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Abstract

This paper considers the problem of stabilizing a plant using a suboptimal stable compensator of fixed order. The resulting equations are a modified form of the optimal projection equations, with the separation principle not holding in either the full- or reduced-order case.

I. Introduction

This paper considers the design of stable fixed-order dynamic compensators in the event that the optimal Linear-Quadratic-Gaussian (LQG) controller is unstable. This is related to the problem of strong stabilization. [1] A condition for the existence of a strong stabilizer and a synthesis procedure are given in [1]. This method may yield high order compensators for some problems. A method proposed by Halevi [2] for finding suboptimal full order compensators uses modifications to LQG synthesis procedures. Ganesh and Pearson [3] employ a frequency domain approach, using the "Q"-parameterization of stabilizing compensators.

II. Problem Statement

The system to be controlled is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t) \tag{1}$$

$$y(t) = Cx(t) + w_2(t) \tag{2}$$

where the A, B, and C matrices are assumed to be known exactly (as in conventional LQG theory). Vectors w_1 and w_2 consist of independent white noise processes with intensities $V_1 \geq 0$ and $V_2 > 0$. The problem is to design a stable, fixed-order, dynamic compensator of order n_c

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{3}$$

$$u(t) = C_c x_c(t) \tag{4}$$

which minimizes

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} E[x^T(t)R_1 x(t) + u^T(t)R_2 u(t)] \tag{5}$$

where x_c is the compensator state of order n_c; A_c, B_c, and C_c are the compensator matrices; $R_1 \geq 0$ and $R_2 > 0$ are the state and control weighting matrices and $E(\cdot)$ denotes the expectation operator.

The expected cost can easily be shown to be

$$J = \mathrm{tr}(\tilde{Q}\tilde{R}) \tag{6}$$

where

$$\tilde{Q} = \lim_{t\to\infty} E(\tilde{x}\tilde{x}^T), \quad \tilde{x} = \begin{bmatrix} x \\ - \\ x_c \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix} \tag{7}$$

The closed loop system may be written as:

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0 \tag{8}$$

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_{12} \\ \tilde{Q}_1^T & \tilde{Q}_2 \end{bmatrix}. \tag{9}$$

III. Compensator Stability Guarantee and Auxiliary Minimization Problem

Theorem 1: Let $\Omega = \Omega^T \geq 0$ be such that

$$\Omega(B_c, Q_{12}) \geq -B_c V_2 B_c^T - B_c C Q_{12} - Q_{12}^T C^T B_c^T \tag{10}$$

and for given A_c, B_c, and C_c, suppose that

$$(\tilde{A}, [\tilde{V} + \tilde{\Omega}]^{1/2}) \text{ is stabilizable} \tag{11}$$

and that there exists $\bar{Q} > 0$ satisfying

$$0 = \tilde{A}\bar{Q} + \bar{Q}\tilde{A}^T + \tilde{V} + \tilde{\Omega} \tag{12}$$

$$\tilde{\Omega} = \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix} \geq 0 \qquad \bar{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \tag{13}$$

Then,

$\tilde{A}$ is asymptotically stable. (14)
$\tilde{Q} \leq \bar{Q}$, (15)
$J \leq \mathrm{tr}(\bar{Q}\tilde{R}) = \bar{J}$, and (16)
A_c is stable. (17)

Proof: See [5].

Equations (6) and (12) may be used to form a new (upper bound) minimization problem as follows:

Upper Bound Minimization Problem: Determine (A_c, B_c, C_c) and $\bar{Q} \geq 0$ that minimize $\bar{J}$ subject to (12). Solution of this problem gives a stable compensator with actual closed loop cost less than or equal to $\bar{J}$. We now choose a form for Ω that satisfies equation (10).

Proposition 1: Let

$$\Omega = \alpha B_c C T C^T B_c^T + \beta Q_{12}^T T^{-1} Q_{12} \tag{18}$$

where T is an arbitrary positive definite matrix; then Ω satisfies (10) when $\alpha = 1$ and $\beta = 1$.

Proof: See [5].

