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# Analytic Gain and Phase Margin Design

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In [5] algorithms are presented for analytic gain and phase margin design. Without special care however, the compensator computed with this algorithm is not a *real* rational function. In [3] it is shown that with some care, a *real* rational compensator for phase margin design can be computed from the theory in [5]. In this paper both gain and phase margin problems are reduced to interpolation problems with *positive-real* functions, which saves a step in the algorithm given in [5], where interpolation is done with *bounded-real* functions, in the case of gain margin design.

## 1 Introduction

In most introductory control textbooks, see for example [4] or [8], the only discussion one finds on robust design is in terms of gain and phase margins, and the only procedures for design are ad-hoc procedures. What is called "analytic design" in these texts is only a one-frequency-point design, which cannot guaranteed closed-loop stability. In practice gain and phase margin designs are also very common. A true "analytic design" procedure should have two elements: 1) an existence result, so that one knows what margins can be achieved for a given plant; and 2) a computable algorithm for a compensator when one is known to exist. In [5] and [6], true analytic procedures are presented for gain and phase margin design. In these references maximum achievable gain and phase margins are computed and interpolation theory is used to compute compensators that realize gain and phase margins that are within the achievable range. In [5] interpolation is done with *Schur* functions (definition follows), and for phase margin design special care is required to guarantee a compensator with *real* coefficients. In this paper we will interpolate with *positive-real* functions, which saves one step in the algorithms of [5]. We also show how to select interpolating functions in the phase-margin case that result in compensators with real coefficients, which, of course, is required for physical realization of the compensator.

This paper is organized as follows. Section 2 contains an outline of the problems and the design procedures. In section 3 we present some illustrative numerical examples and give our conclusions in section 4.

## 2 Outline of The Problems and Main Results

We define first some special functions that will be required in the sequel. In each case the functions in question are assumed to be *rational.* We denote the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $C$. Also $\mathrm{Re}(s)$ denotes the real part of the complex number $s$, $\arg(s)$ denotes the argument of the complex number $s$, and $\|W(s)\|_\infty$ denotes the $\mathcal{H}_\infty$ norm of the function $W(s)$ [5]. Finally we say that a transfer function $T(s)$ is *stable* if it is BIBO stable, i.e. $T(s)$ is proper and analytic in $\mathrm{Re}(s) \geq 0$.

1. A function $W(s)$ is a *strict Schur (SS) function* if it is analytic and $\|W(s)\|_\infty < 1$, for all $s: \ \mathrm{Re}(s) \geq 0$. Note that a SS function may have complex valued coefficients.

2. A function $V(s)$ is a *strictly-bounded-real (SBR) function* if it is a *real* SS function, that is a SS function with only real coefficients.

3. A function $Z(s)$ is a *strictly-positive (SP) function* if it is analytic and $-\pi/2 < \arg(Z(s)) < \pi/2$, or equivalently $\mathrm{Re}(Z(s)) > 0$, for all $s: \ \mathrm{Re}(s) \geq 0$. A function $Z(s)$ is a *strictly-positive-real* function if it a *real* SP function.

4. A function $F(s)$ is an *analytic-positive (AP) function,* [1], if it is analytic and $-\pi < \arg(T(s)) < \pi$ for all $s: \mathrm{Re}(s) \geq 0$.

Our discussion is limited to linear time-invariant single-input-single-output systems with given rational nominal plant transfer function $P(s)$.

The gain margin design problem we will consider can be stated as the problem of finding a proper real rational compensator $C(s)$ such that

$$1 + kC(s)P(s) \neq 0 \text{ for all } s, k : \mathrm{Re}(s) \geq 0, k_1 \leq k \leq \bar{k} \tag{1}$$

where $k_1$ is fixed and we wish to maximize $\bar{k}$. The phase margin problem can be stated as the problem of finding a $C(s)$, as above, such that

$$1 + e^{j\theta}C(s)P(s) \neq 0 \text{ , for all } s, \theta : \mathrm{Re}(s) \geq 0, -\bar{\theta} \leq \theta \leq \bar{\theta} \tag{2}$$

