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Increasing the Accuracy of Cooperative Localization by Controlling the Sensor Graph

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Abstract

We characterize the accuracy of a cooperative localization algorithm based on Kalman Filtering, as expressed by the trace of the covariance matrix, in terms of the algebraic graph theoretic properties of the sensing graph. In particular, we discover a weighted Laplacian in the expression that yields the constant, steady state value of the covariance matrix. We show how one can reduce the localization uncertainty by manipulating the eigenvalues of the weighted Laplacian. We thus provide insight to recent optimization results which indicate that increased connectivity implies higher accuracy and we offer an analysis method that could lead to more efficient ways of achieving the desired accuracy by controlling the sensing network.

1 Introduction

Localization is a process by which a robot estimates its position with respect to a given frame of reference. The general localization problem has many increasingly difficult problem instances, like global localization, position tracking, etc. Global localization is a process in which robot has to estimate its position with no information about its initial location. If the initial position information is available to the robot, the process is called position tracking and is comparatively simple than the global localization process. The "kidnapped robot" problem [9] is considered to be the hardest of all because in this case a fully localized robot is suddenly transferred to another location and in this process loses track of its position information. The above localization problems can become more complicated if the environment is dynamic. Most of the research in this field assumes the environment to be static.

Robot localization is an important aspect of map building [10]. Accurate estimation of robot positions provide accurate maps of the area. Multi-robot localization is gaining popularity due to the fact that the robots can exchange information regarding their position based on the available sensor data and hence can improve their own localization estimates. In an ideal situation, each robot would have its absolute positioning information provided by GPS signals or overhead cameras but in practice, the availability of absolute position information to each robot comes with an extra cost. In addition, GPS information is unavailable indoors or in the vicinity of tall buildings in urban environments. An indoor GPS system called "NorthStar" technology built by Evolutions Robotics can be used in case of an indoor application [11]. The device uses a small inexpensive sensor and an infrared, encrypted light device which helps the robot to estimate its position, but the positional accuracy of this system is not high and it needs an additional powerful and expensive camera.

Deficiencies in the odometry data provided by robot's encoders can be compensated by relative position measurements made by the exteroceptive sensors. Various types of exteroceptive sensors are used, the most common amongst them being laser range finders, sonar sensors, infra-red sensors. Various approaches are adopted by researchers to fuse the measurements obtained by the proprioceptive and exteroceptive sensors like Extended Kalman filtering [2], [4], [5], particle filters [9], [10], [12], [16], grid based methods [18], [17] , expectation maximization algorithms [19], [20], etc.

The following sections give an outline of the problem, describe the filter chosen for data fusion and investigate the factors affecting the accuracy of the system. The contribution of this report is the establishment of an algebraic link between cooperative localization accuracy and the topology of the sensing graph. This connection can lead to additional more efficient ways of improving the accuracy of the cooperative localization algorithm by controlling the topology of the network.

2 Previous work and motivation

The main motivation for this report comes from, and builds on the work in [2]. It is in fact an extension of the work done by Roumeliotis and Mourikis in [2], based on an observation for an interesting particular case, and an associated graph theoretic interpretation. We adopt the same notation to allow for direct comparison and we borrow the initial stages of the analysis in [2].

Consider a group of M mobile robots and N landmarks in an environment. The robots use proprioceptive measurements to propagate their own position estimates and obtain relative position estimates of neighboring robots using their exteroceptive sensors. An Extended Kalman Filter is used to filter out the measurement noise and reduce the uncertainty associated with the interpretation of sensor signals. Also, each robot is provided with a sun sensor or a compass so that the uncertainty in orientation is always bounded. Hence, we are concerned only with the reduction of error in the position estimate of the robots.

Kalman filter estimation can be divided into two cycles: the Position Propagation cycle in which the knowledge about position estimates is propagated to next time step and the Update cycle where the position estimates are updated using relative position information obtained from the exteroceptive sensors.

