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NEW EXTREME-POINT ROBUST STABILITY
RESULTS FOR DISCRETE-TIME POLYNOMIALS

by
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ABSTRACT
This paper provides some new extreme-point robust-stability results for discrete-time polynomials with special uncertainties in the coefficient space. The proofs, obtained using the barycentric coordinates, are simple and the results specialize to existing robust-stability results.

I. INTRODUCTION
The stability of uncertain polynomials has recently become an active area of research. The problem was elegantly solved in the continuous-time case by the celebrated Kharitonov theorem [1]. To date, such a solution does not exist for discrete-time polynomials, although partial results are available in special cases [2,3,4].

The first objective of this paper is to characterize the set of real-coefficient stable polynomials in z using their barycentric coordinates (BC)[5]. By expressing the polynomial using its BC, and using standard results, we were able to generalize some stability results for discrete-time polynomials. In fact, we can prove some existing discrete-time stability results but more importantly introduce and prove more general results. The main contributions of the paper is to affirm the importance of the barycentric coordinates and to extend the class of uncertainties which may be dealt with in the discrete-time case. We point out that the BC have been discussed by some authors [4,5,6] but were never exploited to their full potential.

In section II, we will formulate the problem, review the literature, and introduce our notation. The barycentric coefficients approach will be given in section III. New stability results are given in section IV, and some examples are presented in section V. Our conclusions and directions for future research are discussed in section VI.

II. DEFINITIONS AND PROBLEM STATEMENT
An uncertain polynomial may be described as a member of a set of possible polynomials. The stability of the uncertain polynomial may then be deduced from the stability of the underlying set. The following standard nomenclature is provided for polynomials in z. The set of all univariate real polynomials in the complex variable z is denoted by $\mathbf{R}[z]$. Each polynomial in $\mathbf{R}[z]$ has the form

$$P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n \qquad (2.1)$$

where n is a nonnegative integer, and each a_i is an element of $\mathbf{R}$. If $a_0 \neq 0$, the degree of $P(z)$, denoted by deg $P(z)$ equals n. The set of all real polynomials of degree n is denoted by $\mathbf{R}_n[z]$.

In the following, we provide a review of literature relevant to our work. The parameter space approach to the analysis and design of systems has been advocated by Siljak [7]. Fam [6] presented a geometric approach to the stability problem using the barycentric coordinates which were also used by Ackerman [5]. Cieslik [8] showed via counterexamples that Kharitonov's test does not extend to to the discrete-time case. Mansour et.al. [3]

have presented different Kharitonov-type results for low order polynomials and have contributed to the study of stable discrete-time polynomials with certain perturbations in the coefficient space. Bartlett, Hollot and Lin [2] in their edge theorem, have shown that the stability of the exposed edges of the polytope Γ is a necessary and sufficient condition for the stability of every member of Γ. Although, this results in a huge saving in complexity, one still has to check the stability of $l 2^{l-1}$ exposed edges. The problem of checking such edges is an important open problem. In some cases however, the stability of the two extremes of an edge is sufficient to guarantee the stability of all polynomials on that particular edge [3,13,14]. These are known as the extreme-point results to which this paper is related. These results allow us to develop computationally reasonable methods in the spirit of Kharitonov's seminal work.

In the next section, we introduce the Barycentric coordinates and discuss some of their properties which will be used later. It will be shown that in the BC space, the stability results are simpler to obtain and test.

III. BARYCENTRIC COORDINATES
In this section, we review the barycentric coordinates concept [5] and discuss the stability of $P(z)$. The condition that a polynomial is located inside the convex hull of the stability region, will be reformulated in terms of its barycentric coordinates. This convex hull is the simplex generated by the corner stable polynomials B_i defined in [4,6]. Consider then the bilinear transformation

$$s = \frac{z-1}{z+1} \; ; \; z = \frac{1+s}{1-s} \qquad (3.1)$$

which maps the unit disk of the z-plane to the left half s-plane. Another interpretation of the the the bilinear transformation, corresponds to expressing the polynomial $P(z)$ using the different basis vectors $B_i = (z+1)^i (z-1)^{n-i}$. Using this interpretation, and substituting z from (3.1) into $P(z)$ we get

