6-25-2008

Preliminary Results on Interconnected Hybrid Systems

Chaouki T. Abdallah

Jorge Piovesan

Herbert Tanner

Recommended Citation

Abdallah, Chaouki T.; Jorge Piovesan; and Herbert Tanner. "Preliminary Results on Interconnected Hybrid Systems." *16th Mediterranean Conference on Control and Automation* (2008): 101-106. doi:10.1109/MED.2008.4602157.

16th Mediterranean Conference on Control and Automation
Congress Centre, Ajaccio, France
June 25-27, 2008

Preliminary Results on Interconnected Hybrid Systems

Jorge L. Piovesan, Chaouki T. Abdallah, and Herbert G. Tanner

Abstract— **We present a new framework for describing multi-agent systems with hybrid individual dynamics where the interaction between agents occurs at both the continuous and discrete levels. We formally define these multi-agent systems as Interconnected Hybrid Systems and then recast fundamental hybrid concepts such as a hybrid metric, hybrid execution, and reachability in this new interconnected hybrid systems framework. We then prove a necessary condition for the existence of the interconnected hybrid executions. This work extends results in [10], [16].**

Index Terms— **Interconnected hybrid systems, reachability, metric, execution.**

I. INTRODUCTION

In most of the work reported on cooperative systems, individual models for cooperating agents are described by purely continuous dynamics [2], [4], [6], [11], [13], [17]. There are few exceptions, where discrete event system theory is applied [3]. In exploring new communication network paradigms [7], [15] we sometimes find the use of purely continuous dynamics to be restrictive as explained below.

We envision a network in which functions (e.g. routing) are not fixed to physical nodes, but are instead implemented by software agents that are free to migrate from node to node, depending on resources that they may have to compete for [14]. This approach gives rise to a new type of multi-agent system where agent dynamics are composed by discrete states that represent the location of the agent in the network and its operating mode, and by continuous states that represent the amount of resources that the agent is receiving from the network. The node dynamics are also composed by discrete and continuous states. The discrete states represent changes in the agents hosted by the node, while continuous states represent the evolution of the resource availability due to the competition of agents for such resources. Agents start at initial locations in the network and with a given set of resources. Nodes start at discrete states that reflect the initial distribution of agents and at continuous states corresponding the initial availability of resources. The continuous states of the agents may then evolve according the agents requirements affecting the availability of resources in the nodes. Agents may also jump to different locations depending on the conditions in the nodes. These jumps will affect the continuous evolution of other agents and nodes, and will also cause discrete jumps in the nodes reflecting the new agent distribution. A pictorial example of this situation is depicted in Figure 1.

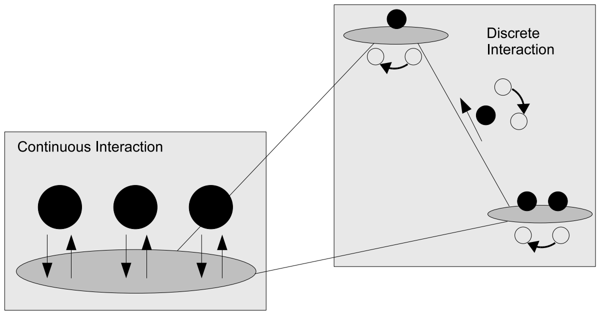

Fig. 1. Example of dynamical behavior of agents and nodes. Agents are as hybrid automata. Each mode in an automaton corresponds to a possible location of an agent in the network (Agents on top). Each transition between modes represents a change of location made by an agent (agent at the bottom). The dynamics of the nodes are also modeled as hybrid systems. Each mode represents a number of agents residing at a node paired with the availability of resources that varies in discrete manner. The agents on top are located on a node, therefore have a discrete state fixed and the continuous dynamics of agents and the nodes that hosts them are interacting. The agent at the bottom is moving between nodes, so a discrete transition is happening.

It is not clear how to capture the operation of such a system with existing hybrid frameworks. The interactions between the hybrid systems that model agents and nodes happen at both the continuous and discrete levels. The continuous and discrete dynamics of the agents depend on both the continuous and discrete states of the nodes and viceversa. We attempt to capture this interaction with a new class of systems: the interconnected hybrid systems. Such systems are not mere parallel compositions, or products, of the component subsystems [18]. The existence and evolution of an individual subsystem can be meaningless if isolated. Moreover, interactions are not limited to common or uncommon events. In our case, the hybrid state in one of the systems modifies the execution in another one. Therefore we formally define the interconnected hybrid system such that the continuous evolution in one agent depends on the continuous states of agents that are connected to it, and similarly the discrete dynamics depend on continuous and discrete dynamics of neighboring agents. This definition also includes a description of the connectivity of the multi-agent system in each agent's hybrid state. We then extend our

Jorge Piovesan (corresponding author), and Chaouki Abdallah are with the Department of Electrical and Computer Engineering, The University of New Mexico, Albuquerque NM 87131 {jlpiovesan,chaouki}@ece.unm.edu, phone: 1-505-277-0298, fax: 1-505-277-1439. Herbert Tanner is with the Department of Mechanical Engineering, The University of New Mexico, Albuquerque NM 87131 tanner@unm.edu, phone: 1-505-277-1493.

Jorge Piovesan and Chaouki Abdallah are supported by NSF award CNS 0626380 under the FIND initiative. Herbert Tanner is supported by the NSF Award 0447898

previous work [16] defining a metric for this new class of systems and explaining the properties of this metric. Finally we recast the reachability and the hybrid execution concepts from hybrid systems theory into the new framework, and provide a necessary condition for the existence of the interconnected hybrid execution (a global concept i.e., related to the whole multi-agent system), in terms of the components of each agent's hybrid model (a local variable) extending some work in [10].

The reminder of this paper is organized as follows: In Section II we define the interconnected hybrid system and explain the key features of this new concept. In Section III we introduce an interconnected hybrid metric and provide its properties, while in Section IV we extend the define the interconnected hybrid execution and prove a necessary condition for its existence. Section V outlines our conclusions.

II. Interconnected Hybrid Systems

A hybrid system is denoted $\mathbf{H}_i$, where $i \in I$ indexes the systems in a group. ν_i denotes dependence of ν on i. ν_{q_i} denotes dependence of ν on both q_i and i. ν^n, denotes the n^{th} element of a sequence in ν, and $\nu(t)$ denotes the value of ν at time t. Finally, with some abuse of notation, ν_0 marks an initial condition.

