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# CONTINUOUS AND DISCRETE TIME SPR DESIGN USING FEEDBACK

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## ABSTRACT

This paper presents necessary and sufficient conditions for the existence of a feedback compensator that will render a given continuous-time or discrete-time linear system SPR. When these conditions hold, the controller is explicitly found.

## 1 INTRODUCTION

The concepts of Positive-Real (PR) and Strictly-Positive-Real (SPR) functions and matrices have been very useful in network theory [1], adaptive control [2] and robust control [3]. These concepts have also been generalized to include discrete-time systems [4] and [5]. The importance of PR and SPR matrices is obvious when dealing with uncertain systems. In this situation, a nominal SPR transfer function allows for large passive uncertainties without the loss of stability [2] and [5]. The standard definition of SPR matrices [6], here termed strong SPR, is usually difficult to apply. Moreover, it was recently shown [7] that the strong SPR definition is overly restrictive for control theory applications. In this paper we will use the term SPR to denote weak SPR matrices as defined in [7] and [8] and reviewed in the next section. On the other hand, if a given transfer matrix is not SPR, the question of whether a feedback controller might make the closed-loop system SPR is of considerable interest. This problem was termed "Almost Strict Positive Real" and studied in [9]. What has been lacking, however, is a set of conditions that will answer the existence question: Given a transfer matrix  $P(s)$ , does a controller that will make it SPR exist?. Moreover, a construction of the controller (when it exists) is desirable. A necessary condition was found in [9] and a partial answer to the existence and construction questions was given in [10] for continuous-time systems. Sufficient existence conditions were also found in [11] for the single-input-single-output (SISO) continuous-time case and in [9] for the Multi-Input-Multi-Output

case. In the present paper, we provide a simple proof of the results in [10] and [11], and extend these results to the discrete-time case and to a more general class of systems. This paper is organized as follows: In Section 2, we review the available SPR definitions for continuous and discrete-time transfer matrices. In Section 3, we define the problem and present our results on designing controllers to make a closed-loop system SPR. Our conclusions are presented in Section 4.

## 2 WHICH SPR?

In order to keep the exposition clear, we will treat the continuous-time case first, then present the discrete-time results.

### 2.1 Continuous-Time Case:

Consider the multi-input-multi-output linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where  $x$  is an  $n$  vector,  $u$  is an  $m$  vector,  $y$  is a  $p$  vector,  $A, B, C$ , and  $D$  are of the appropriate dimensions. The corresponding transfer function matrix is

$$P(s) = C(sI - A)^{-1}B + D \tag{2}$$

We will first assume that the system has an equal number of inputs and outputs, i.e.  $p = m$ . Then define the relative degree  $n^*$  as follows:  $n^* = 0$  if  $\det(D) \neq 0$ , and  $n^* = m$  if  $\det(D) = 0$  but  $\det(CB) \neq 0$ . A formalism for the poles and zeros of multivariable systems is given in [12] and may be used to justify the definition of  $n^*$ . To simplify our notation we will denote the Hermitian part of a real, rational transfer matrix  $T(s)$  by  $He[T(s)] = \frac{1}{2}[T(s) + T^T(s^*)]$  where  $s^*$  is the complex conjugate of  $s$ . A number of definitions have been given for SPR functions and matrices [6] and [8]. It appears that the most useful definition for control applications is the following [7]

**Definition 1** *An  $m \times m$  matrix  $T(s)$  of proper real rational functions which is not identically zero is (weak) SPR if*

1. *All elements of  $T(s)$  are analytic in the closed right half plane, i.e. in the region  $Re(s) \geq 0$ , and*
2. *The matrix  $He[T(s)]$  is positive definite for  $Re(s) \geq 0$ .*

□

As a result of this definition, a necessary condition for a given transfer function to be SPR is that  $n^* = -1, 0, 1$ . The more standard definition of SPR matrices advocated in [6] is more restrictive than Definition 1. In fact, a long-held view was that strong SPR was needed to prove the Meyer-Kalman-Yakubovitch (MKY) lemma, which is, after all, the major application of SPR concepts in control systems. However, As shown in [7], the weak SPR definition is just as useful in this regard and will therefore be adopted in this paper. Note that, from minimum real-part arguments given in [1], condition 2) of Definition 1 is equivalent to  $He[T(jw)] > 0$  for all  $w$ .

