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Finite Time Stability Design via Feedback Linearization

S. Onori, P. Dorato, S. Galeani and C.T. Abdallah

Abstract— **A new nonlinear design technique for Finite-Time Stability for a class of nonlinear systems is developed using feedback linearization. Moreover, a new concept, namely the Finite-Time Contractive Stability with fixed settling time is introduced, giving sufficient conditions for analysis and design. An example illustrates the theoretical results.**

I. Introduction

It is frequently important to determine conditions under which trajectories of a given dynamical system are bounded within a specified region in the state space when the system is operating over a specified finite interval of time.

In fact, in practical cases, such as rockets, airplanes, and space vehicles maneuvering applications, we are only interested in the behavior of the system over a finite time interval, *i.e.* it is required to know its "stability" characteristics not for the time interval $t \geq t_0$, but for some finite interval of time $t_0 \leq t \leq t_0 + T$, that corresponds to the maneuvering time. In other cases, the system under study may exist only for a finite time interval

In order to deal with such situations, Finite-Time Stability (FTS) is a "stability" notion much more natural than the usual Lyapunov stability. We say that a system is Finite-Time stable (FTS) with respect to (w.r.t.) (α, β, T), with $\beta > \alpha$, if starting within α, the norm of the state stays within a β for a time interval of length T. It is important to point out that FTS and Lyapunov stability are independent concepts. A system can be FTS without being Lyapunov stable and vice versa. In addition to the fact that for FTS the state bounds are given a priori, it is important to stress that Lyapunov stability is related to the *local behavior* of a system *around a given motion*, whereas FTS deals with the behavior of a system in a certain region of its state space.

However, while several design results for FTS of linear systems are available, (e.g. [1], [2], [4], [5]), there is a lack of corresponding results for nonlinear systems. A pioneering result appeared in [8], based upon the analysis results developed in [11], in which a set of Multivariate Polynomial Inequalities (MPIs) [6] has to be solved. For many years, further progress was hindered by the lack of effective tools for nonlinear control design; however, the rise of new applications for FTS and the impressive progress of nonlinear control in the 1980's and 1990's set the stage for renewed interest and new results in the field of finite time stabilization. The idea of developing suitably modified FTS versions of nonlinear control designs originally created for Lyapunov stabilization was used in a recent contribution [7], where a constructive FTS design (based on backstepping and the results in [8]) is proposed for nonlinear systems in a "quasi-strict-feedback" form.

The main goal of this paper is to provide a FTS design technique for a class of nonlinear systems affine in the control that are "quasi-feedback linearizable". The approach used consists in splitting the problem in two easier problems: $1)$ transform the nonlinear system into a linear system by a suitable change of coordinates and preliminary feedback; $2)$ solve the FTS design problem by exploiting available results for linear systems. As clarified in the paper, due to the regional characteristics of FTS, the combination of these two steps must be done in a suitable way and with due precaution.

Both the results in [7] and those in the present paper involve coordinate transformations. While coordinate transformations are not a problem when dealing with either local or global stability properties, they must be dealt with special care when the regional stability property considered in FTS is examined. In this respect, there is a substantial difference in the approach taken in this paper with respect to the one taken in [7]. In [7], the difficulties resulting from the coordinate transformation are dealt with by leaving the parameter α in the definition of FTS initially free, and trying to maximize its value during the recursive steps of the design. In the present paper, a more rigorous approach is followed by sticking to the definition of FTS and giving sufficient conditions for the existence of a control law achieving the desired FTS levels exactly.

Furthermore, a novel concept is introduced, i.e. Finite-Time Contractive Stability with fixed settling time (FTC-Swfst). Beyond requiring the norm of the state of the system to be within a certain specified bound for all intervals of time of interest $[0, T]$, and after starting within the initial bound, we also require the state to be within a specified bound, that is smaller than the initial one, at some fixed time $\tau_s < T$. Sufficient conditions for the solvability of the FTCSwfst problem via feedback linearization are also given.

Summarizing, the contribution of the paper is threefold: first, a FT stabilization design based on feedback linearization is proposed, along with a formal result stating sufficient conditions under which it can be successfully applied; then, the FTCSwfst is formally introduced; finally, sufficient

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conditions and a feedback linearization based design for the FTCSwfst problem are provided. Theoretical results are substantiated by an example.

II. Notation and Preliminary Results

Given positive $\alpha, \beta \in \mathbb{R}$, the following notation will be used:

$$\begin{aligned}\mathcal{B}^\beta &= \{\mathbf{x} : \|\mathbf{x}\| < \beta\}, \\ \mathcal{R}_\alpha^\beta &= \{\mathbf{x} : \alpha < \|\mathbf{x}\| < \beta\},\end{aligned}$$

where $\|\mathbf{w}\|$ is the Euclidean norm of the vector $\mathbf{w}$. Since different coordinates $\mathbf{x}$ and $\mathbf{z}$ will be used, when needed for clarity an additional subscript z will be used when referring to the above sets in the $\mathbf{z}$ coordinates, e.g. $\mathcal{R}_{z,\alpha}^\beta = \{\mathbf{z} : \alpha < \|\mathbf{z}\| < \beta\}$. Given a set $V \subset \mathbb{R}^m$, its boundary is denoted by ∂V and its closure by $\bar{V}$.

