7-12-2014

Star Operations and Numerical Semigroup Rings

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Star Operations and Numerical Semigroup Rings

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DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
Mathematics

The University of New Mexico

Albuquerque, New Mexico

May, 2014

Dedication

I dedicate this dissertation to my mother and father who have supported me throughout my mathematical career.

Acknowledgments

The original ideas in this dissertation were developed in part through conversations with my advisor Janet Vassilev. Many thanks! I would also like to thank Janet Vassilev, Alex Buium, Micheal Nakamaye, and Bruce Olberding for their participation on my dissertation committee and Janet Vassilev, Alex Buium, Cristina Pereyra, and Jurg Bolli for writing letters of recommendation on my behalf.

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Abstract

We aim to classify the star and semistar operations on conductive numerical semigroup rings which are of the form $k + x^n k[[x]]$. By classifying the star and semistar operations on conductive numerical semigroup rings we obtain a better understanding of the set of star and semistar operations on general numerical semigroup rings. Here we classify all star and semistar operations on the ring $k + x^4 k[[x]]$ as well as all semistar operations on $k + x^5 k[[x]]$ that are not star. We investigate star operations on $k + x^5 k[[x]]$ with Macaulay 2 and also present several results about general conductive numerical semigroup rings that bring us closer to our goal.

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Glossary

$f(S)$	The Frobenius number of a numerical semigroup S.
$c(S)$	The conductor of a numerical semigroup S.
$e(S)$	The multiplicity of a numerical semigroup S.
$PF(S)$	The set of pseudo-Frobenius numbers on a numerical semigroup S.
$Ap(S; n)$	The Apéry set of a numerical semigroup S with respect to n.
$A(s)$	$\{\alpha \in \{1, 2, \ldots, a_1\} \mid s - \alpha \in S\}$.
$R(S)$	The associated ring of the numerical semigroup S.
$\mathfrak{m}_s(S)$	A semiconductor ideal of $R(S)$.

Chapter 1

Introduction

1.1 Overview

We begin our discussion by defining closure operations on rings in general. A closure operation on a commutative ring with unity R whose set of ideals is denoted $\mathfrak{I}$ is a function $cl : \mathfrak{I} \to \mathfrak{I}$ satisfying the following for all $I \in \mathfrak{I}$,denoting $cl(I) = I^{cl}$:

- $I \subseteq I^{cl}$
- If $I \subseteq J$, then $I^{cl} \subseteq J^{cl}$
- $(I^{cl})^{cl} = I^{cl}$

One of the first papers (or perhaps the first paper) to discuss general closure operations of ideals over a commutative ring with unity was Kirby's 1969 paper [Ki]. A closure operation $\star$ on a domain R is said to be star if for all $x \in R$, $(xI)^\star = xI^\star$. This definition can be extended in an obvious way to define star operations on the set of fractional ideals of the given ring R. In fact, the defining properties of star operations on the set of fractional ideals of a domain were introduced by Krull in his 1935 book, Idealtheorie [Kr], in which he used notation that inspired later mathematicians to call them prime operations. Epstein distinguishes between prime and star operations in his 2011 paper [Ep] and shows that the two definitions coincide in the context of domains.

In their 2011 paper [HMP1], Houston, Mimouni and Park discuss domains that admit at most two star operations. This inquiry was inspired in part by the following result by Bass [Ba, Theorem 6.3] and Matlis [M, Theorem 3.8] done independently. A local Noetherian domain $(R, \mathfrak{m})$ is divisorial (i.e. admits exactly one star operation) if and only if R has dimension one and $\mathfrak{m}^{-1}$ is a two-generated R-module. Bass and Matlis do not talk about star operations explicitly as the result is presented in the context of homological algebra, but the result pertains to star operations nonetheless.

The line of inquiry explored in this paper was inspired primarily by another paper by Houston, Mimouni and Park, namely [HMP2], and a paper by Lance Bryant, namely [Br]. Noetherian domains which have only finitely many star operations are discussed in [HMP2] as the title suggests while numerical semigroups and their associated rings are discussed in [Br]. In [HMP2], two constructions of star operations are used extensively, namely the constructions in [A, Theorem 2] and [HHP, Proposition 3.2]. The first makes use of overrings to construct star operations and the second utilizes fractional ideals I such that $(I : I) = R$.

A partial ordering on the set of star operations on a ring is also introduced in [HMP2], that is if $\star_1$ and $\star_2$ are two star operations on a ring R, then we say that $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for all ideals I of R. The set of star operations on a ring is a lattice under this partial ordering with the infimum being the identity operation, i.e. the operation sending each ideal to itself, and the supremum being the v-operation defined by sending an ideal I to $(I^{-1})^{-1}$. The fact that the identity operation is the infimum is obvious. The fact that the v-operation is a star operation, let alone the supremum of star operations, is not obvious and is proven by Gilmer in [G] and by

Epstein in [Ep]. This fact makes the implication of the theorem by Bass and Matlis to star operations clear.

Here we discuss star operations on numerical semigroup rings which are Noetherian domains (in fact, 1-dimensional domains) that may admit infinitely many star operations. In fact, any numerical semigroup ring with infinite base field k and $\dim_k \mathfrak{m}^{-1}/\mathfrak{m} \geq 4$ (where $\mathfrak{m}$ is the maximal ideal) admits infinitely many star operations as shown in [HMP2, Corollary 2.8]. This includes $k + x^n k[[x]]$ for $n \geq 4$. However, every numerical semigroup ring with finite base field k admits only finitely many star operations as will be discussed in Proposition 2.2.8. In either case, we have classified all of the star operations on the particular ring $k + x^4 k[[x]]$ and have counted them in terms of $|k|$ for finite k.

As previously indicated, we will focus specifically on numerical semigroup rings. A numerical semigroup is a submonoid of $\mathbb{N}_0$ generated by mutually relatively prime numbers (taking $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). The numerical semigroup ring associated with the numerical semigroup $S =< a_1, \ldots, a_\nu >$ is the ring $R(S) = k[[x^{a_1}, \ldots, x^{a_\nu}]]$ where k is any field. We shall assume that $a_i < a_j$ if $i < j$.

We open Chapter 2 with some basic definitions which leads into a short discussion about Goto numbers in Section 2.1. In Section 2.2, we discuss some of the basic properties of numerical semigroups and their associated rings. We also introduce the notions of conductive and semiconductive subrings of numerical semigroup rings and prove that every star operation on a numerical semigroup ring persists on these subrings. That is, if R is a numerical semigroup ring, $\star$ a star operation on R, R' a semiconductive subring, and $\star'$ the star operation on R' defined by setting $I^{\star'} = (IR)^\star \cap I^{v'}$ (where v' is the v-operation on R') for all ideals $I \subseteq R'$, then $I^{\star'} = I^\star$ for every $I \subseteq R, R'$ which is an ideal in both rings. In Chapter 3, we present several results about star operations on conductive numerical semigroup rings. The two most notable of these results give strict restrictions on the actions of star operations on the fractional ideals intermediate between the ring and $k[[x]]$. We also show that, for general domains, if I is a fractional ideal that is also a ring, then $I^\star$ is also a ring. We take that result a bit further by showing that if R is a domain, R' a fractional ideal of R that is also a ring, $\star$ a star operation on R and I a fractional ideal of R that is also an R'-module, then $I^\star$ is an $(R')^\star$-module. In Section 4.1, we classify all star operations on the ring $k + x^4k[[x]]$ using the results from Chapter 3 and count the number of star operations in the case where k is finite and in Section 4.2 we classify the semistar operations on $k + x^4k[[x]]$. In Section 5.1, we classify the semistar operations on $k + x^5k[[x]]$ that are not star and then in Section 5.2, we examine star operations on $k + x^5k[[x]]$ using results from Chapter 3 and the computer program Macaulay 2 [GS]. In Chapter 6, we discuss some goals of future inquiry and examine how the results from Chapter 3 can be applied to general conductive numerical semigroup rings.

Chapter 2

Numerical Semigroup Rings

We begin our discussion with some basic definitions. A numerical semigroup is a subsemigroup of $\mathbb{N}_0 = \mathbb{N} \cup 0$ generated by mutually relatively prime numbers. These are denoted $< a_1, \ldots, a_\nu >$. Note that if we had a subsemigroup of $\mathbb{N}_0$ generated by numbers that weren't mutually relatively prime, then we could obtain a semigroup isomorphism with a numerical semigroup by dividing each generator by the gcd. Also, one may observe that these semigroups are in fact monoids. A numerical semigroup ring is one of the form $R = k[[x^{a_1}, \ldots, x^{a_\nu}]]$ where $< a_1, \ldots, a_\nu >$ is a numerical semigroup. Since a numerical semigroup is by definition generated by mutually relatively prime natural numbers, there is a natural number N such that every natural number $n \geq N$ is in the semigroup. The conductor is defined to be the smallest such number and will be denoted $c(S)$ for a numerical semigroup S. We also have the closely related frobenius which is one less than the conductor (i.e. the largest number not in the semigroup) and will be denoted $f(S)$. Here we present these definitions as well as a few others.

Definition 2.0.1. *Let $S =< a_1, \ldots, a_\nu >$ be a numerical semigroup.*

1. *The **frobenius** of S is defined to be $f(S) = \max \mathbb{N}_0 \setminus S$.*

2. *The* ***conductor*** *of S is defined to be $c(S) = f(S) + 1$.*

3. *The* ***multiplicity*** *of S is defined to be $e(S) = a_1$.*

4. *The* ***Apéry set*** *of S with respect to $n \in \mathbb{N}$ is*

$$Ap(S;n) = \{w \in S \mid w - n \notin S\}.$$

5. *The* ***pseudo-frobenius numbers*** *of S are*

$$PF(S) = \{n \in \mathbb{N}_0 \setminus S \mid n + s \in S \ \forall 0 \neq s \in S\}.$$

6. *Define $A(s) = \{\alpha \in \{1, 2, \ldots, a_1\} \mid s - \alpha \in S\}$.*

The definition of the multiplicity of a numerical semigroup S may seem like an unnecessary renaming of the smallest generator, but the alternative notation for this number is motivated by the fact this number coincides with the usual commutative ring theoretic notion of multiplicity, i.e. the multiplicity of $R(S)$ in the commutative ring theoretic sense is equal to a_1.

Here we introduce several definitions in numerical semigroup rings that are related to the conductor and the frobenius.

Definition 2.0.2. *Let $S =< a_1, \ldots, a_\nu >$ be a numerical semigroup and $R(S) = k[[x^{a_1}, \ldots, x^{a_\nu}]]$ the associated ring.*

1. *Define $\mathfrak{m}_s(S) = x^s k[[x]] \cap R(S)$ for $s \in S$, $s \leq c(S)$. We shall call such an ideal of $R(S)$ a* ***semiconductor ideal*** *of $R(S)$.*

2. *The subring of $R(S)$ whose maximal ideal is $\mathfrak{m}_s(S)$, i.e. $k + \mathfrak{m}_s(S)$, shall be called a* ***semiconductive subring*** *of $R(S)$.*

3. *The* ***conductor ideal*** *of the associated semigroup ring is defined to be $\mathfrak{c} = x^{c(S)} k[[x]]$.*

4. *A numerical semigroup ring is said to be* ***conductive*** *if its maximal ideal is its conductor, i.e.* $R = k + x^n k[[x]]$ *for some* n.

Recall that the pseudo-frobenius numbers of a numerical semigroup are defined as $PF(S) = \{n \in \mathbb{N} \setminus S \mid \text{ for all } s \in S \text{ with } s \neq 0,\ n + s \in S\}$. Note that the frobenius is one of these numbers and that these numbers resemble the frobenius in that they share the above described quality.

In the following set of definitions, we present an analogy to the pseudo-frobenius numbers of a numerical semigroup S and several related definitions in the context of the associated ring $R(S)$.

Definition 2.0.3. *Let* S *be a numerical semigroup and* $R(S)$ *the associated ring. Let* $I \subseteq R$ *be an ideal*

1. *The* ***anchor*** *of* I *is* $a(I) = \max\{a \in \mathbb{N}_0 \mid x^{-a} I \subseteq R(S)\}$.
2. *The* ***pseudo-frobenius numbers*** *of* I *are*

$$PF(I) = \{n \in \mathbb{N} \setminus S \mid x^n x^{-a(I)} I \subseteq R(S)\}.$$

3. *The* ***order*** *of* I *is* $\operatorname{ord}(I) = \min\{\operatorname{ord}(f) \mid f \in I\}$ *where* $\operatorname{ord}(f)$ *is the usual order of* f *as an element of* $k[[x]]$*, i.e. the minimum power of* x *occurring in* f *with non-zero coefficient.*

2.1 Goto Numbers

We begin by discussing Goto numbers in the context of numerical semigroup rings. Here we present the definition of a Goto number of a parameter ideal of a Noetherian local ring.

Definition 2.1.1. *Let Q be a parameter ideal of a Noetherian local ring $(R, \mathfrak{m})$. Then the* ***Goto number*** *is*

$$g(Q) = \max\{q \in \mathbb{N} \mid (Q : \mathfrak{m}^q) \text{ is integral over } Q\}.$$

If $S =< a_1, \ldots, a_\nu >$ is a numerical semigroup and $R = R(S) = k[[x^{a_1}, \ldots, x^{a_\nu}]]$, then for $s \in S$ we define the Goto number $g(s) = g(x^s)$.

Note that in the case of a numerical semigroup ring, an ideal of the form x^s is always a parameter ideal since these rings are 1-dimensional local domains. The following definition is necessary to work with Goto numbers of elements of a numerical semigroup.

Definition 2.1.2. *We define $\sigma(s) = \max\{\mathrm{ord}(w) \mid w \in Ap(S; s)\} =$*

$$\max\{\mathrm{ord}(p + s) \mid s \in PF(S)\}$$

.

Here the order of an element w of the semigroup is the $\mathfrak{m}$-adic order of x^w where $\mathfrak{m}$ is the maximal ideal of $R(S)$. In subsequent sections, $\mathrm{ord}(f)$ shall refer to the (x)-adic order of f in $k[[x]]$.

Proposition 2.1.3 (Bryant)**.** *For $S =< a_1, a_2 >$, $g(ka_2) = a_2 + k - 2 - \lfloor \frac{ka_2}{a_1} \rfloor$ for $1 \leq k \leq a_1 - 1$ and $g(u) = a_1 - 1$ for all other $u \in S$.*

We utilize the above formula to derive another formula.

Proposition 2.1.4. *For $S =< a_1, a_2 >, \sum_{k=1}^{a_1} \sigma(k) = \frac{1}{2}(a_1 - 1)(a_2 + (a_1 - 1))$.*

Proof. We have that $g(ka_2) = a_2 + k - 2 - \lfloor \frac{ka_2}{a_1} \rfloor$. Suppose a_1 is odd.

Then we have

$$
\begin{aligned}
\sum_{k=1}^{a_1-1} \lfloor \frac{ka_2}{a_1} \rfloor &= \sum_{k=1}^{\frac{1}{2}(a_1-1)} (\lfloor \frac{ka_2}{a_1} \rfloor + \lfloor \frac{(a_1-k)a_2}{a_1} \rfloor) \\
&= \sum_{k=1}^{\frac{1}{2}(a_1-1)} (\lfloor \frac{ka_2}{a_1} \rfloor + a_2 - \lfloor \frac{ka_2}{a_1} \rfloor - 1) \\
&= \sum_{k=1}^{\frac{1}{2}(a_1-1)} (a_2 - 1) \\
&= \frac{1}{2}(a_1-1)(a_2-1).
\end{aligned}
$$

Suppose a_1 is even.

Then

$$
\begin{aligned}
\sum_{k=1}^{a_1-1} \lfloor \frac{ka_2}{a_1} \rfloor &= \lfloor \frac{\frac{a_1}{2}a_2}{a_1} \rfloor + \sum_{k=1}^{\frac{1}{2}(a_1-2)} \lfloor \frac{ka_2}{a_1} \rfloor \\
&= \frac{1}{2}(a_2-1) + \sum_{k=1}^{\frac{1}{2}(a_1-2)} (\lfloor \frac{ka_2}{a_1} \rfloor + \lfloor \frac{(a_1-k)a_2}{a_1} \rfloor) \\
&= \frac{1}{2}(a_2-1) + \sum_{k=1}^{\frac{1}{2}(a_1-2)} (\lfloor \frac{ka_2}{a_1} \rfloor + a_2 - \lfloor \frac{ka_2}{a_1} \rfloor - 1) \\
&= \frac{1}{2}(a_2-1) + \sum_{k=1}^{\frac{1}{2}(a_1-1)} (a_2 - 1) \\
&= \frac{1}{2}(a_2-1) + \frac{1}{2}(a_1-2)(a_2-1) \\
&= \frac{1}{2}(a_1-1)(a_2-1).
\end{aligned}
$$

Thus

$$\begin{aligned}
\sum_{k=1}^{a_1} \sigma(k) &= \sum_{k=1}^{a_1} g(ka_2) \\
&= (a_1 - 1) + \sum_{k=1}^{a_1 - 1} (a_2 + k - 2 - \lfloor \frac{ka_2}{a_1} \rfloor) \\
&= (a_1 - 1) + (a_1 - 1)a_2 + \frac{1}{2}a_1(a_1 - 1) - 2(a_1 - 1) \\
&\quad - \frac{1}{2}(a_1 - 1)(a_2 - 1) \\
&= (a_1 - 1)(1 + a_2 + \frac{1}{2}a_1 - 2 - \frac{1}{2}(a_2 - 1)) \\
&= (a_1 - 1)(\frac{1}{2}a_2 + \frac{1}{2}a_1 - \frac{1}{2}) \\
&= \frac{1}{2}(a_1 - 1)(a_2 + (a_1 - 1)).
\end{aligned}$$

□

2.2 Multiplicative Ideal Theory in Numerical Semigroup Rings

The following propositions are well-known and proofs are provided for the purpose of completeness. The ideals we will be most interested in throughout this discussion will be non-principal ideals.

Proposition 2.2.1. *Let R be a domain and $I \subseteq R$ a non-principal ideal. Then for every $f \in I$, $f^{-1} \notin I^{-1}$.*

Proof. Let $f \in I$ and suppose that $f^{-1} \in I^{-1}$. Then for every $g \in I$, $f^{-1}g \in R$. Then we have $g = ff^{-1}g$ which implies that $I = (f)$. □

In the context of numerical semigroup rings, we can see that the previous result tells us that I^{-1} is contained in the integral closure of the ring, namely $k[[x]]$.