## 2.2 Discrete-Time Case:

Consider now the discrete-time multi-input-multi-output linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \quad (3)$$

where  $x(k)$  is an  $n$  vector,  $u(k)$  is an  $m$  vector,  $y(k)$  is a  $p$  vector,  $A, B, C$ , and  $D$  are of the appropriate dimensions. The corresponding transfer function matrix is then

$$P(z) = C(zI - A)^{-1}B + D \quad (4)$$

Similar to the continuous-time case, we assume that the system has an equal number of inputs and outputs i.e.  $p = m$  and define the relative degree  $n^*$  as follows:  $n^* = 0$  if  $D \neq 0$ , and  $n^* = m$  if  $D = 0$  but  $CB \neq 0$ . Also, we denote the Hermitian part of  $T(z)$  by  $He[T(z)] = \frac{1}{2}[T(z) + T^T(z^*)]$  where  $z^*$  is the complex conjugate of  $z$  and  $z = \rho e^{jw}$ . The concept of discrete PR matrices is defined in [4]. The following definition is motivated by [4] and by the continuous-time counterpart.

**Definition 2** *An  $m \times m$  matrix  $T(z)$  of real rational functions is SPR if*

1. *All elements of  $T(z)$  are analytic on and outside the unit circle, i.e. in the region  $|z| \geq 1$ , and*
2. *The matrix  $He[T(e^{jw})]$  is positive definite Hermitian for all real  $w$ .*

□

Note first that a transfer function  $T(z)$  is SPR only if the corresponding  $T(s)$  with  $s = (z - 1)/(z + 1)$  is SPR. In addition, a necessary condition for  $T(z)$  to be SPR is that its relative degree  $n^* = 0$ .

## 3 SPR USING FEEDBACK

We will again separate our results into continuous-time and discrete-time results.

### 3.1 Continuous-Time Case:

The question addressed in this section is to find conditions on (1) or (2) so that a feedback controller will render the closed-loop system SPR. The result of Theorem 1 appeared in [10] for the case of a continuous-time plant and a static output feedback, i.e.  $u = -\gamma Ky + Kr$ . The closed-loop system is then given by

$$\begin{aligned}\dot{x} &= (A - \gamma BKC)x + BKr \\ y &= Cx\end{aligned}\tag{5}$$

or in the frequency-domain

$$Y(s) = [I + \gamma P(s)K]^{-1}P(s)KR(s)\tag{6}$$

We present a simple frequency domain proof to show the existence of  $K$  and  $\gamma$  that will render the closed-loop system SPR.

**Theorem 1** *Let system (1) be stabilizable and detectable and let its relative degree be  $n^* = m$ . Then there exists a nonsingular  $K$  and a positive scalar  $\gamma$  such that the closed-loop system (5) is SPR, if and only if  $P(s)$  is minimum phase.*

**Proof:**

*Sufficiency:* Consider the closed-loop transfer function

$$T(s) = [I + \gamma P(s)K]^{-1}P(s)K$$

or

$$T(s) = [K^{-1}P^{-1}(s) + \gamma I]^{-1}$$

Since  $P(s)$  is minimum phase with a relative degree  $n^* = m$ , its inverse  $P^{-1}(s)$  will be given by

$$P^{-1}(s) = sL + P_1(s)$$

where  $P_1(s)$  is proper and stable, and  $\det(L) \neq 0$ . In fact,  $\det(CB) \neq 0$  and  $L = (CB)^{-1}$ . On the other hand, since  $P(s)$  is minimum phase,  $P_1(s)$  cannot have any poles in  $\text{Re}(s) \geq 0$ . It is now obvious that  $T(s)$  will be stable if and only if  $W(s) = [K^{-1}P^{-1}(s) + \gamma I]$  has no zeros in  $\text{Re}(s) \geq 0$ . Let  $K$  be given by

$$K = (CB)^{-1}$$

then

$$\begin{aligned}W(s) &= sI + CBP_1(s) + \gamma I \\ \text{He}[W(jw)] &= \text{He}[CBP_1(jw)] + \gamma I\end{aligned}$$

