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Well-posedness and Ill-posedness of the Nonlinear Beam Equation

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by

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Abstract

The dissertation consists of two parts, Well-posedness and ill-posedness for the nonlinear beam equation and Strichartz estimates of the beam equation on the domains.

In the first part, we will work to introduce the further studies of Strichartz estimates with initial data both in homogeneous Sobolev spaces $\dot{H}^s \times \dot{H}^{s-2}$ and in inhomogeneous Sobolev space $H^s \times H^{s-2}$. We take advantage of the Strichartz estimates to build well-posedness theorems of the nonlinear beam equations for rough data by the Picard iteration method. We will apply these methods on the nonlinear beam equation with "energy critical, subcritical" and "energy supercritical" cases. Since the beam equation does not satisfy finite speed propagation, we introduce the further result of the fractional chain rule to deal with the "energy super critical" case. We obtain the global well-posedness with initial data in homogeneous Sobolev space $\dot{H}^s \times \dot{H}^{s-2}$ and local well-posedness with initial data in inhomogeneous Sobolev space $H^s \times H^{s-2}$. At the same time, we extend the range of order s. With the global existence for small data, we prove the scattering and asymptotic completeness result for the nonlinear beam equation. Last we prove the nonlinear beam equation is ill-posed in defocusing case $\omega = -1$ when $0 < s < s_c = \frac{n}{2} - \frac{4}{\kappa - 1}$ by small dispersion analysis of M. Christ, J. Colliander and T. Tao.

In the second part, we will study Strichartz estimates on Riemannian manifolds (Ω, g) with boundary, for both the compact case and the case that is the exterior of a smooth, non-trapping obstacle in Euclidean space for the beam equation.

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Chapter 1

Introduction

In recent years, some models involving the beam equations have been studied. Peletier and Troy [19] presented several such nonlinear equation models in physics literature. E.Cordero and D.Zucco [3] studied dispersive properties of the linear beam equation. B. Pausader [17], [18] investigated the well-posedness and scattering theory in the energy space for nonlinear beam equations. In this dissertation, we will mainly consider the Cauchy problem for the nonlinear beam equation with force F as the power-type nonlinearity

$$\begin{cases} \partial_t^2 u(t,x) + \triangle^2 u(t,x) = \omega |u|^{\kappa-1} u(t,x), \\ u\mid_{t=0} = f(x) \\ \partial_t u\mid_{t=0} = g(x) \end{cases}, \tag{1.1}$$

where, $\omega = \pm 1$ and $1 < \kappa < \infty$, and $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$. The equation (1.1) is said to be $defocusing$ when $\omega < 0$, and $focusing$ when $\omega > 0$. We investigate the global and local well-posedness in fractional homogeneous and inhomogeneous Sobolev spaces for the Cauchy problem of this equation under minimal regularity assumptions on the initial data in Euclidean space $\mathbb{R}^n$. We will also study this type of equation on Riemannian manifolds $(\Omega$, g) with boundary for both the compact case and the case that is the exterior of a smooth, non-trapping obstacle in Euclidean space. For the Sobolev space we introduce the following

Definition 1.0.1. The *inhomogeneous Sobolev space* $W^{s,r}$ and the *homogeneous Sobolev space* $\dot{W}^{s,r}$ are defined for $s \in \mathbb{R}$ and $1 < r < \infty$ as the closure of Schwartz functions f under their respective norms

$$\|f\|_{W^{s,r}} = \|\langle D\rangle^s f\|_{L^r},$$

$$\|f\|_{\dot{W}^{s,r}} = \||D|^s f\|_{L^r},$$

where the fractional differentiation operators $\langle D\rangle^s$ and $|D|^s$ are the Fourier multipliers defined by

$$\widehat{\langle D\rangle^s f}(\xi) := \langle \xi\rangle^s \hat{f}(\xi) \quad \text{and} \quad \widehat{|D|^s f}(\xi) := |\xi|^s \hat{f}(\xi),$$

where $\langle \xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$.

In particular, if $s = 2$, then $\langle D\rangle^s = I - \triangle$, where I is the identity operator, and $|D|^s = -\triangle$. If $r = 2$, these spaces are also denoted by H^s and $\dot{H}^s$.

The specific choice of power-type nonlinearity has a number of nice properties. It has the scaling symmetry and it is associated to a Hamiltonian potential. This beam equation enjoys the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{-4}{\kappa-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right); \quad f(x) \mapsto \lambda^{\frac{-4}{\kappa-1}} f\left(\frac{x}{\lambda}\right); \quad g(x) \mapsto \lambda^{\frac{-4}{\kappa-1}-2} g\left(\frac{x}{\lambda}\right). \tag{1.2}$$

This scale invariance predicts a relationship between time existence and regularity of initial data (see Tao[24]). If we compute the initial data $\|\lambda^{\frac{-4}{\kappa-1}} f\left(\frac{x}{\lambda}\right)\|_{\dot{H}^s}$ we see that

$$\|\lambda^{\frac{-4}{\kappa-1}} f\left(\frac{x}{\lambda}\right)\|_{\dot{H}^s} \sim \lambda^{-s+s_c}\|f\|_{\dot{H}^s}, \tag{1.3}$$

where

$$s_c := \frac{n}{2} - \frac{4}{\kappa-1}$$

is the *critical regularity*. For the nonlinear beam equations, to establish local and global properties of the solution, much of the work is in the development of regularity and norm estimates for the solutions. By using the iteration method in various function spaces we establish a local and perturbative theory, combining with frequency analysis and conservation laws we then obtain a global non-perturbative theory.

1.1 Strichartz estimates for the beam equation in $\mathbb{R}^n$

The space-time norm estimates known as Strichartz estimates provide a good quantitative measure of the dispersion phenomena for various dispersive equations. As it turns out, they are very useful in the study of various corresponding nonlinear equations. They play the principal role in the study of the local and global well-posedness of nonlinear equations in Sobolev spaces, scattering theory and nonlinear analysis. The mixed Strichartz space-time norm is defined as the following

$$\|u\|_{L_I^p L^r} = \left[\int_I \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx \right)^{\frac{p}{r}} dt \right]^{\frac{1}{p}}.$$

The works [12],[25] provided us with the fact that the homogeneous beam equation (1.1) can be factorized as the following product

$$(\partial_t^2 + \triangle^2)u = (i\partial_t + \triangle)(-i\partial_t + \triangle)u.$$

Which displays the relation with the Schrödinger equation, suggesting that we can recover Strichartz estimates for the beam equation from the ones for the Schrödinger equation. Some classical references on Strichartz estimates in $\mathbb{R}^n$ for the Schrödinger equation are provided by [5],[7],[11], [23]. In 2007, B. Pausader [17] investigated the Strichartz estimates for nonlinear beam equation in the "energy critical" case. In 2011, E.Cordero, D.Zucco [3] discussed Strichartz estimates for the linear beam equation in homogeneous Sobolev spaces. In order to make full of use Strichartz estimates for the nonlinear beam equation, in this dissertation, we will extend these results, proving estimates with initial data in homogeneous Sobolev spaces $\dot{H}^s \times \dot{H}^{s-2}$ and estimates in inhomogeneous Sobolev spaces $H^s \times H^{s-2}$ for the Cauchy problem of the beam equation

$$\begin{cases} \partial_t^2 u(t,x) + \triangle^2 u(t,x) = F(t,x) \\ u \mid_{t=0} = f(x) \\ \partial_t u \mid_{t=0} = g(x). \end{cases} \tag{1.4}$$

We take the following definitions:

Definition 1.1.1. We say that the exponent pair (p, q) is a Schrödinger-admissible pair if

$$2 \le p, q \le \infty, \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad n \ge 1, \quad (p, q, n) \ne (2, \infty, 2).$$

Definition 1.1.2. We say that the exponent triple (p, r, s) is a beam-admissible triple if

$$2 \le p, r \le \infty, \quad \frac{2}{p} + \frac{n}{r} = \frac{n}{2} - s, \quad n \ge 2, \quad (p, r, n) \ne (2, \infty, 2).$$

We have the following results when $f \in \dot{H}^s, g \in \dot{H}^{s-2}$,

Theorem 1.1.3. *Let $n \ge 1, s \in \mathbb{R}$, I be either the interval $[0, T]$, $T > 0$, or $[0, \infty)$, (p, r, s) be a beam-admissible triple, (a,b) is a Schrödinger-admissible pair, and (a', b') is the Hölder conjugate pair of (a, b). If u is a solution to the Cauchy problem (1.4), then we have the following estimates:*

$$\|u\|_{L_I^p L^r} + \|u(T,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{s-2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|F\|_{L_I^{a'} \dot{W}^{s-2,b'}}, \tag{1.5}$$

with implicit constant independent of T. In particular, when $0 \le s \le 2$, $\tilde{b} = \frac{nb'}{n+(2-s)b'}$.

$$\|u\|_{L_I^p L^r} + \|u(T,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{s-2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|F\|_{L_I^{a'} L^{\tilde{b}}}, \tag{1.6}$$

with implicit constant independent of T.

When $f \in H^s, g \in H^{s-2}$, for p, r, a, b defined as above, we prove the following:

Theorem 1.1.4. *Let $n \geq 1, s \in \mathbb{R}$, I be the interval $[0, T]$, $0 < T < \infty$, (a,b) be Schrödinger-admissible pair, (a', b') be the Hölder conjugate pair of (a,b). If u is a solution to the Cauchy problem (1.4), then we have the following estimates,*

$$\|u\|_{L_I^p L^r} + \|u(T,\cdot)\|_{H^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{H^{s-2}(\mathbb{R}^n)} \lesssim (1 + |T|^{\frac{1}{p}+1})(\|f\|_{H^s} + \|g\|_{H^{s-2}} + \|F\|_{L_I^{a'} W^{s-2,b'}}), \quad (1.7)$$

where (p, r, s) satisfies the following condition

$$2 \leq p, r \leq \infty, \quad \frac{2}{p} + \frac{n}{r} \geq \frac{n}{2} - s, \quad n \geq 2, \quad (p, r, n) \neq (2, \infty, 2).$$

Actually, by counterexamples, we only have this estimate locally.

