{\bar{M} - R_2} + \|A^n\| n 2^{\bar{M} - \frac{R_1}{n}} < 2^{\bar{M} - 1}$  which, when simplified, leads to  $\|A^n\| 2^{-\frac{R_1}{n} + 1} + \|\zeta\| \|\zeta^{-1} A\| 2^{-R_2 + 1} < 1$ . As in previous proofs we now select  $x(k+n)$  as the new initial condition and using the same control law and rates,  $R_1$  and  $R_2$ , the state  $x(k+2n)$  will be a shrunken version of  $x(k+n)$ . Continuing in the same fashion,  $x(k+rn)$  will tend to zero as  $r \in \mathbb{N}$  grows and, therefore  $x(k)$  will tend to zero and asymptotic stability will be achieved. Here again we will need a minimum of  $R_{p1} = \left\lceil \frac{R_1}{D_{Max}} \right\rceil$  packets/time-step for the sensor-controller network and a minimum of  $R_{p2} = \left\lceil \frac{R_2}{D_{Max}} \right\rceil$  packets/time-step in the controller-actuator network. ■

#### IV. SIMULATIONS

To verify some of the results derived previously, we present several numerical examples and simulate them in Matlab®. We want to clarify that in the following plots, although  $x(k)$  is discrete and exists just in the instants  $k = \{0, 1, 2, \dots\}$ , we plotted them like a continuous signal for visualization purposes.

##### A. Example 1

First, we tested the results of Theorem 3.1 for the system

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(k)$$

With initial condition  $x(0) = [-1.33 \quad 3.768 \quad 8.44]'$ . The rate in bits obtained according to Theorem 3.1 is  $R = 18$  bit/time-step and the simulation for such a rate is shown in Figure 3. Note that asymptotic stability is indeed achieved.

##### B. Example 2

In order to test the conservativeness of our results, we considered a system whose eigenvalues are distinct but whose  $A$  matrix is not diagonal. Specifically, we considered a single-input system given by

$$z(k+1) = \begin{bmatrix} 20 & 0 & 10 \\ 0 & 10 & 0 \\ 0 & 10 & 30 \end{bmatrix} z(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

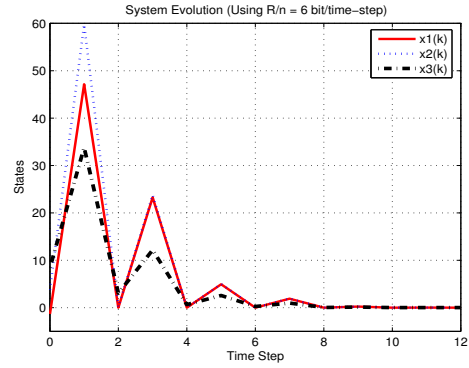


Fig. 3. Closed-loop NCS (Type 1): Multi-Input Case using  $R = 18$  bits/time-step

Using a state-space transformation, we diagonalized the system to obtain

$$x(k+1) = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 30 \end{bmatrix} x(k) + \begin{bmatrix} -1.000 \\ 2.121 \\ 1.225 \end{bmatrix} u(k)$$

We assume the initial condition to be  $x(0) = [1.33 \quad 3.768 \quad 8.44]'$ . Using Corollary 3.1, we have  $R = 48$  bit/time-step. We then verify in Figure 4 the asymptotic stability claims of the corollary. Since our results provide sufficient conditions only, we tried smaller values for  $R$ . We then found out that for this particular example,  $R = 42$  bit/time-step (two bits less per state than the sufficient  $\frac{R}{n}$ ) leads to instability, see Figure 5.

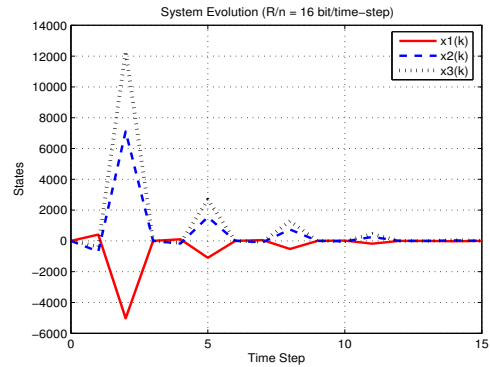


Fig. 4. Closed-loop NCS (Type 1): Single Input Case using  $R = 48$  bit/time-step.

##### C. Example 3

Let us finally consider a system with time-delay  $p = 2$  evolving according to the following dynamics

$$x(k+1) = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k-2)$$

with the initial condition state vector  $x(0) = [1.33 \quad 30.768]'$ . For this system, Theorem 3.2 gives a rate bounded below by  $R = 24$  bit/time-step. The corresponding simulation is shown

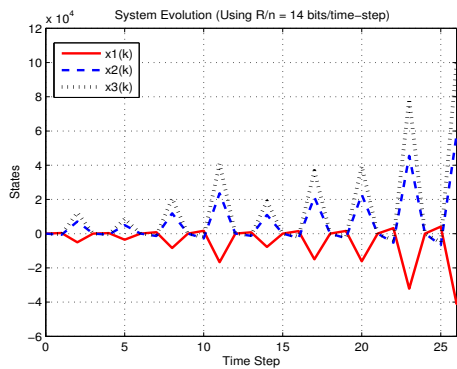


Fig. 5. Closed-loop NCS (Type 1): Single Input Case using  $R = 42$  bit/time-step.