3 Problem description

Consider a group of M mobile robots carry sensors capable of integrating odometry data as well as making relative measurements with respect to other robots. In this report we assume the existence of at least one landmark, the position of which is known exactly. Generally, a robot can make a relative measurement with respect to another robot if the latter is within its sensing radius. At any given moment, based on relative positions, some measurements can be made whereas others cannot, giving rise to a certain sensing network that can be described by a directed graph. The robots and landmarks are the nodes of this directed graph and the relative measurements made by the robots form the directed edges of the graph. The robots that can directly measure their positions with respect to the landmark can estimate their position more accurately. We intend to control the topology of a sensor network in order to improve the localization accuracy of a group of M robots.

The sensing graph is defined as follows:

Definition 3.1. *The sensing graph is a directed graph $\mathcal{X} = \{\mathcal{V}, \mathcal{E}\}$ consisted of*

- *a set of vertices $\mathcal{V} = \{r_1, \ldots, r_M, L_1, \ldots, L_N\}$, indexed by the M robots and N landmarks, and*
- *a set of edges $\mathcal{E}$, containing ordered pairs of the form (r_i, r_j) or (r_i, ℓ_j), with $r_i, r_j, \ell_j \in \mathcal{V}$, representing relative measurements made by the robots.*

In Figure 1, the absolute position information of the landmark is known. Robots $R4$ and $R3$ can measure their relative positions with respect to the landmark whereas $R2$ and $R1$ measure distances with respect to these robots. This scenario seems like a special case of the problem studied in [2]. However, in [2] the sensing topology is assumed time-invariant. The analysis presented in this report seems to suggest that there should not be any constraint on the time dependency of the sensing graph.

3.1 Position propagation

The pose of robot i, $i = 1, \ldots, M$ is given by:

$$X_{r_i} = \begin{bmatrix} x_{r_i} \\ y_{r_i} \\ \phi_i . \end{bmatrix}$$

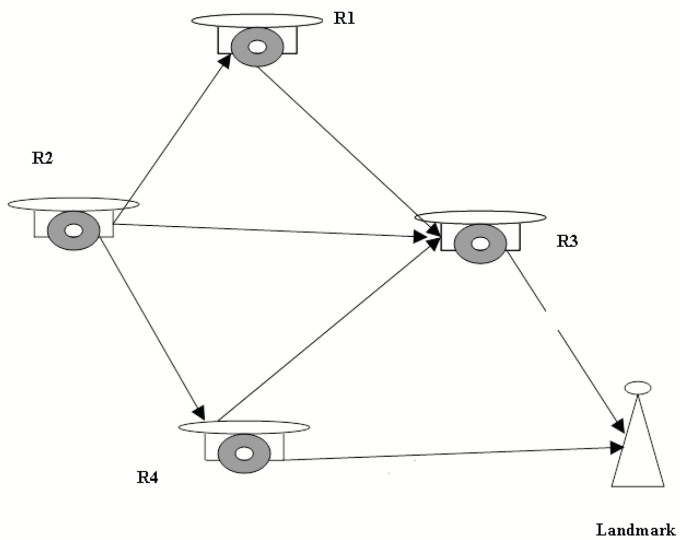

Figure 1: Sensor graph with 4 robots and 1 landmark.

The discrete-time equations approximating the kinematics of the i-th robot are [2]:

$$x_{r_i}(k+1) = x_{r_i}(k) + V_i(k)\delta t\cos(\phi_i(k)) \quad (1)$$
$$y_{r_i}(k+1) = y_{r_i}(k) + V_i(k)\delta t\sin(\phi_i(k)), \quad (2)$$

where δt is sampling period and $V_i(k)$ is the translational velocity at time step k. By linearizing the above equations, the error propagation equation for the robot's position can be derived:

$$\tilde{X}_{r_{i_{k+1|k}}} = I_{2\times 2}\tilde{X}_{r_{i_{k|k}}} + G_{r_i}(k)W_i(k),$$

where $\tilde{X}_{r_{i_{k+1|k}}} = \begin{bmatrix} \tilde{x}^T_{r_{i_{k+1|k}}} & \tilde{y}^T_{r_{i_{k+1|k}}} \end{bmatrix}^T$, and G_{r_i} denotes the error propagation matrix:

$$G_{r_i}(k) = \begin{bmatrix} \delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k)) \\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k)) \end{bmatrix},$$

in which $V_{m_i}(k)$ is the measured robot velocity, and $\hat{\phi}_i(k)$ is the estimate of robot's orientation. The term W_i is defined as:

$$W_i(k) = \begin{bmatrix} w_{V_i}(k) \\ \tilde{\phi}_i(k) \end{bmatrix},$$

where, $w_{V_i}(k)$ is zero-mean, white Gaussian noise sequence of variance $\sigma_{V_i}^2$, σ_{V_i} being the standard deviation of velocity measurement noise for the i-th robot at time step k and $\tilde{\phi}_i(k)$ is the error in robot's orientation at time k.