Consider the class of nonlinear systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state, and $\mathbf{x}' = [x_1, \ldots, x_n]$, with $\mathbf{x}'$ denoting the transpose of $\mathbf{x}$. The vector fields $\mathbf{f}, \mathbf{g}$ are smooth on $\mathcal{R}_\alpha^\beta$ and such that

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \ldots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}, \ \mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1(x_1, \ldots, x_n) \\ g_2(x_1, \ldots, x_n) \\ \ldots \\ g_n(x_1, \ldots, x_n) \end{pmatrix}$$

and u is a scalar control signal.

For such a class of nonlinear systems we would like to solve the nonlinear Finite-Time stabilization problem, i.e. to find a state feedback control law $u = a(\mathbf{x})$ such that the closed loop system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})a(\mathbf{x})$ is FTS w.r.t. (α, β, T), i.e. given α, β, T, with $\beta > \alpha$, if

$$\mathbf{x}'(0)\mathbf{x}(0) < \alpha^2 \Rightarrow \mathbf{x}'(t)\mathbf{x}(t) < \beta^2, \quad \forall\, t \in [0, T].$$

For two vector fields $\mathbf{f}$ and $\mathbf{g}$, the Lie bracket $[\mathbf{f}, \mathbf{g}]$ is a third vector field defined by $[\mathbf{f}, \mathbf{g}](\mathbf{x}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})$ that in compact notation is expressed as $\mathbf{ad_f}\mathbf{g}(\mathbf{x}) = [\mathbf{f}, \mathbf{g}](\mathbf{x})$, and $\mathbf{ad_f}^k\mathbf{g}(\mathbf{x}) = [\mathbf{f}, \mathbf{ad_f}^{k-1}\mathbf{g}](\mathbf{x})$ for $k \geq 1$. Moreover, the derivative of a smooth real valued function $h(x)$ along the vector field $\mathbf{f}(\mathbf{x})$ is expressed by the Lie derivative $L_f h(\mathbf{x}) = \frac{\partial h}{\partial x}\mathbf{f}(\mathbf{x})$.

For the nonlinear system (1) we consider a nonlinear change of coordinates described in the form

$$\mathbf{z} = \Phi(\mathbf{x}) \quad (2)$$

where $\mathbf{z}' = [z_1, \ldots, z_n]$, and $\Phi(\mathbf{x})$ is such that:

(i) $\Phi(\mathbf{x})$ is invertible on $\mathcal{R}_\alpha^\beta$;

(ii) $\Phi(\mathbf{x})$ and $\Phi^{-1}(\mathbf{z})$ are smooth mappings (i.e. have continuous partial derivatives of any order) on $\mathcal{R}_\alpha^\beta$ and $\Phi(\mathcal{R}_\alpha^\beta) := \{\mathbf{z} : \mathbf{z} = \mathbf{\Phi}(\mathbf{x}), \mathbf{x} \in \mathcal{R}_\alpha^\beta\}$, respectively.

We call a transformation Φ with the properties (i) and (ii) an *annulus diffeomorphism* on $\mathcal{R}_\alpha^\beta$.

The solvability of the nonlinear FT stabilization problem is now addressed by splitting it into two subproblems: first studying under which conditions a nonlinear system of the form (1) is feedback equivalent to a linear system, and then applying the results available in the literature to solve the FT stabilization problem for linear system.

A. State Space Exact Linearization Problem

Necessary and sufficient conditions for the solvability of the State Space Exact Linearization Problem (SSELP), i.e. conditions under which a nonlinear system is equivalent (under coordinate transformation and state feedback) to a linear one, can be found e.g. in [9]; they are reported in the following theorem in a form in which we ask for such conditions to hold in the region of interest, i.e. in the annulus ring $\mathcal{R}_\alpha^\beta$.

Theorem 1: Given the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \quad (3)$$

the SSELP is solvable in $\mathcal{R}_\alpha^\beta$ if and only if the following conditions are satisfied

(i) the matrix $G(\mathbf{x}) = [\mathbf{g}(\mathbf{x}), \mathbf{ad_f}\mathbf{g}(\mathbf{x}), \ldots, \mathbf{ad_f}^{n-1}\mathbf{g}(\mathbf{x})]$ has rank $n\ \forall\, \mathbf{x} \in \mathcal{R}_\alpha^\beta$;

(ii) the distribution $\mathcal{D} = span\{\mathbf{g}, \mathbf{ad_f}\mathbf{g}, \ldots, \mathbf{ad_f}^{n-2}\mathbf{g}\}$ is involutive in a neighborhood of every $\mathbf{x} \in \mathcal{R}_\alpha^\beta$.