1.2 Well-posedness and scattering theory for the nonlinear beam equation in $\mathbb{R}^n$

The local and global well-posedness of semilinear dispersive equations has attracted a lot of attention in the past years. In general, when global well-posedness is established, the existence of a scattering operator, comparing the nonlinear dynamics and the linear one, is a direct by-product. H. Lindblad and C. D.Sogge [15] proved existence for semilinear wave equations with low regularity data and determined the minimal Sobolev regularity that is needed to ensure local well-posedness. They took advantage of the Strichartz estimates to prove well-posedness theorems for the nonlinear wave equation with rough initial data by the Picard iteration method. By this method, we will investigate well-posedness with initial data $f(x) \in \dot{H}^s(\mathbb{R}^n), g(x) \in \dot{H}^{s-2}(\mathbb{R}^n)$, and $f(x) \in H^s(\mathbb{R}^n), g(x) \in H^{s-2}(\mathbb{R}^n)$ for "energy critical", "energy subcritical" exponents $\kappa \leq \frac{n+4}{n-4}$ and "energy supercritical" exponents $\kappa > \frac{n+4}{n-4}$, and determine the minimal Sobolev regularity that is needed to ensure local and global well-posedness for the nonlinear beam equation. Since the beam equation does not satisfy finite speed of propagation, we introduce further results on the fractional chain rule to deal with the "energy super critical" case. At the same time we extend the range of regularity s. We will also be concerning the asymptotic completeness and scattering for small amplitude solutions. For "energy critical" and "energy subcritical" exponents $\kappa \leq \frac{n+4}{n-4}$, when $f \in \dot{H}^s, g \in \dot{H}^{s-2}$, we have the following

Theorem 1.2.1. *Set* $s = \frac{n}{2} - \frac{4}{\kappa-1}$*, if* $n \geq 4, max\{\frac{8}{n}+1, \frac{n+1}{n-3}\} < \kappa \leq \frac{n+4}{n-4}$*, then there is a* $T > 0$*, a unique (weak) solution of the nonlinear beam equation (1.1) satisfying*

$$(u, \partial_t u) \in C([0,T]; \dot{H}^s \times \dot{H}^{s-2}) \quad \text{and} \quad u \in L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}. \tag{1.8}$$

Moreover, there is $\epsilon(\kappa) > 0$*, so that if*

$$\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} < \epsilon(\kappa),$$

then one can take $T = \infty$*. When* $n = 3, \kappa > 5$*, we have the results above as* $u \in L^{\frac{(n+2)(\kappa-1)}{4}}([0,T] \times \mathbb{R}^n)$.

When $f \in H^s, g \in H^{s-2}$, we have similarly local well-posedness result since Strcichartz estimates only are available locally with this initial data.

In the "energy supercritical" range $\kappa > \frac{n+4}{n-4}$ case, for small initial data $f \in \dot{H}^s, g \in \dot{H}^{s-2}$, we have the following

Theorem 1.2.2. *Set* $s = \frac{n}{2} - \frac{4}{\kappa-1}, n > 4$*. Suppose there exists an* $l \in \mathbb{N}, l \geq 1$*, with* $\frac{n}{2} - \frac{4}{\kappa-1} - 2 \leq l \leq \kappa - 1$*, then there is a* $T > 0$*, a unique (weak) solution of the nonlinear beam equation (1.1) satisfying*

$$(u, \partial_t u) \in C([0,T]; \dot{H}^s \times \dot{H}^{s-2}) \quad \text{and} \quad u \in L^{2\kappa}([0,T], L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}(\mathbb{R}^n)). \tag{1.9}$$

Moreover, there is $\epsilon(\kappa) > 0$*, so that if*

$$\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} < \epsilon(\kappa),$$

then one can take $T = \infty$*.*

For initial data $f \in H^s, g \in H^{s-2}$ we have the following:

Theorem 1.2.3. *Set* $s = \frac{n}{2} - \frac{4}{\kappa-1}, n > 4$*. Suppose there exists an* $l \in \mathbb{N}$*, with* $\frac{n}{2} - \frac{4}{\kappa-1} - 2 \leq l \leq \kappa$*, then there is a* $T > 0$*, a a unique (weak) solution of the nonlinear beam equation (1.1) satisfying*

$$(u, \partial_t u) \in C([0,T]; H^s \times H^{s-2}) \quad \text{and} \quad u \in L^{\frac{(n+2)(\kappa-1)}{4}}([0,T] \times \mathbb{R}^n). \tag{1.10}$$

For the asymptotic completeness and scattering for small amplitude solutions, the results follows:

Theorem 1.2.4. *For* $\kappa \geq 1$*, consider* u *is the solution of the equation (1.1) with the norm of the data small, namely,*

$$\|f\|_{\dot{H}^{s_c}} + \|g\|_{\dot{H}^{s_c-2}} < \epsilon. \tag{1.11}$$

Then there exists $\epsilon > 0$ *small such that for such data (f,g), there is small data* $(f_+, g_+) \in \dot{H}^{s_c} \times \dot{H}^{s_c-2}$*, so that the solution to the free beam equation with this data,*

$$\begin{cases} \partial_t^2 u_+ + \triangle^2 u_+ = 0, \\ u_+ \mid_{t=0} = f_+ \in \dot{H}^{s_c}, \\ \partial_t u_+ \mid_{t=0} = g_+ \in \dot{H}^{s_c-2} \end{cases} \tag{1.12}$$

satisfies

$$\lim_{T \to +\infty} \|u(T,\cdot) - u_+(T,\cdot)\|_{s(\dot{\kappa})} = 0, \tag{1.13}$$

where

$$\|u(T,\cdot)\|^2_{\dot{s(\kappa)}} = \|u(T,\cdot)\|^2_{\dot{H}^{s_c}} + \|\partial_t u(T,\cdot)\|^2_{\dot{H}^{s_c-2}}.$$

Conversely, if $(f_-, g_-) \in \dot{H}^{s_c} \times \dot{H}^{s_c-2}$ *has sufficiently small norm and* u_- *is the solution to the free beam equation with this data, then there exists a solution* u *to (1.1) satisfying*

$$\lim_{T\to-\infty} \|u(T,\cdot) - u_-(T,\cdot)\|_{\dot{s(\kappa)}} = 0. \tag{1.14}$$

Thus, the scattering operator $S : (f_-, g_-) \to (f_+, g_+)$ *exists in a neighborhood of the origin in* $\dot{H}^{s_c} \times \dot{H}^{s_c-2}$.

1.3 Ill-posedness for the nonlinear beam equation in $\mathbb{R}^n$

There are certain equations and certain regularities for which the Cauchy problem is ill-posed. M. Christ, J. Colliander and T. Tao [4] give examples of solutions to nonlinear wave and Schrödinger equations on $\mathbb{R}^n$ which show that the problem is ill-posed in the Sobolev space when the exponent s is lower than the critical exponent predicted by scaling. In this dissertation, we will discuss the ill-posedness results for the Cauchy problem of the nonlinear beam equation with $0 < s < s_c$ by the small dispersion analysis of M. Christ, J.Colliander and T. Tao.

Theorem 1.3.1. *Let* $n \geq 1$, $\omega = -1$ *and* $\kappa > 1$. *If* κ *is not an odd integer, we assume* $\kappa \geq k+2$ *for some integer* $k > n/2$. *Suppose that* $0 < s < s_c = \frac{n}{2} - \frac{4}{\kappa-1}$. *Then for any* $\epsilon > 0$ *there exist a real-valued solution* u *of the nonlinear beam equation (1.1) and* $t \in \mathbb{R}^+$, *such that* $u(0) \in \mathcal{S}$,

$$\|u(0)\|_{H^s} < \epsilon,$$

$$u_t(0) = 0,$$

$$0 < t < \epsilon,$$

$$\|u(t)\|_{H^s} > \epsilon^{-1}.$$

In particular, for any $t > 0$ the solution map $\mathcal{S} \times \mathcal{S} \ni (u(0), u_t(0)) \to (u(t), u_t(t))$, for Cauchy problem (1.1) fails to be continuous at 0 in the $H^s \times H^{s-2}$ topology.

1.4 Strichartz estimates for the beam equation on compact Riemannian manifolds and exterior domains

Recently, Strichartz estimates have been developed for nontrivial geometries. The Strichartz estimates on Riemannian manifolds (Ω, g) with boundary, for both the compact case and the case that is the exterior of a smooth, non-trapping obstacle in Euclidean space for Schrödinger equation have been established by M. Blair, H. Smith and C. Sogge [1], [2]. O. Ivanovici [9] deduced classical Strichartz estimates for the Schrödinger equation outside a strictly convex obstacle. In this dissertation, we will only discuss the Strichartz estimates of the beam equation in time locally. Consider the homogeneous beam equation (1.1) with Dirichlet boundary conditions

$$u(t,x)|_{x\in\partial\Omega} = 0 \qquad \triangle_g u(t,x)|_{x\in\partial\Omega} = 0.$$

On compact manifolds with boundary we have the following theorem.

Theorem 1.4.1. *Let (Ω, g) be a smooth compact Riemannian manifold with boundary. If u is a solution to the (1.4), then*

$$\|u\|_{L^p([-T,T];L^r(\Omega))} \lesssim \|f\|_{H^{\frac{4}{3p}}} + \|g\|_{H^{\frac{4}{3p}-2}} + \|F\|_{L^1([-T,T];H^{\frac{4}{3p}-2}(\Omega))}, \tag{1.15}$$

where

$$2 \le p, r \le \infty, \quad \frac{2}{p} + \frac{n}{r} = \frac{n}{2}, \quad n \ge 2, \quad (p,r,n) \ne (2,\infty,2).$$

Let $\Omega = \mathbb{R}^n \setminus \Theta$ be the domain exterior to a compact non-trapping obstacle with smooth boundary. Non-trapping means that every unit speed generalized bicharacteristic escapes each compact subset of Ω in finite time. For the Strichartz estimates of the beam equation on these kind domains, we have the following:

Theorem 1.4.2. *let $\Omega = \mathbb{R}^n \setminus \Theta$ be the exterior domain to a compact non-trapping obstacle with smooth boundary, and $\triangle$ the standard Laplace operator on Ω, subject to Dirichlet conditions. Suppose*

$$\begin{cases} \frac{3}{p} + \frac{n}{q} \leq \frac{n}{2}, n = 2, \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, n \geq 3, \end{cases} \tag{1.16}$$

and

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s.$$

Then for the solution of beam equation (1.4) with Dirichlet boundary conditions, the following estimate hold

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T \|u_0\|_{H^s} + \|u_1\|_{H^{s-2}} + \|F\|_{L^1([-T,T];H^{s-2}(\Omega))}. \tag{1.17}$$

When Θ is strictly convex, we have the following:

Corollary 1.4.3. *Let $\Omega = \mathbb{R}^n \backslash \Theta$, where Θ is a compact with smooth boundary. Suppose that $n \geq 2$ and $\partial\Omega$ is strictly geodesically concave throughout. Assume the pair (p, q) satisfying the scaling condition:*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s,$$

then for the solution of beam equation (1.4) with Dirichlet boundary conditions the following estimate holds

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-2}} + \|F\|_{L^1([-T,T];H^{s-2}(\Omega))}.$$

Chapter 2

Strichartz estimates for the beam equation in $\mathbb{R}^n$

In this chapter, we first introduce some notations and definitions that will be frequently used in this dissertation. The expression $X \lesssim Y$ means $X \leq CY$ for some constant C. Consider the linear beam equation,

$$\begin{cases} \partial_t^2 u + \triangle^2 u = F, \\ u \mid_{t=0} = f, \\ \partial_t u \mid_{t=0} = g. \end{cases} \tag{2.1}$$

The solution of this equation can be formally written in the integral form

$$u(t, \cdot) = \cos(t\triangle) f + \frac{\sin(t\triangle)}{\triangle} g + \int_0^t \frac{\sin((t-s)\triangle)}{\triangle} F(s) ds.$$

2.1 Strichartz estimates with initial data in homogeneous Sobolev spaces $\dot{H}^s \times \dot{H}^{s-2}$

We have the following theorem about Strichartz estimates for solutions to the beam equation with initial data $f \in \dot{H}^s, g \in \dot{H}^{s-2}$.