Corollary 2.2.2. *Let R be a numerical semigroup ring and let $I \subseteq R$ be a non-principal ideal. Then $a(I) \leq \operatorname{ord}(I) - e$.*

Proof. Suppose that $a(I) > \operatorname{ord}(I) - e$. Since the only elements $f \in R$ with $\operatorname{ord}(f) < e$ have order 0, we have that there is a $g \in I$ such that $x^{-a(I)}g \in R^{\times}$ yielding $x^{a(I)}g^{-1} \in R^{\times}$. Then we have $x^{a(I)}g^{-1}I = I$ which implies that $g^{-1}I = x^{-a(I)}I \subseteq R$. Thus $g^{-1} \in I^{-1}$ which implies that I is principal. $\square$

The following definition gives a star operation which will be discussed in further detail in Section 3.

Definition 2.2.3. *Let R be a domain. Then the v-operation is defined by $I^v = (I^{-1})^{-1}$ for all ideals $I \subseteq R$.*

A ring in which $I^v = I$ for all fractional ideals I is called divisorial. Conductive numerical semigroup rings are in some sense as far from being divisorial as possible.

Proposition 2.2.4. *Let R be a conductive ring with multiplicity e and $I \subseteq R$ a non-principal ideal. Then $I^v = x^{\operatorname{ord}(I)}k[[x]]$.*

Proof. Since R is conductive, we have $a(I) \geq \operatorname{ord}(I) - e$. Since I is non-principal, we have that $a(I) \leq \operatorname{ord}(I) - e$ by Corollary 2.2.2 and so $a(I) = \operatorname{ord}(I) - e$. Thus it suffices to assume that $\operatorname{ord}(I) = e$. We know that $I \subseteq \mathfrak{c}$ which implies that $k[[x]] \subseteq I^{-1}$, and so it suffices to show that $I^{-1} \subseteq k[[x]]$. Let $f \in I^{-1}$. We know that there is a $g \in I$ with $\operatorname{ord}(g) = e$ and thus $\operatorname{ord}(f) \geq -e$. If $\operatorname{ord}(f) > -e$, then $\operatorname{ord}(fg) > 0$ which implies that $\operatorname{ord}(fg) \geq e$ yielding $\operatorname{ord}(f) \geq 0$ which gives us $f \in k[[x]]$ (since $fg \in R$ by assumption). Thus it suffices to show that $\operatorname{ord}(f) \neq -e$. Suppose that $\operatorname{ord}(f) = -e$. Then $\operatorname{ord}(fg) = 0$ which implies that $fg \in R^{\times}$ which yields an $h \in R$ such that $hfg = 1$ implying that $hf = g^{-1} \in I^{-1}$ and thus I is principal, a contradiction. $\square$

Recall the definition of the Apéry set $Ap(S;n) = \{s \in S : s - n \notin S\}$. The Apéry set will be used in a slightly different way than usual. We want to find the elements $s \in S$ such that $s + n \notin S$ so we replace n with $-n$. The following proposition describes the action of the v-operation on monomial ideals in numerical semigroup rings.

Proposition 2.2.5. *Let S be a numerical semigroup, $R = R(S)$ its associated ring, and $I \subseteq R(S)$ a monomial ideal. Set $s = \mathrm{ord(I)} - a(I)$, let G be the set of monomial generators of $\mathfrak{m}_s(S)$, and J be the ideal generated by*

$$G \setminus \{x^\alpha \mid \alpha \in \bigcup_{n \in PF(I)} Ap(S; -n)\}.$$

Then $I^v = x^{a(I)} J$.

Proof. Since the v-operation is a star operation, it suffices to assume that $a(I) = 0$. By the given definitions, we see that $I^{-1} \supseteq R(S \cup PF(I))$ (this fact holds for general ideals with $a(I) = 0$). We also know that for $n < 0, x^n I \nsubseteq R$ since $a(I) = 0$ by assumption, hence $I^{-1} \subseteq k[[x]]$. We want to show that

$$I^{-1} \subseteq R(S \cup PF(I)).$$

Let $f \in k[[x]]$ such that $fI \subseteq R$, i.e. $f \in I^{-1}$. Suppose $n \in \mathbb{N} \setminus S$ is such that the x^n term of f occurs with a non-zero coefficient. Then we have that

$$\text{for every } m \text{ such that } x^m \in I, x^{m+n} \in R.$$

Then $n \in PF(I)$ and hence $f \in R(S \cup PF(I))$. It remains to be shown that

$$R(S \cup PF(I))^{-1} = J.$$

Suppose $f \in R \setminus J$. Then for some $n \in PF(I)$ and some $m \in Ap(S; -n)$ the x^m term of f occurs with a non-zero coefficient. Then $x^n x^m \notin R$ while $x^n \in R(S \cup PF(I))$ which implies that $fR(S \cup PF(I)) \notin R$ and so

$$f \notin R(S \cup PF(I))^{-1}.$$

Conversely, let $f \in J$. Then for every $m \in PF(I), fx^m \in R$. Thus

$$fR(S \cup PF(I)) \subseteq R \text{ which implies that } f \in R(S \cup PF(I))^{-1}.$$

$\square$

Remark 2.2.6. For general ideals I, Proposition 2.2.5 gives an upper bound for I^v. That is, $I^v \subseteq J$ where J is the ideal generated by

$$G \setminus \{x^\alpha \mid \alpha \in \bigcup_{n \in PF(I)} Ap(S; -n)\}.$$

Corollary 2.2.7. *If $\mathfrak{m}$ is the maximal ideal of $R(S)$, then $\mathfrak{m}^{-1} = R(S \cup PF(S))$.*

Proof. In the proof of Proposition 2.2.5, we showed that for monomial ideals I with $a(I) = 0$, $I^{-1} = R(S \cup PF(I))$. Clearly $PF(\mathfrak{m}) = PF(S)$. $\square$

From [HMP2, Lemma 3.7] we have that every fractional ideal of $(R, \mathfrak{m})$ is isomorphic to one intermediate between R and $\mathfrak{m}^{-1}$ if and only if $\mathfrak{m}^{-1}$ is a PID. Then by Corollary 2.2.7, the only numerical semigroup rings for which every fractional ideal is isomorphic to one intermediate between R and $\mathfrak{m}^{-1}$ are the conductive ones. In this case, $\mathfrak{m} = \mathfrak{c}$ and $\mathfrak{m}^{-1} = k[[x]]$. This means that any star operation on a conductive numerical semigroup ring is completely determined by its action on these intermediate fractional ideals. Furthermore, these fractional ideals are the ones generated by 1 and a set of k-linearly independent polynomials from $k[x]$ of degree no more than $n-1$ as will be discussed in more detail in Chapter 3. One consequence of this fact is given in the following proposition.

Proposition 2.2.8. *Let $R = k + x^n k[[x]]$ be a conductive numerical semigroup ring with finite base field k. Then R admits only finitely many star operations.*

Proof. We know from Proposition 2.2.4 that for any star operation $\star$ on R and any fractional ideal F intermediate between R and $k[[x]]$, $F^\star \subseteq F^v = k[[x]]$. Since every star operation on R is determined by its action on the fractional ideals intermediate between R and $k[[x]]$, it suffices to show that there are only finitely many such fractional ideals. Observe that every such fractional ideal can be generated as an R-module by polynomials in $k[x]$ of degree at most $n-1$. Since k is assumed to be finite, there are only finitely many such polynomials and thus, only finitely many fractional ideals intermediate between R and $k[[x]]$. $\square$

Chapter 3

Star Operations on Numerical Semigroup Rings

Recall that the set of star operations on a ring R is a partially ordered set with the partial ordering given by $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$ for all ideals $I \subseteq R$. It is clear that the identity operation is the infimum of all star operations. The following definition gives us the supremum of all star operations.

The v-operation is the supremum of all star operations on a domain as shown in [G]. Epstein introduces a generalized version of the v-operation that applies to non-domains in [Ep] and proves that it is the supremum of all star operations for any commutative ring with unity. The following propositions help us better understand the v-operation on numerical semigroup rings.

Proposition 3.0.9. *[BDF] Let S be a numerical semigroup and $R(S)$ the associated ring. Let $I \subseteq R(S)$ be an ideal. Then for any star operation $\star$, $\operatorname{ord}(I) = \operatorname{ord}(I^\star)$.*

Remark 3.0.10. Note that every semiconductor ideal is divisorial since $\mathfrak{m}_s(S)$ is maximal over ideals I such that $\operatorname{ord}(I) = s$ and since $\operatorname{ord}(I^v) = \operatorname{ord}(I)$ for all ideals I.

It is known that, in general, principal ideals of a domain are divisorial, i.e. v-closed, and hence, are $\star$-closed for any star operation $\star$. The following proposition shows that in a conductive numerical semigroup ring, we have I^v is a multiple of the conductor for any non-principal ideal I and is, in fact, the largest ideal with the same order as I.

In [HMP2], overrings that are intermediate between R and $\mathfrak{m}^{-1}$ are used extensively to create distinct star operations on the local ring $(R, \mathfrak{m})$. Here we characterize $\mathfrak{m}^{-1}$ for numerical semigroup rings in particular.

We now present a proposition that allows us in some sense to think of the set of star operations on a numerical semigroup ring as a subset of the set of star operations on any of its semiconductive subrings. If R is a numerical semigroup ring and I is any ideal, then I is also a fractional ideal of any subring of R. If $\star$ is a star operation on R, R' is a subring, and v' is the v-operation on R', then we can define the star operation $\star'$ on R' by setting $I^{\star'} = (IR)^{\star} \cap I^{v'}$ as in [A, Theorem 2]. If R' happens to be a semiconductive subring, then for any ideal I of R, $I^{\star} = I^{\star'}$ (where $\star'$ may act on I as a fractional ideal).

Proposition 3.0.11. *Let R be a numerical semigroup ring and $R_s = k + x^s k[[x]] \cap R$ a semiconductive subring. Suppose $\star_1, \star_2$ are distinct star operations on R. For ideals $I \subseteq R_s$, denote $I^{v'} = (R_s : (R_s : I))$. Define $\star_i'$ on R_s by $I^{\star_i'} = (IR)^{\star_i} \cap I^{v'}$. Then the $\star_i'$ are distinct star operations on R_s. Furthermore, if $I \subseteq R$ is an ideal with $I \subseteq R_s$, then $I^{\star_i'} = I^{\star_i}$.*

Proof. The above construction of the $\star_i'$ always yields a star operation so it suffices to prove that these are distinct. Let $I \subseteq R$ be an ideal with $I \subseteq R_s$ such that $I^{\star_1} \neq I^{\star_2}$, and suppose that $I^v \subseteq I^{v'}$. Then $I^{\star_i} \subseteq I^v \subseteq I^{v'}$ and so $I^{\star_i'} = I^{\star_i}$, which implies that the $\star_i'$ define distinct star operations.

We have that for any ideal $I \subseteq R$, there is an $n \in \mathbb{N}$ such that

$x^n I \subseteq \mathfrak{c} \subseteq R_s$. Since there is an ideal $I \subseteq R$ such that $I^{\star_1} \neq I^{\star_2}$, we may assume by the previous statement that $I \subseteq \mathfrak{c}$. Since principal ideals are divisorial, it suffices to consider non-principal ideals. Also, we can choose $f \in k[[x]]^\times$ such that $a'(fI) = \max\{a'(gI) \mid g \in k[[x]]^\times\}$ (where $a'(I) = \max\{a \in \mathbb{N} \mid x^{-a}I \subseteq R_s\}$ is the anchor of I with respect to R_s). Thus it suffices to assume that $a'(I)$ yields the maximum of all such $a'(fI)$ (note that $fI \subseteq \mathfrak{c}$ for any $f \in k[[x]]$).

To show the inclusion $I^v \subseteq I^{v'}$, we shall prove that it holds for $I_0 = x^{-a'(I)}I$. Since $a'(I)$ is maximal, we have that $(R_s : I_0) \subseteq k[[x]]$. Let $f \in I_0^v$. We know that $\mathrm{ord}(f) \geq \mathrm{ord}(I_0)$ since $\mathrm{ord}(I_0^v) = \mathrm{ord}(I_0)$. Then $\mathrm{ord}(f(R_s : I_0)) \geq \mathrm{ord}(I_0) \geq s$. We also have that $f(R_s : I_0) \subseteq f(R : I_0) \subseteq R$. Thus, since $f(R_s : I_0)$ is an R_s-submodule of R, it follows that

$$\mathrm{ord}(f(R_s : I_0)) \geq s \text{ implies that } f(R_s : I_0) \subseteq R_s.$$

□

One particularly useful consequence of the previous proposition is that the set of star operations on a numerical semigroup ring can be realized as a subset of the set of star operations on the conductive subring. Thus, if we can classify all star operations on conductive numerical semigroup rings, we will have classified all star operations on general numerical semigroup rings (although the set of star operations on a particular numerical semigroup ring may only be a small subset of the set of star operations on its conductive subring).

As mentioned previously, every fractional ideal of a numerical semigroup ring is isomorphic to one intermediate between the ring itself and $k[[x]]$. Here we discuss the conductive numerical semigroup rings, i.e. rings of the form $k + x^n k[[x]]$. An example an intermediate fractional ideal of the ring $k + x^5 k[[x]]$ would be $k + xk + x^5 k[[x]]$. Another example would be $k + (x + ax^2)k + x^5 k[[x]]$ for any choice of $a \in k$. If a is not zero, we can add these fractional ideals to get $k + xk + x^2 k + x^5 k[[x]]$. One can see from these examples that in general these fractional ideals are $k + x^n k[[x]]$-modules generated by 1 and some set of polynomials in $k[x]$ which need not be of degree higher than $n - 1$. The following notation allows us to more easily refer to these fractional ideals.

Notation 3.0.12. Let $R = k + x^n k[[x]]$. We denote

$$A_i = \sum_{j \neq i, 0 \leq j < n} x^j R \; for \; 0 < i < n.$$

For example, if $n = 5$, then

$$A_4 = R + xR + x^2 R + x^3 R = k + xk + x^2 k + x^3 k + x^5 k[[x]].$$

We denote $B_i = R + x^i R = k + x^i k + x^n k[[x]]$. We denote

$$A_{(i_1,\dots,i_\mu)} = \sum_{j \in \{0,\dots,n-1\} \setminus \{i_1,\dots,i_\mu\}} x^j R$$

and similarly we denote

$$B_{(i_1,\dots,i_\mu)} = R + \sum_{j \in \{i_1,\dots,i_\mu\}} x^j R.$$

For any $f_1, \dots, f_{n-1} \in k[[x]]^\times$ we denote

$$B_i(f_i) = R + x^i f_i R = k + x^i f_i k + x^n k[[x]].$$

We denote

$$B_{(i_1,\dots,i_\mu)}(f_{i_1}, \dots, f_{i_\mu}) = \sum_{j \in \{i_1,\dots,i_\mu\}} B_j(f_j).$$

We can further condense this notation by denoting

$$B_{(i_1,\dots,i_\mu)}(f_{(i_1,\dots,i_\mu)}) = B_{(i_1,\dots,i_\mu)}(f_{i_1}, \dots, f_{i_\mu}).$$

We will also use the hat notation to denote omission of an index, that is $(i_1, \dots, \hat{i_j}, \dots, i_\mu) = (i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_\mu)$.

This notation is somewhat redundant since, for instance, if $n = 5$ then $B_{(1,3)} = A_{(2,4)}$. We see the usefulness, however, when we observe that $B_{(1,2,3)} = A_4$. We may also find it convenient to extend the above notation in such a manner:

$$A_4(f_1, f_2, f_3) = B_{(1,2,3)}(f_1, f_2, f_3).$$

We begin by constructing some star operations on $R = k + x^n k[[x]]$. The following constructions form star operations on general Noetherian rings R:

If R' is an overring of R, then we can define $\star_{R'}$ by letting $I^{\star_{R'}} = IR' \cap I^v$ for any fractional ideal I of R as constructed in [A, Theorem 2].

If F is a fractional ideal such that $(F : F) = R$, then we can define another star operation $\diamond_F$ by letting $I^{\diamond_F} = (F : (F : I))$ for any fractional ideal I of R as constructed in [HHP, Proposition 3.2].

Given any two star operations $\star_1$ and $\star_2$, we can construct the infimum by defining $\star_1 \cap \star_2$ to be given by $I^{\star_1 \cap \star_2} = I^{\star_1} \cap I^{\star_2}$. We can also construct the supremum by defining $\star_1 \oplus \star_2$ to be given by $I^{\star_1 \oplus \star_2} = \cup_{n \in \mathbb{N}} I^{(\star_2 \circ \star_1)^n}$.

The fact that $\star_1 \oplus \star_2$ gives the supremum of the two star operations was brought to my attention by Jesse Elliott in conversation. He proves this fact in his paper [El]. I include my own proof for the purpose of completeness.

Proposition 3.0.13. *Let R be a Noetherian ring and let $\star_1$ and $\star_2$ be two star operations on R. Then $\star_1 \cap \star_2$ is the infimum of $\star_1$ and $\star_2$ and $\star_1 \oplus \star_2$ is the supremum.*

Proof. To show that $\star_1 \cap \star_2$ is the infimum of $\star_1$ and $\star_2$, it suffices to show that $\star_1 \cap \star_2$ is a star operation since the intersection of two ideals yields the infimum of those ideals. Let I be an ideal of R. Then $I \subseteq I^{\star_1}$ and $I \subseteq I^{\star_2}$ which implies that

$$I \subseteq I^{\star_1} \cap I^{\star_2} = I^{\star_1 \cap \star_2}.$$

If J is an ideal with $I \subseteq J$, then $I^{\star_1} \subseteq J^{\star_1}$ which implies that $I^{\star_1} \cap I^{\star_2} \subseteq J^{\star_1}$ and a similar argument shows that $I^{\star_1} \cap I^{\star_2} \subseteq J^{\star_2}$ and so

$$I^{\star_1 \cap \star_2} = I^{\star_1} \cap I^{\star_2} \subseteq J^{\star_1} \cap J^{\star_2} = J^{\star_1 \cap \star_2}.$$

Observe that, since $I \subseteq I^{\star_1 \cap \star_2} \subseteq I^{\star_1}$, $(I^{\star_1 \cap \star_2})^{\star_1} = I^{\star_1}$ and similarly $(I^{\star_1 \cap \star_2})^{\star_2} = I^{\star_2}$. Thus,

$$(I^{\star_1 \cap \star_2})^{\star_1 \cap \star_2} = (I^{\star_1 \cap \star_2})^{\star_1} \cap (I^{\star_1 \cap \star_2})^{\star_2} = I^{\star_1} \cap I^{\star_2} = I^{\star_1 \cap \star_2}.$$

Let $x \in R$ regular. Then

$$(xI)^{\star_1 \cap \star_2} = (xI)^{\star_1} \cap (xI)^{\star_2} = xI^{\star_1} \cap xI^{\star_2} = x(I^{\star_1} \cap I^{\star_2}) = xI^{\star_1 \cap \star_2}.$$

Thus $\star_1 \cap \star_2$ is a star operation.