IV. Sufficient Conditions for a Stable, Reduced Order Compensator

Recall that a square matrix is nonnegative (positive) semisimple if it has a diagonal Jordan form and nonnegative (positive) eigenvalues. [4]

Lemma 1: Suppose $\hat{P}$, $\hat{Q}$ are nxn nonnegative definite matrices. Then $\hat{Q}\hat{P}$ is nonnegative semisimple. Furthermore, if $\mathrm{rank}(\hat{Q}\hat{P}) = n_c$, then there exist $n_c$xn matrices G, Γ and a positive semisimple n_cxn_c matrix M such that: [4]

$$\hat{Q}\hat{P} = G^T M \Gamma \qquad \Gamma G^T = I_{n_c} \tag{19}$$

Any G, M, and Γ satisfying Lemma 1 will be called a (G, M, Γ) factorization of $\hat{Q}\hat{P}$. The following simplified notation will be used:

$$\Sigma = BR_2^{-1}B^T \quad \bar{\Sigma} = C^T(V_2 + \alpha CTC^T)^{-1}C = C^T V_2'^{-1} C. \tag{20}$$

Theorem 3: Assume that condition (11) holds and that there exist nonnegative definite matrices P, Q, $\hat{P}$, and $\hat{Q}$ satisfying

$$0 = AQ + QA^T + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T + \beta\hat{Q}T^{-1}\hat{Q}. \tag{21}$$

$$0 = A^T P + PA + R_1 - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp + \beta T^{-1}\hat{Q}\hat{P} + \beta\hat{P}\hat{Q}T^{-1} \tag{22}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T - \beta\hat{Q}T^{-1}\hat{Q} \tag{23}$$

$$0 = (A - Q\bar{\Sigma} - \beta\hat{Q}T^{-1})^T\hat{P} + \hat{P}(A - Q\bar{\Sigma} - \beta\hat{Q}T^{-1}) + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp \tag{24}$$

$$\text{rank}(\hat{Q}) = \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\hat{P}) = n_c \tag{25}$$

with $\alpha = \beta = 1$. Then the compensator given by:

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P - \beta\hat{Q}T^{-1})G^T \tag{26}$$

$$B_c = \Gamma QC^T(V_2 + \alpha CTC^T)^{-1} \tag{27}$$

$$C_c = -R_2^{-1}B^T P G^T \tag{28}$$

using a G, M, Γ-factorization of $\hat{Q}\hat{P}$, and with $\tau = G^T\Gamma$ and $\tau_\perp = I_n - \tau$, satisfies conditions (14)-(17).

Conversely if (A_c, B_c, C_c) solves the upper bound minimization problem with A_c a stable matrix and Ω given by (18), then there exist real n x n nonnegative definite matrices P, Q, $\hat{P}$, and $\hat{Q}$ and $0 \le \alpha, \beta \le 1$ that satisfy equations (21)-(25) with A_c, B_c, and C_c given by equations (26)-(28).

Proof: See [5].

Remark 1: These optimal projection equations with a stable compensator guarantee consist of 4 equations that are coupled in both the full ($\tau_\perp = 0$) and reduced-order cases. The separation principle is not valid in either case. The equations reduce to the usual LQG equations when $\alpha = \beta = 1$ and $\tau_\perp = 0$. In Halevi's work [4], the full-order solution is given in terms of 2 decoupled Ricatti equations as compared with the four coupled equations found here.

Remark 2: In [5], the results of this work are extended to the design of fixed-order, strictly positive real compensators.

Remark 3: It is clear that a stable compensator of any order exists if the plant is open loop stable. Reference 5 also demonstrates that (21)-(25) possess a solution if the plant is open loop stable.

V. Example Problem

The example problem was considered by Doyle and Stein [6] and also by Halevi [2] and Ganesh and Pearson [3]. The system is:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w_1 \quad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + w_2 \tag{29}$$

$$R_1 = \begin{bmatrix} 2800 & 473 \\ 473 & 80 \end{bmatrix} \quad V_1 = \begin{bmatrix} 1225 & -2135 \\ -2135 & 3721 \end{bmatrix} \tag{30}$$

with $R_2 = V_2 = 1$. The LQG controller is unstable with a pole at 18.7. For the choice $\alpha = 0.011$, $\beta = 1$, $T = 1000I_2$, the compensator poles are -0.005 and -15.9 with ∞ gain margin, 60° phase margin, cost bound of 4.94×10^5 and actual cost of 4.31×10^5. For the choice $\alpha = \beta = 1$, $T = 62.5I_2$, the compensator poles were $-7.94 \pm j2.98$ with ∞ gain and phase margins, cost bound of 5.93×10^5 and actual cost of 5.51×10^5. For comparison, the lowest cost suboptimal compensator design by Halevi [2] had a cost of 4.4×10^5 with infinite gain margin and 82° phase margin and the design by Ganesh and Pearson [3] had a cost of 3.88×10^5 using a fourth-order compensator. A particular advantage of the method presented here is the ability to handle compensators of order less than the plant order, although no numerical results are given here.

VII. Conclusions

This paper presents a method for designing stable, dynamic compensators of order less than or equal to that of the plant. An overbounding technique on the state covariance guarantees that the compensator is stable if nonnegative definite solutions exist to the design equations.

VIII. References

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