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A SURVEY OF STATE-FEEDBACK SIMULTANEOUS STABILIZATION TECHNIQUES ¹

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ABSTRACT

This paper surveys the control theory literature having to do with the simultaneous stabilization of countably finite sets of systems in the state-space domain. Design methods based upon control parameterization, linear equation solution, and linear matrix inequalities are discussed. The roles of nonlinear programming and convex programming techniques are included, as is a brief description of the applicability of software-based quantifier elimination techniques.

KEYWORDS: Simultaneous stabilization, state feedback control

INTRODUCTION

The problem of simultaneous stabilization of a countably finite number of systems is important in control theory. Applications have been cited in the literature regarding the control of several different linearized operating points of a nonlinear plant and the anticipation of failure modes of a mechanical and/or electronic device. Other situations are also applicable.

The problem, briefly stated, is one of finding a single controller that will stabilize each member of a finite and countable set of plants. In terms of state-feedback, a single controller

$$u_j(t) = -Kx_j(t) , \quad (1)$$

is sought, where $x_j(t) \in \mathbb{R}^n$, $u_j(t) \in \mathbb{R}^q$, and $K \in \mathbb{R}^{q \times n}$, that stabilizes a set of continuous-time state-space linear time-invariant ordinary differential equations

$$\dot{x}_j(t) = A_j x_j(t) + B_j u_j(t) , \quad j \in I_m \triangleq \{1, \dots, m\} . \quad (2)$$

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The papers discussed in this survey consider both the single-input case ($q = 1$ in [1], [14], [18], and [19]) and the multiple-input case ($q > 1$ in [6] and [9]). In the former, the state control vector is $k^T = [k_1, k_2, \dots, k_n]$ and in the latter, the control matrix is a $q \times n$ array.

Blondel [5] demonstrates that it is not possible to rationally decide⁵ whether or not a set of three or more systems is simultaneously stabilizable. Fortunately, however, testable sufficient conditions are available. This means that, as happens frequently in engineering design, the sufficient conditions must be used and the consequent design conservatism accepted.

The papers approach the problem from a number of different directions. Schmittendorf and Hollot [19] show that simultaneous stabilization is possible if there exists a vector c such that systems $c^T(sI - A_j)^{-1}b_j$ satisfy several conditions, the most restrictive of which is that all must be minimum phase. Ackermann [1] considers a space \mathcal{K} containing all linear state-feedback control gain vectors k . He then partitions \mathcal{K} to obtain a subspace guaranteed to simultaneously stabilize systems. Howitt and Luus [14] give a nonlinear programming problem which produces the linear controller. Boyd, et al. [6] show that simultaneous stabilizability can be guaranteed if there exists a single solution to a set of m linear matrix inequalities. Paskota, et al. [18] simultaneously stabilize systems by solving nonlinear Liénard-Chipart constraints. Dorato, et al. [10] apply a relatively new computational technique known as *quantifier elimination* to verify Liénard-Chipart stability conditions. Finally, Chow [9] defines a “multimode” system controllability matrix which simultaneously describes the controllability of all m systems $\{(A_j, B_j)\}_{j \in I_m}$. He gives a sufficient condition to simultaneously place the closed-loop system poles in regions containing the user-specified locations.

MINIMUM PHASE APPROACH

The method due to Schmittendorf and Hollot [19] applies only to those single-input transfer functions with relative degree unity, a significant limitation. They define the j th plant in the frequency domain to be $(sI - A_j)^{-1}b_j = n_j(s)/d_j(s)$ where $n_j(s)$ is a column vector with polynomial entries and the polynomial $d_j(s) = \det(sI - A_j)$.

Theorem 1 (Schmittendorf and Hollot [19]) *If there exists $c \in \mathbb{R}^n$ satisfying, for each $j \in I_m$, (i) $c^T n_j(s)$ is of order $n - 1$, (ii) $c^T n_j(s)$ is Hurwitz, and (iii) the sign of the leading coefficient of $c^T n_j(s)$ is invariant over all j ; then the control $u = -\gamma c^T x = k^T x$, where γ is chosen by a short algorithm omitted here, simultaneously stabilizes the m plants (2). ■*

The proof follows from simple root-locus arguments on the transfer functions $P_j(s) = c^T(sI - A)^{-1}b_j$.