From the above equation, the covariance matrix of system noise for the i-th robot is given by:

$$Q_{r_i}(k) = EG_{r_i}(k)W_i(k)W_i^T(k)G_{r_i}^T(k) = C(\hat{\phi}_i(k))DC(\hat{\phi}_i(k))^T,$$

where $C(\hat{\phi}_i(k))$ denotes the 2×2 rotation matrix. Matrix D is defined as:

$$D = \begin{bmatrix} \delta_t^2\sigma_{V_i}^2 & 0 \\ 0 & \delta_t^2 V_{m_i}^2(k)\sigma_{\phi_i}^2 \end{bmatrix},$$

where σ_{V_i} is the standard deviation of the noise in the velocity measurements, and σ_{ϕ_i} is the standard deviation of the noise contaminating the orientation estimates.

Landmarks are considered stationary objects in the environment, and thus their state is same at all times. By collecting the terms, the covariance propagation equations of the state error can be given by the following equation [2]:

$$P_{k+1|k} = P_{k|k} + GQ_r(k)G^T,$$

where G and Q_r are the block diagonal matrices constructed using G_{r_i} and Q_{r_i}, respectively.

3.2 Position update

A robot performs relative position measurements whenever a robot or a landmark comes within its sensor range. The information obtained from these relative measurements is used in the update phase to reduce the position uncertainty of the robot. The relative position measurements obtained when robot r_i observes robot r_m are given by:

$$z_{r_i,r_m} = C^T(\phi_i)(X_{r_m} - X_{r_i}) + n_{z_{r_i,r_m}} \tag{3}$$

where $X_{r_m} = \begin{bmatrix} x_{r_m}^T & y_{r_m}^T \end{bmatrix}^T$, $X_{r_i} = \begin{bmatrix} x_{r_i}^T & y_{r_i}^T \end{bmatrix}$, and $n_{z_{r_i,r_m}}$ is the noise contaminating the above measurement. A similar equation can be obtained for robot r_i observing landmark L_n:

$$z_{r_i,L_n} = C^T(\phi_i)(X_{L_n} - X_{r_i}) + n_{z_{r_i,L_n}} \tag{4}$$

Equations (3) and (4), for the j-th observation made by the i-th robot can be written in one form as:

$$z_{ij} = C^T(\phi_i)(X_{ij} - X_{r_i}) + n_{z_{ij}} \tag{5}$$

where X_{ij} denotes the position of the robot or landmark observed by robot i and $n_{z_{ij}}$ is the noise affecting this measurement.

The measurement error equation can be written as:

$$\begin{aligned}\tilde{z}_{ij}(k+1) &= z_{ij}(k+1) - \hat{z}_{ij}(k+1)\\ &= C^T(\hat{\phi}_i(k+1))\begin{bmatrix}0 & \dots & -I_{2\times 2} & \dots & I_{2\times 2} & 0 & \dots\end{bmatrix}\tilde{X}_{k+1|k}\\ &\quad + \begin{bmatrix}I_{2\times 2} & -C^T(\hat{\phi}_i(k+1))J\Delta\hat{p}_{ij}(k+1)\end{bmatrix}\begin{bmatrix}n_{z_{ij}}(k+1)\\ \tilde{\phi}_i(k+1)\end{bmatrix}\\ &= H_{ij}(k+1)\tilde{X}_{k+1|k} + \Gamma_{ij}(k+1)n_{ij}(k+1)\end{aligned}$$

where $\tilde{X}$ is the combined stack vector of all the estimates for the robot and landmark positions:

$$\tilde{X}^T = \begin{bmatrix}\dots & \tilde{X}_{r_i}^T & \dots & \tilde{X}_{T_{ij}}^T & \dots\end{bmatrix},$$