$$P(z) = b_0(z+1)^n + b_1(z+1)^{n-1}(z-1) + ... + b_n(z-1)^n$$

$$P(z) = b_0 B_n + b_1 B_{n-1} + ... + b_n B_0$$

which may be converted to

$$P(s) = b_0 s^n + b_1 s^{n-1} + ... + b_n \qquad (3.2)$$

where $b_0, b_1, \cdots, b_n$ are the barycentric coordinates of $P(z)$ [5]. It is easy to deduce that $P(z)$ has all its roots inside the unit disk in the z-plane if and only if $P(s)$ has all its roots in the left-half s-plane. Now, let the vector $\mathbf{b}_n$ be defined as

$$\mathbf{b}_n = [b_0, b_1, ..., b_n]^T \qquad (3.3)$$

then, it was shown in [9] that

$$\mathbf{a}_n = M_n \mathbf{b}_n, \qquad (3.4\text{-}a)$$

$$\mathbf{b}_n = \frac{1}{2^n} M_n \mathbf{a}_n, \qquad (3.4\text{-}b)$$

$$b_0 + b_1 + ... + b_n = \sum_{i=0}^{n} b_i = 1 \qquad (3.4\text{-}c)$$

where M_n is an $(n+1)\text{x}(n+1)$ invertible matrix described by

$$M_n = [m_{ij}] \; ; \; i,j = 0,1,\ldots,n \tag{3.5}$$

Letting the ith row of M_n be given by the row vector $\mathbf{m}_i^r$ and the jth column of M_n be given by the column vector $\mathbf{m}_j^c$, i.e.

$$\mathbf{m}_i^r = [m_{i0} \quad m_{i1} \quad \ldots \quad m_{in}], \tag{3.6-a}$$

$$\mathbf{m}_j^c = [m_{0j} \quad m_{1j} \quad \ldots \quad m_{nj}]^T. \tag{3.6-b}$$

The following recursions are then obtained

$$M_{n+1} = \begin{bmatrix} M_n & \mathbf{m}_n^c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ M_n & -\mathbf{m}_n^c \end{bmatrix}, \tag{3.7-a}$$

with

$$M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}. \tag{3.7-b}$$

The following two lemmas which further describe M_n were given in [12] and will be useful in proving the robust stability results of the following sections.

Lemma 3.1: M_n is an $(n+1)\text{x}(n+1)$ matrix with elements m_{ij} such that for all i,j

$$m_{ij} = (-1)^j m_{n-i,j} = (-1)^i m_{i,n-j}.$$

$\square$

Lemma 3.2: Given M_n with n even, then

$$m_{2k+1,n/2} = 0 \; ; \; k = 0,\ldots,n/2-1$$

$\square$

Lemma 3.3: (The Interlacing Property or Hermite-Bieler Theorem) Let

$$P(s) = b_0 s^n + b_1 s^{n-1} + \cdots b_n = P_e(s) + s P_o(s) \tag{3.8}$$

where $P_e(s)$ and $P_o(s)$ are even-degree polynomials. Then, the polynomial $P(s)$ is stable if and only if $b_1/b_0 > 0$ and the roots of $P_e(s)$ and $P_o(s)$ all lie on the imaginary axis, are different and interlace. Note that the root closest to the origin must be one of $P_e(s)$.

$\square$

The proof of lemma 3.3 can be found in [10]. As a result of the Hermite-Bieler theorem we have the following corollary.

Corollary 3.1: Let

$$P(s,\lambda) = P_0(s) + \lambda P_d(s) \tag{3.9}$$

where $P_d(s)$ is a polynomial with all even or all odd-degree terms and $0 \le \lambda \le \lambda_{max}$. Then, $P(s,\lambda)$ is stable for all $0 \le \lambda \le \lambda_{max}$ if and only if $P(s,0)$ and $P(s,\lambda_{max})$ are both stable.

$\square$

In the next section, we use these results to provide simple proofs of the so-called "Weak Kharitonov Theorem" [3], and to introduce some new stability results.

