8-27-2009

Control and filtering of time-varying linear systems via parameter dependent Lyapunov functions

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Renato Alves Borges

B.S., Electrical Engineering, Federal University of Goiás, Brazil, 2002

M.S., Electrical Engineering, University of Campinas, Brazil, 2004

DISSERTATION

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Engineering

The University of New Mexico

Albuquerque, New Mexico

June, 2009

Dedication

To all that use the scientific method as the basic tool in changing the world.

Acknowledgments

First of all, I would like to thank my advisers Dr. Pedro Luis Dias Peres from FEEC/UNICAMP and Dr. Chaouki Tanios Abdallah from ECE/UNM. You both have provided priceless experiences and opportunities to my carrier that I would definitely not forget. I respect and admire the readiness with which I have always been treated.

Moreover, I thank all who supported me in any way in the development of this work, among which I highlight:

- the Brazilian agencies FAPESP and CAPES and the Electrical and Computer Engineering Department of UNM for the financial support during my doctoral studies in Brazil and in the United States;
- the members of my PhD committee for their comments, suggestions and contributions, which helped to improve the quality in the final version of the manuscript;
- Ricardo C. L. F. Oliveira for the discussions and partnership in the work done. In particular, I emphasize the contributions in the development of computer codes and numerical simulations;
- Taís Ruoso, Vinicius Montagner and Giorgio Valmórbida, directly related to this PhD dissertation, for the discussions, suggestions and contributions;
- my co-workers and friends from FEEC/UNICAMP/Brazil and from ECE/UNM /USA for the shared experiences and knowledge on several occasions;
- Hugo Murici, Marcelo Castro and Leonardo Queiroz for the conviviality and solidarity;

- Quintin Ndibongo, the Stephens family and the couple Ivan and Alma for the essential support during my stay in the U.S.A.;
- my parents Genésio and Leci, my sister Juliana, my brother-in-law Fernando for their unconditional support and my beautiful niece Lana for the constant joy;
- my lovely wife Mireille, for the indescribable companionship and unwavering presence along this path.

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ABSTRACT OF DISSERTATION

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Abstract

The main contribution of this dissertation is to propose conditions for linear filter and controller design, considering both robust and parameter dependent structures, for discrete time-varying systems. The controllers, or filters, are obtained through the solution of optimization problems, formulated in terms of bilinear matrix inequalities, using a method that alternates convex optimization problems described in terms of linear matrix inequalities. Both affine and multi-affine in different instants of time (path dependent) Lyapunov functions were used to obtain the design conditions, as well as extra variables introduced by the Finsler's lemma. Design problems that take into account an $\mathscr{H}_\infty$ guaranteed cost were investigated, providing robustness with respect to unstructured uncertainties. Numerical simulations show the efficiency of the proposed methods in terms of $\mathscr{H}_\infty$ performance when compared with other strategies from the literature.

Contents

Glossary

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
LPV	Linear Parameter-Varying
$(\star)$	indicates symmetric block in the LMIs/BMIs
$L > 0$	indicates that matrix L is symmetric positive definite
$L \geq 0$	indicates that matrix L is symmetric positive semi-definite
A	matrices (capital letters from Roman alphabet)
A', $(')$	indicates the transposition operation of a matrix
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}_+$	set of nonnegative integer numbers
$\mathbf{I}$	identity matrix of appropriate dimension
$\mathbf{0}$	zero matrix of appropriate dimension

Glossary

N	indicates the number of vertices of a polytope
n	represents the order of a square matrix
$\alpha(\cdot)$	represents the uncertainties of a system
$diag(\cdot)$	indicates a block-diagonal matrix with diagonal given by $[\cdot]$
$\ell_2[0,\infty)$	space formed by finite energy sequences

Introduction

Similarly to all processes in nature, the scientific method is in constant evolution. This does not occur without cause or reason, but is related to the struggle for survival waged by human beings over their history and to the constant adaptation and development of their work tools. Due to the imperative need to understand and accommodate nature, humans developed and improved their communication media, ranging from cave paintings to the highest level of abstraction found in Mathematics. It is obvious that the development of language is essential for the transmission and recording of knowledge, and is therefore crucial to the advancement of the human over nature.

Due to the dual character of nature, in which predominates ultimately a game of exchange between attraction and repulsion, it is inconsistent to conceive matter devoid of movement. And it is this movement that drives evolution, including the scientific method itself, which evolves following the natural movement of nature. Not surprisingly, Theory and practice also share the same duality, being impossible to conceive one without the other. Hence, the validation of theory occurs in practice and practice advances with the support of the theory. To paraphrase Carl Sagan, science is a self-correcting process, to be accepted, new ideas must survive the most rigorous standards of evidence and scrutiny. Chebyshev also remarks that the closer the points of view of theory and practice are, the more benefits result in both arena.

Although the motivation of this work is concerned with practical applications such as networked control of electrical and mechanical devices, the reported experimentations were limited to computer simulations. Possibly, the ideas proposed in this dissertation may not survive the rigorous standards of evidence and scrutiny of practice, or may need adjustments for such end, since this thesis does not cover the two poles of the scientific movement: theory and practice. Such limitation however does not make the research less important or the results less interesting.

The aim of this work is to advance theoretical aspects in the context of control and filtering problems of dynamic systems. The study of control systems seeks to establish the necessary conditions for a precise action, and the satisfactory control of a process. Thus, the improvement of control techniques is an essential step in the constant technological refinement, and more specifically in the constant development of productive forces, which in turn determine the successive changes in the social relations of humans. Hence the importance of the issues addressed in this thesis and their proper placement in the social-economic context.

Due to the dynamic and transient nature of physical problems mentioned above, the description of physical processes by models with constant parameters cannot explain the highly variable behavior of the phenomena observed in practice. This simple but essential detail reveals the inevitable presence of uncertainties in dynamic models. Among the techniques used to treat uncertain models, the main ones are based in adaptive and robust control. An adaptive controller changes its behavior to comply with new or changed circumstances while a robust controller, for the purpose of this study, are fixed yet able to tolerate limited parametric changes and uncertainties.

Regarding the strategies for synthesis of such controllers, they are mostly divided into two classes: the ones that directly solve the set of differential, or difference, equations which describe the dynamics of the system, and those that indirectly analyze the behavior of the output based on both the parameters used in the model and in the set of input and output data. Of particular interest is the second group of strategies, because by not requiring the solution of the set of differential or difference equations, we obtain flexible conditions for control synthesis, particularly in the context of uncertain systems. The second method of Lyapunov, used throughout this work, falls within the scope of the indirect methods.

Lyapunov theory is based on the concept of energy dissipation. Lyapunov studied the phenomena of contraction and expansion of the movement in an autonomous mechanical system, analyzing the asymptotic behavior of the state around an equilibrium point. The central idea of the theory is based on the fact that if an equilibrium of a dynamic system is the local minimum of an energy function and the system is dissipative, then this equilibrium is locally stable, as mathematically described below.

[62, Theorem 4.1]. *Let* $x = 0$ *be an equilibrium point of an autonomous system and* $\mathscr{D}$ *a domain containing this point. If there exists a continuously differentiable function* $\vartheta : \mathscr{D} \to \mathbb{R}$ *such that*

$$\vartheta(0) = 0 \quad \textit{and} \quad \vartheta(x) > 0 \quad \textit{in} \quad \mathscr{D} - \{0\}$$
$$\dot{\vartheta}(x) \leq 0 \quad \textit{in} \quad \mathscr{D}$$

then $x = 0$ *is stable. Moreover, if*

$$\dot{\vartheta}(x) < 0 \quad \textit{in} \quad \mathscr{D} - \{0\}$$

then $x = 0$ *is asymptotically stable.*

Specifically, the analysis of stability and the synthesis of controllers and filters by the second method of Lyapunov are based on the construction of these energy functions, also called Lyapunov functions [62], [106]. A key point in the Lyapunov method is the definition of the class from which the Lyapunov function candidate will be chosen. Several classes of functions are available nowadays, with the quadratic ones being the most popular for linear systems.

Concerning the class of dynamic systems considered in this dissertation, we focus on those described by differential equations in the form

$$\dot{x}(t) = f(x(t), u(t), \alpha(t)) \tag{1}$$

$$y(t) = h(x(t), u(t), \alpha(t)) \tag{2}$$

$$z(t) = r(x(t), u(t), \alpha(t)) \tag{3}$$

or by difference equations in the form

$$x(k+1) = f(x(k), u(k), \alpha(k))$$

$$y(k) = h(x(k), u(k), \alpha(k))$$

$$z(k) = r(x(k), u(k), \alpha(k))$$

where $x(\cdot) \in \mathbb{R}^n$ is the vector of state variables, $u(\cdot) \in \mathbb{R}^m$ the vector of control inputs, $y(\cdot) \in \mathbb{R}^p$ and $z(\cdot) \in \mathbb{R}^r$ the outputs of the system, $\alpha(\cdot)$ an input representing the model variations, and $f(\cdot)$, $h(\cdot)$ linear functions. As an example, consider a continuous-time system described by

$$\dot{x}(t) = A(\alpha)x(t) \tag{4}$$

where $A(\alpha) \in \mathbb{R}^{n \times n}$ is a time-invariant uncertain matrix belonging to a polytope $\mathscr{A}$. A possible way to check the robust stability of (4) is by using quadratic Lyapunov functions in the form $\vartheta = x(t)'Px(t)$ (quadratic stability). The stability analysis problem is then reduced to the search of a matrix $P = P' > 0$ such that

$$A(\alpha)'P + PA(\alpha) < 0, \quad \forall A(\alpha) \in \mathscr{A} \tag{5}$$

Many advances have been provided by the so-called quadratic stability [7] approach. Several results for the analysis, robust control, and filtering with performance indices such as the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms were developed, see for example [24], [40] and the references therein. In several cases, the design conditions may be formulated in terms of linear matrix inequalities (LMIs), which are solved numerically using specialized computer packages such as the LMI Control Toolbox [44] and SeDuMi [101], both implemented in the Matlab software.

Note however that condition (5) is only sufficient to verify the asymptotically stability of a dynamical. There are examples of stable uncertain systems for which there is no positive definite symmetric matrix P satisfying the inequality above. In this case, the search for Lyapunov functions that provide less conservative conditions must be undertaken. The use of parameter-dependent Lyapunov matrices, $P(\alpha)$, leads to conditions less conservative than the quadratic stability, as shown in [90] and [91] for the following choice

$$P(\alpha) = \sum_{j=1}^{N} \alpha_j P_j \ , \ \sum_{j=1}^{N} \alpha_j = 1 \ , \ \alpha_j \geq 0, \quad j = 1, \ldots, N \tag{6}$$

Extensions of these stability conditions to deal with the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ guaranteed-cost problems may be found in [34] and [35]. We mention specifically the approaches presented in [41], [45], [80], [92], [102], [111]. Some of these conditions still require a high computational effort, or that the uncertainties satisfy a given structure, justifying the search for broader terms that can be formulated in terms of a finite set of LMIs.

Robust stability tests based on affine parameter-dependent Lyapunov functions appeared in [31], [32], [33], [51] and [88]. Results containing the previous ones and providing less conservative analysis appeared in [65] for time-invariant uncertainties and in [72] for the time-varying case.

For the robust stability of discrete-time systems based on parameter-dependent Lyapunov functions, we mention the LMI approaches presented in [32], [33] and [88]. Robust design conditions of controllers and filters appear in [4], [33] and [53]. More general classes of functions $P(\alpha(\cdot))$, polynomial in $\alpha(\cdot)$, for which the condition (5) approaches necessity may be seen in [84] and [85].

The controllers considered in this dissertation are described by

$$u(t) = h_c(x(t), \alpha(t)) \tag{7}$$

and the filters by

$$\dot{x}_f(t) = f_f(x_f(t), y(t), \alpha(t)) \tag{8}$$

$$z_f(t) = h_f(x_f(t), y(t), \alpha(t)) \tag{9}$$

where $h_c(\cdot)$, $f_f(\cdot)$ and $h_f(\cdot)$ are linear functions. The connection of the filter (8)-(9) with the system (1)-(3) results in an augmented dynamical system whose state variables are given by the augmented vector $\tilde{x}(t) = [x(t) \ \ x_f(t)]'$, and the output by the estimation error $e(t) = z(t) - z_f(t)$. The discrete-time case follows in a similar way.

With the advance of parameter-dependent Lyapunov functions, linear parameter varying (LPV) systems have received an increasing attention and attracted many research efforts. LPV systems have occupied a prominent role in the scientific community, mainly because they are useful not only in the representation of dynamic models affected by time-varying parameters, but also to describe certain classes of nonlinear systems [95], [97]. The use of a scheduling rule for the controller gain results, in several cases, in a better performance of the closed-loop system when compared with the results obtained by robust structures, that is, with fixed gain. This fact is related to the adaptive characteristic of the LPV control. For example, consider the uncertain time-varying discrete-time system

$$x(k+1) = A(\alpha(k))x(k) + B_u(\alpha(k))u(k) \tag{10}$$

$$y(k) = C(\alpha(k))x(k) \tag{11}$$

The vector of time-varying parameters $\alpha(k)$ belongs to the unit simplex

$$\mathscr{U} = \left\{ \delta \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1, \ \delta_i \geq 0 \ , \ i = 1, \ldots, N \right\}$$

for all $k \geq 0$ with limited rates of variation, given by

$$-g(\alpha_i(k)) \leq \Delta\alpha_i(k) \leq q(\alpha_i(k)), \tag{12}$$

where $g(\cdot)$ and $q(\cdot)$ are functions to be defined and $\Delta\alpha_i(k) = \alpha_i(k+1) - \alpha_i(k)$, $i = 1,\ldots,N$.

A possible structure for the state feedback control signal is an LPV gain, given by

$$u(\alpha, x(t)) = K(\alpha(t))x(t) \tag{13}$$

$$K(\alpha(t)) = \sum_{j=1}^{N} \alpha_j K_j \ , \ \sum_{j=1}^{N} \alpha_j = 1 \ , \ \alpha_j \geq 0, \quad j = 1,\ldots,N \tag{14}$$

LMI conditions to obtain the matrices K_j of the controller (13) that guarantee an upper bound to the $\mathscr{H}_\infty$ performance of the system may be obtained by the discrete parameter-dependent version of the bounded real lemma [37], [117]. The conservatism of these conditions can be reduced by exploiting the extra variables introduced by using Finsler's lemma [31].

Finally, with recent technological advances, the extensive use of communication channels in the control of dynamical systems have become increasingly frequent, [108], [115]. By using a real-time communication network for exchanging information between the control components (sensors, actuators, filters, etc), these structures represent an attractive approach in the implementation of distributed and interconnected control systems. The study of strategies for networked control systems (NCS) has received considerable attention [69], [71], [81], [82], [83], [104], [114] and will also be addressed in this work.

Detailed Description of the Contents

Chapter 1 presents the preliminary considerations, emphasizing the results used throughout the other chapters.

Chapter 2 considers the problem of gain-scheduled state feedback control for discrete-time systems with time-varying parameters. The time-varying parameters are assumed to belong to the unit simplex and to have bounded rates of variation, which depend on the values of the parameters and can vary from slow to arbitrarily fast. An augmented state vector is defined to take into account possible time-delayed inputs, allowing a simplified closed-loop analysis by means of parameter-dependent Lyapunov functions. A gain-scheduled state feedback controller that minimizes an upper bound of the $\mathscr{H}_\infty$ performance of the closed-loop system is proposed. The design conditions are expressed in terms of bilinear matrix inequalities (BMI) due to the use of extra variables introduced by the Finsler's lemma. By fixing some of the extra variables, the BMIs reduce to LMIs that can be solved using an algorithm based on the alternation of convex problems. Robust controllers for time-invariant uncertain parameters, as well as gain-scheduled controllers for arbitrarily time-varying parameters can be obtained as particular cases of the proposed conditions. As illustrated by numerical examples, the extra variables in the BMIs can provide better results in terms of the closed-loop $\mathscr{H}_\infty$ performance. This chapter is a preprint of a paper accepted by the *International Journal of Robust and Nonlinear Control* and is subject to the *John Wiley & Sons, Inc.* copyright, `http://www.wiley.com/WileyCDA/Section/id-301464.html`. Once the final version is published, the copy of record will be available at `http://www3.interscience.wiley.com/journal/5510/home`.

Chapter 3 investigates the problem of controller design for systems with uncertain sampling rates. The system is controlled through a communication network. The sampling period, within a given interval, is assumed to be time-varying and a simplified framework for the networked-induced delay is considered. The overall system is thus described by an uncertain discrete-time model with time-varying parameters inside a polytope whose vertices are obtained by means of the Cayley-Hamilton theorem. A digital robust controller that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the closed-loop networked control system is determined.

The design conditions rely on a particular parameter-dependent Lyapunov function and are expressed as bilinear matrix inequalities in terms of extra matrix variables, which may be explored in the search for a better system behavior. Numerical examples illustrate the results. This chapter is a preprint of a paper accepted by the *IET Control Theory & Applications* and is subject to the *IET* copyright, `http://www.theiet.org/help/legalnotices.cfm`. Once the final version is published, the copy of record will be available at `http://www.ietdl.org/IET-CTA`.

Chapter 4 deals with the problem of LPV filter design for time-varying discrete-time polytopic systems with bounded rates of variation. The design conditions, expressed as BMIs, are obtained by using a parameter-dependent Lyapunov function and extra variables for the filter design. An LPV filter, that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the estimation error, is obtained as the solution of an optimization problem. A more precise geometric representation of the parameter time variation was used in order to obtain less conservative design conditions. Robust filters for time-varying polytopic systems can be obtained as a particular case of the proposed method. Numerical examples illustrate the results. This chapter is a preprint of a paper accepted by the *Signal Processing* and is subject to the *Elsevier B.V.* copyright, `http://www.elsevier.com/wps/find/termsconditions.cws_home/termsconditions`. Once the final version is published, the copy of record will be available at `http://www.elsevier.com/wps/find/journaldescription.cws_home/505662/description#description`.

Finally, Chapter 5 concludes the dissertation summarizing the results obtained during this research and presenting some perspectives for future work.

Chapter 1

Preliminary Results

The aim of this chapter is to introduce the definitions and concepts useful in understanding the subject of this thesis. As in any systematic study of physical phenomena, it is necessary to present and classify the tools used to obtain the main results that are, in the case of this work, the mathematical models, the concept of stability and performance criteria. In general, the theoretical basis presented in this chapter is described with a usual notation, consisting of basic facts from the calculus, analysis and linear algebra. In summary, the techniques employed throughout this work reduce the analysis and synthesis of controllers and filters to optimization problems involving matrix inequalities.

1.1 Concepts and General Definitions

1.1.1 System Description

Consider the linear discrete-time time-varying system

$$x(k+1) = A(\alpha(k))x(k) + B_u(\alpha(k))u(k) + B_w(\alpha(k))w(k) \tag{1.1}$$

$$y(k) = C(\alpha(k))x(k) + D_u(\alpha(k))u(k) + D_w(\alpha(k))w(k) \tag{1.2}$$

$$z(k) = C_1(\alpha(k))x(k) + D_{w1}(\alpha(k))w(k) \tag{1.3}$$

where $x(k) \in \mathbb{R}^n$ represents the vector of state variables, $u(k) \in \mathbb{R}^m$ the vector of control inputs and $w(k) \in \mathbb{R}^r$ the vector of $\ell_2[0,\infty)$ noise. In control problems, $y(k) \in \mathbb{R}^p$ is the controlled output, $z(k) \in \mathbb{R}^q$ is not used, and it is assumed full access to the states. In filtering problem, $y(k)$ is the measured output, $B_u(\alpha(k))$ and $D_u(\alpha(k))$ are zero, and $z(k)$ is the signal to be estimated. The matrices are real, with appropriate dimensions, and belonging to the polytope

$$\mathscr{P} \triangleq \left\{ \left[\begin{array}{c|c|c} A(\alpha(k)) & B_u(\alpha(k)) & B_w(\alpha(k)) \\ \hline C(\alpha(k)) & D_u(\alpha(k)) & D_w(\alpha(k)) \\ \hline C_1(\alpha(k)) & D_{w1}(\alpha(k)) & - \end{array} \right] = \sum_{i=1}^{N} \alpha_i(k) \left[\begin{array}{c|c|c} A_i & B_{ui} & B_{wi} \\ \hline C_i & D_{ui} & D_{wi} \\ \hline C_{1i} & D_{w1i} & - \end{array} \right] \right\} \tag{1.4}$$

The vector of time-varying parameters $\alpha(k)$ belongs to the unit simplex

$$\mathscr{U} = \left\{ \delta \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0\ ,\ i = 1,\ldots,N \right\}$$

for all $k \geq 0$ with rates of variation given by

$$-g(\alpha_i(k)) \leq \Delta\alpha_i(k) \leq q(\alpha_i(k)), \tag{1.5}$$

where $\Delta\alpha_i(k) = \alpha_i(k+1) - \alpha_i(k),\ i = 1,\ldots,N$. The functions $g(\cdot)$ and $q(\cdot)$, to be appropriately defined later on, are used to model the cases where the matrices of the plant may vary in an arbitrary, or limited, way within the polytope. The time-invariant case is modeled by $g(\alpha_i(k)) = q(\alpha_i(k)) = 0$ for all $i = 1,\ldots,N$.

For controller design, the signal $z(k)$ is not used and the general structure of the control signal is given by

$$u(k) = K(\alpha(k))x(k) \tag{1.6}$$

The gain $K(\alpha(k))$ is designed to insure satisfactory performance of the closed-loop system in the presence of the disturbance $w(k)$.

In the filtering case, the structure of the filter is given by

$$\begin{aligned} x_f(k+1) &= A_f(\alpha(k))x_f(k) + B_f(\alpha(k))y(k) \\ z_f(k) &= C_f(\alpha(k))x_f(k) + D_f(\alpha(k))y(k) \end{aligned} \tag{1.7}$$

where $x_f(t) \in \mathbb{R}^n$ represents the vector of the filter state variables and $z_f(t) \in \mathbb{R}^q$ the signal to be estimated. The filter matrices are real, with appropriate dimensions, belonging to the polytope

$$\mathscr{F} \triangleq \left\{ \left[\begin{array}{c|c} A_f(\alpha(k)) & B_f(\alpha(k)) \\ \hline C_f(\alpha(k)) & D_f(\alpha(k)) \end{array} \right] = \sum_{i=1}^{N} \alpha_i(k) \left[\begin{array}{c|c} A_{fi} & B_{fi} \\ \hline C_{fi} & D_{fi} \end{array} \right], \ \alpha(k) \in \mathscr{U} \right\} \tag{1.8}$$

The vertices of the filter are designed to ensure satisfactory performance of the dynamics of the estimation error, $z(k) - z_f(k)$, in the presence of the disturbance $w(k)$.

1.1.2 Stability

Consider the system (1.1)-(1.2) in the form

$$x(k+1) = A(\alpha(k))x(k), \quad \alpha(k) \in \mathscr{U} \tag{1.9}$$

Definition 1.1 ([8]). *The system* (1.9) *is said to be globally asymptotically stable if, for all* $\alpha(k) \in \mathscr{U}$*, it is:*

i) Locally Stable: For all $\upsilon > 0$ *there exists* $\varsigma > 0$ *such that if* $||x(0)|| \leq \varsigma$ *then*

$$||x(k)|| \leq \upsilon, \quad \forall k \geq 0$$

ii) Globally Attractive: For all $\mu > 0$ *and* $\varepsilon > 0$ *there exists* $T(\mu, \varepsilon) > 0$ *such that if* $||x(0)|| \leq \mu$ *then*

$$||x(k)|| \leq \varepsilon, \quad \forall k \geq T(\mu, \varepsilon)$$

Definition 1.1 refers to robust stability since the local stability and the global attractiveness (item *i)* and *ii)*) must be checked for every $\alpha(k) \in \mathscr{U}$. Using Lyapunov functions, the stability of system (1.9) may be checked without knowing its trajectory $x(k)$. The lemma below provides a result along this line.

Lemma 1.1. *The null solution,* $x(k)=0$, *of system* (1.9) *is globally asymptotically stable if there exists a quadratic in the state Lyapunov function*

$$\vartheta(x(k),\alpha(k)) = x(k)'P(\alpha(k))x(k)$$

such that

$$\vartheta(x(k),\alpha(k)) > 0$$

and the values of $\vartheta(\cdot)$ *are decreasing along the trajectory of the system* (1.9), *that is*

$$\Delta\vartheta(x(k),\alpha(k)) = \vartheta(x(k+1),\alpha(k+1)) - \vartheta(x(k),\alpha(k)) < 0$$

for all $x(k) \in \mathbb{R}^n$ *and* $\alpha(k) \in \mathscr{U}$.

Although Lemma 1.1 has been presented as a sufficient condition, less conservative versions may be obtained by exploring the structure of the Lyapunov matrix $P(\alpha(k))$, [64], [84], [85]. This work will consider the forms affine in $\alpha(k)$

$$P(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)P_i, \quad \alpha(k) \in \mathscr{U} \tag{1.10}$$

and dependent on two instants of time (path-dependent), $\alpha(k)$ and $\alpha(k+1)$

$$P(\alpha(k),\alpha(k+1)) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i(k)\alpha_j(k+1)P_{ij}, \quad \alpha(k) \in \mathscr{U} \tag{1.11}$$

1.1.3 $\mathscr{H}_\infty$ Index of Performance

The $\mathscr{H}_\infty$ index of performance of a dynamic system coincides, in the time domain, with its ℓ_2 gain. For system (1.1)-(1.2), the $\mathscr{H}_\infty$ performance with respect to the input $w(k)$ is given by

$$\gamma^\star \triangleq \sup_{w\neq 0} \frac{\|y\|_2^2}{\|w\|_2^2} \tag{1.12}$$

where $w \in \ell_2[0,\infty)$.

The design conditions considered in this dissertation are based on the minimization of an upper bound γ to (1.12). The motivation is to address the worst scenario in which no information on the frequency range where the disturbance acts is available. Moreover, it is known that the $\mathscr{H}_\infty$ index of performance is related to the small gain theorem which states that a feedback loop is stable with a finite ℓ_2 gain if the product between the gains of each subsystem (the one in the open loop and the one in the feedback loop) is less than or equal to one ([25]). Thus, controllers and filters designed considering the $\mathscr{H}_\infty$ index of performance will present a certain degree of robustness against unstructured transfer functions that can model, as for instance, parametric variations and time delays.

1.2 LMIs and Auxiliary Lemmas

1.2.1 Linear Matrix Inequalities

In general, the use of LMIs in system and control theory is motivated by two facts: convex formulation and solution of problems by computer algorithms in polynomial time. Its origin is associated with the work of Lyapunov in 1890 (the Lyapunov equation) and its peak occurred at the beginning of the 1990s after the advent of interior point algorithms for LMIs [24].

The general structure of an LMI is given by

$$L(x) \triangleq L_0 + \sum_{i=1}^{m} x_i L_i > 0 \tag{1.13}$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrices $L_i \in \mathbb{R}^{n\times n}$, $i = 0,\dots,m$ are known. However, it is more common to fiund LMI problems in which the variables are matrices, as in the Lyapunov inequality below

$$A'PA - P < 0 \tag{1.14}$$

where $P = P' > 0$ is the matrix variable. As mentioned in [24], the condensed form (1.14), in addition to saving notation, may lead to more efficient computation.

1.2.2 Schur's Complement

The Schur's complement, widely used in this work, is useful in transforming a set of non-linear matrix inequalities into LMIs, as described below.

Lemma 1.2 (Schur's Complement)**.** *The set of non-linear matrix inequalities*

$$\left\{A_{11} > 0, \quad A_{22} > A_{12}' A_{11}^{-1} A_{12}\right\}$$

in which $A_{11} = A_{11}'$ *and* $A_{22} = A_{22}'$*, is equivalent to the following LMI*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}' & A_{22} \end{bmatrix} > 0 \tag{1.15}$$

Exchanging rows and columns in the matrix above, it follows that (1.15) *is equivalent to the set*

$$\left\{A_{22} > 0, \quad A_{11} > A_{12} A_{22}^{-1} A_{12}'\right\}$$

1.2.3 Stability and Index of Performance

Lemma 1.1 can be rewritten as shown below.