We first prove that $\star_1 \oplus \star_2$ is a star operation. Clearly $I \subseteq I^{\star_1 \oplus \star_2}$. If J is an ideal such that $I \subseteq J$, then $I^{\star_1} \subseteq J^{\star_1}$ which implies that $(I^{\star_1})^{\star_2} \subseteq (J^{\star_1})^{\star_2}$. Thus, for all $n \in \mathbb{N}$, $I^{(\star_2 \circ \star_1)^n} \subseteq J^{(\star_2 \circ \star_1)^n}$ which yields $I^{\star_1 \oplus \star_2} \subseteq J^{\star_1 \oplus \star_2}$. To show that $(I^{\star_1 \oplus \star_2})^{\star_1 \oplus \star_2} = I^{\star_1 \oplus \star_2}$, we use the Noetherian condition on R. Observe that

$$I \subseteq I^{\star_2 \circ \star_1} \subseteq I^{(\star_2 \circ \star_1)^2} \subseteq \ldots \subseteq I^{(\star_2 \circ \star_1)^n} \subseteq \ldots$$

forms an ascending chain and, since R is Noetherian, there exists $N \in \mathbb{N}$ such that $I^{(\star_2 \circ \star_1)^n} = I^{(\star_2 \circ \star_1)^N}$ for every $n \geq N$. Thus $I^{\star_1 \oplus \star_2} = I^{(\star_2 \circ \star_1)^N}$ and it is clear that $(I^{\star_1 \oplus \star_2})^{\star_1 \oplus \star_2} = I^{\star_1 \oplus \star_2}$. Let $x \in R$ regular. We see that

$$(xI)^{\star_2 \circ \star_1} = ((xI)^{\star_1})^{\star_2} = (xI^{\star_1})^{\star_2} = x(I^{\star_1})^{\star_2} = xI^{\star_2 \circ \star_1}.$$

Then for all $n \in \mathbb{N}$, $(xI)^{(\star_2 \circ \star_1)^n} = xI^{(\star_2 \circ \star_1)^n}$ and, in particular,

$$(xI)^{(\star_2 \circ \star_1)^N} = xI^{(\star_2 \circ \star_1)^N} \text{ thus } (xI)^{\star_1 \oplus \star_2} = xI^{\star_1 \oplus \star_2}.$$

Finally, we show that $\star_1 \oplus \star_2$ is the supremum of $\star_1$ and $\star_2$. Let $\star_{sup}$ be the supremum of $\star_1$ and $\star_2$. Observe that

$$I^{\star_1} \subseteq (I^{\star_1})^{\star_2} \text{ and so } I^{\star_1} \subseteq I^{\star_1 \oplus \star_2}, \text{ i.e. } \star_1 \leq \star_1 \oplus \star_2.$$

Similarly, since $I \subseteq I^{\star_1}$,

$$I^{\star_2} \subseteq (I^{\star_1})^{\star_2} \text{ so that } I^{\star_2} \subseteq I^{\star_1 \oplus \star_2}, \text{ i.e. } \star_2 \leq \star_1 \oplus \star_2.$$

We have established that $\star_1 \oplus \star_2$ is an upper bound for $\star_1$ and $\star_2$ so that certainly $\star_{sup} \leq \star_1 \oplus \star_2$. It remains to be shown that $\star_1 \oplus \star_2 \leq \star_{sup}$. Let J be an ideal such that $I \subseteq J \subseteq I^{\star_{sup}}$. Then $J^{\star_{sup}} = I^{\star_{sup}}$ which yields the inclusions

$$J^{\star_1} \subseteq I^{\star_{sup}} \text{ and } J^{\star_2} \subseteq I^{\star_{sup}}.$$

Since $I \subseteq J^{\star_1} \subseteq I^{\star_{sup}}$, $I \subseteq J^{\star_2 \circ \star_1} \subseteq I^{\star_{sup}}$. In particular, $I \subseteq I^{\star_2 \circ \star_1} \subseteq I^{\star_{sup}}$. If $I \subseteq I^{(\star_2 \circ \star_1)^n} \subseteq I^{\star_{sup}}$, then $I \subseteq I^{(\star_2 \circ \star_1)^{n+1}} \subseteq I^{\star_{sup}}$ so that

$$\text{for all } n \in \mathbb{N}, \; I \subseteq I^{(\star_2 \circ \star_1)^n} \subseteq I^{\star_{sup}}.$$

In particular, $I \subseteq I^{(\star_2 \circ \star_1)^N} \subseteq I^{\star_{sup}}$ and so

$$I^{\star_1 \oplus \star_2} \subseteq I^{\star_{sup}}, \text{ i.e. } \star_1 \oplus \star_2 \leq \star_{sup}.$$

□

For an arbitrary collection of star operations $\{\star_\gamma\}_{\gamma \in \Gamma}$, we can construct the infimum $\bigcap\limits_{\gamma \in \Gamma} \star_\gamma$ by setting $I^{\bigcap_{\gamma \in \Gamma} \star_\gamma} = \bigcap\limits_{\gamma \in \Gamma} I^{\star_\gamma}$. This is constructed and proven in [A, Theorem 2]. For the supremum, a slightly different construction is needed. We start by taking $\sum\limits_{\gamma \in \Gamma} I^{\star_\gamma}$. Let I_1 be this ideal. Define an ascending chain of ideals recursively by setting $I_{n+1} = \sum\limits_{\gamma \in \Gamma} I_n^{\star_\gamma}$. If $|\Gamma| = 2$, then this construction coincides with the construction used in Proposition 3.0.13 since $I_n \subseteq I^{(\star_2 \circ \star_1)^n} \subseteq I_{n+1}$.

By observing that every fractional ideal of R is isomorphic to one intermediate between R and $k[[x]]$, we see that any star operation on R is completely determined by its action on such fractional ideals. For example, if $R = k + x^4 k[[x]]$, it suffices to consider the fractional ideals A_1, $A_2(f_2)$, $A_3(f_1, f_2)$, $B_1(f_1)$, $B_2(f_2)$, and B_3. The following propositions eliminate many of the possible actions of star operations on these fractional ideals.

Proposition 3.0.14. *Let $R = k + x^n k[[x]]$. Let $\star$ be a star operation on R. Then if $A_i^\star = A_i$, then $A_j^\star = A_j$ for any j such that $j \leq i$.*

Proof. Observe that for any $i \in \{1, \ldots, n-1\}$, $A_i^\star = A_i$ or $A_i^\star = k[[x]]$. Suppose that $A_i^\star = A_i$ and that $A_j^\star = k[[x]]$ for some $j < i$. Then we have $x^{i-j} A_j \subseteq A_i$ which implies that $x^{i-j} k[[x]] \subseteq A_i$ which gives us $x^i \in A_i^\star$ since $x^j \in k[[x]]$ yielding the desired contradiction. $\square$

Figure 3.2 illustrates how the previous proposition applies to the ring $k + x^4 k[[x]]$.

We shall use a dotted arrow like in Figure 3.1 to indicate the implication $J^\star = J$

$$\begin{tikzcd} \boxed{I} \arrow[r, dashed] & \boxed{J} \end{tikzcd}$$

Figure 3.1: Implication on actions of star operations.

only if $I^\star = I$ for any star operation $\star$.

$$\begin{tikzcd}
 & \boxed{k[[x]]} & \\
\boxed{A_1} \arrow[r, dashed] & \boxed{A_2} \arrow[r, dashed] & \boxed{A_3} \\
\boxed{B_3} & \boxed{B_2} & \boxed{B_1} \\
 & \boxed{k + x^4 k[[x]]} &
\end{tikzcd}$$

Figure 3.2: Implication by Proposition 3.0.14 applied to $k + x^4 k[[x]]$.

Proposition 3.0.15. *Let $R = k + x^n k[[x]]$ and let $A = A_i(f_{(1,\ldots,\hat{i},\ldots,n-1)})$ for some $i \in \{1, \ldots, n-1\}$ and $f_j \in k[[x]]^\times$. Then there is an $f \in k[[x]]^\times$ such that $fA = \sum_{j \neq i, j < n} x^j R = A_i$.*

Proof. Without loss of generality, we may assume that $f_j = 1$ for $j > i$ and that $f_j = 1 + a_j x^{i-j}$ for $j < i$ and for some $a_j \in k$. To see this, we construct an $(n-1) \times n$ matrix whose rows are given by the coefficients of the polynomial generators of A. If we perform elementary row operations on this matrix (with scalars coming from the base field k), the result will have rows that give coefficients for alternate polynomial generators of A. By the way we constructed A, we see that the rank of this matrix is $n-1$ and, when put into reduced row echelon form, will have its free variable in the $i+1^{\text{st}}$ column (the one corresponding to the coefficient for the x^i term of the polynomial). We wish to construct the desired $f \in k[[x]]^\times$. Let $f = 1 + \sum_{l=1}^{i-1} -a_{i-l}x^l$. Then $fA = \sum_{l \neq i, l<n} f f_j x^j R$. For $j < i$ we have

$$\begin{aligned}
f f_j x^j &= x^j + \sum_{l=1}^{i-1} -a_{i-l}x^{l+j} + a_j x^i (1 + \sum_{l=1}^{i-1} -a_{i-l}x^l) \\
&= x^j + \sum_{l=1}^{i-j-1} -a_{i-l}x^{l+j} + -a_{i-(i-j)}x^{(i-j)+j} + \\
&\quad \sum_{l=i-j+1}^{i-1} -a_{i-l}x^{l+j} + a_j x^i + \sum_{l=1}^{i-1} -a_{i-l}a_j x^{l+i} \\
&= x^j + \sum_{l=1}^{i-j-1} -a_{i-l}x^{l+j} + (a_j - a_j)x^i + \\
&\quad \sum_{l=i-j+1}^{i-1} -a_{i-l}x^{l+j} + \sum_{l=1}^{i-1} -a_{i-l}a_j x^{l+i} \\
&= x^j + \sum_{l=1}^{i-j-1} -a_{i-l}x^{l+j} + \sum_{l=i-j+1}^{i-1} -a_{i-l}x^{l+j} + \sum_{l=1}^{i-1} -a_{i-l}a_j x^{l+i}.
\end{aligned}$$

If $j > i$, then

$$f f_j x^j = x^j + \sum_{l=1}^{i-1} -a_{i-l}x^{l+j}.$$

We know that $x^n k[[x]] \subseteq A$ which gives us that $f x^n k[[x]] \subseteq fA$ yielding the inclusion $x^n k[[x]] \subseteq fA$. We shall show by induction that $x^{i+1}k[[x]] \subseteq fA$. We have already shown this in the case where $i = n-1$. If $i < n-1$, let $m \geq i+1$ and suppose that

$x^{m+1}k[[x]] \subseteq fA$. Then

$$ff_m x^m = x^m + \sum_{l=1}^{i-1} -a_{i-l}x^{l+m} \in fA$$

which implies that $x^m \in fA$ since

$$\sum_{l=1}^{i-1} -a_{i-l}x^{l+m} \in x^{m+1}k[[x]] \subseteq fA.$$

Thus the claim holds. It remains to be shown that $x^m \in fA$ for $m < i$. We may ignore the case where $i = 1$. If $m = i - 1$, then

$$ff_{i-1}x^{i-1} = x^{i-1} + \sum_{l=2}^{i-1} -a_{i-l}x^{l+i-1} + \sum_{l=1}^{i-1} -a_{i-l}a_{i-1}x^{l+i}$$

which implies that $x^{i-1} \in fA$ since $\sum_{l=2}^{i-1} -a_{i-l}x^{l+i-1} + \sum_{l=1}^{i-1} -a_{i-l}a_{i-1}x^{l+i}$ is an element of $x^{i+1}k[[x]]$ which is contained in fA. Let $m < i$ and suppose that $\sum_{l=m+1}^{i-1} x^l R \subseteq fA$. We have

$$ff_m = x^m + \sum_{l=1}^{i-m-1} -a_{i-l}x^{l+m} + \sum_{l=i-m+1}^{i-1} -a_{i-l}x^{l+m} + \sum_{l=1}^{i-1} -a_{i-l}a_m x^{l+i}.$$

Thus $x^m \in fA$ since

$$\sum_{l=1}^{i-m-1} -a_{i-l}x^{l+m} + \sum_{l=i-m+1}^{i-1} -a_{i-l}x^{l+m} + \sum_{l=1}^{i-1} -a_{i-l}a_j x^{l+i}$$

is an element of $\sum_{l=m+1}^{i-1} x^l R + x^{i+1}k[[x]]$ which is contained in fA. This concludes the proof. $\square$

To illustrate how this process is performed in practice, we consider the following example.

Example 3.0.16. Let $R = k + x^4k[[x]]$ and consider the fractional ideal $A_3(1 + x^2, 1 - 2x) = R + (x + x^3)R + (x^2 - 2x^3)R$. We take f to be $f = 1 + 2x - x^2$.

Then we have

$$\begin{aligned} fA_3(1+x^2,&1-2x) \\ &= (1+2x-x^2)R + (x+2x^2+2x^4-x^5)R + (x^2-5x^4+2x^5)R \\ &= (1+2x-x^2)R + (x+2x^2)R + x^2R \\ &= R + xR + x^2R = A_3. \end{aligned}$$

To see the second to last equality, observe that $x^2 \in fA_3(1+x^2, 1-2x)$ and $x+2x^2 \in fA_3(1+x^2, 1-2x)$ by Line 3 so $x \in fA_3(1+x^2, 1-2x)$. Likewise, $x^2 \in fA_3(1+x^2, 1-2x)$, $x \in fA_3(1+x^2, 1-2x)$ and $1+2x-x^2 \in fA_3(1+x^2, 1-2x)$ so $1 \in fA_3(1+x^2, 1-2x)$.

One can easily draw many quick conclusions from Propositions 3.0.14 and 3.0.15. For example, if $\star$ is a star operation on $R = k + x^4k[[x]]$ and if $A_2^\star = A_2$, then $A_1^\star = A_1$ by Proposition 3.0.14. By Proposition 3.0.15 we could also conclude that $A_2(f_1)^\star = A_2(f_1)$ for any $f_1 \in k[[x]]^\times$. A somewhat less obvious conclusion that could be drawn is that for any star operation $\star$, $B_3^\star \neq A_2$ despite the fact that $B_3 \subsetneq A_2$. Suppose $B_3^\star = A_2$. Then $A_2^\star = B_3^\star \subseteq A_1^\star$ which implies that $A_1^\star = k[[x]]$, since $k[[x]] = A_1 + A_2 \subseteq (A_1+A_2)^\star = (A_1^\star + A_2^\star)^\star = (A_1^\star)^\star = A_1^\star$. This implies by Proposition 3.0.14 that $A_2^\star = k[[x]]$. By our supposition, however, $A_2^\star = (B_3^\star)^\star = B_3^\star = A_2 \neq k[[x]]$. This observation is generalized in the following corollary.

Corollary 3.0.17. *Let $\star$ be a star operation on $R = k + x^nk[[x]]$, $\mu \in \{1, \ldots, n-2\}$ and $f_1, \ldots, f_{n-2} \in k[[x]]^\times$. Then for any $j \in \{1, \ldots, n-1\} \setminus \{i_1, \ldots, i_\mu\}$ with $j \neq$* $\min(\{1, \ldots, n-1\} \setminus \{i_1, \ldots, i_\mu\})$, $B_{(i_1,\ldots,i_\mu)}(f_{(i_1,\ldots,i_\mu)})^\star \neq A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)})$.

Proof. Suppose that $B_{(i_1,\ldots,i_\mu)}(f_{(i_1,\ldots,i_\mu)})^\star = A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)})$. Let

$$M = \min(\{1, \ldots, n-1\} \setminus \{i_1, \ldots, i_\mu\}).$$

We have that

$$B_{(i_1,\ldots,i_\mu)}(f_{(i_1,\ldots,i_\mu)}) \subseteq A_M(f_{(1,\ldots,\hat{M},\ldots,n-2)})$$

which implies that

$$B_{(i_1,\ldots,i_\mu)}(f_{(i_1,\ldots,i_\mu)})^\star \subseteq A_M(f_{(1,\ldots,\hat{M},\ldots,n-2)})^\star$$

yielding the inclusion

$$A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)}) + A_M(f_{(1,\ldots,\hat{M},\ldots,n-2)}) \subseteq A_M(f_{(1,\ldots,\hat{M},\ldots,n-2)})^\star$$

which gives us that $A_M(f_{(1,\ldots,\hat{M},\ldots,n-2)})^\star = k[[x]]$. Then by Proposition 3.0.14, $A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)})^\star = k[[x]]$. By Proposition 3.0.15, $A_M^\star = k[[x]] = A_j^\star$. By supposition, $B_{(i_1,\ldots,i_\mu)}(f_{(i_1,\ldots,i_\mu)})^\star = A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)})$ which implies that $A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)})^\star = A_j(f_{(1,\ldots,\hat{j},\ldots,n-2)})$. Then by Proposition 3.0.15,

$$A_j^\star = A_j \neq k[[x]].$$

□

Figure 3.3 shows how the results of Corollary 3.0.17 apply to the case where $n = 4$. Here, the arrows indicate the possible actions of star operations on these fractional ideals.