$\Gamma_{ij}(k)$ can be given by:

$$\Gamma_{ij}(k) = \begin{bmatrix}I_{2\times 2} & -C^T(\hat{\phi}_i(k+1))J\Delta\hat{p}_{ij}(k+1)\end{bmatrix},$$

where

$$J = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \qquad n_{ij}(k) = \begin{bmatrix}n_{z_{ij}}(k)\\ \phi_i(k)\end{bmatrix}, \qquad \Delta\hat{p}_{ij}(k+1) = \hat{X}_{ij_{k+1|k}} - \hat{X}_{r_{i_{k+1|k}}},$$

$\hat{X}_{r_i}$ denotes the estimate for the position of robot i, and $\hat{X}_{ij}$ is the position of robot j as estimated by robot i. The matrix H_{ij} is given as $H_{ij}(k+1) = C^T(\hat{\phi}_i(k+1))H_{o_{ij}}$, in which:

$$H_{o_{ij}} = \begin{bmatrix}0 & \dots & -I_{2\times 2} & \dots & I_{2\times 2} & 0 & \dots & 0\end{bmatrix}. \tag{6}$$

If robot i observes robot j, $H_{o_{ij}}$ is a $2\times(2M+2N)$ matrix with i-th entry as $-I_{2\times 2}$ and j-th entry as $I_{2\times 2}$, and the rest of the entries are zero.

The measurement matrix of the entire system, $H(k+1)$, is a block matrix with block rows $H_i(k+1)$. Each $H_i(k+1)$ is given as

$$H_i(k+1) = \Xi_{\hat{\phi}_i}^T H_{o_i},$$

where H_{o_i} is a constant matrix with block rows $H_{o_{ij}}$, j varies from 1 to M_i, M_i is the number of observations made by robot i, and

$$\Xi_{\hat{\phi}_i}^T = I_{M_i\times M_i}\otimes C(\hat{\phi}_i(k+1)),$$

with $\otimes$ denoting the Kronecker matrix product. This measurement matrix for robot i, $H_i(k+1)$, stores the M_i relative position measurements made by robot i.

The covariance update equation of the EKF can be written as [2]

$$\begin{aligned}P_{k+1|k} &= P_{k+1|k} - P_{k+1|k}H^T(k+1)(H(k+1)P_{k+1|k}H^T(k+1) + R(k+1))^{-1}H(k+1)P_{k+1|k}\\ &= P_{k+1|k} - P_{k+1|k}H_o^T\Xi(k+1)(\Xi^T(k+1)H_oP_{k+1|k}H_o^T\Xi(k+1)\\ &\quad + \Xi^T(k+1)R_o\Xi(k+1))^{-1}\Xi^T(k+1)H_oP_{k+1|k}\\ &= P_{k+1|k} - P_{k+1|k}H_o^T(H_oP_{k+1|k}H_o^T(H_oP_{k+1|k}H_o^T + R_o(k+1))^{-1}H_oP_{k+1|k}.\end{aligned}$$

In the above,

- H_o is a matrix having as block rows H_{o_i},

- R_o is a block diagonal matrix with elements R_{o_i} [2]:

$$R_{o_i} = \sigma^2_{\rho_i} I_{2M_i \times 2M_i} - D_i \mathrm{diag}\left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}}\right) D_i^T + \sigma^2_{\theta_i} D_i D_i^T + \sigma^2_{\phi_i} D_i 1_{M_i \times M_i} D_i^T,$$

  where D_i is the block diagonal matrix with diagonal blocks $J\Delta\hat{p}_{ij}$, and M_i are the number of observations made by robot i,
- $R(k+1)$ is a block diagonal matrix with elements $R_i(k+1)$,
- R_i is a block matrix in which the 2×2 (diagonal) (l,m) block is:

$${}^iR_{lm}(k+1) = \sigma^2_{\phi_i} C^T(\hat{\phi}_i) J \Delta\hat{p}_{ij} \Delta\hat{p}^T_{il} J^T C(\hat{\phi}_i), \text{ for } \quad l,m = 1,\ldots,M_i,$$

  where $\Delta\hat{p}_{ij}(k+1) = \hat{X}_{ij_{k+1|k}} - \hat{X}_{r_{i_{k+1|k}}}$, and $\Delta\hat{p}_{il}(k+1) = \hat{X}_{il_{k+1|k}} - \hat{X}_{r_{i_{k+1|k}}}$.