Since  $P_1(jw)$  has no poles on the  $iw$  axis,  $He[W(jw)]$  may be made positive-definite by a large enough positive scalar  $\gamma$ . This then implies that  $W(s)$  is weak SPR. Since  $T(s)$  is the inverse of  $W(s)$ , it is also weak SPR [6].

*Necessity:* Suppose now that a nonsingular  $K$  and a  $\gamma$  were found to make the closed-loop system  $T(s)$  SPR and that  $D = 0$ . Then

$$W(s) = [K^{-1}P^{-1}(s) + \gamma I]$$

is also SPR. Writing  $P^{-1}(s)$  as  $sL + P_1(s)$ , with  $L = (CB)^{-1}$  we get

$$W(s) = K^{-1}(CB)^{-1} + K^{-1}P_1(s) + \gamma I$$

Since  $W(s)$  is SPR,  $P_1(s)$  must be stable, hence  $P(s)$  must be minimum-phase.

□

This result indicates that with the given assumptions on  $P(s)$ , static output feedback can always be found to stabilize the closed-loop system  $T(s)$ . Moreover,  $T(s)$  can also be made SPR to give the desired robustness against passive uncertainties. It can also be seen that a dynamic output feedback compensator will not relax the conditions of the theorem since output compensation can not move the open-loop zeros nor change the relative degree of the plant. The choice of  $K = (CB)^{-1}$  in the proof of the theorem is not unique. In fact, it is sufficient to choose  $K = Q(CB)^{-1}$  where  $Q$  is any symmetric positive-definite matrix. Next, note that the condition  $\det(CB) \neq 0$  (or that  $P(s)$  has a relative degree  $n^* = m$ ), also reveals that the system (1) has an inverse obtained by cascading one differentiator and a dynamical system [13]. Note that the inverse system given in the proof of Theorem 1 may be written in state-space as

$$\begin{aligned} \dot{x} &= [A - B(CB)^{-1}CA]x + B(CB)^{-1}\dot{y} \\ u &= -(CB)^{-1}CAx + (CB)^{-1}\dot{y} \end{aligned} \quad (7)$$

Now recall that the invertibility of the system (1) may still be inferred even though  $\det(CB) = 0$ . In fact, a sufficient condition for the inverse to exist is that the first nonzero matrix in the sequence,  $D, CB, CAB, CA^2B, \dots, CA^{n-1}B$ , be nonsingular [13]. It is then obvious that for a nonzero matrix  $D$ , the condition for  $T(s)$  to be SPR is that  $D$  be invertible and  $P(s)$  be minimum phase, i.e. an exactly-proper, minimum-phase transfer function may be made SPR with a static output feedback if its high frequency gain is nonsingular. On the other hand, the following general result may be established.

**Theorem 2** *Suppose that (1) is both stabilizable and detectable, and  $\det(CA^iB) \neq 0$  where  $CA^iB$  is the first nonzero matrix in the sequence*

$$D, CB, CAB, CA^2B, \dots, CA^{n-1}B$$

Then the closed-loop system from  $r$  to  $\frac{d^i y}{dt^i}$  given by

$$T_i(s) = CA^i(sI - A + \gamma BKC A^i)^{-1}BK$$

is SPR if and only if  $P(s)$  is minimum phase.