LINEAR EQUATION SOLUTION FOR FEEDBACK GAIN

Ackermann Formula Approach

⁵“... rationally undecidable: it is not possible to find a general criterion that involves only the coefficients of three or more linear systems, rational operations, logical operations (‘and’ and ‘or’) and sign tests operations (equal to, greater than, greater than or equal to, etc.) and that is necessary and sufficient for simultaneous stabilizability of the systems.”

Ackermann [1] considers the problem of bounding the single-input system closed-loop eigenvalues with a region Γ in the complex plane which depends explicitly on system design specifications. Let Γ be the space of complex scalars containing all possible user-specified closed-loop poles. When simultaneous stabilization is considered, Γ is the open left hand plane (LHP). When stabilization is extended and simultaneous performance design is considered (such as system overshoot and settling time responses) then Γ is a subset of the open LHP.

Define $\mathcal{K} \subset \mathbb{R}^n$ to be the space containing all static feedback gain vectors $k \in \mathbb{R}^n$. Let \mathcal{K}_Γ be the set of all gains k which place closed-loop poles inside Γ . The idea is to map the system eigenvalue constraints Γ into the space \mathcal{K} , thus defining the subspace \mathcal{K}_Γ that is equivalent (via an affine transformation) to Γ .

First, the given region Γ is used to construct an equivalent space based on the values of the coefficients of the equivalent closed-loop characteristic polynomials. Consider the characteristic polynomials for each of the m systems

$$\det [sI - (A_j - b_j k^T)] = \lambda^n + a_{1j} \lambda^{n-1} + \dots + a_{nj}, \quad j \in I_m \quad (3)$$

and define for each a vector of coefficients $p_j = [a_{nj}, a_{n-1,j}, \dots, a_{1j}]$. Let \mathcal{P} denote the space of all vectors p_j and let $\mathcal{P}_\Gamma \subset \mathcal{P}$ denote the space of all vectors p_j that result in closed-loop system eigenvalues being contained by Γ .

Then define W_j to be the matrix that transforms each system (A_j, b_j) into controllable canonic form. It is then possible to use the affine mapping $k_j^T = k_0^T + p_j^T W_j^{-1}$ to characterize a space \mathcal{K}_{Γ_j} for each system j that corresponds to characteristic polynomial coefficient space \mathcal{P}_Γ . The bias vector k_0 is found with a short algorithm omitted here. After spaces \mathcal{K}_{Γ_j} have been found for each of the systems, define the total space as $\mathcal{K}_\Gamma = \bigcap_{j=1}^m \mathcal{K}_{\Gamma_j}$. This means that in the context of nonlinear programming the optimization variables will include the components of the gain vector k . The constraints to be satisfied will include the mathematical characterization of region \mathcal{K}_Γ .

Simultaneous Stability Design

Howitt and Luus [14] use the brute force idea of minimizing the scalar objective γ subject to nonlinear inequality constraints on closed-loop eigenvalues $\Re[\lambda_{ij}(A_j - b_j k)] \leq \gamma, \forall i \in I_n, j \in I_m$ as their point of departure. They note the difficulty in optimizing such systems, specifically the nonlinear eigenvalue constraints so they add the eigenvalues to the list of optimization variables and constrain system modes linearly. Additional constraints are also required to enforce the conjugacy of complex eigenvalues. They point out that their method can be derived from that of Ackermann [1].

The nonlinear programming constraints that explicitly describe the relationship between feedback vector k and the eigenvalues can be posed by relating k to the coefficients of the characteristic polynomial. Let e_j^T be the bottom row of the inverse of the controllability matrix for the j th system and define arrays

$$\begin{aligned} G_j &= - \left[e_j \quad A_j^T e_j \quad \dots \quad (A_j^{n-1})^T e_j \right]^{-1} \in \mathbb{R}^{n \times n} \\ h_j &= G_j (A_j^n)^T e_j \in \mathbb{R}^n . \end{aligned}$$

It turns out that the relationship between the eigenvalues of each of the m closed-loop systems on the one hand, and the feedback vector k being sought on the other, can be written as $G_j k + h_j = \delta_j$ for all $j \in I_m$.