Lemma 1.3. *The null solution of system* (1.9) *is globally asymptotically stable if there exists a parameter-dependent Lyapunov matrix*

$$P(\alpha(k)) = P(\alpha(k))' > 0 \tag{1.16}$$

such that

$$A(\alpha(k))' P(\alpha(k+1)) A(\alpha(k)) - P(\alpha(k)) < 0 \tag{1.17}$$

for all $\alpha(k) \in \mathscr{U}$*.*

Proof: Multiply inequality (1.17) on the right by $x(k)$ and on the left by $x(k)'$ in order to obtain

$$\underbrace{x(k)'A(\alpha(k))'}_{x(k+1)'}P(\alpha(k+1))\underbrace{A(\alpha(k))x(k)}_{x(k+1)} - x(k)'P(\alpha(k))x(k) < 0$$

which is equivalent to $\Delta\vartheta(x(k),\alpha(k)) < 0$ with $\vartheta(x(k),\alpha(k)) = x(k)'P(\alpha(k))x(k)$. Finally, $P(\alpha(k)) = P(\alpha(k))' > 0$ assures $\vartheta(x(k),\alpha(k)) > 0$ and, in accordance with Lemma 1.1, the null solution of (1.9) is globally asymptotically stable for all $\alpha(k) \in \mathscr{U}$. ∎

LMI conditions for calculating the $\mathscr{H}_\infty$ performance of system (1.1)-(1.2) with respect to the input $w(k)$ may be obtained by the discrete parameter-dependent version of the bounded real lemma as shown in [29], [37]. A variant of this result is given below.

Lemma 1.4. *For a given γ, if there exist a matrix $Q(\alpha(k))' = Q(\alpha(k)) > 0$ such that*

$$\begin{bmatrix} -Q(\alpha(k)) & Q(\alpha(k))A(\alpha(k))' & Q(\alpha(k))C(\alpha(k))' & \mathbf{0} \\ (\star) & -Q(\alpha(k+1)) & \mathbf{0} & B_w(\alpha(k)) \\ (\star) & (\star) & -\gamma\mathbf{I} & D_w(\alpha(k)) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{1.18}$$

for all $\alpha(k) \in \mathscr{U}$, then system (1.1)-(1.2), with $B_u(\alpha(k)) = 0$ and $D_u(\alpha(k)) = 0$, is globally asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof: The guarantee of global asymptotic stability of system (1.1)-(1.2) with an upper bound on the $\mathscr{H}_\infty$ performance can be summarized by the asymptotic stability of $A(\alpha(k))$ and by the existence of a Lyapunov function $\vartheta(x(k),\alpha(k))$ such that

$$\Delta\vartheta(x(k),\alpha(k)) < -\gamma^{-1}y(k)'y(k) + \gamma w(k)'w(k) \tag{1.19}$$

for all $x(k) \in \mathbb{R}^n$ e $w(k) \in \mathbb{R}^r$. The validity of (1.18) ensures

$$\begin{bmatrix} -Q(\alpha(k)) & Q(\alpha(k))A(\alpha(k))' \\ (\star) & -Q(\alpha(k+1)) \end{bmatrix} < 0$$

that is equivalent, applying the Schur's complement, to the inequalities (1.16)-(1.17) with $P(\cdot) = Q(\cdot)^{-1}$. Therefore, according to Lemma 1.3, (1.18) ensures global asymptotic stability of system (1.1)-(1.2).

By appropriately, exchanging rows and columns in (1.18) it follows that

$$\begin{bmatrix} -Q(\alpha(k)) & \mathbf{0} & Q(\alpha(k))C(\alpha(k))' & Q(\alpha(k))A(\alpha(k))' \\ (\star) & -\gamma\mathbf{I} & D_w(\alpha(k))' & B_w(\alpha(k))' \\ (\star) & (\star) & -\gamma\mathbf{I} & \mathbf{0} \\ (\star) & (\star) & (\star) & -Q(\alpha(k+1)) \end{bmatrix} < 0 \tag{1.20}$$

Multiplying both sides of (1.20) by

$$T = \text{diag}([P(\alpha(k)), \quad \mathbf{I}, \quad \mathbf{I}, \quad P(\alpha(k+1))])$$

with $Q(\alpha(k)) = P(\alpha(k))^{-1}$ and $Q(\alpha(k+1)) = P(\alpha(k+1))^{-1}$, one has

$$\begin{bmatrix} -P(\alpha(k)) & \mathbf{0} & C(\alpha(k))' & A(\alpha(k))'P(\alpha(k+1)) \\ (\star) & -\gamma\mathbf{I} & D_w(\alpha(k))' & B_w(\alpha(k))'P(\alpha(k+1)) \\ (\star) & (\star) & -\gamma\mathbf{I} & \mathbf{0} \\ (\star) & (\star) & (\star) & -P(\alpha(k+1)) \end{bmatrix} < 0 \tag{1.21}$$

Applying the Schur's complement in (1.21) it follows that

$$\begin{bmatrix} m_{11} & m_{12} \\ (\star) & m_{22} \end{bmatrix} < 0 \tag{1.22}$$

with

$$m_{11} = A(\alpha(k))'P(\alpha(k+1))A(\alpha(k)) - P(\alpha(k)) + C(\alpha(k))'\gamma^{-1}C(\alpha(k))$$
$$m_{12} = A(\alpha(k))'P(\alpha(k+1))B_w(\alpha(k)) + C(\alpha(k))'\gamma^{-1}D_w(\alpha(k))$$
$$m_{22} = B_w(\alpha(k))'P(\alpha(k+1)B_w(\alpha(k))) + D_w(\alpha(k))'\gamma^{-1}D_w(\alpha(k)) - \mathbf{I}\gamma$$

Finally, considering $\vartheta(x(k), \alpha(k)) = x(k)'P(\alpha(k))x(k)$, multiply inequality (1.22) on the right by $[x(k)' \; w(k)']'$ and on the left by its transpose, in order to yield (1.19) concluding the proof. ■

It is worth emphasizing that Lemma 1.4 provides a sufficient condition to verify the stability of a time-varying discrete-time system with a given upper bound on the $\mathscr{H}_\infty$ performance with respect to $\ell_2[0,\infty)$ disturbances. This is the starting point for the results obtained in the subsequent chapters.

1.2.4 Finsler's Lemma and Extensions

The reduction of the conservatism in the conditions presented in the previous subsection may be obtained by exploiting the extra variables introduced when using the Finsler's lemma.

Lemma 1.5 ([31]). *Let $\xi \in \mathbb{R}^a$, $\mathscr{Q} = \mathscr{Q}' \in \mathbb{R}^{a\times a}$, $\mathscr{B} \in \mathbb{R}^{b\times a}$ with $rank(\mathscr{B}) < a$, and $\mathscr{B}^\perp$ a basis for the null space of $\mathscr{B}$ (that is, $\mathscr{B}\mathscr{B}^\perp = 0$). The following statements are equivalents:*

i) $\xi'\mathscr{Q}\xi < 0,\ \forall \mathscr{B}\xi = 0,\ \xi \neq 0$;

ii) $\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^\perp < 0$;

iii) $\exists\ \mu \in \mathbb{R} : \mathscr{Q} - \mu\mathscr{B}'\mathscr{B} < 0$;

iv) $\exists\ \mathscr{X} \in \mathbb{R}^{a\times b} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0$.

Finsler's lemma has been widely used in control theory. As mentioned in [31] and [99], in many cases the motivation is to eliminate design variables in matrix inequalities, which may be done for example, by using the equivalence *iv*) $\Rightarrow$ *ii*) in which the variable $\mathscr{X}$ is eliminated. In other cases, such as those studied in this thesis, the aim is exactly the opposite: to introduce extra variables in order to increase the degree of freedom during the search for feasible solutions. In such situations, one translates from the conditions described in *i*), *ii*) to the conditions in the form of *iii*) and *iv*) with the introduction of the extra variables μ and $\mathscr{X}$. In particular, Lemma 1.4 may be rewritten using item *ii*) of Finsler's lemma as follows.

Lemma 1.6. *For a given γ, if there exists a matrix $P(\alpha(k))' = P(\alpha(k)) > 0$ such that*

$$\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^{\perp} < 0 \tag{1.23}$$

for all $\alpha(k) \in \mathscr{U}$, with

$$\mathscr{B}^{\perp} = \begin{bmatrix} A(\alpha(k)) & B_w(\alpha(k)) \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\mathscr{Q} = \begin{bmatrix} P(\alpha(k)) & \mathbf{0} & \mathbf{0} \\ (\star) & -P(\alpha(k)) + C(\alpha(k))'C(\alpha(k))\gamma^{-1} & C(\alpha(k))'D(\alpha(k))\gamma^{-1} \\ (\star) & (\star) & D(\alpha(k))'D(\alpha(k))\gamma^{-1} - \gamma\mathbf{I} \end{bmatrix}$$

then system (1.1)-(1.2), *with $B_u(\alpha(k)) = 0$ and $D_u(\alpha(k)) = 0$, is globally asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.*

Proof: Developing the matrix product in (1.23) one has

$$\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^{\perp} = \begin{bmatrix} \mathscr{V}_{11} & A(\alpha(k))'P(\alpha(k+1))B_w(\alpha(k)) + C(\alpha(k))'D_w(\alpha(k))\gamma^{-1} \\ (\star) & B_w(\alpha(k))'P(\alpha(k+1))B_w(\alpha(k)) + D(\alpha(k))'D(\alpha(k))\gamma^{-1} - \gamma\mathbf{I} \end{bmatrix} < 0 \tag{1.24}$$

where

$$\mathscr{V}_{11} = A(\alpha(k))'P(\alpha(k+1))A(\alpha(k)) - P(\alpha(k)) + C(\alpha(k))'C(\alpha(k))\gamma^{-1}$$

Applying the Schur's complement in (1.24) it follows that

$$\begin{bmatrix} -P(\alpha(k)) & \mathbf{0} & C(\alpha(k))' & A(\alpha(k))' \\ (\star) & -\gamma\mathbf{I} & D(\alpha(k))' & B_w(\alpha(k))' \\ (\star) & (\star) & -\gamma\mathbf{I} & \mathbf{0} \\ (\star) & (\star) & (\star) & -P(\alpha(k+1))^{-1} \end{bmatrix} < 0 \tag{1.25}$$

Multiplying both sides of (1.25) by

$$T = \text{diag}([P(\alpha(k))^{-1}, \ \mathbf{I}, \ \mathbf{I}, \ \mathbf{I}])$$

with $Q(\alpha(k)) = P(\alpha(k))^{-1}$ and $Q(\alpha(k+1)) = P(\alpha(k+1))^{-1}$, one has

$$\begin{bmatrix} -Q(\alpha(k)) & \mathbf{0} & Q(\alpha(k))C(\alpha(k))' & Q(\alpha(k))A(\alpha(k))' \\ (\star) & -\gamma\mathbf{I} & D(\alpha(k))' & B_w(\alpha(k))' \\ (\star) & (\star) & -\gamma\mathbf{I} & \mathbf{0} \\ (\star) & (\star) & (\star) & -Q(\alpha(k+1))^{-1} \end{bmatrix} < 0 \tag{1.26}$$

that is equivalent, exchanging appropriately rows and columns, to inequality (1.18) of Lemma 1.4. ■

Finally, by exploring the extra variable $\mathscr{X}$ in Finsler's lemma (item *iv*)), it is possible to obtain less conservative conditions for Lemma 1.6. The main idea is to define particular structures, dependent on the parameter $\alpha(k)$, for matrix $\mathscr{X}$. Note that the equivalence between the items in Finsler's lemma (that is, its necessary and sufficient characteristic) loses its validity if the variable structure is constrained. It will however be possible to obtain sufficient conditions that provide satisfactory results for design problems (as will be clarified in next chapters). The next lemma provides a condition less conservative than Lemma 1.6 by using condition *iv*) of Finsler's lemma.

Lemma 1.7. *For a given γ, if there exist matrices $P(\alpha(k))' = P(\alpha(k)) > 0$ and $\mathscr{X}$ such that*

$$\mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0 \tag{1.27}$$

for all $\alpha(k) \in \mathscr{U}$, where $\mathscr{Q}$ is given as in Lemma 1.6 and

$$\mathscr{B} = \begin{bmatrix} -\mathbf{I} & A(\alpha(k)) & B_w(\alpha(k)) \end{bmatrix}$$

then system (1.1)-(1.2)*, with $B_u(\alpha(k)) = 0$ and $D_u(\alpha(k)) = 0$, is globally asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.*

Proof: Multiplying inequality (1.27) on the right by $\mathscr{B}^\perp$ and on the left by $\mathscr{B}^{\perp\prime}$, with $\mathscr{B}^\perp$ defined as in Lemma 1.6, one has

$$\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^\perp < 0$$

since $\mathscr{B}\mathscr{B}^{\perp} = 0$. ■

1.2.5 General Overview of the Results

In general, the contributions of this work are summarized in three main concepts: the use of parameter-dependent Lyapunov functions, the use of extra variables introduced by Finsler's lemma and in the parametric variation modeling of $\alpha(k)$. The results are illustrated on the problem of synthesis of state feedback controllers with fixed and LPV gains, and the problem of synthesis of robust and LPV filters.

For the synthesis of LPV controllers, we use an affine Lyapunov function, as in (1.10), the extra variable $\mathscr{X}$ with the structure

$$\mathscr{X} = [F(\alpha(k))' \;\; F(\alpha(k))'G(\alpha(k+1))' \;\; F(\alpha(k))'H(\alpha(k+1))']' \tag{1.28}$$

and a model for the parametric variation given by

- for $0 \leq b \leq 0.5$

$$g(\alpha_i(k)) = \begin{cases} \alpha_i & \text{if } \alpha_i \leq b \\ b & \text{if } b \leq \alpha_i \leq 1-b \\ b & \text{if } 1-b \leq \alpha_i \end{cases} \quad q(\alpha_i(k)) = \begin{cases} b & \text{if } \alpha_i \leq b \\ b & \text{if } b \leq \alpha_i \leq 1-b \\ 1-\alpha_i & \text{if } 1-b \leq \alpha_i \end{cases}$$

- for $1 \geq b \geq 0.5$

$$g(\alpha_i(k)) = \begin{cases} \alpha_i & \text{if } \alpha_i \leq 1-b \\ \alpha_i & \text{if } 1-b \leq \alpha_i \leq b \\ b & \text{if } b \leq \alpha_i \end{cases} \quad q(\alpha_i(k)) = \begin{cases} b & \text{if } \alpha_i \leq 1-b \\ 1-\alpha_i & \text{if } 1-b \leq \alpha_i \leq b \\ 1-\alpha_i & \text{if } b \leq \alpha_i \end{cases}$$

where $b \in [0,1]$ is a given constant. Note that $b = 0$ models time-invariant systems and $b = 1$ arbitrarily time-varying systems. A simplified structure, based on a memory device, is considered in order to deal with possible delays in the control input.

The procedure consists of augmenting the state vector, adding the values stored in memory, and proceeding with the analysis for the augmented system. This approach is feasible for small magnitudes of delay. This topic is covered in Chapter 2, which deals with LPV systems.

For the synthesis of robust controllers (fixed gain), we use a Lyapunov function dependent on two instants of time, as in (1.11), the extra variable $\mathscr{X}$ with the structure

$$\mathscr{X} = [F' \;\; F'G(\alpha(k+1))' \;\; F'H(\alpha(k+1))']' \tag{1.29}$$

and a model for the parametric variation given by $g(\alpha_i(k)) = \alpha_i(k)$ and $q(\alpha_i(k)) = (1-\alpha_i(k))$, that is, arbitrarily rates of variation. The aim is to stabilize dynamic systems across communication networks. We consider time-varying sampling rates, and time delays that can be treated with the use of memory controllers (similar to the previous case). This topic is covered in Chapter 3, which deals with systems with uncertain sampling rates.

For the synthesis of LPV filters we use both affine Lyapunov functions, as well as Lyapunov functions that depend two instants of time. The structure chosen for the extra variable $\mathscr{X}$ is shown below

$$\mathscr{X} = [F' \;\; F'TG(\beta)'T^{-1} \;\; F'TH(\beta)']' \tag{1.30}$$

and the model for the parametric variation is the same as it was in the case of LPV controllers. This topic is covered in Chapter 4, which also presents conditions for the design of robust filters.

Note that all results use the extra variable $\mathscr{X}$ introduced by Finsler's lemma. Specifically, the design conditions presented in the subsequent chapters are based on Lemma 1.7 and lead to BMIs due to the structure chosen for the variable $\mathscr{X}$. However, LMI conditions may be obtained with particular choices for matrices $G(\cdot)$ and $H(\cdot)$ in $\mathscr{X}$. This fact is explored in the search for lower values of the upper bound γ to the $\mathscr{H}_\infty$ performance, in an algorithmic process that consists of alternating between convex optimization problems described by LMIs.

Finally, the main difference between the results presented in chapters 2, 3 and 4 is in the combination used for the structure of $\mathscr{X}$, the type of Lyapunov functions, and the model for the rates of variation in each specific application: robust networked control, gain-scheduled control and robust and LPV filtering.

Chapter 2

A BMI approach for $\mathscr{H}_\infty$ gain scheduling of discrete time-varying systems

2.1 Introduction

One cannot deny the fact that gain scheduling has become an important topic within control system theory [67], [95]. As shown in [62], this technique can extend the validity of the linearization approach of nonlinear systems to a range of operating points. Consequently, gain scheduled controllers are guaranteed to work in a larger region instead of only in a certain neighborhood of a single operating point. The main idea is to model the system in such a way that the different operating points are parametrized by one or more variables, commonly called scheduling variables [62]. Stability is then assured by a closed-loop Lyapunov function and a family of linear controllers, whose parameters are changed in accordance with the scheduling rules. Although there are many articles addressing the topic of gain scheduling, we call the readers' attention to the references [2], [3], [98].

The use of linear parameter varying (LPV) structures to model certain classes of nonlinear systems has provided an interesting framework for gain scheduling control by means of convex optimization [23], [93], [95], [97]. It is worth mentioning that the state-space dynamic matrices of LPV systems depend on time-varying parameters that are assumed to be measured online. The use of such parameters in defining scheduling rules, brings extra information during the design step that may lead to less conservative results when compared to robust control strategies.

Lyapunov theory has been extensively used as the main tool to deal with the synthesis of gain-scheduled controllers. In many cases, it might be possible to express the design conditions as an optimization problem in terms of linear matrix inequalities (LMIs), that can be numerically handled by specific software packages [24], [70], [101]. To guarantee robustness against disturbances, the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms have been frequently used as performance indices. Recent works include: [73] where the problem of stabilizability and $\mathscr{H}_\infty$ control of discrete-time LPV systems is investigated by means of gain scheduled state feedback controllers, [110] in which gain scheduled controllers for linear fractional transformation (LFT) systems is designed using parameter-dependent Lyapunov functions, [36] where gain scheduled $\mathscr{H}_2$ controllers for affine LPV systems are proposed, [38] in which robust and gain scheduled controllers for LFT parameter-dependent systems are designed using duality theory, and [113] where switching $\mathscr{H}_\infty$ controllers for a class of LPV systems scheduled along a measurable parameter trajectory are addressed.

Bilinear matrix inequalities (BMIs) have also been applied in the study of control of LPV systems. It is well-known that optimization problems expressed in terms of BMIs are non-convex. Nevertheless, the use of BMIs may represent a good strategy for problems with either no solution, or where only sufficient conditions available in the literature, as well as to improve the closed-loop performance. See, for instance, [43], [61], [103], [116] and references therein.

Another important aspect observed in a large number of dynamic systems, including LPV plants, is the presence of time delays. In many cases, a good characterization of time delays is required since they may represent a source of instability to the system trajectories. When the delay is known, a simple strategy consists in defining an augmented state vector, and then to design a standard controller that takes into account the delayed states (*i.e.* a memory controller). Other approaches could be used to cope with time delays, as for instance the ones based on the Lypuanov-Krasovskii functionals, which result in general in more complex conditions that demand a higher computational effort.

The aim of this chapter is to provide gain-scheduled memory controllers to stabilize discrete time-varying linear systems with bounded rates of parameter variation. A simplified framework for possible time delays is assumed, where the delay is constant and a memory is used to store the delayed information. The use of a memory in the feedback loop allows one to cope with time delays without making use of more complex Lyapunov functionals. All the system matrices are assumed to be affected by the time-varying parameters, which are assumed to lie inside a known polytope. An $\mathscr{H}_\infty$ guaranteed cost, which reflects the worst-case energy gain of the system, provides robustness with respect to unmodeled uncertainties. A preliminary version of the work presented in this chapter appeared in [14], where the time-varying parameters were allowed to vary arbitrarily fast inside the polytope. Here, a more precise parameter variation modeling is used to take into account the bounds on the rates of parameter variation, providing synthesis procedures to cope with parameters that can be frozen or can vary slowly or arbitrarily fast. Lyapunov theory is applied to assure the closed-loop stability with $\mathscr{H}_\infty$ disturbance attenuation, with a parameter-dependent Lyapunov function that reduces the conservatism of the proposed method, resulting in a more general approach when compared to methods based on quadratic stability. Extra variables introduced by Finsler's lemma may be freely explored in the search for better performance of the LPV system, and lead to design conditions expressed in terms of BMIs. The gain-scheduled memory controller is then obtained through the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of BMI constraints formulated only in terms of the vertices of the polytopic model. An iterative scheme is proposed exploiting the fact that the BMIs reduce to LMIs by fixing some variables. Some results from the literature concerned with stability without time delays may be obtained as a particular case of the proposed method. Numerical examples illustrate the proposed conditions. The strategy proposed here could also be adapted to cope with the design of gain-scheduling controllers based on other types of storage functions, such as Lyapunov-Krasovskii functionals.

2.2 Preliminaries and Problem Statement

Consider the time-varying discrete-time system,

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B_{du}(\alpha(k))u(k-\tau) + B_u(\alpha(k))u(k) + B_w(\alpha(k))w(k),\ x(0)=0 \\ y(k) &= C(\alpha(k))x(k) + D_{du}(\alpha(k))u(k-\tau) + D_u(\alpha(k))u(k) + D_w(\alpha(k))w(k) \end{aligned} \tag{2.1}$$

where τ represents the discrete-time delay, $x(k) \in \mathbb{R}^n$ is the state space vector, $u(k) \in \mathbb{R}^m$ is the control signal, $w(k) \in \mathbb{R}^r$ is the $l_2[0,\infty)$ noise and $y(k) \in \mathbb{R}^q$ is the controlled output. The time delay τ is an integer number assumed to be known and constant. The time-varying vector of parameters $\alpha(k)$ belongs to the unit simplex

$$\mathscr{U}_N = \left\{ \psi \in \mathbb{R}^N : \sum_{i=1}^{N} \psi_i = 1,\ \psi_i \geq 0\ ,\ i = 1,\ldots,N \right\}$$

for all $k \geq 0$ with bounded rates of parameter variation of percentage $b \in [0,1]$. For instance, $b = 0.05$ indicates that the parameters are constrained to vary only 5% of their original values between two instants of time. The time-invariant case is modeled by $b = 0$ and arbitrarily fast variations by $b = 1$.

All matrices are real, with appropriate dimensions, belonging to the polytope

$$\hat{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha(k)) & B_u(\alpha(k)) \\ \hline B_{du}(\alpha(k)) & B_w(\alpha(k)) \\ \hline C(\alpha(k)) & D_u(\alpha(k)) \\ \hline D_{du}(\alpha(k)) & D_w(\alpha(k)) \end{array}\right] = \sum_{i=1}^{N} \alpha_i(k) \left[\begin{array}{c|c} A_i & B_{ui} \\ \hline B_{dui} & B_{wi} \\ \hline C_i & D_{ui} \\ \hline D_{dui} & D_{wi} \end{array}\right],\ \alpha(k) \in \mathscr{U}_N \right\} \tag{2.2}$$

More specifically, the system matrices are given, for any time $k \geq 0$, by the convex combination of the well-defined vertices of the polytope $\hat{\mathscr{P}}$. As usual in gain-scheduling control, it is also assumed that the parameters $\alpha(k)$ are measured online.

In order to guarantee the stability of system (2.1), a memory state feedback controller with a parameter-dependent gain is designed. Using extra state variables $z(k)$ to store the delayed values of the control signal, system (2.1) can be rewritten as follows [5]

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\alpha(k))\tilde{x}(k) + \tilde{B}_u(\alpha(k))u(k) + \tilde{B}_w(\alpha(k))w(k),\ \tilde{x}(0) = 0 \\ y(k) &= \tilde{C}(\alpha(k))\tilde{x}(k) + \tilde{D}_u(\alpha(k))u(k) + \tilde{D}_w(\alpha(k))w(k) \end{aligned} \tag{2.3}$$

where $\tilde{x}(k) = [x(k)'\ z(k)']' \in \mathbb{R}^{n+m\tau}$ and

$$\tilde{A}(\alpha(k)) = \begin{bmatrix} A(\alpha(k)) & B_{du}(\alpha(k)) & 0 & \dots & 0 \\ 0 & 0 & \mathbf{I} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{I} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},\ \tilde{B}_u(\alpha(k)) = \begin{bmatrix} B_u(\alpha(k)) \\ 0 \\ 0 \\ \vdots \\ \mathbf{I} \end{bmatrix},$$

$$\begin{aligned} \tilde{B}_w(\alpha(k))' &= \begin{bmatrix} B_w(\alpha(k))' & 0 & 0 & \dots & 0 \end{bmatrix},\ \tilde{D}_w(\alpha(k)) = D_w(\alpha(k)) \\ \tilde{C}(\alpha(k)) &= \begin{bmatrix} C(\alpha(k)) & D_{du}(\alpha(k)) & 0 & \dots & 0 \end{bmatrix},\ \tilde{D}_u(\alpha(k)) = D_u(\alpha(k)) \end{aligned} \tag{2.4}$$

The memory control law is given by

$$u(k) = \begin{bmatrix} K_x(\alpha(k)) & K_d(\alpha(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} = K(\alpha(k))\tilde{x}(k) \tag{2.5}$$

where $K(\alpha(k)) = [K_x(\alpha(k)) \;\; K_d(\alpha(k))]$, yielding the closed-loop system

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}_{cl}(\alpha(k))\tilde{x}(k) + \tilde{B}_w(\alpha(k))w(k), \; \tilde{x}(0) = 0 \\ y(k) &= \tilde{C}_{cl}(\alpha(k))\tilde{x}(k) + \tilde{D}_w(\alpha(k))w(k) \end{aligned} \tag{2.6}$$

with $\tilde{x}(k) \in \mathbb{R}^{n+m\tau}$, $w(k) \in \mathbb{R}^r$, $y(k) \in \mathbb{R}^q$ and

$$\tilde{A}_{cl}(\alpha(k)) = \tilde{A}(\alpha(k)) + \tilde{B}_u(\alpha(k))K(\alpha(k)) \; , \;\; \tilde{C}_{cl}(\alpha(k)) = \tilde{C}(\alpha(k)) + \tilde{D}_u(\alpha(k))K(\alpha(k)) \tag{2.7}$$

The control problem to be dealt with may be stated as follows.

Problem 1. *Find parameter-dependent matrices $K_x(\alpha(k)) \in \mathbb{R}^{m\times n}$ and $K_d(\alpha(k)) \in \mathbb{R}^{m\times m\tau}$ of the control law (2.5), such that the closed-loop system (2.6) is asymptotically stable, and an upper bound $\gamma > 0$ to the $\mathscr{H}_\infty$ performance is minimized, that is*

$$\sup_{w\neq 0} \frac{\|y\|_2^2}{\|w\|_2^2} < \gamma^2 \tag{2.8}$$

with $w \in l_2[0,\infty)$.

Condition (2.8) for a given closed-loop discrete time-varying linear system may be characterized by the discrete-time version of the *bounded real lemma* in terms of parameter-dependent LMIs, as was presented in [29], [37]. The result is extended here in the context of parameter-dependent time-varying systems, as follows.

Lemma 2.1. *For a given γ, if there exists a bounded matrix sequence $P(\alpha(k))' = P(\alpha(k)) > 0$ such that*[1]

$$\begin{bmatrix} -P(\alpha(k)) & P(\alpha(k))\tilde{A}(\alpha(k))' & P(\alpha(k))\tilde{C}(\alpha(k))' & \mathbf{0} \\ (\star) & -P(\alpha(k+1)) & \mathbf{0} & \tilde{B}(\alpha(k)) \\ (\star) & (\star) & -\gamma\mathbf{I} & \tilde{D}(\alpha(k)) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{2.9}$$

[1]The symbol $(\star)$ indicates symmetric blocks in the LMIs.