IV. NEW STABILITY RESULTS

In this section we consider that a particular coefficient a_i varies in the following closed interval

$$a_i^- \le a_i \le a_i^+. \tag{4.1}$$

Let us also define the following extreme polynomials:

$$P_1(z) = a_0 z^n + \cdots + a_i^- z^{n-i} + \cdots + a_{n-i}^- z^i + \cdots + a_n.$$
$$P_2(z) = a_0 z^n + \cdots + a_i^+ z^{n-i} + \cdots + a_{n-i}^+ z^i + \cdots + a_n.$$
$$P_3(z) = a_0 z^n + \cdots + a_i^+ z^{n-i} + \cdots + a_{n-i}^- z^i + \cdots + a_n.$$
$$P_4(z) = a_0 z^n + \cdots + a_i^- z^{n-i} + \cdots + a_{n-i}^+ z^i + \cdots + a_n.$$
$$\tag{4.2}$$

The main problem is to obtain conditions under which the stability of an edge can be easily derived from the stability of the extreme polynomials. This will significantly simplify the procedure of determining the stability of the entire edge and of course, of possible convex combinations of such edges.

We will use the following theorem which states that if the coefficients of $P(z)$ are such that a_i and a_{n-i} vary on a line with slope $\pi/4$ or $3\pi/4$ radians in the parameter space, while the other coefficients remain constant, it is necessary and sufficient to check the stability at the extremes of the line in order to guarantee the stability of the whole family.

Theorem 4.1: Given the family of polynomials $P(z)$ as defined in (2.1), where for some i, $0 \le i \le n$,

$$a_i - a_i^- = a_{n-i} - a_{n-i}^- \tag{4.3-a}$$

Then, the entire family of polynomials is stable if and only if the two extreme polynomials are stable, i.e., if and only if the two corresponding polynomials $P_1(z)$ and $P_2(z)$ are stable. For the case when

$$a_i^+ - a_i = a_{n-i} - a_{n-i}^- \tag{4.3-b}$$

the entire family of polynomials is stable if and only if

$P_3(z)$ and $P_4(z)$ are stable.

Proof: The proof of this theorem may be found in [3,11].

$\square$

It should be noted that Theorem 4.1 contains two important constraints:

i) The coefficients must be coupled as (a_i, a_{n-i}). This means that different types of coupling are not allowed nor is a single coefficient allowed to vary on its own.

ii) The edges allowed in the pairwise variation defined in i) have $\pi/4$ and $3\pi/4$ slopes in the parameter space.

Now we will show that Theorem 4.1 and Theorem 2 in [4] are obtained as particular cases of Theorem 4.2 which allows a larger class of perturbations in the coefficient space. We start by introducing Theorem 4.2 below which partially removes constraint i), by allowing coupling between a_i and a_{n+j-i} where $0 \le j \le i$. In order to prove Theorem 4.2, we introduce the following notation for some extreme polynomials.

$$P_{1j}(z) = P(z,a_i^-,a_{n+j-i}^-)$$
$$P_{2j}(z) = P(z,a_i^+,a_{n+j-i}^+)$$
$$P_{3j}(z) = P(z,a_i^+,a_{n+j-i}^-)$$
$$P_{4j}(z) = P(z,a_i^-,a_{n+j-i}^+) \tag{4.4}$$

Similar definitions of $P_{kj}(s)$; $k = 1, 2, 3, 4$ will also be used. We also let $\{x\}$ be the largest integer less than or equal to x.

Theorem 4.2: Consider the family of polynomials

$$P(z) = P(z,a_i,a_{n+j-i}) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \tag{4.5}$$

where for some i, $0 \le i \le n$ and some $0 \le j \le i$

$$a_i - a_i^- = a_{n+j-i} - a_{n+j-i}^- \tag{4.6-a}$$

Then, the entire family of polynomials is stable if and only if the corresponding corner polynomials $P_{1j}(z)$ and $P_{2j}(z)$ are stable. Similarly, for the case when

$$a_i^+ - a_i = a_{n+j-i} - a_{n+j-i}^- \qquad (4.6\text{-b})$$

the stability of the whole family is equivalent to that of $P_{3j}(z)$ and $P_{4j}(z)$

Proof: See Appendix A.

