10-5-2003

Networked control systems: a sampled-data approach

Chaouki T. Abdallah

Peter E. Hokayem

Recommended Citation

Abdallah, Chaouki T. and Peter E. Hokayem. "Networked control systems: a sampled-data approach." *2003 IEEE International Symposium on Intelligent Control* (2003): 415-419. doi:10.1109/ISIC.2003.1254671.

Proceedings of the 2003 IEEE
International Symposium on Intelligent Control
Houston, Texas • October 5-8, 2003

Networked Control Systems: A Sampled-Data Approach†

Peter F. Hokayem and Chaouki T. Abdallah
EECE Department, MSC01 1100
1 University of New Mexico
Albuquerque, NM 87131-0001, USA
{hokayem, chaouki}@eece.unm.edu

Abstract— **In this paper we present a novel modelling method for networked control systems, motivated from a sampled-data approach. We study sufficient conditions that guarantee exponential stability for the closed-loop system and illustrate our results via a numerical example.**

Index Terms— **Networked Control Systems, sampled-data systems, lifting, exponential stability.**

I. Introduction

Over the past decade, major advancements in the area of communication and computer networks [9] have made it possible for control engineers to include them in feedback systems in order to achieve real-time requirements. This gave rise to a new paradigm in control systems where instantaneous flow of the control signals is no longer sufficient, and the feedback loop is closed through a real-time network. Such control systems that utilize networks to achieve closed loop performance are called *Networked Control Systems* (NCS). Several examples of NCSs are available in automobile industry, teleoperation of robots, and automated manufacturing systems. Including the networks into the design of such systems has made it possible to increase mobility, reduce the cost of dedicated cabling, and render easier and cheaper maintenance.

This paper starts by reviewing some basic trends in the study of stability of networked control systems in Section II. Then we present our new approach for modelling such systems in Section III. In Section IV, we address the issue of stability, of such models, using Lyapunov techniques for discrete-time systems. Finally, we illustrate our results via a numerical example in Section V.

II. Review of Previous Work

In the past decade, several methods of modelling networked control systems have been proposed, and the stability of such models was the main concern of their analysis. In this section we provide an overview of basic approaches and results.

A. Structural Approach

The authors of [7] present an extended structural analysis of networked control systems, using an eigenvalue approach. In their model, the network resides between the sensors that are attached to the plant, and the actuators. The network is modelled as a fixed-rate sampling of the continuous plant. They also present a model plant that provides state estimate, and the error between the actual plant and the model plant is used to augment the state-vector. Then, the analysis is applied to the augmented system in order to obtain necessary conditions for guaranteeing stability of the closed-loop system. They analyze the performance of the system when full state and partial state are available for feedback.

B. Perturbation Approach

In [10], a try-once-discard (TOD) protocol is introduced, where the next node to transmit data on a multi-node network is decided dynamically based on the highest weighted error from the last transmission. The goal is to find a maximum transmission interval that guarantees satisfactory stability performance. The network resides between the plant and controller and introduces the error between successive transmissions. The resulting state-space system is comprised of the plant state-vector, and the error state-vector. The error is considered as a perturbation of the original plant, and methods presented in [5] are utilized to derive conditions for the stability of the closed-loop system.

C. Delay Approach

Nilsson [8] includes the following cases for modeling the effects of introducing the network into the control-loop, rendering an NCS:

- Constant delay
- Random independent delays
- Random delays governed by an underlying Markov chain

Then for each model he solves an LQG optimal control problem, to generate a controller that guarantees stability.