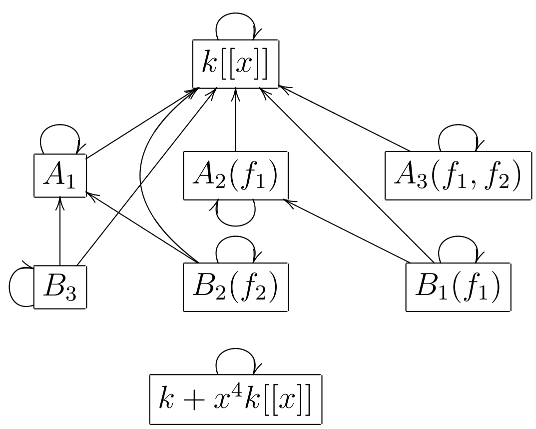

Figure 3.3: All possible actions of star operations on $k + x^4k[[x]]$.

One might think that Corollary 3.0.17 could be generalized by replacing j with a vector of indices and demanding that these indices are the minimum possible such indices but the corollary fails to hold if such a substitution were made in the statement as shown in the following example.

Example 3.0.18. Let $R = k+x^5k[[x]]$. Note that $A_{(1,2)}$ and $A_{(1,3)}$ are rings. Observe that $B_4^{\star_{A_{(1,2)}}} = A_{(1,2)}$ while $B_4^{\star_{A_{(1,3)}}} = A_{(1,3)}$.

The following two propositions eliminate possibilities for star operations on $R = k + x^n k[[x]]$ where $n \geq 5$.

Proposition 3.0.19. *Let R be any domain. Let R' be a fractional ideal of R that is also a ring and let $\star$ be a star operation on R. Then $(R')^\star$ is also a ring.*

Proof. First we show that $(R')^\star$ is an R'-module. Since we know that $(R')^\star$ is an R-module, it must be closed under addition, thus it suffices to show that $(R')^\star$ is closed under R'-scaling. Let $f \in R'$. Since R' is a ring, $fR' \subseteq R'$ which implies that $(fR')^\star \subseteq (R')^\star$ which yields the inclusion $f(R')^\star \subseteq (R')^\star$.

Now let $f \in (R')^\star$. Since $(R')^\star$ is an R'-module, $fR' \subseteq (R')^\star$ implying that $(fR')^\star \subseteq (R')^\star$ yielding $f(R')^\star \subseteq (R')^\star$. □

We give an example of how Proposition 3.0.19 can be used to narrow down the possible actions of star operations on these rings.

Example 3.0.20. Consider $R = k + x^5k[[x]]$. We have $B_3 \subseteq A_{(1,2)},\ A_{(1,4)},\ A_{(2,4)}$ so that a priori $B_3^\star$ could be any of these for an arbitrary star operation $\star$. However, $B_3^\star \neq A_{(1,4)},\ A_{(2,4)}$ since $B_3^\star$ must be a ring by Proposition 3.0.19 while $A_{(1,4)}$ and $A_{(2,4)}$ are not rings.

Proposition 3.0.21. *Let R, R', and $\star$ be as in Proposition 3.0.19, and let I be a fractional ideal of R that is also an R'-module. Then $I^\star$ is an $(R')^\star$-module.*

Proof. We begin by showing that $I^\star$ is an R'-module. As in Proposition 3.0.19, it suffices to show that $I^\star$ is closed under R' scaling. Let $f \in R'$. Then $fI \subseteq I$ which implies that $(fI)^\star \subseteq I^\star$ which yields the inclusion $fI^\star \subseteq I^\star$.

To show that I is an $(R')^\star$-module, it suffices once again to show that $I^\star$ is closed under $(R')^\star$ scaling. Let $f \in I^\star$. Then $fR' \subseteq I^\star$ implying that $(fR')^\star \subseteq (I^\star)^\star = I^\star$ yielding $f(R')^\star \subseteq I^\star$. □

In the following example, we use a sub-result of Proposition 3.0.21, that is if I is an R'-module, then so is $I^\star$, to eliminate some possible actions of star operations on $k + x^7k[[x]]$.

Example 3.0.22. Let $R = k + x^7k[[x]]$. Note that B_4 is a ring and $B_{(3,4)}$ is a B_4-module (and not a ring) while $B_{(2,3,4)}$ is not a B_4-module. Thus, by Proposition 3.0.21, $B^\star_{(3,4)} \neq B_{(2,3,4)}$ for any star operation $\star$.

The following proposition from Gilmer helps eliminate more possibilities of star operations on $R = k + x^nk[[x]]$ and, along with the Propositions 3.0.14, 3.0.15, and Corollary 3.0.17, allows us to classify all star operations on $k + x^4k[[x]]$.

Proposition 3.0.23. *[G, Section 32, Exercise 1] Let R be a domain and let $\star$ be a star operation on R. Suppose that A is a $\star$-closed fractional ideal of R. Then for any fractional ideal B of R, $(A : B)$ is $\star$-closed.*

Proof. It suffices to show that $(A : B)^\star \subseteq (A : B)$. Let $f \in B$. Then $f(A : B) \subseteq A$ which implies that $(f(A : B))^\star \subseteq A^\star$ yielding the inclusion $f(A : B)^\star \subseteq A$. Since this holds for all $f \in B$, it follows that $(A : B)^\star \subseteq (A : B)$. □

Note the following two consequences of the previous proposition:

Observe that if $R = k + x^4k[[x]]$, then $(B_1(f_1) : B_3) = xf_1B_3$. It follows that if $B_1(f_1)$ is $\star$-closed, then so is B_3. We also have that $(B_2(f_2) : B_3) = x^2f_2A_1$ which tells us that if $B_2(f_2)$ is $\star$-closed, then so is A_1. These results can be generalized in the following proposition:

Proposition 3.0.24. *Let $R = k+x^n k[[x]]$, $\star$ a star operation on R, $i \in \{1,\dots,n-2\}$, and $f_i \in k[[x]]^\times$. Then if $B_i(f_i)$ is $\star$-closed, then so is $B_{(n-i,\dots,n-1)}$.*

Proof. First we show that $(B_i(f_i) : B_{n-1}) = x^i f_i k + x^n k[[x]]$. Since $1 \in B_{n-1}$, $(B_i(f_i) : B_{n-1}) \subseteq B_i(f_i)$. We see that for any $a \in k$, $ax^i f_i B_{n-1} = x^i f_i k + x^{n+i-1}k[[x]] \subseteq B_i(f_i)$, so $x^i f_i k \subseteq (B_i(f_i) : B_{n-1})$. Clearly

$$x^n k[[x]] \subseteq (B_i(f_i) : B_{n-1}) \text{ so } x^i f_i k + x^n k[[x]] \subseteq (B_i(f_i) : B_{n-1}).$$

Suppose $f \in B_i(f_i) \setminus x^i f_i k + x^n k[[x]]$. Then $\operatorname{ord}(f) = 0$ which implies that $fx^{n-1} \in fB_{n-1}$. Since $\operatorname{ord}(fx^{n-1}) = n-1$, $fx^{n-1} \notin B_i(f_i)$ and so $f \notin (B_i(f_i) : B_{n-1})$. Then we have that $(B_i(f_i) : B_{n-1}) = x^i f_i k + x^n k[[x]]$ which is isomorphic to $B_{(n-i,\dots,n-1)}$ since

$$x^{-i} f_i^{-1}(x^i f_i k + x^n k[[x]]) = k + x^{n-i} f_i^{-1} k[[x]] = k + x^{n-i} k[[x]] = B_{(n-i,\dots,n-1)}.$$

$\square$

In Proposition 3.0.15, we observed that star operations act in essentially the same way on fractional ideals of the form $A_i(f_{(1,\dots,\hat{i},\dots,n-2)})$ for a particular i. The following equivalence relation describes the extent to which this phenomenon occurs for other fractional ideals intermediate between $k + x^n k[[x]]$ and $k[[x]]$.

Definition 3.0.25. *Let I be a fractional ideal of $R = k + x^n k[[x]]$ intermediate between R and $k[[x]]$. We define the following relation:*

We say that $I \sim J$ if there exists $f \in I \cap k[[x]]^\times$ such that $f^{-1}I = J$.

Proposition 3.0.26. *The relation defined in Definition 3.0.25 is an equivalence relation.*

Proof. Let I, J and L be fractional ideals intermediate between R and $k[[x]]$.

1. Since $1 \in I$, $I \sim 1^{-1}I = I$.

2. Suppose $I \sim J$. Then for some $f \in I \cap k[[x]]^\times$, $f^{-1}I = J$. Since $1 \in I$, $f^{-1} \in J$ and then $(f^{-1})^{-1}J = fJ = I$. Thus $J \sim I$.

3. Suppose $I \sim J$ and $J \sim L$. Then there exist $f \in I \cap k[[x]]^\times$ and $g \in J \cap k[[x]]^\times$ such that $f^{-1}I = J$ and $g^{-1}J = L$. Then $f^{-1}g^{-1}I = L$ so it remains to be shown that $fg \in I$. We know that $g \in J = f^{-1}I$ and so $fg \in I$ as desired.

□

Note that Proposition 3.0.15 shows that $A_i(f_{(1,\ldots,\hat{i},\ldots,n-2)}) \sim A_i(g_{(1,\ldots,\hat{i},\ldots,n-2)})$ for any $f_{(1,\ldots,\hat{i},\ldots,n-2)}$ and $g_{(1,\ldots,\hat{i},\ldots,n-2)}$.

Remark 3.0.27. There is an equivalence relation on general fractional ideals defined by $I \sim J$ if there is an $f \in K$ such that $fI = J$ where K is the fraction field of R. Here we define an equivalence relation on specifically the fractional ideals intermediate between R and $k[[x]]$. Note that if $f \notin I$, then $1 \notin f^{-1}I$ so $f^{-1}I$ does not contain R.

Recall that for any fractional ideal F satisfying $(F : F) = k + x^4k[[x]]$, we can define the star operation $\diamond_F$ by setting $I^{\diamond_F} = (F : (F : I))$ for every fractional ideal I of $k + x^4k[[x]]$. In particular, every fractional of the form $B_1(f_1)$ satisfies $(B_1(f_1) : B_1(f_1)) = k + x^4k[[x]]$ so we can define $\diamond_{B_1(f_1)}$ in this fashion.

Example 3.0.28. Let $R = k + x^4k[[x]]$ and $k = \mathbb{F}_2$. Consider the fractional ideals $B_1(1 + ax + bx^2)$. We have $(1 + x) \in B_1$ which yields $(1 + x)^{-1}B_1 = B_1(1 + x + x^2)$ so that $B_1 \sim B_1(1 + x + x^2)$. Similarly we have $(1 + x + x^2) \in B_1(1 + x)$ yielding $(1 + x + x^2)^{-1}B_1(1 + x) = B_1(1 + x^2)$ so that $B_1(1 + x) \sim B_1(1 + x^2)$.

In the following proposition, we show that those fractional ideals intermediate between R and $k[[x]]$ that are rings are equivalent only to themselves under $\sim$.

Proposition 3.0.29. *Let R' be an overring of R intermediate between R and $k[[x]]$. Then the equivalence class of R' under $\sim$ consists only of R'.*

Proof. Let $f \in R' \cap k[[x]]^{\times}$. Since R' is a ring, $fR' \subseteq R'$ and so $R' \subseteq f^{-1}R'$. If we write R' in terms of its generators as an R-module, i.e. $R' = R + \sum_{j=1}^{\nu} f_{i_j} x^{i_j} R$, we see that R' is completely determined by the quotient $R'/x^n k[[x]]$ which is a finite-dimensional k-vector space. Since $f^{-1} \in k[[x]]^{\times}$, we have that $f^{-1}R'/x^n k[[x]] = f^{-1}k + \sum_{j=1}^{\nu} f^{-1} f_{i_j} x^{i_j}$ which is at most ν-dimensional and contains the ν-dimensional k-vector space $R'/x^n k[[x]]$. Thus $f^{-1}R'/x^n k[[x]] = R'/x^n k[[x]]$ and so $R' = f^{-1}R'$. $\square$

The following result classifies the equivalence classes of fractional ideals of the form $B_1(f)$ in the case where $R = k + x^4 k[[x]]$.

Proposition 3.0.30. *Let $R = k + x^4 k[[x]]$ and $f = 1 + ax + bx^2$ and consider $B_1(f)$. Then the function $\phi : k \to \widetilde{B_1(f)}$ defined by*

$$\phi(\alpha) = B_1(1 + (a - \alpha)x + (b - 2a\alpha + \alpha^2)x^2),$$

where $\widetilde{B_1(f)}$ is the equivalence class of $B_1(f)$ under $\sim$, is a bijection.

Proof. If $\phi(\alpha_1) = \phi(\alpha_2)$ then $a - \alpha_1 = a - \alpha_2$ which implies that $\alpha_1 = \alpha_2$ and so ϕ is injective. Now let $B \in \widetilde{B_1(f)}$. Then there is a $g \in B_1(f) \cap k[[x]]^{\times}$ such that $g^{-1}B_1(f) = B$. We may assume that g is of the form

$$g = 1 + \alpha x + \alpha a x^2 + \alpha b x^3 + h$$

for some $h \in x^4 k[[x]]$ and some $\alpha \in k$. Then we can write g^{-1} as

$$1 - \alpha x + (\alpha^2 - \alpha a)x^2 + (2\alpha^2 a - \alpha b - \alpha^3)x^3 + l$$

for some $l \in x^4k[[x]]$. Since $R \subseteq B$, B can be generated by 1 and

$$-\alpha x + (\alpha^2 - \alpha a)x^2 + (2\alpha^2 a - \alpha b - \alpha^3)x^3.$$

Then we have that $B = B_1(1 + (a - \alpha)x + (b - 2\alpha a + \alpha^2)x^2) = \phi(\alpha)$. $\square$

One consequence of this is that if k is a finite field, then $|\widetilde{B_1(f)}| = |k|$ and since there are exactly $|k|^2$ fractional ideals of this type, there are exactly $|k|$ such equivalence classes. We see that if $B_1(f) \sim B_1(g)$ then $B_1(f)$ is $\diamond_{B_1(g)}$-closed. In the following proposition, we show that these star operations are actually the same.

Proposition 3.0.31. *Let $R = k + x^4k[[x]]$ and I be fractional ideal intermediate between R and $k[[x]]$. Suppose there is $g \in I$ with $1 \leq \operatorname{ord}(g) \leq 2$ and that $I \notin \widetilde{B_1(f)}$. Then $I^{\diamond_{B_1(f)}} = k[[x]]$.*

Proof. It suffices to show that $(B_1(f) : I) \subseteq x^4k[[x]]$. Since $1 \in I$, we observe that $(B_1(f) : I) \subseteq B_1(f)$. Let $h \in (B_1(f) : I)$. We have that either $\operatorname{ord}(hg) \geq 4$ or $\operatorname{ord}(hg) = 1$. If $\operatorname{ord}(hg) \geq 4$, then $\operatorname{ord}(h) \geq 3$ which gives us $\operatorname{ord}(h) \geq 4$ since $h \in B_1(f)$. If $\operatorname{ord}(g) = 2$ then $\operatorname{ord}(hg) \neq 1$ and we are done. If $\operatorname{ord}(g) = 1$, then $\operatorname{ord}(h) = 0$ and so $hI/x^4k[[x]]$ is a k-vector space of dimension at least 2 which is contained in $B_1(f)/x^4k[[x]]$. Since $B_1(f)/x^4k[[x]]$ is a 2-dimensional k-vector space, we have that $B_1(f)/x^4k[[x]]hI/x^4k[[x]]$ which implies that $B_1(f) = hI$ yielding $h^{-1}B_1(f) = I$. Since $h \in B_1(f)$, it follows that $I \in \widetilde{B_1(f)}$ which contradicts our hypotheses. $\square$

The only fractional ideals intermediate between R and $k[[x]]$ that don't satisfy the hypotheses of the above proposition are those in $\widetilde{B_1(f)}$ and B_3. We know that every $I \in \widetilde{B_1(f)}$ is $\diamond_{B_1(f)}$-closed. By Proposition 3.0.24 we know that B_3 is $\diamond_{B_1(f)}$-closed. This leads us to the conclusion that if $I \in \widetilde{B_1(f)}$, then $\diamond_{B_1(f)} = \diamond_I$.

Chapter 4

Star and Semistar Operations on $k + x^4k[[x]]$

4.1 Classification Theorem for Star Operations on $k + x^4k[[x]]$

We examine the star operations on the ring $k + x^4k[[x]]$ by examining their actions on the fractional ideals intermediate between $k + x^4k[[x]]$ and $k[[x]]$. Recall that if R' is an overring of $k + x^4k[[x]]$, then we can define the star operation $\star_{R'}$ by setting $I^{\star_{R'}} = IR' \cap I^v$ for every fractional ideal I of $k + x^4k[[x]]$. In particular, we have that the fractional ideals of the form $B_2(f_2)$ are overrings of $k + x^4k[[x]]$ and so we can define $\star_{B_2(f_2)}$ in this fashion. Similarly, we have B_3 and A_1 as overrings of $k + x^4k[[x]]$ and we define the corresponding star operations $\star_{B_3}$ and $\star_{A_1}$. Recall also that for any fractional ideal F satisfying $(F : F) = k + x^4k[[x]]$, we can define the star operation $\diamond_F$ by setting $I^{\diamond_F} = (F : (F : I))$ for every fractional ideal I of $k + x^4k[[x]]$. In particular, every fractional of the form $B_1(f_1)$ satisfies $(B_1(f_1) : B_1(f_1)) = k + x^4k[[x]]$ so we can define $\diamond_{B_1(f_1)}$ in this fashion. As we will see in the statement of Theorem 4.1.1, we can construct all star operations on $k + x^4k[[x]]$ by building suprema and taking intersections of the above star operations. The following proposition classifies all star operations on $k + x^4k[[x]]$.