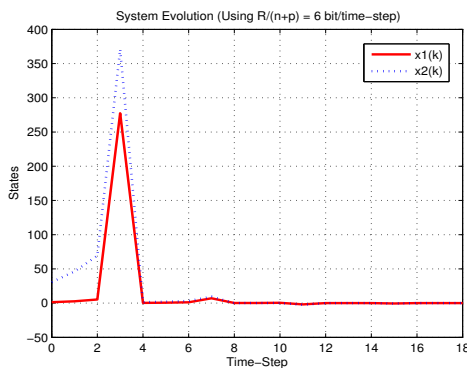


Fig. 6. Closed-loop NCS with Time-Delay: Type 1

in Figure 6. It is important to note again that for our simulations, we have assumed that the encoder is synchronized with the decoder, so it knows exactly both the sign and the position of each significant bit when it is encoded. In a real implementation, this will represent extra bits of information or more computational power at the encoder and decoder sites to keep tracking of the evolution of the system.

## V. CONCLUSIONS AND FUTURE WORK

This paper has provided extensions of previous results on determining the sufficient rate of a packet-based networked control system. The condition of  $n$  inputs has been relaxed to the general case of  $m < n$  inputs. Constant time delay was also considered in one version of the Network Control System. The rates for Network Type I without time-delay are much higher than the limits shown in previous works since we encoded the state itself and not the error between the state and its encoded version.

Future work will include the inclusion of time delays in a Network Control System Type II, and the extension of the general case of  $m$  inputs of this type of closed-loop system. Other ideas for future work include dealing with noise in the loop, the compensation in the networks rates for the extra information required by the decoder, and the generalization of previous research that has already considered packet drops and saturation in the control signal.

## REFERENCES

- [1] P. Antsaklis, and A. N. Michel, "Linear Systems", Mc Graw Hill, pp. 283-284, 1997.
- [2] N. Elia, and S. Mitter, Stabilization of Linear Systems With Limited Information, IEEE Transactions on Automatic Control, Vol. 46(9), 1384-1400, September 2001.
- [3] W. Zhang, M.S. Branicky, and S.M. Philips, Stability of Networked Control systems, IEEE Control Systems Magazine, Vol. 21 (1), pp. 84-99, February 2001.
- [4] R. Sandoval-Rodriguez, Teleautonomy of mobile robots, Ph.D. Dissertation, Department of Electrical and Computer Engineering, University of New Mexico, Albuquerque, USA, 2004.
- [5] F. Lian, J. Moyné, and D. Tilbury, Time Delay Modeling and Sample Time Selection for Networked Control Systems, Proceedings of ASME-DSC, Vol. 10, pp. 1-8, New-York, 2001.
- [6] G.C. Walsh, H. Ye, and L. Bushnell, Stability Analysis of Networked Control Systems, Proceedings of the American Control Conference, Vol. 4, pp. 2876-2880, San Diego, June 1999.
- [7] J. Nilsson, Real-Time Control Systems with Delays, Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden, 1998.
- [8] G.N. Nair, and R.J. Evans, Communication Limited Stabilization of Linear Systems, Proceedings of the IEEE Conference on Decision and Control, Vol. 1, pp. 1005-1010, 2000.
- [9] S. Mitter, Control with limited information: the Role of Systems Theory and Information Theory, ISIT 2000 Plenary Talk, IEEE Information Theory Society Newsletter, Eur. Jrn. Control, Vol. 7, pp. 1-23, 2001
- [10] A. Sahai, Evaluating Channels for Control: Capacity Reconsidered, Proceedings of the American Control Conference, Vol. 4, pp. 2358-2362, 2000.
- [11] S. Tatikonda, and S. Mitter, Control Under Communication Constraints, IEEE Transactions on Automatic Control, Vol. 49(7), pp. 1056-1068, July 2004.
- [12] S. Tatikonda, and S. Mitter, Control Over Noisy Channels, IEEE Transactions on Automatic Control, Vol. 49(7), pp. 1196-1201, July 2004.
- [13] L. Shi, and R. Murray, Towards a Packet-based Control Theory - Part I: Stabilization Over a Packet-based Network, Proceedings of the American Control Conference, Vol. 2, pp. 1251-1256, Portland, June 2005.
- [14] L. Shi, and R. Murray, Towards a Packet-based Control Theory - Part II: Rates Issues, Proceedings of the American Control Conference, Minneapolis, pp. 3482-3487, 2006.
- [15] L. Shi, M. Epstein and R. Murray, Networked Control Systems with Norm Bounded Uncertainties: A Stability Analysis, Proceedings of the American Control Conference, Minneapolis, pp. 2783-2788, 2006.
- [16] W.S. Wong and R.W. Brockett, Systems with Finite Communication Bandwidth-part II: Stabilization with Limited Information feedback, IEEE Transactions on Automatic Control, Vol. 44 (5), pp. 1049-1053, May 1999.