Definition 1 (Interconnected Hybrid System) *An* Interconnected Hybrid System *(IHS) is a set $\mathbf{H}^* = \{\mathbf{H}_i\}_{i\in I}$ of Controlled Hybrid Dynamical Systems [1] $\mathbf{H}_i$ indexed by the set I. For each $i \in I$, $\mathbf{H}_i = [Q_i, \mathbf{\Sigma}_i, \mathbf{G}_i, \mathbf{Z}_i, \mathbf{S}_i]$, we have that*

- *Q_i is the set of discrete states: $Q_i = O_i \times D_i$, where O_i is the set of operating states and D_i is the set of connectivity states.*
- *$\mathbf{\Sigma}_i = \{\Sigma_{q_i}\}_{q_i \in Q_i}$ where $\Sigma_{q_i} = (X_{q_i}, f_{q_i}, U_{q_i}, \mathbb{R}^+)$ is a dynamical system that corresponds to $q_i \in Q_i$ with X_{q_i} being the continuous state space, f_{q_i} the continuous dynamics, U_{q_i} the set of continuous controls, and $\mathbb{R}^+ = [0, \infty)$ the time set.*
- *$\mathbf{S}_i = \{S_{q_i}\}_{q_i\in Q_i}$ is the set of discrete transition labels of $\mathbf{H}_i \in \mathbf{H}^*$.*
- *$\mathbf{G}_i = \{G_{q_i}\}_{q_i\in Q_i}$ is the set of guard conditions for $\mathbf{H}_i \in \mathbf{H}^*$.*
- *$\mathbf{Z}_i = \{Z_{q_i}\}_{q_i\in Q_i}$ is the set of transition maps of $\mathbf{H}_i \in \mathbf{H}^*$.*

The state space of the IHS $\mathbf{H}^*$ is $H^* = \prod_{i\in I} H_i$ where $H_i = Q_i \times \bigcup_{q_i\in Q_i} X_{q_i}$ is the state space of hybrid system $\mathbf{H}_i$, and the state of the IHS is denoted as $\vec{h} = (\vec{q}, \vec{x}_{\vec{q}})$ where $\vec{q} = (q_i)^T_{i\in I}$, and $\vec{x}_{\vec{q}} = (x^T_{q_i})^T_{i\in I}$, where $q_i \in Q_i$ for all $i \in I$, and $x_{q_i} \in X_{q_i}$ for all $q_i \in Q_i$ and for all $i \in I$.

We refine the definition by the following remarks:

- Each $o_i \in O_i$ represents a different operating condition of $\mathbf{H}_i$. Each $d_i \in D_i$, represents different connectivity conditions. $(o_i, d_i) \in Q_i$ is denoted as q_i. Each q_i has an associated set $V(q_i) \subseteq I\ \forall q_i \in Q_i$ and $\forall i \in I$, which stores the indexes of the systems that are connected to $\mathbf{H}_i$, i.e., if $j \in V(q_i)$ then $\mathbf{H}_j$ is connected to $\mathbf{H}_i$. Note that $V(q) = V(q')$ for all $q = (o, d), q' = (o', d') \in Q_i$ that satisfy $d = d'$.
- For all $\mathbf{H}_i \in \mathbf{H}^*$ the continuous control inputs in U_{q_i} are obtained with the function function $u_{q_i} : X_{q_i} \times \prod_{j\in V(q_i)} \left(\bigcup_{q_j\in Q_j} X_{q_j}\right) \to U_{q_i}$. Therefore, the continuous controls of any system $\mathbf{H}_i \in \mathbf{H}^*$, are obtained as functions of the continuous states of the systems that are connected to $\mathbf{H}_i$.
- Symbol $s_{q_i} \in S_{q_i}$ determines the discrete state after a transition from $q_i \in Q_i$ in system $\mathbf{H}_i$. We consider two types of transitions: Transitions triggered by local external events and transitions that are functions of the states of the local system and the systems connected to it.
- A guard condition for an event-triggered transition is denoted as $G^{\mathrm{E}}_{q_i}$. This guard must satisfy a condition on the state of the system(s) and on the existence of an event i.e, $G^{\mathrm{E}}_{q_i} : S_{q_i} \to E_i \times X_{q_i} \times \prod_{j\in V(q_i)} H_j$ where E_i is the set of possible events of $\mathbf{H}_i \in \mathbf{H}^*$. A guard condition for a state-based transition, denoted $G^{\mathrm{S}}_{q_i}$ needs to satisfy a condition on the state of the system(s) only i.e, $G^{\mathrm{S}}_{q_i} : S_{q_i} \to X_{q_i} \times \prod_{j\in V(q_i)} H_j$.
- $Z_{q_i} : G_{q_i} \times S_{q_i} \to \bigcup_{p_i\in Q_i}\{X_{p_i}\}$ determines the continuous state of $\mathbf{H}_i \in \mathbf{H}^*$ after a transition $s_{q_i} \in S_{q_i}$.

We highlight the following features in Definition 1: The discrete states of the systems are divided into operating states, which are used to describe modes of operation of each individual agent in the system, and connectivity states, which describe the possible configurations for information exchange between agents in the system. If one thinks in the usual graph theoretic argument that describes the connectivity between agents in multi-agent systems literature [2], [4], [6], [8], [11]–[13], [17], [19] different connectivity states in each agent correspond to its different possible neighborhoods. We however, do not limit the connectivity description of the IHS to the use of graph theory. Also note that no assumptions are made about symmetry on the connectivity, so this definition includes the possibility of agent $i \in I$ being connected to agent $j \in I : j \neq i$ without j being connected to i, which corresponds to a directed graph on the graph theoretic argument.

The interactions between the agents in the systems are achieved in the continuous dynamics through the continuous control inputs. The continuous control inputs of agent $i \in I$ in the IHS are functions of the continuous state of agent $i \in I$ and the continuous states of the agents that are directly connected to agent $i \in I$. Therefore the continuous evolution of each agent is influenced by the continuous dynamics of the agents that are connected to it.