Theorem 2.1.1. *Let $n \geq 1, s \in \mathbb{R}$, I be either the interval $[0,T]$, $T > 0$, or $[0,\infty)$, (p,r,s) be a beam-admissible triple, (a,b) is a Schrödinger-admissible pair, and (a',b') is conjugate pair of (a,b). If u is a solution to the Cauchy problem (2.1), then we have the following estimates:*

$$\|u\|_{L^p_I L^r}+\|u(T,\cdot)\|_{L^\infty_I \dot{H}^s(\mathbb{R}^n)}+\|\partial_t u(T,\cdot)\|_{L^\infty_I \dot{H}^{s-2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s}+\|g\|_{\dot{H}^{s-2}}+\|F\|_{L^{a'}_I \dot{W}^{s-2,b'}}, \tag{2.2}$$

with implicit constant independent of T. In particular, when $0 \leq s \leq 2$, $\tilde{b} = \frac{nb'}{n+(2-s)b'}$,

$$\|u\|_{L^p_I L^r}+\|u(T,\cdot)\|_{L^\infty_I \dot{H}^s(\mathbb{R}^n)}+\|\partial_t u(T,\cdot)\|_{L^\infty_I \dot{H}^{s-2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s}+\|g\|_{\dot{H}^{s-2}}+\|F\|_{L^{a'}_I L^{\tilde{b}}}, \tag{2.3}$$

with implicit constant independent of T.

Proof. By the work of E.Cordero, D.Zucco [3], the following estimates hold

$$\|u\|_{L^p_I \dot{W}^{s,q}} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|F\|_{L^{a'}_I \dot{W}^{s-2,b'}}, \tag{2.4}$$

where (p,q) and (a,b) are Schrödinger-admissible pairs. For fixed t, by Sobolev embedding, we have

$$\|u(t,\cdot)\|_{L^r} \lesssim \|u(t,\cdot)\|_{\dot{W}^{s,q}},$$

when $\frac{1}{q} = \frac{1}{r} + \frac{s}{n}$, combining with

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

we have

$$\frac{2}{p} + \frac{n}{r} = \frac{n}{2} - s.$$

Therefore we have the estimate

$$\|u\|_{L^p_I L^r} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|F\|_{L^{a'}_I \dot{W}^{s-2,b'}}. \tag{2.5}$$

Let v be the solution of (2.1) with $F(t,x) = 0$, w be the solution of (2.1) with vanishing initial data. Then the solution of (2.1) is $u = v + w$. By the energy inequality for the linear Cauchy problem, we have

$$\|v(T,\cdot)\|_{L^\infty_I \dot{H}^s(\mathbb{R}^n)} + \|\partial_t v(T,\cdot)\|_{L^\infty_I \dot{H}^{s-2}(\mathbb{R}^n)} \leq 2(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}}). \tag{2.6}$$

For $w(t,\cdot) = \int_0^t \frac{\sin((t-s)\triangle)}{\triangle} F(s)ds$, we have the inhomogeneous Strichartz estimates from [3],

$$\left\| \int_0^t \frac{e^{i(t-s)\triangle}}{\triangle} F(s)ds \right\|_{L^p_I \dot{W}^{s,q}} \lesssim ||F||_{L^{a'}_I \dot{W}^{s-2,b'}},$$

where $(p,q), (a,b)$ are Schrödinger-admissible pairs. When $p = \infty$, $q = 2$, by the definition of the homogeneous Sobolev space, we have

$$\|(-\triangle)^{\frac{s}{2}} w(T,\cdot)\|_{L^\infty_I L^2(\mathbb{R}^n)} \lesssim \|F\|_{L^{a'}_I \dot{W}^{s-2,b'}}.$$

Then we have,

$$\|w(T,\cdot)\|_{L^\infty_I \dot{H}^s(\mathbb{R}^n)} + \|\partial_t w(T,\cdot)\|_{L^\infty_I \dot{H}^{s-2}(\mathbb{R}^n)} \lesssim \|F\|_{L^{a'}_I \dot{W}^{s-2,b'}}. \tag{2.7}$$

Combining with (2.5),(2.6),(2.7), we have the estimates (2.2). Since

$$\|F\|_{L^{a'}_I \dot{W}^{s-2,b'}} = \|(-\triangle)^{\frac{s-2}{2}} F\|_{L^{a'}_I L^{b'}},$$

now assume $s \leq 2$, for fixed t, by the Theorem 1 of chapter 5 in [22] (which is equivalent to Sobolev embedding),

$$\|(-\triangle)^{\frac{s-2}{2}} F(t,\cdot)\|_{L^{b'}} \lesssim \|F(t,\cdot)\|_{L^{\tilde{b}}},$$

where, $\frac{1}{\tilde{b}} = \frac{1}{b'} + \frac{2-s}{n}$. Then by the same way of proving (2.2), we have the Strichartz estimates (2.3) where ,

$$\frac{2}{p} + \frac{n}{r} = \frac{n}{2} - s = \frac{2}{a'} + \frac{n}{\tilde{b}} - 4. \tag{2.8}$$

□

2.2 Strichartz estimates in inhomogeneous Sobolev space $H^s \times H^{s-2}$

Now we consider the Strichartz estimates for solutions to the beam equation with initial data $f \in H^s, g \in H^{s-2}$ (inhomogeneous Sobolev space), we have the following

Theorem 2.2.1. *Let $n \geq 1, s \in \mathbb{R}$, I be the interval $[0, T]$, $0 < T < \infty$, (a,b) be Schrödinger-admissible pair, (a', b') be the conjugate pair of (a,b). If u is a solution to the Cauchy problem (2.1), then we have the following estimates,*

$$\|u\|_{L_I^p L^r} + \|u(T, \cdot)\|_{H^s(\mathbb{R}^n)} + \|\partial_t u(T, \cdot)\|_{H^{s-2}(\mathbb{R}^n)} \lesssim (1 + |T|^{\frac{1}{p}+1})(\|f\|_{H^s} + \|g\|_{H^{s-2}} + \|F\|_{L_I^{a'} W^{s-2,b'}}), \tag{2.9}$$

where (p, r, s) satisfies the following condition

$$2 \leq p, r \leq \infty, \quad \frac{2}{p} + \frac{n}{r} \geq \frac{n}{2} - s, \quad n \geq 2, \quad (p, r, n) \neq (2, \infty, 2).$$

Proof. Let $\beta(\xi)$ be a smooth cutoff function with support

$$supp(\beta) \subset B_2(0),$$

$$supp(1 - \beta) \subset \{|\xi| \geq 1\}.$$

Let $u_0 = \beta(D)u$, $F_0 = \beta(D)F$, $u_1 = (1 - \beta(D))u$, $F_1 = (1 - \beta(D))F$, where u is the solution of (2.1), then we have

$$\partial_t^2 u_0 + \triangle^2 u_0 = F_0, \tag{2.10}$$

$$\partial_t^2 u_1 + \triangle^2 u_1 = F_1. \tag{2.11}$$

Since

$$|\xi| \geq 1 \Longrightarrow |\xi|^{s-2} \approx (1+|\xi|^2)^{\frac{s-2}{2}},$$

by Theorem 2.1.1 we have

$$\|u_1\|_{L_I^p L^r} + \|u_1(T,\cdot)\|_{H^s(\mathbb{R}^n)} + \|\partial_t u_1(T,\cdot)\|_{H^{s-2}(\mathbb{R}^n)} \lesssim (\|f\|_{H^s} + \|g\|_{H^{s-2}} + \|F_1\|_{L_I^{q'} W^{s-2,b'}}). \tag{2.12}$$

Since, $|\xi| \leq 1 \Longrightarrow (1+|\xi|^2)^{s_1} \approx (1+|\xi|^2)^{s_2}$ for any s_1, s_2, then for u_0, first, by Sobolev embedding we have,

$$\|u_0\|_{L_I^p L^r} \lesssim (1+|T|^{\frac{1}{p}})\|u_0\|_{L_I^\infty H^s} \approx (1+|T|^{\frac{1}{p}})\|u_0\|_{L_I^\infty L^2}. \tag{2.13}$$

We define the energy of u_0 by

$$E(u_0;t) = \int \frac{1}{2}|\partial_t u_0(t,x)|^2 + \frac{1}{2}|\triangle u_0(t,x)|^2 dx. \tag{2.14}$$

The energy identity gives us

$$\partial_t E(u_0;t) = \int \partial_t u_0(t,x) F_0(t,x) dx. \tag{2.15}$$

By Cauchy-Schwarz inequality

$$|\partial_t E^{\frac{1}{2}}(u_0;t)| \lesssim \|F_0(t,x)\|_{L^2} \lesssim \|F_0(t,x)\|_{H^{s-2}}.$$

By the fundamental theorem of calculus,

$$\|u_0(t,\cdot)\|_{L^2} \leq \|u_0(0,\cdot)\|_{L^2} + \int_0^t \|\partial_t u_0(\tau,\cdot)\|_{L^2} d\tau \leq \|u_0(0,\cdot)\|_{L^2} + \int_0^t E^{\frac{1}{2}}(u_0;\tau) d\tau$$
$$\leq \|u_0(0,\cdot)\|_{L^2} + \int_0^t \|F_0(\tau,\cdot)\|_{L^2} d\tau \leq \|u_0(0,\cdot)\|_{L^2} + \|F_0\|_{L_I^1 H^{s-2}}. \tag{2.16}$$

Therefore we have

$$\|u_0\|_{L_I^\infty H^s} + \|\partial_t u_0\|_{L_I^\infty L^2} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-2}} + \|F_0\|_{L_I^1 H^{s-2}}. \tag{2.17}$$

Since $F_0(x) = \beta(x)^\vee * F(x)$, by Young's inequality,

$$\|F_0\|_{H^{s-2}} = ||\beta^\vee * \langle D\rangle^{s-2} F\|_{L^2} \le \|\beta^\vee\|_{L^{b_1}} \|\langle D\rangle^{s-2} F\|_{b'} \lesssim \|F\|_{W^{s-2,b'}}, \tag{2.18}$$

where, $\frac{1}{2} = \frac{1}{b_1} + \frac{1}{b'} - 1$. Combining with (2.13) (2.16) (2.17) and (2.18), we have

$$\begin{aligned} &\|u_0\|_{L_I^p L^r} + \|u_0(T,\cdot)\|_{H^s(\mathbb{R}^n)} + \|\partial_t u_0(T,\cdot)\|_{H^{s-2}(\mathbb{R}^n)} \\ &\qquad \lesssim (1 + |T|^{\frac{1}{p}+1})(\|f\|_{H^s} + \|g\|_{H^{s-2}} + \|F\|_{L_I^{a'} W^{s-2,b'}}). \end{aligned} \tag{2.19}$$

Therefore combines (2.12) and (2.19), we have the Strichartz estimates (2.9).