**Proof:** Given system (1), repeated here for convenience

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

let us define an output  $z_i$  by

$$z_i = \frac{d^i y}{dt^i} = y^{(i)}$$

then

$$z_{i+1} = CA^{i+1}x + CA^iBu$$

If  $\det(CA^iB) \neq 0$ , the inverse system of  $P(s)$  is given by

$$P^{-1}(s) = (CA^iB)^{-1}s^{i+1} + P_2(s)$$

where  $P_2(s)$  is stable. Repeating the arguments of theorem 3.1 and using the controller

$$u = -\gamma Ky^{(i)} + Kr$$

we obtain the desired result. □

### 3.2 Discrete-Time Case:

Next, we turn our attention to the discrete-time case. Specifically, we find conditions on (3) or (4) so that a feedback controller will render the closed-loop system SPR. Consider then the static output feedback, i.e.  $u(k) = -\gamma Ky(k) + Kr(k)$ . The closed-loop system is then given by

$$\begin{aligned}x(k+1) &= (A - \gamma BKC)x(k) + BKr(k) \\ y(k) &= Cx(k) + Dr(k)\end{aligned}\tag{8}$$

or in the  $z$ - domain

$$Y(z) = [I + \gamma P(z)K]^{-1}P(z)KR(z)\tag{9}$$

The following theorem parallels theorem ref1 and shows the existence of  $K$  and  $\gamma$  that will render the closed-loop system (8) SPR if  $D$  is invertible.

**Theorem 3** *Let system (3) be stabilizable and detectable and let its relative degree be  $n^* = 0$ . Then there exist a nonsingular  $K$  and a positive scalar  $\gamma$  such that the closed-loop system (8) is SPR, if and only if  $P(z)$  is minimum phase.*

**Proof:**

*Sufficiency:* Consider the closed-loop transfer function

$$\begin{aligned} T(z) &= [I + \gamma P(z)K]^{-1}P(z)K \\ &= [K^{-1}P^{-1}(z) + \gamma I]^{-1} = W^{-1}(z) \end{aligned}$$

Since  $P(z)$  is minimum phase with a relative degree  $n^* = 0$ , its inverse  $P^{-1}(z)$  will be given by

$$P^{-1}(z) = P_1(z)$$

where  $P_1(z)$  is proper and stable. Note that  $W(z) = K^{-1}P^{-1}(z) + \gamma I$  can be made SPR by choosing  $\gamma$  positive and large enough since

$$\|P^{-1}(z)\| \leq c \infty; \text{ for all } |z| \geq 1$$

Since the inverse of an SPR matrix is SPR,  $T(z)$  is SPR and therefore analytic in  $|z| \geq 1$ .

*Necessity:* Similar to the continuous-time case.

□

Note that the discrete-time case requires that  $P(z)$  be of relative degree  $n^* = 0$ . Thus for example, and unlike the continuous-time plant  $\frac{1}{s-a}$ , the discrete-time plant  $\frac{1}{z-a}$  may not be made SPR using the output feedback suggested. Next, we discuss the inverse-system interpretation of Theorem 3. Note that the condition  $\det D \neq 0$  (or that  $P(z)$  has a relative degree  $n^* = 0$ ), also reveals that the system (3) has an inverse system given in state-space by

$$\begin{aligned} x(k+1) &= [A - BD^{-1}C]x(k) + BD^{-1}y(k) \\ u(k) &= -D^{-1}Cx(k) + D^{-1}y(k) \end{aligned} \tag{10}$$

The following general result may then be established.

**Theorem 4** *Suppose that (3) is both stabilizable and detectable, and  $\det(CA^iB) \neq 0$  where  $CA^iB$  is the first nonzero matrix in the sequence*

$$D, CB, CAB, CA^2B, \dots, CA^{n-1}B$$

*Then the closed-loop system from  $r(k)$  to  $y(k+i)$  given by*

$$T_i(z) = CA^i(zI - A + \gamma BKCA^i)^{-1}BK$$

*is SPR if and only if  $P(z)$  is minimum phase.*

**Proof:** See the proof of Theorem 2.

□



## 4 CONCLUSIONS

In this paper we found necessary and sufficient conditions for a transfer function to be rendered SPR using output feedback. These results generalize a previously published result and establish a connection with the invertibility problem. The design is useful when a passive uncertainty enters the system such as in the Lure's problem [6].

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