3.3 Covariance matrix when a single landmark is accurately known

The system consisting of M robots and one landmark becomes observable if the absolute position information of the landmark is available. In this case, the steady-state solution to Riccati recursion becomes [2],

$$P^{u(0)}_\infty = \begin{bmatrix} P^u_{rr\infty} & 0_{2M\times 2} \\ 0_{2\times 2M} & 0_{2\times 2} \end{bmatrix}. \tag{7}$$

One measure of localization accuracy is the trace of the covariance matrix, $P^{u(0)}_\infty$. To increase the localization accuracythe trace of $P^{u(0)}_\infty$ should decrease. It can be seen from the above equation that,

$$\mathrm{trace}(P^{u(0)}_\infty) = \mathrm{trace}(P^u_{rr\infty}) \tag{8}$$

The matrix $P^u_{rr\infty}$ is given by,

$$P^u_{rr\infty} = Q^{\frac{1}{2}}_{r_u} U \mathrm{diag}(\frac{1}{2} + (\frac{1}{4} + \frac{1}{\lambda_i})^{\frac{1}{2}}) U^T Q^{\frac{1}{2}}_{r_u}. \tag{9}$$

In the above equation, U and λ_i, $i = 1,\ldots,2M$, are the matrix of eigenvectors of another matrix Ψ and the eigenvalues of Ψ, respectively, where

$$\Psi = Q^{\frac{1}{2}}_{r_u} I_r Q^{\frac{1}{2}}_{r_u} \tag{10}$$

Q_{r_u} is a diagonal matrix with elements q_i:

$$q_i = \max(\delta t^2 \sigma^2_{V_i}, \delta t^2 V_i^2 \sigma^2_{\phi_i}). \tag{11}$$

In equation (10), the matrix I_r is given by:

$$I_r = I_y^T H_o^T R_u^{-1} H_o I_y, \tag{12}$$

where

$$I_y = \begin{bmatrix} I_{2M\times 2M} \\ 0_{2\times 2M} \end{bmatrix}.$$

In the above equation, R_u is a diagonal matrix with (block) elements:

$$r_i I_{2M_i\times 2M_i} = (\sigma^2_{\rho_i} + M_i \sigma^2_{\phi_i} \rho_0^2 + \sigma^2_{\theta_i} \rho_0^2) I_{2M_i \times 2M_i}.$$

4 Algebraic graph theoretic characterization

It can be seen from equation (6) that $H_{o_{ij}}$ is in fact the Kronecker matrix product of the row of the incidence matrix of the sensor graph $\mathcal{X}$ (Definition 3.1) corresponding to edge (i,j), B_{ij}, with $I_{2\times 2}$:

$$H_{o_{ij}} = B_{ij} \otimes I_{2\times 2}$$

Then,

$$H_o = B \otimes I_{2\times 2}$$

Matrix R_u^{-1} can be thought of as a weight matrix. Weights corresponding to each node change, whenever the degree of each node does. The Laplacian matrix for graph $\mathcal{X}$ is a positive semidefinite matrix obtained by multiplying the incidence matrix B with its transpose:

$$L = B^T B.$$

A weighted Laplacian matrix is obtained in a similar way, by multiplying the weight matrix with the incidence matrix from the right, and transpose of incidence matrix from the left. If W denotes a weight matrix, then the weighted Laplacian of a graph can be written as,

$$L_w = B^T W B = H_o^T R_u^{-1} H_o$$

The matrix $H_o^T R_u^{-1} H_o$ is thus a weighted Laplacian matrix of $\mathcal{X}$, with incidence matrix H_o and weight matrix R_u^{-1}. The smallest eigenvalue of a weighted Laplacian matrix is zero and the rest are always positive. By multiplying the weighted Laplacian matrix of the sensor graph $H_o^T R_u^{-1} H_o$ by I_y from right and I_y^T from the left, we obtain I_r in equation (12).