The interactions between the discrete dynamics of the agents in the system are achieved through the transition guards. In both cases (the event-triggered, and the state-based transition) the transition guards of agent $i \in I$ set conditions on the continuous states of agent $i \in I$ and on the hybrid states of the agents that are connected to agent $i \in I$. So, for the case of state-based transitions, a discrete transition may occur when both the continuous state of agent i and the hybrid states the the agents connected to $i \in I$ reach a guard condition. In the event-triggered case, a discrete transition occurs on agent $i \in I$ when this agent experiences an external event if the condition on the states of agent i and the agents connected to it is satisfied. Therefore in both state-based and event-triggered transitions of agent $i \in I$, the discrete dynamics are influenced by the hybrid states of the agents that are connected to agent $i \in I$. Note that the events are assumed to be local, i.e an event in agent $i \in I$ has direct influence only on this agent's dynamics. However, since an event will generate an state change in agent $i \in I$, such state change will potentially affect the dynamics of the agents that are connected to agent $i \in I$. For this reason we believe that the assumption of the events being local should not represent a restriction.

To summarize, Definition 1 essentially extends to the hybrid level, the standard multi-agent setting [2], [4], [6], [8], [11]–[13], [17], [19] where each agent uses the states of its neighbors to update its own evolution.

III. A METRIC FOR INTERCONNECTED HYBRID SYSTEMS

In [16] we introduced a new notion of hybrid metric. We extend this concept for interconnected hybrid systems. Let the directed graph that represents the hybrid system $\mathbf{H}$ [9] be denoted as $\mathcal{G}_H$.

Assumption 1 *For each $i \in I$, there exist a vector space X_i such that $X_{q_i} \subseteq X_i$ for all $q_i \in Q_i$.*

Definition 2 (Discrete Distance [16]) *Let the distance between two discrete states of a hybrid system q and q' be the length of the shortest path[1] from node q to node q' in $\mathcal{G}_H$. This distance is denoted by $d_D(q, q')$.*

Definition 3 (Interconnected hybrid distance (IHD)) *Let the distance from $\vec{h} \in H^*$ to $\vec{h}' \in H^*$ be:*

$$d_H^*(\vec{h}, \vec{h}') = \max_{i \in I}(d_D(q_i, q_i')) + \tanh(\|\vec{x}_{\vec{q}} - \vec{x}'_{\vec{q}'}\|)$$

where for each $i \in I$, q_i and q_i' are the components of $\vec{q} = (q_i)_{i\in I}^T$ and $\vec{q}' = (q_i')_{i\in I}^T$.

Note that $\vec{x}_{\vec{q}} = (x_{q_i}^T)_{i\in I}^T$ and $\vec{x}'_{\vec{q}'} = (x'^T_{q_i'})_{i\in I}^T$ where each $x_{q_i}^T$ and $x'^T_{q_i'}$ is a vector. Then $\vec{x}$ and $\vec{x}'$ are vectors formed by concatenating the vector states of each individual system in $\mathbf{H}^*$. Therefore the norm $\|\vec{x}_{\vec{q}} - \vec{x}'_{\vec{q}'}\|$ is well defined on $\prod_{i\in I} X_i$. In the reminder of this section we drop the subindex notation on $\vec{x}_{\vec{q}}$ for simplicity because the correspondence between $\vec{x}_{\vec{q}}$ and $\vec{q}$ is clear from the context.

Remark 1 (Use of the hyperbolic tangent) *The $\tanh(.)$ function of the norm in the interconnected hybrid distance provides a mechanism to distinguish the discrete and the continuous parts of the distance between two interconnected hybrid states: The interconnected hybrid distance is composed by an integer and a fractionary part. The integer part provides the exact number of discrete transitions that the system needs to experience to reach one discrete state from another, while the fractionary part results from the application of an invertible function to the standard notion of distance between two continuous states.*

[1]For a definition of a path, see [5].

Theorem 1 (Properties of the IHD) *Given three interconnected hybrid states $\vec{h} = (\vec{q}, \vec{x}), \vec{h}' = (\vec{q}', \vec{x}'), \vec{h}'' = (\vec{q}'', \vec{x}'') \in H^*$, the following properties hold:*

1) $d_H^*(\vec{h}, \vec{h}') \geq 0$ *for all* $h, h' \in H^*$.
2) $d_H^*(\vec{h}, \vec{h}') = 0$ *if and only if* $\vec{h} = \vec{h}'$.
3) $d_H^*(\vec{h}, \vec{h}'') \leq d_H^*(\vec{h}, \vec{h}') + d_H^*(\vec{h}', \vec{h}'')$ *for all* $h, h', h'' \in H^*$.

Proof:

1) From Def. 2 here, and Def. 11 and Prop. 2 in [16] $d_D(q_i, q_i') \geq 0 \ \forall i \in I$, then $\max_{i\in I} d_D(q_i, q_i') \geq 0$. By properties of norm and of tanh, $\tanh\|\vec{x} - \vec{x}'\| \geq 0$ for all $\vec{x}, \vec{x}' \in \prod_{i\in I} X_i$. Thus $d_H^*(\vec{h}, \vec{h}') \geq 0$ for all $h, h' \in H^*$.
2) ($\Rightarrow$) If $\vec{h} = \vec{h}'$, $\vec{q} = \vec{q}'$ and $\vec{x} = \vec{x}'$. $\vec{q} = \vec{q}'$ implies $q_i = q_i' \ \forall i \in I$. Then $d_D(q_i, q_i') = 0 \ \forall i \in I$, which implies $\max_{i\in I}(d_D(q_i, q_i')) = 0$. $\vec{x} = \vec{x}'$ implies $\tanh(\|\vec{x} - \vec{x}'\|) = 0$. Thus $\vec{h} = \vec{h}' \Rightarrow d_H^*(\vec{h}, \vec{h}') = 0$.
   ($\Leftarrow$) Since $\max_{i\in I} d_D(q_i, q_i') \geq 0$ and $\tanh(\|\vec{x} - \vec{x}'\|) \geq 0 \ \forall \vec{h}, \vec{h}' \in H^*$, $d_H^*(\vec{h}, \vec{h}') = 0$ implies that $\max_{i\in I} d_D(q_i, q_i') = 0$ and $\tanh(\|\vec{x} - \vec{x}'\|) = 0$. From Def. 2 $\max_{i\in I} d_D(q_i, q_i') = 0$ implies $d_D(q_i, q_i') = 0 \ \forall i \in I$, which together with Prop. 1 in [16] implies $q_i = q_i' \ \forall i \in I$, which implies $\vec{q} = \vec{q}'$. $\tanh(\|\vec{x} - \vec{x}'\|) = 0$ implies $\|\vec{x} - \vec{x}'\| = 0$, which implies $\vec{x} = \vec{x}'$. Thus $d_H^*(\vec{h}, \vec{h}') = 0 \Rightarrow \vec{h} = \vec{h}'$.
   ($\Rightarrow$) and ($\Leftarrow$) imply $d_H^*(\vec{h}, \vec{h}') = 0 \iff \vec{h} = \vec{h}'$.
3) **Discrete:** From Lemma 2 in [16] $d_D(q_i, q_i'') \leq d_D(q_i, q_i') + d_D(q_i', q_i'')$. Suppose $\exists \vec{q}, \vec{q}', \vec{q}'' \in \prod_{i\in I} Q_i$ such that $\max_{i\in I} d_D(q_i, q_i'') > \max_{i\in I} d_D(q_i, q_i') + \max_{i\in I} d_D(q_i', q_i'')$. Then, $\exists i, j, k \in I$ such that $d_D(q_i, q_i'') > d_D(q_j, q_j') + d_D(q_k', q_k'')$. Note that this implies $d_D(q_j, q_j') \geq d_D(q_i, q_i')$ and $d_D(q_k', q_k'') \geq d_D(q_i', q_i'')$. This implies $d_D(q_i, q_i'') > d_D(q_i, q_i') + d_D(q_i', q_i'')$, which contradicts Lemma 2 in [16]. Therefore $\max_{i\in I} d_D(q_i, q_i'') \leq \max_{i\in I} d_D(q_i, q_i') + \max_{i\in I} d_D(q_i', q_i'') \ \forall \vec{q}, \vec{q}', \vec{q}'' \in \prod_{i\in I} Q_i$.
   Continuous: $\tanh(\|x - x''\|) \leq \tanh(\|x - x'\|) + \tanh(\|x -' x''\|)$ follows from Lemma 3 in [16]. **Discrete** and **Continuous** parts imply the claim.

■

Remark 2 (Asymmetric distance) *The interconnected hybrid distance does not satisfy the symmetry property that metrics usually do because of the use of the discrete distance of Definition 2. However, we believe that the absence of this property is actually desirable because the number of transitions that are required to reach $\vec{q}$ from $\vec{q}'$ may be different from the number of transitions required to reach $\vec{q}'$ from $\vec{q}$.*

It is possible to reformulate Definition 3 and Theorem 1 to prevent simultaneous discrete transitions among different individual systems. In such case a more meaningful notion of distance would be $d^*_H(\vec{h},\vec{h}') = \sum_{i\in I}(d_D(q_i,q'_i)) + \tanh(\|\vec{x}-\vec{x}'\|)$.

IV. Interconnected Hybrid Execution

In this section we introduce the Interconnected Hybrid Execution (IHE) based on the concept of hybrid execution in [10]. The IHE is the analog of the state evolution of a continuous multi-agent dynamical system, and captures the system's hybrid behavior with respect to both discrete and continuous interactions of the agents among themselves and with and with its environment. Then we provide a necessary condition for the existence of an infinite IHE. This condition is stated as a function of each agent in the system. Therefore the desired global behavior of the system (existence of its execution), can be guaranteed by the specification of local design variables inside each agents dynamics.

A *Time Trajectory* is a sequence $\bar{\tau} = \{\bar{\tau}^0,\bar{\tau}^1,\ldots,\bar{\tau}^n,\ldots,\bar{\tau}^{\bar{N}}\}$, where $\bar{\tau}^n \leq \bar{\tau}^{n+1}$ for all $n=\{0,1,\ldots,\bar{N}-1\}$. $\bar{\tau}$ is infinite if $\bar{N}=\infty$ and is finite otherwise. τ is an *Interconnected Hybrid Time Trajectory* (IHTT) if τ is a time trajectory and if 1) τ^0 is the time when $\mathbf{H}^*$ starts its evolution, 2) τ^n is the time at which there is a system $\mathbf{H}_i \in \mathbf{H}^*$ that makes a discrete transition from q_i^n to q_i^{n+1} for $n=\{0,1,\ldots,N-1\}$, such that the Interconnected Hybrid System $\mathbf{H}^*$ makes a discrete transition from $\vec{q}^n$ to $\vec{q}^{n+1}$, and 3) τ^N is the time when $\mathbf{H}^*$ ends its evolution. $\hat{\tau}$ is an *Event Time Trajectory* (ETT), if $\hat{\tau}$ is a time trajectory and $\hat{\tau}^n$ is the time when there is a system $\mathbf{H}_i \in \mathbf{H}^*$ that experiments a discrete event $e_i^n \in E_i$ for all $n \in \{0,1,\ldots,\hat{N}\}$ where $\hat{N}$ is the number of events that $\mathbf{H}^*$ experiments.

The IHTT and the ETT are used to encode timing information for the continuous and discrete dynamics of the IHS $\mathbf{H}^*$. The IHTT stores the times when a discrete transition takes place at least on one of the agents in the system. As a consequence the IHTT also specifies time intervals between two consecutive elements in the sequence where uninterrupted continuous evolution takes place. On the other hand the ETT stores information about the specific times that events happen somewhere in the system. Note that these two sequences are considered completely independent. This is useful because the occurrence of an event does not necessarily imply that a discrete transition takes place.

The IHTT and the ETT as defined above allow to have more than one hybrid system in the overall IHS taking a discrete transition or experimenting an event at the same time. These definitions may be reformulated to exclude this possibility. Any time trajectory $\bar{\tau}$ is linearly ordered by the relation $\prec$ defined by $t_1 \prec t_2$ for $t_1 \in [\bar{\tau}_i,\bar{\tau}_{i+1}]$ and $t_2 \in [\bar{\tau}_j,\bar{\tau}_{j+1}]$ if $t_1 < t_2$ or $i<j$. We say $\bar{\tau}=\{\bar{\tau}^0,\bar{\tau}^1,\ldots,\bar{\tau}^{\bar{N}}\}$ is a prefix of $\tilde{\tau}=\{\tilde{\tau}^0,\tilde{\tau}^1,\ldots,\tilde{\tau}^{\tilde{N}}\}$ (written $\bar{\tau}\sqsubseteq\tilde{\tau}$) if either they are identical, or $\bar{\tau}$ is finite, $\bar{N}\leq\tilde{N}$, $\bar{\tau}_n=\tilde{\tau}_n$ for all $n\in\{0,1,\ldots,\bar{N}-1\}$, and $[\bar{\tau}_{N-1},\bar{\tau}_N[\subseteq [\tilde{\tau}_{N-1},\tilde{\tau}_N[$, where [is either] or).