Notice that the above conditions are equivalent to the existence of an "output" function $\lambda(\mathbf{x})$ for which the system has relative degree n on the considered annuls ring. If the SSEPL is solvable, there exists a diffeomorphism $\mathbf{z} = \Phi(\mathbf{x})$ such that (1) is feedback linearizable, *i.e.* in the new coordinates $z_i = \phi_i(\mathbf{x}) = L_f^{i-1}\lambda(\mathbf{x})$, $i = 1, \ldots, n$, the system will be described by equations of the form

$$\begin{aligned}\dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= b(\mathbf{z}) + a(\mathbf{z})u\end{aligned}$$

with $[z_1, \ldots, z_n]'$ the new state space vector. If the following state feedback control law is chosen

$$u = \frac{1}{a(\mathbf{z})}(-b(\mathbf{z}) + v) \quad (4)$$

the resulting system, governed by the equation $\dot{\mathbf{z}} = A_0\mathbf{z} + B_0 v$ with

$$A_0 := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_0 := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (5)$$

is linear and controllable, and it is said to be in the *normal form*. So, at this point one can easily find a vector K such that the feedback control $v = K\mathbf{z}$ makes the closed loop linear system FTS; hence under the nonlinear control law $u = \frac{1}{a(\Phi(\mathbf{x}))}(-b(\Phi(\mathbf{x})) + K\Phi(\mathbf{x}))$ the closed loop nonlinear system is FTS, as wanted.

Thus any nonlinear system of the form (1) with relative degree n at any point $\mathbf{x}_0 \in \mathcal{R}_\alpha^\beta$ can be transformed into a system which, in a neighborhood of the point $\mathbf{z}_0 = \Phi(\mathbf{x}_0)$, is linear and controllable.

B. FTS results for linear systems

Consider the linear time-invariant (LTI) system described by the state equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{6}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state, $\mathbf{u} \in \mathbb{R}^m$ is the system input, and the matrices A, B have dimension $n \times n$ and $n \times m$ respectively. The linear FT stabilization problem consists in finding a state feedback control law $u = K\mathbf{x}$ such that the closed loop system

$$\dot{\mathbf{x}} = (A + BK)\mathbf{x} \tag{7}$$

is FTS w.r.t. (α, β, T).

The results in [5], [2] use the Gronwall-Bellman inequality, [10], for dealing with FT stabilization of linear systems, to force the FTS bounds on the state. The main result in [2], reported next, yields the one in [5] by choosing $P = I$ in the Lyapunov-like function.

Theorem 2: [2] The linear system (7) is FTS w.r.t. (α, β, T) if there exists a positive scalar δ, a symmetric positive matrix P, and a matrix K such that

$$A'P + PA + K'B'P + PBK - \delta P < 0 \tag{8a}$$

$$cond(P) < \frac{\beta^2}{\alpha^2} e^{-\delta T} \tag{8b}$$

where $cond(P) = \frac{\lambda_{max}(P)}{\lambda_{min}(P)}$ is the condition number of P.

Although (8a) and (8b) are not LMIs, their feasibility can be efficiently checked using a LMI solver and a bisection search over δ for $\delta \in (0, \frac{1}{T} \ln \frac{\beta^2}{\alpha^2})$, after rewriting (8) as

$$AX + XA' + Y'B' + BY - \delta X < 0 \tag{9a}$$

$$\frac{\alpha^2}{\beta^2 e^{-\delta T}} I < X < I \tag{9b}$$

where $X := \rho P^{-1}$, $Y := K\rho P^{-1}$ and $\rho := \lambda_{\min}(P) > 0$, and noticing that (9) are LMIs when δ is fixed.

III. FTS VIA FEEDBACK LINEARIZATION

The results on the SSELP and on the linear FT stabilization problem presented so far, represent the main ingredients used to deal with the nonlinear FT stabilization problem whose solution is discussed in this section.

Theorem 3: Consider the nonlinear system (1) and the triplet (α, β, T).
If the following conditions are satisfied:

- $\text{rank}([\mathbf{g}(\mathbf{x}), \mathbf{ad_f}\mathbf{g}(\mathbf{x}), \ldots, \mathbf{ad_f}^{n-1}\mathbf{g}(\mathbf{x})]) = n, \ \forall \mathbf{x} \in \mathcal{R}_\alpha^\beta$;
- the distribution $\mathcal{D} = span\{\mathbf{g}, \mathbf{ad_f}\mathbf{g}, \ldots, \mathbf{ad_f}^{n-2}\mathbf{g}\}$ is involutive in a neighborhood of every $\mathbf{x} \in \mathcal{R}_\alpha^\beta$;
- the inequality $0 < \bar{\alpha} < \bar{\beta}$ holds, where

$$\bar{\alpha} := \max_{\|\mathbf{x}\|=\alpha} \Phi(\mathbf{x}), \qquad \bar{\beta} := \min_{\|\mathbf{x}\|=\beta} \Phi(\mathbf{x}), \tag{10}$$

  and $\Phi(\cdot)$ is the coordinate transformation used for feedback linearization;
- there exist a positive scalar $\bar{\delta}$, a symmetric positive matrix P, and a vector K such that

$$A_0'P + PA_0 + K'B_0'P + PB_0K - \bar{\delta}P < 0 \tag{11a}$$

$$cond(P) < \frac{\bar{\beta}^2}{\bar{\alpha}^2} e^{-\bar{\delta}T} \tag{11b}$$

then (1) can be made FTS w.r.t. (α, β, T) by using the control law

$$u = \frac{1}{L_g L_f^{n-1}\lambda(\mathbf{x})}(-L_f^n\lambda(\mathbf{x}) + K\Phi(\mathbf{x})) \tag{12}$$