□

Obviously, if we set $j = 0$ in (4.6), we obtain Theorem 4.1. Also, if we set $j = 2i - n$, $i > \{(n-1)/2\}$, we find that the allowed variation is $a_i^- \le a_i \le a_i^+$. Thus, only the upper-half of the coefficients are allowed to vary. Combining this with the Edge Theorem in [2], we obtain Theorem 2 of [4] which states that independent variations in the upper-half of the coefficients may be treated using a Kharitonov-like test.

The next theorem will relax constraint ii) above by allowing variations in directions other than $\pi/4$ and $3\pi/4$. In fact, variations in the sector $\pi/4 \le \theta \le 3\pi/4$ may be handled.

Theorem 4.3 : Consider the family of polynomials

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad (4.7)$$

where for some i , $0 \le i \le (n+1)/2$ and some β , $0 \le \beta < 1$

$$a_i - a_i^- = \beta\,(a_{n-i} - a_{n-i}^-) \qquad (4.8\text{-a})$$

Then, the entire family of polynomials is stable if and only if the corresponding polynomials $P_1(z)$ and $P_2(z)$ defined in (4.2) are stable. Similarly, if

$$a_i^+ - a_i = \beta\,(a_{n-i} - a_{n-i}^-) \qquad (4.8\text{-b})$$

then, the entire family of polynomials is stable if and only if the corresponding polynomials $P_3(z)$ and $P_4(z)$ defined in (4.2) are stable.

Proof: See Appendix B.

□

Even though the case $\beta = 1$ is not included in Theorem 4.3, it is clear that it is also valid, since that is exactly Theorem 4.1. The set of allowed variations in the parameter space, as it can be inferred from Theorem 4.3 is represented in Fig. 4.1, where (a_i^*, a_{n-i}^*) is the starting point of the line segment and the ending point can be anywhere in the shadowed region.

Corollary 4.1: The family of polynomials

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

where for some i, $0 \le i \le n$, some j, $\max\{2i-n, 0\} \le j \le i$ and some β, $0 \le \beta \le 1$

$$a_i - a_i^- = \beta\,(\,a_{n+j-i} - a_{n+j-i}^-\,)$$

is stable if and only if the corresponding extreme polynomials $P_{1j}(z)$ and $P_{2j}(z)$ defined in (4.4) are stable. Similarly, if

$$a_i^+ - a_i = \beta\,(\,a_{n+j-i} - a_{n+j-i}^-\,)$$

then, the stability of the entire family of polynomials is equivalent to the stability of the corresponding extremes, $P_{3j}(z)$ and $P_{4j}(z)$ as defined in (4.4).

Proof: The proof is obtained by applying Theorems 4.2 and 4.3 to $P(z)$ successively.

□

By combining the previous results, we obtain our most general test.

Theorem 4.4: Consider the polytope in the coefficients space where each pair $(\,a_i, a_k\,)$, $0 \le i \le n$, $n-i \le k \le n$ is varying inside a polytope with edges sloped in the closed interval $[\,\pi/4, 3\pi/4\,]$ and where each a_i can only be combined with one a_k and vice-versa, i.e., the pairwise variations $(\,a_i, a_k\,)$ and $(\,a_i, a_l\,)$, $k \ne l$ are not allowed simultaneously. Then,

every polynomial in the polygon will be stable if and only if all the polynomials obtained by combining all the polygon corners are stable.

Proof: The result is a direct consequence of the *Edge Theorem* [2] and Corollary 4.1.

□

Note that the so-called "Weak-Kharitonov Theorem" [3] is a particular case of Theorem 4.4, where the polygons are "rotated" boxes and the coefficients are combined as in Theorem 4.1. In this sense, the result obtained can be considered as a Weak-Kharitonov for the class of polynomials under consideration.

V. NUMERICAL EXAMPLES.

In this section we will provide counterexamples to the extension of Theorems 4.2 and 4.3 and an example of the utilization of these results.