A *Group Event Sequence* of $\mathbf{H}^*$ is a collection $\mathcal{E}^* = (\hat{\tau},\mathrm{Es})$ where $\hat{\tau}$ is an ETT and $\mathrm{Es}=(e^0_{\alpha^0},e^1_{\alpha^1},\ldots,e^{\hat{N}}_{\alpha^{\hat{N}}})$ is the sequence of events that $\mathbf{H}^*$ experiments, where $\alpha^n \in I$ for all $n\in\{0,1,\ldots\hat{N}\}$, such that $e^n_{\alpha^n}\in E_{\alpha^n}$ specifies the event that occurs at $\hat{\tau}^n$, and the individual system $\mathbf{H}_{\alpha^n}$ that experiments such event for all $n\in\{0,1,\ldots\hat{N}\}$.

In the following, in order to simplify the description of our results we divide the transition guards into a local part, a remote part, and an event part when needed. The local part verifies that the state of the agent experimenting a discrete transition satisfies the transition guard. The remote part verifies that the states of the agents connected to the one that is experimenting the transition satisfy the transition guard, and finally the event part (in the case of an event-triggered transition) verifies that the agent experimenting the transition has also experimented a discrete event that enables such transition. Let $G^{\mathrm{S/Local}}_{q_i}(s)\subseteq X_{q_i}$ denote the first element in the cartesian product of the state-based transition guard $G^{\mathrm{S}}_{q_i}(s)$. Let $G^{\mathrm{S/Remote}}_{q_i}(s)\subseteq\prod_{j\in V(q_i)}H_j$ denote the reminder of the elements of the cartesian product of the state-based transition guard $G^{\mathrm{S}}_{q_i}(s)$. Let $G^{\mathrm{E/E}}_{q_i}(s)\subseteq E_i$ denote the first element in the cartesian product of the event-triggered transition guard $G^{\mathrm{E}}_{q_i}(s)$. Let $G^{\mathrm{E/Local}}_{q_i}(s)\subseteq X_{q_i}$ denote the second element in the cartesian product of the event-triggered transition guard $G^{\mathrm{E}}_{q_i}(s)$. Finally let $G^{\mathrm{E/Remote}}_{q_i}(s)\subseteq\prod_{j\in V(q_i)}H_j$ denote the reminder of the elements of the cartesian product of the event-triggered transition guard $G^{\mathrm{E}}_{q_i}(s)$. We also use the following notation: $q_i\bar{\in}\vec{q}$ if q_i is a component of the vector $\vec{q}$. $x_{q_i}\bar{\in}\vec{x}_{\vec{q}}$ if x_{q_i} is a component of $\vec{x}_{\vec{q}}$ where $q_i\bar{\in}\vec{q}$ (Similarly for $s_{q_i}\bar{\in}\vec{s}$ and $u_{q_i}\bar{\in}\vec{u}$). $h_i\bar{\in}\vec{h}$ if h_i is a component of $\vec{h}$. Finally since $\vec{h}=(\vec{q},\vec{x}_{\vec{q}})$ we also say $q_i\bar{\in}\vec{h}$ if $q_i\bar{\in}\vec{q}$ and $x_{q_i}\bar{\in}\vec{h}$ if $x_{q_i}\bar{\in}\vec{x}_{\vec{q}}$.

Definition 4 (Interconnected Hybrid Execution) *An* Interconnected Hybrid Execution *(IHE) $\chi(\vec{h}_0,\mathcal{E}^*)$ with initial conditions $\vec{h}_0$ and group event sequence $\mathcal{E}^*$ is a collection $(\tau,\mathbf{q},\mathbf{s},\mathbf{x},\mathbf{u})$, where:*

- *τ is an interconnected hybrid time trajectory.*
- *$\mathbf{q}=\{\vec{q}^0,\vec{q}^1,\ldots,\vec{q}^n,\ldots,\vec{q}^N\}$ is a sequence of vectors of discrete locations $\vec{q}^n=(q_i^n)^T_{i\in I}$ where q_i^n is the discrete mode of system $\mathbf{H}_i$ at the n step on the sequence.*
- *$\mathbf{s}=\{\vec{s}^0,\vec{s}^1,\ldots,\vec{s}^n,\ldots,\vec{s}^N\}$ is a sequence of vectors of switching labels $\vec{s}^n=(s^n_{q_i})^T_{i\in I}$ where $s^n_{q_i}$ is the switching label of system $\mathbf{H}_i$ at n step in the execution.*
- *$\mathbf{x}=\{\vec{x}^0,\vec{x}^1,\ldots,\vec{x}^n,\ldots,\vec{x}^N\}$ is a sequence of continuous evolution $\vec{x}^n=(x^T_{q_i^n})^T_{i\in I}$ where $x_{q_i^n}$ is an absolutely differentiable map $x_{q_i^n}:[\tau^n,\tau^{n+1})\to X_{q_i^n}$ of system $\mathbf{H}_i$ at the n step on the sequence for all $i\in I$.*
- *$\mathbf{u}=\{\vec{u}^0,\vec{u}^1,\ldots,\vec{u}^n,\ldots,\vec{u}^N\}$ is a sequence of continuous control inputs $\vec{u}^n=\left(u_{q_i^n}(t)\right)^T_{i\in I}$ where $u_{q_i^n}(t)\in U_{q_i^n}$ (Definition 1 and following remarks[2]) for all $t\in$*

[2]Note that $u_{q_i^n}$ is originally defined as $u_{q_i}: X_{q_i}\times\prod_{j\in V(q_i)}\left(\bigcup_{q_j\in Q_j}X_{q_j}\right)\to U_{q_i}$ in Definition 1 and the following remarks, but since $x_{q_i^n}$ is defined above as a function of time, function composition allows us to express $u_{q_i^n}$ as a function of time.