Multiplication by I_r results in the removal of the last row (and column) from L_w. This is the row and column corresponding to the (known) landmark. Due to the matrix tree theorem, I_r becomes a positive definite matrix with all positive eigenvalues [1]. We will write :

$$I_r = L_w[u]$$

according to the notation in [1], using u to represent the node associated with the landmark.

It can be seen from equation (8) that the trace of covariance matrix depends on the trace of matrix $P^u_{rr_\infty}$, which in turn depends on matrices Q_{r_u} and U, and the eigenvalues λ_i:

$$\mathrm{trace}(P^u_{rr_\infty}) = \mathrm{trace}\left(Q_{r_u}^{\frac{1}{2}} U \mathrm{diag}\left(\left[\frac{1}{4} + \frac{1}{\lambda_i}\right]^{\frac{1}{2}} \right) U^T Q_{r_u}^{\frac{1}{2}} \right).$$

Using the property [7]:

$$\mathrm{trace}(ABC) = \mathrm{trace}(BCA) = \mathrm{trace}(CAB)$$

we can write

$$\begin{aligned}
\mathrm{trace}(P^u_{rr_\infty}) =& \mathrm{trace}\left(Q_{r_u}^{\frac{1}{2}} U \mathrm{diag}\left(\left[\frac{1}{4} + \frac{1}{\lambda_i}\right]^{\frac{1}{2}} \right) U^T Q_{r_u}^{\frac{1}{2}} \right) \\
=& \mathrm{trace}\left(U^T Q_{r_u}^{\frac{1}{2}} Q_{r_u}^{\frac{1}{2}} U \mathrm{diag}\left(\left[\frac{1}{4} + \frac{1}{\lambda_i}\right]^{\frac{1}{2}} \right) \right) \\
=& \mathrm{trace}\left(U^T Q_{r_u} U \mathrm{diag}\left(\left[\frac{1}{4} + \frac{1}{\lambda_i}\right]^{\frac{1}{2}} \right) \right)
\end{aligned}$$

Note that $U^T Q_{r_u} U$ is a positive definite matrix, since U is the orthogonal matrix of eigenvectors of Ψ and Q_{r_u} is a diagonal matrix with positive elements. Hence, $U^T Q_{r_u} U$ is positive definite and its diagonal elements are always positive. Let α_i denote the i-th diagonal element of matrix $U^T Q_{r_u} U$ and μ_i denote the i-th diagonal element of matrix $\mathrm{diag}\left(\left[\frac{1}{4}+\frac{1}{\lambda_i}\right]^{\frac{1}{2}}\right)$. Then,

$$\mathrm{trace}(P^u_{rr_\infty}) = \sum_{i=1}^{2M} \alpha_i \mu_i \Rightarrow \frac{\partial(\mathrm{trace}(P^u_{rr_\infty}))}{\partial \mu_i} = \frac{\partial(\sum_{i=1}^{2M} \alpha_i \mu_i)}{\partial \mu_i} = \alpha_i > 0$$

Since α_i is always positive and depends on the sensors parameters, the trace of $P^u_{rr_\infty}$ increases monotonically with each μ_i.

In order to decrease the trace of $P^u_{rr_\infty}$, the value of μ_i should be minimized. The values of μ_i depend upon the eigenvalues of matrix Ψ. The larger the eigenvalues of Ψ, the smaller the trace of $P^u_{rr_\infty}$ will be. Hence, to increase accuracy of the system, one should try to achieve a matrix Ψ with as large eigenvalues as possible.

The trace of the matrix $\Psi = Q_{r_u}^{\frac{1}{2}} I_r Q_{r_u}^{\frac{1}{2}}$ can also be written as

$$\mathrm{trace}(\Psi) = \mathrm{trace}(Q_{r_u} I_r)$$

Let q_i denote the i-th diagonal element of Q_{r_u} and i_{ii} denote the i-th diagonal element of matrix I_r. Then note that:

$$\mathrm{trace}(Q_{r_u}^{\frac{1}{2}} I_r Q_{r_u}^{\frac{1}{2}}) = \mathrm{trace}(Q_{r_u} I_r) = \sum_{i=1}^{2M} q_i i_{ii} > 0,$$
$$\frac{\partial(\mathrm{trace}(Q_{r_u} I_r))}{\partial i_{ii}} = \frac{\partial(\sum_{i=1}^{2M} q_i i_{ii})}{\partial i_{ii}} = q_i > 0$$