Theorem 4.1.1. *Every star operation on $k + x^4k[[x]]$ other than the identity and v is one of the following for some index sets Γ and Δ and some collections $\{f_\gamma \in k[[x]]^\times \mid \gamma \in \Gamma\}$ and $\{f_\delta \in k[[x]]^\times \mid \delta \in \Delta\}$:*

1. $\star_{B_3} \cap (\bigcap_{\delta\in\Delta} \diamond_{B_1(f_\delta)}) \cap (\bigcap_{\gamma\in\Gamma} \star_{B_2(f_\gamma)})$
2. $(\star_{B_3} \oplus \star_{B_2}) \cap (\bigcap_{\delta\in\Delta} \diamond_{B_1(f_\delta)}) \cap (\bigcap_{\gamma\in\Gamma} \star_{B_2(f_\gamma)})$
3. $(\star_{B_3} \oplus \diamond_{B_1}) \cap (\bigcap_{\delta\in\Delta} \diamond_{B_1(f_\delta)}))$

Proof. Propositions 3.0.14 and 3.0.15 yield the following implication: If $A_3(f_1, f_2)$ is $\star$-closed, then $\star$ is the identity operation. Thus, it suffices to assume that $A_3(f_1, f_2)^\star = k[[x]]$. We will consider three possible cases:

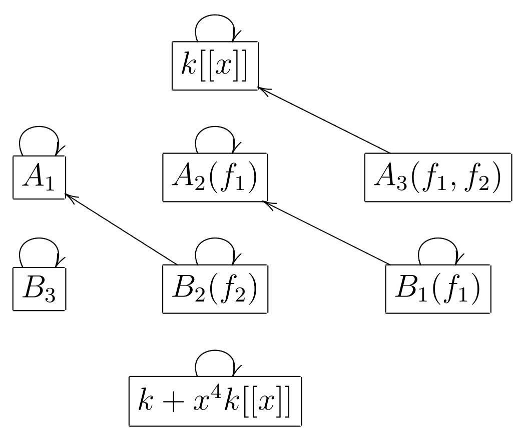

Figure 4.1: All possible star operations on $k + x^4k[[x]]$ assuming A_1 and A_2 are $\star$-closed.

1. A_1 and $A_2(f_1)$ are both $\star$-closed.

2. A_1 is $\star$-closed and $A_2(f_1)^\star = k[[x]]$.

3. $A_1^\star = k[[x]] = A_2(f_1)^\star$.

We first examine all possible star operations $\star$ such that A_1 and $A_2(f_1)$ are $\star$-closed. The possibilities in this situation are described by the diagram in 4.1.

We have one of these diagrams for each pair $f_1, f_2 \in k[[x]]^\times$. Note that in this case, B_3 must be closed because if $B_3^\star = A_1$, then $A_1 \subseteq A_2^\star$ which implies that $A_2^\star = k[[x]]$, contrary to our supposition.

Consider the star operation $\star_{B_3}$. The action of this star operation is described by the diagram of Figure 4.2.

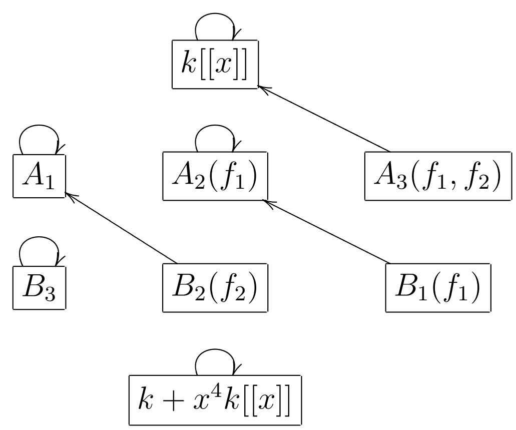

Figure 4.2: Diagram of $\star_{B_3}$.

One can see that this star operation is the supremum over all of those that satisfy the hypothesis. Given any $f_2 \in k[[x]]^\times$, we can construct a star operation that fixes $B_2(f_2)$ but does not fix $B_1(f_1)$ for any $f_1 \in k[[x]]^\times$ nor $B_2(f)$ for any $f \in k[[x]]^\times$ such that $B_2(f) \neq B_2(f_2)$, namely, $\star_{B_2(f_2)}$. The action of this star operation is

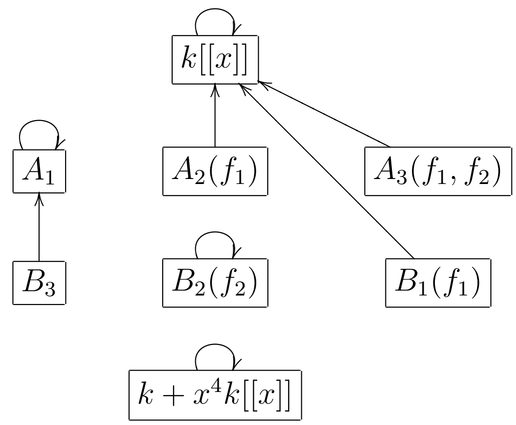

Figure 4.3: Diagram of $\star_{B_2(f_2)}$, main slice.

demonstrated in Figures 4.3 and 4.4.

Figure 4.4 demonstrates a different "slice" of the lattice.

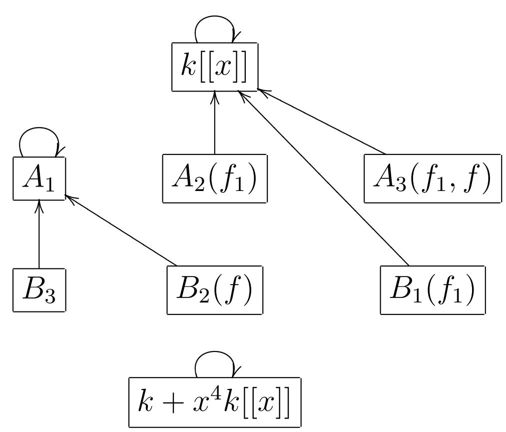

Figure 4.4: Diagram of $\star_{B_2(f_2)}$, auxiliary slice.

The intersection of $\star_{B_3}$ and $\star_{B_2(f_2)}$, namely $\star_{B_3} \cap \star_{B_2(f_2)}$, yields the diagrams

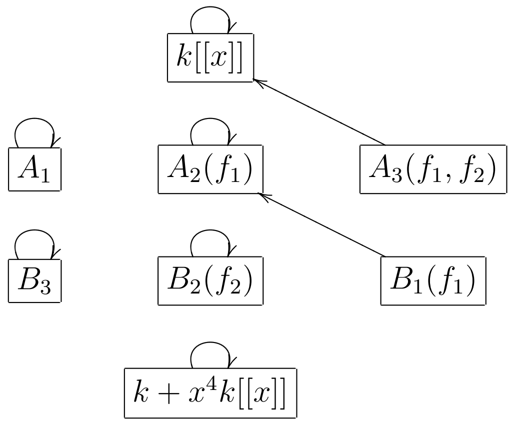

Figure 4.5: Diagram of $B_3 \cap B_2(f_2)$, main slice.

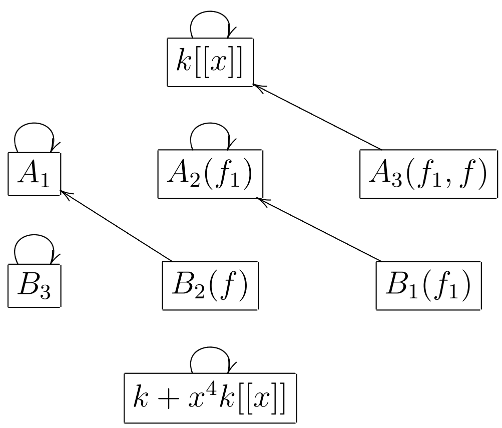

Figure 4.6: Diagram of $B_3 \cap B_2(f_2)$, auxiliary slice.

shown in Figures 4.5 and 4.6.

Figure 4.6 shows what a different ”slice” would look like.

One can see easily that if we intersect many of these star operations, then we can construct a star operation that fixes as many of the $B_2(f)$’s as we wish pushing all of the $B_1(f)$’s up and satisfying the hypotheses.

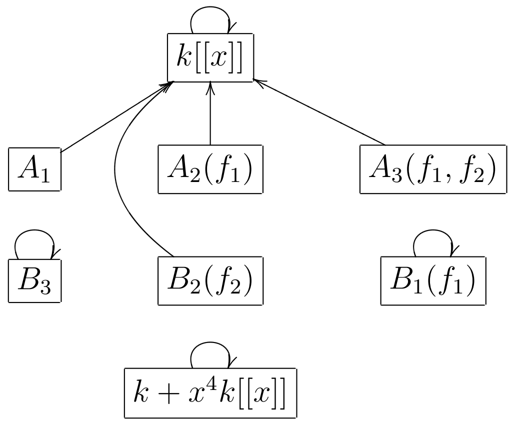

Figure 4.7: Diagram of $\star_{B_1(f_1)}$, main slice.

We now examine the star operations of the form $\diamond_{B_1(f_1)}$. The diagrams are shown in Figures 4.7 and 4.8.

If f is such that $B_1(f) \not\sim B_1(f_1)$, we obtain a different "slice" of the lattice that is shown in Figure 4.8.

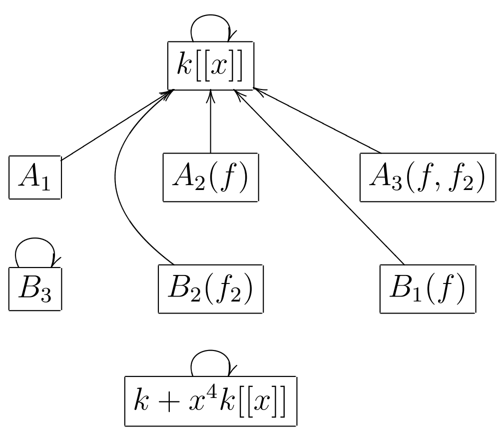

Figure 4.8: Diagram of $\star_{B_1(f_1)}$, auxiliary slice.

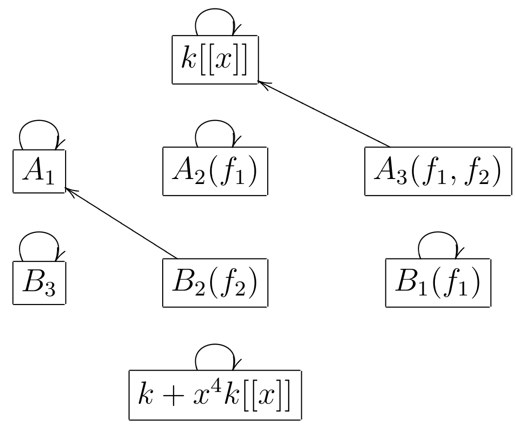

Figure 4.9: Diagram of $\star_{B_3} \cap \star_{B_1(f_1)}$, main slice.

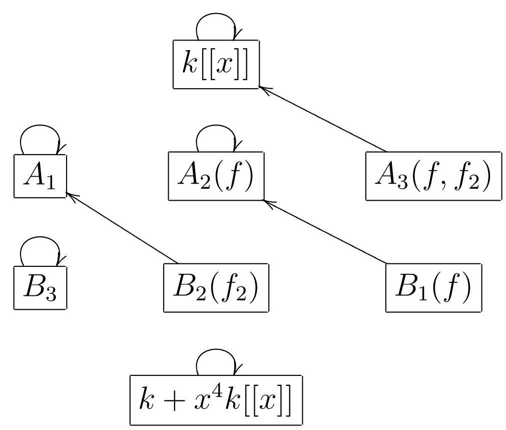

Figure 4.10: Diagram of $\star_{B_3} \cap \star_{B_1(f_1)}$, auxiliary slice.

So if we intersect this star operation with $\star_{B_3}$, namely $\star_{B_3} \cap \diamond_{B_1(f_1)}$, we obtain the diagrams in Figures 4.9 and 4.10.

A different "slice" is shown in 4.10.

We see that we can intersect these with the $\star_{B_2(f_2)}$'s to fix as many of the $B_2(f_2)$'s and $B_1(f_1)$'s as we want. This suffices to cover all possibilities in this case.

Now suppose that A_1 is $\star$-closed and that $A_2(f_1)^\star = k[[x]]$. The possibilities are described in Figure 4.11.

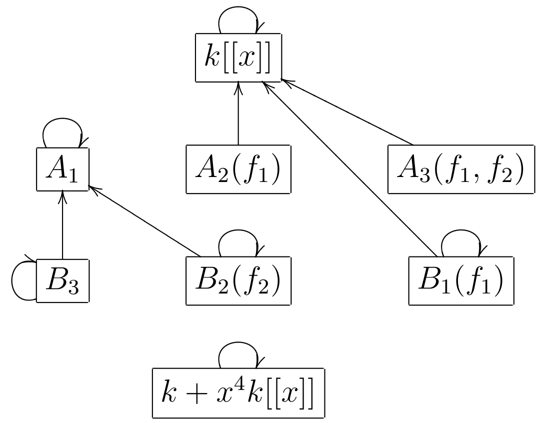

Figure 4.11: All possible actions of star operations on $k + x^4k[[x]]$ assuming that $A_1^\star = A_1$ and $A_2^\star = k[[x]]$.

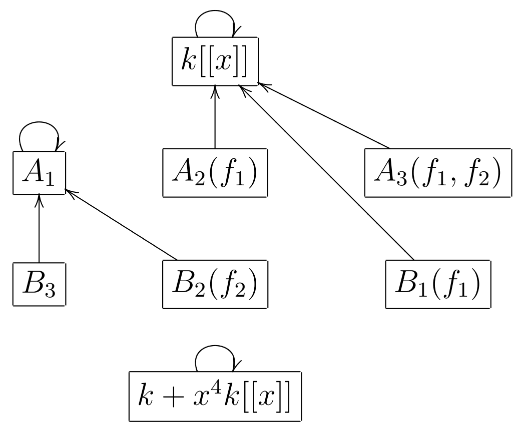

Figure 4.12: Diagram of $\star_{A_1} = \star_{B_3} \oplus \star_{B_2}$.

First consider the star operation $\star_{A_1}$. Note that $\star_{A_1} = \star_{B_3} \oplus \star_{B_2}$ which is the result of a more general phenomenon described in Proposition 6.0.2. The diagram for this star operation is exhibited in Figure 4.12.

It is clear from the diagram that this is the supremum of all star operations satisfying these hypotheses. Observing the diagram of $\star_{B_2(f_2)}$, we obtain the two diagrams for $\star_{B_2(f_2)} \cap \star_{A_1}$ shown in Figures 4.13 and 4.14.

By intersecting these we can construct star operations that fix as many of the

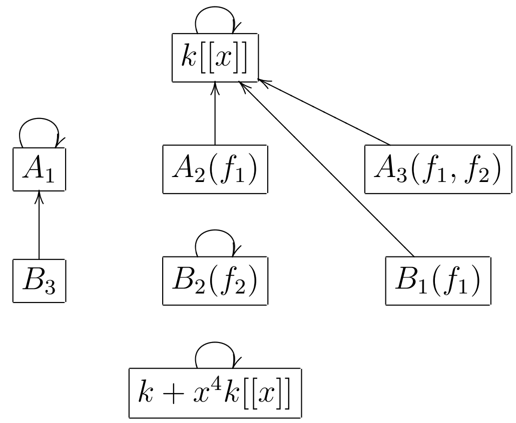

Figure 4.13: Diagram of $\star_{B_2(f_2)} \cap \star_{A_1}$, main slice.

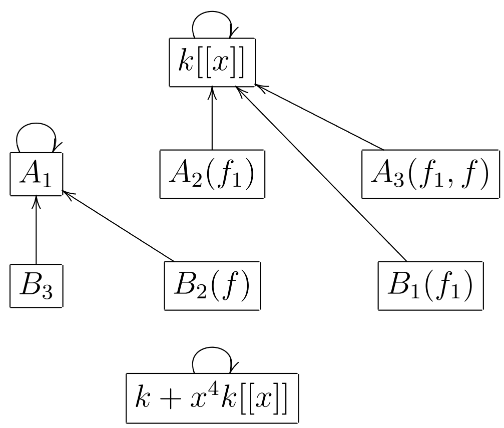

Figure 4.14: Diagram of $\star_{B_2(f_2)} \cap \star_{A_1}$, auxiliary slice.

$\star_{B_2(f_2)}$'s as we wish while simultaneously pushing B_3 and all of the $B_1(f_1)$'s up.

Now consider the star operation $\star_{B_3} \oplus \diamond_{B_1}$. The diagram is shown in Figure 4.15.

We see that this star operation pushes everything up to $k[[x]]$ except for B_3 which it fixes. Thus, if we intersect this star operation with the ones discussed above, we can fix as many of the $B_2(f_2)$'s as we want while fixing B_3 and pushing all of the $B_1(f_1)$'s up.

Recall the following implication: If $B_1(f_1)$ is $\star$-closed for any $f_1 \in k[[x]]^\times$, then so is B_3. With this fact, we see that B_3 must be $\star$-closed for the remainder of the

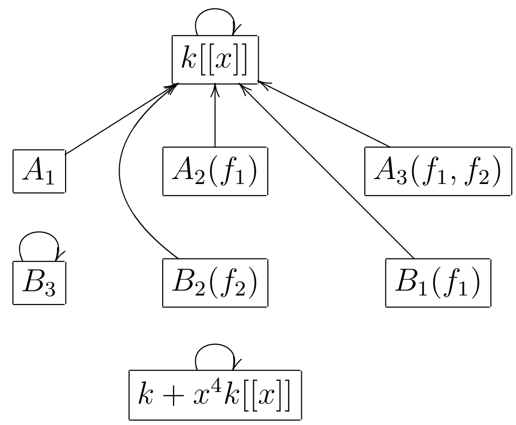

Figure 4.15: Diagram of $\star_{B_3} \oplus \star_{B_1}$.

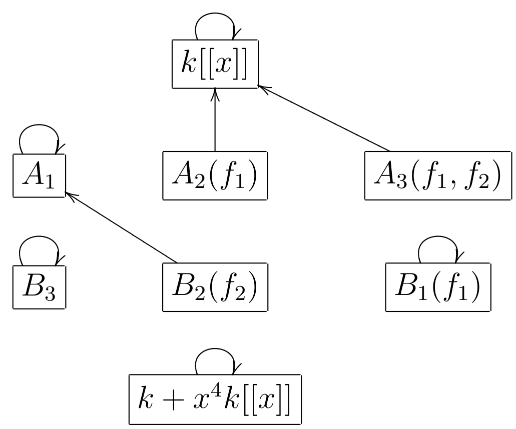

Figure 4.16: Diagram of $\star_{B_1(f_1)} \cap \star_{A_1}$, main slice.

discussion of this particular case.

Observing the diagrams for $\diamond_{B_1(f_1)}$, we see that the star operation $\diamond_{B_1(f_1)} \cap \star_{A_1}$ yields the diagrams shown in Figures 4.16 and 4.17

For f such that $B_1(f_1) \not\sim B_1(f)$ we obtain the diagram in Figure 4.17.