$[\tau^n, \tau^{n+1})$, *and* $i \in I$.

The interconnected hybrid execution $\chi(\vec{h}_0, \mathcal{E}^*)$ *satisfies the following conditions:*

- *Initial Condition:* $\vec{h}_0 = (\vec{q}^0, \vec{x}^0(0))$ *is an initial condition of* $\mathbf{H}^*$.
- *Continuous Dynamics: for all* $t \in [\tau^n, \tau^{n+1})$, *all* $n \in \{0, 1, 2, \ldots, N-1\}$, *and all* $i \in I$, $\dot{x}_{q_i^n}(t) = f_{q_i^n}(x_{q_i^n}, u_{q_i^n}, t)$, $x_{q_i^n} \in X_{q_i^n}$ *and* $u_{q_i^n} \in U_{q_i^n}$ *where* $q_i^n \bar{\in} \vec{q}^n$, $x_{q_i^n} \bar{\in} \vec{x}^n$, *and* $u_{q_i^n} \bar{\in} \vec{u}^n$.
- *Discrete Dynamics: Either the* ***event-triggered transition conditions*** *or the* ***state-based transition conditions*** *hold for each* $n \in \{0, 1, 2, \ldots, N-1\}$ *and for all* $i \in I$.
  The ***event-triggered transition conditions*** *are:*
  - $q_i^{n+1} = s_{q_i^n} \in S_{q_i^n}$, *where* $q_i^{n+1} \bar{\in} \vec{q}^{n+1}$ *and* $s_{q_i^n} \bar{\in} \vec{s}$,
  - *There exists a* $(\hat{\tau}^{\hat{n}}, e^{\hat{n}}_{\alpha^{\hat{n}}}) \in \mathcal{E}^*$ *such that* $\alpha^{\hat{n}} = i$, $\hat{\tau}^{\hat{n}} = \tau^{n+1}$, *and* $e_i^{\hat{n}} \in G^{\mathrm{E/E}}_{q_i^n}(s_{q_i^n})$,
  - $x_{q_i^n}(\tau^{n+1}) \in G^{\mathrm{E/Local}}_{q_i^n}(s_{q_i^n})$, *and* $(h_{q_j^n})_{j \in V(q_i^n)}(\tau^{n+1}) \in G^{\mathrm{E/Remote}}_{q_i^n}(s_{q_i^n})$, *where* $x_{q_i^n} \bar{\in} \vec{x}_{\vec{q}}$ *and* $h_{q_j^n} \bar{\in} \vec{h}^n$,
  - $x_{q_i^{n+1}}(\tau^{n+1}) \in Z_{q_i^n}(G^E_{q_i^n}, s_{q_i^n})$.

  The ***state-based transition conditions*** *are:*
  - $q_i^{n+1} = s_{q_i^n} \in S_{q_i^n}$, *where* $q_i^{n+1} \bar{\in} \vec{q}^{n+1}$ *and* $s_{q_i^n} \bar{\in} \vec{s}$,
  - $x_{q_i^n}(\tau^{n+1}) \in G^{\mathrm{S/Local}}_{q_i^n}(s_{q_i^n})$ *and* $(h_{q_j^n})_{j \in V(q_i^n)}(\tau^{n+1}) \in G^{\mathrm{S/Remote}}_{q_i^n}(s_{q_i^n})$, *where* $x_{q_i^n} \bar{\in} \vec{x}_{\vec{q}}$ *and* $h_{q_j^n} \bar{\in} \vec{h}^n$, *and*
  - $x_{q_i^{n+1}}(\tau^{n+1}) \in Z_{q_i^n}(G^{\mathrm{S}}_{q_i^n}, s_{q_i^n})$.

The IHE provides the information about the continuous and discrete states and inputs of the system at each instant of its evolution. It is the analog of the state-input trajectory in continuous time systems. The conditions imposed in the second part of Definition 4 are required for it to be valid to $\mathbf{H}^*$. Therefore an IHE should start at a valid initial condition. The continuous evolution between two times in the interconnected hybrid time trajectory should satisfy the continuous dynamics of each agent, and the discrete transitions should have valid transition guards and transition maps.

Note that we used $\chi(\vec{h}_0, \mathcal{E}^*)$ to denote an IHE with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$. We say that an IHE $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ of $\mathbf{H}^*$ is a prefix of another IHE $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*) = (\tilde{\tau}, \tilde{\mathbf{q}}, \tilde{\mathbf{s}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}})$ of $\mathbf{H}^*$ (written $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$) if $\tau \sqsubseteq \tilde{\tau}$, and for all $n \in \{0, 1, \ldots, N\}$ and for all $t \in [\tau^n, \tau^{n+1}[$ $(\vec{q}^n, \vec{s}^n, \vec{x}^n(t), \vec{u}^n(t)) = (\vec{\tilde{q}}^n, \vec{\tilde{s}}^n, \vec{\tilde{x}}^n(t), \vec{\tilde{u}}^n(t))$. We say that $\chi(\vec{h}_0, \mathcal{E}^*)$ is a strick prefix of $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$ (written $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubset \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$) if $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$, and $\chi(\vec{h}_0, \mathcal{E}^*) \neq \tilde{\chi}(\vec{h}_0, \mathcal{E}^*)$.

An IHE $\chi(\vec{h}_0, \mathcal{E}^*)$ is called maximal if it is not a strick prefix of any other execution. An IHE $\chi(\vec{h}_0, \mathcal{E}^*)$ is finite if τ is a finite sequence and the last elements of $\mathbf{u}$ and $\mathbf{x}$ are defined over compact intervals of time, i.e. $\vec{u}^N : [\tau^{N-1}, \tau^N] \to \prod_{i \in I} U_{q_i^n}$, and $\vec{x}^N : [\tau^{N-1}, \tau^N] \to \prod_{i \in I} X_{q_i^n}$. $\chi(\vec{h}_0, \mathcal{E}^*)$ is infinite if τ is an infinite sequence or if $\tau^N = \infty$.

$\chi^S(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all IHEs with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$, and similarly $\chi^F(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all finite IHEs, $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all infinite IHEs, and $\chi^M(\vec{h}_0, \mathcal{E}^*)$ denotes the set of all maximal IHEs with initial condition $\vec{h}_0$ and group event sequence $\mathcal{E}^*$. Init denotes the set of all initial conditions, and ESS denotes the set of all possible group event sequences.