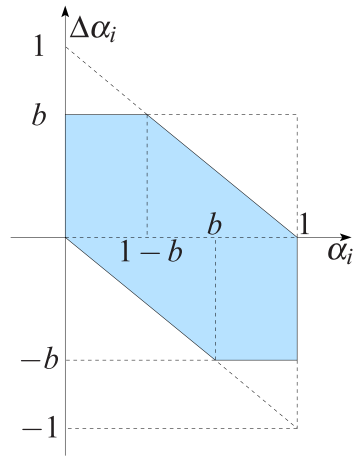

Figure 2.1: Region on the plane $\Delta\alpha_i \times \alpha_i$ where $\Delta\alpha_i$ can assume values as a function of α_i (dark region).

for all $\alpha(k) \in \mathscr{U}_N$, then the closed-loop system (2.6) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Note that, since the parameters lie inside a unit simplex, the rates of variation are intrinsically lower bounded by $-b$ and upper bounded by b, $b \in [0,1]$. In order to develop a model[2] for the parameter variation when $-b < \Delta\alpha_i(k) < b,\ b \neq 0$, note that the feasible values of $\Delta\alpha_i(k)$ depend on the actual values of $\alpha_i(k)$, as shown in Figure 2.1 (darken area). Thus, any pair $(\alpha_i, \Delta\alpha_i)$ belongs to the polytope Λ_i, $i = 1,\ldots,N$ given by

$$\begin{aligned} \Lambda_i &\triangleq \left\{ \delta \in \mathbb{R}^2 : \delta = \sum_{j=1}^{6} \lambda_j s_j, \quad \lambda \in \mathscr{U}_6 \right\}, \\ S = [s_1 \cdots s_6] &= \left[\begin{array}{c:c:c:c:c:c} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{array} \right], \end{aligned} \tag{2.10}$$

that is, Λ_i represents the convex combination of the extremes (vertices) of the feasible area.

[2]For simplicity, the same b is considered for all $\alpha_i,\ i = 1,\ldots,N$.

To construct the $(\alpha, \Delta\alpha)$-space, the Cartesian product of all Λ_i, $i = 1, \ldots, N$ must be considered, taking into account that the new vertices must satisfy $\alpha_1 + \cdots + \alpha_N = 1$ and $\Delta\alpha_1 + \cdots + \Delta\alpha_N = 0$. The resulting polytope, called Λ, is then given by

$$\Lambda \triangleq \left\{ \delta \in \mathbb{R}^{2N} : \delta = \sum_{i=1}^{M} \lambda_i q_i, \quad \lambda \in \mathscr{U}_M \right\}, \tag{2.11}$$

where $q_i \in \mathbb{R}^{2N}$ are given vectors. Thus, the first step to search for a solution to any LMI/BMI depending on both α and $\Delta\alpha$ is to lift the inequalities to the λ-space, by observing that from (2.11) one has

$$\begin{bmatrix} \alpha \\ \Delta\alpha \end{bmatrix} = Q\lambda, \quad Q = [q_1 \cdots q_M] \in \mathbb{R}^{2N \times M}, \quad \lambda \in \mathscr{U}_M. \tag{2.12}$$

Therefore, in the case of parameter-dependent matrices affine on $\alpha(k)$, that is

$$X(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) X_i, \quad \alpha_i(k) = \sum_{j=1}^{M} \lambda_j Q_{ij} \tag{2.13}$$

$$X(\alpha(k+1)) = \sum_{i=1}^{N} (\alpha_i(k) + \Delta\alpha_i(k)) X_i, \quad \Delta\alpha_i(k) = \sum_{j=1}^{M} \lambda_j Q_{(i+N)j} \tag{2.14}$$

it follows that

$$X(\alpha(k)) = \bar{X}(\lambda) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_j Q_{ij} X_i = \sum_{j=1}^{M} \lambda_j \bar{X}_j \tag{2.15}$$

$$X(\alpha(k+1)) = \tilde{X}(\lambda) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_j (Q_{ij} + Q_{(i+N)j}) X_i = \sum_{j=1}^{M} \lambda_j \tilde{X}_j \tag{2.16}$$

where

$$\bar{X}_j = \sum_{i=1}^{N} Q_{ij} X_i \tag{2.17}$$

$$\tilde{X}_j = \sum_{i=1}^{N} (Q_{ij} + Q_{(i+N)j}) X_i \tag{2.18}$$

Another preliminary result, Finsler's lemma, is reproduced here for convenience.

Lemma 2.2. *Let $\xi \in \mathbb{R}^a$, $\mathscr{Q} = \mathscr{Q}' \in \mathbb{R}^{a \times a}$, $\mathscr{B} \in \mathbb{R}^{b \times a}$ with $rank(\mathscr{B}) < a$, and $\mathscr{B}^\perp$ a basis for the null-space of $\mathscr{B}$ (i.e. $\mathscr{B}\mathscr{B}^\perp = 0$). The following statements are equivalent.*

i) $\xi'\mathscr{Q}\xi < 0,\ \forall \mathscr{B}\xi = 0,\ \xi \neq 0$;

ii) $\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^{\perp} < 0$;

iii) $\exists\ \mu \in \mathbb{R} : \mathscr{Q} - \mu\mathscr{B}'\mathscr{B} < 0$;

iv) $\exists\ \mathscr{X} \in \mathbb{R}^{a\times b} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0$.

Proof: See [31]. ∎

The variables μ and $\mathscr{X}$ in statements *iii)* and *iv)* of Lemma 2.2 allow one to present a more general version of Lemma 2.1. As pointed out in [31], these variables represent extra degrees of freedom that may be exploited for design purposes. By considering the particular structure

$$\mathscr{X} = [F(\alpha(k))' \ \ F(\alpha(k))'G(\alpha(k+1))' \ \ F(\alpha(k))'H(\alpha(k+1))']' \tag{2.19}$$

the following condition is obtained.

Theorem 2.1. *For a given $\gamma > 0$, if there exists a bounded matrix sequence $F(\alpha(k))$, $G(\alpha(k))$, $P(\alpha(k))' = P(\alpha(k)) > 0$ and $H(\alpha(k))$, such that*

$$\begin{bmatrix} P(\alpha(k+1)) - F(\alpha(k)) - F(\alpha(k))' & \hat{\mathscr{F}}_{12} & \hat{\mathscr{F}}_{13} & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & \hat{\mathscr{F}}_{23} & \tilde{B}_{wcl}(\alpha(k)) \\ (\star) & (\star) & \hat{\mathscr{F}}_{33} & \tilde{D}_{wcl}(\alpha(k)) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{2.20}$$

$$\begin{aligned} \hat{\mathscr{F}}_{12} &= F(\alpha(k))\tilde{A}_{cl}(\alpha(k))' - F(\alpha(k))'G(\alpha(k+1))' \\ \hat{\mathscr{F}}_{13} &= F(\alpha(k))\tilde{C}_{cl}(\alpha(k))' - F(\alpha(k))'H(\alpha(k+1))' \\ \hat{\mathscr{F}}_{22} &= G(\alpha(k+1))F(\alpha(k))\tilde{A}_{cl}(\alpha(k))' + \tilde{A}_{cl}(\alpha(k))F(\alpha(k))'G(\alpha(k+1))' - P(\alpha(k)) \\ \hat{\mathscr{F}}_{23} &= G(\alpha(k+1))F(\alpha(k))\tilde{C}_{cl}(\alpha(k))' + \tilde{A}_{cl}(\alpha(k))F(\alpha(k))'H(\alpha(k+1))' \\ \hat{\mathscr{F}}_{33} &= H(\alpha(k+1))F(\alpha(k))\tilde{C}_{cl}(\alpha(k))' + \tilde{C}_{cl}(\alpha(k))F(\alpha(k))'H(\alpha(k+1))' - \gamma\mathbf{I} \end{aligned}$$

for all $(\alpha(k), \Delta\alpha(k)) \in \Lambda$, then the closed-loop system (2.6) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof: Firstly, using Schur complement, inequality (2.20) can be rewritten as follows

$$\begin{bmatrix} P(\alpha(k+1)) - F(\alpha(k)) - F(\alpha(k))' & \hat{\mathscr{F}}_{12} & F\tilde{C}_{cl}(\alpha(k))' - F'H(\alpha(k+1))' \\ (\star) & \hat{\mathscr{F}}_{22} & \hat{\mathscr{F}}_{23} \\ (\star) & (\star) & \hat{\mathscr{F}}_{33} \end{bmatrix} + \gamma^{-1}\hat{\mathscr{F}}_4(\alpha(k))\hat{\mathscr{F}}_4(\alpha(k))' < 0 \quad (2.21)$$

where

$$\hat{\mathscr{F}}_4(\alpha(k)) = \begin{bmatrix} \mathbf{0} & \tilde{B}_{wcl}(\alpha(k))' & \tilde{D}_{wcl}(\alpha(k))' \end{bmatrix}'$$

Secondly, by setting

$$\mathscr{Q} = \begin{bmatrix} P(\alpha(k+1)) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^{-1}\tilde{B}_{wcl}(\alpha(k))\tilde{B}_{wcl}(\alpha(k))' - P(\alpha(k)) & \gamma^{-1}\tilde{B}_{wcl}(\alpha(k))\tilde{D}_{wcl}(\alpha(k))' \\ \mathbf{0} & \gamma^{-1}\tilde{D}_{wcl}(\alpha(k))\tilde{B}_{wcl}(\alpha(k))' & \gamma^{-1}\tilde{D}_{wcl}(\alpha(k))\tilde{D}_{wcl}(\alpha(k))' - \gamma\mathbf{I} \end{bmatrix}$$

$$\mathscr{B} = \begin{bmatrix} -\mathbf{I} & \tilde{A}_{cl}(\alpha(k))' & \tilde{C}_{cl}(\alpha(k))' \end{bmatrix}, \; \xi = \begin{bmatrix} \tilde{x}(k+1)' & \tilde{x}(k)' & w(k)' \end{bmatrix}',$$

with $\mathscr{X}$ given by (2.19), inequality (2.21) yields statement *iv)* of Lemma 2.2. Finally, if statement *iv)* of Lemma 2.2 holds then statement *ii)* also holds and Lemma 2.1 follows immediately. The fact that *iv)* $\Rightarrow$ *ii)* can be verified by multiplying (2.21) on the left by $\mathscr{B}^{\perp}$ and on the right by $\mathscr{B}^{\perp\prime}$, where

$$\mathscr{B}^{\perp} = \begin{bmatrix} \tilde{A}_{cl}(\alpha(k))' & \tilde{C}_{cl}(\alpha(k))' \\ \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}.$$

■

The conditions of Theorem 2.1 exhibit nonlinearities and must be tested at all points of the unit simplex $\mathscr{U}_N$, *i.e.*, for an infinite number of points. Moreover, the unknown parameter-dependent matrices appear as functions of both $\alpha(k+1)$ and $\alpha(k)$. Hence, the main goal hereafter is to obtain finite-dimensional conditions in terms of the vertices of the polytope $\hat{\mathscr{P}}$ to solve Problem 1, considering the particular structure for the Lyapunov matrix (similar structures for $F(\alpha(k))$, $G(\alpha(k+1))$ and $H(\alpha(k+1))$ have been used)

$$P(\alpha(k)) = \alpha_1(k)P_1 + \alpha_2(k)P_2 + \ldots + \alpha_N(k)P_N,\ \alpha(k) \in \mathscr{U}_N. \tag{2.22}$$

More complex structures, as for instance with polynomial dependence on $\alpha(k)$, could be used following the ideas depicted in [85], yielding BMI conditions that would be more precise at the expense of being much more involved. Now, considering the λ-space presented, using the Schur complement, a change of variables and exploring the extra variables provided by Lemma 2.2, BMI conditions assuring the existence of such controllers are given in the next section.

2.3 Main Results

Theorem 2.2. *Given the augmented discrete-time system (2.3) and matrix Q as in* (2.12)*, if there exist matrices L_i, H_i, F_i, G_i, $P_i = P_i' > 0$, with appropriate dimensions, for $i = 1,\ldots,N$ and a scalar $\gamma > 0$, the control law (2.5), with matrices given by*

$$K(\alpha(k)) = \begin{bmatrix} K_x(\alpha(k)) & K_d(\alpha(k)) \end{bmatrix} = L(\alpha(k))(F(\alpha(k))')^{-1} \tag{2.23}$$

where

$$L(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)L_i, \quad F(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)F_i,\ \alpha(k) \in \mathscr{U}_N \tag{2.24}$$

assures the asymptotic stability of the closed-loop system (2.6) and an $\mathscr{H}_\infty$ guaranteed cost γ provided that, for matrices $\bar{L}_i$, $\bar{H}_i$, $\bar{F}_i$, $\bar{G}_i$, $\bar{P}_i$, $\hat{A}_i$, $\hat{B}_{ui}$, $\hat{B}_{wi}$, $\hat{C}_i$, $\hat{D}_{ui}$ and $\hat{D}_{wi}$ given as in (2.17) *and $\tilde{H}_i$, $\tilde{G}_i$, $\tilde{P}_i$ as in* (2.18)

$$\Xi_i \triangleq \begin{bmatrix} \tilde{P}_i - \bar{F}_i - \bar{F}_i' & \mathscr{F}_{12} & \bar{F}_i\hat{C}_i' + \bar{L}_i'\hat{D}_{ui}' - \bar{F}_i'\tilde{H}_i' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & \mathscr{F}_{23} & \hat{B}_{wi} \\ (\star) & (\star) & \mathscr{F}_{33} & \hat{D}_{wi} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{2.25}$$

$$i = 1,\ldots,M$$

$$
\begin{aligned}
\mathscr{F}_{12} &= \bar{F}_i\hat{A}_i' + \bar{L}_i'\hat{B}_{ui}' - \bar{F}_i'\tilde{G}_i', \\
\mathscr{F}_{22} &= \tilde{G}_i\bar{F}_i\hat{A}_i' + \hat{A}_i\bar{F}_i'\tilde{G}_i' + \tilde{G}_i\bar{L}_i'\hat{B}_{ui}' + \hat{B}_{ui}\bar{L}_i\tilde{G}_i' - \bar{P}_i, \\
\mathscr{F}_{23} &= \tilde{G}_i\bar{F}_i\hat{C}_i' + \tilde{G}_i\bar{L}_i'\hat{D}_{ui}' + \hat{A}_i\bar{F}_i'\tilde{H}_i' + \hat{B}_{ui}\bar{L}_i\tilde{H}_i', \\
\mathscr{F}_{33} &= \tilde{H}_i\bar{F}_i\hat{C}_i' + \hat{C}_i\bar{F}_i'\tilde{H}_i' + \tilde{H}_i\bar{L}_i'\hat{D}_{ui}' + \hat{D}_{ui}\bar{L}_i\tilde{H}_i' - \gamma\mathbf{I}
\end{aligned}
$$

$$
\Xi_{ik} \triangleq \begin{bmatrix} \bar{\mathscr{F}}_{11} & \bar{\mathscr{F}}_{12} & \bar{\mathscr{F}}_{13} & \mathbf{0} \\ (\star) & \bar{\mathscr{F}}_{22} + \bar{\mathscr{F}}_{22}' - 2\bar{P}_i - \bar{P}_k & \bar{\mathscr{F}}_{23} & 2\hat{B}_{wi} + \hat{B}_{wk} \\ (\star) & (\star) & \bar{\mathscr{F}}_{33} + \bar{\mathscr{F}}_{33}' - 3\gamma\mathbf{I} & 2\hat{D}_{wi} + \hat{D}_{wk} \\ (\star) & (\star) & (\star) & -3\gamma\mathbf{I} \end{bmatrix} < 0 \tag{2.26}
$$

$$
i = 1, \ldots, M, \; k = 1, \ldots M, \; i \neq k
$$

$$
\begin{aligned}
&\bar{\mathscr{F}}_{11} = 2\tilde{P}_i + \tilde{P}_k - 2\bar{F}_i - 2\bar{F}_i' - \bar{F}_k - \bar{F}_k', \; \bar{\mathscr{F}}_{12} = \bar{F}_i\hat{A}_i' + \bar{F}_i\hat{A}_k' + \bar{F}_k\hat{A}_i' + \bar{L}_i'\hat{B}_{ui}' + \bar{L}_i'\hat{B}_{uk}' + \bar{L}_k'\hat{B}_{ui}' \\
&-\bar{F}_i'\tilde{G}_i' - \bar{F}_i'\tilde{G}_k' - \bar{F}_k'\tilde{G}_i', \; \bar{\mathscr{F}}_{13} = \bar{F}_i\hat{C}_i' + \bar{F}_i\hat{C}_k' + \bar{F}_k\hat{C}_i' + \bar{L}_i'\hat{D}_{ui}' + \bar{L}_i'\hat{D}_{uk}' + \bar{L}_k'\hat{D}_{ui}' - \bar{F}_i'\tilde{H}_i' - \bar{F}_i'\tilde{H}_k' \\
&-\bar{F}_k'\tilde{H}_i', \; \bar{\mathscr{F}}_{22} = \tilde{G}_i\bar{F}_i\hat{A}_k' + \tilde{G}_k\bar{F}_i\hat{A}_i' + \tilde{G}_i\bar{F}_k\hat{A}_i' + \tilde{G}_i\bar{L}_i'\hat{B}_{uk}' + \tilde{G}_k\bar{L}_i'\hat{B}_{ui}' + \tilde{G}_i\bar{L}_k'\hat{B}_{ui}', \; \bar{\mathscr{F}}_{23} = \tilde{G}_i\bar{F}_i\hat{C}_k' \\
&+\tilde{G}_k\bar{F}_i\hat{C}_i' + \tilde{G}_i\bar{F}_k\hat{C}_i' + \tilde{G}_i\bar{L}_i'\hat{D}_{uk}' + \tilde{G}_k\bar{L}_i'\hat{D}_{ui}' + \tilde{G}_i\bar{L}_k'\hat{D}_{ui}' + \hat{A}_i\bar{F}_i'\tilde{H}_k' + \hat{A}_k\bar{F}_i'\tilde{H}_i' + \hat{A}_i\bar{F}_k'\tilde{H}_i' \\
&+\hat{B}_{ui}\bar{L}_i\tilde{H}_k' + \hat{B}_{uk}\bar{L}_i\tilde{H}_i' + \hat{B}_{ui}\bar{L}_k\tilde{H}_i', \; \bar{\mathscr{F}}_{33} = \tilde{H}_i\bar{F}_i\hat{C}_k' + \tilde{H}_k\bar{F}_i\hat{C}_i' \\
&+\tilde{H}_i\bar{F}_k\hat{C}_i' + \tilde{H}_i\bar{L}_i'\hat{D}_{uk}' + \tilde{H}_k\bar{L}_i'\hat{D}_{ui}' + \tilde{H}_i\bar{L}_k'\hat{D}_{ui}'
\end{aligned}
$$

$$
\Xi_{ik\ell} \triangleq \begin{bmatrix} \tilde{\mathscr{F}}_{11} & \tilde{\mathscr{F}}_{12} & \tilde{\mathscr{F}}_{13} & \mathbf{0} \\ (\star) & \tilde{\mathscr{F}}_{22} + \tilde{\mathscr{F}}_{22}' - 2\bar{P}_i - 2\bar{P}_k - 2\bar{P}_\ell & \tilde{\mathscr{F}}_{23} & \hat{B}_{wi} + \hat{B}_{wk} + \hat{B}_{w\ell} \\ (\star) & (\star) & \tilde{\mathscr{F}}_{33} + \tilde{\mathscr{F}}_{33}' - 6\gamma\mathbf{I} & \hat{D}_{wi} + \hat{D}_{wk} + \hat{D}_{w\ell} \\ (\star) & (\star) & (\star) & -6\gamma\mathbf{I} \end{bmatrix} < 0
$$

$$
i = 1, \ldots, M-2, \; k = i+1, \ldots, M-1, \; \ell = k+1, \ldots, M, \tag{2.27}
$$

$$
\begin{aligned}
&\tilde{\mathscr{F}}_{11} = 2\tilde{P}_i + 2\tilde{P}_k + 2\tilde{P}_\ell - 2\bar{F}_i - 2\bar{F}_i' - 2\bar{F}_k - 2\bar{F}_k' - 2\bar{F}_\ell - 2\bar{F}_\ell', \; \tilde{\mathscr{F}}_{12} = \bar{F}_i\hat{A}_k' + \bar{F}_k\hat{A}_i' + \bar{F}_i\hat{A}_\ell' \\
&+\bar{F}_\ell\hat{A}_i' + \bar{F}_\ell\hat{A}_k' + \bar{F}_k\hat{A}_\ell' + \bar{L}_i'\hat{B}_{uk}' + \bar{L}_k'\hat{B}_{ui}' + \bar{L}_i'\hat{B}_{u\ell}' + \bar{L}_\ell'\hat{B}_{ui}' + \bar{L}_\ell'\hat{B}_{uk}' + \bar{L}_k'\hat{B}_{u\ell}' - \bar{F}_i'\tilde{G}_k' - \bar{F}_k'\tilde{G}_i' \\
&-\bar{F}_i'\tilde{G}_\ell' - \bar{F}_\ell'\tilde{G}_i' - \bar{F}_\ell'\tilde{G}_k' - \bar{F}_k'\tilde{G}_\ell', \; \tilde{\mathscr{F}}_{13} = \bar{F}_i\hat{C}_k' + \bar{F}_k\hat{C}_i' + \bar{F}_i\hat{C}_\ell' + \bar{F}_\ell\hat{C}_i' + \bar{F}_\ell\hat{C}_k' + \bar{F}_k\hat{C}_\ell' + \bar{L}_i'\hat{D}_{uk}' \\
&+\bar{L}_k'\hat{D}_{ui}' + \bar{L}_i'\hat{D}_{u\ell}' + \bar{L}_\ell'\hat{D}_{ui}' + \bar{L}_\ell'\hat{D}_{uk}' + \bar{L}_k'\hat{D}_{u\ell}' - \bar{F}_i'\tilde{H}_k' - \bar{F}_k'\tilde{H}_i' - \bar{F}_i'\tilde{H}_\ell' - \bar{F}_\ell'\tilde{H}_i' - \bar{F}_\ell'\tilde{H}_k' \\
&-\bar{F}_k'\tilde{H}_\ell', \; \tilde{\mathscr{F}}_{22} = \tilde{G}_i\bar{F}_k\hat{A}_\ell' + \tilde{G}_i\bar{F}_\ell\hat{A}_k' + \tilde{G}_k\bar{F}_i\hat{A}_\ell' + \tilde{G}_k\bar{F}_\ell\hat{A}_i' + \tilde{G}_\ell\bar{F}_i\hat{A}_k' + \tilde{G}_\ell\bar{F}_k\hat{A}_i' + \tilde{G}_i\bar{L}_k'\hat{B}_{u\ell}'
\end{aligned}
$$

$$+\tilde{G}_i\bar{L}'_\ell\hat{B}'_{uk}+\tilde{G}_k\bar{L}'_i\hat{B}'_{u\ell}+\tilde{G}_k\bar{L}'_\ell\hat{B}'_{ui}+\tilde{G}_\ell\bar{L}'_i\hat{B}'_{uk}+\tilde{G}_\ell\bar{L}'_k\hat{B}'_{ui},\ \tilde{\mathscr{F}}_{23}=\tilde{G}_i\bar{F}_k\hat{C}'_\ell+\tilde{G}_i\bar{F}_\ell\hat{C}'_k+\tilde{G}_k\bar{F}_i\hat{C}'_\ell$$
$$+\tilde{G}_k\bar{F}_\ell\hat{C}'_i+\tilde{G}_\ell\bar{F}_i\hat{C}'_k+\tilde{G}_\ell\bar{F}_k\hat{C}'_i+G_iL'_k\hat{D}'_{u\ell}+G_iL'_\ell\hat{D}'_{uk}+G_kL'_i\hat{D}'_{u\ell}+G_kL'_\ell\hat{D}'_{ui}+G_\ell L'_i\hat{D}'_{uk}$$
$$+G_\ell L'_k\hat{D}'_{ui}+\hat{A}_i\bar{F}'_k\tilde{H}'_\ell+\hat{A}_i\bar{F}'_\ell\tilde{H}'_k+\hat{A}_k\bar{F}'_i\tilde{H}'_\ell+\hat{A}_k\bar{F}'_\ell\tilde{H}'_i+\hat{A}_\ell\bar{F}'_i\tilde{H}'_k$$
$$+\hat{A}_\ell\bar{F}'_k\tilde{H}'_i+\hat{B}_{ui}\bar{L}_k\tilde{H}'_\ell+\hat{B}_{ui}\bar{L}_\ell\tilde{H}'_k+\hat{B}_{uk}\bar{L}_i\tilde{H}'_\ell+\hat{B}_{uk}\bar{L}_\ell\tilde{H}'_i$$
$$+\hat{B}_{u\ell}\bar{L}_i\tilde{H}'_k+\hat{B}_{u\ell}\bar{L}_k\tilde{H}'_i,\ \tilde{\mathscr{F}}_{33}=\tilde{H}_i\bar{F}_k\hat{C}'_\ell+\tilde{H}_i\bar{F}_\ell\hat{C}'_k+\tilde{H}_k\bar{F}_i\hat{C}'_\ell+\tilde{H}_k\bar{F}_\ell\hat{C}'_i+\tilde{H}_\ell\bar{F}_i\hat{C}'_k$$
$$+\tilde{H}_\ell\bar{F}_k\hat{C}'_i+\tilde{H}_i\bar{L}'_k\hat{D}'_{u\ell}+\tilde{H}_i\bar{L}'_\ell\hat{D}'_{uk}+\tilde{H}_k\bar{L}'_i\hat{D}'_{u\ell}+\tilde{H}_k\bar{L}'_\ell\hat{D}'_{ui}+\tilde{H}_\ell\bar{L}'_i\hat{D}'_{uk}+\tilde{H}_\ell\bar{L}'_k\hat{D}'_{ui}$$

Proof: Applying the following operation [89]

$$\Xi(\lambda)=\sum_{i=1}^{M}\lambda_i^3\Xi_i+\sum_{i=1}^{M}\sum_{k=1,k\neq i}^{M}\lambda_i^2\lambda_k\Xi_{ik}+\sum_{i=1}^{M-2}\sum_{k=i+1}^{M-1}\sum_{\ell=k+1}^{M}\lambda_i\lambda_k\lambda_\ell\Xi_{ik\ell} \tag{2.28}$$

to the BMIs (2.25), (2.26) and (2.27) inequality (2.20) follows immediately considering the particular structure (2.22) for the Lyapunov matrix, the change of variables $L(\alpha(k))=K(\alpha(k))F(\alpha(k))'$ and the lift of the BMI to the λ-space. Note that the choice of $P(\alpha(k))$ given by (2.22) with $P_i>0$ assures a lower bound to the sequence. Lastly, the parameter-dependent gain $K(\alpha(k))$ is obtained by the change of variables given in (2.23), what concludes the proof. ■

Note that the actual variables are L_i, H_i, F_i, G_i, $P_i=P_i'>0$, but the BMIs (2.25)-(2.27) are written in terms of $\bar{L}_i$, $\bar{H}_i$, $\bar{F}_i$, $\bar{G}_i$, $\bar{P}_i$, $\tilde{H}_i$, $\tilde{G}_i$ and $\tilde{P}_i$.

Corollary 2.1. *The minimum γ attainable by the conditions of Theorem 2.2 is given by the optimization problem*

$$\min\gamma\quad s.t.\ (2.25),(2.26),(2.27) \tag{2.29}$$

Note that as the problem is non-convex, only sub-optimal solutions can be obtained. The use of a memory controller brings some advantages when dealing with discrete time delay systems. Using extra variables to store the past values of the control signal, it is possible to cope with Problem 1 without applying more complex Lyapunov methods, (for instance, the Lyapunov-Krasovskii functional). Sophisticated Lyapunov functionals may lead to conditions that require a larger computational effort to be solved. Whenever possible, the use of memory controllers is suggested when dealing with discrete time delay systems since it simplifies the analysis. Nevertheless, the method could be adapted to cope with other Lyapunov functions, as the Lyapunov-Krasovskii one.

Gain scheduled control of discrete-time systems with time-varying parameters was also addressed by means of affinely parameter-dependent Lyapunov functions in [28], [29] and improved in [73] to cope with systems in which all state space matrices are supposed to be affected by time-varying parameters. In the above works, the design conditions are given in terms of LMIs. In this chapter, however, statement *iv)* in Lemma 2.2 is applied to reach more general BMI conditions with multiplier defined as in (2.19). The advantages of this approach are due to the extra variables that can be used in the search for better performance of the closed-loop system. For example, lower $\mathscr{H}_\infty$ guaranteed costs may be obtained exploring the new variables $G(\alpha(k+1))$ and $H(\alpha(k+1))$. In this sense, Theorem 2.1 encompasses the conditions in [28].

The computational time necessary to solve the sufficient BMI conditions presented here man be estimated in terms of the number of scalar variables V and the number of BMIs L. These two parameters are written as a function of $\tilde{n}$ (number of augmented states) and N (number of vertices) as follows.

$$V = N\left(\frac{\tilde{n}(\tilde{n}+1)}{2} + 2\tilde{n}^2 + \tilde{n}(q+m)\right) + 1, \quad L = \frac{(M^4 + 3M^3 + 2M^2 + 6M)}{6}, \quad \tilde{n} = n + m\tau$$

When dealing with problems that take uncertainties into account, it is clear that the difficulty in solving the problem increases with the number of uncertain parameters. In the framework studied in this chapter, this fact can be particularly illustrated by the number of BMIs in Theorem 2.2. Considering a system with a large number of uncertainties, the number of vertices used to describe the whole of possible system outcomes will also be large, yielding a large number of inequalities in the conditions of Theorem 2.2. Naturally, the computation time will also increase since for the LMI/BMI solvers available nowadays, the computational time depends on the number of LMIs/BMIs, on the number of variables of the problem to be solved and, of course, on the computer hardware used.