Intersecting these and the $\star_{B_2(f_2)}$'s, we can fix as many of the $B_2(f_2)$'s and classes $\widetilde{B_1(f_1)}$'s as we like provided that B_3 is $\star$-closed whenever any of the $B_1(f_1)$'s are.

Now suppose that $A_1^\star = k[[x]] = A_2(f_1)^\star$. Recall the following implication: If

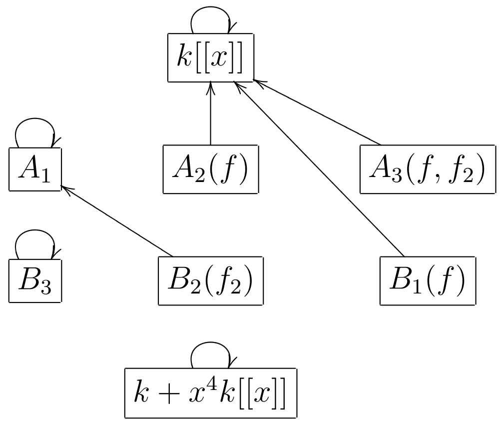

Figure 4.17: Diagram of $\star_{B_1(f_1)} \cap \star_{A_1}$, auxiliary slice.

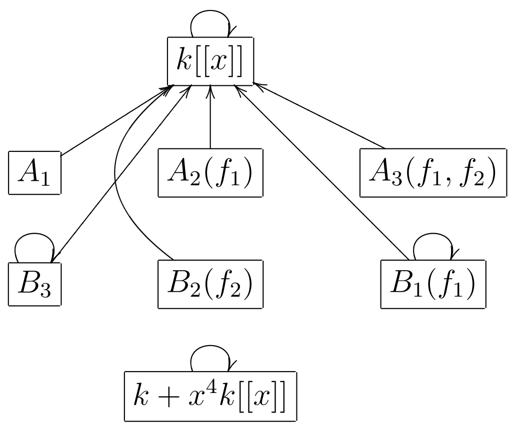

Figure 4.18: All possible actions of star operations on $k + x^4k[[x]]$ assuming that $A_1^\star = A_2^\star = k[[x]]$.

$B_2(f_2)$ is $\star$-closed, then A_1 is $\star$-closed.

With this fact in mind, we have the diagram in Figure 4.18 to describe the set of all possibilities in this case.

Note that $B_2(f_2)$ is never fixed in this scenario because of the previously discussed implication. The supremum of all such star operations is the v-operation whose diagram is shown in Figure 4.19.

We have already observed that the star operation $\star_{B_3} \oplus \diamond_{B_1}$ pushes everything

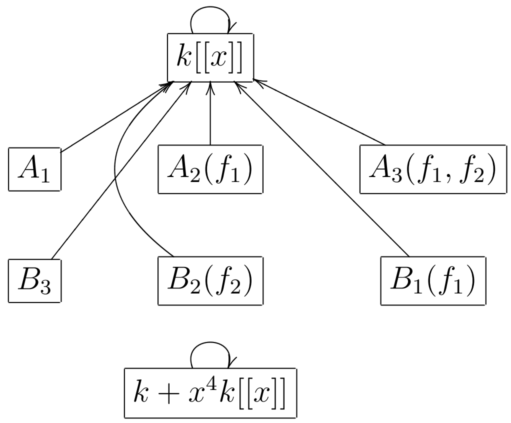

Figure 4.19: Diagram of the v-operation on $k + x^4k[[x]]$.

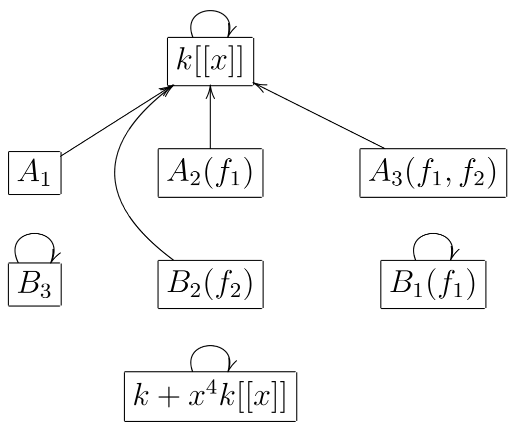

Figure 4.20: Diagram of $\star_{B_1(f_1)}$, main slice.

up to $k[[x]]$ except for B_3 which it fixes.

Recall once again the following implication:

If $B_1(f_1)$ is $\star$-closed for any $f_1 \in k[[x]]^\times$, then so is B_3. In light of this, we may assume that B_3 is $\star$-closed for the remainder of the discussion of this case. Recall the star operation $\diamond_{B_1(f_1)}$ whose diagrams are those given in 4.20 and 4.21 for f such that $B_1(f_1) \not\sim B_1(f)$.

Intersecting these, we can fix as many of the classes $\widetilde{B_1(f_1)}$'s as we like which

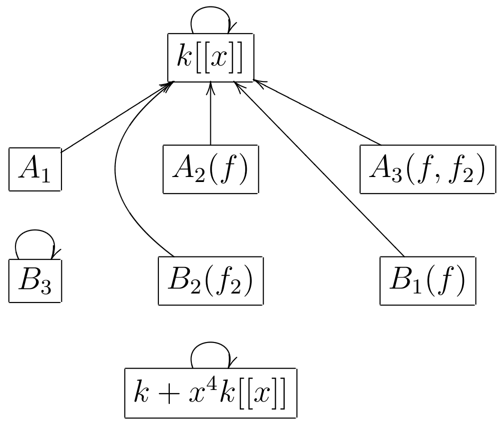

Figure 4.21: Diagram of $\star_{B_1(f_1)}$, auxiliary slice.

covers all of the remaining possibilities. We have now constructed all possible star operations on the ring $k + x^4k[[x]]$.

$\square$

It has already been shown by Houston, Mimouni and Park in [HMP2] that if $(R, \mathfrak{m})$ is a local ring with $\dim_k \mathfrak{m}^{-1}/\mathfrak{m} \geq 4$, then there are at least $\frac{1}{2}|k| + 3$ star operations on R by [HMP2, Theorem 2.8]. In light of this result, we see that if k is infinite, then there are infinitely many star operations on R. The ring $k + x^4k[[x]]$ satisfies the hypotheses of the previous proposition, and one can see that $2^{|k|}$ star operations can be constructed by intersecting, for example, operations of the form $\star_{B_2(f_2)}$. However, if k is finite, then we can use the classification of star operations on $k + x^4k[[x]]$ in Theorem 4.1.1 to count the exact number of star operations on this ring.

Corollary 4.1.2. *Suppose k is finite. Then $k + x^4k[[x]]$ has exactly*

$$2^{2|k|+1} + 2^{|k|+1} + 2 \text{ star operations.}$$

Proof. We begin by counting the star operations $\star$ such that $A_3^\star = k[[x]]$ and A_1 and A_2 are $\star$-closed. Recall that all such star operations fix B_3 and are thus completely determined by their action on the fractional ideals of the form $B_2(f_2)$ and and the equivalence classes $\widetilde{B_1(f_1)}$. Furthermore, if we let $\mathfrak{B}_1 = \{\widetilde{B_1(f_1)} \mid f_1 \in k[[x]]^\times\}$, $\mathfrak{B}_2 = \{B_2(f_2) \mid f_2 \in k[[x]]^\times\}$, and $\mathfrak{F} = \mathfrak{B}_1 \cup \mathfrak{B}_2$, then for any subset $\mathfrak{S} \subseteq \mathfrak{F}$ we can construct a star operation that fixes all of the fractional ideals in $\mathfrak{S}$ while not fixing those in $\mathfrak{F} \setminus \mathfrak{S}$. We constructed these by intersecting star operations of the form $\star_{B_3} \cap \star_{B_2(f_2)}$ and $\star_{B_3} \cap \diamond_{B_1(f_1)}$. Thus, to count all of the star operations satisfying these hypotheses, it suffices to count the number of subsets of $\mathfrak{F}$. To count the fractional ideals of the form $B_2(f_2)$, we need only count the polynomials of the form $1 + ax$ since $x^4k[[x]] \subseteq B_2(f_2)$. Similarly, to count the fractional ideal classes of the form $\widetilde{B_1(f_1)}$, it suffices to count the polynomials of the form $1 + ax + bx^2$ and divide by the number fractional ideals in each class, i.e. $|k|$. In light of this observation, we see that there are $|k|$ fractional ideals of the form $B_2(f_2)$ and $|k|^2$ fractional ideals of the form $B_1(f_1)$ giving $|k|$ classes of the form $\widetilde{B_1(f_1)}$. Thus, there are $|k| + |k| = 2|k|$ elements of $\mathfrak{F}$ yielding $2^{2|k|}$ subsets corresponding to the star operations satisfying these hypotheses.

Next we count the star operations $\star$ such that $A_3^\star = k[[x]] = A_2^\star$ and A_1 is $\star$-closed. Recall that if any fractional ideal of the form $B_1(f_1)$ is $\star$-closed, then so is B_3. In other words, if B_3 is not $\star$-closed, then neither is any fractional ideal of the form $B_1(f_1)$. We shall begin by counting the star operations satisfying these hypotheses that do not fix B_3. Under this additional assumption, we have that B_3 and the fractional ideals of the form $B_1(f_1)$ are not fixed by $\star$, i.e. $B_3^\star = A_1$, and $B_1(f_1)^\star = k[[x]]$. For any subset $\mathfrak{S} \subseteq \mathfrak{B}_2$, we can construct such a star operation that fixes all of the fractional ideals in $\mathfrak{S}$ while not fixing any of the fractional ideals in $\mathfrak{F} \setminus \mathfrak{S}$ by intersecting operations of the form $\star_{B_2(f_2)}$. To count these star operations, we need only count the subsets of $\mathfrak{B}_2$ of which there are $2^{|k|}$. Now suppose that B_3 is $\star$-closed. For any subset $\mathfrak{S} \subseteq \mathfrak{F}$, we can construct a star operation that fixes all of the fractional ideals in $\mathfrak{S}$ but does not fix any of the fractional ideals in $\mathfrak{F} \setminus \mathfrak{S}$ by intersecting operations of the form $\star_{B_2(f_2)}$ and $(\star_{B_3} \oplus \star_{B_2}) \cap \diamond_{B_1(f_1)}$. To count these star operations, we need only count the subsets of $\mathfrak{F}$ of which there are $2^{|k|+|k|} = 2^{2|k|}$. Thus, we have counted $2^{2|k|} + 2^{|k|}$ star operations $\star$ such that $A_3^\star = k[[x]] = A_2^\star$ and A_1 is $\star$-closed.

Finally, we count the star operations $\star$ such that $A_1^\star = A_2^\star = A_3^\star = k[[x]]$. Recall that if A_1 is not $\star$-closed, then neither is $B_2(f_2)$ for any $f_2 \in k[[x]]^\times$. Thus, for any star operation satisfying these hypotheses, $B_2(f_2)^\star = k[[x]]$ for all $f_2 \in k[[x]]^\times$. Suppose that $B_3^\star = k[[x]]$. Then $B_1(f_1)^\star = k[[x]]$ for every $f_1 \in k[[x]]^\times$ and we find that the only such star operation is the v-operation adding one more to our total count. Now suppose that B_3 is $\star$-closed. Then for any subset $\mathfrak{S} \subseteq \mathfrak{B}_1$, we can construct such a star operation that fixes all of the fractional ideals in $\mathfrak{S}$ while not fixing those in $\mathfrak{B}_1 \setminus \mathfrak{S}$ by intersecting operations of the form $\diamond_{B_1(f_1)}$ (or take $\star_{B_3} \oplus \star_{B_1}$ in the case that $\mathfrak{S}$ is empty). Thus, to count these it suffices to count the subsets of $\mathfrak{B}_1$ of which there are $2^{|k|}$.

The only case we have not considered is the case where A_1, A_2, and A_3 are $\star$-closed. As discussed earlier, the only such star operation is the identity operation which adds one more to our total count. If we add up all the star operations we have counted, we obtain

$$2^{2|k|} + 2^{|k|} + 2^{2|k|} + 1 + 2^{|k|} + 1 = 2^{2|k|+1} + 2^{|k|+1} + 2.$$

$\square$

4.2 Classification of Semistar Operations on $k + x^4k[[x]]$

Previously we classified all star operations on $k + x^4k[[x]]$. With little effort, we can classify all semistar operations on this ring as well. First we shall define semistar operations.

Definition 4.2.1. *Let R be a domain, K its field of fractions and $\bar{F}(R)$ the set of R-submodules of K. A star operation $\star$ is a map $\star : \bar{F}(R) \to \bar{F}(R)$ written $F \mapsto F^\star$ satisfying the following for any $F, G \in \bar{F}(R)$ and any $x \in K$.*

1. $F \subseteq F^\star$
2. *If* $F \subseteq G$, *then* $F^\star \subseteq G^\star$.
3. $(F^\star)^\star = F^\star$
4. $(xF)^\star = xF^\star$

This definition is very similar to the definition of a star operation the key differences being that a semistar operation is defined on the entire set of R-submodules of K instead of just being defined on the fractional ideals and we no longer demand that $R^\star = R$. In the case of numerical semigroup rings, $\bar{F}(R) = F(R) \cup \{k((x))\}$ where $F(R)$ is the set of fractional ideals of R. It was shown by Anderson and Anderson in (reference) that for any semistar operation $\star$, $R^\star$ is an overring of R. Thus, if we restrict $\star$ to the fractional ideals of $R^\star$, we obtain a star operation on $R^\star$. We now examine the semistar operations on $R = k+x^4k[[x]]$. Proposition 3.0.23 by Gilmer still holds when $\star$ is only assumed to be semistar so the restrictions on the actions of star operations that came from this proposition still hold for semistar operations. For example, we determined that if $B_1(f_1)$ is $\star$-closed, then so is B_3. We also have that Propositions 3.0.14 and 3.0.24 still hold in the semistar case so all of the possible actions of semistar operations on the fractional ideals not isomorphic to R have already been determined assuming that no fractional ideal is sent to a submodule not contained in $k[[x]]$. There is only one semistar operation which sends a fractional ideal to such a module and that is e defined by $I^e = k((x))$ for all submodules I as is shown in the following proposition.

Proposition 4.2.2. *Let $\star$ be a semistar operation on a numerical semigroup ring R. Suppose that $I^\star \not\subseteq k[[x]]$ for some fractional ideal I intermediate between R and $k[[x]]$. Then $\star = e$.*

Proof. We know that $R^\star$ is an overring of R so either $R^\star$ is intermediate between R and $k[[x]]$, or $R^\star = k((x))$ since there are no rings in the fraction field strictly between $k[[x]]$ and $k((x))$. If $R^\star = k((x))$, then $\star = e$. If $R^\star$ is an overring of R intermediate between R and $k[[x]]$, then $\star$ is a star operation when restricted to the fractional ideals of $R^\star$. Thus $k[[x]]$ is $\star$-closed and so $I^\star \subseteq k[[x]]$ for all I intermediate between R and $k[[x]]$. □

From this proposition we can conclude that every semistar operation other than e on $R = k + x^4k[[x]]$ will coincide with some star operation on the non-principal fractional ideals. Thus we shall examine the semistar operations by considering their actions on R. We begin by constructing some semistar operations that are not star. Previously, we had constructed star operations from overrings in the following way. If R' is an overring of R, define $\star_{R'}$ by $I^{\star_{R'}} = IR' \cap I^v$ for every fractional ideal I. We shall define the semistar operation $\bar{\star}_{R'}$ by $I^{\bar{\star}_{R'}} = IR'$. In particular, this semistar operation sends R to R' so if $R \neq R'$, then $\bar{\star}_{R'}$ is not star. We will need one more construction to complete our list of semistar operations on $k + x^4k[[x]]$. If R' is an overring of R and $\star$ a semistar operation on R', then $\star(\bar{\star}_{R'})$ defined by $I^{\star(\bar{\star}_{R'})} = (IR')^\star$ is a semistar operation on R. The following proposition classifies all semistar operations on $k + x^4k[[x]]$.

Proposition 4.2.3. *Every semistar operation on $k + x^4k[[x]]$ is a star operation or is of one of the following forms:*

1. e

2. $\bar{\star}_{B_2(f_2)}$

3. $\bar{\star}_{B_3}$

4. $\bar{\star}_{B_3} \oplus \diamond_{B_1}$

5. $\star_{A_1}(\bar{\star}_{B_3})$

6. $\bar{\star}_{A_1}$

7. $\bar{\star}_{k[[x]]}$

Proof. Suppose $R^\star = k[[x]]$. Then $\star = \bar{\star}_{k[[x]]}$.

Suppose $R^\star = A_1$. Then $A_1^\star \subseteq I^\star$ for every fractional ideal I intermediate between R and $k[[x]]$. Thus $A_2^\star \supseteq A_2 + A_1 = k[[x]]$ and so $A_2^\star = A_3^\star = k[[x]]$. We also have $A_1 \subseteq B_3^\star \subseteq A_1^\star = A_1$ and similarly for $B_2(f_2)$. Finally, $B_1(f_1)^\star \supseteq B_1(f_1)+A_1 = k[[x]]$. Thus $\star = \bar{\star}_{A_1}$.

Suppose that $R^\star = B_2(f_2)$. Then $B_2(f_2)$ is $\star$-closed which implies that A_1 is $\star$-closed. If $f \in k[[x]]^\times$ is such that $B_2(f) \neq B_2(f_2)$, then $A_1 = B_2(f) + B_2(f_2) \subseteq B_2(f)^\star \subseteq A_1$. Similarly, $B_3^\star = A_1$. We also have that $A_2^\star \supseteq A_2 + B_2(f_2) = k[[x]]$ so $A_2^\star = A_3^\star = k[[x]]$. Finally $B_1(f_1)^\star \supseteq B_2(f_2)$ so $B_1(f_1)$ is not $\star$-closed which implies that $B_1(f_1)^\star = A_2(f_1)$ or $B_1(f_1)^\star = k[[x]]$. Since A_2 is not $\star$-closed, we must have $B_1(f_1) = k[[x]]$. Thus $\star = \bar{\star}_{B_2(f_2)}$.

Suppose $R^\star = B_3$. We have three cases under this assumption.

1. $A_1^\star = k[[x]] = A_2^\star$.

2. $A_1^\star = A_1$ and $A_2^\star = k[[x]]$.