□

2.3 Counterexample

The Theorem above tells us the Strichartz estimates of the beam equation exist locally in inhomogeneous Sobolev space $H^s \times H^{s-2}$, actually, the following counterexample tells us this Strichartz estimate is only valid locally.

Theorem 2.3.1. *For T sufficiently large, we have*

$$\sup_{g \in \mathcal{S}} \frac{\left\| \frac{\sin(t\triangle)}{\triangle} g \right\|_{L^\infty([0,T];H^s)}}{\|g\|_{H^s}} \ge c|T|.$$

Proof. Take $f = 0$, then the solution of the homogeneous beam equation will have the form $u(t,\cdot) = \frac{\sin(t\triangle)}{\triangle} g$. Let $\beta_\epsilon(\xi)$ be a smooth cut off supported in

$$supp(\beta_\epsilon) \subset \left\{ \xi : \frac{\epsilon^2}{2} \le |\xi|^2 \le \frac{3\epsilon^2}{2} \right\}, \quad \epsilon \ll 1.$$

Set $\widehat{g(\xi)} = \beta_\epsilon(\xi)$, then

$$\|u(t,\cdot)\|_{H^s}^2 = \int \left| \frac{\sin(t|\xi|^2)}{|\xi|^2} \beta_\epsilon(\xi) \langle \xi \rangle^s \right|^2 d\xi.$$

Therefore, at $t = \frac{\pi}{2}\epsilon^{-2}$, $\|u(t,\cdot)\|_{H^s} \approx \epsilon^{-2}(\int |\beta_\epsilon(\xi)|^2 d\xi)^{\frac{1}{2}}$. Also

$$\|g\|_{H^{s-2}} \approx (\int |\beta_\epsilon(\xi)|^2 d\xi)^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned} \frac{\|u(\frac{\pi}{2}\epsilon^{-2},\cdot)\|_{H^s}}{\|g\|_{H^{s-2}}} &\geq \frac{\|u(\frac{\pi}{2}\epsilon^{-2},\cdot)\|_{H^s}}{\|g\|_{H^{s-2}}} \\ &\gtrsim \epsilon^{-2} \frac{(\int |\beta_\epsilon(\xi)|^2 d\xi)^{\frac{1}{2}}}{(\int |\beta_\epsilon(\xi)|^2 d\xi)^{\frac{1}{2}}} = \epsilon^{-2} \approx t. \end{aligned}$$

Therefore, we have

$$\sup_{g \in \mathcal{S}} \frac{\left\| \frac{\sin(t\triangle)}{\triangle} g \right\|_{L^\infty([0,T];H^s)}}{\|g\|_{H^s}} \geq c|T|,$$

for $T \gg 1$. □

Chapter 3

Well-posedness and scattering for the nonlinear beam equation in $\mathbb{R}^n$

Consider the Cauchy problem for the nonlinear beam equation with force F which is the power-type nonlinearity function

$$\begin{cases} \partial_t^2 u(t,x) + \triangle^2 u(t,x) = \omega|u|^{\kappa-1}u(t,x), \\ u\mid_{t=0}= f(x) \\ \partial_t u\mid_{t=0}= g(x), \end{cases} \tag{3.1}$$

where $\omega = \pm 1$ and $1 < \kappa < \infty$, and $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$. The equation (3.1) is said to be $defocusing$ when $\omega < 0$, and $focusing$ when $\omega > 0$. We are concerned in this chapter with proving local well-posedness and global well-posedness for small data in $\dot{H}^s \times \dot{H}^{s-2}$ and local well-posedness in $H^s \times H^{s-2}$.

To prove the existence of the solution, we use Picard iteration argument. First we define $F_\kappa(u) = \omega|u|^{\kappa-1}u$. Set $u_{-1} \equiv 0$, and define $u_m, m = 0, 1, 2, ...,$ by

$$\begin{cases} (\partial_t^2 + \triangle^2)u_m = F_\kappa(u_{m-1}), \\ u_m \mid_{t=0} = f, \\ \partial_t u_m \mid_{t=0} = g. \end{cases} \tag{3.2}$$

At last we need show that there is a $0 < T \le \infty$ and a function u so that

$$u_m \to u \quad \text{and} \quad F_\kappa(u_m) \to F_\kappa(u), \quad \text{in} \quad \mathcal{D}(S_T) \quad \text{with} \quad S_T = [0,T] \times \mathbb{R}^n. \tag{3.3}$$

3.1 Well-posedness Theorems for "energy critical" and "energy subcritical" cases

For "energy critical" and "energy subcritical" exponents $\kappa \le \frac{n+4}{n-4}$, when $f \in \dot{H}^s, g \in \dot{H}^{s-2}$, we have the following

Theorem 3.1.1. *Set $s = \frac{n}{2} - \frac{4}{\kappa-1}$, if $n \ge 4, max\{\frac{8}{n}+1, \frac{n+1}{n-3}\} < \kappa \le \frac{n+4}{n-4}$, then there is a $T > 0$, a unique (weak) solution of the nonlinear beam equation (3.1) satisfying*

$$(u, \partial_t u) \in C([0,T]; \dot{H}^s \times \dot{H}^{s-2}) \quad \textit{and} \quad u \in L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}. \tag{3.4}$$

Moreover, there is $\epsilon(\kappa) > 0$,so that if

$$\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} < \epsilon(\kappa),$$

then one can take $T = \infty$. When $n = 3, \kappa > 5$, we have the results above with $u \in L^{\frac{(n+2)(\kappa-1)}{4}}([0,T] \times \mathbb{R}^n)$.

Because the main step is to show that the nonlinear mapping $u_m \to u_{m+1}$ is a contraction for the proof of the existence, we set the following lemma.

Lemma 3.1.2. *For given* $n \geq 4$, *max*$\{\frac{8}{n}+1, \frac{n+1}{n-3}\} < \kappa \leq \frac{n+4}{n-4}$, $s = \frac{n}{2} - \frac{4}{\kappa-1}$, *then for* $I = [0,T]$ *if we set,*

$$A_m(T) = \|u_m\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} and B_m(T) = \|u_m - u_{m-1}\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}}, \tag{3.5}$$

there is an $\epsilon_0 > 0$, *so that if* $2A_0(T) \leq \epsilon_0$ *and if* $m = 0, 1, 2, ...$

$$A_m(T) \leq 2A_0(T), \quad B_{m+1}(T) \leq \frac{1}{2} B_m(T). \tag{3.6}$$

Proof. Suppose that u is a weak solution of the linear equation (3.1), by Theorem 2.1.1, if $0 \leq s \leq 2$ and $\frac{n(\kappa-1)}{3\kappa+1} > 1$ (which imply $\max\{\frac{8}{n}+1, \frac{n+1}{n-3}\} < \kappa \leq \frac{n+4}{n-4}$), for every $T > 0$, we have the following Strichartz estimate for any $I \subseteq \mathbb{R}$

$$\|u\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|u(T,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{s-2}(\mathbb{R}^n)}$$
$$\lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|F\|_{L_I^2 L^{\frac{n(\kappa-1)}{3\kappa+1}}}. \tag{3.7}$$

Then if we write

$$(\partial_t^2 + \triangle^2)(u_{m+1} - u_{j+1}) = V_\kappa(u_m, u_j)(u_m - u_j)$$

with

$$V_k(u,v) = \frac{F_\kappa(u) - F_\kappa(v)}{u - v},$$

then by (3.7), the Hölder inequality and the fact that $V_\kappa(u_m, u_j) = O(|u_m|^{\kappa-1} + |u_j|^{\kappa-1})$,

$$\begin{aligned}\|u_{m+1} - u_{j+1}\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} &\leq C\|V_\kappa(u_m,u_j)(u_m-u_j)\|_{L_I^2 L^{\frac{n(\kappa-1)}{3\kappa+1}}} \\ &\leq C'\|V_\kappa(u_m,u_j)\|_{L_I^{\frac{2\kappa}{\kappa-1}} L^{\frac{n\kappa}{(3\kappa+1)}}} \|u_m - u_j\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \\ &\leq C''(\|u_m\|^{\kappa-1}_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \\ &+ \|u_j\|^{\kappa-1}_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}})\|u_m - u_j\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}}.\end{aligned}$$

Take $j=-1$

$$\|u_{m+1}-u_0\|_{L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \le C''\|u_m\|^{\kappa}_{L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}}. \tag{3.8}$$

Thus, if $\epsilon_0^{\kappa-1}$ is small enough so that there exist a constant C'' such that $\epsilon_0^{\kappa-1}C''<\frac{1}{4}$ and if we assume that $A_m(T)\le 2A_0(T)$ then by (3.8) we get

$$A_{m+1}(T)\le A_0(T)+\frac{1}{2}A_m(T),$$

by induction we get the result. Taking $j=m-1$ gives $B_{m+1}(T)\le\frac{1}{2}B_m(T)$. □

Proof of Theorem 3.1.1. First of all, by (3.7) we have,

$$\|u_0\|_{L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}}\le C(\|f\|_{\dot{H}^s}+\|g\|_{\dot{H}^{s-2}}).$$

Therefore if the right side is sufficiently small for all T take $T=\infty$. Otherwise the dominated convergence theorem furnishes Tsufficiently small such that

$$2\|u_0\|_{L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}}=2A_0\le\epsilon_0.$$

Since $B_0(T)=A_0(T)$, using the lemma result, it follows that u_m converges to a limit $u\in L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}(S_T)$ and hence in the sense of distributions. Since

$$\begin{aligned}&\|F_\kappa(u_{m+1})-F_\kappa(u_m)\|_{L_I^2L^{\frac{n(\kappa-1)}{3\kappa+1}}}\\&\qquad\le C'\|V_\kappa(u_{m+1},u_m)\|_{L_I^{\frac{2\kappa}{\kappa-1}}L^{\frac{n\kappa}{(3\kappa+1)}}}\|u_{m+1}-u_m\|_{L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}}.\end{aligned}\tag{3.9}$$

By the lemma, we have $F_\kappa(u_m)\to F_\kappa(u)$ in $L_I^2L^{\frac{n(\kappa-1)}{3\kappa+1}}$. Meanwhile, if we assume the initial data belong to C_0^∞, by (3.6) and (3.7), there exists a v, $(u_m,\partial_t u_m)$ must be a Cauchy sequence in $C([0,T];\dot{H}^s\times\dot{H}^{s-2})$ converging to (u,v). An examination of the Duhamel formula reveals that $v=\partial_t u$, where

$$u(t,\cdot)=\cos(t\triangle)f+\frac{\sin(t\triangle)}{\triangle}g+\int_0^t\frac{\sin((t-s)\triangle)}{\triangle}F(u(s))ds.$$

Hence the proof of existence part of Theorem 3.1.1 with $\kappa \leq \frac{n+4}{n-4}$ is completed.