D. Hybrid Systems Approach

Zhang *et. al* [11], [12] utilize results previously derived for the stability of hybrid systems, to find bounds on the delay introduced by the network. In particular, [11] models the network as a constant delay introduced into the full state feedback as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BK\hat{x}(t), \quad t \in [kh+\tau, (k+1)h+\tau] \\ \hat{x}(t^+) &= x(t-\tau), \qquad\quad t \in [kh+\tau, k=0,1,2,\dots] \end{aligned} \tag{1}$$

where h is the sampling period. Then the trajectory of the delayed state vector $x(t-\tau)$ is solved for, in terms of $x(t)$

†This work has been supported by NSF Grant INT-9818312 and NSF Grant 0233205.

and $\hat{x}(t)$. The bound on the delay τ results from imposing Schur stability conditions on the following matrix.

$$H = \begin{pmatrix} e^{Ah} & -E(h)BK \\ e^{A(h-\tau)} & -e^{-A\tau}(E(h) - E(\tau))BK \end{pmatrix} \quad (2)$$

where for a given matrix M, $E(h)M \equiv \int_0^h e^{A(h-\eta)} M d\eta$.

An extensive study has recently appeared in [4] where the NCS has limited data rate available in order to maintain stability. The problem is tackled from different perspectives: Variable-rate sampling, various quantization schemes, distributed control, and switching control with sufficient dwell-time. The main objective is to reduce the amount of data to be transmitted via the network.

III. New Modelling of NCS

As seen in the previous sections, there are several trends in modeling networked control systems. In this section we are going to introduce yet another modelling method and manipulate it to obtain a generalized LTI sampled-data system. The proposed model allows us to avoid the tedious analysis of the effect of the delay introduced by the network. This is achieved through incorporating the delay into the model of the system, and it is sufficient to study the stability of the overall system, without explicitly addressing the actual value and nature of the delay. Before we introduce the new model, we present few assumptions:

I. The controller and actuators are directly attached to the plant, i.e. no transport delay exists between the controller and plant actuators.
II. The sensors are part of the plant model.
III. The network effect is recognized only between the sensors and controller.

Proposition 1: We model the network as a variable-rate ideal sampler (S_{τ_k}), between the plant (G) and the controller (C), and a corresponding zero-order hold (H_{τ_k}), as shown in Figure 1.

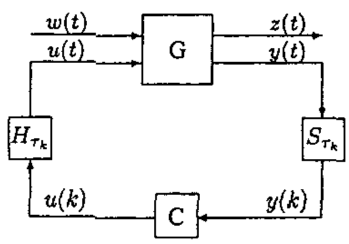

Fig. 1. System Model

Consider the following plant model,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) \end{aligned} \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $w(t) \in \mathbb{R}^l$ is the vector of exogenous inputs, $z(t) \in \mathbb{R}^p$ is the vector of controlled outputs, and $y(t) \in \mathbb{R}^q$ is the vector of measurable outputs. Finally,

$$G = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \quad (4)$$

We assume that $D_{21} = D_{22} = 0$, i.e. the transfer functions from the control input, $u(t)$, and from the exogenous input, $w(t)$, to the measured output, $y(t)$, are strictly proper. The latter condition provides continuity in the measured output vector [1], i.e. avoiding impulses in the output.

The above framework results in a time-varying system, that has both continuous and discrete signals, hence a hybrid system. The study of such systems is in general complex, and a unified theory for such systems is not yet available [6]. For such reasons, we need to manipulate the model in order to obtain a generalized LTI sampled-data system. In order to do so, we employ the *lifting* technique [1], [2], and incorporate the ideal sampler and hold devices into the plant model in the following manner:

$$\begin{aligned} \tilde{G} &= \begin{pmatrix} L_{\tau_k} & 0 \\ 0 & S_{\tau_k} \end{pmatrix} G \begin{pmatrix} L_{\tau_k}^{-1} & 0 \\ 0 & H_{\tau_k} \end{pmatrix} \\ &= \begin{pmatrix} L_{\tau_k} G_{11} L_{\tau_k}^{-1} & L_{\tau_k} G_{12} H_{\tau_k} \\ S_{\tau_k} G_{21} L_{\tau_k}^{-1} & S_{\tau_k} G_{22} H_{\tau_k} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix} \end{aligned} \quad (5)$$

where $\tau_k = t_k - t_{k-1}$ is the variable sampling-rate, L_{τ_k} and $L_{\tau_k}^{-1}$ are the lifting and inverse lifting operators, respectively. The transformed system is shown in Figure 2.