Since A_j and b_j are real, the eigenvalues will be complex conjugate pairs and/or reals as $\lambda_{ij} = \alpha_{ij} + j\beta_{ij}$ for even numbers of eigenvalues. When λ_{ij} is real the corresponding $\beta_{ij} = 0$. It can be shown that the nonlinear programming problem will require the following additional constraints to enforce this behavior: $g_j = [\beta_{1j} + \beta_{2j}, \alpha_{1j}\beta_{2j} + \alpha_{2j}\beta_{1j}, \beta_{3j} + \beta_{4j}, \alpha_{3j}\beta_{4j} + \alpha_{4j}\beta_{3j}, \dots, \beta_{N-1,j} + \beta_{Nj}, \alpha_{N-1,j}\beta_{Nj} + \alpha_{Nj}\beta_{N-1,j}]^T = 0$ where $N = n$ if n is even and $N = n - 1$ if n is odd. The numerical nonlinear programming Problem 1 can be used to construct a static k if one exists (that is, if $\gamma < 0$).

Nonlinear Programming Problem 1 (Howitt and Luus [14])

Minimize the scalar objective function γ where the feedback gains k , bound γ , and eigenvalues λ_{ij} are the optimization variables, subject to the state equations (2), the control equation (1) and the following equality and inequality constraints $\alpha_{ij} \leq \gamma$, $G_j k + h_j = \delta_j$, and $g_j = 0$, $\forall i \in I_n, j \in I_m$. ■

In a subsequent paper [15], Howitt and Luus present an algorithm based on the positive-definite secant BFGS (Broyden, Fletcher, Goldfarb, and Shanno) algorithm.

LINEAR MATRIX INEQUALITIES

Boyd, et al. [6] discuss the simultaneous stabilization of m systems in the context of quadratic stabilizability of a continuum of systems. It can be shown that all multi-input systems $\{(A_j, B_j)\}_{j \in I_m}$ can be simultaneously stabilized by a single static feedback gain K if there exists a matrix solution $P = P^T > 0$ to the set of m matrix Lyapunov inequalities

$$(A_j - B_j K)^T P + P (A_j - B_j K) + W < 0 \quad (4)$$

for some $W > 0$ dictated by the particular application.

The intent is to solve these inequalities for P and K , but they are not convex in those matrix variables. But it is possible to use a change of variables to arrive at a convex reformulation. As suggested by Bernussou, et al. [4], let $P = Y^{-1}$ and $K = XY^{-1}$, then pre- and post-multiply each term in inequalities (4) by $Y = Y^T > 0$ to obtain

$$-Y A_j^T + X^T B_j^T - A_j Y + B_j X - Y W Y > 0. \quad (5)$$

This inequality is now quadratic in Y and can also be linearized through the invocation of the LMI Lemma which states:

Theorem 2 (LMI Lemma, Boyd, et al. [7])

Consider matrices $Q = Q^T \geq 0$ and $R = R^T > 0$. Then

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0 \quad \Leftrightarrow \quad \begin{array}{l} R > 0 \\ Q - S R^{-1} S^T > 0. \end{array}$$

■

Let $Q_j = -Y A_j^T - A_j Y + X^T B_j^T - B_j X$, $S = Y$, and $R = W^{-1}$. This allows inequalities (5) to be rewritten as $Q_j - S R^{-1} S > 0$; and $W > 0$ implies $R = W^{-1} > 0$. Via the LMI Lemma, each of the quadratic matrix inequalities can be rewritten as the convex and linear matrix inequalities

$$\begin{bmatrix} -Y A_j^T - A_j Y + X^T B_j^T + B_j X & Y \\ Y & W^{-1} \end{bmatrix} > 0 .$$

If there is a single solution (X, Y) to each of the j LMIs, there exists a simultaneously stabilizing static feedback gain controller $K = XY^{-1}$. That is, to prove simultaneous stabilizability, one need only look for solutions to this collection of m different LMIs.