To prove the uniqueness, we first define $w(t,\cdot) = u_1(t,\cdot) - u_2(t,\cdot)$, where $u_1(t,\cdot)$, $u_2(t,\cdot)$ are two solutions of (3.1) satisfying (3.7), then $w(t,\cdot)$ is the solution of $(\partial_t^2 + \triangle^2)w(t,\cdot) = V_\kappa(u_1(t,\cdot), u_2(t,\cdot))w(t,\cdot)$ with zero inital data. We consider the following equation

$$(\partial_t^2 + \triangle^2)w(t,\cdot) = V_\kappa(u_1, u_2)w(t,\cdot), \tag{3.10}$$

where $V_\kappa(u_1,u_2) \in L_I^{\frac{2\kappa}{\kappa-1}} L^{\frac{n\kappa}{3\kappa+1}}$. Let T be the largest number such that

$$\|V_\kappa(u_1,u_2)\|_{L_I^{\frac{2\kappa}{\kappa-1}} L^{\frac{n\kappa}{3\kappa+1}}} < \epsilon_s, \qquad \text{for,} \qquad t \leq T,$$

where ϵ_s is a universal constant to be determined. In particular, for some constant C, if $\epsilon_s \leq C^{-1}/2$, then by (3.7) and Hölder inequality

$$\|w\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \leq \frac{1}{2}\|w\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}},$$

which implies $w(t,\cdot) = 0$, this implies uniqueness of solutions $u_1(t,\cdot) = u_2(t,\cdot) \in L^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}([0,T] \times \mathbb{R}^n)$. For $n = 3, \kappa > 5$, we can adjust Strichartz norm of u as $u \in L^{\frac{(n+2)(\kappa-1)}{4}}([0,T] \times \mathbb{R}^n)$ that we could use to get Picard iterates to converge by taking advantage of Strichartz estimates as above and prove the results. □

3.2 Well-posedness Theorems for "energy supercritical" case

For the "energy supercritical" range $\kappa > \frac{n+4}{n-4}$, we have two cases to discuss.

(1) Small initial data $f \in \dot{H}^s, g \in \dot{H}^{s-2}$, we have the following

Theorem 3.2.1. *Set* $s = \frac{n}{2} - \frac{4}{\kappa-1}$ *and assume* $n > 4$. *Suppose there exists an* $l \in \mathbb{N}, l \geq 1$ *with* $\frac{n}{2} - \frac{4}{\kappa-1} - 2 \leq l \leq \kappa - 1$, *then there is a* $T > 0$ *a unique (weak)* *solution of the nonlinear beam equation (3.1) satisfying*

$$(u, \partial_t u) \in C([0,T]; \dot{H}^s \times \dot{H}^{s-2}) \quad \text{and} \quad u \in L^{2\kappa}([0,T], L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}(\mathbb{R}^n)). \tag{3.11}$$

Moreover, there is $\epsilon(\kappa) > 0$,so that if

$$\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} < \epsilon(\kappa),$$

then one can take $T = \infty$.

To show that the nonlinear mapping $u_m \to u_{m+1}$ is a contraction for the proof of the existence of this theorem requires a different argument from Lemma 3.1.2. We have to use a specific inequality which comes from Strichartz estimates as the following:

Theorem 3.2.2. *Suppose that u is a solution of (2.1). Then,*

$$\|u\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|(\sqrt{-\triangle})^{s-2} u\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}} + \|u(T,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{s-2}(\mathbb{R}^n)}$$

$$\lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|(\sqrt{-\triangle})^{s-2} F\|_{L_I^2 L^{\frac{2n}{n+2}}}, \tag{3.12}$$

with $S_T = [0,T] \times \mathbb{R}^n$.

Proof. We assume $v = (\sqrt{-\triangle})^{s-2} u$, then

$$(\partial_t^2 + \triangle^2) v = (\sqrt{-\triangle})^{s-2} (\partial_t^2 + \triangle^2) u = (\sqrt{-\triangle})^{s-2} F,$$

and

$$v|_{t=0} = (\sqrt{-\triangle})^{s-2} f \in \dot{H}^2,$$

$$v_t|_{t=0} = (\sqrt{-\triangle})^{s-2} g \in L^2.$$

By (2.3) with $s=2$, we have,

$$\|(\sqrt{-\triangle})^{s-2}u\|_{L_I^{2\kappa}L^{\frac{2n\kappa}{(n-4)\kappa-2}}} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|(\sqrt{-\triangle})^{s-2}F\|_{L_I^2 L^{\frac{2n}{n+2}}}. \quad (3.13)$$

Choose $p=2\kappa, r=\frac{n\kappa(\kappa-1)}{3\kappa+1}$ for (2.2), we have

$$\|u\|_{L_I^{2\kappa}L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|u(T,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{s-2}(\mathbb{R}^n)}$$
$$\lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}} + \|(\sqrt{-\triangle})^{s-2}F\|_{L_I^2 L^{\frac{2n}{n+2}}}. \quad (3.14)$$

Combining with (3.13) and (3.14), we have (3.12). □

We first introduce fractional chain rule lemma,

Lemma 3.2.3. *Let $F\in C^{l+1}(\mathbb{C};\mathbb{C}), l\in\mathbb{N}$. Assume that there is $\kappa\ge l$ such that*

$$|\nabla^i F(z)| \le |z|^{\kappa-i}, \quad i=1,2,...,l.$$

If $\kappa>2, 0\le s\le l$, $1<q<r<\infty$ obey the scaling condition $\frac{n}{q}=\frac{n\kappa}{r}-(\kappa-1)s$, then

$$\|F(f)-F(g)\|_{\dot{W}^{s,q}(\mathbb{R}^n)} \lesssim (\|f\|_{\dot{W}^{s,r}(\mathbb{R}^n)} + \|g\|_{\dot{W}^{s,r}(\mathbb{R}^n)})^{\kappa-1}\|f-g\|_{\dot{W}^{s,r}(\mathbb{R}^n)}, \quad (3.15)$$

for all $f,g\in\dot{W}^{s,r}$.

Proof. By the fundamental theorem of calculus we write

$$F(f)-F(g) = \int_0^1 DF((1-\theta)f+\theta g)(f-g)d\theta.$$

Let $V(f,g)=\int_0^1 DF((1-\theta)f+\theta g)d\theta$, we have $F(f)-F(g)=(f-g)V(f,g)$. By the generalized Leibniz rule (see Theorem, 5 A. Gulisashvili and M.A. Kon [8])

$$\|F(f)-F(g)\|_{\dot{W}^{s,q}} \lesssim \|f-g\|_{\dot{W}^{s,r}}\|V(f,g)\|_{L^p} + \|f-g\|_{L^a}\|V(f,g)\|_{\dot{W}^{s,b}},$$

where $\frac{1}{q}=\frac{1}{r}+\frac{1}{p}$, $\frac{1}{q}=\frac{1}{a}+\frac{1}{b}$.

Since

$$\|V(f,g)\|_{L^p} \lesssim \|f\|^{\kappa-1}_{L^{p(\kappa-1)}} + \|g\|^{\kappa-1}_{L^{p(\kappa-1)}},$$

by Sobolev embedding, we have

$$\|V(f,g)\|_{L^p} \lesssim \|f\|_{\dot{W}^{s,r}}^{\kappa-1} + \|g\|_{\dot{W}^{s,r}}^{\kappa-1},$$

if $\frac{1}{r} - \frac{1}{p(\kappa-1)} = \frac{s}{n}$. Combining with $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, we have

$$\frac{n}{q} = \frac{n\kappa}{r} - (\kappa-1)s.$$

Therefore

$$\|f-g\|_{\dot{W}^{s,r}} \|V(f,g)\|_{L^p} \lesssim \|f-g\|_{\dot{W}^{s,r}} (\|f\|_{\dot{W}^{s,r}}^{\kappa-1} + \|g\|_{\dot{W}^{s,r}}^{\kappa-1}). \tag{3.16}$$

By Leibnitz rule for fractional derivatives (see Lemma A3, T. Kato[10]), and Sobolev embedding,

$$\|V(f,g)\|_{\dot{W}^{s,b}} \lesssim \int_0^1 \|(1-\theta)f + \theta g\|_{\dot{W}^{s,c}}^{\kappa-1} d\theta \lesssim \|f\|_{\dot{W}^{s,c}}^{\kappa-1} + \|g\|_{\dot{W}^{s,c}}^{\kappa-1}$$

$$\|f-g\|_{L^a} \lesssim \|f-g\|_{\dot{W}^{s,r}},$$

where $\frac{1}{b} = \frac{\kappa-1}{c} - (\kappa-2)\frac{s}{n}$, $\quad \frac{1}{a} = \frac{1}{r} - \frac{s}{n}$, combining with $\frac{1}{q} = \frac{1}{a} + \frac{1}{b}$ and $\frac{n}{q} = \frac{n\kappa}{r} - (\kappa-1)s$, we have $c = r$. Therefore we have

$$\|f-g\|_{L^a} \|V(f,g)\|_{\dot{W}^{s,b}} \lesssim \|f-g\|_{\dot{W}^{s,r}} (\|f\|_{\dot{W}^{s,r}}^{\kappa-1} + \|g\|_{\dot{W}^{s,r}}^{\kappa-1}), \tag{3.17}$$

with

$$\frac{n}{q} = \frac{n\kappa}{r} - (\kappa-1)s.$$

Combining with (3.16) and (3.17), we have the result.