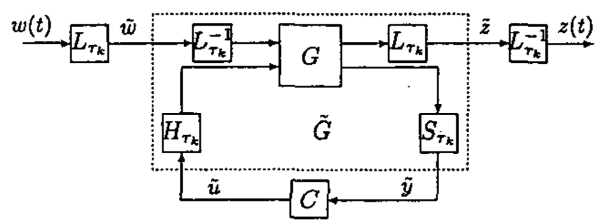

Fig. 2. The Reconfigured NCS

Next we present the above transformations mathematically.

i. $\underline{G_{11} \to \tilde{G}_{11}}$
The transfer function G_{11} relates $w(t)$ to $z(t)$, in continuous time. $\tilde{G}_{11}$ on the other hand relates $\tilde{w}$ to $\tilde{z}$ both being the lifted signals, corresponding to $w(t)$ and $z(t)$. Consequently the linear operators of $\tilde{G}_{11}$ are given as follows:

$$\begin{aligned} \tilde{A} &= e^{A\tau_k} \\ \tilde{B}_1 \tilde{w} &= \int_0^{\tau_k} e^{A(\tau_k - \eta)} B_1 w(\eta) d\eta \\ (\tilde{C}_1 x)(t) &= C_1 e^{At} x \\ (\tilde{D}_{11} \tilde{w})(t) &= D_{11} w(t) + C_1 \int_0^t e^{A(t-\eta)} B_1 w(\eta) d\eta \end{aligned} \quad (6)$$

ii. $\underline{G_{12} \to \tilde{G}_{12}}$
In a similar fashion, we transform B_{12} and D_{12} into B_{12} and D_{12}, respectively. And $\tilde{G}_{12}$ relates the discrete input u_k and the lifted output z_k.

$$\begin{aligned} \tilde{B}_2 &= \int_0^{\tau_k} e^{A\eta} d\eta B_2 \\ (\tilde{D}_{12}\tilde{u})(t) &= D_{12}\tilde{u} + C_1 \int_0^t e^{A\eta} d\eta B_2 \tilde{u} \tilde{C}_2 \end{aligned} \tag{7}$$

iii. $\underline{G_{21} \to \tilde{G}_{21}}$ and $\underline{G_{22} \to \tilde{G}_{22}}$
Both transformations follow from (6) and (7).

After applying the above transformations to (4) we obtain an LTI sampled-data system ($\tilde{G}$), which is shown in Figure 2. Then we refer back to the usual $\mathcal{H}^\infty$ (see [2]) design to obtain the controller (C). Assuming that the controller (C) has been designed, we present stability analysis results of the overall system in the next section.

IV. Stability Analysis

In this section we study the stability of the model presented in the previous section. We shall start by deriving the closed-loop system that involves $\tilde{G}_{22}$ and the controller C. Note that we only need to stabilize $\tilde{G}_{22}$ due to the following theorem.

Theorem 1: [1] The controller $\mathcal{C}$ internally stabilizes the hybrid system in figure 2, if and only if it internally stabilizes the discrete-time system $\tilde{G}_{22}$ in (5). ∎

The plant model of $\tilde{G}_{22}$ is described as follows,

$$\begin{aligned} x_{k+1} &= \tilde{A}x_k + \tilde{B}_2 u_k \\ y_k &= \tilde{C}_2 x_k = C_2 x_k \end{aligned} \tag{8}$$

and the controller C is described by the following state-space realization

$$\begin{aligned} v_{k+1} &= A_c v_k + B_c y_k \\ u_k &= C_c v_k + D_c y_k \end{aligned} \tag{9}$$

Combining (8) and (9) we get the following augmented state space representation

$$\begin{aligned} s_{k+1} &\doteq \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A} + \tilde{B}_2 D_c C_2 & \tilde{B}_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} \\ &\doteq H_k s_k \end{aligned} \tag{10}$$

Notice that the above system does not take into account the effects of disturbances. Consequently, we shall introduce the effects of disturbances, through $w(t)$ in (3), into (10) as follows

$$s_{k+1} = H_k s_k + \begin{pmatrix} \tilde{B}_1 \tilde{w} \\ 0 \end{pmatrix} \doteq H_k s_k + \Gamma_k \tag{11}$$

Before we plunge into the stability analysis, we shall present a general formal definition of *exponential stability* for discrete-time systems.

