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ABSTRACT

This paper deals with the class of nonlinear systems described by the equation $M(q(t))\ddot{q}(t)=f(t) - N(q(t),\dot{q}(t))$ with f(t) a control input. We employ a simple method of control design which has two stages. First, a global linearization is performed to yield a decoupled controllable linear system. Then a controller is designed for this linear system.

We provide a rigorous analysis of the effects of uncertain dynamics, which we study using robustness results in the time domain based on a Lyapunov equation and the total stability theorem. Using this approach we are able to give meaningful robustness bounds which justify assumptions that are currently made in the literature in an ad hoc fashion.

I. INTRODUCTION

A class of nonlinear systems that has been extensively studied is described by the equation

$$M(q)\ddot{q}= f - N(q,\dot{q}) \tag{1.1}$$

where f is a control vector, and M(q) and $N(q,\dot{q})$ are a matrix and a vector containing system parameters. The argument t has been dropped for convenience. This class of systems includes robotic systems in the Lagrange-Euler formulation [7]. The contributions of this paper are a simple control design scheme and a rigorous analysis of the effects of uncertainties present in the entries of M(q) and $N(q,\dot{q})$.

The proposed control design technique has two steps and is similar to other approaches in the literature [8]. A global linearization, is first performed to yield a controllable and decoupled linear system. We then present two controllers, the first being a pole-placement design and the second a linear-quadratic tracking design. The results are easily interpreted since the states of the linearized system and those of (1.1) are the same.

A rigorous time-domain robustness analysis based on a Lyapunov equation [11] and the total stability theorem [2] is then carried out to find bounds on the uncertainties in M(q) and $N(q,\dot{q})$ for closed-loop stability. Unlike the analysis in [3,9], our results allow the inclusion of <u>structured</u> uncertainties. We are also able to vary two design parameter "a" and "b" which affect a trade-off between the required accuracy of M(q) and that of $N(q,\dot{q})$. Our result gives bounds for stability on the uncertainties in the <u>individual entries</u> of M(q) and $N(q,\dot{q})$, yielding results which are easily interpreted from a practical standpoint. This short paper is a preliminary exposition of results to appear in more detail in [12], to which we defer for all proofs.

II. CONTROLLER DESIGN

Let a nonlinear system be described by the differential equation

$$M(q)\ddot{q}= f - N(q,\dot{q}) \tag{2.1}$$

with $q(t)\in R^n$ and the control $f(t)\in R^n$. M and N contain the system parameters, some of which may be unknown. We assume that M(q) is invertible for all q.

Define a state by $x^T= [q^T \ \dot{q}^T]^T$ and the desired trajectory x_d by $x_d^T= [q_d^T \ \dot{q}_d^T]^T$. Defining the error as $e= (x_d-x)$, the error dynamics become

$$\dot{e}= \begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix} e + \begin{bmatrix}0\\ I\end{bmatrix} u \equiv Ae + Bu, \tag{2.2}$$

where

$$u= M^{-1}N - M^{-1}f + \ddot{q}_d . \tag{2.3}$$

Note that we have simply described an implicit global linearization of the kind used in [5] and that the resulting linear system is decoupled. Since the input transformation (2.3) is one-to-one, f(t) can be recovered from u(t) by using

$$f= M(\ddot{q}_d - u) + N. \tag{2.4}$$

which does not require the inversion of M.

The problem of regulating the original nonlinear system has therefore been transformed into the simpler problem of stabilizing the (controllable) decoupled linear system (2.2). The control (2.4) has been called partitioned control, or computed torque control [7]. In the following, we give two methods of finding the control u(t) for (2.2).

First, let the control objective be the pole placement of the linear system in (2.2). Consider the closed-loop system

$$\dot{e}= A_c e \tag{2.5}$$

where

$$u= -Ke \tag{2.6}$$

so that

$$A_c= A - BK = \begin{bmatrix}0 & I\\ -K_1 & -K_2\end{bmatrix} . \tag{2.7}$$

Let the 2n desired eigenvalues of A_c be : $\mu_1, \ \mu_2, \ldots, \mu_{2n}$ and define D and E by

$$D= \text{diag}(\mu_i),\ E= \text{diag}(\mu_{i+n});\ i=1,..,n \tag{2.8}$$

<u>Theorem 2.1</u>

The gain $K= [K_1 \ \ K_2]$ required to place the poles of (2.2) has K_1, K_2 diagonal and given by:

$$K_1= ED \tag{2.9}$$
$$K_2= -(D + E) \tag{2.10}$$

∎

Next, consider the control objective to be the minimization of the linear-quadratic (LQ) performance index

$$J= \frac{1}{2}\int_0^\infty (e^TQe + u^TRu)\, dt \tag{2.11}$$

where

$$Q = \mathrm{diag}(Q_i),\ i = 1,..,n$$

with $Q \geq 0$, $R > 0$, and Q_1, Q_2, and R nxn diagonal (to obtain simple explicit formulas in the sequel). Note that the state of the nonlinear system is not changed by the linearization, so that performance index (2.11) is meaningful in terms of real performance objectives for (2.1).