□

Then we give the contraction lemma as the following:

Lemma 3.2.4. *Given $s = \frac{n}{2} - \frac{4}{\kappa-1}, n > 4$, there exists an $l \in \mathbb{N}$, when $\frac{n}{2} - \frac{4}{\kappa-1} - 2 \leq l \leq \kappa - 1$, and $l \geq 1$, then if we set,*

$$A_m(T) = \|u_m\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|(\sqrt{-\triangle})^{s-2} u_m\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}},$$

and,

$$B_m(T) = \|(\sqrt{-\triangle})^{s-2}(u_m - u_{m-1})\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}}, \tag{3.18}$$

there is an $\epsilon_0 > 0$ so that if $m = 0, 1, 2, \ldots$

$$A_m(T) \leq 2A_0(T), \quad B_{m+1}(T) \leq \frac{1}{2} B_m(T), \quad \text{if} \quad 2A_0 \leq \epsilon_0. \tag{3.19}$$

Proof. By the Leibniz rule for fractional derivatives (see Lemma A3, T.Kato[10]) with $0 \leq s-2 \leq l$,

$$\|(\sqrt{-\triangle})^{s-2} F(u)\|_{L_I^2 L^q} \lesssim \|u\|^{\kappa-1}_{L_I^{2\kappa} L^p} \|(\sqrt{-\triangle})^{s-2} u\|_{L_I^{2\kappa} L^r}, \tag{3.20}$$

where $\frac{1}{q} = \frac{\kappa-1}{p} + \frac{1}{r}$. We apply (3.20) with $q = \frac{2n}{n+2}$, $p = \frac{n\kappa(\kappa-1)}{3\kappa+1}$, $r = \frac{2n\kappa}{(n-4)\kappa-2}$. Specifically , this inequality along with (3.12) applied to the equation

$$(\partial_t^2 + \triangle^2)(u_{m+1} - u_0) = F_\kappa(u_m),$$

gives

$$\begin{aligned} &\|u_{m+1}\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|(\sqrt{-\triangle})^{s-2} u_{m+1}\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}} \\ &\leq C \|u_m\|^{\kappa-1}_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \|(\sqrt{-\triangle})^{s-2} u_m\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}} \\ &\qquad + \|u_0\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|(\sqrt{-\triangle})^{s-2} u_0\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}}. \end{aligned} \tag{3.21}$$

So we have

$$\begin{aligned} A_{m+1} &\leq C' \|u_m\|^{\kappa-1}_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} A_m + A_0 \\ &\leq C'' A_m^\kappa + A_0. \end{aligned}$$

From (3.12),

$$A_0(T) = \|u_0\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} + \|(\sqrt{-\triangle})^{s-2} u_0\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}} \leq C(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-2}}). \quad (3.22)$$

Then we choose a proper ϵ_0 with a constant C such that $C2^\kappa \epsilon_0^{\kappa-1} < 1$, then we could get $A_{m+1} \leq 2A_0$ by induction. By Hölder inequality for

$$\begin{aligned} B_{m+1}(T) &= \|(\sqrt{-\triangle})^{s-2}(u_{m+1} - u_m)\|_{L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}} \\ &\leq C\|(\sqrt{-\triangle})^{s-2}(F_\kappa(u_m) - F_\kappa(u_{m-1}))\|_{L_I^2 L^{\frac{2n}{n+2}}}. \quad (3.23) \end{aligned}$$

By (3.15) with $0 \leq s-2 \leq l$, we have

$$\begin{aligned} &\|(F_\kappa(u_m) - F_\kappa(u_{m-1}))\|_{L_I^2 \dot{W}^{s-2,\frac{2n}{n+2}}} \\ &\leq C'(\|u_m\|_{L_I^{2\kappa} \dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}} \\ &+ \|u_{m-1}\|_{L_I^{2\kappa} \dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}})^{\kappa-1} \|u_m - u_{m-1}\|_{L_I^{2\kappa} \dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}} \\ &\leq C'' \epsilon_0^{\kappa-1} B_m(T), \end{aligned}$$

leading to the desired bound if $C''\epsilon_0^{\kappa-1} < \frac{1}{2}$. □

With this contraction lemma, we finish the following

Proof of Theorem 3.2.1. Arguing as before for $2A_0 \leq \epsilon_0$ holds. Since $B_0(T) \leq A_0(T)$, then by (3.19), u_m must tend to a limit in $L_I^{2\kappa} \dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}$. Similarly, we see that $F_\kappa(u_m)$ converges to a limit in $L_I^2 \dot{W}^{s-2,\frac{2n}{n+2}}$. By Fatou's lemma,

$$\|u\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \leq \liminf_{m\to\infty} \|u_m\|_{L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}} \leq 2A_0(T) < \infty, \quad (3.24)$$

then $u \in L_I^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}$. By (3.18), (3.19) we have $(\sqrt{-\triangle})^{s-2} u \in L_I^{2\kappa} L^{\frac{2n\kappa}{(n-4)\kappa-2}}$, and by the fractional chain rule we have $(\sqrt{-\triangle})^{s-2} F_\kappa(u) \in L_I^2 L^{\frac{2n}{n+2}}$. Applying Theorem 3.2.2, we proved $(u, \partial_t u) \in C([0,T]; \dot{H}^s \times \dot{H}^{s-2})$, then the existence proof of Theorem 3.2.1 with $\kappa > \frac{n+4}{n-4}$ is completed.

To prove the uniqueness part of the theorem, we assume $u_1(t,\cdot)$ and $u_2(t,\cdot)$ are two solutions of (3.1) satisfying (3.11) then the difference $w(t,\cdot) = u_1(t,\cdot) - u_2(t,\cdot)$ satisfies the equation

$$(\partial_t^2 + \triangle^2)w(t,\cdot) = V(t,\cdot),$$

$$w(0,x) = \partial_t w(0,x) = 0,$$

$$V(t,\cdot) = F_\kappa(u_1(t,\cdot)) - F_\kappa(u_2(t,\cdot)).$$

By the Strichartz estimates (2.4), we have

$$\|w\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}} \leq C\|V\|_{L_I^2\dot{W}^{s-2,\frac{2n}{n+2}}}.$$

By (3.15),

$$\|(F_\kappa(u_1) - F_\kappa(u_2))\|_{L_I^2\dot{W}^{s-2,\frac{2n}{n+2}}}$$

$$\leq C'(\|u_1\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}} + \|u_2\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}})^{\kappa-1}\|u_1 - u_2\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}}.$$

Then we have,

$$\|w\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}}$$

$$\leq C'(\|u_1\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}} + \|u_2\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}})^{\kappa-1}\|w\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}}$$

If we choose T sufficiently small, $\|w\|_{L_I^{2\kappa}\dot{W}^{s-2,\frac{2n\kappa}{(n-4)\kappa-2}}} = 0$ in $[0,T]$, iterating the argument it follows that $w = 0$ in $[0,T]$ for any fixed $T > 0$ and this proves uniqueness.

□

(2) For initial data $f \in H^s, g \in H^{s-2}$ and we have the following

Theorem 3.2.5. *Set $s = \frac{n}{2} - \frac{4}{\kappa-1}$ and assume $n > 4$. Suppose there exists an $l \in \mathbb{N}$ with $\frac{n}{2} - \frac{4}{\kappa-1} - 2 \leq l \leq \kappa$, there is a $T > 0$, a unique (weak) solution of the nonlinear beam equation (3.1) satisfying*

$$(u, \partial_t u) \in C([0,T]; H^s \times H^{s-2}) \quad and \quad u \in L^{\frac{(n+2)(\kappa-1)}{4}}([0,T] \times \mathbb{R}^n) \tag{3.25}$$

Similar to the proof of Theorem 3.2.1, we also need specific Strichartz inequality as the following theorem.

Theorem 3.2.6. *Suppose that u is a solution of (2.1). Then,*

$$\|u\|_{L^{\frac{(n+2)(\kappa-1)}{4}}(S_T)} + \|(\sqrt{I-\triangle})^{s-2}u\|_{L^{\frac{2(n+2)}{n-4}}(S_T)} + \|u(T,\cdot)\|_{H^s(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{H^{s-2}(\mathbb{R}^n)}$$

$$\lesssim \|f\|_{H^s} + \|g\|_{H^{s-2}} + \|(\sqrt{I-\triangle})^{s-2}F\|_{L^{\frac{2(n+2)}{n+4}}(S_T)}, \tag{3.26}$$

with $S_T = [0,T]\times\mathbb{R}^n, T<\infty$.

The proof of this theorem is similar to the proof of Theorem 3.2.2. We also need the following

Lemma 3.2.7. *Given* $s = \frac{n}{2} - \frac{4}{\kappa-1}, n > 4$, *there exists an* $l \in \mathbb{N}$, *when* $\frac{n}{2} - \frac{4}{\kappa-1} - 2 \leq l \leq \kappa$, $q = \frac{(n+2)(\kappa-1)}{4}$ *then if we set,*

$$A_m(T) = \|u_m\|_{L^q(S_T)} + \|(\sqrt{I-\triangle})^{s-2}u_m\|_{L^{\frac{2(n+2)}{n-4}}(S_T)},$$

and

$$B_m(T) = \|u_m - u_{m-1}\|_{L^{\frac{2(n+2)}{n-4}}(S_T)}, \tag{3.27}$$

there is an $\epsilon_0 > 0$ *so that if* $2A_0(T) \leq \epsilon_0, B_0(T) \lesssim A_0(T)$ *and if* $m = 0,1,2,...$

$$A_m(T) \leq 2A_0(T), \quad B_{m+1}(T) \leq \frac{1}{2}B_m(T). \tag{3.28}$$

Proof. By the same way to prove Lemma 3.2.4, we easily have

$$\begin{aligned} A_{m+1} &\leq C'\|u_m\|^{\kappa-1}_{L^{\frac{(n+2)(\kappa-1)}{4}}} A_m + A_0 \\ &\leq C'' A_m^{\kappa} + A_0. \end{aligned}$$

From (3.26),

$$A_0(T) = \|u_0\|_{L^q(S_T)} + \|(\sqrt{I-\triangle})^{s-2}u_0\|_{L^{\frac{2(n+2)}{n-4}}} \leq C(\|f\|_{H^s} + \|g\|_{H^{s-2}}). \quad (3.29)$$

Then we want to choose ϵ_0 small enough with a constant C'' such that $C''2^\kappa\epsilon_0^{\kappa-1} < 1$, for then $A_{m+1} \leq 2A_0$ by induction. Similarly, by Hölder's inequality, Strichartz inequality and (3.20)(which also works for inhomogeneous spaces), for

$$\begin{aligned} B_{m+1}(T) &= \|u_{m+1} - u_m\|_{L^{\frac{2(n+2)}{n-4}}} \\ &\leq C\|F_\kappa(u_m) - F_\kappa(u_{m-1})\|_{L^{\frac{2(n+2)}{n+4}}} \\ &\leq C'(\|u_m\|^{\kappa-1}_{L^{\frac{(n+2)(\kappa-1)}{4}}} + \|u_{m-1}\|^{\kappa-1}_{L^{\frac{(n+2)(\kappa-1)}{4}}})B_m(T) \\ &\leq C''\epsilon_0^{\kappa-1}B_m(T). \end{aligned}$$

If we choose a ϵ_0 such that $C''\epsilon_0^{\kappa-1} < \frac{1}{2}$, we have $B_{m+1}(T) \leq \frac{1}{2}B_m(T)$. □

With this contraction lemma, and since $\|u_0\|_{L^{\frac{2(n+2)}{n-4}}} \lesssim \|(\sqrt{I-\triangle})^{s-2}u_0\|_{L^{\frac{2(n+2)}{n-4}}}$, then we have $B_0(T) \lesssim A_0(T)$. By the same way to prove Theorem 3.2.1 (Fatou's Lemma), we have $u \in L^{\frac{(n+2)(\kappa-1)}{4}}([0,T]\times\mathbb{R}^n)$. Also if $\phi \in C_0^\infty$, $\langle u_m, \phi\rangle \to \langle u, \phi\rangle$ as $m \to \infty$. Therefore, by Hölder inequality

$$|\langle u_m, \phi\rangle| \leq \|(\sqrt{I-\triangle})^{s-2}u_m\|_{L^{\frac{2(n+2)}{n-4}}} \|(\sqrt{I-\triangle})^{2-s}\phi\|_{L^{\frac{2(n+2)}{n+8}}} \quad (3.30)$$

$$\leq 2A_0\|(\sqrt{I-\triangle})^{2-s}\phi\|_{L^{\frac{2(n+2)}{n+8}}}, \quad (3.31)$$

we have

$$|\langle u, \phi\rangle| \leq 2A_0\|(\sqrt{I-\triangle})^{s-2}\phi\|_{L^{\frac{2(n+2)}{n+8}}},$$

and hence $(\sqrt{I-\triangle})^{s-2}u \in L^{\frac{2(n+2)}{n-4}}$. By Strichartz estimates and the fractional chain rule, we have $(\sqrt{I-\triangle})^{s-2}F_\kappa(u) \in L^{\frac{2(n+2)}{n+4}}$. Applying Theorem 3.2.6, we proved $(u, \partial_t u) \in C([0,T]; H^s \times H^{s-2})$, then the existence proof of Theorem 3.2.5 with $\kappa > \frac{n+4}{n-4}$ is completed. By the same way of the uniqueness proof in the previous theorem, we get the uniqueness of the solution.