Definition 1: The origin of the system $x_{k+1} = A_k x_k$ is exponentially stable if there exists an $\alpha > 0$, and for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$, such that

$$||x_k|| \le \epsilon e^{-\alpha(t_k - t_0)} ||x_0|| \tag{12}$$

whenever $||x_0|| < \delta(\epsilon)$ and $t_0 \ge 0$. If $\delta(\epsilon) \to \infty$ then the system is exponentially stable in the large. ∎

The following theorem utilizes results in [3], and specializes them to solve the problem at hand.

Theorem 2: The origin of the closed loop discrete-time system (10) is exponentially stable in the large provided,

i. $\sup_{\forall k \in N} \tau_k < \infty$
ii. $||H_k|| < \frac{1}{\sqrt{2}}, \forall k \in N$

Proof. Given $||H_k|| < a < 1, \forall k \in N$, then there exist a symmetric matrix $P_k > 0$, such that $H_k^T P_k H_k - P_k = -I$. Then $||P_k|| \le ||I|| + ||H_k^T P_K H_k|| \le 1 + a^2 ||P_k|| \Rightarrow 1 \le ||P_k|| \le \frac{1}{1-a^2}$, since $0 < a < 1$.
Let $V(s_k) \doteq s(k)^T P_{k-1} s(k)$, then

$$\begin{aligned} \Delta V &\doteq V(s_{k+1}) - V(s_k) \\ &= s_{k+1}^T P_k s_{k+1} - s_k^T P_{k-1} s_k \\ &= s_k^T (H_k^T P_k H_k - P_k) s_k + s_k^T (P_k - P_{k-1}) s_k \\ &= -s_k^T I s_k + s_k^T (P_k - P_{k-1}) s_k \\ &\le -||s_k||^2 + \left(\frac{a^2}{1-a^2} \right) ||s_k||^2 \\ &= \left(\frac{2a^2 - 1}{1 - a^2} \right) ||s_k||^2 \end{aligned} \tag{13}$$

Since $||P_k - P_{k-1}||_{max} = \frac{1}{1-a^2} - 1 = \frac{a^2}{1-a^2}$. For the system to be stable, ΔV must be less than zero. Therefore, $\left(\frac{2a^2-1}{1-a^2} \right) < 0$ $\Rightarrow a < \frac{1}{\sqrt{2}}$.
The above result guarantees that the system (10) is stable. Still required to prove that it is exponentially stable. Since $V(s_k) \doteq s(k)^T P_{k-1} s(k)$ then

$$||s_k||^2 \le V(s_k) \le \frac{1}{1-a^2} ||s_k||^2 \tag{14}$$

Using (13), $V(s_{k+1}) \le V(s_k) + \left(\frac{2a^2-1}{1-a^2} \right) ||s_k||^2 \le (2a^2 - 2)V(s_k)$. But $||s_0||^2 \le V(s_0) \le \frac{1}{1-a^2} ||s_0||^2$ then

$$V(s_k) \le (2a^2)^k . \left(\frac{1}{1-a^2} \right) ||s_0||^2 \tag{15}$$

Combining (14) and (15) we get,

$$||s_k|| \le \sqrt{\frac{1}{1-a^2}} . (\sqrt{2a^2})^k ||s_0|| \tag{16}$$

Let $\alpha = min\{1, -\ln(\sqrt{2a^2})\}$ and $\epsilon = \sqrt{\frac{1}{1-a^2}}$, the result follows. ∎

In the above analysis we have ignored the effect of the disturbances on the system. So we are going to extend the result of Theorem 2 to compensate for bounded and vanishing, state-bounded disturbances and in what follows.