It is well known [1,6] that the solution to the LQ problem described by (2.2) and (2.11) is given by the state feedback (2.6) where

$$K = R^{-1}B^T S \qquad (2.12)$$
$$-Q = A^T S + SA - SBR^{-1}B^T S.$$

The simple structure of our linear system leads to an explicit expression for K, as we now show.

Theorem 2.2

The feedback gain $K = [K_1\ \ K_2]$ minimizing (2.11) is given by

$$K_1 = (Q_1R^{-1})^{1/2}, \quad K_2 = (2K_1 + Q_2R^{-1})^{1/2} \qquad (2.13)$$

Moreover, the closed-loop poles are described by (2.8) with

$$Q_1R^{-1} = D^2E^2, \qquad Q_2R^{-1} = D^2 + E^2 \qquad (2.14)$$

∎

It is therefore possible in this special case to relate the weighting matrices Q_1, Q_2 and R to meaningful physical parameters such as the damping ratios and natural frequencies of the closed-loop poles. Using the expression for K given in equations (2.9),(2.10) in evaluating f as in (2.4) one gets:

$$f = -M[D^2E^2 \quad -(D^2 + E^2)]e + M\ddot{q}_d + N \qquad (2.15)$$

This nonlinear feedback law will make the nonlinear system (2.1) follow the desired trajectory x_d.

III. ROBUSTNESS ANALYSIS

In practice, the system described by (2.1) suffers from uncertainty in the entries of M(q) and $N(q,\dot{q})$. This will cause the calculated control law f_c to be different from the one found when M and N are completely known [3,9].

One of our contributions is the capability to deal with structured uncertainties in M(q) and $N(q,\dot{q})$ and so obtain tighter bounds than those in [9]. Let the calculated control for the nonlinear system be given by

$$f_c = M_c(\ddot{q}_d - u) + N_c \qquad (3.1)$$

where M_c and N_c are the calculated values of M and N. M_c and N_c differ from M and N due to simplifying assumptions and/or uncertain values of some parameters. Let u be given by (2.6) so that

$$f_c = M_c(\ddot{q}_d + Ke) + N_c \qquad (3.2)$$

One therefore obtains a calculated version u_c of the input u to the linear system (2.2). Let u_c be given by

$$u_c = \ddot{q}_d + M^{-1}N - M^{-1}f_c \qquad (3.3)$$

or

$$u_c = \triangle\ddot{q}_d + \delta + (\triangle - I)Ke \qquad (3.4)$$

where

$$\triangle = (I - M^{-1}M_c),\ \delta = M^{-1}(N - N_c).$$

One is now concerned with the stability of the system

$$\dot{e} = Ae + Bu_c \qquad (3.5)$$

or

$$\dot{e} = (A - BK + B\triangle K)e + B(\triangle\ddot{q}_d + \delta) \qquad (3.6)$$

$$\dot{e} = (A_c + B\triangle K)e + B(\triangle\ddot{q}_d + \delta)$$

$$\dot{e} = \underline{A}e + Bv \qquad (3.7)$$

The objective is to find bounds on $\triangle$ and δ to keep the above system stable given that (2.5) is stable. Consider the system (3.7). It is clear that its stability is dependent on $\underline{A}$ and on v. The effect of $\underline{A}$ is the stability robustness problem studied in [4,10,11]. We can therefore use the results from [11] which allow for structured uncertainties, taking advantage of the special form of the matrices A and B, to define

$$F = B\triangle K = \begin{bmatrix} 0 & 0 \\ F_1 & F_2 \end{bmatrix} \qquad (3.8)$$

where

$$F_i = \triangle K_i\ ;\ i = 1,\ 2. \qquad (3.9)$$

Let

$$F_{ij} \leq |F_{ij}|_{max} \equiv \varepsilon_{ij} \text{ and } \varepsilon \equiv \max \varepsilon_{ij}. \qquad (3.10)$$

where F_{ij} denotes the (i,j)th term of matrix F.

Suppose that K has been selected so that A_c has a stability margin of -a, where the constant "a" is a design parameter which may be selected to trade off the stability of A_c (guaranteed by the feedback K) against the required accuracy in M_c and N_c, as will become clear in our development.

Theorem 3.1

Let $(A_c - aI)$ be asymptotically stable (AS). Then the solution P of the Lyapunov equation

$$(A_c - aI)^T P + P(A_c - aI) + 2I = 0 \qquad (3.11)$$

is given by $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$

where P_i, i = 1,2,3 are diagonal and

$$P_2 = [4a^3I + 4a^2K_2 + a(K_2^2 + 3K_1) + K_1K_2]^{-1}[K_2 + a(2I - K_1)]$$

$$P_3 = (K_2 + 2aI)^{-1}(I + P_2), \quad P_1 = (K_2 + 2aI)P_2 + P_3K_1 \qquad (3.12)$$

∎

In the next theorem, conditions on $\triangle$ are given for the eigenvalues of $(A_c + F)$ to be less than -b where b is a positive design parameter (b<a) selected to trade off requirements on the accuracy of M_c against requirements on the accuracy of N_c, as shown next. Let us define

$$\varepsilon' \equiv \max|F_{ij} + (a-b)|_{max}. \qquad (3.13)$$

Theorem 3.2

Let A_c have eigenvalues with real parts less than -a. Then the eigenvalues of $(A_c + F)$ have real parts less than -b if

$$\varepsilon' < (s_{max}(P_mU)_s)^{-1} \equiv 1/s \qquad (3.14)$$

where:

$s_{max}(.)$ = maximum singular value of (.)