3.3 Scattering Theory

In this section we consider the existence of scattering operators for nonlinear beam equation (3.1) with initial data $f \in \dot{H}^s, g \in \dot{H}^{s-2}$.

Theorem 3.3.1. *For $\kappa \geq 1$, consider u is the solution of the equation (3.1) with the norm of the data is small, namely,*

$$\|f\|_{\dot{H}^{s_c}} + \|g\|_{\dot{H}^{s_c-2}} < \epsilon. \tag{3.32}$$

Then there exists $\epsilon > 0$ small such that for such data (f, g), there is small data $(f_+, g_+) \in \dot{H}^{s_c} \times \dot{H}^{s_c-2}$ so that the solution to the free beam equation with this data,

$$\begin{cases} \partial_t^2 u_+ + \triangle^2 u_+ = 0, \\ u_+ \mid_{t=0} = f_+ \in \dot{H}^{s_c}, \\ \partial_t u_+ \mid_{t=0} = g_+ \in \dot{H}^{s_c-2}, \end{cases} \tag{3.33}$$

satisfies

$$\lim_{T \to +\infty} \|u(T,\cdot) - u_+(T,\cdot)\|_{\dot{s(\kappa)}} = 0, \tag{3.34}$$

where

$$\|u(T,\cdot)\|^2_{\dot{s(\kappa)}} = \|u(T,\cdot)\|^2_{\dot{H}^{s_c}} + \|\partial_t u(T,\cdot)\|^2_{\dot{H}^{s_c-2}}.$$

Conversely, if $(f_-, g_-) \in \dot{H}^{s_c} \times \dot{H}^{s_c-2}$ has sufficiently small norm and u_- is the solution to the free beam equation with this data, then there exists a solution u to (3.1) satisfying

$$\lim_{T \to -\infty} \|u(T,\cdot) - u_-(T,\cdot)\|_{\dot{s(\kappa)}} = 0. \tag{3.35}$$

Thus, the scattering operator $S : (f_-, g_-) \to (f_+, g_+)$ exists in a neighborhood of the origin in $\dot{H}^{s_c} \times \dot{H}^{s_c-2}$.

In the proof, we will only consider $\kappa \leq \frac{n+4}{n-4}$, $n > 4$ case, because for $\kappa > \frac{n+4}{n-4}$ case, the method is the same, provided l satisfies hypothesis of Theorem 3.2.5.

Proof. To prove (3.38), first, by the proof of Theorem 3.1.1, we have $u \in L^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}$ and $F_\kappa(u) \in L^2 L^{\frac{n(\kappa-1)}{3\kappa+1}}$. It follows that there is an increasing sequence of times,T_j, for which

$$(\int_{T_j}^{\infty} (\int_{\mathbb{R}^n} |F_\kappa(u)|^{\frac{n(\kappa-1)}{3\kappa+1}} dx)^{\frac{2(3\kappa+1)}{n(\kappa-1)}} dt)^{\frac{1}{2}} < 2^{-j}. \tag{3.36}$$

Then we let u_j solve the free beam equation with the same data as u at $t = T_j$:

$$\begin{cases} \partial_t^2 u_j + \triangle^2 u_j = 0, \\ u_j \mid_{t=0} = u(T_j, \cdot), \quad \partial_t u_j \mid_{t=0} = \partial_t u(T_j, \cdot) \end{cases}.$$

Then $u - u_j$ has zero data at $t = T_j$ and satisfies

$$(\partial_t^2 + \triangle^2)(u - u_j) = F_\kappa(u).$$

Then by the Strichartz estimates (2.2) and (3.40), we have for $T > T_j$,

$$\|u(T, \cdot) - u_j(T, \cdot)\|_{\dot{s(\kappa)}} \leq C(\int_{T_j}^{\infty} (\int_{\mathbb{R}^n} |F_\kappa(u)|^{\frac{n(\kappa-1)}{3\kappa+1}} dx)^{\frac{2(3\kappa+1)}{n(\kappa-1)}} dt)^{\frac{1}{2}} < 2^{-j}. \tag{3.37}$$

Since u and u_k have a the same initial data at $t = T_k$, if $k > j$ this implies

$$\|u_k(T_k, \cdot) - u_j(T_k, \cdot)\|_{\dot{s(\kappa)}} = \|u(T_k, \cdot) - u_j(T_k, \cdot)\|_{\dot{s(\kappa)}} \leq 2^{-j}.$$

Consequently, the energy inequality yields

$$\|u_k(0, \cdot) - u_j(0, \cdot)\|_{\dot{s(\kappa)}} \leq C 2^{-j}.$$

Therefore $f_j = u_j(0, \cdot), g_j = \partial_t u_j(0, \cdot)$, is a Cauchy sequence of initial data in $\dot{H}^{s_c} \times \dot{H}^{s_c - 2}$. If we let $\lim_{j\to\infty} f_j = f_+, \lim_{j\to\infty} g_j = g_+$, then (3.41) and the energy inequality yield

$$\lim_{T\to+\infty} \|u(T, \cdot) - u_+(T, \cdot)\|_{\dot{s(\kappa)}} = 0.$$

To prove the second part of the theorem we define u_- so that the solution to the free beam equation with initial data $(f_-, g_-) \in \dot{H}^{s_c} \times \dot{H}^{s_c-2}$, has small norm. We let

$u_{-1} = 0$ and $u_0 = u_-$ be defined by

$$\begin{cases} \partial_t^2 u_- + \triangle^2 u_- = 0, \\ u_- \mid_{t=0} = f_- \in \dot{H}^{s_c}, \\ \partial_t u_- \mid_{t=0} = g_- \in \dot{H}^{s_c - 2}, \end{cases} \tag{3.38}$$

and define $u_m, m = 1, 2, ...$ by

$$u_m(t, \cdot) = u_0(t, \cdot) + \int_{-\infty}^{t} \frac{\sin((t-s)\triangle)}{\triangle} F_\kappa(u_{m-1})(s, \cdot) ds, \tag{3.39}$$

which means that u_m solves $\partial_t^2 u_m + \triangle^2 u_m = F_\kappa(u_{m-1})$ with initial data (f_-, g_-). Then use the Picard iteration argument. Similar to Lemma 3.1.2, we have u_m converges to a solution u of

$$u(t, \cdot) = u_0(t, \cdot) + \int_{-\infty}^{t} \frac{\sin((t-s)\triangle)}{\triangle} F_\kappa(u)(s, \cdot) ds, \tag{3.40}$$

where $u \in L^{2\kappa} L^{\frac{n\kappa(\kappa-1)}{3\kappa+1}}$, $F_\kappa(u) \in L^2 L^{\frac{n(\kappa-1)}{3\kappa+1}}$ and for any T, $(u, \partial_t u) \in C([0,T]; \dot{H}^{s_c} \times \dot{H}^{s_c-2})$. Then we have (3.39), therefore, the scattering operator $S : (f_-, g_-) \to (f_+, g_+)$ exists in a neighborhood of the origin in $\dot{H}^{s_c} \times \dot{H}^{s_c-2}$. □

Chapter 4

Ill-posedness for the nonlinear beam equation in $\mathbb{R}^n$

We now consider ill-posedness of the nonlinear beam equation (1.1) in the defocusing case.

We analyze the small dispersion approximation for the beam equation (1.1),

$$\begin{cases} \partial_\tau^2 \phi(\tau, y) + \nu^4 \triangle^2 \phi(\tau, y) = \omega |\phi|^{\kappa-1}\phi, \\ \phi(0, y) = \phi_0(y), \\ \partial_s \phi(0, y) = 0, \end{cases} \tag{4.1}$$

in the zero-dispersion limit $\nu \to 0$. Then for time t define

$$u(t, x) = \phi(t, \nu x). \tag{4.2}$$

For fixed initial datum ϕ_0 in the small dispersion regime $\nu \to 0$, (4.1) can be transformed back into (1.1). Indeed, for any solution ϕ of (4.1), by the scaling symmetry,

$$\lambda^{\frac{-4}{\kappa-1}} \phi(\lambda^{-2} t, \lambda^{-1} \nu x) \tag{4.3}$$

also defines a solution of (1.1). Setting $\nu = 0$ in (4.1) gives the ODE

$$\begin{cases} \partial_s^2 \phi(\tau, y) = \omega |\phi|^{\kappa-1}\phi, \\ \phi(0, y) = \phi_0(y), \\ \partial_\tau \phi(0, y) = 0. \end{cases} \tag{4.4}$$

We define ϕ^0 as this ODE solution. In the defocusing case $\omega = -1$, we give the solution formula as the following

$$\phi^0(\tau, y) = \mathcal{C}(|\phi_0(y)|^{\frac{\kappa-1}{2}}\tau)\phi_0(y), \tag{4.5}$$

where $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ is the unique solution to the ODE

$$-\mathcal{C}''(\tau) = |\mathcal{C}(\tau)|^{\kappa-1}\mathcal{C}(\tau); \qquad \mathcal{C}(0) = 1; \qquad \mathcal{C}'(0) = 0.$$

This is the Hamiltonian flow on a two dimensional phase space with Hamiltonian

$$H : \frac{1}{2}|\mathcal{C}'(\tau)|^2 + \frac{1}{\kappa+1}|\mathcal{C}(\tau)|^{\kappa+1}.$$

It can be seen that $\mathcal{C}$ is a bounded nonconstant periodic function and $C^{\kappa+2}$ function since $F = \omega|\phi|^{\kappa-1}\phi \in C^\kappa$. To avoid causing some problems with smoothness of $|\phi_0(y)|$, we let $\phi_0(y) = (\psi(y))^{2l}$, where $\psi(y)$ is Schwartz function.

We now use the following lemma to see that the solution of (4.1) ϕ may stay close to the ODE solution ϕ^0, when $\nu > 0$ but small.