Theorem 3: (Bounded Disturbance) Given that the origin of the discrete-time system (10) is exponentially stable, and that $||\Gamma_k|| \leq \gamma < +\infty$ (bounded-input), then the system (11) has a bounded-state output.

Proof. The proof is simple through analyzing the time progression of the state-vector.

$x(k+1) = \prod_{i=0}^{k} H_i x(0) + \sum_{j=0}^{k} \left(\prod_{i=j+1}^{k} H_i\right) .\Gamma_j$. Taking the limit of k on both sides: $\lim_{k\to\infty}(x(k+1)) = \lim_{k\to\infty}\left(\prod_{i=0}^{k} H_i.x(0)\right) + \lim_{k\to\infty}\left(\sum_{j=0}^{k}\left(\prod_{i=j+1}^{k} H_i\right).\Gamma_j\right)$

$\Rightarrow ||x(\infty)|| \leq ||\Gamma||.\left(\frac{1}{1-a}\right) = \frac{\gamma}{1-a} < \infty$. Since the first limit tends to zero as $k \to \infty$ and $||H_k|| < a < 1 \Rightarrow$ we take the maximum of $H_k = a$ and form a geometric progression whose answer is $\left(\frac{1}{1-a}\right)$. ■

Theorem 4: (Vanishing Disturbance) The origin of the closed loop discrete-time system (11) is exponentially stable in the large provided,

i. $\sup_{\forall k \in N} \tau_k < \infty$
ii. $||H_k|| < \frac{1}{\sqrt{2}}, \forall k \in N$
iii. $||\Gamma_k|| < ||s_k||, \forall k \in N$

Proof. We follow a similar analysis as in Theorem 2. Let $||\Gamma_k|| < \gamma||s_k||$, where $\gamma > 0$.

$$\begin{aligned}
\Delta V &\doteq V(s_{k+1}) - V(s_k) \\
&= (H_k s_k + \Gamma_k)^T P_k (H_k s_k + \Gamma_k) - s_k^T P_{k-1} s_k \\
&= s_k^T (H_k^T P_k H_k - P_k) s_k + s_k^T (P_k - P_{k-1}) s_k \\
&\quad +2 s_k^T H_k^T P_k \Gamma_k + \Gamma_k^T P_k \Gamma_k \\
&\leq \left(\frac{2a^2-1}{1-a^2}\right)||s_k||^2 \\
&\quad + \left(\frac{2a\gamma}{1-a^2} + \frac{\gamma^2}{1-a^2}\right)||s_k||^2
\end{aligned} \quad (17)$$

And ΔV in (17) is always negative provided that $\gamma < 1$. The rest follows as in Theorem 2. ■

V. Numerical Example

In this section we will consider a numerical example to illustrate the theoretical stability results derived in Section IV, specifically in Theorem 2.

Consider the following scalar continuous-time LTI plant model

$$\begin{aligned}
\dot{x}(t) &= 0.5x(t) + 10u(t) \\
y(t) &= x(t)
\end{aligned} \quad (18)$$

whose discrete version is that described in (8). Consider also the following discrete-time LTI controller C

$$\begin{aligned}
v_{k+1} &= 0.1v_k - 0.5y_k \\
u_k &= -0.5v_k - y_k
\end{aligned} \quad (19)$$

Consequently, the closed-loop system matrix H_k in (10), corresponding to (18) and (19), is given by

$$H_k = \left(\begin{array}{c|c} e^{0.5\tau_k} - 20(e^{0.5\tau_k} - 1) & -10(e^{0.5\tau_k} - 1) \\ \hline -0.5 & 0.1 \end{array}\right) \quad (20)$$