Hence, in order to minimize $P^u_{rr_\infty}$, the eigenvalues of I_r should be increased. Matrix I_r is the matrix obtained from the weighted Laplacian matrix of graph $\mathcal{X}$ after the removal of the column and row which correspond to the landmark. Therefore, all the eigenvalues of I_r and Ψ are greater than zero. Thus, by increasing the eigenvalues of $H_o^T R_u^{-1} H_o$ we can reduce the trace of covariance matrix.

In case of a Laplacian matrix of an unweighted graph, it has been shown in [1] that its eigenvalues always increase with the addition of an edge. In the following section, we prove that the eigenvalues of a weighted graph Laplacian also interlace the eigenvalues of the new graph Laplacian obtained by addition of edges to the old graph.

5 Interlacing for Weighted Graphs

Interlacing is a relationship between the eigenvalues of a matrix and those of its submatrix. Suppose A is a real, symmetric $m \times m$ matrix and B is a real, symmetric $n \times n$ submatrix of A. Then the eigenvalues of B interlace the eigenvalues of A. Let $\{\lambda_1, ..., \lambda_n\}$ denote the eigenvalues of A and $\{\theta_1, ..., \theta_n\}$ be the eigenvalues of B, then, [1]

$$\lambda_{n-m+i}(A) \le \theta_i(B) \le \lambda_i(A) \tag{13}$$

In this section we iterate interlacing results on the Laplacian spectrum of *weighted* graphs; we give detailed proofs for known results on interlacing that appear in [21]. This section is not introducing new theory, but merely provides insight into how these known results apply to the particular case of cooperative localization using relative measurements. First we start with proving that the smallest positive eigenvalue of the new weighted Laplacian matrix obtained by adding an edge to the original one is always greater than the smallest positive eigenvalue of the original graph.

Theorem 5.1. *Let X be a graph with V vertices and E edges with weights w_{ij} on each edge and Q(X) be the weighted Laplacian matrix of X. Let Y be obtained by adding an edge to X by joining two distinct vertices of X, then,*

$$\lambda_2(X) \leq \lambda_2(Y)$$

Proof. Let Y be obtained by joining the vertices r and s of X. Q(Y) denotes the weighted Laplacian of Y. For any eigenvector z orthogonal to 1, we have from [1] and [3],

$$\begin{aligned}
\lambda_2(Y) &= \min_{z^T 1=0} \frac{z^T Q(Y) z}{z^T z} \\
&= \min_{z^T 1=0} \frac{w_{rs}(z_r - z_s)^2 + \sum_{uv \in E(X)} w_{uv}(z_u - z_v)^2}{z^T z} \\
&= \min_{z^T 1=0} \frac{w_{rs}(z_r - z_s)^2}{z^T z} + \min_{z^T 1=0} \frac{\sum_{uv \in E(X)} w_{uv}(z_u - z_v)^2}{z^T z} \\
&= \min_{z^T 1=0} \frac{w_{rs}(z_r - z_s)^2}{z^T z} + \lambda_2(X)
\end{aligned}$$

Hence,

$$\lambda_2(Y) \geq \lambda_2(X)$$

□

It can be seen from the above equation that the smallest eigenvalue always increases with addition of an edge to the original graph. Increasing λ_2 will only affect one of the terms that are added to yield the trace of $P^u_{rr_\infty}$. In the next theorem we prove that all the eigenvalues of the new graph Laplacian increase with addition of an edge. We will make use of the following lemma from [7] to prove the theorem.

Lemma 5.2. *Let A, B be Hermitian matrices. For a positive definite matrix B, the following is always true [7],*

$$\lambda_k(A) \leq \lambda_k(A+B)$$

The next theorem essentially extends the interlacing property to weighted graphs.