Using the theory in [5], condition (1) can be shown to be equivalent to the condition that the closed-loop transfer function $T_g(s) = \frac{k_1 C(s)P(s)}{1+k_1 C(s)P(s)}$ be BIBO stable and avoid the region

$$\mathbb{E} = \left\{ s \in \mathbb{C} \,\middle|\, s = -\frac{k_1}{k - k_1}, k_1 \leq k \leq \bar{k} \right\}$$

and condition (2 can be shown to be equivalent to the condition that the closed-loop transfer function $T_p(s) = \frac{C(s)P(s)}{1+C(s)P(s)}$ be BIBO stable and avoid the region

$$\mathbb{F} = \left\{ s \in \mathbb{C} \,\middle|\, s = \frac{1}{2} \pm j\frac{\sin(\theta)}{2(1-\cos(\theta))}, 0 \leq \theta \leq \bar{\theta} \right\}$$

Again using the theory in [5] it can be shown that the functions $T_g(s)$ and $T_p(s)$ avoid the regions $\mathbb{E}$ and $\mathbb{F}$, respectively, if the functions $F_g(s)$ and $F_p(s)$, where

$$F_g(s) = \frac{k_1}{\bar{k} - k_1} + T_g(s) \tag{3}$$

and

$$F_p(s) = \frac{1}{2} + \frac{a - j\left(T_p(s) - \frac{1}{2}\right)}{a + j\left(T_p(s) - \frac{1}{2}\right)}, \tag{4}$$

where $a = \dfrac{\sin(\bar{\theta})}{2(1-\cos(\bar{\theta}))} = \dfrac{1}{2}\left(\tan\left(\dfrac{\bar{\theta}}{2}\right)\right)^{-1}$, both avoid the line segment shown in Figure 1, i.e. the line segment consisting of the whole negative real axis. Note that if $F_g(s)$ or $F_p(s)$ are AP functions, the line segment in Figure 1 is indeed avoided, since $-\pi < arg(F) < \pi$ guarantees that $F$ can never assume negative real values. However an AP function can always be written as the square of an SP function, i.e.

$$F_g(s) = Z_g^2(s), \;\; F_p(s) = Z_p^2(s), \tag{5}$$

where $Z_g(s)$ and $Z_p(s)$ are SP functions. Now for internal stability $T_g(s)$ and $T_p(s)$ must satisfy certain interpolation conditions, in particular at the unstable poles

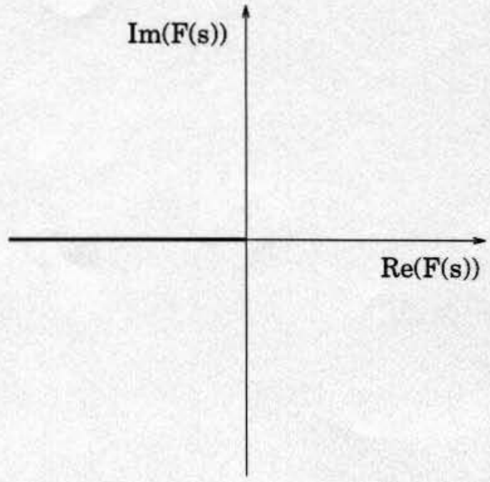


Figure 1: Region to be avoided by $F_g(s)$ and $F_p(s)$ for all $s : \mathrm{Re}(s) \geq 0$.

and zeros of the nominal plant $P(s)$, denoted $a_i$ and $b_i$ respectively (all assumed to have multiplicity one for simplicity), we must have

$$T_g(a_i) = T_p(a_i) = 1, \; i = 1, ..., n$$

and

$$T_g(b_i) = T_p(b_i) = 0, \; i = 1, ..., m$$

where $n$ is the number of unstable poles and $m$ is the number of unstable zeros, including infinity. These interpolation conditions then translate, via the mappings (3), (4), and (5) into

$$Z_g(a_i) = \sqrt{\frac{\bar{k}}{\bar{k} - k_1}}, \; Z_g(b_i) = \sqrt{\frac{k_1}{\bar{k} - k_1}} \tag{6}$$

and

$$Z_p(a_i) = \frac{1}{2} + \frac{a - j}{a + J} = e^{-j\bar{\theta}/2}, \; Z_p(b_i) = \frac{1}{2} + \frac{a + j}{a - j} = e^{j\bar{\theta}/2} \tag{7}$$

Gain and phase margin design is thus reduced to interpolation with an SP function. The maximum values of $\bar{k}$ and $\bar{\theta}$ are fixed by the requirement that $Z_g(s)$ and $Z_p(s)$ be strictly positive functions. In [9] algorithms are given for interpolation with *positive-real* functions which relate directly to interpolation with SP functions. In [5], sections 11.13 and 11.14, the maximum values of $\bar{k}$ and $\bar{\theta}$ are computed from the minimal value of the $H_\infty$ norms of $T_g(s)$ and $T_p(s)$. Once $Z_g(s)$ and $Z_p(s)$ are computed, $T_g(s)$ and $T_p(s)$ may be computed from equations (5), (3), and (4). The respective compensators are then given by

$$C_{g,p} = \frac{T_{g,p}(s)}{P(s)(1 - T_{g,p}(s))} \tag{8}$$

For phase margin design $T_p(s)$ may not be a *real* function since for real unstable poles and zeros the interpolation values, see (7), are not real. However as shown in [3] is is possible to compute a complex $Z_p(s)$ which results in a real $T_p(s)$.