Although other methods could be applied to problem (2.29), the following algorithm is proposed.

Algorithm 1. *Let* $G_i = \mathbf{0}$ *and* $H_i = \mathbf{0}$, $i = 1, \ldots, N$. *Let* ε *be given. Set* $k = 1$ *and iterate:*

1. *Fix the variables* H_i *and* G_i, *minimize w.r.t.* γ_k *and determine* F_i, L_i *and* P_i.
2. *Fix the variables* F_i *and* L_i, *minimize w.r.t.* γ_k *and obtain* H_i, G_i *and* P_i.
3. *If* $|\gamma_k - \gamma_{k-1}| < \varepsilon$, *then stop (no significant changes).*
4. *Set* $k = k + 1$ *and go to step 1.*

This approach is sometimes called an Alternating Semi-Definite Programming method [43]. At each step, a convex optimization problem in terms of LMI conditions is solved. It is worth stressing that the aim here is not to develop new strategies to solve BMIs. Whenever feasible, other methods from the literature may be applied to solve Corollary 2.1, such as the methods in [43], [61], [103], [116]. Concerning the convergence aspect, the proposed algorithm is a heuristic approach and consequently there is no guaranteed convergence result to the local optimum. However, since steps 1 and 2 are *convex* optimization problems, the resulting $\mathscr{H}_\infty$ cost is non-increasing with the iterations.

An important aspect of Algorithm 1 is the choice of the initial values of the variables G_i and H_i. Initializing them as null matrices produces LMI conditions in step 1 of the first iteration similar to the ones presented in [29], [73] in terms of stabilization, since the only extra variables in the LMIs are F_i. In this case, the extra degree of freedom provided by G_i and H_i cannot be explored. As a remedy, an alternative structure to matrices G_i and H_i is proposed:

$$G_i = \varsigma \mathbf{I}, \qquad H_i = [h_{rs}]_i,\ h_{rs} = \varsigma, \quad i = 1, \dots, N \tag{2.30}$$

where ς is a real number. In this case, the conditions of Theorem 2.2 can be tested as LMIs through line searches.

Corollary 2.2. *Given the augmented discrete-time system (2.3) and a scalar $\varsigma \in \mathbb{R}$, if there exist matrices L_i, F_i, $P_i = P_i' > 0$, with appropriate dimensions, $i = 1, \dots, N$ and a scalar $\gamma > 0$ such that (2.25) and (2.26) hold with G_i and H_i given by (2.30), then there exists a memory control law (2.5), ensuring the asymptotic stability of the closed-loop system (2.6) and an $\mathscr{H}_\infty$ guaranteed cost γ, with $K(\alpha)$ given as in (2.23) and (2.24).*

Through a line search on ς, the conditions of Corollary 2.2 can be used to search for stabilizing controllers even when the conditions [29], [73] and the first iteration of Algorithm 1 fail. Moreover, if Corollary 2.2 provides a feasible solution, the respective ς can be used to initialize G_i and H_i as in (2.30), assuring a feasible solution to the first iteration of Algorithm 1.

By fixing the variable matrices $F_i = F$ and $L_i = L$ (not depending on $\alpha(k)$), $\mathscr{H}_\infty$ robust memory controllers can be obtained using the conditions of Theorem 2.2, as stated in the next corollary.

Corollary 2.3. *Given the augmented discrete-time system (2.3), if BMI (2.25), for $i = 1, \dots, M$, and BMI (2.26), for $i = 1, \dots, M-1, \quad j = i+1, \dots, M$, of Theorem 2.2 are feasible with fixed variable matrices L and F then the closed-loop system (2.6) is asymptotically stable with a robust memory controller $K = L(F')^{-1}$ and an $\mathscr{H}_\infty$ guaranteed cost γ.*

Note that BMI (2.27) is not necessary in this case, since it would produce redundant conditions. The line search strategy may also be applied in this context, similarly to Corollary 2.2.

It is worth stressing that for time-varying discrete-time systems, robust stabilizability implies gain scheduling stabilizability, but the converse is not true [9]. In other words, there may exist systems for which Theorem 2.2 and Corollary 2.2 provide feasible solutions but Corollary 2.3 is unfeasible. This fact points out the importance of studying and improving gain scheduling strategies for control systems, specially in the discrete-time domain.

Finally, the novelties presented here consist especially in the use of modeling parameter variations in the $\boldsymbol{\lambda}$-space within the gain-scheduling framework and in the use of BMIs as a tool in the search of better $\mathscr{H}_\infty$ performance. To the best of the authors' knowledge, the use of Lemma 2.2 with the particular structure (2.19) (that results in Theorem 2.1) has never been seen in the literature in the context of gain-scheduled control. Consequently, Theorem 2.2, obtained through Theorem 2.1 and expressions (2.17), (2.18) and (2.22) represents a novel strategy to face the problem of feedback control for discrete time-varying systems. The conditions provide good results when compared to other recent methods in the control literature, as shown in the numerical experiments, and represents a flexible strategy in the sense that it can be used in four different contexts, namely, LPV or robust control of time-varying systems with bounded or unbounded rates of variation.

2.4 Numerical Experiments

All the experiments have been performed in a PC equipped with: Athlon 64 X2 6000+ (3.0 GHz), 2GB RAM (800 MHz), using Linux (Ubuntu), Matlab (7.0.1) and the SDP solver SeDuMi [101] interfaced by the parser YALMIP [70]. The numerical complexity associated with the proposed conditions and the ones from the literature used for comparison purposes are estimated in terms of the computational times given in seconds. Only the time required to solve the LMIs is considered, since the time necessary to build the set of LMIs is highly dependent on the LMI parser interface.

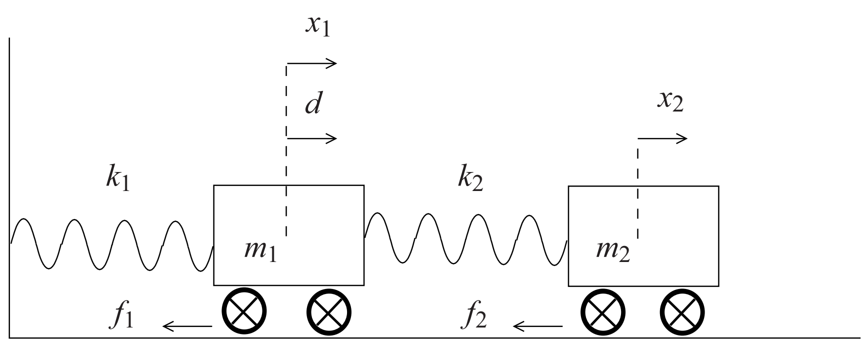

Figure 2.2: Mass-spring system.

Particularly with respect to the iterative procedure given in Algorithm 1, the time of the i-th iteration is the cumulated total time.

Example I: This example is concerned with the fourth order two-mass-spring system presented in [59] that is reproduced here in Figure 2.2. The same transfer function is considered, i.e. from the input force d applied to mass m_1 to the error signal $e = x_2$ (position of mass m_2).

The masses and the stiffness of the second spring are assumed constant as $m_1 = 2$, $m_2 = 1$, $k_2 = 0.5$. The friction forces f_1 and f_2 are associated to the viscous friction coefficient c_0. The stiffness of the first spring and the viscous friction coefficient are assumed to be time-varying in the ranges

$$1 \leq k_1(k) \leq 13, \qquad 1 \leq c_0(k) \leq 13$$

resulting in a polytope of $N = 4$ vertices, obtained by evaluating the following discrete-time equation at the extreme values of the parameters.

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ \frac{-0.1(k_1+k_2)}{m_1} & \frac{0.1k_2}{m_1} & 1 - \frac{0.1c_0}{m_1} & 0 \\ \frac{0.1k_2}{m_2} & \frac{-0.1k_2}{m_2} & 0 & 1 - \frac{0.1c_0}{m_2} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ \frac{0.1}{m_1} \\ 0 \end{bmatrix} u(k) \qquad (2.31)$$

The sampled version (2.31) of the two-mass-spring system was obtained using the

Euler's first-order approximation for the derivative with a sampling time of 0.1 s. The other system matrices are $C_i = [0 \ \ 1 \ \ 0 \ \ 0]$, $B_{wi} = [0 \ \ 0.1 \ \ 0.1 \ \ 0]'$, $D_{wi} = 0.01$, $D_{ui} = 0$, $i = 1,\ldots,4$. Additionally, it is also investigated the situation where the model is affected by a one-step-delayed input, considering $B_{dui} = [0 \ \ 0 \ \ 1 \ \ 0]'$ and $D_{dui} = 0$, $i = 1,\ldots,4$. The results obtained by the methods [29, Theorem 4] (gain scheduling control), [29, Theorem 5] (robust control), Theorem 2.2 and Corollary 2.3 are shown in Table 2.1 for the case of arbitrarily fast variations of the parameters ($b = 1$) and for slow variations ($b = 0.05$), i.e. the value of the parameters are constrained to vary only 5% from the instant k to the instant $k+1$.

Table 2.1: Results and elapsed time associated to the methods of [29] and the conditions of Theorem 2.2 (T2.2) and Corollary 2.3 (C2.3) for the control design problem in Example I.

Method	[29, T4]	[29, T5]	$T2.2_{it=1}$	$C2.3_{it=2}$	$T2.2_{it=1}$	$C2.3_{it=2}$
b	1	1	1	1	0.05	0.05
γ ($\tau = 0$)	0.80	1.50	0.79	1.48	0.43	1.03
Time (s)	0.5	0.4	460.8	69.7	810.5	63.2
γ ($\tau = 1$)	1.40	2.52	1.39	2.49	0.63	1.55
Time (s)	0.6	0.4	1415.5	92.7	1351.0	116.3

In the case $b = 1$, the conditions of [29] and the ones proposed in this chapter produce practically the same $\mathscr{H}_\infty$ guaranteed costs. On the other hand, for $b = 0.05$ (slow parameter variation), the method proposed yields significantly less conservative results. Such results illustrate that the proposed approach may be beneficial when bounds on the parameters variation are known and accounted for. In general, this is the case for mechanical systems, as in this example, where the parameters c_0 and k_2 are assumed to vary slowly. The improvements in the $\mathscr{H}_\infty$ guaranteed costs, when considering bounds on the rates of variation, obtained by Theorem 2.2 and Corollary 2.3 were 47% and 30%, respectively, for the delay-free case. For the delayed input case, the improvements are larger, i.e. 54% and 40%, respectively, for

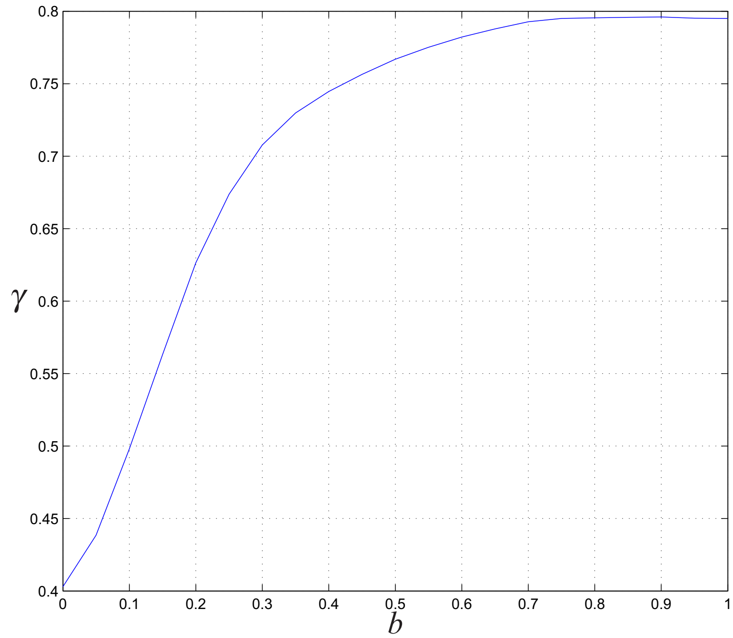

Figure 2.3: $\mathscr{H}_\infty$ attenuation level γ as a function of bound b.

Theorem 2.2 and Corollary 2.3. Concerning the computational complexity, the time demanded by the proposed approach is higher due to the conversion of the parameters to the λ-space domain. In this example, the four parameters in the original polytopic domain yield twenty eight vertices in the λ-space domain. This is the price to be paid in order to take into account limited rates of variation. Figure 2.3 illustrates the $\mathscr{H}_\infty$ attenuation level γ as a function of bound b obtained with Theorem 2.2 for the delay free case ($\tau = 0$) with one iteration (it=1).

Finally, a time simulation has been performed for the delayed input case with the gain-scheduled controllers obtained through the proposed conditions. The parameters $k_2(k)$ and $c_o(k)$ vary $\approx 4\%$ per instant of time, starting from their minumum values until their maximum. The input noise was generated using the Matlab command $w(k) = 0.2 * \texttt{randn}$ for $0 \leq k \leq 100$. The noise and the outputs (considering

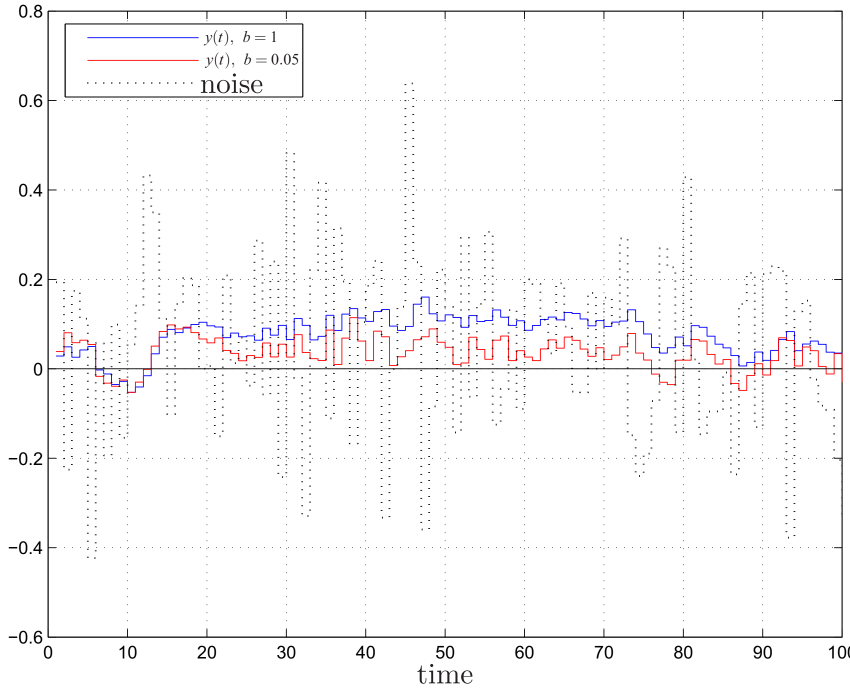

Figure 2.4: Time simulation of the mass-spring system, with a one step delay, stabilized through the conditions of Theorem 2.2 for the cases $b = 1$ and $b = 0.05$.

$D_w = 0$ and $x_0 = 0$) of the system, using the synthesized gain-scheduling controllers for $b = 1$ and $b = 0.05$, are depicted in Figure 2.4. Clearly, the case $b = 0.05$ presents a better disturbance rejection. In fact, the total error $e = \sum_{i=1}^{100} |y(k)|$ is $e = 7.91$ and $e = 4.29$ for the cases $b = 1$ and $b = 0.05$, respectively, yielding an improvement of 45%. Note that the error is attenuated in both cases due to the $\mathscr{H}_\infty$ guaranteed cost.

Example II: Consider system (2.3) with vertices (borrowed from [73, Example 2]) given by

$$\tilde{A}_1 = \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix}, \quad \tilde{B}_{u1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{B}_{u2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$\tilde{B}_{wi} = [1 \ \ 0]'$, $\tilde{C}_i = [1 \ \ 0]$ and $\tilde{D}_{wi} = \tilde{D}_{ui} = 0$, $i = 1, 2$. This system with arbitrarily fast parameters was also studied in [29], but in a simpler case where matrix $\tilde{B}_u$ was fixed and time-invariant (i.e. $\tilde{B}_{u1} = \tilde{B}_{u2}$). The aim here is to compare the gain scheduling design conditions from [73] (capable to cope with time-varying $B_u(\alpha(k))$) with the BMI approach proposed in Theorem 2.2. Additionally, it is considered that the system is affected by one single delay (one step) with $B_{dui} = [0 \ \ 1]'$ and $D_{dui} = 0$, $i = 1,2$. Table 2.2 shows the improvements due to the BMI approach over [73] as the number of iterations (it) evolves. As can be seen in Table 2.2, the $\mathscr{H}_\infty$ upper bound

Table 2.2: Results and elapsed time associated to the method [73] and the conditions of Theorem 2.2 (T2.2) in the gain scheduling control design given in Example II.

Method	[73]	$T2.2_{it=1}$	$T2.2_{it=2}$	$T2.2_{it=3}$	$T2.2_{it=4}$	$T2.2_{it=5}$	$T2.2_{it=6}$
γ	20.09	14.39	9.63	8.60	8.27	8.14	8.06
Improvement	–	28.33%	52.04%	57.15%	58.82%	59.46%	59.87%
Time (s)	0.12	1.12	2.07	2.98	3.91	4.80	5.70

γ was reduced in approximately 59.87% with 6 iterations, providing better rejection of disturbances.

2.5 Conclusion

The $\mathscr{H}_\infty$ gain scheduled memory controller for LPV discrete-time systems, with bounded rates of variation, belonging to a polytope has been designed in this chapter. The memory of the controller, used to store the previous values of the control signal, was modeled as a new state-space variable leading to an augmented system representation. A sufficient condition has been proposed in terms of BMIs described only at the vertices of the polytope. Extra variables provided by Finsler's lemma were used to derive the BMI conditions. The controller design is accomplished by means of an optimization problem that combines convex optimization and line searches. An extension to deal with the design of $\mathscr{H}_\infty$ robust memory controllers has also been given. The conditions compare favorably with other methods appeared recently in the literature, as shown in the numerical experiments.

Chapter 3

Robust $\mathscr{H}_\infty$ networked control for uncertain sampled-data systems

3.1 Introduction

The control community has struggled for decades to find solutions to problems concerned with the perfect operation of dynamical systems immersed in hostile environments. There is no denying that it is wise to seek better characterizations of model uncertainties, to guarantee not only stability but also robustness against practical disturbances and perturbations. Within this framework, networked control system (NCS) architecture has recently received considerable attention.

Technological advances have enabled the extensive use of communication channels in the control of dynamical systems [108], [115]. Using a real-time network to exchange information among control system components (sensors, actuators, controllers, etc.), NCSs are a good alternative to implementing distributed control and interconnected systems. To illustrate the usefulness of NCSs one can cite the following benefits: reduced system wiring, plug and play devices, and ease of system diagnosis and maintenance [115]. Unfortunately there are also some drawbacks: systems with loops closed over communication networks become more complex and require sophisticated control techniques. Among the main issues arising in NCSs deserving special attention are network-induced delays, packet dropouts, multiple-packet transmission, and bandwidth requirements.

Network-induced delays occur whenever data are exchanged through a communication channel and, in general, can be broken into three parts: time delays at the source node, across the network channel, and at the destination node, [68]. As pointed out in [115], their nature is related to the medium access control (MAC) protocol and may be constant, time-varying, or random. Packet dropouts may occur whenever more than one node tries to transmit simultaneously, leading to message collisions, or they may occur because of node failures. Although retransmission is an option, there are some cases where it may be disadvantageous or even impossible to re-transmit. Multiple-packets instead of single-packet transmission may be needed for many reasons, such as bandwidth and packet size constraints, which in some sense increase the chances of packet dropouts and network-induced delays. Bandwidth usage has a direct impact on system stability and performance. From the control point of view, it is known that a faster sampling rate is required to guarantee that the behavior of sampled data models approximates that of continuous-time systems. In NCSs this implies a high network load and consequently larger bandwidth requirements.

The study of control strategies to overcome these difficulties has received considerable recent attention [26], [69], [71], [79], [83], [104], [114]. Lyapunov theory, which has been one of the main tools for dealing with the stability analysis and synthesis of controllers, is being used within the NCS framework. Recent works include [109], where a feedback controller is constructed for a discrete-time Markovian jump system with random delays via a set of linear matrix inequality (LMI) conditions; [48], where the control problem is solved for the multipoint-packet system using $\mathscr{H}_2$ optimization techniques; [112], where stabilization of an NCS is achieved by means of a packet-loss dependent Lyapunov function; and [58], where a Lyapunov-Krasovskii functional is used to design a state feedback controller for a time delay sampled system. As can be seen from these references, much effort has been expanded to bring together advances in control theory and the benefits of communication networks.

Depending on the system to be controlled, some networks may be more suitable than others. For instance, Ethernet-based network solutions may be more appropriate for NCSs operating at low network loads, since in this case the induced time delay is very small, whereas ControlNet network solutions equipped with a token bus protocol perform well at high network loads when the percentage of packets discarded is at issue, as discussed in [68]. It is important to point out that control strategies based on a simplified framework, such as a constant delay or even zero delay, may display reliable behavior when applied in specific cases. In any case, a controller design method that takes into account all the characteristics of a network which impact system stability still remains a challenge.

This chapter addresses the design of robust controllers to stabilize NCSs subject to time-varying sampling rates. The stability of this type of system is important within the NCS framework, especially in the context of dynamic bandwidth allocation and bandwidth usage control. A simplified framework for the networked-induced delay is assumed. The uncertain sampling period is taken to lie inside a known interval. The sampled data system is represented by an uncertain discrete-time linear model with time-varying parameters lying inside a polytope whose vertices are determined through the Cayley-Hamilton theorem, without using approximations or truncation. The proposed approach complements and extends the results of [79], [104] in two directions: index of performance and the stability of sampled data systems with time-varying sampling periods. Specifically, the stability conditions of the closed-loop system are certified by a parameter-dependent Lyapunov function and the robustness of the controller using an $\mathscr{H}_\infty$ guaranteed cost, as proposed in the preliminary version of the results presented in this chapter [15].

An improved strategy is used in which a more general parameter-dependent Lyapunov function is applied to provide less conservative stability conditions. As shown in [63], [64], this class of *path-dependent* Lyapunov functions can provide necessary and sufficient conditions for robust stability analysis of arbitrarily time-varying discrete-time systems. Extra matrix variables are introduced in the bounded real lemma conditions, producing design conditions that are expressed in terms of bilinear matrix inequalities (BMIs). A robust memory controller is then obtained from the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of BMI constraints formulated only in terms of the vertices of a polytope. As illustrated by means of numerical examples, the use of BMIs can improve the NCS performance. Furthermore, the conditions may be reduced to a set of LMIs by a convenient choice of the extra variables. At each step of the algorithm, a convex optimization problem with LMI constraints is solved, providing nonincreasing values for the bounds on the $\mathscr{H}_\infty$ index of performance. Even when no communication channel is considered, the proposed approach improves some of the results in the literature concerned with the robust control of time-varying discrete-time systems [28], [29].

3.2 Preliminaries and Problem Statement

The NCS model considered is described in Figure 3.1.

The continuous-time plant is given by the following equations, for $t \geq 0$,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) \\ y(t) &= Cx(t) + Du(t) + D_d u(t-\tau) \\ x(0) &= 0,\ u(\varsigma) = 0,\ \varsigma \in \{-\tau, 0\} \end{aligned} \tag{3.1}$$

where τ represents the network-induced time delay, $x(t) \in \mathbb{R}^n$ is the state space vector, $u(t) \in \mathbb{R}^m$ is the control signal, and $y(t) \in \mathbb{R}^q$ is the output. All matrices are real,

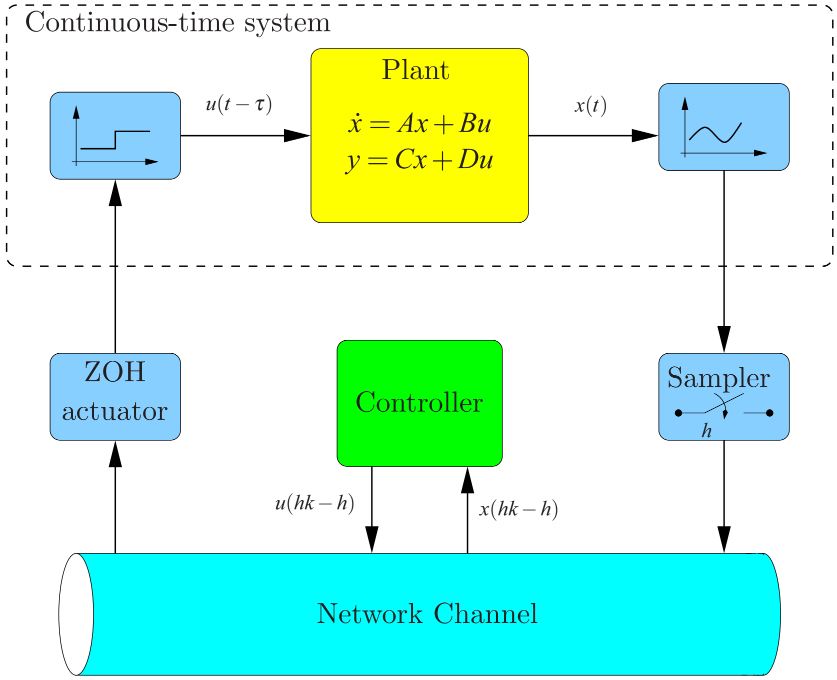

Figure 3.1: NCS Model.

with appropriate dimensions.

The total networked-induced delay τ is broken into two parts: the delay that occurs when data are transmitted from the sensor to the controller τ_{sc} and the delay when the data are transmitted from the controller to the actuator τ_{ca}. As mentioned in [115], the delay due to computations in the controller can be modeled into either τ_{sc} or τ_{ca}. Note that the delays are not being used to model network scheduling. Depending on the MAC protocol of the network, network-induced delays may be constant, time-varying, or random. Under the assumption of a scheduling MAC protocol, the delays occur while waiting for the token, or time slot. In this case, it can be said that a scheduling network is an example of a situation in which the delay can be bounded and made constant by transmitting packets periodically [115]. Concerned with small delays, Ethernet-based networks experience almost no delay at low network loads [68]. Furthermore, if the controller is time-invariant (such as the one disucssed in this chapter), these sources of delays can be added for analysis purposes, for example, $\tau = \tau_{sc} + \tau_{ca}$. For simplicity, τ is considered constant and known, a situation that may occur when static scheduling network protocols are implemented. Moreover, τ is assumed to be less than one sampling period. A buffer in the controller node is used to store the delayed information.

System (3.1) is sampled with a period $h > \tau$, yielding the discrete-time model [5], for $k \in \mathbb{Z}_+$, $x(0) = 0$, and $u(\varsigma) = 0,\ \varsigma \in \{-h, 0\}$:

$$\begin{aligned} x(kh+h) &= A_s(h)x(kh) + B_{su0}(h,\tau)u(kh) + B_{su1}(h,\tau)u(kh-h) + B_{sw}w(kh) \\ y(kh) &= C_s x(kh) + D_{su}u(kh) + D_{sd}u(kh-h) + D_{sw}w(kh) \end{aligned} \tag{3.2}$$

where $w(kh) \in \mathbb{R}^r$ is an extra input, belonging to $l_2[0,\infty)$, used to model, through matrices B_{sw} and D_{sw}, possible noise in the process. The system matrices $A_s(h)$, $B_{su0}(h,\tau)$, $B_{su1}(h,\tau)$, C_s, D_{su}, and D_{sd} are given by

$$\begin{aligned} &A_s(h) = \exp(Ah), \quad B_{su0}(h,\tau) = \int_0^{h-\tau} \exp(As)dsB, \quad D_{sd} = D_d \\ &B_{su1}(h,\tau) = \exp\left(A(h-\tau)\right) \int_0^{\tau} \exp(As)dsB, \quad C_s = C, \quad D_{su} = D \end{aligned} \tag{3.3}$$

As discussed in [79], [104], the sampling period h may change its value at runtime for different reasons, for example, dynamic bandwidth allocation and scheduling decisions. By considering the sampling period as a time-varying parameter, it is possible to reduce the flow of information between sensor and actuator. Nevertheless, bounds on such variations can be determined, guaranteeing that the actual values of h at each instant k, namely, h_k, lie inside a finite discrete set as specified below:

$$h_k \in \{h_{min}, \ldots, h_{max}\},\ h_k = \kappa \cdot g,\ \kappa \in \mathbb{N} \tag{3.4}$$

It is assumed that the real values of h_k are not known at the instant of time k, but only that they belong to (3.4) and $h_{min} \geq \tau$. The number of possible values of these sets depends on the processor/network clock granularity g, as discussed in [104]. The clock granularity is related to processor frequency and $\kappa \in \mathbb{N}$ is a function of time that specifies how many times g the sampling period h will be at instant k.