Example 5.1: Consider the family of polynomials

$$P(z) = a_0 z^4 - 3.5 z^3 - a_2 z^2 - 0.3 z + 1$$

where the couple $(\,a_0, a_2\,)$ is varying in the following way

$$a_0 = -2.7 + \lambda$$
$$a_2 = -2.7 + \lambda$$

with $0 \le \lambda \le 6.6$. Note that this variation would correspond to $i = 0$ and $j = -2$ in (4.6-a). As can be verified, both the extreme polynomials are stable, while for $\lambda = 0.5$, the resulting polynomial is unstable. This proves that the results in Theorem 4.2 can not be extended to any j, $i-n \le j < 0$.

Example 5.2: Consider the family of polynomials

$$P(z) = a_0 z^4 + 1.2 z^3 - 0.025 z^2 + 1.75 z + a_4$$

where the couple $(\,a_0, a_4\,)$ is varying in the following way

$$a_0 = 2.7 - \lambda$$
$$a_4 = 1.98 + \lambda\,/\,1.5$$

with $0 \le \lambda \le 15.4$. Note that this case would correspond to $i = 0$ and $\beta = 1.5$ in (4.8-b). It can be verified that while the extreme polynomials are stable, the polynomial obtained for $\lambda=1$ is not. In fact, the counterexample given in [8] corresponds to $\beta = \infty$, i.e., to a horizontal edge (0 slope) in the parameter space.

Example 5.3: Consider the family of polynomials

$$P(z) = a_0 z^4 + 3.2 z^3 + a_2 z^2 + a_3 z + a_4$$

where the pairs $(\,a_0, a_4\,)$ and $(\,a_2, a_3\,)$ are varying in the parameter space as is shown in Figure 5.1. It can be seen that the 12 extreme polynomials represented by all the possible corner combinations are stable and that the conditions of Theorem 4.4 are accomplished, so it follows that every polynomial belonging to the family is stable.

VI. CONCLUSIONS

In this paper we have reviewed the Barycentric coordinates of a discrete-time polynomial. We then used some recursive relationships, developed in [9,12] to obtain some new stability results. In fact, we showed that for special coefficients variations in the sector bounded by the $\pi/4$ and the $3\pi/4$ sloped lines, the stability of the family of polynomials may be reduced to a check of the stability of the corner polynomials. These results are in the spirit of the extreme-point theorems.

It is obvious by now that a counterpart of Kharitonov's theorem (with necessary and sufficient conditions) does not exist in the discrete polynomial case. We have then attempted to obtain necessary and sufficient conditions for the stability of special polynomials, with simple tests in the spirit of Kharitonov's Theorem. Our results generalize some of the existing ones, allowing a wider class of perturbations in the parameter space.

We are currently applying the tests developed in this paper to the design of static and first-order compensators of single-input-single-output discrete-time systems. The results of this line of research will be reported shortly.
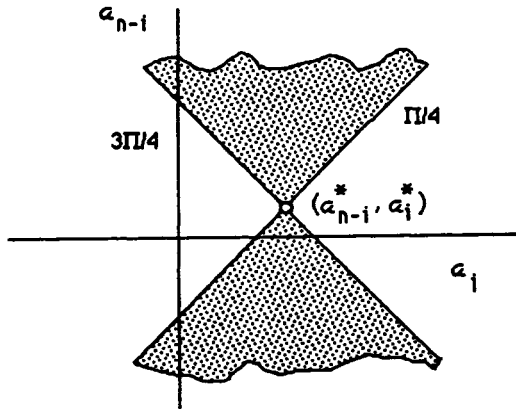
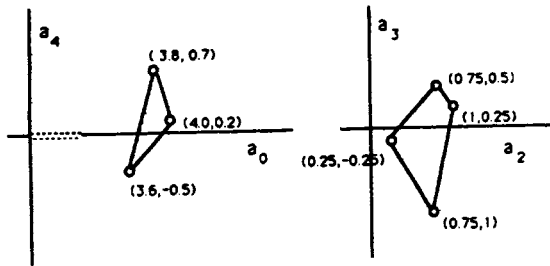Figure 4.1: Allowed slopes in parameter space, after Theorem 4.3.

