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ON THE FRAGILITY OF HIGH-DIMENSIONAL CONTROLLERS

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ABSTRACT

In this paper we study the fragility of controllers designed to optimize some performance indices. We trace the fragility problem to the dimension of the resulting controllers, and use results from high-dimensional geometry to analyze the problem both in the continuous and discrete domains.

One of the purposes of this paper is to understand the effects of uncertainties in the implementation of controllers which optimize some performance and robustness criteria in

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linear systems theory, and as such turn out to have high dimensions. Most control design algorithms do not consider the problems introduced by implementing uncertain controllers. We first note that it is reasonable to consider only structured uncertainties in the controller since by design, one can choose the exact structure of the controller. The controllers obtained using most robust design approaches are thus *optimal* if implemented exactly. There are however many reasons to believe that one can never exactly implement a compensator which theoretically meets all objectives. For example, the controller may be implemented digitally, even though it was design using analog models of the plant. Moreover, it is easy to argue that even when exact implementation is possible, some tuning by the control engineer is required on the actual controller.

In a recent paper, Keel and Bhattacharyya [6] have shown that, in the case of unstructured uncertainties in the plant, and using $\mathcal{H}_\infty$, $\mathcal{H}_2$ and l_1 synthesis, the resulting controllers exhibit a poor stability margin [6]. This so-called "fragility" is displayed despite (or because of?) the fact that these controllers are optimal when implemented using their nominal parameters. Paper [6] ends with some considerations which attempt to overcome the fragility problem. Among the suggestions given, are the following:

1. Developing synthesis algorithms which take into account some structured uncertainties inside the controllers and searching for the "best" solution that guarantees a compromise between optimality and fragility,

2. Examining at the structure of the controller in order to parameterize it in a useful way (lower-order or fixed-structure controller).

In this paper, we concentrate on issue 2) above and show that as the controller order increases, fragility becomes more prominent.

In a recent paper, Haddad and Corrado [4] address and solve the fragility problem by considering a *structured uncertain* dynamic compensator for a noise-driven linear plant. They obtain sufficient conditions by bounding the uncertainties in the controller using classical quadratic Lyapunov bounds [2]. The resulting controllers are proven to be "resilient" in the sense that even when they are not exactly implemented, stability and some measure of performance are guaranteed.

It is true that other authors have hinted at the problem of fragility [1] and that many critics have dismissed the issue, since robust controllers are not designed to be resilient. On the other hand, the problem is reminiscent of the LQG optimal controllers which were only useful when implemented on the exact plant, and had no robustness margins if the plant was uncertain. This lack of robustness was corrected using LQG/LTR [3]. In addition, and as mentioned above, even robust controllers will eventually have to be implemented on an actual system using digital hardware and should be resilient both to implementation errors (discretization, round-off errors, etc) and to tuning.

The current paper concentrates on one possible cause of fragility namely, the dimension of the resulting controller. We therefore cite results form high-dimensional geometry to show that fragility is more prevalent than previously noticed. The following discussion is taken from [5]. Consider a vector p contained in the n-dimensional vector space $\mathbb{R}^n$ equipped with the Euclidean norm, and let the ball of radius r be defined as

$$B_r = \{x \in \mathbb{R}^n; \quad x_1^2 + \cdots + x_n^2 \leq r^2\}$$

It is then known that the volume of such a ball is given by

$$\begin{aligned} V_n(r) &= C_n r^n \\ C_n &= \frac{2\pi}{n} C_{n-2}; \quad C_1 = 2; \quad C_2 = \pi \end{aligned}$$

In particular, for $n = 2k$, the volume becomes $V_{2k}(r) = (\pi r^2)^k/(k!)$. This clearly shows that the volume starts decreasing as $k > \pi r^2$. In fact, and regardless whether n is even or not, C_n will start decreasing after reaching its maximum at $n = 5$. This then leads to an apparent paradox as the volume of the ball shrinks to zero as n goes to infinity. In fact, if we calculate the portion V_ϵ of the volume of the sphere within $\epsilon < r$ from its surface to the total volume $V_n(r)$ of the ball, we obtain

$$\frac{V_\epsilon}{V_n(r)} = 1 - (1 - \frac{\epsilon}{r})^n$$

which goes to 1 as n goes to infinity. This then leads to the conclusion that most of the volume of the sphere is on the surface. In fact, one can show that for any convex figure "almost all of the volume is on the surface". These conclusions will also hold in any other norm and in fact, if one were to choose an n-dimensional vector in B_r at random, then more likely than not, such a vector will be of length r.

Further insight into the potential consequences of the dimensionality increase is obtained by considering what happens when the ball is embedded in a quantized domain. Consider a unit hypercube S_n in the n-dimensional space $\mathbb{R}^n$ and let $\Lambda \subset \mathbb{R}^n$ be a lattice defined as

$$\Lambda = \{x \in \mathbb{R}^n; x_i = k_i \cdot \Delta, \text{for } k_i \in \mathbb{Z} \text{and for all } i = 1, \cdots, n\} \tag{1}$$

for some real and positive $\Delta < 1$. Define $\Lambda_S = \Lambda \bigcap S_n$, that is, Λ_S contains those discrete points in the lattice Λ that are inside the unit hypercube. With these definitions at hand,

and noting that (without loss of generality) the volume of S_n is normalized to 1, it is possible to compute the expected number of points $E_n(r)$ in Λ_S that are contained in the ball B_r assuming that the center of the ball follows a uniform distribution in S_n:

$$E_n(r) = \frac{V_n(r)}{\Delta^n} \tag{2}$$

As it can be seen from the discussion above, $E_n(r)$ goes to zero as n goes to infinity, meaning that the expected number of lattice points inside the ball B_r also goes to zero. Another way of seeing this is to compute the probability $p_n(r)$ of finding at least one lattice point inside the ball when its center follows a uniform distribution. This probability has the following properties:

$$p_n(r) = \begin{cases} V_n(r), & \text{whenever } r < \Delta/2 \\ 1, & \text{whenever } r > \sqrt{n} \end{cases} \tag{3}$$

The first probability in (3) approaches zero as n goes to infinity. The condition for the second probability in (3) (i.e. $r > \sqrt{n}$ will be violated for a sufficiently large n. The intermediate cases for r, $\sqrt{n} \leq r \leq \Delta/2$ are much more involved, however one can show that the probability will go to zero as n grows, regardless of r and Δ. The conclusion that can be drawn from these facts is that when the Euclidean space is quantized and for a sufficiently large n, the probability of finding a quantized vector inside the ball is arbitrarily small.

It is not our contention that fragility is only present in higher-order controllers but that when dealing with such controllers, extra caution must be exerted in order to avoid the problem. The final version of the paper will further elaborate on these points and present simulation and hardware implementations of controllers which illustrate the effect of high-dimensionality in controller fragility.

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