In order to guarantee the stability of the networked system shown in Figure 3.1, a state feedback controller is designed. Using an extra state variable $z(kh) = u(kh-h)$ to store the last value of the control signal, the dynamics of system (3.2) can be represented by the following difference equations [5]:

$$\begin{aligned} \tilde{x}(kh+h) &= \tilde{A}(h)\tilde{x}(kh) + \tilde{B}_u(h)u(kh) + \tilde{B}_w w(kh) \\ y(kh) &= \tilde{C}\tilde{x}(kh) + D_{su}u(kh) + D_{sw}w(kh) \end{aligned} \tag{3.5}$$

where $\tilde{x}(kh) = [x(kh)'\ z(kh)']'$ and

$$\begin{aligned} \tilde{A}(h) &= \begin{bmatrix} A_s(h) & B_{su1}(h,\tau) \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_u(h) = \begin{bmatrix} B_{su0}(h,\tau) \\ \mathbf{I} \end{bmatrix}, \\ \tilde{B}_w &= \begin{bmatrix} B_{sw} \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_s & D_{sd} \end{bmatrix} \end{aligned} \tag{3.6}$$

In the case where there is no time delay ($\tau = 0$), the state space vector becomes $\tilde{x}(kh) = x(kh)$ and the augmented system matrices simplify in a standard way.

The control signal is given by

$$u(kh) = K_x x(kh) + K_d u(kh-h) = \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} x(kh) \\ z(kh) \end{bmatrix} = K\tilde{x}(kh) \tag{3.7}$$

A discrete-time polytopic model is used to represent the set of all possible matrices in system (3.5) due to the uncertain time-varying sampling periods h_k given by (3.4). More specifically, the system matrices $(\tilde{A}(h), \tilde{B}_u(h))$, for any $k \geq 0$, are described as a convex combination of well-defined vertices $(\tilde{A}_j, \tilde{B}_{uj})$. The main difficulty in defining the vertices is related to the exponential terms in (3.3), which need to be computed for all h_k in (3.4). By using the Cayley-Hamilton theorem [1], these terms can be written as

$$\exp(Ah) = \sum_{i=0}^{n-1} \rho_i(h) A^i \tag{3.8}$$

$$\int_0^{h-\tau} \exp(As)ds = \int_0^{h-\tau} \Big(\sum_{i=0}^{n-1} \rho_i(s) A^i \Big) ds = \sum_{i=0}^{n-1} \Big(\int_0^{h-\tau} \rho_i(s) ds \Big) A^i = \sum_{i=0}^{n-1} \eta_i(h) A^i \tag{3.9}$$

where

$$\eta_i(h) = \int_0^{h-\tau} \rho_i(s)ds$$

The coefficients $\rho_i(h)$ and $\eta_i(h)$ may be determined for each value of h_k by solving a set of linear equations defined in terms of the eigenvalues of matrix A. For instance, the first block of the matrix $\tilde{A}(h_k)$ in (3.6) is given by

$$A_s(h) = \exp(Ah) = \sum_{i=0}^{n-1} \rho_i(h)A^i = \sum_{i=0}^{n-1} \theta_i(h_k)\Omega_i \tag{3.10}$$

where the coefficients $\theta_i(h_k)$, $i = 0, \ldots, n-1$, are obtained from the modes associated with the eigenvalues of A and matrices $\Omega_i \in \mathbb{R}^{n\times n}$ are determined by collecting terms in the above equality. Similarly, $B_{su1}(h_k, \tau)$ and $B_{su0}(h_k, \tau)$ can be computed as a linear combination of matrices, following (3.3), (3.8)-(3.9) and, in some cases, can be described in terms of the same parameters $\theta_i(h_k)$, $i = 0, \ldots, n-1$.

Since $\rho_i(h)$, $i = 0, \ldots, n-1$, are written as linear combinations of terms $h^k \exp(\lambda h)$, where λ is an eigenvalue of matrix A, and h_k satisfies (3.4), the minimum and maximum values of $\theta_i(h_k)$, $i = 0, \ldots, n-1$, can be determined in such a way that

$$\underline{\theta}_i \leq \theta_i(h_k) \leq \overline{\theta}_i, \quad i = 0, \ldots, n-1 \tag{3.11}$$

All possible outcomes for $\tilde{A}(h_k)$ and $\tilde{B}_u(h_k)$ are then given by

$$\tilde{A}(\alpha(k)) = \sum_{j=1}^{N} \alpha_j(k)\tilde{A}_j, \quad \tilde{B}_u(h_k) = \sum_{j=1}^{N} \alpha_j(k)\tilde{B}_{uj}$$

with $N = 2^n$ and the time-varying vector $\alpha(k)$ lying inside the unit simplex

$$\mathscr{U} = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0\ ,\ i = 1, \ldots, N \right\} \tag{3.12}$$

for all $k \geq 0$. The vertices $(\tilde{A}_j, \tilde{B}_{uj})$ of the polytope are obtained by all possible combinations of $\underline{\theta}_i$ and $\overline{\theta}_i$ in (3.11). In the numerical example presented later, the above computation is given in details.

The uncertain polytopic closed-loop system is then given by

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}_{cl}(\alpha(k))\tilde{x}(k) + \tilde{B}_w w(k) \\ y(k) &= \tilde{C}_{cl}\tilde{x}(k) + \tilde{D}_w w(k) \end{aligned} \tag{3.13}$$

with

$$\tilde{A}_{cl}(\alpha(k)) = \tilde{A}(\alpha(k)) + \tilde{B}_u(\alpha(k))K, \qquad \tilde{C}_{cl} = \tilde{C} + D_{su}K, \qquad \tilde{D}_w = D_{sw} \tag{3.14}$$

and the uncertain matrices $\left(\tilde{A}(\alpha(k)), \tilde{B}_u(\alpha(k))\right)$ belong to the polytope

$$\mathscr{P} \triangleq \left\{ \left(\tilde{A}(\alpha(k)), \tilde{B}_u(\alpha(k))\right) = \sum_{j=1}^{N} \alpha_j \left(\tilde{A}_j, \tilde{B}_{uj}\right), \quad \alpha \in \mathscr{U} \right\} \tag{3.15}$$

for all $k \geq 0$.

The control problem to be dealt with is stated as follows.

Problem 2. *Find constant matrices $K_x \in \mathbb{R}^{m\times n}$ and $K_d \in \mathbb{R}^{m\times m}$ of the state feedback control (3.7) such that the closed-loop system (3.13) is asymptotically stable and an upper bound γ to the $\mathscr{H}_\infty$ performance is minimized, that is*

$$\sup_{w\neq 0} \frac{\|y\|_2^2}{\|w\|_2^2} < \gamma^2 \tag{3.16}$$

with $w \in l_2[0,\infty)$.

In the literature, an LMI characterization of such an $\mathscr{H}_\infty$ disturbance attenuation for a precisely known closed-loop system is given by the discrete-time version of the bounded real lemma [24], with extensions to uncertain systems [87] and to the time-varying case [37]. A slightly modified version, motivated by a quadratic in the state path-dependent Lyapunov function [63] is presented in the next lemma.

Lemma 3.1. *The closed-loop system (3.13) is asymptotically stable with an $\mathscr{H}_\infty$ disturbance attenuation given by $\gamma > 0$ if there exists a symmetric parameter-dependent* *matrix* $P(\alpha(k),\alpha(k+1))$ *such that*[1]

$$\begin{bmatrix} P(\alpha(k+1),\alpha(k+2)) & \tilde{A}_{cl}(\alpha(k))'P(\alpha(k),\alpha(k+1)) & \tilde{C}_{cl}' & 0 \\ (\star) & P(\alpha(k),\alpha(k+1)) & 0 & P(\alpha(k),\alpha(k+1))\tilde{B}_w \\ (\star) & (\star) & \gamma\mathbf{I} & \tilde{D}_w \\ (\star) & (\star) & (\star) & \gamma\mathbf{I} \end{bmatrix} > 0 \tag{3.17}$$

Proof: Note that the feasibility of (3.17) assures $P(\alpha(k),\alpha(k+1)) > 0$. Multiply on the left and on the right in (3.17) by $\mathrm{diag}\{P(\alpha(k+1),\alpha(k+2))^{-1}, P(\alpha(k),\alpha(k+1))^{-1}, \mathbf{I}, \mathbf{I}\}$ and apply the Schur complement to obtain

$$\begin{bmatrix} P(\alpha(k),\alpha(k+1))^{-1} - \tilde{A}_{cl}(\alpha(k))P(\alpha(k+1),\alpha(k+2))^{-1}\tilde{A}_{cl}(\alpha(k))' - \gamma^{-1}\tilde{B}_w\tilde{B}_w' & \tilde{A}_{cl}(\alpha(k))P(\alpha(k+1),\alpha(k+2))^{-1}\tilde{C}_{cl}' + \gamma^{-1}\tilde{B}_w\tilde{D}_w' \\ (\star) & \gamma\mathbf{I} - \tilde{C}_{cl}P(\alpha(k+1),\alpha(k+2))^{-1}\tilde{C}_{cl}' - \gamma^{-1}\tilde{D}_w\tilde{D}_w' \end{bmatrix} > 0$$

which is the discrete-time version of the bounded real lemma for time-varying systems. As a matter of fact, the above condition may be obtained by defining the Lyapunov function

$$\vartheta(x(k)) = x(k)'P(\alpha(k),\alpha(k+1))^{-1}x(k) \tag{3.18}$$

and imposing

$$\Delta\vartheta(x(k)) + \gamma^{-1}y(k)'y(k) - \gamma w(k)'w(k) < 0$$

on the dual of system (3.13). ∎

Lemma 3.2. *For a given* $\gamma > 0$*, if there exist a symmetric parameter-dependent matrix* $P(\alpha(k),\alpha(k+1)) > 0$ *and a parameter-dependent matrix* $\mathscr{X}(\alpha(k),\alpha(k+1))$

[1] The symbol $(\star)$ indicates symmetric blocks in the LMIs.

such that

$$\begin{bmatrix} P(\alpha(k+1),\alpha(k+2)) & 0 & 0 \\ (\star) & -P(\alpha(k),\alpha(k+1))+\gamma^{-1}\tilde{B}_w\tilde{B}_w' & \gamma^{-1}\tilde{B}_w\tilde{D}_w' \\ (\star) & (\star) & \gamma^{-1}\tilde{D}_w\tilde{D}_w'-\gamma\mathbf{I} \end{bmatrix}$$
$$+\mathscr{X}(\alpha(k),\alpha(k+1))\mathscr{B}(\alpha(k))+\mathscr{B}(\alpha(k))'\mathscr{X}(\alpha(k),\alpha(k+1))'<0 \quad (3.19)$$

where

$$\mathscr{B}(\alpha(k)) = \begin{bmatrix} -\mathbf{I} & \tilde{A}_{cl}(\alpha(k))' & \tilde{C}_{cl}' \end{bmatrix}$$

then the closed-loop system (3.13) is asymptotically stable with an upper bound $\gamma > 0$ *to the* $\mathscr{H}_\infty$ *performance.*

Proof: Suppose there exist $P(\alpha(k),\alpha(k+1))$ and $\mathscr{X}(\alpha(k),\alpha(k+1))$ such that (3.19) is verified. Then, multiply (3.19) by $(\mathscr{B}^\perp(\alpha(k)))'$ on the left and by $\mathscr{B}^\perp(\alpha(k))$ on the right with

$$\mathscr{B}^\perp(\alpha(k)) = \begin{bmatrix} \tilde{A}_{cl}(\alpha(k))' & \tilde{C}_{cl}' \\ \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \quad \mathscr{B}(\alpha(k))\mathscr{B}^\perp(\alpha(k)) = 0$$

Considering the dual system (i.e., $\tilde{A}_{cl} = \tilde{A}_{cl}'$, $\tilde{B}_w = \tilde{C}_{cl}'$, $\tilde{C}_{cl} = \tilde{B}_w'$, and $\tilde{D}_w = \tilde{D}_w'$) and using the Schur complement, inequality (3.17) follows in a straightforward way. ■

Lemma 3.2 provides a sufficient condition that assures robust asymptotic stability with γ disturbance attenuation to the uncertain time-varying closed-loop system (3.13) in terms of the existence of a symmetric parameter-dependent matrix $P(\alpha(k),\alpha(k+1))$ and an extra variable $\mathscr{X}(\alpha(k),\alpha(k+1))$ that must verify inequality (3.19) for $\alpha(k) \in \mathscr{U}$, $\alpha(k+1) \in \mathscr{U}$. As has been presented, Lemma 3.2 cannot be used to solve Problem 2, since the decision variables do not have a known structure, the control gains K_x and K_d in the time-varying closed-loop matrix $\tilde{A}_{cl}(\alpha(k))$ appear in nonlinear terms, and the parameter-dependent condition (3.19) must be tested for all $\alpha(k) \in \mathscr{U}$, $k \geq 0$.

The main purpose of this chapter is to provide finite-dimensional LMI based conditions, formulated in terms of the vertices of the polytope $\mathscr{P}$, to solve Problem 2. For that, two main facts are exploited:

- The time-varying parameter of the polytopic model $\alpha(k)$ is allowed to vary arbitrarily fast inside the polytope, that is, $\alpha(k+1) \in \mathscr{U}$ is independent of $\alpha(k) \in \mathscr{U}$.

- Lemma 3.2 provides a sufficient condition for the closed-loop system asymptotic stability with γ disturbance attenuation independently of matrix $\mathscr{X}(\alpha(k), \alpha(k+1))$, which represents an important degree of freedom. The result can be viewed as an extension of Finsler's lemma [31]. Several different sufficient conditions could be derived by imposing particular choices to $\mathscr{X}(\alpha(k), \alpha(k+1))$. As an example, the particular choice

$$\mathscr{X} = \begin{bmatrix} F(\alpha(k))' & 0 & 0 \end{bmatrix}'$$

  produces a result which is similar to the one in [37, Theorem 1], but with inconvenient products of terms depending on $\alpha(k)$. To avoid the product of parameter-dependent terms occurring at the same instant of time, some blocks could be made constant, zeroed out, or constrained to depend only on $\alpha(k+1)$.

By making $\alpha(k+2) = \delta(k) \in \mathscr{U}$, $\alpha(k+1) = \beta(k) \in \mathscr{U}$ and by imposing a special structure to the the extra variable $\mathscr{X}(\alpha(k), \alpha(k+1)) = \mathscr{X}(\beta(k))$ in Lemma 3.2, BMI conditions assuring the existence of K_x and K_d that solve Problem 2 are given in the next section.

3.3 Main Results

Theorem 3.1. *($\mathscr{H}_\infty$ Robust Controller) For a given $\gamma > 0$, if there exist matrices $L \in \mathbb{R}^{m\times(n+m)}$, $H_i \in \mathbb{R}^{q\times(n+m)}$, F, G_i, $P_{ij} = P_{ij}' > 0 \in \mathbb{R}^{(n+m)\times(n+m)}$, $i = 1, \ldots, N$ and*

$j = 1, \ldots, N$, *such that*

$$\begin{bmatrix} P_{jk} - F - F' & F\tilde{A}_i' + L'\tilde{B}_{ui}' - F'G_j' & F\tilde{C}' + L'\tilde{D}_u - F'H_j' & 0 \\ (\star) & G_jF\tilde{A}_i' + \tilde{A}_iF'G_j' + G_jL'\tilde{B}_{ui}' + \tilde{B}_{ui}LG_j' - P_{ij} & G_jF\tilde{C}' + G_jL'\tilde{D}_u' + \tilde{A}_iF'H_j' + \tilde{B}_{ui}LH_j' & \tilde{B}_w \\ (\star) & (\star) & H_jF\tilde{C}' + \tilde{C}F'H_j' + H_jL'\tilde{D}_u' + \tilde{D}_uLH_j' - \gamma\mathbf{I} & \tilde{D}_w \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \quad (3.20)$$

$$i = 1, \ldots, N, \; j = 1, \ldots, N, \; k = 1, \ldots, N$$

then the memory state feedback control gain that solves Problem 2 is given by

$$K = \begin{bmatrix} K_x & K_d \end{bmatrix} = L(F')^{-1} \quad (3.21)$$

assuring that the closed-loop system (3.13) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof: Multiplying (3.20) by α_i, β_j, and δ_k, summing for $i, j, k = 1, \ldots, N$, and letting $L = KF'$, one obtains

$$\begin{bmatrix} P(\beta,\delta) - F - F' & F(\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)' - F'G(\beta)' & F(\tilde{C} + \tilde{D}_uK)' - F'H(\beta)' & 0 \\ (\star) & G(\beta)F(\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)' + (\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)F'G(\beta)' - P(\alpha,\beta) & G(\beta)F(\tilde{C} + \tilde{D}_uK)' + (\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)F'H(\beta)' & \tilde{B}_w \\ (\star) & (\star) & H(\beta)F(\tilde{C} + \tilde{D}_uK)' + (\tilde{C} + \tilde{D}_uK)F'H(\beta)' - \gamma\mathbf{I} & \tilde{D}_w \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \quad (3.22)$$

which is exactly the parameter-dependent condition (3.19) of Lemma 3.2 with

$$\alpha(k) = \alpha \in \mathscr{U} \ , \ \alpha(k+1) = \beta \in \mathscr{U} \ , \ \alpha(k+2) = \delta \in \mathscr{U} \quad , \ \forall k \geq 0$$

$$P(\alpha(k), \alpha(k+1)) = P(\alpha, \beta) = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \beta_j P_{ij}$$

$$P(\alpha(k+1), \alpha(k+2)) = P(\beta, \delta) = \sum_{j=1}^{N} \sum_{k=1}^{N} \beta_j \delta_k P_{jk}$$

$$\mathscr{X}(\alpha(k), \alpha(k+1)) = \sum_{k=1}^{N} \beta_j \begin{bmatrix} F \\ G_j F \\ H_j F \end{bmatrix} = \begin{bmatrix} F \\ G(\beta) F \\ H(\beta) F \end{bmatrix}$$

and the closed-loop matrices $\tilde{A}_{cl}(\alpha)$ and $\tilde{C}_{cl}$ as in (3.13). Finally, the control gain is obtained from the change of variables $L = KF'$, yielding $K = L(F')^{-1}$. ■

Corollary 3.1. *The minimum γ attainable by the conditions of Theorem 3.1 is given by the optimization problem*

$$\min \gamma \ \ s.t. \ (3.20) \tag{3.23}$$

3.3.1 Remarks and Extensions

The first important remark is that by fixing $G = H = 0$, the conditions of Theorem 3.1 reduce to LMIs. Consequently, Corollary 3.1 in this case is a convex optimization problem that can be efficiently handled by semidefinite programming algorithms, see for instance SeDuMi [101]. Although several methods may be applied for the solution of the BMI problem (3.23), the following algorithm is suggested. Fix the variables $H_i = 0$ and $G_i = 0$ and minimize γ with respect to F, L and P_{ij}. Then, fix the variables F, L and P_{ij}, minimize γ with respect to H_i and G_i, and obtain the new values of H_i and G_i. Repeat this procedure until no significant changes in the value of γ occur. This algorithm is sometimes called the alternating semidefinite programming method and consists of fixing some variables and solving for others in such a way that at each step one has a convex optimization problem. Despite the fact that there is no guarantee of convergence to a local minimum in a general BMI setting, these methods are easy to implement and provide good results in many cases, as illustrated by the examples presented in Section 3.4.

It is important to emphasize at this point that the BMI conditions are used to improve the quality of the $\mathscr{H}_\infty$ attenuation level γ, that is, to make it tighter. At each step of the proposed algorithm, a *convex* optimization problem with LMI constraints is solved. More specifically, in the first step of this algorithm the matrices $G(\cdot)$ and $H(\cdot)$ are set to zero and the initial solution is obtained from a convex LMI optimization problem (Corollary 3.1). The start of the algorithm with zeroed matrices turns out to be a good option since it reproduces the convex controller design conditions appearing in the literature for discrete-time systems with time-varying parameters [28]. Other choices for initial values $G(\cdot)$ and $H(\cdot)$, although possible, do not have a correspondence with existing conditions. Moreover, by fixing some variables while searching for others, one is always solving LMI problems that assure nonincreasing values of γ. As can be seen from the numerical examples, the algorithm provides very good results.

The fact that the conditions of Theorem 3.1 need to be satisfied by constant matrices L and F guarantees the existence of a robust state feedback gain $K = L(F')^{-1}$. Other choices may be used, resulting in different structures for $\mathscr{X}(\alpha(k),\alpha(k+1))$ that would, in general, lead to parameter-dependent feedback gains. In particular, the choices made in Theorem 3.1 assure that the extra variable $\mathscr{X}(\alpha(k),\alpha(k+1))$ depends only on $\alpha(k+1)$, in other words, that all the products between the uncertain time-varying matrix $\tilde{A}(\alpha(k))$ and $G(\cdot)$, $H(\cdot)$ in Lemma 3.2 occur at different instants of time. Products of time-varying matrices at the same instant of time $\alpha(k)$ in Lemma 3.2 would require more involved manipulations, such as, for instance, the ones proposed in [91].

A state feedback controller has been chosen to illustrate the possibilities of the proposed approach. Sufficient conditions for decentralized or output feedback control may be obtained by imposing block diagonal structures to the matrices L and F in Theorem 3.1, following the ideas in [50], [66].

The results of Lemma 3.2 and Theorem 3.1 may be improved by considering a larger *path* in the Lyapunov function (3.18) of Lemma 3.2, that is, $P(\alpha(k),\ldots,\alpha(k+L))$. Larger paths (not necessarily of the same size) and other structures may also be used in the extra matrix $\mathscr{X}(\cdot)$ of Lemma 3.2. At the expense of a larger computational effort, lower values for γ may be obtained. Note that the LMI conditions for a path of size $L+1$ provide at least the same values of γ obtained with L.

On the other hand, simpler design conditions based on a Lyapunov matrix $P(\alpha(k))$ can be obtained as a particular case of Theorem 3.1. This preliminary result, stated in the next corollary, appears in [15].

Corollary 3.2. *A sufficient condition for the existence of a memory state feedback control gain that solves Problem 2 is obtained by solving Theorem 3.1 with matrices* $P_{ij} = P_i$ *and* $P_{jk} = P_j$*, that is*

$$P(\alpha(k)) = \sum_{i=1}^{N} \alpha_i P_i, \quad P(\alpha(k+1)) = \sum_{j=1}^{N} \beta_j P_j, \quad \alpha,\ \beta \in \mathscr{U}$$

Finally, it is important to emphasize that using the Cayley-Hamilton theorem to deal with the matrix exponential in (3.3) provides a systematic way to obtain the vertices of polytope (3.15). It is also helpful when bounded rates of variation are involved, since in this case an explicit expression for the variation rate may be obtained. Additionally, the use of a polytope to model the time-varying parameter h_k represents an interesting strategy for solving Problem 2. First, it does not require a knowledge of the processor/network clock granularity g, since the only information used to derive the polytopic model is the extreme values of sets (3.4). Second, the time-varying uncertainties, introduced during the sampling stage, can be completely modeled by a polytope of the form (3.15). Once one has defined the vertices of the closed-loop polytope, there will exist a vector $\alpha(k)$ such that (3.15) holds for each instant of time $k \geq 0$. The only condition on vector $\alpha(k)$ is that it belongs to the unit simplex $\mathscr{U}$ for all $k \geq 0$. Furthermore, the number of values in the set (3.4) does not influence the computational burden; in other words, a larger number of h_k does not imply a greater computational effort, which allows clock granularity to be as small as possible.

3.3.2 More complex NCS scenarios

The controller design method addressed here is mainly concerned with a time-varying sampling period motivated by applications to reduce bandwidth usage. As pointed out in [79], the bandwidth may be reduced by controlling the values that h_k assumes as time evolves in order to reduce the flow of information between the sensor and the controller/actuator. Since robust control is at issue, the sampling period is considered to be uncertain and Lyapunov theory is used for the purpose of synthesis. Although the proposed approach simplifies, or even neglects some aspects of the NCS (the assumptions here being constant time delay, no packet dropouts, single-packet transmission, and infinite sensor precision), some ideas are proposed on how to deal with more complex scenarios.

When the time delay is considered constant and longer than h, system (3.5) has to be slightly modified and more state variables are used to describe the delay, as proposed in [5]. In this case, the matrices in (3.6) become

$$\tilde{A}(h) = \begin{bmatrix} A_s(h) & B_{su1}(h,\tau) & 0 & \dots & 0 \\ 0 & 0 & \mathbf{I} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{I} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \tilde{B}_u(h) = \begin{bmatrix} B_{su0}(h,\tau) \\ 0 \\ \vdots \\ 0 \\ \mathbf{I} \end{bmatrix}, \quad \tilde{B}_w(h) = \begin{bmatrix} B_{sw}(h,\tau) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} C_s & D_{sd} & 0 & \dots & 0 \end{bmatrix}$$

Assuming an event-driven controller and actuator, Theorem 3.1 could be applied when the delay is time-varying within an interval, but in this case the information on the bounds of τ_k is used to derive the vertices of the polytope — in this case, $\eta_i(\cdot)$ in (3.9) would be a function of both h_k and τ_k. Whenever possible, the use of a memory controller through the simplified analysis presented here is suggested, but the method could be adapted to use more complex Lyapunov functions, such as Lyapunov-Krasovskii functionals.

Data packet dropout and multiple-packet transmission in NCS can be modeled as an asynchronous dynamic system (ADS) with rate constraints on events [115]. A simplified ADS with rate constraints can be written as a set of difference equations, as proposed in [115]:

$$x(k+1) = f_s(x(k)), \quad s = 1, 2, \dots, N \tag{3.24}$$

where each discrete state $f_s(\cdot)$ occurs in a fraction of time r_s, $\sum_{s=1}^{N} r_s = 1$. The stability of such a class of systems is studied in [55], as reproduced in the following lemma.

Lemma 3.3 ([55]). *Given an ADS as (3.24), if there exist a Lyapunov function $\vartheta(x(k)) : \mathbb{R}^n \to R_+$ and scalars $\xi_1, \xi_2, \dots, \xi_N$ corresponding to each rate such that*

$$\xi_1^{r_1} \xi_2^{r_2} \cdots \xi_N^{r_N} > \xi > 1 \tag{3.25}$$

$$\vartheta(x(k+1)) - \vartheta(x(k)) \leq (\xi_s^{-2} - 1)\vartheta(x(k)), \quad s = 1, 2, \dots, N \tag{3.26}$$

then the ADS remains exponentially stable, with a decay rate greater than ξ.

By using Lemma 3.3, Theorem 3.1 can be extended to deal with packet dropout and multiple-packet transmission. The NCS is modeled by a set of difference equations activated by a switch that closes at a certain rate r. The packet dropout effect, or the multiple-packet transmission, is then represented by an augmented system, as done in [115], and Lemma 3.3 is applied in the study of stability.

Finally, concerning the infinite sensor precision, the effect of quantizers can be modeled using the sector bound approach. This strategy treats the quantization error as a nonlinearity that lies inside a sector bound. It is a simple and classic approach to study quantization effects and is closely related to absolute stability theory [62]. The approaches discussed in [42] could be explored in this direction.

It is worth mentioning that the extensions proposed in this subsection may be involved or may introduce some conservatism in the results. The aim here is to point out that Theorem 3.1 is not restricted to a simplified framework and may be adapted to deal with different situations. These topics will be further investigated in future work.

3.4 Numerical Experiments

Example I

The aim here is to illustrate the proposed method and to show in detail the steps based on the Cayley-Hamilton theorem to obtain the vertices of the polytopic model.

This example, borrowed from [1], is a simplified model of an armature voltage-controlled DC servo motor consisting of a stationary field and a rotating armature and load. All effects of the field are neglected. The aim is to design $\mathscr{H}_\infty$ robust memory control of the speed of the shaft. All information is sent through a communication network. The behavior of the DC servo motor shown in Figure 3.2 is described by the differential equations

$$\begin{bmatrix} \ddot{\varphi} \\ \ddot{\rho}_a \end{bmatrix} = \begin{bmatrix} -\dfrac{b}{J} & \dfrac{K_T}{J} \\ -\dfrac{K_\varphi}{L_a} & -\dfrac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\rho}_a \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} e_a(t) \tag{3.27}$$

where e_a is the externally applied armature voltage, ρ_a is the armature current, R_a is the resistance of the armature winding, L_a is the armature winding inductance, e_m is the back emf voltage induced by the rotating armature winding ($e_m = K_\varphi \dot{\varphi}, K_\varphi > 0$), b is the viscous damping due to bearing friction, J is the moment of inertia of the armature and load, and φ is the shaft position. The torque generated by the motor is given by $T = K_T i_a$ and $J = 0.01 kgm^2/s^2$, $b = 0.1 Nms$, $K_T = K_\varphi = 0.01 Nm/Amp$, $R_a = 1\Omega$, and $L_a = 0.5H$.