$P_{mij} = |P_{ij}|$

$$U_{ij} = \begin{cases} 0, & \text{if } F_{ij} = 0. \\ 1, & \text{otherwise.} \end{cases}$$

$(P_m U)_s$ = Symmetric part of $P_m U$.

∎

Now that the global asymptotic stability of $\underline{A}$ has been guaranteed with a desired margin of stability -b, we shift our attention to the effect of the disturbance v(t) on the closed-loop behavior. The following lemma is needed in the sequel.

Lemma 3.3

Let $\dot{e} = \underline{A}e$ where $\underline{A} = A_c + F(t)$, with $\underline{A}$ and A_c satisfying the conditions of Theorem 3.2. The state transition matrix $X(t,0)$ is then bounded as follows,

$$||X(t,0)|| \le \sigma e^{-\alpha t}, \quad \alpha \equiv -(2ns\varepsilon' + \mu + |a-b|) > 0 \text{ and } \sigma \ge 1 \tag{3.15}$$

where $||X(t,0)|| = (\sum_{i,j} x_{ij}^2)^{1/2}$,

$$\mu \equiv \text{max real part of eigenvalues of } A_c,\ \mu < -a, \tag{3.16}$$

ε' and s defined in (3.13) and (3.14).

∎

In the next theorem the error is bounded, when v(t) is not zero.

Theorem 3.4

Let $\dot{e}(t) = \underline{A}e(t) + Bv(t)$ where $\underline{A}$ is asymptotically stable with a margin -b and $||Bv(e(t))|| \le Lr$ for some constant L when $||e(t)|| \le r$. If $||e(0)|| < r/\sigma$ and $(L\sigma/\alpha) < 1$ there exists a unique solution e(t) of (3.7) and

$$||e(t)|| \le \sigma e^{-\alpha t}||e(0)|| + (L\sigma/\alpha) r(1 - e^{-\alpha t}) \le r \tag{3.17}$$

for all $t \ge 0$.

∎

Lemma 3.5

If in Theorem 3.4 one has $||Bv(e(t))|| \le L||e||$ then under the conditions of that Theorem

$$||e(t)|| \le \sigma e^{-\alpha' t}||e(0)|| \text{ where } \alpha' \equiv 2nL\varepsilon\sigma + \mu. \tag{3.18}$$

Moreover, e(t) is asymptotically stable if $\alpha' > 0$.

∎

Note that $s > -(\mu+b)/(2n\varepsilon' + |a-b|)$ is always guaranteed because of Theorem 3.2. Therefore, $(L\sigma/\alpha) < 1$ is satisfied if $L < -\alpha(2n\varepsilon' + |a-b|)/(\mu+a)$. This gives an upper bound on the magnitude of v(t) for stability. In order to get a bound on v(t) using a norm of the type defined in (3.10) let us note that $||Bv|| \le \sqrt{n}V$ where $V_i \equiv |Bv|_{max}$ and $V \equiv \max V_i$. Next, we will assume that $\sqrt{n}V \le Lr$ and find stability robustness bounds on the differences between the <u>individual</u> true entries of M and N and their calculated ones. These bounds are useful in practical applications.

Theorem 3.6

Let the computed entries of M and N in (3.1) be $\{M_{cij}\}$ and $\{N_{ci}\}$ and the true ones be $\{M_{ij}\}$ and $\{N_i\}$. If a= b, then the error in (2.2) is bounded as in (3.18) if

$$|M_{ij} - M_{cij}| < \frac{1}{sk} \sum_{m=1}^{n} |M_{im}| \tag{3.19}$$

where

$k = \max k_i$, $i = 1,..,2n$

k_i = ith diagonal element of K_1 if $i \le n$
(i - n)th diagonal element of K_2 if $n+1 \le i \le 2n$,

and

$$|N_i - N_{ci}| \le \delta' \sum_{m=1}^{n} |M_{im}| \tag{3.20}$$

where $\delta' \equiv \max|\delta_i|_{max}$.

∎

IV. CONCLUSION

Using a global linearization, a nonlinear system was transformed into a decoupled linear system. A controller was designed for the latter by two methods, pole-placement and LQR theory, and from this was derived the control for the nonlinear system.

Controllers in robotics, for example, are often designed by ad hoc means which amount to omitting certain nonlinear terms and assuming a decoupled dynamical description of the system, but little work has been done on rigorously justifying the simplification, notable exceptions being found in [3,9]. In this paper, the robustness of the closed-loop system was studied using a time-domain Lyapunov approach, and stability robustness bounds were found in terms of meaningful physical parameters. Our approach exploits the <u>structure</u> of the disturbances, and presents bounds on the actual magnitude of the disturbances rather than on their L_∞ norm.

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