Proof: The first two hypotheses guarantee the existence of a coordinate transformation $\Phi(\mathbf{x}) = \mathbf{z}$ mapping $\mathcal{R}_\alpha^\beta$ into $\Phi(\mathcal{R}_\alpha^\beta)$ and ensuring that in such regions (that are the only ones we need to worry about) the nonlinear system is feedback equivalent to a linear system in normal form. After a linear system is obtained, we need to see how the original FTS levels, α and β, are mapped under Φ and then choose new FTS levels $\bar{\alpha}$ and $\bar{\beta}$ to be imposed in the linear FT stabilization problem. Since the compact set $\partial\mathcal{B}^\beta$ under the continuous transformation Φ will result in the compact set $\Phi(\partial\mathcal{B}^\beta)$, by Weierstrass theorem the continuity of the norm function guarantees that $\min_{\|\mathbf{x}\|=\beta}\Phi(x)$ exists and is finite; a similar reasoning holds for proving existence and finiteness of $\max_{\|\mathbf{x}\|=\alpha}\Phi(x)$. The existence of $\min_{\|\mathbf{x}\|=\beta}\Phi(x)$ implies the existence of $\bar{\beta} > 0$ such that $\mathcal{B}_z^{\bar{\beta}} \subseteq \Phi(\mathcal{B}^\beta)$. As far as α is concerned, choosing the one-to-one mapping Φ such that $\Phi(0) = 0$ assures that $\bar{\alpha} = \max_{\|\mathbf{x}\|=\alpha}\Phi(x)$ will be strictly positive. So, using $\bar{\alpha}$ and $\bar{\beta}$ as new FTS levels for a linear FT stabilization problem, as long as $\bar{\beta} > \bar{\alpha}$ and the last hypothesis holds, a controller $v = Kz$ can be designed to guarantee FTS w.r.t. (α, β, T) for the linear system; then by the choice done for $\bar{\alpha}$ and $\bar{\beta}$, (16) guarantees, in a conservative way, the FTS w.r.t. (α, β, T) for the nonlinear system. ■

IV. FINITE-TIME CONTRACTIVE STABILITY WITH FIXED SETTLING TIME

In this section we will introduce a new concept, namely the Finite-Time Contractive Stability with fixed settling time for systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{13}$$

defined over a finite interval of time $\mathcal{I} = [t_0, t_0 + T]$, with $\mathbf{x} \in \mathbb{R}^n$ the system state, and $\mathbf{f}$ assumed to be smooth enough in $\mathbf{x}$ and t, over $\mathcal{R}$ and $\mathcal{I}$, to assure the existence and uniqueness of solutions over $\mathcal{R}$ and $\mathcal{I}$ as well as the continuous dependence of the solutions on initial conditions at t_0.

Definition 1: System (13) is Finite-Time Contractively Stable with fixed settling time (FTCSwfst) w.r.t. $(\alpha, \beta, \gamma, \tau_s, T)$, with $\gamma < \alpha \leq \beta$, if $\mathbf{x}'(t_0)\mathbf{x}(t_0) < \alpha^2$ implies

$$\mathbf{x}'(t)\mathbf{x}(t) < \beta^2 \qquad \forall\, t \in [t_0,\, t_0 + T]$$
$$\mathbf{x}'(t)\mathbf{x}(t) < \gamma^2 \qquad \forall\, t \in [t_0 + \tau_s,\, t_0 + T]$$

The idea is to have a contraction of the norm of the state of the system, within the bound γ, for all instants of time in the interval $[t_0 + \tau_s,\, t_0 + T]$. Moreover, as for FTS, there is no interest on what happens after the time $t_0 + T$, as shown in Fig. 1.

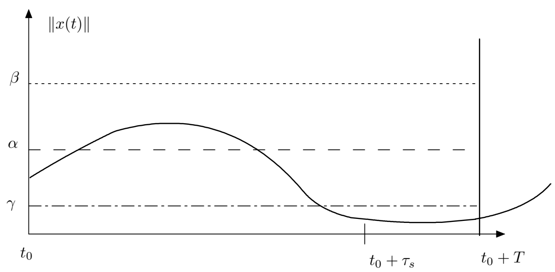

Fig. 1. Finite-Time Contractive Stability behaviour.

The concept of Finite-Time Contractive Stability was first introduced in [11], but without fixing a priori the time of the state staying below the pre-specified bound $\gamma < \alpha$.

The sufficient conditions we give next, are an extension of the ones in [11] and allow to fix the time of the state contraction.

The following notation will be used: $V : \mathcal{R} \times \mathcal{I} \to \mathbb{R}$,

$$V_M^{\alpha}(t) = \max_{\|\mathbf{x}\|=\alpha} V(t,\mathbf{x}), \qquad V_m^{\beta}(t) = \min_{\|\mathbf{x}\|=\beta} V(t,\mathbf{x})$$

$$\dot{V}(t,\mathbf{x}) = \frac{\partial V}{\partial t} + \left(\frac{\partial V(\mathbf{x})}{\partial x}\right)' \dot{\mathbf{x}},\ V_{M_0} = \max_{\mathbf{x}\in(\bar{\mathcal{B}}^{\alpha}-\mathcal{B}^{\gamma})} V(t_0,\mathbf{x})$$