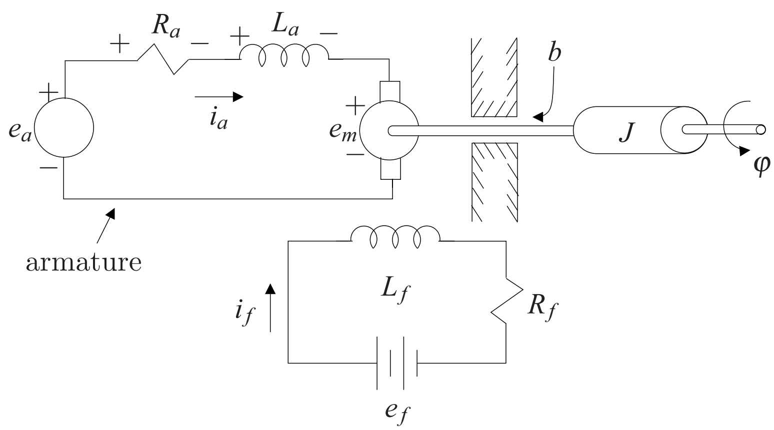

Figure 3.2: DC Servo motor as presented in [1].

System (3.27) was also studied in [104], assuming zero delay, time-varying sampling rates in the sensor, and no index of performance. Although this system is already stable, Corollary 3.1 was applied in order to provide a gain matrix that guarantees robustness against unmodeled $l_2[0,\infty)$ perturbations by minimizing the $\mathscr{H}_\infty$ index of performance of the closed-loop system. Furthermore, a nonzero delay is considered, $\tau = 0.5ms$, and the sampling rate is allowed to vary within the interval $h_k \in [0.001\ 0.099]$.

Closing the loop with (3.7), one can express system (3.27) by the polytope (3.15) with four vertices ($N = 4$) obtained through Cayley-Hamilton theorem as follows. First, to compute $A_s(h) = \exp(Ah)$, obtain $\rho_0(h)$ and $\rho_1(h)$ by solving the linear system

$$\begin{bmatrix} 1 & -9.9975 \\ 1 & -2.0025 \end{bmatrix} \begin{bmatrix} \rho_0(h) \\ \rho_1(h) \end{bmatrix} = \begin{bmatrix} \theta_1(h) \\ \theta_2(h) \end{bmatrix}$$

$$\theta_1(h) = \exp(-9.9975h), \quad \theta_2(h) = \exp(-2.0025h)$$

Then, express $\exp(Ah)$ as

$$\exp(Ah) = \rho_0(h)\mathbf{I} + \rho_1(h)A = \begin{bmatrix} 1.0003 & -0.1251 \\ 0.0025 & -0.0003 \end{bmatrix} \theta_1(h) + \begin{bmatrix} -0.0003 & 0.1251 \\ -0.0025 & 1.0003 \end{bmatrix} \theta_2(h)$$

By evaluating $\theta_1(h)$ and $\theta_2(h)$ at the extreme values of h, one has

$$0.3717 \leq \theta_1(h) \leq 0.9901, \qquad 0.8202 \leq \theta_2(h) \leq 0.9980$$

and the polytopic model with $N = 4$ vertices (obtained by collecting terms) is given by

$$\exp(Ah) = \alpha_1 \begin{bmatrix} 0.3715 & 0.0561 \\ -0.0011 & 0.8203 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.9901 & -0.0212 \\ 0.0004 & 0.8201 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0.3715 & 0.0783 \\ -0.0016 & 0.9982 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0.9900 & 0.0010 \\ -0.0000 & 0.9980 \end{bmatrix}$$

Similarly, to compute $B_{su0}(h,\tau)$, use Cayley-Hamilton to obtain $\eta_0(h)$ and $\eta_1(h)$ such that

$$\int_0^{h-\tau} \exp(As)ds = \eta_0(h)\mathbf{I} + \eta_1(h)A$$

by solving the linear system

$$\begin{bmatrix} \eta_0(h) \\ \eta_1(h) \end{bmatrix} = \begin{bmatrix} 0.0252 & -0.6251 \\ 0.0126 & -0.0625 \end{bmatrix} \begin{bmatrix} \theta_1(h) \\ \theta_2(h) \end{bmatrix} - \begin{bmatrix} -0.5994 \\ -0.0500 \end{bmatrix}$$

Then, using the extreme values for $\theta_1(h)$ and $\theta_2(h)$ above and collecting terms, one obtains

$$B_{su0}(h,\tau) = \int_0^{h-\tau} \exp(As)dsB = \alpha_1 \begin{bmatrix} 0.0067 \\ 0.1788 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.0222 \\ 0.1788 \end{bmatrix} + \alpha_3 \begin{bmatrix} -0.0155 \\ 0.0010 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0.0010 \end{bmatrix}$$

Matrix $B_{su1}(h,\tau)$ can be evaluated from $\exp(Ah)$ since

$$B_{su1}(h,\tau) = \exp(Ah)\Big(\exp(-A\tau)\int_0^{\tau} \exp(As)dsB\Big)$$

yielding (from similar steps) the vertices

$$B_{su1_1} = \begin{bmatrix} 0.0001 \\ 0.0008 \end{bmatrix}, \; B_{su1_2} = \begin{bmatrix} 0 \\ 0.0008 \end{bmatrix}, \; B_{su1_3} = \begin{bmatrix} 0.0001 \\ 0.0010 \end{bmatrix}, \; B_{su1_4} = \begin{bmatrix} 0 \\ 0.0010 \end{bmatrix}$$

The polytopic model for $\tilde{A}(\alpha)$ is then given by

$$\tilde{A}_1 = \left[\begin{array}{cc|c} 0.3715 & 0.0561 & 0.0001 \\ -0.0011 & 0.8203 & 0.0008 \\ \hline 0 & 0 & 0 \end{array}\right], \; \tilde{A}_2 = \left[\begin{array}{cc|c} 0.9901 & -0.0212 & 0 \\ 0.0004 & 0.8201 & 0.0008 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\tilde{A}_3 = \left[\begin{array}{cc|c} 0.3715 & 0.0783 & 0.0001 \\ -0.0016 & 0.9982 & 0.0010 \\ \hline 0 & 0 & 0 \end{array}\right], \; \tilde{A}_4 = \left[\begin{array}{cc|c} 0.9900 & 0.0010 & 0 \\ 0 & 0.9980 & 0.0010 \\ \hline 0 & 0 & 0 \end{array}\right]$$

System (3.27) is then rewritten as in (3.5) with matrices $D_{sw} = [1]$, $B'_{sw} = [0.1 \; 0]$ and $D_{sd} = [0]$.

Corollaries 3.1 and 3.2 are aplied using alternating semidefinite programming. Each iteration consists of two steps. First, the problem is solved with $G(\cdot) = 0$ and $H(\cdot) = 0$ (in this case, the problem is convex) and, second, $G(\cdot)$ and $H(\cdot)$ are explored in the search for a better $\mathscr{H}_\infty$ upper bound γ. The results after five iterations are shown in Table 3.1.

Table 3.1: $\mathscr{H}_\infty$ Robust Memory Controller for Example I.

Method	$\mathscr{H}_\infty$ Upper Bound γ	Gain Matrix K
Corollary 3.1	10.87	$[-1.8822 - 9.6684 - 0.0117]$
Corollary 3.2	10.90	$[-1.5670 - 9.8076 - 0.0150]$

Sufficient conditions for the existence of a decentralized or a static output feedback control gain may be obtained from Theorem 3.1 by simply imposing to matrices L and F in (3.21) a fixed structure, following the lines in [50], [66]. For instance, suppose that the first state variable is not available for feedback. By imposing

$$L = \begin{bmatrix} 0 & \ell_2 & \ell_3 \end{bmatrix}, \; F = \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{bmatrix}$$

to matrices L and F in Theorem 3.1 the following result is obtained (after five iterations)

$$K = \begin{bmatrix} 0 & 0.8002 & -7.5613 \times 10^{-6} \end{bmatrix}, \quad \gamma = 11.09$$

Example II

This example is intended to point out the quality of the proposed method when no communication channel is considered. Consider an uncertain time-varying discrete-time system with vertices given by

$$\tilde{A}_1 = \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \; \tilde{A}_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix}, \; \tilde{B}_{u1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; \tilde{B}_{u2} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}$$

$$\tilde{B}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \tilde{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; \tilde{D}_w = \tilde{D}_u = \begin{bmatrix} 0 \end{bmatrix}$$

This system is also studied in [29] but for a simpler case, where the parameters of matrix $\tilde{B}_u$ are time-invariant. Here, the results from Theorem 3.1 are compared to

[73, Remark 2]. In order to illustrate the efficiency of the proposed method due to the use of a path-dependent Lyapunov function, Corollary 3.1 is contrasted with Corollary 3.2. The results can be seen in Table 3.2.

Table 3.2: $\mathscr{H}_\infty$ Robust Memory Controller for Example II.

Method	Iteration	γ	Improvement	Time (sec)
[73]	–	67.33	–	0.09
Corollary 3.1	1	31.37	53.40%	0.17
Corollary 3.1	2	23.10	65.69%	0.33
⋮	⋮	⋮	⋮	
Corollary 3.1	10	19.21	71.47%	1.55
Corollary 3.2	1	30.39	54.87%	0.20
Corollary 3.2	2	17.10	74.61%	0.39
⋮	⋮	⋮	⋮	
Corollary 3.2	10	11.25	83.29%	1.72

3.5 Conclusion

This chapter addressed the $\mathscr{H}_\infty$ robust controller for NCSs with uncertain time-varying sampling rates. A new state space variable, representing the buffer of the controller, was added to model a time-delay in the control signal. A polytope with vertices determined by Cayley-Hamilton theorem was used to model the system. Using an approach based on path-dependent Lyapunov functions, theoretical conditions were formulated for the existence of a state feedback control assuring an $\mathscr{H}_\infty$ attenuation level for the closed-loop system. Then, sufficient conditions for the existence of the memory controller are derived in terms of BMIs described only at the vertices of the polytope. An algorithm exploiting appropriate choices of the extra variables is used to solve the problem through a sequence of convex optimization procedures, providing lower levels for the $\mathscr{H}_\infty$ performance of the closed-loop system. When no communication channel is considered, the proposed conditions can also provide better results when compared to other methods in the literature dealing with time-varying discrete-time systems. Some remarks on possible extensions to more complex NCS scenarios were presented and numerical experiments were provided to illustrate different aspects of the proposed approach.

Chapter 4

$\mathscr{H}_\infty$ filtering for discrete-time systems with bounded time-varying parameters

4.1 Introduction

Technological advances have always pushed the control community to face more complex problems in several different frameworks. Concerning the linear filtering problem, that extends from the original work by Kalman [60], a large number of papers dealing with deterministic and stochastic scenarios has appeared in the literature. More sophisticated structures are needed when dealing with signal recovery and estimation under time-varying or constant uncertainties.

In this context, Lyapunov theory has been extensively applied as a tool to deal with the synthesis of filters that guarantee the stability of the estimation error dynamics, while guaranteeing a certain level of performance. For example, quadratic Lyapunov functions have been used to deal with time-invariant or arbitrarily time-varying systems as can be seen in [30], [49], [52] concerning the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ robust filtering. Improvements of these results may be obtained using parameter-dependent Lyapunov functions, as proposed in [53] for the time-invariant case and in [37] for the time-varying case with bounded rates of variation. Recent published works have also dealt with robust filtering.

Considering the case where the time-varying parameters, although unknown *a priori*, may still be measured online, gain scheduling techniques represent an interesting option for filtering or control of dynamic systems when contrasted with robust methods. Furthermore, as discussed in [62], gain-scheduling strategies extend the validity of the linearization approach of nonlinear systems to a range of operating points. As mentioned in [95], gain scheduling is an effective and economical method for nonlinear control design in practice. In the filtering framework, recent works include [6], [96] where affine parameter varying filters, with limited rate of variation, are obtained, [57] in the context of parameter-dependent filters by means of nonlinear fractional transformation and quadratic stability, [46] concerned with LPV filtering for slowly varying systems and [18] where the LPV filtering for arbitrarily time-varying systems in polytopic domain is addressed.

Extending the powerful features of gain scheduling (well presented in [95]) to deal with the filtering problem is of great importance especially within the class of time-varying discrete-time systems. It is known from [9] that for time-varying discrete-time systems, robust stabilizability implies gain scheduling stabilizability, but the converse is not true. These facts motivate the results and effort of the present work.

This chapter investigates the LPV filtering of time-varying systems with bounds on the rate of variation. A preliminary version of the results presented in this chapter have appeared in [16] considering only the robust filter design, and applications in the context of networked robust filtering in [21]. Here, the proposed approach complements and extends previous results in the literature by presenting a systematic procedure for filtering design that can be applied in four different frameworks, namely, LPV or robust filtering of time-varying systems with bounded or unbounded rates of variation. Lyapunov theory is applied in order to obtain the design conditions of the filter. A parameter-dependent Lyapunov function is used to reduce the conservatism of the proposed method, resulting in a more general approach when compared to methods based on quadratic stability. All system matrices are assumed to be affected by time-varying parameters, which are assumed to lie inside a polytope. A more precise parameter variation modeling is applied to give a better description of the uncertainty domain and an $\mathscr{H}_\infty$ guaranteed cost is used as a performance index. The $\mathscr{H}_\infty$ filtering limits the maximum possible variance of the error signal over all exogenous inputs with bounded variance [94], *i.e.* the $\mathscr{H}_\infty$ norm reflects the worst-case energy gain of the system and does not require statistical assumptions on the exogenous input (a situation in which the Kalman filtering cannot be employed, [105]). Furthermore, it can provide robustness with respect to unmodeled uncertainties. The LPV filter is then obtained from the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of bilinear matrix inequality (BMI) constraints formulated only in terms of the vertices of a polytope. No grids in the parametric space are used. Extra variables introduced through the BMI conditions can be explored in the search for better $\mathscr{H}_\infty$ performance of the estimation error dynamics giving more flexibility to the design process. Robust filters for time-invariant and arbitrarily time-varying uncertain systems may be obtained as particular cases of the proposed method. Numerical examples illustrate the efficacy of the proposed results.

4.2 Problem statement and preliminary results

Consider the time-varying discrete-time system, for $k \geq 0$

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B(\alpha(k))w(k) \\ z(k) &= C_1(\alpha(k))x(k) + D_1(\alpha(k))w(k) \\ y(k) &= C_2(\alpha(k))x(k) + D_2(\alpha(k))w(k) \end{aligned} \tag{4.1}$$

where $x(k) \in \mathbb{R}^n$ is the state space vector, $w(k) \in \mathbb{R}^m$ is the noise input belonging to $l_2[0,\infty)$, $z(k) \in \mathbb{R}^p$ is the signal to be estimated and $y(k) \in \mathbb{R}^q$ is the measured output. The time-varying vector of parameters $\alpha(k)$ belongs to the unit simplex (for all $k \geq 0$)

$$\mathscr{U}_N = \left\{ \delta \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0\ ,\ i = 1,\ldots,N \right\}$$

and has bounded rates of variation of percentage $b \in [0,1]$. For instance, $b = 0.05$ indicates that the parameters are constrained to vary only 5% of their original values between two instants of time. The time-invariant case is modeled by $b = 0$ and arbitrarily fast variations by $b = 1$.

All matrices are real, with appropriate dimensions, belonging to the polytope[1]

$$\tilde{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C_1(\alpha) & D_1(\alpha) \\ \hline C_2(\alpha) & D_2(\alpha) \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_{1i} & D_{1i} \\ \hline C_{2i} & D_{2i} \end{array} \right],\ \alpha \in \mathscr{U}_N \right\}. \tag{4.2}$$

More specifically, the system matrices are given, for any time $k \geq 0$, by the convex combination of the well-defined vertices of the polytope $\mathscr{P}$.

A full-order, proper, LPV filter is investigated here, as given by:

$$\begin{aligned} x_f(k+1) &= A_f(\alpha)x_f(k) + B_f(\alpha)y(k),\ x_f(0) = 0 \\ z_f(k) &= C_f(\alpha)x_f(k) + D_f(\alpha)y(k) \end{aligned} \tag{4.3}$$

where $x_f(t) \in \mathbb{R}^n$ is the filter state space vector and $z_f(t) \in \mathbb{R}^p$ the estimated signal. All filter matrices are real, with appropriate dimensions, belonging to the polytope

$$\hat{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A_f(\alpha) & B_f(\alpha) \\ \hline C_f(\alpha) & D_f(\alpha) \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_{fi} & B_{fi} \\ \hline C_{fi} & D_{fi} \end{array} \right],\ \alpha \in \mathscr{U}_N \right\}. \tag{4.4}$$

The estimation error dynamics is given by

$$\begin{aligned} \varsigma(k+1) &= \hat{A}(\alpha)\varsigma(k) + \hat{B}(\alpha)w(k), \varsigma(0) = 0 \\ e(k) &= \hat{C}(\alpha)\varsigma(k) + \hat{D}(\alpha)w(k) \end{aligned} \tag{4.5}$$

[1]The time dependence of α(k) will be omitted to lighten the notation.

where $\varsigma(k) = [x(k)' \;\; x_f(k)']'$, $e(k) = z(k) - z_f(k)$ and

$$\hat{A}(\alpha) = \begin{bmatrix} A(\alpha) & \mathbf{0} \\ B_f(\alpha)C_2(\alpha) & A_f(\alpha) \end{bmatrix}, \; \hat{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ B_f(\alpha)D_2(\alpha) \end{bmatrix}$$

$$\begin{aligned} \hat{C}(\alpha) &= \begin{bmatrix} C_1(\alpha) - D_f(\alpha)C_2(\alpha) & -C_f(\alpha) \end{bmatrix}, \\ \hat{D}(\alpha) &= \begin{bmatrix} D_1(\alpha) - D_f(\alpha)D_2(\alpha) \end{bmatrix} \end{aligned} \tag{4.6}$$

The filtering problem is stated as follows.

Problem 3. *Find matrices $A_{fi} \in \mathbb{R}^{n\times n}$, $B_{fi} \in \mathbb{R}^{n\times q}$, $C_{fi} \in \mathbb{R}^{p\times n}$ and $D_{fi} \in \mathbb{R}^{p\times q}$ $i = 1,\ldots,N$, of the filter (4.3), such that the estimation error system (4.5) is asymptotically stable, and an upper bound γ to the $\mathscr{H}_\infty$ estimation error performance is minimized, that is*

$$\sup_{w\neq 0} \frac{\|e\|_2^2}{\|w\|_2^2} < \gamma^2 \tag{4.7}$$

with $w \in l_2[0,\infty)$.

Note that, since the parameters lie inside a unit simplex, the rates of variation are intrinsically lower bounded by $-b$ and upper bounded by b, $b \in [0,1]$. In order to model[2] the parameter variation when $-b < \Delta\alpha_i(k) < b, \;\; b \neq 0$, it must be taken into account that the feasible values of $\Delta\alpha_i(k)$ depend on the actual values of $\alpha_i(k)$, as show in Figure 4.1 (darken area).

Thus, any pair $(\alpha_i, \Delta\alpha_i)$ belongs to the polytope Λ_i, $i = 1,\ldots,N$ given by

$$\begin{aligned} \Lambda_i &\triangleq \left\{ \delta \in \mathbb{R}^2 : \delta = \sum_{j=1}^{6} \lambda_j r_j, \quad \lambda \in \mathscr{U}_6 \right\}, \\ [r_1 \cdots r_6] &= \left[\begin{array}{c:c:c:c:c:c} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{array} \right], \end{aligned} \tag{4.8}$$

that is, Λ_i is the convex combination of the extremes (vertices) of the feasible area.

[2]For simplicity, the same b is considered for all α_i, $i = 1,\ldots,N$.

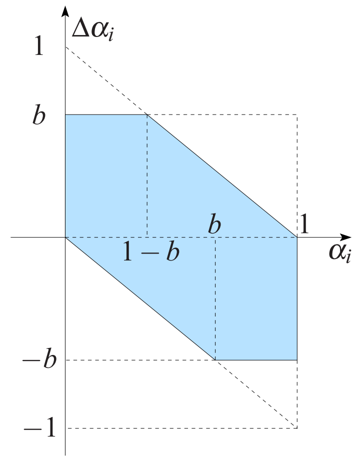

Figure 4.1: Region on the plane $\Delta\alpha_i \times \alpha_i$ where $\Delta\alpha_i$ can assume values as a function of α_i (dark region).

To construct the $(\alpha, \Delta\alpha)$-space, the Cartesian product of all Λ_i, $i = 1, \ldots, N$ must be considered, taking into account that the new vertices must satisfy $\alpha_1 + \cdots + \alpha_N = 1$ and $\Delta\alpha_1 + \cdots + \Delta\alpha_N = 0$. The resulting polytope, called Λ, is then given by

$$\Lambda \triangleq \left\{ \delta \in \mathbb{R}^{2N} : \delta = \sum_{i=1}^{M} \lambda_i s_i, \quad \lambda \in \mathscr{U}_M \right\}, \tag{4.9}$$

where $s_i \in \mathbb{R}^{2N}$ are given vectors. As a consequence, the first step to search for a solution to any LMI/BMI depending on both α and $\Delta\alpha$ is to make a lifting to the λ-space. From (4.9) one has

$$(\alpha', \Delta\alpha')' = S\lambda, \quad S = [s_1 \cdots s_M] \in \mathbb{R}^{2N \times M}, \quad \lambda \in \mathscr{U}_M. \tag{4.10}$$

In the case of affine parameter-dependent matrices, that is

$$X(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) X_i, \quad \alpha_i(k) = \sum_{j=1}^{M} \lambda_j S_{ij}, \tag{4.11}$$

$$X(\alpha(k+1)) = \sum_{i=1}^{N} (\alpha_i(k) + \Delta\alpha_i(k)) X_i, \quad \Delta\alpha_i(k) = \sum_{j=1}^{M} \lambda_j S_{(i+N)j}, \tag{4.12}$$

it follows that

$$\bar{X}(\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{M} \lambda_j S_{ij} X_i = \sum_{j=1}^{M} \lambda_j \bar{X}_j, \quad \tilde{X}(\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{M} \lambda_j (S_{ij} + S_{(i+N)j}) X_i = \sum_{j=1}^{M} \lambda_j \tilde{X}_j, \tag{4.13}$$

where[3]

$$\bar{X}_j = \sum_{i=1}^{N} S_{ij} X_i, \tag{4.14}$$

$$\tilde{X}_j = \sum_{i=1}^{N} (S_{ij} + S_{(i+N)j}) X_i. \tag{4.15}$$

Theorem 4.1. *(Stability Analysis) For a given γ, if there exists bounded matrix sequences $\mathscr{P}(\alpha)' = \mathscr{P}(\alpha) > 0$, $G(\zeta)$, $H(\zeta)$, matrix F and full rank matrix T, with appropriate dimensions, such that (the term $(\star)$ indicates symmetric blocks in the matrix inequality)*

$$\begin{bmatrix} \mathscr{P}(\alpha_+) - F - F' & F\hat{A}(\alpha)' - F'TG(\zeta)'T^{-1} & F\hat{C}(\alpha)' - F'TH(\zeta)' \\ (\star) & \mathscr{L}_{22} & \mathscr{L}_{23} \\ (\star) & (\star) & \mathscr{L}_{33} \end{bmatrix} < 0, \tag{4.16}$$

$$\begin{aligned} \mathscr{L}_{22} &= (T')^{-1}G(\zeta)T'F\hat{A}(\alpha)' + \hat{A}(\alpha)F'TG(\zeta)'T^{-1} - \mathscr{P}(\alpha) + \gamma^{-1}\hat{B}(\alpha)\hat{B}(\alpha)', \\ \mathscr{L}_{23} &= (T')^{-1}G(\zeta)T'F\hat{C}(\alpha)' + \hat{A}(\alpha)F'TH(\zeta)' + \gamma^{-1}\hat{B}(\alpha)\hat{D}(\alpha)', \\ \mathscr{L}_{33} &= H(\zeta)T'F\hat{C}(\alpha)' + \hat{C}(\alpha)F'TH(\zeta)' - \gamma\mathbf{I} + \gamma^{-1}\hat{D}(\alpha)\hat{D}(\alpha)', \end{aligned}$$

for all $\alpha, \zeta \in \mathscr{U}_N$, where $\alpha_+ = \alpha(k+1)$, and bounded $\Delta\alpha$, then the error dynamics (4.5) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof: Firstly, multiply the inequality (4.16) on left by $\mathscr{T}'$ and on right by $\mathscr{T}$, with

$$\mathscr{T}' = \begin{bmatrix} \hat{A}(\alpha) & I & \mathbf{0} \\ \hat{C}(\alpha) & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

[3]The same conversion is applied in the system and filter matrices.

in order to obtain

$$\begin{bmatrix} \hat{A}(\alpha)\mathscr{P}(\alpha_+)\hat{A}(\alpha)' - \mathscr{P}(\alpha) + \gamma^{-1}\hat{B}(\alpha)\hat{B}(\alpha)' & \hat{A}(\alpha)'\mathscr{P}(\alpha_+)\hat{C}(\alpha)' + \gamma^{-1}\hat{B}(\alpha)\hat{D}(\alpha)' \\ (\star) & \hat{C}(\alpha)\mathscr{P}(\alpha_+)\hat{C}(\alpha)' + \gamma^{-1}\hat{D}(\alpha)\hat{D}(\alpha)' - \gamma\mathbf{I} \end{bmatrix} < 0 \quad (4.17)$$

If (4.17) holds, it follows that $\hat{A}(\alpha)\mathscr{P}(\alpha_+)\hat{A}(\alpha)' - \mathscr{P}(\alpha) < 0$ and, from the Lyapunov theory [62], the system is asymptotically stable. Secondly, by choosing $\vartheta(k) = \varsigma(k)'\mathscr{P}(\alpha)\varsigma(k)$ as a parameter-dependent Lyapunov function and considering the dual system (*i.e.* $\hat{A} = \hat{A}'$, $\hat{B} = \hat{C}'$, $\hat{C} = \hat{B}'$ and $\hat{D} = \hat{D}'$), it follows, after some algebraic manipulation, that inequality (4.17) implies

$$\Delta\vartheta(k) < -\gamma^{-1}e(k)'e(k) + \gamma w(k)'w(k).$$

Therefore, system (4.5) has an upper bound γ to the $\mathscr{H}_\infty$ performance. ■

It is important to stress that the extra variables F, $G(\zeta)$ and $H(\zeta)$ in (4.16) represent extra degrees of freedom in the search for a feasible solution of Theorem 4.1. As pointed out in [31], these variables may be identified as Lagrangian multipliers and can be explored for design purposes. In this sense, different structures of matrices F, $G(\zeta)$ and $H(\zeta)$ can be used yielding different sufficient conditions for stability analysis. For instance, assuming polytopic structures, $G(\cdot)$ and $H(\cdot)$ c maye parametrized in α or α_+ as used throughout this chapter.

The nonlinear inequality conditions of Theorem 4.1 must be tested at all points of the simplex $\mathscr{U}_N$, *i.e.*, at an infinite number of points. Hence, the main goal hereafter is to obtain finite-dimensional BMI conditions in terms of the vertices of the polytope $\mathscr{P}$ to solve Problem 3. Using Schur complement and a change of variables, finite-dimensional BMIs assuring the existence of such filters are given in the next section.

4.3 Main Results

By considering the particular structure

$$\mathscr{P}(\alpha) = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_N P_N,\ \alpha \in \mathscr{U}_N, \tag{4.18}$$

lifted to the λ-space, the following sufficient condition can be obtained.