It has been reported in the literature that convex optimization methods known as *interior-point programming* (see Nesterov and Nemirovskii [16]) are particularly adept at numerically solving such LMI-constrained convex programming problems. Computational tools specifically designed to solve LMI problems are available (LMI-TOOL, in El Ghaoui, et al. [12] and Nikoukhah, et al. [17]) and nonlinear convex optimization problems in general (SP, Vandenberghe and Boyd [20]; and SDPSOL, Boyd and Wu [8]). There is also an LMI toolbox for MATLAB.

Quantifier Elimination

Paskota, et al. [18] describe the simultaneous stabilization of single-input systems (2) by enforcing nonlinear Liénard-Chipart conditions (see Gantmacher [11]) on the coefficients of each of the corresponding characteristic polynomial equations (3). Barnett and Cameron [3] provide four different sets of conditions, each of which is necessary and sufficient for the characteristic polynomials to be Hurwitz. One of those sets is $a_{nj}(k) > 0$, $a_{n-2,j}(k)$, \dots and $H_{1j} > 0$, $H_{3j} > 0 \dots$ where H_{ij} is the i th leading principal minor of an $n \times n$ Hurwitz matrix, for all $j \in I_m$. Coefficients of the j th characteristic polynomial can be calculated with Leverrier's method (Ackermann [1]) $a_{ij}(k) = -\text{tr}\{\hat{A}_j^i + a_{1j}\hat{A}_j^{i-1} + \dots + a_{i-1,j}\hat{A}_j\}/i$ where $\hat{A}_j = A_j - b_j k^T$ for all $i \in I_n$. It turns out that $a_{rj} = 0$ for all $r > n$.

Dorato, et al. [10] extend similar work by Anderson, et al. [2]. They apply a relatively new computational method known as *quantifier elimination*⁶ to the necessary and sufficient Liénard-Chipart conditions. Until recently such decision theoretic problems have been essentially intractable due to computational complexity. They discuss the robust stabilization of a single system so their results are extended herein. If the parameters in feedback gain vector k enter the coefficients of the characteristic polynomial as polynomial functions, then the Liénard-Chipart inequality constraints can be thought of as polynomial inequality constraints. Denote those inequalities as $u_{ij}(k) > 0$. Then simultaneous stabilizability is equivalent to the quantified formulae: $(\forall i, j)(\exists K) u_{ij}(K) > 0$. This quantifier-based expression can then be processed with software known as QEPCAD (see Hong [13]) to automatically produce statements with some of the quantifiers eliminated. The statements can be used to establish the existence of a solution and to obtain sets of admissible k .

SIMULTANEOUS PLACEMENT OF POLES IN DISKS

⁶That is, the elimination of all universal \forall and existential \exists quantifiers to produce an equivalent quantifier-free expression.