Theorem 5.3. *Let X be a weighted graph with n vertices and Y be obtained from X by adding an edge joining two distinct vertices of X. Then, for all i,*

$$\lambda_i(X) \leq \lambda_i(Y)$$

Proof. Suppose a new edge e is obtained by joining vertices i and j of the graph X, when i-th robot measures relative distance with respect to the j-th robot. Associated with this measurement we have a new $(2M+2N)\times 2$ matrix p_j with i-th entry as $-I_{2\times 2}$ and j-th entry as $I_{2\times 2}$ and all the other entries zero.

$$p_j = \begin{bmatrix} 0 \\ -I_{2\times 2} \\ \dots \\ I_{2\times 2} \\ \dots \end{bmatrix}$$

Let $p_1,...,p_{M_i}$ represent similar $(2M+2N)\times 2$ matrices obtained as a result of the various relative measurements made by the robot i, for example, matrix p_1 is obtained when robot i makes relative measurements with respect to robot 1. The i-th entry in matrix p_1 is $-I_{2\times 2}$ and the first entry is $I_{2\times 2}$.

The weight matrix changes when a new edge is added to the graph. If originally robot i made M_i relative measurements, then, the i-th block in the weight matrix R_u^{-1} is given by a diagonal matrix of size $2M_i \times 2M_i$ with diagonal elements $\frac{1}{a_i}$, where:

$$a_i = \sigma^2_{\rho_i} + M_i\sigma^2_{\phi_i}\rho_0^2 + \sigma^2_{\theta_i}\rho_0^2$$

When robot i makes an additional observation, say robot j, the i-th block in the weight matrix becomes an $2(M_i+1)\times 2(M_i+1)$ diagonal matrix with elements $\frac{1}{b_i}$, where b_i is given by:

$$b_i = \sigma^2_{\rho_i} + (M_i+1)\sigma^2_{\phi_i}\rho_0^2 + \sigma^2_{\theta_i}\rho_0^2$$

The weighted Laplacian matrix of the new graph, Y, can be expressed in terms of weighted Laplacian of X as shown below:

$$Q(Y) = Q(X) + PWP^T$$

where P is an $2(M+N)\times 2(M_i+1)$ matrix consisting of matrices $p_1,...p_{M_i+1}$, $P = \begin{bmatrix} p_1 & p_2 & \dots & p_{M_i} \end{bmatrix}$ and W is an $2(M_i+1)\times 2(M_i+1)$ diagonal matrix with the i-th element as $(b_i - a_i)$ and rest elements as b_i.

It can be observed from the above equation that the term PWP^T is another weighted Laplacian matrix that corresponds to the subgraph of Y consisting of all the vertices present in the graph but only the edges obtained by measurements made by robot i. Laplacian matrix is always positive semidefinite, hence, by Lemma 5.2 we can conclude that the eigenvalues of a graph always increase when an edge is added to the original graph. □

Motivated by the fact that the spectrum of the weighted Laplacian behaves in a similar manner as that of the unweighted Laplacian of the same graph, allows us to concentrate on the sensing network topology as a means of optimizing the performance of the EKF algorithm. This is especially true for the case that sensors are homogeneous, and as a result the Laplacian gains are the same. While some edges (those with larger weights, or specifically smaller error variances) might contribute more significantly toward a more accurate estimate, the general direction is to establish sensing links in such a way that graph eigenvalues are highest. This could be important if conflicting objectives need to be met, for instance: spread the network over a wide area, while keeping localization error below a certain threshold.

6 Conclusion

Addition of edges results in improvement in the accuracy of localization. This report provides insight to the way in which network connectivity affects the cooperative localization accuracy. We have demonstrated that by adding an edge to a sensing graph with robots and a known landmark as nodes, the trace of the covariance matrix associated with robot position estimates can be decreased. By connecting more and more robots through relative sensor measurements, a dense sensor network can be established. This is in agreement with a recent result obtained by Mourikis and Roumeliotis [8] that suggested a complete sensor graph to be "optimal" for localization accuracy. The point we make in this report is that by linking the steady state value of covariance matrix to the eigenvalues of the weighted sensor graph Laplacian, we can investigate more efficient ways of improving accuracy. It suggests that to achieve the maximum possible accuracy with the smallest computational load, we should be looking out for the "sparsest" sensing network topologies, with highest Laplacian spectrum.

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