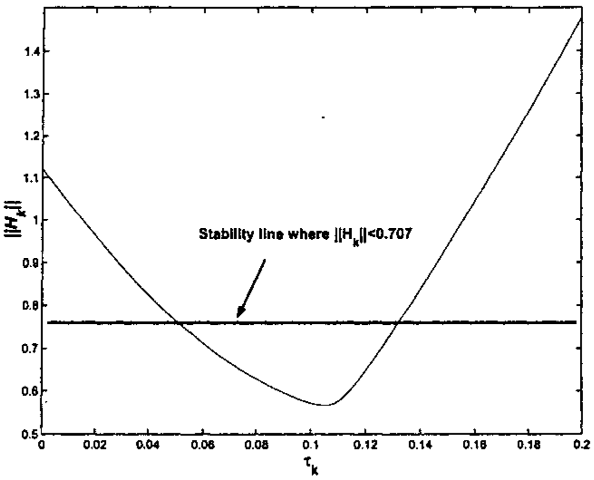

Fig. 3. Range of τ_k that corresponds to $||H_k|| < \frac{1}{\sqrt{2}}$

By Theorem 2, we need to keep the norm of H_k less than $\frac{1}{\sqrt{2}}$. Since we fixed the values for the controller parameters, we can vary τ_k to meet the required condition on H_k. The range of τ_k for which the induced Euclidean norm[1] of H_k is less than $\frac{1}{\sqrt{2}}$ is shown in Figure 3, where

$$0.061 < \tau_k < 0.126. \quad (21)$$

In order to fully understand the implications of varying the sampling time τ_k on the stability of the system, we will first study the behavior of the closed loop system in (20) given a constant τ_k.

The response of the closed-loop system at the boundary of the range given in (21), i.e. $\tau_k = 0.126$, is shown in Figure 4 where the system retains its stability.

We further increase the value to τ_k beyond 0.126 until we hit the first instability point. As seen in Figure 5, the response of the closed loop system diverges for $\tau_k = 0.164$. This conveys the conservativeness of the stability analysis, since the results are sufficient but not necessary.

Finally, we test the system response for a variable sampling time given by

$$\tau_k = 0.126 + \epsilon \times U \quad (22)$$

where U is a uniformly distributed random number between 0 and 1, and $\epsilon \in \mathbb{R}$. This representation of τ_k allows us to see how far can we sample randomly beyond the theoretical bound and still maintain stability. As seen in Figure 6, the system diverges for $\epsilon = 0.076$.

It is interesting to compare the two results presented in Figures 5 and 6. For the random case, the value of τ_k depends on the outcome of the random number U given in (22), whose mean is 0.038 for the simulation in Figure 6. Hence, average$(\tau_k) = 0.126 +$ average$(\epsilon \times U) = 0.164$ which is the

[1]The induced Euclidean norm of any matrix M is given by $\lambda_{max}(M^T M)^{1/2}$, where λ_{max} denotes the maximum eigenvalue.