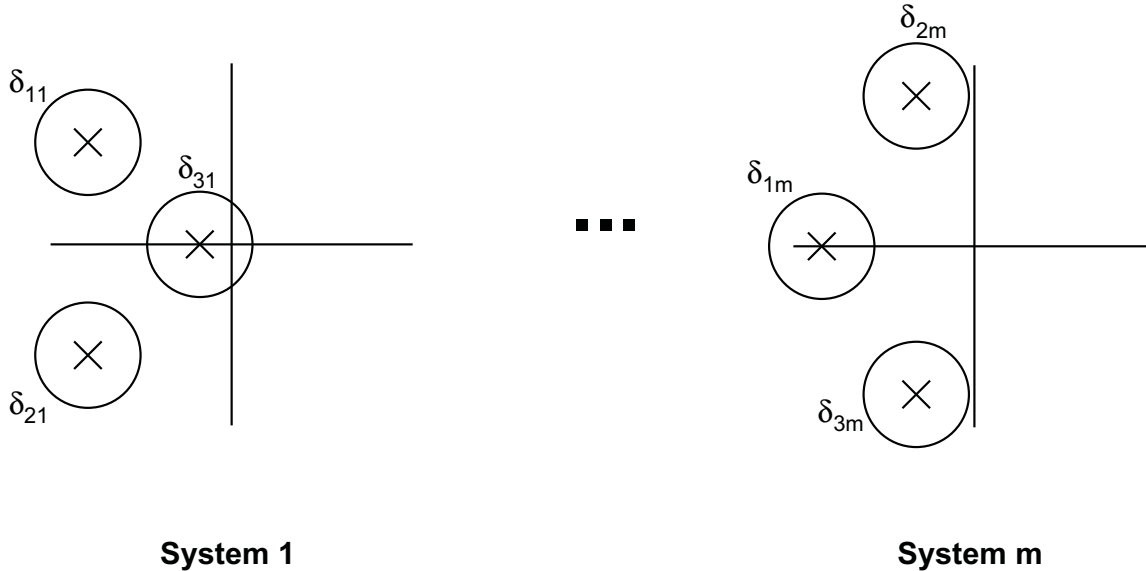


Figure 1: Disks δ_{ij} in the Complex s -Plane

Several papers have been written on the problem of pole-placement through feedback design. Chow [9] wrote in 1990 about using pole placement for multiple-input “systems with multiple operating conditions” (read: “simultaneous pole-placement”). He constructs a controller that places poles “locally” near design specifications

$$\lambda(A_j - B_j K) = \sigma_{ij}, \forall j \in I_m,$$

σ_{ij} being the n -vector of desired eigenvalues for the j th system. There exists a solution to the *precise* simultaneous pole placement problem if and only if there exists K satisfying $\det[sI - (A_j - B_j K)] = \rho_j(s)$, $j \in I_m$ where $\rho_j(s)$ is the characteristic polynomial corresponding to the user-specified poles. But this is unlikely so the placement of the poles in discs centered at points σ_{ij} is sought instead.

Let $\mathcal{C}_{j\ell} = [b_{j\ell}, A_j b_{j\ell}, \dots, A_j^{n-1} b_{j\ell}]$ be the controllability matrix of the j th system with respect to the ℓ th control. Note that $u \in \mathbb{R}^q$ and that the control coefficient matrix B_j for the j th system is considered to be composed of q column vectors as $B_j = [b_{j1}, b_{j2}, \dots, b_{jq}]$. Let the multimode controllability matrix \mathcal{C} associated with the (A_j, B_j) -controllable pairs be a block matrix with the (j, ℓ) th block being $\mathcal{C}_{j\ell}$ as $\mathcal{C} = [\mathcal{C}_{j\ell}]_{j \in I_m, \ell \in I_q}$. Let δ_{ij} , $i \in I_n$ be a set of n disks in the complex s -plane centered at the eigenvalues of the j th open-loop system $\lambda(A_j)$, for *each* of the open-loop systems. In Figure 1, each disk δ_{ij} has fixed radius ε . Chow then proves a *local pole-placement* Theorem 3.

Theorem 3 (Chow [9]) (*Sufficiency*) *If rank $\mathcal{C} = mn$ then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, there exists a gain K placing the eigenvalues of the closed-loop systems (A_j, B_j) , $j \in I_m$ within disks δ_{ij} (not necessarily completely contained by the left-half plane) for all $j \in I_m$, $i \in I_n$. ■*

The condition of exact placement of poles is therefore relaxed to one of placing poles in

regions δ_{ij} provided that one eigenvalue is located in each disk and that the conjugacy of complex poles is satisfied.

SUMMARY AND CONCLUSIONS

This paper surveys the control theory literature on methods useful for the simultaneous stabilization of an integer number of dynamical systems. It concentrates on methods having to do with state-space descriptions of systems, rather than the input-output frequency domain descriptions. Based on the numbers of papers counted on each side, it seems that we move in a relatively less congested direction. The areas of linear static state feedback are covered, especially those involving control parameterization, mapping techniques, and nonlinear and convex programming design.

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