3. $A_1^\star = A_1$ and $A_2^\star = A_2$.

If $A_1^\star = k[[x]] = A_2^\star$, then $B_2(f_2)^\star = k[[x]]$ since A_1 is not $\star$-closed. Similarly, $B_1(f_1)^\star = k[[x]]$. Thus $\star = \bar{\star}_{B_3} \oplus \diamond_{B_1}$.

Now suppose that $A_1^\star = A_1$ and $A_2^\star = k[[x]]$. Then $B_3 \subseteq B_2(f_2)^\star \subseteq A_1$ and so $B_2(f_2)^\star = A_1$. Also, $B_3 \subseteq B_1(f_1)^\star$ which implies that $B_1(f_1)^\star = k[[x]]$ since A_2 is not $\star$-closed. Thus, $\star = \star_{A_1}^{B_3}$.

Finally, suppose that $A_1^\star = A_1$ and $A_2^\star = A_2$. Then $B_3 \subseteq B_2(f_2)^\star \subseteq A_1$ and $B_3 \subseteq B_1(f_1)^\star \subseteq A_2(f_1)$ and so $B_2(f_2)^\star = A_1$ and $B_1(f_1)^\star = A_2(f_1)$. Thus $\star = \bar{\star}_{B_3}$. $\square$

Chapter 5

Star and Semistar Operations on $k + x^5k[[x]]$

5.1 Classification of Semistar Operations on $k + x^5k[[x]]$

The classification of all star operations on $k + x^5k[[x]]$ is a work in progress but the semistar operations on $k + x^5k[[x]]$ which are not star are classified by Proposition 5.1.2. Before we classify these, we need the following proposition.

Proposition 5.1.1. *The ring $k[[x^3 + ax^4, x^5, x^7]]$ has exactly four star operations. The identity, the v-operation, $\star_{B_4}$, and $v(\star_{\bar{B}_4}) \cap v$.*

This was proven in [HMP2] where $a = 0$. The proof easily extends to the slightly more general case. The diagram of the lattice of intermediate fractional ideals is given in Figure 5.1.

This diagram simplifies, however, under the observation that $< 1, x >=$

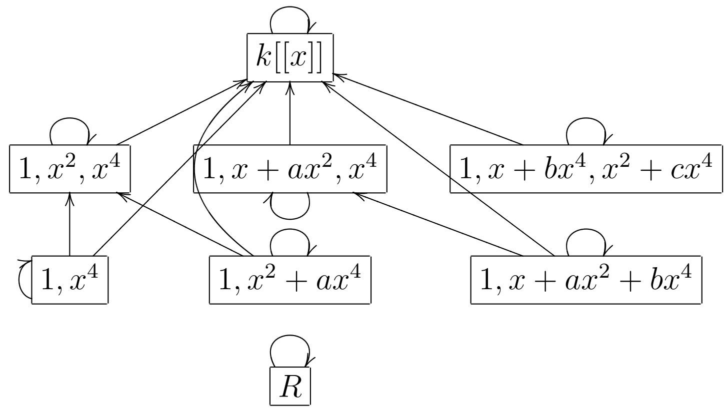

Figure 5.1: All possible actions of star operations on $R = k[[x^3, x^5, x^7]]$.

$< 1, x, x^4 >$ and $< 1, x, x^2 >= k[[x]]$ as shown in Figure 5.2.

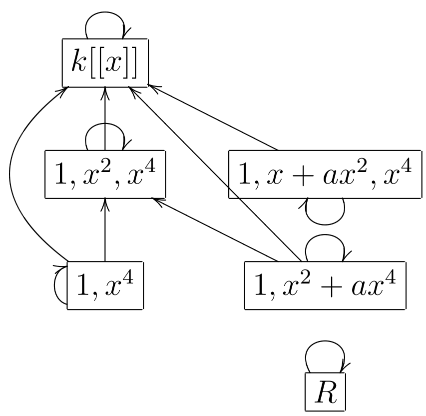

Figure 5.2: All possible actions of star operations on $R = k[[x^3, x^5, x^7]]$ revised.

Proposition 5.1.2. *Every semistar operation on $k + x^5k[[x]]$ is a star operation or is of one of the following forms where $\star_4$ is any of the star operations described in Theorem 4.1.1.*

1. e

2. $\bar{\star}_{B_3(f_3)}$

3. $v(\bar{\star}_{B_3(f_3)})$

4. $v(\bar{\star}_{B_3(f_3)}) \cap \star_{B_{(3,4)}}(\bar{\star}_{B_3(f_3)})$

5. $\star_{A_1}(\bar{\star}_{B_3(f_3)})$

6. $\star_4(\bar{\star}_{B_4})$

7. $\bar{\star}_{A_1}$

8. $\bar{\star}_{k[[x]]}$

Proof. Suppose $R^\star = k[[x]]$. Then $\star = \bar{\star}_{k[[x]]}$.

Suppose $R^\star = A_1$. Then $A_1^\star \subseteq I^\star$ for every fractional ideal I intermediate between R and $k[[x]]$. Thus $A_2^\star \supseteq A_2 + A_1 = k[[x]]$ and so $A_2^\star = A_3^\star = k[[x]]$. We also have $A_1 \subseteq B_3^\star \subseteq A_1^\star = A_1$ and similarly for $B_2(f_2)$. Finally, $B_1(f_1)^\star \supseteq B_1(f_1) + A_1 = k[[x]]$. Thus $\star = \bar{\star}_{A_1}$.

Suppose $R^\star = B_4$. Then $\star$ coincides with a star operation on $B_4 = k + x^4k[[x]]$ which were given in Theorem 4.1.1.

Suppose $R^\star = B_3(f_3)$. Then $\star$ coincides with a star operation on $B_3(f_3)$ as discussed previously.

Suppose $R^\star = B_{(3,4)}$. Then $\star$ coincides with a star operation on $B_{(3,4)}$ all three of which were determined by Houston, Mimouni and Park in [HMP2].

□

5.2 Star Operations on $k + x^5k[[x]]$

The star operations on $k + x^5k[[x]]$ are still not yet classified. However, we at least have the following constructions.

1. $\star_{A_1}$
2. $\star_{B_{(3,4)}}$
3. $\star_{B_{(2,4)}(f_2)}$
4. $\star_{B_4}$
5. $\star_{B_3(f_3)}$
6. $\diamond_{B_1(f_1)}$
7. $\diamond_{B_2(f_2)}$
8. $\diamond_{B_{(1,2)}(f_1,f_2)}$
9. $\diamond_{B_{(1,3)}(f_1,f_3)}$

In Example 3.0.20, we saw how Proposition 3.0.19 becomes useful. We examine a very similar example here.

Example 5.2.1. Consider $R = k + x^5k[[x]]$. We have $B_4 \subseteq A_{(1,2)},\ A_{(1,3)},\ A_{(2,3)}$ but $B_4^\star \neq A_{(2,3)}$ since $B_4^\star$ must be a ring by Proposition 3.0.19 while $A_{(2,3)}$ is not a ring.

If we put Propositions 3.0.14, 3.0.15, and 3.0.19, together with Corollary 3.0.17, we can eliminate many possible actions of star operations immediately. The diagram in Figure 5.3 describes the possible actions of star operations on the monomial fractional ideals intermediate between $k + x^5k[[x]]$ and $k[[x]]$ in light of Propositions 3.0.14, 3.0.15, 3.0.19 and Corollary 3.0.17. This suffices to represent the possible actions on all such fractional ideals. We omit the arrows indicating the possibility of each fractional ideal being fixed by the star operation and denote $R = k + x^5k[[x]]$.

Here we have seen that many possible actions of star operations on the set of fractional ideals intermediate between R and $k[[x]]$ have been eliminated. We can

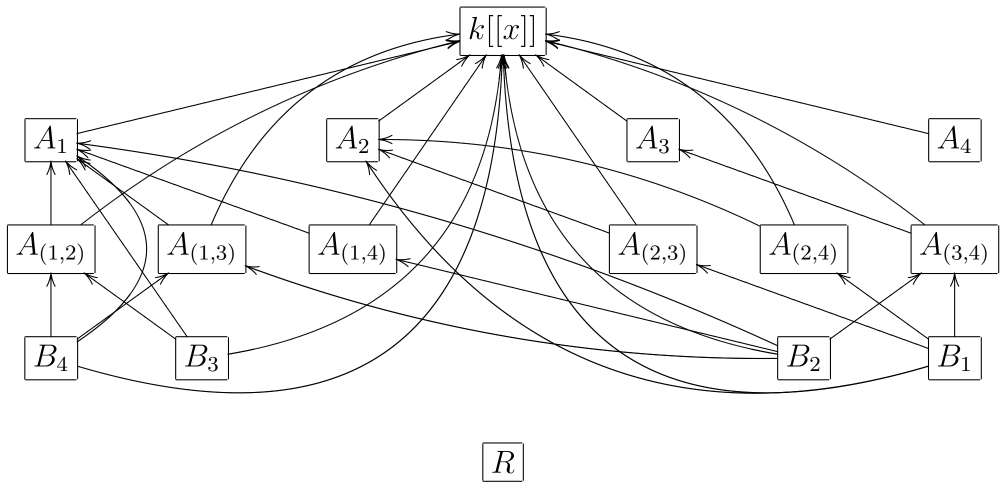

Figure 5.3: All possible star operations on $k + x^5k[[x]]$.

eliminate some more possibilities with Proposition 3.0.24. That is, for any $f_1, f_2, f_3 \in k[[x]]^\times$ and any star operation $\star$ on R, we have that

1. If $B_1(f_1)$ is $\star$-closed, then so is B_4.
2. If $B_2(f_2)$ is $\star$-closed, then so is $A_{(1,2)}$.
3. If $B_3(f_3)$ is $\star$-closed, then so is A_1.

As far as constructing star operations, we already have $\star_{B_4}$, $\star_{B_3(f_3)}$, $\diamond_{B_2(f_2)}$, and $\diamond_{B_1(f_1)}$. We also have $\star_{B_{(2,4)}}$ as discussed earlier. In fact we have $\star_{B_{(2,4)}(f_2)}$. We also have $\star_{B_{(3,4)}}$ although we can construct this by taking $\star_{B_3} \oplus \star_{B_4}$ and similarly we have $\star_{A_1} = \star_{B_3} \oplus \star_{B_{(2,4)}}$. We can also construct more "diamond" operations as we have that $(A_{(2,4)}(f_1, f_3) : A_{(2,4)}(f_1, f_3)) = R$ and likewise $(A_{(3,4)}(f_1, f_2) : A_{(3,4)}(f_1, f_2)) = R$ yielding the operations $\diamond_{A_{(2,4)}(f_1,f_3)}$ and $\diamond_{A_{(3,4)}(f_1,f_2)}$ respectively. The remaining fractional ideals fail to generate star operations since they are neither rings nor satisfy $(F : F) = R$ except for A_4. We have that $(A_4 : A_4) = R$ which yields a $\diamond_{A_4}$, however, we have that $A_4^{\diamond_{A_4}} = A_4$ which implies that $\diamond_{A_4}$ is the identity operation by Propositions 3.0.14 and 3.0.15. This phenomenon also occurred in the case of $k+x^4k[[x]]$ with A_3. In fact, for $k+x^nk[[x]]$, we have that $(A_{n-1} : A_{n-1}) = k+x^nk[[x]]$ and that $\diamond_{A_{n-1}}$ is the identity operation.

The actions of the $\star$ constructions are fairly obvious but it can be difficult to determine what the $\diamond$ operations will do, especially $\diamond_{B_{(1,2)}(f_1,f_2)}$, and $\diamond_{B_{(1,3)}(f_1,f_3)}$. For this reason, we utilize the computer program Macaulay 2 [GS] to gather some data about these star operations for specific fields k. The following data were collected on the action of $\diamond_{B_{(1,2)}}$ on fractional ideals of the form $B_1(f_1)$ setting $k = \mathbb{Z}/2\mathbb{Z}$.

1. $B_1^{\diamond_{B_{(1,2)}}} = B_1$
2. $B_1(1+x+x^2+x^3)^{\diamond_{B_{(1,2)}}} = B_1(1+x+x^2+x^3)$
3. $B_1(1+x)^{\diamond_{B_{(1,2)}}} = B_{(1,2)}$
4. $B_1(1+x^2+x^3)^{\diamond_{B_{(1,2)}}} = B_{(1,2)}(1+x^2+x^3, 1+x)$
5. $B_1(1+x^2)^{\diamond_{B_{(1,2)}}} = B_{(1,2)}(1+x^2, 1+x^2)$
6. $B_1(1+x+x^3)^{\diamond_{B_{(1,2)}}} = B_{(1,2)}(1+x^2+x^3, 1+x)$
7. $B_1(1+x+x^2)^{\diamond_{B_{(1,2)}}} = B_{(1,4)}(1+x+x^2)$
8. $B_1(1+x^3)^{\diamond_{B_{(1,2)}}} = B_{(1,4)}$

Chapter 6

Future Work

Here we have classified all star operations on the ring $k + x^4k[[x]]$, but the long term goal of this line of inquiry is to classify all star operations on general numerical semigroup rings. In the case of $k + x^4k[[x]]$, we could construct all of the star operations with $\star_{B_3}$, $\star_{B_2(f_2)}$, and $\diamond_{B_1(f_1)}$. In the general case of $k+x^nk[[x]]$, we have that $B_j(f_j)$ is a ring if $j \geq \frac{n}{2}$ and $(B_j(f_j) : B_j(f_j)) = k + x^nk[[x]]$ if $0 < j < \frac{n}{2}$. Thus, $\star_{B_j(f_j)}$ is defined for each j with $\frac{n}{2} \leq j < n$ and every $f_j \in k[[x]]^\times$ and similarly $\diamond_{B_j(f_j)}$ is defined for each j with $0 \leq j < \frac{n}{2}$ and every $f_j \in k[[x]]^\times$. It is reasonable to conjecture that every star operation on $k + x^nk[[x]]$ can be constructed from these in a fashion similar to Theorem 4.1.1. However, in the case of $R = k + x^5k[[x]]$, we have $\star_{B_{(2,4)}}$ which cannot be constructed from $\star_{B_4}$, $\star_{B_3(f_3)}$, $\diamond_{B_2(f_2)}$, and $\diamond_{B_1(f_1)}$ since $B_4^{\star_{B_{(2,4)}}} = B_{(2,4)}$ while the only star operation out of $\star_{B_4}$, $\star_{B_3(f_3)}$, $\diamond_{B_2(f_2)}$, and $\diamond_{B_1(f_1)}$ that fix $B_{(2,4)}$ is $\star_{B_4}$ which also fixes B_4. The immediate next goal is to classify all the star operations on $k + x^5k[[x]]$. We can already see that, for example, $\star_{B_{(3,4)}} = \star_{B_3} \oplus \star_{B_4}$. In fact this observation can be made general by the following proposition.

Proposition 6.0.2. *Let R be a conductive numerical semigroup ring and let*

$\frac{n}{2} \leq i_1 < i_2 < n$. *Then*

$$\star_{B_{(i_1,i_2)}(f_{i_1},f_{i_2})} = \star_{B_{i_1}(f_{i_1})} \oplus \star_{B_{i_2}(f_{i_2})}.$$

Proof. To show this, we need only show that for any overring R' intermediate between R and $k[[x]]$ and any ideal I of R, $IR' \subseteq I^v$. Proposition 2.2.4 gives us that $I^v = x^{\mathrm{ord}(I)}k[[x]]$. Since $R' \subseteq k[[x]]$, $\mathrm{ord}(IR') = \mathrm{ord}(I)$ which implies that $IR' \subseteq I^v$. □

One might wonder what conditions a fractional ideal of this type are necessary or sufficient for it to be a ring. One condition we have is that if $R = k + x^n k[[x]]$, then $B_{(i_1,\ldots,i_\mu)}$ is a ring if and only if $\{i_1, \ldots, i_\mu\} \cup n + \mathbb{N}$ is a numerical semigroup. It would be nice, however, to find some conditions that were easier to check in a general setting. We would also like to find necessary or sufficient conditions for one of these fractional ideals to satisfy $(F : F) = R$. We can easily see that if F is one of these fractional ideals, then F is a ring if and only if $(F : F) = F$, and so F cannot be a ring and satisfy $(F : F) = R$ unless $F = R$. More investigation will hopefully reveal more widely applicable conditions. We saw how Proposition 3.0.23 yielded Proposition 3.0.24. Proposition 3.0.23 can give us even more information. For example, in the case where $R = k + x^5k[[x]]$, we have that $(B_{(2,3)} : B_4) = x^2k + x^3k + x^5k[[x]]$ which is isomorphic to A_2 implying that if $\star$ is a star operation on R, then A_2 is $\star$-closed if $B_{(2,3)}$ is. It seems likely that there is some phenomenon at work here that is a generalization of Proposition 3.0.24. We ask two final question: Are all star operations on conductive numerical semigroup rings able to be constructed from these "star" and "diamond" constructions using intersections and the $\oplus$ operation? What are the minimal sets of star operations necessary to construct all of them in this fashion? Hopefully, these questions will be answered upon further investigation.

Appendices

Appendix A

Macaulay 2 Code

In Chapter 3.0.20, we discussed the use of the computer program Macaulay 2 to compute the $\star$-closures of ideals. Macaulay 2 has a built in function for computing the ideal $(I : J)$ where I and J are ideals of the ring R. The difficult part of using Macaulay 2 to do this in the ring $R = k + x^5k[[x]]$ is constructing the ring within the confines of the Macaulay 2 environment. Actually, Macaulay 2 does not yet have a package for formal power series rings but as shown in Chapter 3.0.20, the ring $k + x^5k[x]$ will suffice for our purposes. Macaulay 2 is limited to quotients of polynomial rings so to construct $k + x^5k[x]$, we must represent this ring as a quotient of $k[x_1, x_2, x_3, x_4, x_5]$. In other words, we must produce sufficient relations among the variables to produce a ring that is isomorphic to the one desired. The following proposition establishes such an isomorphism writing $k[x_1, x_2, x_3, x_4, x_5]$ as $k[x_5, x_6, x_7, x_8, x_9]$ to indicate which variable is to correspond to which power of x.