Figure 5.1: Variation regions for example 5.3

APPENDIX A

Proof of Theorem 4.2: We will provide the proof for case 4.6-a, which can be easily adapted to case 4.6-b. The Schur stability of $P(z)$ does not change if we multiply this polynomial by z^j, $0 \leq j \leq i$. In this case, a new polynomial $P'(z)$ of degree $n+j$ is obtained.

$$P'(z) = d'_0 z^{n+j} + d'_1 z^{n+j-1} + \cdots + d'_{n+j}$$

where:

$$d'_i = a_i \; ; \; i \leq n$$

$$d'_i = 0 \; ; \; n < i \leq n+j.$$

The proof can now be completed by applying Theorem 4.1 to $P'(z)$.

APPENDIX B

Proof of Theorem 4.3: We will derive the result for case 4.8-a, since case 4.8-b is similar. The Schur stability of $P(z)$ is maintained if we multiply it by the term $(z^k + \beta)$, where $0 \leq \beta < 1$, and $k = n-2i$. In this case, a new polynomial

$$P'(z) = (z^k + \beta) P(z)$$

or

$$P'(z) = a_0 z^{n+k} + \cdots + a_i z^{n+k-i} + \cdots + a_{n-i} z^i + \cdots + \beta a_n$$

is obtained, where we have used the fact that $(n+k)/2 = n-i$. Let

$$\lambda = a_i - a_i^- = \beta(a_{n-i} - a_{n-i}^-) \tag{B.1}$$

so that $0 \leq \lambda \leq a_i^+ - a_i^- = \beta(a_{n-i}^+ - a_{n-i}^-)$. If we write

$$P'(z) = d'_0 z^{n+k} + d'_1 z^{n+k-1} + \dots + d'_{n+k},$$

it becomes clear that the only coefficients of $P'(z)$ that are varying with λ are d'_{n+k-i}, $d'_{(n+k)/2}$ and d'_i. Identifying terms and using (B.1), the variations are

$$d'_i = a_i = a_i^- + \lambda = d'_i{}^- + \lambda$$

$$d'_{(n+k)/2} = a_{n-i} + \beta a_i = a_{n-i}^- d'_{(n+k)/2}^- + \left(\frac{\beta^2+1}{\beta}\right)\lambda$$

$$d'_{n+k-i} = \beta a_{n-i} = \beta a_{n-i}^- + \lambda \tag{B.2}$$

If we denote by $P'(s)$ the result of applying the bilinear transformation to $P'(z)$, and discarding the factor $1/2^{n+k}$, we can write

$$P'(s) = \sum_{j=0}^{n+k} \left(\sum_{l=0}^{n+k} m_{jl} d'_l \right) s^{n+k-j} \tag{B.3}$$

Substituting (B.2) into (B.3), we can write

$$P'(s) = P'(s, d'_i{}^-, d'_{(n+k)/2}^-, d'_{n+k-i}^-) + \lambda \sum_{j=0}^{n+k} m_{ji} s^{n+k-j} \frac{\beta^2+1}{\beta} \sum_{j=0}^{n+k} m_{j,(n+k)/2} s^{n+k-j} \tag{B.3}$$

Then, using lemmas 3.1 and 3.2 and noting that the degree of $P'(s)$, $n+k$ is always even, we obtain

$$P'(s) = P'(s, d'_i{}^-, d'_{(n+k)/2}^-, d'_{n+k-i}^-) + \lambda \sum_{r=0}^{(n+k)/2} 2m_{2r,i} s^{n+k-2r}$$

$$+ \lambda \sum_{r=0}^{(n+k)/2} \frac{\beta^2+1}{\beta} m_{2r,(n+k)/2} s^{n+k-2r}$$

$$= P'(s, d'_i{}^-, d'_{(n+k)/2}^-, d'_{n+k-i}^-) + \lambda P_d(s)$$

where $P_d(s)$ is a polynomial with only even powers of s. The proof is then completed by corollary 3.1

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