Theorem 4: System (13) is FTCSwfst w.r.t. $(\alpha,\beta,\gamma,\tau_s,T)$, with $\gamma < \alpha \leq \beta$, and $0 < \tau_s \leq T$, if there exists a real-valued function $V(t,\mathbf{x})$ and two functions $\psi_1(t)$ and $\psi_2(t)$ which are integrable over $\mathcal{I}$ such that

(i) $\dot{V}(t,\mathbf{x}) < \psi_1(t), \quad t \in \mathcal{I}, \quad \forall\, \mathbf{x} \in (\bar{\mathcal{B}}^{\beta} - \mathcal{B}^{\alpha})$
(ii) $\dot{V}(t,\mathbf{x}) < \psi_2(t), \quad t \in \mathcal{I}, \quad \forall\, \mathbf{x} \in (\bar{\mathcal{B}}^{\beta} - \mathcal{B}^{\gamma})$
(iii) $\int_{t_a}^{t_b} \psi_1(\tau)d\tau \leq V_m^{\beta}(t_b) - V_M^{\alpha}(t_a) \quad \forall\, t_a, t_b \in \mathcal{I},\, t_b > t_a$
(iv) $\int_{\tau}^{t_0+T} \psi_2(\rho)d\rho \leq V_m^{\gamma}(t_0+T) - V_M^{\gamma}(\tau), \quad \forall\, \tau \in \mathcal{I}$
(v) $\int_{t_0}^{t_0+\tau_s} \psi_2(\tau)d\tau < V_m^{\gamma}(t_0+\tau_s) - V_{M_0}$,
(vi) $V(t_0+\tau_s, \mathbf{x}(t_0+\tau_s)) > V_m^{\gamma}(t_0+\tau_s), \quad \forall\, \mathbf{x} \in (\bar{\mathcal{B}}^{\beta} - \mathcal{B}^{\gamma})$
(vii) $V(t_0+T, \mathbf{x}(t_0+T)) > V_m^{\gamma}(t_0+T), \quad \forall\, \mathbf{x} \in (\bar{\mathcal{B}}^{\beta} - \mathcal{B}^{\gamma})$

Proof: By (i) and (iii), the system is FTS w.r.t. (α, β, T) [11]. Following [11], we show that there exists a time $t_1 < \tau_s$ for which $\|\mathbf{x}(t_0 + t_1)\| < \gamma$ and then we show that there is containment, *i.e.* the state keeps staying in the region $\mathcal{B}^{\bar{\gamma}}$ for all $t \in [t_0 + t_1, t_0 + T]$.

Thus, consider an arbitrary trajectory $\mathbf{x}(t)$ of (13), such that $\|\mathbf{x}(t_0)\| < \alpha$ and suppose, by contradiction, $\|\mathbf{x}(t)\| > \gamma$ for all $t \leq t_0 + \tau_s$. Then,

$$\begin{aligned} V(t,\mathbf{x}(t)) &= V(t_0,\mathbf{x}(t_0)) + \int_{t_0}^{t} \dot{V}(\tau,\mathbf{x}(\tau))d\tau \\ &< V_{M_0} + \int_0^t \psi_2(\tau)d\tau \end{aligned}$$

Hence, from hypothesis (5), at $t = t_0 + \tau_s$

$$\begin{aligned} V(t_0+\tau_s, \mathbf{x}(t_0+\tau_s)) &\leq V_{M_0} + \int_{t_0}^{t_0+\tau_s} \psi_2(\tau)d\tau \\ &< V_{M_0} + V_m^{\gamma}(t_0+\tau_s) - V_{M_0} = V_m^{\gamma}(t_0+\tau_s) \end{aligned}$$

But, by hypothesis (vi) this is a contradiction: hence there exists $t_1 < t_0 + \tau_s$ for which $\|\mathbf{x}(t_1)\| < \gamma$, *i.e.* the state is forced to undergo the contraction at $t_1 < t_0 + \tau_s$.

In order to prove that there is also containment, *i.e.* $\|\mathbf{x}(t)\| < \gamma$ for $t \in [t_0 + t_1, t_0 + T]$, we will proceed showing that if $\|\mathbf{x}(t)\| < \gamma$, for $t = t_0 + t_1$, then the state can never leave the region $\bar{\mathcal{B}}^{\gamma}$ at future times, *i.e.* $\bar{\mathcal{B}}^{\gamma}$ is a positively invariant set (for $t \in [t_0 + t_1,\ t_0 + T]$). Assuming, by contradiction, $\|\mathbf{x}(t)\| > \gamma$ for some $t \in (t_0 + t_1, t_0 + T)$, then there exists $\tau \in (t_0 + t_1,\, t_0 + T)$ such that $\|\mathbf{x}(\tau)\| = \gamma$, hence

$$V(t,\mathbf{x}(t)) = V(\tau,\mathbf{x}(\tau)) + \int_{\tau}^{t} \dot{V}(\rho,\mathbf{x}(\rho))d\rho$$

at $t = t_0 + T$, using hypotheses (ii) and (iv)

$$\begin{aligned} V(t_0+T, \mathbf{x}(t_0+T)) &\leq V_M^{\gamma}(\tau) + \int_{\tau}^{t_0+T} \psi_2(\rho)d\rho \\ &\leq V_M^{\gamma}(\tau) + V_m^{\gamma}(t_0+T) - V_M^{\gamma}(\tau) = V_m^{\gamma}(t_0+T) \end{aligned}$$