Theorem 4.2. *($\mathscr{H}_\infty$ LPV FILTERING) Given the system (4.1) and matrix S as in* (4.10)*, if there exist matrices Z, Y, R, $Q_i \in \mathbb{R}^{n\times n}$, $L_i \in \mathbb{R}^{n\times q}$, $J_i \in \mathbb{R}^{p\times n}$, $D_{fi} \in \mathbb{R}^{p\times q}$, G, $M_i = M_i' > 0 \in \mathbb{R}^{2n\times 2n}$, $H \in \mathbb{R}^{p\times 2n}$, $i = 1,\ldots,N$ and a scalar $\gamma > 0$ such that, for matrices $\bar{Q}_i$, $\bar{L}_i$, $\bar{J}_i$, $\bar{D}_{fi}$, $\bar{M}_i$, $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_{1i}$, $\bar{C}_{2i}$, $\bar{D}_{1i}$ and $\bar{D}_{2i}$ given as in* (4.14) *and $\tilde{M}_i$ as in* (4.15)

$$\Xi_i \triangleq \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} & \hat{F}_{3i} - \hat{F}_1' H' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & G\hat{F}_{3i} + \hat{F}_{2i}' H' & \hat{F}_{4i} \\ (\star) & (\star) & H\hat{F}_{3i} + \hat{F}_{3i}' H' - \gamma \mathbf{I} & \mathscr{F}_{34} \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0,\ i = 1,\ldots,M, \tag{4.19}$$

$$\mathscr{F}_{11} = \tilde{M}_i - \hat{F}_1 - \hat{F}_1',\ \mathscr{F}_{12} = \hat{F}_{2i} - \hat{F}_1' G',$$
$$\mathscr{F}_{22} = G\hat{F}_{2i} + \hat{F}_{2i}' G' - \bar{M}_i,\ \mathscr{F}_{34} = \bar{D}_{1i} - \bar{D}_{fi}\bar{D}_{2i},$$
$$\hat{F}_1 = \begin{bmatrix} Z & Y' + R' \\ Z & Y' \end{bmatrix},\ \hat{F}_{2i} = \begin{bmatrix} \bar{A}_i' Z & \bar{A}_i' Y' + \bar{C}_{2i}' \bar{L}_i' + \bar{Q}_i' \\ \bar{A}_i' Z & \bar{A}_i' Y' + \bar{C}_{2i}' \bar{L}_i' \end{bmatrix},$$
$$\hat{F}_{3i} = \begin{bmatrix} \bar{C}_{1i}' - \bar{C}_{2i}' \bar{D}_{fi}' - \bar{J}_i' \\ \bar{C}_{1i}' - \bar{C}_{2i}' \bar{D}_{fi}' \end{bmatrix},\ \hat{F}_{4i} = \begin{bmatrix} Z' \bar{B}_i \\ Y\bar{B}_i + \bar{L}_i \bar{D}_{2i} \end{bmatrix},$$

$$\Xi_{ik} \triangleq \begin{bmatrix} \hat{\mathscr{F}}_{11} & \hat{\mathscr{F}}_{12} & \hat{F}_{3ik} - 2\hat{F}_1' H' & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & G\hat{F}_{3ik} + \hat{F}_{2ik}' H' & \hat{F}_{4ik} \\ (\star) & (\star) & H\hat{F}_{3ik} + \hat{F}_{3ik}' H' - 2\gamma \mathbf{I} & \hat{\mathscr{F}}_{34} \\ (\star) & (\star) & (\star) & -2\gamma \mathbf{I} \end{bmatrix} < 0, \begin{cases} i = 1,\ldots,M-1 \\ k = i+1,\ldots,M \end{cases}, \tag{4.20}$$

$$\hat{\mathscr{F}}_{11} = \tilde{M}_i + \tilde{M}_k - 2\hat{F}_1 - 2\hat{F}_1',\ \hat{\mathscr{F}}_{12} = \hat{F}_{2ik} - 2\hat{F}_1'G',$$
$$\hat{\mathscr{F}}_{22} = G\hat{F}_{2ik} + \hat{F}_{2ik}'G' - \bar{M}_i - \bar{M}_k,\ \hat{\mathscr{F}}_{34} = \bar{D}_{1i} + \bar{D}_{1k} - \bar{D}_{fi}\bar{D}_{2k} - \bar{D}_{fk}\bar{D}_{2i},$$

$$\hat{F}_{2ik} = \begin{bmatrix} (\bar{A}_i' + \bar{A}_k')Z & (\bar{A}_i' + \bar{A}_k')Y' + \bar{C}_{2i}'\bar{L}_k' + \bar{C}_{2k}'\bar{L}_i' + \bar{Q}_i' + \bar{Q}_k' \\ (\bar{A}_i' + \bar{A}_k')Z & (\bar{A}_i' + \bar{A}_k')Y' + \bar{C}_{2i}'\bar{L}_k' + \bar{C}_{2k}'\bar{L}_i' \end{bmatrix},$$
$$\hat{F}_{3ik} = \begin{bmatrix} \bar{C}_{1i}' + \bar{C}_{1k}' - \bar{C}_{2i}'\bar{D}_{fk}' - \bar{C}_{2k}'\bar{D}_{fi}' - \bar{J}_i' - \bar{J}_k' \\ \bar{C}_{1i}' + \bar{C}_{1k}' - \bar{C}_{2i}'\bar{D}_{fk}' - \bar{C}_{2k}'\bar{D}_{fi}' \end{bmatrix},\ \hat{F}_{4ik} = \begin{bmatrix} Z'(\bar{B}_i + \bar{B}_k) \\ Y(\bar{B}_i + \bar{B}_k) + \bar{L}_i\bar{D}_{2k} + \bar{L}_k\bar{D}_{2i} \end{bmatrix},$$

then there exists an LPV filter in the form of (4.3), ensuring the asymptotic stability of the estimation error dynamic (4.5) and an $\mathscr{H}_\infty$ *guaranteed cost* γ*, for all* $(\alpha, \Delta\alpha) \in \Lambda$ *with vertices given by*

$$\begin{aligned} A_{fi} &= \hat{V}^{-1}Q_i(UZ)^{-1},\ B_{fi} = \hat{V}^{-1}L_i, \\ C_{fi} &= J_i(UZ)^{-1},\ D_{fi}, \end{aligned} \tag{4.21}$$

where $U \in \mathbb{R}^{n\times n}$ *and* $\hat{V} \in \mathbb{R}^{n\times n}$ *are matrices arbitrarily chosen such that* $R = \hat{V}UZ$*.*

Proof: Applying the following operation

$$\Xi(\lambda) = \sum_{i=1}^{N} \lambda_i^2 \Xi_i + \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \lambda_i\lambda_k\Xi_{ik}, \tag{4.22}$$

to the BMIs (4.19) and (4.20) one gets

$$\Xi(\lambda) = \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} & \mathscr{F}_{13} & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & \mathscr{F}_{23} & \hat{F}_4 \\ (\star) & (\star) & \mathscr{F}_{33} & \mathscr{F}_{34} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0, \tag{4.23}$$

$$\mathscr{F}_{11} = \tilde{M}(\lambda) - \hat{F}_1 - \hat{F}_1',\ \mathscr{F}_{12} = \hat{F}_2(\lambda) - \hat{F}_1'G',\ \mathscr{F}_{13} = \hat{F}_3(\lambda) - \hat{F}_1'H',$$
$$\mathscr{F}_{22} = G\hat{F}_2(\lambda) + \hat{F}_2(\lambda)'G' - \bar{M}(\lambda),\ \mathscr{F}_{23} = G\hat{F}_3(\lambda) + \hat{F}_2(\lambda)'H',$$
$$\mathscr{F}_{33} = H\hat{F}_3(\lambda) + \hat{F}_3(\lambda)'H' - \gamma\mathbf{I},\ \mathscr{F}_{34} = \bar{D}_1(\lambda) - \bar{D}_f(\lambda)\bar{D}_2(\lambda),$$

where

$$\hat{F}_2(\lambda) = \begin{bmatrix} \bar{A}(\lambda)'Z & \bar{A}(\lambda)'Y' + \bar{C}_2(\lambda)'\bar{L}(\lambda)' + \bar{Q}(\lambda)' \\ \bar{A}(\lambda)'Z & \bar{A}(\lambda)'Y' + \bar{C}_2(\lambda)'\bar{L}(\lambda)' \end{bmatrix},$$
$$\hat{F}_3(\lambda)' = \begin{bmatrix} \bar{C}_1(\lambda) - \bar{D}_f(\lambda)\bar{C}_2(\lambda) - \bar{J}(\lambda) & \bar{C}_1(\lambda) - \bar{D}_f(\lambda)\bar{C}_2(\lambda) \end{bmatrix},$$
$$\hat{F}_4(\lambda)' = \begin{bmatrix} \bar{B}(\lambda)'Z & \bar{B}(\lambda)'Y' + \bar{D}_2(\lambda)'\bar{L}(\lambda)' \end{bmatrix}.$$

Then, define the partitioned matrices [27]

$$F = \begin{bmatrix} X' & U' \\ \hat{U}' & \hat{X}' \end{bmatrix},\ F^{-1} = \begin{bmatrix} Y & \hat{V} \\ V & \hat{Y} \end{bmatrix},\ T = \begin{bmatrix} X^{-1} & Y' \\ \mathbf{0} & \hat{V}' \end{bmatrix},\ T^{-1} = \begin{bmatrix} X & -XY'(\hat{V}')^{-1} \\ \mathbf{0} & (\hat{V}')^{-1} \end{bmatrix}$$

together with the following variable transformation

$$\begin{bmatrix} \bar{Q}(\lambda) & \bar{L}(\lambda) \\ \bar{J}(\lambda) & \bar{D}_f(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{A}_f(\lambda) & \bar{B}_f(\lambda) \\ \bar{C}_f(\lambda) & \bar{D}_f(\lambda) \end{bmatrix} \begin{bmatrix} UZ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},\ R = \hat{V}UZ, \tag{4.24}$$

where $Z = X^{-1}$. Using the above change of variable, multiply inequality (4.23) to the left by $\hat{S}'$ and to the right by $\hat{S}$ with

$$\hat{S} = \begin{bmatrix} \mathscr{S} & \mathbf{0} \\ \mathbf{0} & \mathscr{I} \end{bmatrix},\ \mathscr{S} = \begin{bmatrix} T^{-1} & \mathbf{0} \\ \mathbf{0} & T^{-1} \end{bmatrix},\ \mathscr{I} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

yielding the following inequality

$$\begin{bmatrix} \tilde{\mathscr{P}}(\lambda) - F - F' & F\hat{A}(\lambda)' - F'TG'T^{-1} & F\hat{C}(\lambda)' - F'TH' & \mathbf{0} \\ (\star) & \hat{\mathscr{L}}_{22} & \hat{\mathscr{L}}_{23} & \hat{B}(\lambda) \\ (\star) & (\star) & \hat{\mathscr{L}}_{33} & \hat{D}(\lambda) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0, \tag{4.25}$$

$$\hat{\mathscr{L}}_{22} = (T')^{-1}GT'F\hat{A}(\lambda)' + \hat{A}(\lambda)F'TG'T^{-1} - \bar{\mathscr{P}}(\lambda),$$
$$\hat{\mathscr{L}}_{23} = (T')^{-1}GT'F\hat{C}(\lambda)' + \hat{A}(\lambda)F'TH',$$
$$\hat{\mathscr{L}}_{33} = HT'F\hat{C}(\lambda)' + \hat{C}(\lambda)F'TH' - \gamma\mathbf{I},$$

where $\mathscr{P}(\lambda) = (T')^{-1}M(\lambda)T^{-1}$ and the matrices $\hat{A}(\cdot)$, $\hat{B}(\cdot)$, $\hat{C}(\cdot)$ and $\hat{D}(\cdot)$ have the same structure of (4.6), but in the λ-space. Finally, considering the lift of the BMI to the λ-space and applying Schur complement, inequality (4.25) reduces to (4.16) of Theorem 4.1 with $G(\zeta) = G$ and $H(\zeta) = H$. The filter matrices are obtained by the change of variables (4.24), which concludes the proof. ■

Corollary 4.1. *The minimum γ attainable by the conditions of Theorem 4.2 is given by the optimization problem*

$$\min \gamma \quad \text{s.t. (4.19)–(4.20)} \tag{4.26}$$

Theorem 4.2 is presented in terms of BMI constraints due to the use of extra variables F, G and H. The advantages of this approach are due the fact that such variables may be used in the search for better performance of the closed-loop system. For instance, a lower $\mathscr{H}_\infty$ guaranteed cost may be obtained by exploring the new variables G and H. Nevertheless, by choosing $G = \mathbf{0}$ and $H = \mathbf{0}$ the conditions of Theorem 4.2 reduce to LMIs, and, in this case, Corollary 4.1 becomes a convex optimization problem that can be handled by Semi-Definite Programming (SDP) algorithms.

In order to solve Corollary 4.1 within the BMI framework, many methods from the literature may be applied, such as the two following algorithms. The first one is sometimes called Alternating SDP (or Gauss-Seidel) method [43] and consists of fixing some variables and solving for others in such a way that at each step a convex optimization problem is solved. The second one is called path-following method [56] and consists of linearizing the BMIs. Although in both cases there is no guarantee of convergence, these methods are easy to implement and provide good results. In this chapter, the first approach is used and the algorithm is as follows.

Algorithm 2. *Let $G = \mathbf{0}$ and $H = \mathbf{0}$. Let ε and k_{max} be given. Set $k = 1$ and iterate:*

1. *Fix the variables H and G, minimize w.r.t. γ_k, Z, Y, R, Q_i, L_i, J_i, D_{fi} and M_i. Get the new values of Z, Y, R, Q_i, L_i, J_i and D_{fi}.*

2. *Fix the variables* Z, Y, R, Q_i, L_i, J_i *and* D_{fi}, *minimize w.r.t.* γ_k, H, G *and* M_i. *Get the new values of* H *and* G.

3. *If* $|\gamma_k - \gamma_{k-1}| < \varepsilon$, *then stop (no significant changes).*

4. *Set* $k = k+1$ *and go to step 1 if* $k \leq k_{max}$. *Otherwise stop.*

In order to reduce the number of BMIs and the computational time required to solve the optimization problem (4.26), the conditions of Theorem 4.2 were obtained with $G(\zeta) = G$ and $H(\zeta) = H$. If $G(\zeta)$ and $H(\zeta)$ were parametrized with α a more sophisticated procedure, such as the one proposed in [89], should be applied.

If $b = 0$, Problem 3 corresponds to the filtering problem of time-invariant uncertain systems. In this case, Theorem 4.2 provides sufficient conditions to design filters for uncertain discrete-time systems in polytopic domains. In the case $b = 1$, *i.e.* the parameters may vary arbitrarily inside the unit simplex $\mathscr{U}_N$, the conditions of Theorem 4.2 encompass the ones provided in [11, Theorem 2] leading to less conservative results when contrasted with LPV filters designed through quadratic Lyapunov functions.

4.3.1 Robust filtering

For the robust case, consider $P(\alpha)$ as in (4.18) and the particular structures

$$G(\zeta) = G(\alpha) = \sum_{i=1}^{N} \alpha_i G_i, \quad H(\zeta) = H(\alpha) = \sum_{i=1}^{N} \alpha_i H_i, \ \alpha \in \mathscr{U}_N,$$

lifted to the λ-space, yielding the following result.

Theorem 4.3. *(*$\mathscr{H}_\infty$ *Robust Filtering) Given the system (4.1) and matrix* S *as in* (4.10), *if there exist matrices* $Z, Y, R, Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times q}$, $J \in \mathbb{R}^{p\times n}$, $D_f \in \mathbb{R}^{p\times q}$, G_i, $M_i = M_i' > 0 \in \mathbb{R}^{2n\times 2n}$, $H_i \in \mathbb{R}^{p\times 2n}$, $i = 1,\dots,N$ *and a scalar* $\gamma > 0$ *such that, for matrices* $\bar{Q}$, $\bar{L}$, $\bar{J}$, $\bar{D}_f$, $\bar{G}_i$, $\bar{H}_i$, $\bar{M}_i$, $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_{1i}$, $\bar{C}_{2i}$, $\bar{D}_{1i}$ *and* $\bar{D}_{2i}$ *given as in* (4.14) *and* $\tilde{M}_i$ *as in* (4.15)

$$\Xi_i \triangleq \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} & \hat{F}_{3i} - \hat{F}_1'\bar{H}_i' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & \bar{G}_i\hat{F}_{3i} + \hat{F}_{2i}'\bar{H}_i' & \hat{F}_{4i} \\ (\star) & (\star) & \bar{H}_i\hat{F}_{3i} + \hat{F}_{3i}'\bar{H}_i' - \gamma\mathbf{I} & \mathscr{F}_{34} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0, \quad i = 1, \dots, M, \tag{4.27}$$

$$\mathscr{F}_{11} = \tilde{M}_i - \hat{F}_1 - \hat{F}_1', \ \mathscr{F}_{12} = \hat{F}_{2i} - \hat{F}_1'\bar{G}_i',$$
$$\mathscr{F}_{22} = \bar{G}_i\hat{F}_{2i} + \hat{F}_{2i}'\bar{G}_i' - \bar{M}_i, \ \mathscr{F}_{34} = \bar{D}_{1i} - \bar{D}_f\bar{D}_{2i},$$
$$\hat{F}_1 = \begin{bmatrix} Z & Y' + R' \\ Z & Y' \end{bmatrix}, \ \hat{F}_{2i} = \begin{bmatrix} \bar{A}_i'Z & \bar{A}_i'Y' + \bar{C}_{2i}'\bar{L}' + \bar{Q}' \\ \bar{A}_i'Z & \bar{A}_i'Y' + \bar{C}_{2i}'\bar{L}' \end{bmatrix},$$
$$\hat{F}_{3i} = \begin{bmatrix} \bar{C}_{1i}' - \bar{C}_{2i}'\bar{D}_f' - \bar{J}' \\ \bar{C}_{1i}' - \bar{C}_{2i}'\bar{D}_f' \end{bmatrix}, \ \hat{F}_{4i} = \begin{bmatrix} Z'\bar{B}_i \\ Y\bar{B}_i + \bar{L}\bar{D}_{2i} \end{bmatrix},$$

$$\Xi_{ik} \triangleq \begin{bmatrix} \hat{\mathscr{F}}_{11} & \hat{\mathscr{F}}_{12} & \hat{F}_{3i} + \hat{F}_{3k} - \hat{F}_1'(\bar{H}_i' + \bar{H}_k') & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & \bar{G}_i\hat{F}_{3k} + \bar{G}_k\hat{F}_{3i} + \hat{F}_{2i}'\bar{H}_k' + \hat{F}_{2k}'\bar{H}_i' & \hat{F}_{4i} + \hat{F}_{4k} \\ (\star) & (\star) & \bar{H}_i\hat{F}_{3k} + \bar{H}_k\hat{F}_{3i} + \hat{F}_{3i}'\bar{H}_k' + \hat{F}_{3k}'\bar{H}_i' - 2\gamma\mathbf{I} & \hat{\mathscr{F}}_{34} \\ (\star) & (\star) & (\star) & -2\gamma\mathbf{I} \end{bmatrix} < 0, \tag{4.28}$$

$$i = 1, \dots, M-1, \ k = i+1, \dots, M,$$
$$\hat{\mathscr{F}}_{11} = \tilde{M}_i + \tilde{M}_k - 2\hat{F}_1 - 2\hat{F}_1', \ \hat{\mathscr{F}}_{12} = \hat{F}_{2i} + \hat{F}_{2k} - \hat{F}_1'(\bar{G}_i' + \bar{G}_k'),$$
$$\hat{\mathscr{F}}_{22} = \bar{G}_i\hat{F}_{2k} + \bar{G}_k\hat{F}_{2i} + \hat{F}_{2i}'\bar{G}_k' + \hat{F}_{2k}'\bar{G}_i' - \bar{M}_i - \bar{M}_k, \ \hat{\mathscr{F}}_{34} = \bar{D}_{1i} + \bar{D}_{1k} - \bar{D}_f(\bar{D}_{2i} + \bar{D}_{2k}),$$

then there exists a robust filter in the form of (4.3), ensuring the asymptotic stability of the estimation error dynamics (4.5) and an $\mathscr{H}_\infty$ *guaranteed cost* γ*, for all* $(\alpha, \Delta\alpha) \in \Lambda$ *with vertices given as in* (4.21).

Proof: Similar to the proof of Theorem 4.2. ■

The remarks relevant for Theorem 4.2 also hold for Theorem 4.3. Additionally, for $b = 0$, $G(\alpha) = \mathbf{0}$ and $H(\alpha) = \mathbf{0}$, the conditions of Theorem 4.3 reduce to the $\mathscr{H}_\infty$ extension of the results in [53, Theorem 5.1].

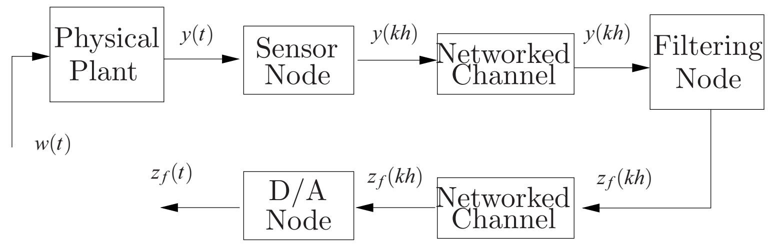

Figure 4.2: Networked Filtering Model.

4.3.2 Practical appeal and possible extensions

The filter design method presented in this section may be applied to all types of dynamical processes that may be written as (4.1). It encompasses the cases of time-invariant ($b = 0$), bounded time-varying ($0 < b < 1$) and arbitrarily time-varying ($b = 1$) systems. Consequently, it can be used in many different practical situations, including systems that exchange information through a communication channel, commonly known as networked control systems (NCSs). The usefulness and importance of NCS architectures is largely due to advances in digital control and computer interfaced structures. Drawbacks associated with NCS are discussed in [68], [107], [115]. In the filtering framework, the problem of estimating a signal of a precisely known continuous-time system, sampled by a zero-order hold with a time-varying sampling period, through an NCS can be studied using the proposed technique. By using the Cayley-Hamilton theorem or Taylor series expansion, the time-varying sampled-data matrices can be rewritten as in (4.2) and Theorem 4.2 may be applied to provide the filter matrices. More specifically, consider a time-invariant continuous-time system sampled by a zero-order hold with a period h. The structure of the filtering model is illustrated in Figure 4.2. Assuming that h may change its value at run-time due to various reasons, such as bandwidth allocation and scheduling decisions, let the actual value of h at each instant k (i.e., h_k) lie inside a finite discrete set as specified below

$$h_k \in \{h_{min}, \ldots, h_{max}\},\ h_k = \kappa \cdot g,\ \kappa \in \mathbb{N}. \tag{4.29}$$

The parameter g is known as the processor/network clock granularity, [104]. The clock granularity is related with the processor frequency and $k \in \mathbb{N}$ is a function of time that specifies how many times g the sampling period h will be at instant k.

To represent the set of all possible sampled-data system matrices due to uncertain sampling rates, a polytopic model is considered. In this case, the system matrices, for any time $k \geq 0$, are described as a convex combination of well-defined vertices, which are given by the arrangements of the extreme values of (4.29) with the help of the Cayley-Hamilton theorem or Taylor series expansion [1]. The sampled system is then rewritten as (4.1) and Theorem 4.2 (or Theorem 4.3) may be used to provide a networked filter such that the estimation error is asymptotically stable under time-varying sampling rates. This problem is of great interest specially when dealing with scheduling or dynamic bandwidth allocation for bandwidth reduction [79].

Other improvements of theorems 4.2 and 4.3 may be obtained by exploring the structure of the Lyapunov matrix $\mathscr{P}(\alpha)$ and the extra variables F, $G(\zeta)$ and $H(\zeta)$ of Theorem 4.1. As can be seen in (4.18), the Lyapunov matrix used in Theorem 4.2 is affine in α. More sophisticated structures may lead to better results, for example, the polynomially parameter dependent Lyapunov (PPDL) functions used in [85] can be explored for $b < 1$. The case $b = 1$ (arbitrarily parameter variation) seems to be more involved. Considering stability analysis and Lyapunov matrices that depend only on the parameter at the actual instant of time (*i.e.* $\alpha(k)$) it turns out that the positiveness of the affine Lyapunov matrix (4.18) is a necessary condition for the positiveness of PPDL functions with degree greater than one. Consequently, whether or not PPDL functions with higher degree will help to improve the performance when compared to affine functions for synthesis purpose with $b = 1$ is still an open question. Nevertheless, parameter dependent Lyapunov matrices that depend on more than one instant of time, as the path dependent Lyapunov function proposed in [63], [64], can provide better results for $b=1$ when contrasted to the affine Lyapunov matrix.

Changes in the structure of matrices F, $G(\cdot)$ and $H(\cdot)$ have appeared in (4.16) may also lead to better results, as given in [39], [47]. A result for arbitrarily time-varying systems, obtained with the path dependent Lyapunov matrix[4]

$$\mathscr{P}(\alpha) = P(\alpha, \alpha_+) = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_{+_j} P_{ij}, \ \alpha, \ \alpha_+ \in \mathscr{U}_N, \tag{4.30}$$

and the particular choices

$$G(\zeta) = G(\rho) = \sum_{i=1}^{N} \rho_i G_i, \ \ H(\zeta) = H(\rho) = \sum_{i=1}^{N} \rho_i H_i, \ \rho \in \mathscr{U}_N,$$

with $\rho \in \mathscr{U}_N$, is presented in the next theorem. Note that, since $b=1$ (arbitrarily fast rates of variation), there is no need to lift the matrices to the λ-space.

Theorem 4.4. *(*Path dependent approach*) Given the system (4.1), if there exist matrices Z, Y, R, $Q_i \in \mathbb{R}^{n\times n}$, $L_i \in \mathbb{R}^{n\times q}$, $J_i \in \mathbb{R}^{p\times n}$, $D_{fi} \in \mathbb{R}^{p\times q}$, G_i, $M_{ij} = M_{ij}' > 0 \in \mathbb{R}^{2n\times 2n}$, $H_i \in \mathbb{R}^{p\times 2n}$ $i,j = 1,\ldots,N$ and a scalar $\gamma > 0$ such that*

$$\Xi_{ij\ell} \triangleq \begin{bmatrix} M_{j\ell} - \hat{F}_1 - \hat{F}_1' & \hat{F}_{2i} - \hat{F}_1' G_j' & \hat{F}_{3i} - \hat{F}_1' H_j' & \mathbf{0} \\ (\star) & G_j \hat{F}_{2i} + \hat{F}_{2i}' G_j' - M_{ij} & G_j \hat{F}_{3i} + \hat{F}_{2i}' H_j' & \hat{F}_{4i} \\ (\star) & (\star) & H_j \hat{F}_{3i} + \hat{F}_{3i}' H_j' - \gamma \mathbf{I} & D_{1i} - D_{fi} D_{2i} \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0, \tag{4.31}$$

$$i = 1,\ldots,N, \ j = 1,\ldots,N, \ \ell = 1,\ldots,N,$$

$$\Xi_{ikj\ell} \triangleq \begin{bmatrix} 2M_{j\ell} - 2\hat{F}_1 - 2\hat{F}_1' & \hat{F}_{2ik} - 2\hat{F}_1' G_j' & \hat{F}_{3ik} - 2\hat{F}_1' H_j' & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & G_j \hat{F}_{3ik} + \hat{F}_{2ik}' H_j' & \hat{F}_{4ik} \\ (\star) & (\star) & H_j \hat{F}_{3ik} + \hat{F}_{3ik}' H_j' - 2\gamma \mathbf{I} & \hat{\mathscr{F}}_{34} \\ (\star) & (\star) & (\star) & -2\gamma \mathbf{I} \end{bmatrix} < 0, \tag{4.32}$$

[4] The Lyapunov matrix can also be generalized for any number of instants ahead following the lines given in [63], at the price of a quick increase in the computational effort.

$$i = 1, \ldots, N-1,\ k = i+1, \ldots, N\ j = 1, \ldots, N,\ \ell = 1, \ldots, N,$$
$$\hat{\mathscr{F}}_{22} = G_j \hat{F}_{2ik} + \hat{F}'_{2ik} G'_j - M_{ij} - M_{kj},\ \hat{\mathscr{F}}_{34} = D_{1i} + D_{1k} - D_{fi} D_{2k} - D_{fk} D_{2i},$$

where $\hat{F}_1$*,* $\hat{F}_{2i}$*,* $\hat{F}_{3i}$*,* $\hat{F}_{4i}$*,* $\hat{F}_{2ik}$*,* $\hat{F}_{3ik}$ *and* $\hat{F}_{4ik}$ *have the same structure of the ones from Theorem 4.2 but in the* α *domain, then there exists an LPV filter in the form of (4.3), ensuring the asymptotic stability of the estimation error dynamic (4.5) and an* $\mathscr{H}_\infty$ *guaranteed cost* γ*, for all* $\alpha \in \mathscr{U}_N$ *with arbitrary rates of variation and vertices given as in* (4.21)*.*

Proof: Similar to the proof of Theorem 4.2 except that now there is no lift to the λ-space and the operation (4.22) becomes

$$\Xi(\alpha,\rho,\eta) = \sum_{\ell=1}^{N} \eta_\ell \left\{ \sum_{j=1}^{N} \rho_j \left\{ \sum_{i=1}^{N} \alpha_i^2 \Xi_{ij\ell} + \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \alpha_i \alpha_k \Xi_{ikj\ell} \right\} \right\}.$$

■

Note that the Lyapunov matrix (4.30) would imply on three instants of time $\alpha(k)$, $\alpha(k+1)$ and $\alpha(k+2)$ in Theorem 4.1. Since these values are completely independent when $b = 1$, they are represented respectively by α, ρ and η (all of them belonging to unit simplexes, for all $k \geq 0$), yielding matrix $\Xi(\alpha,\rho,\eta)$ in Theorem 4.4. The robust version of Theorem 4.4 can be obtained in a similar way of Theorem 4.3.