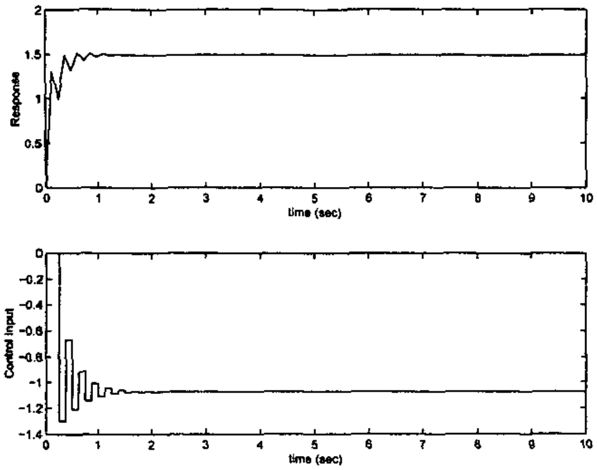

Fig. 4. System response for $\tau_k = 0.126$

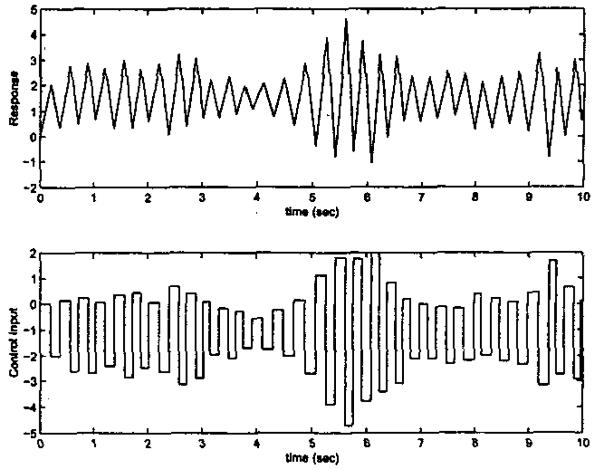

Fig. 6. System response for $\tau_k = 0.126 + \epsilon \times U$

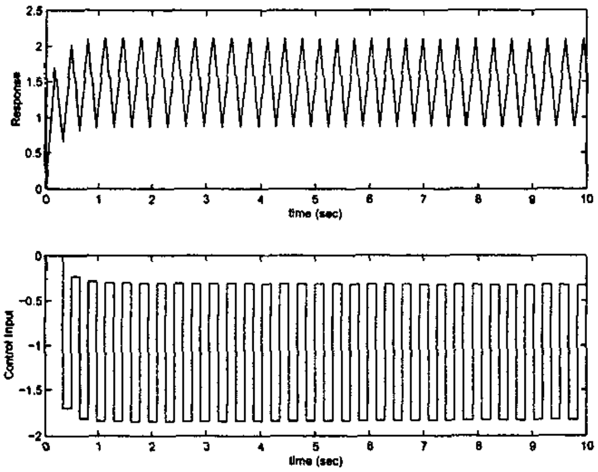

Fig. 5. System response for $\tau_k = 0.164$

same as the fixed τ_k in Figure 5. Consequently, the random τ_k behaves like the fixed one on average.

VI. Conclusion

In this paper we have presented a new method for modelling Networked Control Systems. The main idea is viewing NCS as a variable-rate, sampled-data system. Then, we utilized some results pertaining to the stability of such sampled-data systems and extended them to the problem at hand. The bounds derived for guaranteeing stability are conservative, and further work should aim at developing new bounds that eliminate that kind of conservativeness.

References

[1] Bamieh, B.A. and Pearson, J.B., "A General Framework for Linear Periodic Systems with Applications to $\mathcal{H}^\infty$ Sampled-Data Control," *IEEE Transactions on Automatic Control*, vol. 37, pp. 418-435, April 1992.

[2] Chen, T. and Francis, B., *Optimal Sampled-Data Control Systems*, Springer, London, 1995.

[3] Hu, B. and Michel, A.N., "Stability Analysis of Digital Feedback Control Systems with Time-Varying Sampling Periods," *Automatica*, vol. 36, pp. 890-905, 2000.

[4] Ishii, H. and Francis, B.A., *Limited Data Rate in Control Systems with Networks*, Springer-Verlag, Berlin Heidelberg, Germany, 2002.

[5] Khalil, H., *Nonlinear Systems, 3rd Edition*, Prentice Hall, Upper Saddle River, NJ, 2002.

[6] Malmborg, J., *Analysis and Design of Hybrid Control Systems*, Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden, 1998.

[7] Montestruque, L.A. and Antsaklis, P.J., *Model-Based Networked Control System - Stability*, ISIS Technical Report ISIS-2002-001, Notre Dame, IN, January 2001.

[8] Nilsson, J., *Real Time Control Systems with Delays*, Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden, 1998.

[9] Raji, R.S., "Smart networks for control," *IEEE Spectrum*, pp. 49-55, June 1994.

[10] Walsh, G.C., Ye, H., and Bushnell, L., "Stability Analysis of Networked Control Systems," *Proc. American Control Conference*, pp. 2876-288-, San Diego, CA, June, 1999.

[11] Zhang, W., *Stability Analysis of Networked Control Systems*, Ph.D. dissertation, Department of Electrical Engineering and Computer Science, Case Western Reserve University, Cleveland, OH USA, August 2001.

[12] Zhang, W., Branicky, M.S. and Phillips, S.M., "Stability Analysis of Networked Control Systems," *IEEE Control Systems Magazine*, pp. 84-99, February 2001.