We say that $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$ if $\tau = \{\tau^0, \tau^1, \ldots, \tau^N\}$ and $\vec{h} = (\vec{q}^N, \vec{x}^N(\tau^N))$. An interconnected hybrid state $\vec{h} \in Reach(\vec{h}_0, \mathcal{E}^*)$ if there exists a finite IHE $\chi(\vec{h}_0, \mathcal{E}^*) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ that maps $\vec{h}_0$ to $\vec{h}$ with group event sequence $\mathcal{E}^*$. The set of states $\vec{h}$ that can be reached from any initial condition and with any group event sequence $Reach_{\mathbf{H}^*} = \bigcup_{(\vec{h}_0, \mathcal{E}^*) \in \mathrm{Init} \times \mathrm{ESS}} Reach(\vec{h}_0, \mathcal{E}^*)$ is called *Interconnected Reachable Set*.

Let $\psi(q_i, x_{q_i}, u_{q_i}, t)$ denote the flow of $f_{q_i}(x_{q_i}, u_{q_i}, t)$ for all $i \in I$. We define the set for which continuous evolution is impossible as $Out_{\mathbf{H}^*} = \{\vec{h} \in \prod_{i \in I} X_i \times \prod_{i \in I} Q_i; \forall \epsilon > 0, \exists t \in [0, \epsilon)$ and $\exists i \in I$, such that $\psi(q_i, x_{q_i}, u_{q_i}, t) \notin X_{q_i}$, where $q_i \bar{\in} \vec{h}, x_{q_i} \bar{\in} \vec{h}\}$.

We say that $\mathbf{H}^*$ is deterministic if given $\vec{h}_0$ and $\mathcal{E}^*$, $\chi^M(\vec{h}_0, \mathcal{E}^*)$ contains at most one element.

Theorem 2 (Existence of infinite IHE) *Suppose* $\mathbf{H}^*$ *is deterministic. Then given an initial condition* $\vec{h}_0$ *and a group event sequence, if* $\mathcal{E}^*$, $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ *is nonempty then for all* $\vec{h} \in Reach_{\mathbf{H}^*} \bigcap Out_{\mathbf{H}^*}$ *either one of the following conditions holds:*

1) *There exist a* $\mathbf{H}_i \in \mathbf{H}^*$ *s.t. there exists a* $s \in S_{q_i}$ *with* $x_{q_i} \in G^{\mathrm{S/Local}}_{q_i}(s)$, *and* $(h_{q_j})_{j \in V(q_i)} \in G^{\mathrm{S/Remote}}_{q_i}(s)$ *where* $q_i \bar{\in} \vec{h}$, $x_{q_i} \bar{\in} \vec{h}$, *and* $h_{q_i} \bar{\in} \vec{h}$ *for all* $i \in I$.
2) *There exist a* $\mathbf{H}_i \in \mathbf{H}^*$ *and an element* $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$ *with* $\alpha^k = i$ *s.t. there exists a* $s \in S_{q_i}$ *with* $x_{q_i} \in G^{\mathrm{E/Local}}_{q_i}(s)$, $(h_{q_j})_{j \in V(q_i)} \in G^{\mathrm{E/Remote}}_{q_i}(s)$, *and* $\hat{\tau}^k = \tau^N$ *and* $e^k_{\alpha^k} \in G^{\mathrm{E/E}}_{q_i}(s)$ *where* τ^N *is the time of the last element of the finite execution* $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ *that maps the system* $\mathbf{H}^*$ *from* $\vec{h}_0$ *to* $\vec{h}$ *with group event sequence* $\mathcal{E}^*$, *and where* $q_i \bar{\in} \vec{h}$, $x_{q_i} \bar{\in} \vec{h}$, *and* $h_{q_i} \bar{\in} \vec{h}$ *for all* $i \in I$..

Proof: Suppose for the sake of contradiction that $\mathbf{H}^*$ is deterministic, and for any $\vec{h}_0$ and $\mathcal{E}^*$ $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is nonempty, but there is a $\vec{h} \in Reach_{\mathbf{H}^*} \bigcap Out_{\mathbf{H}^*}$ for which none of 1) or 2) hold. Since $\vec{h} \in Reach_{\mathbf{H}^*}$ there is a finite execution $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u}) \in \chi^F(\vec{h}_0, \mathcal{E}^*)$ such that $\tau = \{\tau^0, \tau^1, \ldots, \tau^N\}$ and $\vec{h} = (\vec{q}^N, \vec{x}^N(\tau^N))$.

a) Suppose there exists another execution $\check{\chi}(\vec{h}_0, \mathcal{E}^*) = (\check{\tau}, \check{\mathbf{q}}, \check{\mathbf{s}}, \check{\mathbf{x}}, \check{\mathbf{u}})$ such that $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \check{\chi}(\vec{h}_0, \mathcal{E}^*)$ and $\check{\tau} = \{\tau^0, \tau^1, \ldots, \tau^{N-1}, \tau^N + \epsilon\}$ for some $\epsilon > 0$. Then there exists $t \in [0, \epsilon)$ such that $\psi(q_i, x_{q_i}, u_{q_i}, t) \in X_{q_i}$ for all $i \in I$, violating $\vec{h} \in Out_{\mathbf{H}^*}$.

b) Suppose there exists $\check{\chi}(\vec{h}_0, \mathcal{E}^*) = (\check{\tau}, \check{\mathbf{q}}, \check{\mathbf{s}}, \check{\mathbf{x}}, \check{\mathbf{u}})$ such that $\chi(\vec{h}_0, \mathcal{E}^*) \sqsubseteq \check{\chi}(\vec{h}_0, \mathcal{E}^*)$ and $\check{\tau} = \{\tau^0, \tau^1, \ldots, \tau^N, \check{\tau}^{N+1}\}$, then there exists $\mathbf{H}_i \in \mathbf{H}^*$ that executes either a state-based transition or a an event-triggered transition at τ^N, therefore one of the following holds:

- If $\mathbf{H}_i$ executes a state-based transition, Definition 4 implies there exists a $s \in S_{q_i^{N-1}}$ such that $x_{q_i^{N-1}}(\tau^N) \in G^{\mathrm{S/Local}}_{q_i^{N-1}}(s)$, $(h_{q_j^{N-1}})_{j \in V(q_i^{N-1})}(\tau^N) \in G^{\mathrm{S/Remote}}_{q_i^{N-1}}(s)$, and $x_{q_i^N}(\tau^N) \in Z_{q_i^{N-1}}(G^{\mathrm{S}}_{q_i^{N-1}}, s)$ where $q_i^n \bar{\in} \vec{h}^n$, $x_{q_i^n} \bar{\in} \vec{h}^n$, $h_{q_j^n} \bar{\in} \vec{h}^n$ for all $i, j \in I$ and for all $n \in \{N, N-1\}$. Note that this violates assumption that 1) does not hold.
- If $\mathbf{H}_i$ executes an event-triggered transition, Definition 4 implies there exists a $s \in S_{q_i^{N-1}}$ and a $(\hat{\tau}^k, e^k_{\alpha^k}) \in \mathcal{E}^*$, such that $\alpha^k = i$, $\hat{\tau}^k = \tau^N$, $e^k_{\alpha^k} \in G^{\mathrm{E/E}}_{q_i^{N-1}}(s)$, $x_{q_i^{N-1}(\tau^N)} \in G^{\mathrm{E/Local}}_{q_i^{N-1}}(s)$, $(h_{q_j^{N-1}})_{j \in V(q_i^{N-1})}(\tau^N) \in G^{\mathrm{E/Remote}}_{q_i^{N-1}}(s)$, and $x_{q_i^N}(\tau^N) \in Z_{q_i^{N-1}}(G^{\mathrm{E}}_{q_i^{N-1}}, s)$, where $q_i^n \bar{\in} \vec{h}^n$, $x_{q_i^n} \bar{\in} \vec{h}^n$, $h_{q_j^n} \bar{\in} \vec{h}^n$ for all $i, j \in I$ and for all $n \in \{N, N-1\}$. Note that this violates assumption that 2) does not hold.

a) and b) imply that $\chi(\vec{h}_0, \mathcal{E}^*) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ is maximal. However by assumption $\chi^\infty(\vec{h}_0, \mathcal{E}^*)$ is nonempty, therefore there exists an infinite execution $\tilde{\chi}(\vec{h}_0, \mathcal{E}^*) \in \chi^\infty(\vec{h}_0, \mathcal{E}^*)$. This execution is also maximal and different from $\chi(\vec{h}_0, \mathcal{E}^*)$, which implies that $\chi^M(\vec{h}_0, \mathcal{E}^*)$ has at least two different elements violating the assumption that $\mathbf{H}^*$ is deterministic, which proves our claim. ∎

Note that Theorem 2 states the necessary condition for the existence of an IHE in terms of each agent's model. We are currently working on the formal proof for the sufficient condition. These two conditions may be used together to design the dynamics of each agent in local form such that the existence of the multi-agent system's execution is guaranteed globally.

V. Conclusion

We have presented an interconnected hybrid systems framework: a set of hybrid systems with interweaved continuous and discrete dynamics that form a multi-agent system with hybrid individual dynamics. We extended the work in [10], [16] defining a metric, reachable sets, and executions for interconnected hybrid systems. We explained the properties of the new metric and proved a necessary condition for the existence of interconnected hybrid executions that is written in terms of the local model of each hybrid agent.

We are currently working on the sufficient condition for existence and on the necessary and sufficient conditions for the uniqueness of such execution. We expect hat this new theoretical framework will enable us to analyze, control and perform abstractions on multi-agent systems with hybrid individual dynamics.

References

[1] M. Branicky, V. Borkar, and S. Mitter. A unified framework for hybrid control: Model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, Jan. 1998.

[2] J. Cortez and F. Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization*, 44(5):1543–1574, 2005.

[3] J. Finke, K. Passino, and A. Sparks. Stable task load balancing strategies for cooperative control of networked autonomoues vehicles. *IEEE Transactions on Control Systems Technology*, 14(5):789–803, September 2006.

[4] V. Gazi and M. Passino. Stability analysis of swarms. *IEEE Transactions on Automatic Control*, 48(4):692–697, Apr. 2003.

[5] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, NY, USA, 2001.

[6] A. Jadbabaie, J. Lin, and S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, June 2003.

[7] H. Jerez, C. Abdallah, and J. Khoury. A mobile transient network architecture. Pre-print available at `http://hdl.handle.net/2118/hj_tran_06`, 2006.

[8] Y. Liu, K. Passino, and M. Polycarpou. Stability analysis of m-dimensional asynchronous swarms with a fixed communication topology. *IEEE Transactions on Automatic Control*, 48(1):76–95, Jan. 2003.

[9] J. Lygeros. Lecture notes on hybrid systems. Notes for an ENSIETA workshop, February–June 2004.

[10] J. Lygeros, K. Johansson, S. Simić, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48(1):2–16, Jan. 2003.

[11] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, Feb. 2005.

[12] R. Olfati-Saber. Consensus problems in networks of agents with switching topology and time delay systems. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, Sept. 2004.

[13] R. Olfati-Saber, J. Fax, and R. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, January 2007.

[14] J. Piovesan, C. Abdallah, and H. Tanner. A hybrid framework for resource allocation among multiple agents moving on discrete environments. To appear in the Special Issue on Collective Behavior and Control of Multi-Agent Systems of the Asian Journal of Control, 2008.

[15] J. Piovesan, C. Abdallah, H. Tanner, H. Jerez, and J. Khoury. Resource allocation for multi-agent problems in the design of future communication networks. UNM Technical Report EECE-TR-07-001, Dept. Electrical and Computer Engineering, Univeristy of New Mexico, Albuquerque, NM, 87131, April 2007. `http://hdl.handle.net/1928/2973`.

[16] J. Piovesan, H. Tanner, and C. Abdallah. Discrete asymptotic astractions of hybrid systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 917–922, San Diego, CA, USA, December 2006.

[17] W. Ren, R. Beard, and E. Atkins. Information consensus in multi-vehicle cooperative control. *IEEE Control Systems Magazine*, 27(2):71–82, April 2007.

[18] M. Shields. *An introduction to automata theory*. Blackwell Scientific Publications, 1987.

[19] H. Tanner, A. Jadbabaie, and G. Pappas. Flocking in fixed and switching networks. *IEEE Transactions on Automatic Control*, 52(5):863–868, May 2007.