But, by hypotheses (vii), this is a contradiction; hence, $\|\mathbf{x}(t)\| < \gamma$ for all $t \in (t_0 + \tau_s,\ t_0 + T)$ and the system is FTCSwfst τ_s and $\bar{\mathcal{B}}^{\gamma}$ is an invariant set. ∎

Remark: It is worth noting that the application of such conditions is not straightforward because of the choice of the functions $\psi_1(t)$ and $\psi_2(t)$. The problem is that we would like the bounding functions $\psi_1(t)$ and $\psi_2(t)$ to be as tight as possible in (i) and (ii). In the case of autonomous nonlinear system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, Theorem 4 can be restated using either a time-varying function $V(t,\mathbf{x})$ or a time-invariant function $V(\mathbf{x})$. In the latter case, the functions $\psi_1(t)$ and $\psi_2(t)$ can be replaced by constants ψ_1 and ψ_2.

A more constructive approach to deal with FTCSwfst is given by the following theorem.

Theorem 5: The nonlinear system (13) is FTCSwfst w.r.t. $(\alpha, \beta, \gamma, \tau_s, T)$ if there exists a function $V(\mathbf{x}) > 0$ such that

(i) $k_1 \|\mathbf{x}\|^a \leq V(t,\mathbf{x}) \leq k_2 \|\mathbf{x}\|^a,\ t \in \mathcal{I}, \quad \forall\, \mathbf{x} \in \bar{\mathcal{B}}^{\beta} - \mathcal{B}^{\gamma}$;
(ii) $\dot{V}(t,\mathbf{x}) \leq -k_3 \|\mathbf{x}\|^a, \quad t \in \mathcal{I}, \quad \forall\, \mathbf{x} \in \bar{\mathcal{B}}^{\beta} - \mathcal{B}^{\gamma}$;
(iii) $-k_3 < \frac{k_2 \ln \frac{k_1 \gamma^2}{k_2 \alpha^2}}{\tau_s}$;
(iv) $\frac{k_2}{k_1}\alpha^2 < \beta^2$.

where k_1, k_2, k_3 and a are positive constants.

Proof: Without loss of generality let us consider $a = 2$. i) and ii) show that V satisfies the differential inequality $\dot{V} \leq -\frac{k_3}{k_2}V$. By the Comparison Lemma, [10],

$$V(t,\mathbf{x}(t)) \leq V(t_0,\mathbf{x}(t_0))e^{(-k_3/k_2)(t-t_0)}$$

Hence, starting with an initial condition within a bound α,

$$\|\mathbf{x}(t)\|^2 \leq \frac{k_2}{k_1} e^{(-k_3/k_2)(t-t_0)} \alpha^2$$

is within the bound β, for all $t \in [t_0,\, t_0 + T]$, for (iv). Moreover, since we are asking for the norm of the state to be bounded by γ for all $t \geq t_0 + \tau_s$, from (iii) we have

$$\frac{k_2}{k_1} e^{(-k_3/k_2)\tau_s} \alpha^2 < \gamma^2$$

hence $\frac{k_2}{k_1}\alpha^2 < e^{\frac{k_3}{k_2}\tau_s}\gamma^2$ and

$$\|\mathbf{x}(t)\|^2 \leq e^{\frac{k_3}{k_2}(\tau_s+t_0-t)}\gamma^2 < \gamma^2$$

because $\frac{k_3}{k_2} > 0$ and $t \geq t_0 + \tau_s$. So, the ball of radius γ is an invariant set over the interval $[t_0 + \tau_s,\, t_0 + T]$. ■

Remark: It is worth noting that if we had asked i) and ii) of Theorem 5 to hold in $\bar{\mathcal{B}}^\beta$ rather than in the annulus ring $\bar{\mathcal{B}}^\beta - \mathcal{B}^\gamma$, we would have required exponential stability of the origin. But, from the foregoing theorem, the state is kept in the invariant set $\bar{\mathcal{B}}^\gamma$ for all $t \in [t_0 + \tau_s,\, t_0 + T]$, and, after that, can have the behaviour shown Fig. 1. Furthermore, the origin does not need to be an equilibrium point of our system.

The fact that (i) and (ii) hold with the same exponent a, is not restrictive, as long as $V \in C^1$ (*i.e.* V is continuously differentiable) and V and $\dot{V}$ are different from zero in the region of interest that excludes an open neighborhood of the origin.

V. Finite-Time Contractive Stability with fixed settling time design

Design for FTCSwfst of nonlinear systems follows the same approach seen for FTS. The solvability of SSELP reduces the nonlinear system into a feedback equivalent linear system and then the linear Finite-Time Contractive stabilization problem is addressed with the tools presented in the previous section.