1. **Gain Margin Design:** From the SPR function $Z_g(s)$ which meets the interpolation conditions in (6), one can compute $T_g(s)$ from

$$T_g(s) = Z_g^2(s) - \frac{k_1}{\bar{k} - k_1} \tag{9}$$

Then the compensator $C_g(s)$ may be computed from (8).

2. **Phase Margin Design:** From the SP function $Z_p(s)$ which meets the interpolation conditions in (7), one can compute $T_p(s)$ from

$$T_p(s) = \frac{1}{2} + \frac{a}{j}\frac{1 - Z_p^2(s)}{1 + Z_p^2(s)} \tag{10}$$

Then compensator $C_p(s)$ may be computed from (8). However to insure a compensator with *real* coefficients, one should compute $T_p(s)$, as noted in [3], from

$$T_p(s) = V(s)\frac{1 - \sin\left(\frac{\bar{\theta}}{2}\right)V(s)}{\sin\left(\frac{\bar{\theta}}{2}\right)(1 - V^2(s))} \tag{11}$$

where $V(s)$ is an SBR function which satisfies the following interpolation conditions

$$V(a_i) = |\beta| = \sin\left(\frac{\bar{\theta}}{2}\right) \quad i = 1, \ldots, n,$$
$$V(b_i) = 0 \qquad\qquad i = 1, \ldots, m.$$

where $V(s)$ is related to $Z_p(s)$ from

$$W(s) = \frac{Z_p(s) - e^{j\frac{\bar{\theta}}{2}}}{Z_p(s) + e^{-j\frac{\bar{\theta}}{2}}}, \text{ and } W(s) = -je^{j\frac{\bar{\theta}}{2}}V(s)$$

## 3 Examples

### *3.1 Example 1*

Consider the gain margin design problem of example 1 in reference [7]. The plant is the following

$$P(s) = \frac{(s-1)}{(s+1)(s-2)},$$

here the maximum permissible value of $\bar{k}$ is 4. As in [7] we take $\bar{k} = 3.5$ and $k_1 = 1$. The interpolation conditions in this case are, from (6),

$$Z_g(1) = Z_g(\infty) = 0.63245, \quad Z_g(2) = 1.63245$$

Following the above steps, the computed compensator is

$$C(s) = 1488\frac{(s+1)^2(s+60)}{(s-123)(16.4833s^2 + 723.379s + 1271.1)}$$

Note that the compensator obtained here is of third order, compared to a sixth order compensator reported in [7]. This plant cannot be stabilized with a *stable* compensator since the plant does not satisfy *p.i.p.* (See reference [10]). In this case the compensator has a single unstable pole at $s = 123$. Since the plant has one unstable pole, at $s = 2$ the Nyquist plot for the compensated system must have two counter-clockwise encirclements. The Nyquist plot of the loop-gain transfer function is shown in figure 2 and there, the necessary encirclements may be noted. As in [7], this compensator is very fragile with respect to *decreasing* gain margin. With a different choice of $k_1$, this problem can be avoided. However the poor phase margin that results from this gain-margin design (See the stretched-out shape of the Nyquist plot) is not avoidable, it illustrates the problem one has when optimizing with respect to a single performance index.