4.4 Numerical Experiments

All the experiments have been performed in a PC equipped with: Athlon 64 X2 6000+ (3.0 GHz), 2GB RAM (800 MHz), using the SDP solver SeDuMi [101] interfaced by the parser YALMIP [70]. The numerical complexity is estimated in terms of the computational times given in seconds. Particularly to the iterative procedure given in Algorithm 2, the time of the i-th iteration is the total time accumulated up to this iteration.

Example I: Consider the following time-varying discrete-time system borrowed from [37]

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\theta(k) \end{bmatrix} x(k) + \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix} w(k), \\ z(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k), \\ y(k) &= \begin{bmatrix} -100 & 10 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(k), \end{aligned} \tag{4.33}$$

where $\underline{\theta} \leq \theta(k) \leq \overline{\theta}$ and $|\Delta\theta(k)| \leq \delta$. The equivalent polytopic representation of system (4.33) is obtained by the change of variables $\theta(k) = \alpha_1(k)\underline{\theta} + \alpha_2(k)\overline{\theta}$ and $|\Delta\alpha_1(k)| = |\Delta\alpha_2(k)| \leq \delta/|\overline{\theta} - \underline{\theta}| = b$. With respect to the ranges of the time-varying parameters, the case to be investigated is $\overline{\theta} = -\underline{\theta} = 0.4$ and $0 \leq \delta \leq 0.8$ (corresponding to $0 \leq b \leq 1$).

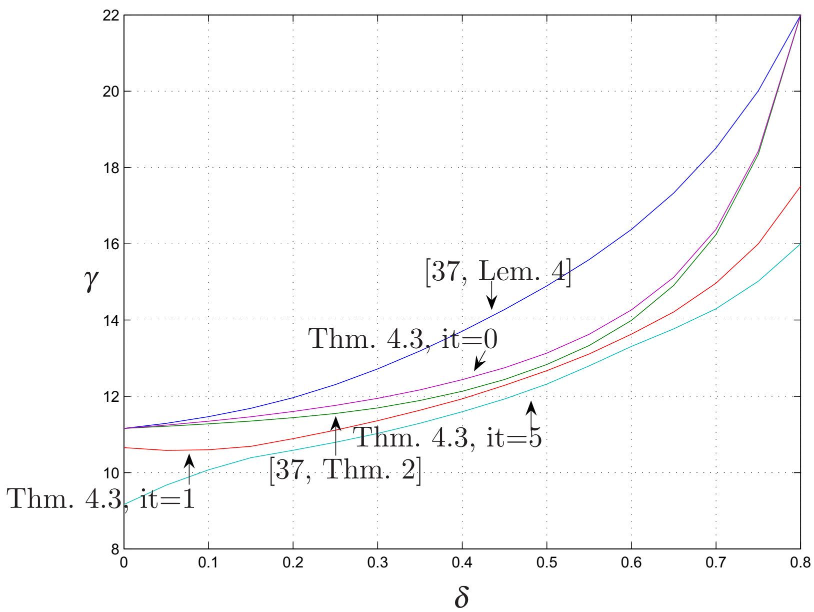

Figure 4.3: $\mathscr{H}_\infty$ upper bound attained by using strictly proper robust filters in the design problem of Example I.

The first task is to synthesize robust filters using Algorithm 2 with Theorem 4.3 and the approaches from [37, Lemma 4] (Lyapunov matrix affine in $\theta(k)$) and [37,

Table 4.1: $\mathscr{H}_\infty$ guaranteed costs and computational times obtained in the design problem of Example I for time-invariant ($\delta = 0$) and arbitrarily fast ($\delta = 0.8$) parameters.

Method	Filter	δ	γ	Time
[37, Lem. 4]	robust	0	11.16	0.44
[37, Thm. 2]		0	11.16	0.44
[39, Thm. 2]$_{it=5}$		0	9.30	1.45
Theorem 4.3$_{it=0}$		0	11.16	0.38
Theorem 4.3$_{it=1}$		0	10.65	0.78
Theorem 4.3$_{it=5}$		0	9.16	3.55
[37, Lem. 4]	robust	0.8	21.99	0.57
[37, Thm. 2]		0.8	21.99	0.58
Theorem 4.3$_{it=0}$		0.8	21.99	0.46
Theorem 4.3$_{it=1}$		0.8	17.72	0.95
Theorem 4.3$_{it=5}$		0.8	16.04	5.15
Theorem 4.4$_{it=0}$		0.8	17.59	0.22
Theorem 4.4$_{it=1}$		0.8	15.68	0.44
Theorem 4.4$_{it=5}$		0.8	14.52	2.18
[118, Thm. 3]	LPV	0.8	8.49	0.42
Theorem 4.2$_{it=0}$		0.8	8.49	0.33
Theorem 4.2$_{it=1}$		0.8	8.49	0.48
Theorem 4.4$_{it=0}$		0.8	8.49	0.24
Theorem 4.4$_{it=1}$		0.8	8.49	0.33

Theorem 2] (Lyapunov matrix quadratic in $\theta(k)$). Algorithm 2 is performed twice, considering the maximum number of iterations as $k_{max} = 1$ and $k_{max} = 5$. Figure 4.3 shows the minimum γ achieved with strictly proper filters (D_f=$\mathbf{0}$). Note that with only one iteration, where in fact the conditions of Theorem 4.3 reduce to LMIs, the proposed approach based on affine parameter-dependent Lyapunov matrix outperforms the best method of [37] that is based on a Lyapunov matrix quadratic in $\theta(k)$. The zero iteration case (it = 0) shown in the figure was obtained without introducing the extra variables $G(\cdot)$ and $H(\cdot)$. Smaller guaranteed costs can be obtained through the iterative procedure given in Algorithm 2 at the price of a higher computational effort.

The second part of the experiment concerns a more detailed comparison between the proposed design conditions and the methods from the literature for the specific cases $\delta = 0$ (time-invariant parameter) and $\delta = 0.8$ (arbitrarily fast). In the case $\delta = 0$ the nonconvex procedure from [39, Theorem 2] is also included in the comparisons. For $\delta = 0.8$, the LPV filter design conditions proposed in the chapter are compared to [118, Theorem 3]. The results are shown in Table 4.1, where $it = 0$ means without the extra variables $G(\cdot)$ and $H(\cdot)$. In the robust filtering case, the proposed conditions provide the best $\mathscr{H}_\infty$ guaranteed costs with five iterations at the price of slightly higher computational efforts. In the LPV filtering case the proposed conditions presented the same $\mathscr{H}_\infty$ guaranteed costs than [118, Theorem 3] for the case $\delta = 0.8$. Note that, differently from [118, Theorem 3], the proposed conditions could still synthesize LPV filters for the range $0 < \delta < 0.8$.

Example II: Consider a time-varying system with state-space matrices given by

$$A = \begin{bmatrix} 0.265 - 0.1650\theta(k) & 0.45(1+\theta(k)) \\ 0.5(1-\theta(k)) & 0.265 - 0.215\theta(k) \end{bmatrix}, \quad B = \begin{bmatrix} 1.5 - 0.5\theta(k) \\ 0.1 \end{bmatrix}, \quad C_2' = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $D_2 = 1$, $C_1 = \mathbf{I}_2$, $D_1 = \mathbf{0}_2$ and $-1 \le \theta(k) \le 1$ is an arbitrarily fast time-varying parameter ($\Delta\theta(k) = 2$). The polytopic representation of the system is obtained as in Example I. The aim is to synthesize robust and LPV $\mathscr{H}_\infty$ filters using the conditions proposed in the chapter and the ones from [37] and [118]. For the LPV case, only Theorem 4.2 and Theorem 4.4 were able to provide a feasible solution. In the robust case, all methods failed except the robust version of Theorem 4.4. The results can be seen in Table 4.2. The robust filter matrices after one iteration are given by

$$A_f = \begin{bmatrix} 0.5238 & 1.5844 \\ -0.0406 & -0.0927 \end{bmatrix}, \ B_f = \begin{bmatrix} -0.7852 \\ 0.8116 \end{bmatrix}, \ C_f = \begin{bmatrix} 0.0070 & -0.0002 \\ 0.5238 & 1.7974 \end{bmatrix},$$
$$D_f' = \begin{bmatrix} 0.9931 & -0.7029 \end{bmatrix},$$

and after six iterations, with an improvement of approximately 67%, by

$$A_f = \begin{bmatrix} 0.2508 & 0.3667 \\ -0.0201 & -0.0447 \end{bmatrix}, \; B_f = \begin{bmatrix} -6.3140 \\ 4.1041 \end{bmatrix}, \; C_f = \begin{bmatrix} -0.0045 & -0.0135 \\ 0.1703 & 0.5597 \end{bmatrix},$$

$$D_f' = \begin{bmatrix} 1.0012 & -0.3848 \end{bmatrix}.$$

As expected, the $\mathscr{H}_\infty$ guaranteed cost associated with the LPV filter was better but no improvement was obtained with the BMI iterations. This example illustrates the fact that there may exist systems where robust filters can only be designed by using path-dependent Lyapunov matrices, that encompass the methods based on Lyapunov matrices depending (affinely, quadratically or polynomially) on parameters only at the current instant of time k.

Table 4.2: $\mathscr{H}_\infty$ guaranteed costs and computational times obtained in the design problem of Example II. The computational time (in seconds) is the cumulated time as the number of BMI iterations evolves.

Method	Filter	γ	Improvement	Time
$T4.4_{it=1}$	robust	19.41	–	0.89
$T4.4_{it=2}$		9.10	53.10 %	2.03
$T4.4_{it=3}$		7.55	61.07 %	3.17
$T4.4_{it=4}$		6.82	64.85 %	4.21
$T4.4_{it=5}$		6.56	66.18 %	5.30
$T4.4_{it=6}$		6.26	67.75 %	6.41
$T4.4_{it=1}$	LPV	1.22	–	0.85
$T4.4_{it=2}$		1.22	0.00 %	1.96

Figure 4.4 shows the results for the noise input generated by the Matlab command $w(k) = 0.3{*}\texttt{randn}$, for $0 \leq k \leq 50$, and zero initial condition. After six iterations, the first state of the error vector had an improvement of 2.29% and the second state of 40.05%.

Example III: This example, borrowed from [1], consists of a simplified model of an armature voltage-controlled DC servo motor, consisting of a stationary field and

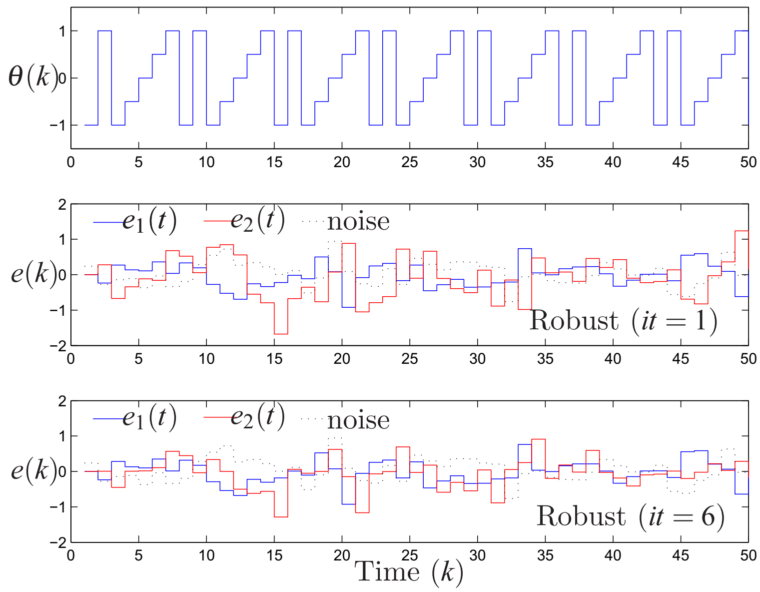

Figure 4.4: Time-domain analysis. The first graph illustrates the parameter variation in time while the others show the estimation errors for two robust filters designed in Example II.

a rotating armature and load. All effects of the field are neglected. The aim is to design an $\mathscr{H}_\infty$ robust filter to estimate the armature current given the speed of the shaft. All information is sent through a communication network. The behavior of the DC servo motor shown in Figure 4.5 can be described by

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\rho}_a \end{bmatrix} = \begin{bmatrix} -\dfrac{b_v}{J} & \dfrac{K_T}{J} \\ -\dfrac{K_\theta}{L_a} & -\dfrac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\rho}_a \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \omega, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\rho}_a \end{bmatrix}, \tag{4.34}$$

where e_a is the externally applied armature voltage, ρ_a is the armature current, R_a the resistance of the armature winding, L_a the armature winding inductance, e_m the back-emf voltage induced by the rotating armature winding ($e_m = K_\theta \dot{\theta}, K_\theta > 0$), b_v the viscous damping due to bearing friction, J the moment of inertia of the armature and load and θ the shaft position. Further, the torque generated by the motor is given by $T = K_T i_a$. For $J = 0.01 kgm^2/s^2$, $b_v = 0.1 Nms$, $K_T = K_\theta = 0.01 Nm/Amp$,

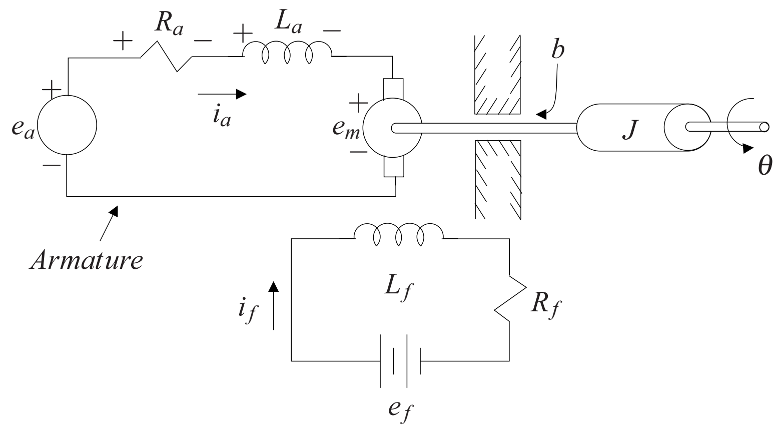

Figure 4.5: DC Servo motor as presented in [1].

$R_a = 1\Omega$ and $L_a = 0.5H$, system (4.34) can be rewritten in the form (4.1) with the following sampled-data matrices, presented as a function of h_k,

$$\begin{aligned} A_s &= \begin{bmatrix} \exp(-10h_k) - 0.0003\exp(-2h_k) & 0.125(\exp(-2h_k) - \exp(-10h_k)) \\ 0.002(\exp(-10h_k) - \exp(-2h_k)) & -0.0003\exp(-10h_k) + \exp(-2h_k) \end{bmatrix}, \\ B_s &= \begin{bmatrix} 0.025\exp(-10h_k) - 0.125\exp(-2h_k) + 0.099 \\ 0.0000626\exp(-10h_k) - 0.99\exp(-2h_k) + 0.99 \end{bmatrix}, \\ C_{1s} &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \; C_{2s} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_{1s} = 0, \; D_{2s} = 0. \end{aligned} \tag{4.35}$$

The sampling rate is allowed to vary within the interval $h_k \in [0.001\ 0.099]$. The system is then expressed by polytope (4.2) with four vertices ($N = 4$), obtained by evaluating $\exp(-10h_k)$ and $\exp(-2h_k)$ at the extreme values of h_k, where the parameters α_i are related to h_k and $b = 1$. Theorem 4.3 provided a robust filter after one iteration with $\mathscr{H}_\infty$ upper bound $\gamma = 1.1519$ and matrices

$$\begin{aligned} A_f &= \begin{bmatrix} 9.4526 & 76.4453 \\ -1.1618 & -9.3958 \end{bmatrix}, \; B_f = \begin{bmatrix} -1917.2236 \\ 253.1882 \end{bmatrix}, \\ C_f &= \begin{bmatrix} 0.0214 & 0.1722 \end{bmatrix}, \; D_f = \begin{bmatrix} 5.1008 \end{bmatrix}. \end{aligned}$$

4.5 Conclusion

The $\mathscr{H}_\infty$ LPV filtering for uncertain discrete-time systems with bounded time-varying parameters has been addressed in this chapter, where all system matrices are affected by time-varying parameters. With a convex description of the parameter time variation, a less conservative design condition was obtained. Extra variables were used to derive BMI conditions that may be explored in the search for a better $\mathscr{H}_\infty$ performance. The filter design is accomplished by means of an optimization problem, formulated only in terms of the vertices of the polytope. The proposed approach provides improvements and advantages when compared to other methods from the literature, as illustrated by examples.

Chapter 5

Conclusion and Perspectives

This dissertation presented contributions to the control and filtering of discrete time-varying systems in terms of parameter-dependent Lyapunov functions. The main contributions are as follows.

Concerning the LPV control problems, the novelties rely mainly on the use of a more accurate model for the parametric variations of the system, and in the use of BMIs for the improvement of the $\mathscr{H}_\infty$ performance. No prior results used Lemma 1.7 with the structure (1.28) for the synthesis of LPV controllers. Consequently, the theorems obtained in Chapter 2, represent a new strategy for the synthesis of feedback controllers for discrete time-varying systems. The conditions presented provide better performance when compared with recent techniques published in literature, as verified through numerical simulations. The proposed method represents a flexible option since it can be applied in four different contexts, namely, LPV or robust control of time-varying systems with bounded or unbounded rates of variation.

With respect to the networked control problem, a solution was proposed to stabilize systems with time-varying, possibly uncertain, sampling rates, a relevant issue in the context of dynamic bandwidth allocation. By considering time-varying sampling rates, it is possible to control and to reduce the flow of information between sensors and actuators. This is an important issue when dealing with NCS design, which in general is based on the cost-benefit ratio between stability/performance and the usage of bandwidth. Although a simplified NCS structure was considered, as modeled by a discrete time-varying uncertain system, a solution based on parameter-dependent Lyapunov functions was presented. Furthermore, a precise characterization of the sampling rate in terms of a polytope, with a systematic procedure for obtaining the vertices using Cayley-Hamilton theorem was presented. In addition to providing a complete theoretical characterization for robust control systems with uncertain sampling rates, the proposed conditions can effectively improve the performance of the closed-loop system. Therefore, by minimizing an upper bound to the $\mathscr{H}_\infty$ performance, controllers designed with the aid of the theorems proposed in Chapter 3 present a certain degree of robustness to unstructured uncertainties. The results proposed can also be used when no communication networks are considered.

Similarly to the case of LPV control, the main contributions in the context of filtering problems rely on the use of Lemma 1.7, using the structure (1.30), and for the model used for parametric variation. The performance achieved with filters designed using the theorems presented in Chapter 4 was superior when compared with the main techniques published recently in the literature for filtering of discrete systems with bounded rates of variation, particularly the ones presented in [37]. Moreover, in the case of arbitrarily fast rates of variation, the use of path-dependent Lyapunov functions was very efficient, providing conditions for synthesis of LPV filters never seen in this context before. Applications to filtering through communication networks were also presented.

Perspectives

- A natural step, and a promising one from the practical point of view, is the extension of the results to cope with the synthesis of dynamic output feedback controllers. It is known that most real physical systems have a larger number of states than the number of outputs. In this sense, the use of output feedback in the context of networked control systems is attractive since it provides a way to reduce the volume of information to be sent through the network and therefore the usage of network bandwidth;

- Another perspective is to consider, in the context of LPV systems, the availability of the parameter $\alpha(k)$ only at the instant $k+1$. In this case, the LPV controller, or filter, is implemented using only the information given by $\alpha(k-1)$. It is expected an increase in the number of BMIs/LMIs, and that the arbitrarily time-varying case is easier to be faced than the case with bounded rates of variation. Moreover, concerning the uncertainty modeling, another promising approach is the use of multi-simplex [86] for the cases with more than one uncertain parameter. This is an interesting strategy, especially for discrete-time systems with bounded rates of variation, because it provides a direct interpretation of the parameters in the polytopic model with respect to the parameters of the real plant;

- Other possibilities for future work include: the extension of the results of Chapter 2 to cope with networked control systems with different delays in the sensor/controller and controller/actuator paths, what may be done using the Lyapunov-Krasovskii functional and the techniques proposed in [54] and [100]; the minimization of the upper bound γ to the $\mathscr{H}_\infty$ performance within specific frequency ranges using weighting functions established in accordance with the characteristics of the real system; and lastly explore other structures for the Lyapunov function, such as polynomial functions, [84], [85] or dependent on more instants of time [64].

Publications

Directly related to the dissertation: [12], [15], [16], [17] and [21].

Indirectly related to the dissertation: [10], [11], [13], [18], [74], [75], [76], [77] and [78]. The abstracts of these papers are presented below.

- **BP06a**: "Filtragem LPV com desempenho $\mathscr{H}_2$ para sistemas lineares politópicos variantes no tempo" — In this paper, the continuous and discrete-time filtering problems for linear time-varying systems are investigated. Convex conditions to design linear parameter varying filters which minimize an upper bound to the $\mathscr{H}_2$ estimation error performance are provided. Both system and filter matrices are considered to be affected by arbitrarily time-varying parameters belonging to a polytope. Different from other strategies in the literature, the filter design is accomplished by means of a convex optimization procedure, formulated only in terms of the vertices of the polytope, avoiding the use of exhaustive gridding in the parameter space. Numerical examples illustrate the efficiency of the proposed approach.

- **BP06b**: "$\mathscr{H}_\infty$ LPV filtering for linear systems with arbitrarily time-varying parameters in polytopic domains" — In this paper, the problem of $\mathscr{H}_\infty$ filtering for linear systems affected by arbitrarily time-varying parameters in polytopic domains is investigated. A linear parameter-varying filter which minimizes an upper bound to the $\mathscr{H}_\infty$ estimation error performance is determined for both continuous and discrete-time cases. Different from other strategies in the literature, the filter design is accomplished by means of a convex optimization procedure and the time-varying parameters are supposed to affect all systems matrices. The LPV filter is obtained from the optimal solution of a convex linear matrix inequality problem formulated only in terms of the vertices of the polytope. There is no use of exhaustive gridding in the parameter space. Numerical examples illustrate the efficiency of the proposed approach.

- **BMV$^+$08**: "Filtragem LPV $\mathscr{H}_\infty$ de sistemas contínuos variantes no tempo com atraso no estado: uma abordagem por relaxações LMIs" — In this paper, the problem of $\mathscr{H}_\infty$ filtering for continuous time systems with state delays and time-varying parameters in polytopic domains is investigated. The time-varying parameters can affect all system matrices and are supposed to be available (measured) online. A linear parameter varying filter, which minimizes an upper bound to the $\mathscr{H}_\infty$ estimation error performance, is determined considering that both filter and system state space variables are affected by constant time delay. The filter design is accomplished by means of a convex optimization procedure formulated using LMIs and parameter-dependent Lyapunov functionals, without using exhaustive gridding in the parameter space. Both delay-dependent and independent conditions are presented. LMI relaxations for the filtering problem are also considered. Numerical examples illustrate the efficiency of the proposed approach.

- **BMO$^+$08**: "Parameter-dependent $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for linear systems with arbitrarily time-varying parameters in polytopic domains" — In this paper, the problem of filter design for linear continuous-time systems with arbitrarily fast time-varying parameters is investigated. The time-varying parameters belong to a polytope with known vertices, affect all the system matrices and are assumed to be available online for implementation of the filters. Necessary and sufficient parameter-dependent LMI conditions for the existence of a parameter-dependent filter assuring that the estimation error dynamics is quadratically stable and satisfies bounds to the $\mathscr{H}_2$ or to the $\mathscr{H}_\infty$ norms are given. A sequence of standard LMI conditions assuring the existence of homogeneous polynomially parameter-dependent (HPPD) solutions to the parameter-dependent LMIs for filter design is provided in terms of the vertices of the polytope (no gridding is required), yielding parameter-dependent filters of arbitrary degree assuring quadratic stability of the error dynamics for the $\mathscr{H}_2$ or the $\mathscr{H}_\infty$ cases. As the degree of the HPPD solutions increases, less and less conservative LMI conditions are obtained, tending to the necessary conditions that assure optimal values for the $\mathscr{H}_2$ or the $\mathscr{H}_\infty$ performance of the estimation error dynamics under quadratic stability. Numerical examples illustrate the results, showing that parameter-dependent filters can provide better attenuation levels than the ones obtained with robust filters, at the price of a more complex filtering strategy.

- **MBP06**: "Projeto de controladores escalonados $\mathscr{H}_2$ por realimentação de estados para sistemas politópicos variantes no tempo" — This paper focuses on the control of linear time-varying systems affected by parameters which can vary arbitrarily inside a polytope. The proposed conditions are sufficient to ensure quadratic stability, exploiting the duality in systems and being described in terms of convex optimization problems subject to linear matrix inequalities of finite dimension whose solutions provide parameter-dependent state feedback gains (gain-scheduled controllers) which ensure the stability of the system with a prescribed $\mathscr{H}_2$ guaranteed cost. Differently from other approaches in the literature, the conditions proposed do not use exhaustive discretization in the space of the parameters to compute a family of controllers, do not assume that some of the system matrices are fixed (time-invariant) neither assume special structures for the time-varying parameters. Numerical examples including the design of controllers subject to failures of actuators for an aircraft model are given, illustrating the efficiency of the proposed conditions.

- **MBOP06**: "Síntese convexa de controladores para sistemas arbitrariamente variantes no tempo: estabilidade com dissipatividade garantida" — This paper provides a convex condition to compute linearly parameter-dependent state feedback gains (gain-scheduling) which stabilize the closed-loop system with guaranteed dissipativity when the system is affected by arbitrarily time-varying parameters belonging to a polytope. The proposed conditions are written as a convex optimization problem with linear matrix inequality constraints. A sequence of relaxations, based on Pólya's Theorem, provides conditions that are progressively more accurate, ensuring a dissipation that tends to the maximum value of dissipation which one can obtain using quadratic stability and linearly parameter-dependent state feedback gains, as illustrated by numerical examples.

- **MBP07**: "$\mathscr{H}_2$ Dynamic output feedback scheduled controllers for linear time-varying polytopic systems: a convex LMI approach" — This paper provides a convex condition to design dynamic output feedback scheduled controllers which ensure the closed-loop stability and minimize an upper bound to the $\mathscr{H}_2$ norm for linear systems whose matrices are affected by arbitrarily time-varying parameters belonging to a polytope. Differently from the conditions for the design of robust $\mathscr{H}_2$ dynamic controllers, which are non-convex, the proposed design is entirely based on a convex linear matrix inequality optimization procedure. Moreover, in this paper, all the system matrices are affected by the vector of time-varying parameters which can vary arbitrarily fast inside the polytope. By means of variable elimination and also by exploiting the positivity of the parameters, it is shown that the design problem can be expressed as a convex optimization problem subject to a finite number of linear matrix inequality constraints formulated only in terms of the vertices of the polytope, avoiding the use of exhaustive gridding in the parameter space to compute a family of controllers. Numerical examples, including an application to the control of a model of a helicopter subject to abrupt failures of actuator, illustrate the efficiency of the proposed approach.

- **MOC$^+$07**: "Robust absolute stability and stabilization based on homogeneous polynomially parameter-dependent Lur'e functions" — This paper provides finite dimensional convex conditions to construct homogeneous polynomially parameter-dependent Lur'e functions which ensure the stability of nonlinear systems with state-dependent nonlinearities lying in general sectors and which are affected by uncertain parameters belonging to the unit simplex. The proposed conditions are written as linear matrix inequalities parameterized in terms of the degree g of the parameter-dependent solution and in terms of the relaxation level d of the inequality constraints, based on an extension of Pólya's Theorem. As g and d increase, progressive less conservative solutions are obtained. The results in the paper include as special cases existing conditions for robust stability analysis and for absolute stability. A convex solution for control design is also provided. Numerical examples illustrate the efficiency of the proposed conditions

- **MOC$^+$09**: "Robust absolute stability and nonlinear state feedback stabilization based on polynomial Lur'e functions" — This is a more detailed version of the paper MOC$^+$07, presenting a discussion of systems with sector bounded nonlinearities and parameter dependent Lyapunov functions, published in the international journal *Nonlinear Analysis. Theory, Methods and Applications.*

Accepted for publication

Directly related to the dissertation: [19], [20] and [22].

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