The Finite-Time Contractive stabilization problem with fixed settling time for the linear system (6) consists in finding a state feedback control law $u = K\mathbf{x}$ such that the closed loop system $\dot{\mathbf{x}} = (A + BK)\mathbf{x}$ is FTCSwfst w.r.t. $(\alpha, \beta, \gamma, \tau_s, T)$. Applying Theorem 5 to the closed loop system $\dot{\mathbf{x}} = (A + BK)\mathbf{x}$ and choosing a Lyapunov-like function $V = \mathbf{x}'P\mathbf{x}$, we end up with the following result:

Theorem 6: The LTI system (7) is FTCSwfst w.r.t. $(\alpha, \beta, \gamma, \tau_s, T)$ if there exists a positive scalar δ, a symmetric positive matrix P, and a matrix K such that

$$A'P + PA + K'B'P + PBK + \delta P < 0 \quad (14a)$$
$$cond(P) < \frac{\gamma^2}{\alpha^2}e^{\delta\tau_s} \quad (14b)$$
$$cond(P) < \frac{\beta^2}{\alpha^2} \quad (14c)$$

Similarly to (8), also (14) can be rewritten in a form such that their feasibility can be checked by a LMI solver and a bisection algorithm.

Theorem 7: Consider the nonlinear system (1) and the set $(\alpha, \beta, \gamma, \tau_s, T)$.
If the following conditions are satisfied:

- $\text{rank}([\mathbf{g}(\mathbf{x}), \mathbf{ad_f g}(\mathbf{x}), \ldots, \mathbf{ad_f}^{n-1}\mathbf{g}(\mathbf{x})]) = n$, $\forall \mathbf{x} \in \mathcal{R}_\alpha^\beta$;
- the distribution $\mathcal{D} = span\{\mathbf{g}, \mathbf{ad_f g}, \ldots, \mathbf{ad_f}^{n-2}\mathbf{g}\}$ is involutive in a neighborhood of every $\mathbf{x} \in \mathcal{R}_\alpha^\beta$;
- the inequality $0 < \bar{\gamma} < \bar{\alpha} < \bar{\beta}$ holds, where

$$\bar{\gamma} := \min_{\|\mathbf{x}\|=\gamma} \Phi(\mathbf{x}),\ \bar{\alpha} := \max_{\|\mathbf{x}\|=\alpha} \Phi(\mathbf{x}),\ \bar{\beta} := \min_{\|\mathbf{x}\|=\beta} \Phi(\mathbf{x})$$

  and $\Phi(\cdot)$ is the coordinate transformation used for feedback linearization;
- there exist a positive scalar $\bar{\delta}$, a symmetric positive matrix P, and a vector $\bar{K}$ such that

$$A_0'P + PA_0 + \bar{K}'B_0'P + PB_0\bar{K} + \bar{\delta}P < 0 \quad (15a)$$
$$cond(P) < \frac{\bar{\gamma}^2}{\bar{\alpha}^2}e^{\bar{\delta}\tau_s} \quad (15b)$$
$$cond(P) < \frac{\bar{\beta}^2}{\bar{\alpha}^2} \quad (15c)$$

then (1) can be made FTCSwfst w.r.t. $(\alpha, \beta, \gamma, \tau_s, T)$ by using the control law

$$u = \frac{1}{L_g L_f^{n-1}\lambda(\mathbf{x})}(-L_f^n\lambda(\mathbf{x}) + \bar{K}\Phi(\mathbf{x})) \quad (16)$$

Proof: The proof follows the same lines of the proof of Theorem 3 with the difference that now we have to take also in account $\bar{\mathcal{B}}^\gamma$ and its transform under $\Phi(x)$. So, with the same reasoning as before the existence of $\bar{\mathcal{B}}^{\bar{\gamma}}$ is guaranteed, and as long as $\bar{\gamma} < \bar{\alpha} < \bar{\beta}$, and (15b) and (15c) are satisfied, the closed loop nonlinear system is FTCSwfst w.r.t. $(\alpha, \beta, \gamma, \tau_s, T)$. ■

VI. Example

A. FTS design

The use of the FT stabilization results via feedback linearization is shown for the system

$$\dot{x}_1 = x_1^2 + x_2 \quad (17a)$$
$$\dot{x}_2 = Dz_{0.3}(x_1^2 + x_2^2)u \quad (17b)$$

where $Dz_c(v) := v - Sat_c(v)$, with

$$Sat_c(v) := \begin{cases} v & \text{if } |v| < c, \\ sgn(v)c & \text{if } |v| \geq c. \end{cases}$$

We would like to design for FTS w.r.t. (0.8,5,12), applying the procedure discussed in Theorem 3. In the annulus ring of interest, $\mathcal{R}_{0.8}^5$, the system has the required regularity property. Therefore, we can proceed solving the SSELP, by finding a diffeomorphic transformation $\Phi(\mathbf{x}) = \mathbf{z}$ that maps the annulus ring $\mathcal{R}_{0.8}^5$ into the region $\Phi(\mathcal{R}_{0.8}^5)$. Since the state space has dimension $n = 2$, we only need to check the rank condition of the matrix $G(\mathbf{x}) = [\mathbf{g}, \mathbf{ad_f g}]$ in $\mathcal{R}_{0.8}^5$.