Proposition A.0.3. *The ring $R = k+x^5k[[x]]$ (resp. $R = k+x^5k[x]$) is isomorphic to the ring $S = k[[x_5, x_6, x_7, x_8, x_9]]/I$ (resp. $S = k[x_5, x_6, x_7, x_8, x_9]/I$) where*

$$I = (x_5x_7 - x_6^2, x_5x_8 - x_6x_7, x_5x_9 - x_6x_8, x_5x_9 - x_7^2, x_5^3 - x_7x_8, x_5^3 - x_6x_9,$$
$$x_5^2x_6 - x_7x_9, x_5^2x_6 - x_8^2, x_5^2x_7 - x_8x_9, x_5^2x_8 - x_9^2).$$

Proof. First note that the same proof works for both the polynomial rings and the formal power series rings so we'll just prove it for the power series rings. The map we claim to be an isomorphism is $\phi : S \to R$ defined by $\phi(a) = a$ for all $a \in k$ and $\phi(x_i) = x^i$. Define the map $\tilde{\phi} : k[[x_5, x_6, x_7, x_8, x_9]] \to k + x^5 k[[x]]$ in the same way as ϕ. It is easy to see that $\tilde{\phi}$ is a surjective homomorphism. It is also not difficult to check that $I \subseteq ker\tilde{\phi}$ and so the map ϕ is well-defined. It remains to be shown that ϕ is injective. This breaks down to a semigroup-theoretic argument. The map $\tilde{\phi}$ induces the obvious analog from $\mathbb{N}_0^5$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) to $< 0, 5, 6, 7, 8, 9 >$. We simply need to show that for any $n \in < 5, 6, 7, 8, 9 >$, every $\mathbb{N}_0$-linear combination of $5, 6, 7, 8$ and 9 yielding n can be reduced to one particular such linear combination using the relations analogously obtained from I. For instance $5, 6, 7, 8, 9, 10$ and 11 each only have one such $\mathbb{N}_0$-linear combination representing them. However, $5 + 7 = 12$ and $2 \cdot 6 = 12$ so that $\tilde{\phi}$ maps two elements onto x^{12}, namely x_5x_7 and x_6^2. However, since $x_5x_7 - x_6^2$ is a generator of I, ϕ only maps one element to x^12. We will proceed by showing that all possible $\mathbb{N}_0$-linear combinations are equivalent under the relations given by I for each natural number from 12 to 19. Then we will use the fact that the conductor of the semigroup $< 5, 6 >$ is 20 to conclude that every natural number greater than or equal to 20 can be written as an $\mathbb{N}_0$-linear combination of 5 and 6. Then we will show that the relations given by I are sufficient to rewrite any $\mathbb{N}_0$-linear combination representing one of these numbers in the form $n \cdot 5 + i \cdot 6$ where $n \in \mathbb{N}$ and $i \in \{0, 1, 2, 3, 4\}$. We have already taken care of 12. For 13 we have $5 + 8 = 13$ and $6 + 7 = 13$ which are identified by the relation $x_5x_8 - x_6x_7$. 14 can be written as $5 + 9$, $6 + 8$, or $2 \cdot 7$ which are covered by the relations $x_5x_9 - x_6x_8$ and $x_5x_9 - x_7^2$. We have $15 = 3 \cdot 5 = 6 + 9 = 7 + 8$ which are covered by the relations $x_5^3 - x_6x_9$ and $x_5^3 - x_7x_8$. For 16 we have $2 \cdot 5 + 6 = 7 + 9 = 2 \cdot 8$ which are identified by the relations $x_5^2x_6 - x_7x_9$ and $x_5^2x_6 - x_8^2$.

For the numbers 17 through 19, we can partially reduce the problem to one of the previous numbers. For example, we have that $17 = 5 + 12 = 6 + 11 = 7 + 10 =$

$8+9$. We already have that 10 and 11 each have only one $\mathbb{N}_0$-linear combination representing them. We found two for 12 and a relation from I identifying them so we can multiply this relation by x_5 to identify the two $\mathbb{N}_0$-linear combinations of the form $5+12$ representing 17. Thus we need only find relations identifying one of the $\mathbb{N}_0$-linear combinations of the form $5+12$ to one of each of the other forms. Of course $6+11=6+6+5=5+12$ so the first and second type are really the same. We identify the last two $\mathbb{N}_0$-linear combinations with the relation $x_5^2x_7-x_8x_9$ and we have $7+10=7+5+5=12+5$ so this one was of the first type in the first place.

We have $18=5+13=6+12=7+11=8+10=2\cdot 9$. We have that $8+10=8+5+5=13+5$ equating these two types of $\mathbb{N}_0$-linear combinations and similarly $7+11=7+6+5=13+5$. We also have $6+12=6+7+5=13+5$ so we need only identify $2\cdot 9$ to one of these which we do with the relation $x_5^2x_8-x_9^2$. For 19 we have $19=5+14=6+13=7+12=8+11=9+10$. We have $9+10=9+5+5=14+5$, $8+11=8+6+5=14+5$, $7+12=7+7+5=14+5$, and $6+13=6+8+5=14+5$ which equates all of these $\mathbb{N}_0$-linear combinations.

Every number from here can be written as an $\mathbb{N}_0$-linear combination of 5 and 6. We also obtain $x_6^5=x_5^6$ from $x_6^5=x_6^2x_6^3=x_5x_7x_5x_7x_6=x_5^2x_7x_8x_5=x_5^3x_5^3=x_5^6$. Thus we have sufficient relations to reduce any $\mathbb{N}_0$-linear combination of 5 and 6 to one where the 6 coefficient is from 0 to 4. Suppose we have $n_5\cdot 5+n_6\cdot 6+n_7\cdot 7n_8\cdot 8n_9\cdot 9$ an $\mathbb{N}_0$-linear combination of a number that is at least 20. Since we have the relation $x_5^2x_8-x_9^2$, we may assume that $n_9=1$ or $n_9=0$. We also have $x_5^2x_6-x_8^2$ which allows us to assume that $n_8=1$ or $n_8=0$. We also have the relation $x_5x_9-x_7^2$ so if we reduce our $\mathbb{N}_0$-linear combination with this relation first we may also assume that $n_7=1$ or $n_7=0$. Suppose $n_7=n_8=n_9=1$. Then we have the relations $x_7x_9=x_5^2x_6$ and $x_5^2x_8=x_6^3$ so we have reduced this to an $\mathbb{N}_0$-linear combination of 5 and 6. Suppose that $n_7=n_8=1$ and $n_9=0$. Then the relation $x_7x_8=x_5^3$ allows us to reduce this to an $\mathbb{N}_0$-linear combination of 5 and 6. Suppose $n_7=n_9=1$ and $n_8=0$. Then we have $x_7x_9 = x_5^2x_6$ reducing this to an $\mathbb{N}_0$-linear combination of 5 and 6. Suppose $n_8 = n_9 = 1$ and $n_7 = 0$. Then $x_8x_9 = x_5^2x_7 = x_5x_6^2$ allowing this to be written as an $\mathbb{N}_0$-linear combination of 5 and 6. Suppose that $n_7 = 1$ and $n_8 = n_9 = 0$. Since the number is assumed to be at least 20, $n_5 + n_6 \geq 3$. If $n_5 \geq 1$ then we have $x_5x_7 = x_6^2$. If $n_5 = 0$, then $n_6 \geq 4$ so we have $x_6^3x_7 = x_6^2x_5x_8 = x_6x_5x_5x_9 = x_5^2x_6x_9 = x_5^2x_5^3$. Thus we can reduce this to an $\mathbb{N}_0$-linear combination of 5 and 6. Suppose $n_8 = 1$ and $n_7 = n_9 = 0$. Then $n_5 + n_6 \geq 2$. If $n_6 = 0$ then $n_5 \geq 2$ so $x_5^2x_8 = x_5x_6x_7 = x_6x_6^2$. If $n_5 = 0$ then $n_6 \geq 2$ and so $x_6^2x_8 = x_6x_5x_9 = x_5x_5^3$. If $n_6 = 1$ then $n_5 \geq 2$ and if $n_5 = 1$ then $n_6 \geq 2$. Thus this can be reduced to an $\mathbb{N}_0$-linear combination of 5 and 6. Suppose that $n_9 = 1$ and $n_7 = n_8 = 0$. If $n_6 \geq 1$ then $x_6x_9 = x_5^3$. If $n_6 = 0$ then $n_5 \geq 3$ so $x_5^3x_9 = x_5^2x_7^2 = x_6^4$. Thus we can reduce this to an $\mathbb{N}_0$-linear combination of 5 and 6. $\square$

This proposition justifies us in using the following code to represent $k + x^5k[x]$ in Macaulay 2.

```
R=ZZ/2[x_5,x_6,x_7,x_8,x_9,Degrees=>{5,6,7,8,9}]
I=ideal(x_5*x_7-x_6^2, x_5*x_8-x_6*x_7, x_5*x_9-x_6*x_8,
x_5*x_9-x_7^2, x_5^3-x_7*x_8, x_5^3-x_6*x_9,
x_5^2*x_6-x_7*x_9, x_5^2*x_6-x_8^2, x_5^2*x_7-x_8*x_9,
x_5^2*x_8-x_9^2)
S=R/I
```

Note that we may replace $\mathbb{Z}/2\mathbb{Z}$ with any field Macaulay 2 is programmed with including $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or any finite field achievable within the physical constraints of the hardware. Once we have constructed the desired ring in Macaulay 2, we may proceed to compute $(I : J)$ but we must be careful since Macaulay 2's built in colon function works over the ring itself, not the ambient fraction field (or total ring of fractions if the ring is not a domain). To account for this, we use the fact that

$(aI : bJ) = ab^{-1}(I : J)$. For example, if we wish to compute $(B_{(1,2)} : B_1)$, we compute $(x^{10}B_{(1,2)} : x^5B_1) = x^5(B_{(1,2)} : B_1)$ since $(B_{(1,2)} : B_1) \subseteq k[[x]]$ which implies that $x^5(B_{(1,2)} : B_1) \subseteq x^5k[[x]] \subseteq k + x^5k[[x]]$. For example, if we wish to compute $B_1^{\diamond_{B_{(1,2)}}}$, we compute $(x^{10}B_{(1,2)} : (x^{10}B_{(1,2)} : x^5B_1)) = x^{10}(B_{(1,2)} : x^5(B_{(1,2)} : B_1)) = x^5(B_{(1,2)} : (B_{(1,2)} : B_1)) = x^5B_1^{\diamond_{B_{(1,2)}}}$. We see what this computation yields in the following example.

Example A.0.4. *To compute $x^5B_1^{\diamond_{B_{(1,2)}}}$, we define the following two ideals in R.*

```
Bd_12=ideal(x_5^2,x_5*x_6,x_5*x_7)
B_1=ideal(x_5,x_6)
```

This defines the ideals $x^{10}B_{(1,2)}$ and x^5B_1. Now we compute $x^5B_1^{\diamond_{B_{(1,2)}}}$ as follows.

```
input:Bd_12:(Bd_12:B_1)
output:ideal(x_5,x_6)
```

In other words, $x^5B_1^{\diamond_{B_{(1,2)}}} = x^5B_1$ and so $B_1^{\diamond_{B_{(1,2)}}} = B_1$ so B_1 is $\diamond_{B_{(1,2)}}$-closed.

The following code defines all fractional ideals intermediate between $k + x^5k[[x]]$ and $k[[x]]$ for $k = \mathbb{Z}/2\mathbb{Z}$ multiplied by x^5 (except for $A_i(f_{(1,\ldots,i-1)})$).

```
B_1=ideal(x_5,x_6)
Bf_1=ideal(x_5,x_6+x_7)
Bg_1=ideal(x_5,x_6+x_7+x_8)
Bh_1=ideal(x_5,x_6+x_7+x_8+x_9)
Bi_1=ideal(x_5,x_6+x_8+x_9)
Bj_1=ideal(x_5,x_6+x_9)
Bk_1=ideal(x_5,x_6+x_7+x_9)
```

```
Bl_1=ideal(x_5,x_6+x_8)

B_2=ideal(x_5,x_7)
Bf_2=ideal(x_5,x_7+x_8)
Bg_2=ideal(x_5,x_7+x_8+x_9)
Bh_2=ideal(x_5,x_7+x_9)

B_3=ideal(x_5,x_8)
Bf_3=ideal(x_5,x_8+x_9)

B_4=ideal(x_5,x_9)

B_12=ideal(x_5,x_6,x_7)
Bf_12=ideal(x_5,(x_6+x_8+x_9),x_7)
Bg_12=ideal(x_5,(x_6+x_8),x_7)
Bh_12=ideal(x_5,(x_6+x_9),x_7)
Bi_12=ideal(x_5,x_6,(x_7+x_8))
Bj_12=ideal(x_5,(x_6+x_8+x_9),(x_7+x_8))
Bk_12=ideal(x_5,(x_6+x_8),(x_7+x_8))
Bl_12=ideal(x_5,(x_6+x_9),(x_7+x_8))
Bm_12=ideal(x_5,x_6,(x_7+x_8+x_9))
Bn_12=ideal(x_5,(x_6+x_8+x_9),(x_7+x_8+x_9))
Bo_12=ideal(x_5,(x_6+x_8),(x_7+x_8+x_9))
Bp_12=ideal(x_5,(x_6+x_9),(x_7+x_8+x_9))
Bq_12=ideal(x_5,x_6,(x_7+x_9))
Br_12=ideal(x_5,(x_6+x_8+x_9),(x_7+x_9))
Bs_12=ideal(x_5,(x_6+x_8),(x_7+x_9))
Bt_12=ideal(x_5,(x_6+x_9),(x_7+x_9))
```

```
B_13=ideal(x_5,x_6,x_8)
Bf_13=ideal(x_5,(x_6+x_7),x_8)
Bg_13=ideal(x_5,(x_6+x_9),x_8)
Bh_13=ideal(x_5,(x_6+x_7+x_9),x_8)
Bi_13=ideal(x_5,x_6,(x_8+x_9))
Bj_13=ideal(x_5,(x_6+x_7),(x_8+x_9))
Bk_13=ideal(x_5,(x_6+x_9),(x_8+x_9))
Bl_13=ideal(x_5,(x_6+x_7+x_9),(x_8+x_9))

B_14=ideal(x_5,x_6,x_9)
Bf_14=ideal(x_5,(x_6+x_7),x_9)
Bg_14=ideal(x_5,(x_6+x_8),x_9)
Bh_14=ideal(x_5,(x_6+x_7+x_8),x_9)

B_23=ideal(x_5,x_7,x_8)
Bf_23=ideal(x_5,(x_7+x_9),x_8)
Bg_23=ideal(x_5,x_7,(x_8+x_9))
Bh_23=ideal(x_5,(x_7+x_9),(x_8+x_9))

B_24=ideal(x_5,x_7,x_9)
Bf_24=ideal(x_5,(x_7+x_8),x_9)

B_34=ideal(x_5,x_8,x_9)

A_1=ideal(x_5,x_7,x_8,x_9)
A_2=ideal(x_5,x_6,x_8,x_9)
A_3=ideal(x_5,x_6,x_7,x_9)
```

```
A_4=ideal(x_5,x_6,x_7,x_8)
```

In order to apply the various $\diamond_I$ operations to these ideals, we must define ideals of the form $x^{10}I$. Here we do so for ideals of the form $B_{(1,2)}(f_1, f_2)$ and $B_{(1,3)}(f_1, f_3)$.

```
Bd_12=ideal(x_5^2,x_5*x_6,x_5*x_7)
Bdf_12=ideal(x_5^2,x_5*(x_6+x_8+x_9),x_5*x_7)
Bdg_12=ideal(x_5^2,x_5*(x_6+x_8),x_5*x_7)
Bdh_12=ideal(x_5^2,x_5*(x_6+x_9),x_5*x_7)
Bdi_12=ideal(x_5^2,x_5*x_6,x_5*(x_7+x_8))
Bdj_12=ideal(x_5^2,x_5*(x_6+x_8+x_9),x_5*(x_7+x_8))
Bdk_12=ideal(x_5^2,x_5*(x_6+x_8),x_5*(x_7+x_8))
Bdl_12=ideal(x_5^2,x_5*(x_6+x_9),x_5*(x_7+x_8))
Bdm_12=ideal(x_5^2,x_5*x_6,x_5*(x_7+x_8+x_9))
Bdn_12=ideal(x_5^2,x_5*(x_6+x_8+x_9),x_5*(x_7+x_8+x_9))
Bdo_12=ideal(x_5^2,x_5*(x_6+x_8),x_5*(x_7+x_8+x_9))
Bdp_12=ideal(x_5^2,x_5*(x_6+x_9),x_5*(x_7+x_8+x_9))
Bdq_12=ideal(x_5^2,x_5*x_6,x_5*(x_7+x_9))
Bdr_12=ideal(x_5^2,x_5*(x_6+x_8+x_9),x_5*(x_7+x_9))
Bds_12=ideal(x_5^2,x_5*(x_6+x_8),x_5*(x_7+x_9))
Bdt_12=ideal(x_5^2,x_5*(x_6+x_9),x_5*(x_7+x_9))

Bd_13=ideal(x_5^2,x_5*x_6,x_5*x_8)
Bdf_13=ideal(x_5^2,x_5*(x_6+x_7),x_5*x_8)
Bdg_13=ideal(x_5^2,x_5*(x_6+x_7+x_9),x_5*x_8)
Bdh_13=ideal(x_5^2,x_5*(x_6+x_9),x_5*x_8)
Bdi_13=ideal(x_5^2,x_5*x_6,x_5*(x_8+x_9))
Bdj_13=ideal(x_5^2,x_5*(x_6+x_7),x_5*(x_8+x_9))
Bdk_13=ideal(x_5^2,x_5*(x_6+x_7+x_9),x_5*(x_8+x_9))
```

```
Bdj_13=ideal(x_5^2,x_5*(x_6+x_9),x_5*(x_8+x_9))
```

With this code we are able to determine actions of star operations like $B_1(1+x)^{\diamond_{B_{(1,2)}}}$ as in the following example.

Example A.0.5. *We compute $B_1(1+x)^{\diamond_{B_{(1,2)}}}$ in the following manner.*

```
input: Bd_12:(Bd_12:Bf_1)
```

```
output: ideal(x_5, x_6, x_7)
```

This tells us that $x^5 B_1(1+x)^{\diamond_{B_{(1,2)}}} = x^5 B_{(1,2)}$ and so $B_1(1+x)^{\diamond_{B_{(1,2)}}} = B_{(1,2)}$.

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