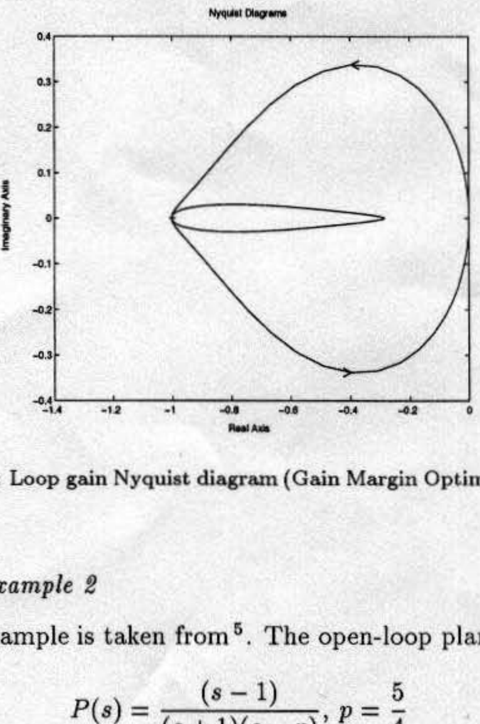


Figure 2: Loop gain Nyquist diagram (Gain Margin Optimization)

### *3.2 Example 2*

This example is taken from [5]. The open-loop plant is

$$P(s) = \frac{(s-1)}{(s+1)(s-p)}, \; p = \frac{5}{4}$$

¿From the theory in [5] we find that the maximum possible phase margin is 12.7587°. We select the guaranteed phase margin to be $\bar{\theta} = 10° = \frac{\pi}{18}$. This plant has a simple zero at infinity, hence a first-order roll-off term of the form

$$\frac{1}{\tau s + 1}$$

is required, with $\tau$ chosen small enough so that the $\mathcal{H}_\infty$-norm $W(s)$ remains less than 1. In this case $W(s)$ is exactly

$$W(s) = -1440je^{j\pi/36}\sin\left(\frac{\pi}{36}\right)\frac{(s-1)}{(s+1)(4s+155)}, \left(\tau = \frac{4}{155}\right)$$

resulting in the controller

$$C(s) = 360\frac{4s^2 + 148.06158s + 165.93841}{4s^2 - 1121s - 49445}.$$

For this plant the *p.i.p.* condition is not satisfied, so that an unstable controller is expected. In particular, the controller designed above has one unstable pole, so that for closed-loop stability the Nyquist diagram should encircle the $-1$ point twice (one unstable pole in the plant and one unstable pole in the controller). The Nyquist plot shown in Figure 3 has the correct number of encirclements. The Bode plots of the loop gain are shown in Figure 4 and the computed increasing-gain and phase margins are GM $= 0.3972\,\text{dB}$, $\tilde{\theta} = 10.23° \geq \bar{\theta}$. Note that in order to meet a near-optimal phase margin, the Nyquist diagram is distorted in such a way that a very small gain margin results. This implies a very fragile/non-robust controller with respect to gain perturbations, and again illustrates the robustness and fragility problems that typically result when a single optimization criterion is used for design.

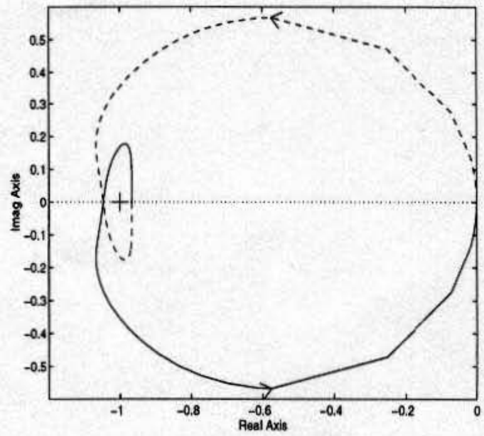


Figure 3: Loop gain Nyquist diagram (Example 2)

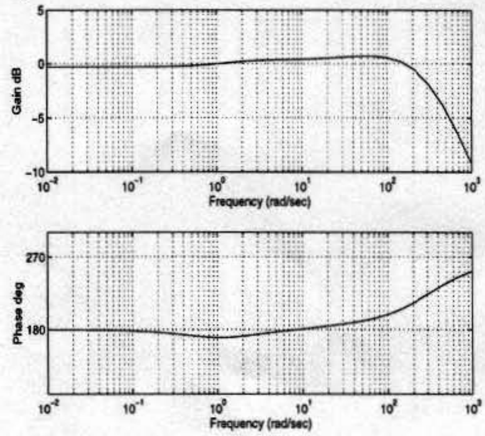


Figure 4: Loop gain Bode diagram (Example 2)

## 4 Conclusions

The results in this paper should be useful in organizing procedures for analytic gain and phase margin design. This should be of interest to practicing engineers. Of special interest is an understanding of the achievable margins for given plants. The results presented here indicate directly how *real* compensators can be synthesized that satisfy feasible phase margins. Hopefully, the theory of analytic gain and phase margin design presented in[5], and expanded upon here, will appear in future introductory textbooks.

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