In the annulus ring $\mathcal{R}_{0.8}^5$, $G(\mathbf{x})$ has rank equal to two, so we have an annulus diffeomorphism $\Phi(\mathbf{x})$, given by

$$z_1 = x_1$$
$$z_2 = x_1^2 + x_2$$

to transform (17) into the system

$$\dot{z}_1 = z_2 \quad (18a)$$
$$\dot{z}_2 = 2z_1z_2 + (z_1^2 + z_2^2 + z_1^4 - 2z_2z_1^2 - 0.4)u \quad (18b)$$

By choosing the control u such that

$$u = \frac{1}{(z_1^2 + z_2^2 + z_1^4 - 2z_2z_1^2 - 0.4)}(-2z_1z_2 + v) \quad (19)$$

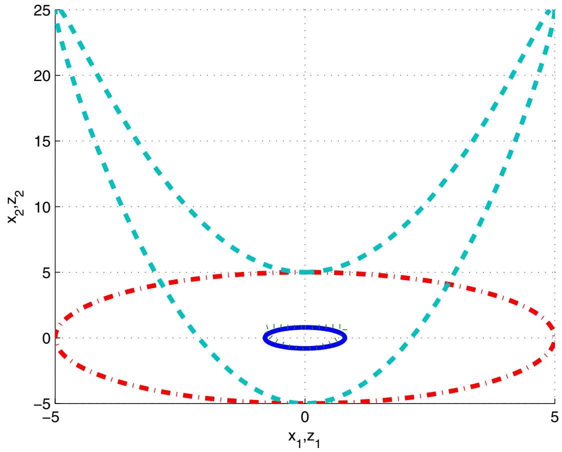

Fig. 2. Modification of the level curves after applying $\Phi(x)$. Level Curve $||x|| = 0.8$ (-), Transformation of $||x|| = 0.8$ (:), Level Curve $||x|| = 5$ (-.), Transformation of $||x|| = 5$ (- -).

system (18) is transformed into the linear system

$$\dot{z}_1 = z_2 \tag{20a}$$

$$\dot{z}_2 = v \tag{20b}$$

After applying the transformation Φ to (17), the given FTS levels undergo distortions as shown in Fig. 2. We can obtain the new FTS levels $\bar{\alpha}$ and $\bar{\beta}$, as suggested from (10) and shown in Fig. 3.

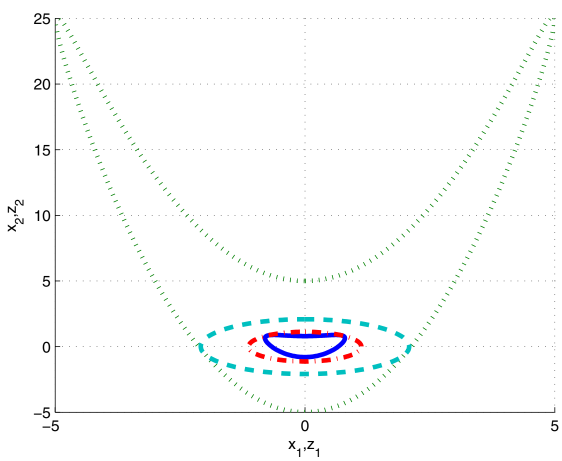

Fig. 3. New FTS levels. Transformation of $||x|| = 0.8$ (-), Transformation of $||x|| = 5$ (:), Level Curve $\|z\| = \bar{\alpha}$ (-.), level Curve $\|z\| = \bar{\beta}$ (- -).

Since we obtain $\bar{\beta} = 2.086 > \bar{\alpha} = 1.1283$, we can proceed to solving the linear FT stabilization problem for the system (20) w.r.t. $(1.1283, 2.086, 12)$. The solution of (11a) and the verification of (11b) leads to the vector $K = [-1.3044, \ -0.9254]$. So, from (16) we obtain

$$u = \frac{1}{x_1^2 + x_2^2 - 0.16}(-1.3044x_1 - (2x_1 + 0.9254)(x_1^2 + x_2))$$

that makes the closed loop system FTS w.r.t (0.8,5,12).

B. FTCS design

For the system (17), we want now to design for FTCSwrfst w.r.t. (0.8,5,0.4,8,12), i.e. keeping the same bounds for α, β and T as for FTS design, we also require the norm of the state to be less than the prespecified value of $\gamma = 0.4$ at the fixed settling time $\tau_s = 8$. Since all the sufficient conditions of Theorem 7 are satisfied, with $\bar{\gamma} = 0.3471 < \bar{\alpha} = 1.1283 < \bar{\beta} = 2.086$ (by exploiting the same diffeomorphic transformation $\Phi(\mathbf{x})$ as above), then we can make the system FTCSwrfst w.r.t. (0.8,5,0.4,8,12) with the control law:

$$u = \frac{1}{x_1^2 + x_2^2 - 0.16}(-2.0053x_1 - (2x_1 + 1.6755)(x_1^2 + x_2))$$

VII. Conclusions and Future Research

In this paper we have presented a new design technique for Finite Time Stabilization of nonlinear systems with single input. The structure of the nonlinear system is crucial in applying the proposed design, since we require the system to be "quasi-feedback linearizable".

Moreover, a novel concept has been introduced for continuous time-varying nonlinear systems, namely Finite Time Contractive Stability with fixed settling time, for which sufficient conditions have been provided and a Finite Time Contractive Stabilization problem with fixed settling time has been addressed based on "quasi-feedback linearization".

As for future research, it would be interesting to investigate the robustness properties of compensators designed by the proposed approach, and to address the problem of Finite Time stabilization of uncertain nonlinear plants. Also, it would be interesting to study the case when only input-output feedback linearization is achievable, and the possible interpretations, in the finite time stability framework, of the minimum phase concept.

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