Lemma 4.0.2. *Let $n \geq 1, \kappa \geq 1, k > \frac{n}{2}$ be an integer, and if κ is not an odd integer, then $\kappa \geq k+2$. Let $\phi_0(y) = (\psi(y))^{2l}$, where $\psi(y)$ is a Schwartz function, and l is sufficiently large, so ϕ_0 is the square of a Schwartz function. Then there exist C, c, such that for each sufficiently small real number $0 < \nu \leq c$, there exists a solution $\phi(\tau, y)$ of (4.1) for all $|\tau| \leq c|\ln \nu|^c$ such that*

$$||\phi(\tau) - \phi^0(\tau)||_{H^k} + ||\phi_\tau(\tau) - \phi_\tau^0(\tau)||_{H^k} \leq C|\nu|, \tag{4.6}$$

with ϕ^0 as in (4.5).

Proof. We define the function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = |z|^{\kappa-1} z,$$

and plug F in (4.4), thus

$$\partial_\tau^2 \phi^0 = \omega F(\phi^0),$$

and the equation to be solved is

$$\partial_\tau^2 \phi + \nu^4 \triangle_y^2 \phi = \omega F(\phi),$$

then with the ansatz

$$\phi = \phi^0 + w$$

w is a solution of the Cauchy problem

$$\begin{cases} \partial_\tau^2 w + \nu^4 \triangle^2 w = \nu^4 \triangle^2 \phi^0 + \omega(F(\phi^0 + w) - F(\phi^0)), \\ w(0, y) = 0, \\ \partial_\tau w(0, y) = 0. \end{cases} . \tag{4.7}$$

Since $\kappa \geq k + 2$, it is guaranteed that F is a C^{k+2} function with all k derivatives locally Lipschitz and hence $\mathcal{C}$ is C^{k+4}. Define the energy of w by

$$E_\nu(w(\tau)) = \int \frac{1}{2} |w_\tau(\tau, y)|^2 + \frac{\nu^4}{2} |\triangle w(\tau, y)|^2 dy, \tag{4.8}$$

if we have $\partial_\tau^2 w + \nu^4 \triangle^2 w = \mathcal{F}$, then the energy identity gives

$$\partial_\tau E_\nu(w(\tau)) = \int w_\tau(\tau, y) \mathcal{F}(\tau, y) dy. \tag{4.9}$$

By Cauchy-Schwarz inequality

$$|\partial_\tau E_\nu^{\frac{1}{2}}(w(\tau))| \leq C \|\mathcal{F}(\tau)\|_2.$$

Similarly, if we define

$$E_{\nu,k}(w(\tau)) = \sum_{j=0}^{k} \sum_{|\alpha| \leq j} E_\nu(\partial_y^\alpha w(\tau)).$$

Then

$$|\partial_\tau E^{\frac{1}{2}}_{\nu,k}(w(\tau))| \leq C\|\mathcal{F}(\tau)\|_{H^k} \tag{4.10}$$

and

$$E^{\frac{1}{2}}_{\nu,k}(w(\tau)) = \int_0^\tau \partial_\tau E^{\frac{1}{2}}_{\nu,k}(w(\tau'))d\tau' \leq C\int_0^\tau \|\mathcal{F}(\tau')\|_{H^k}d\tau'. \tag{4.11}$$

Since $\phi_0 = \psi(y)^{2l}, \psi(y)$ is Schwartz, F is C^{k+2} and $\mathcal{C}$ is C^{k+4} ,

$$\|\phi^0\|_{H^k} + \|\phi^0\|_{C^k} \leq C(1+|\tau|)^k, \tag{4.12}$$

and

$$\|\nu^4\triangle^2\phi^0\|_{H^k} \leq C\nu^4(1+|\tau|)^{k+4}. \tag{4.13}$$

Using Taylor formula and the fact that H^k is an algebra since $k > \frac{n}{2}$, we have

$$\|F(\phi^0+w)(\tau) - F(\phi^0)(\tau)\|_{H^k} \lesssim \|w(\tau)\|_{H^k}(\|w(\tau)\|^{\kappa-1}_{H^k} + \|\phi^0(\tau)\|^{\kappa-1}_{H^k}).$$

Define

$$e(\tau) := \sup_{0\leq\tau'\leq\tau} E^{\frac{1}{2}}_{\nu,k}(w(\tau')),$$

which is a non-decreasing function. By the fundamental theorem of calculus

$$\|w(\tau)\|_{H^k} \leq \int_0^\tau \|w_\tau(\tau')\|_{H^k}d\tau' \quad \leq \int_0^\tau E^{\frac{1}{2}}_{\nu,k}(w(\tau'))d\tau' \quad \leq C\tau e(\tau). \tag{4.14}$$

Under the assumption that $w(\tau)$ is bounded in H^k, e.g. $\|w(\tau)\|_{H^k} \leq 1$ and combining with (4.13), (4.14), we have

$$\|\nu^4\triangle^2\phi^0 + F(\phi^0+w)(\tau) - F(\phi^0)(\tau)\|_{H^k} \leq C(1+|\tau|)^c(\nu^4 + e(\tau) + e(\tau)^\kappa).$$

Then combining with (4.12), we have the differential inequality

$$e(\tau) \leq C\int_0^\tau (1+|\tau'|)^C(\nu^4 + e(\tau') + e(\tau')^\kappa)d\tau'.$$

Since $e(0) = 0$, by Gronwall's inequality , for $|\tau| \leq c|\ln\nu|^c$, then we have $e(\tau) \leq C\nu^{\frac{7}{2}}$, and the claim follows from (4.14) if ν is sufficiently small. □

Now we give the ill-posedness result about the nonlinear beam equation (1.1) as the following

Theorem 4.0.3. *Let $n \geq 1$, $\omega = -1$ and $\kappa > 1$, if κ is not an odd integer, we assume $\kappa \geq k+2$ for some integer $k > n/2$. Suppose that $0 < s < s_c = \frac{n}{2} - \frac{4}{\kappa-1}$. Then for any $\epsilon > 0$, there exist a real-valued solution u of the nonlinear beam equation (1.1) and $t \in \mathbb{R}^+$ such that $u(0) \in \mathcal{S}$,*

$$\|u(0)\|_{H^s} < \epsilon,$$

$$u_t(0) = 0,$$

$$0 < t < \epsilon,$$

$$\|u(t)\|_{H^s} > \epsilon^{-1}.$$

In particular, for any $t > 0$ the solution map $\mathcal{S} \times \mathcal{S} \ni (u(0), u_t(0)) \to (u(t), u_t(t))$, for Cauchy problem (1.1) fails to be continuous at 0 in the $H^s \times H^{s-2}$ topology.

Remark 4.0.4. Since when time progresses, the function $\phi^\nu(\tilde{t})$ transfers its energy to increasingly higher frequencies, then we take considering H^s instead of $\dot{H}^s$.

Proof. Let $0 < \nu \ll 1$ be a parameter. We will construct solutions of (1.1) which are depending on ν, and analyze them quantitatively as $\nu \searrow 0$. By the lemma above, for $\nu \leq c$ there exists a solution $\phi^\nu(\tau, y) = \phi(\tau, y)$ to the equation (4.1) with initial data,

$$\phi^\nu(0, y) := \phi_0(y), \qquad \phi^\nu_\tau := 0, \tag{4.15}$$

and we have for $|\tau| \leq C|\ln \nu|^c$,

$$||\phi^\nu(\tau) - \phi^0(\tau)||_{H^k} + ||\phi^\nu_\tau(\tau) - \phi^0_\tau(\tau)||_{H^k} \leq C|\nu|. \tag{4.16}$$

Applying the scaling symmetry gives then solutions $u(t,x) = u^{(\nu,\lambda)}(t,x)$ to (1.1) defined by

$$u^{(\nu,\lambda)}(t,x) = \lambda^{\frac{-4}{\kappa-1}} \phi^\nu(\lambda^{-2}t, \lambda^{-1}\nu x). \tag{4.17}$$

In particular, we have the initial data

$$u^{(\nu,\lambda)}(0,x) = \lambda^{\frac{-4}{\kappa-1}}\phi_0(\lambda^{-1}\nu x); \qquad u_t^{(\nu,\lambda)}(0,x) = 0. \tag{4.18}$$

Assume $0 < \lambda \le \nu \ll 1$, and observe

$$[u^{(\nu,\lambda)}(0)]^\wedge(\xi) = \lambda^{\frac{-4}{\kappa-1}}(\frac{\lambda}{\nu})^n\hat{\phi}_0(\frac{\lambda}{\nu}\xi).$$

Hence

$$\|u^{(\nu,\lambda)}(0)\|_{H^s}^2 = \lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^{2n}\int|\hat{\phi}_0(\frac{\lambda}{\nu}\xi)|^2(1+|\xi|^2)^s d\xi,$$

define $\eta = \frac{\lambda}{\nu}\xi$,

$$\|u^{(\nu,\lambda)}(0)\|_{H^s}^2 = \lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^{n}\int|\hat{\phi}_0(\eta)|^2(1+|\frac{\nu}{\lambda}\eta|^2)^s d\eta$$

$$\approx \lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^{n-2s}\int_{|\eta|\ge\lambda\nu^{-1}}|\hat{\phi}_0(\eta)|^2|\eta|^{2s}d\eta + \lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^{n}\int_{|\eta|\le\lambda\nu^{-1}}|\hat{\phi}_0(\eta)|^2 d\eta$$

$$= \lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^{n-2s}\int_{\mathbb{R}^n}|\hat{\phi}_0(\eta)|^2|\eta|^{2s}d\eta + \lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^{n-2s}\int_{|\eta|\le\lambda\nu^{-1}}|\hat{\phi}_0(\eta)|^2((\frac{\lambda}{\nu})^{2s}-|\eta|^{2s})d\eta.$$

Then for some constant C, we have

$$\|u^{(\nu,\lambda)}(0)\|_{H^s} \le C\lambda^{\frac{-4}{\kappa-1}}(\frac{\lambda}{\nu})^{\frac{n}{2}-s} = C\lambda^{s_c-s}\nu^{s-\frac{n}{2}}.$$

Given ν, define

$$\lambda^{s_c-s}\nu^{s-\frac{n}{2}} = \epsilon. \tag{4.19}$$

Consider the behavior of $u^{(\nu,\lambda)}(\tilde{t})$ for $\tilde{t} > 0$, starting with the analysis of $\phi^0(\tilde{t},x)$ for $\tilde{t} \gg 1$, gives,

$$\partial_x^j\phi^0(\tilde{t},y) = \phi_0(x)\tilde{t}^j(\nabla_y|\phi_0(y)|^{\frac{p-1}{2}})^j\mathcal{C}^{(j)}(\tilde{t}|\phi_0(y)|^{\frac{p-1}{2}}) + O(\tilde{t}^{j-1}),$$

for $j = 0, 1, ..., k$. Since $\mathcal{C}$ and its derivatives vanish on a countable set we have

$$\|\phi^0(\tilde{t})\|_{H^j} \sim \tilde{t}^j.$$

In particular, since the Sobolev norms H^s are interpolation spaces,

$$\|\phi^0(\tilde{t})\|_{H^s} \sim \tilde{t}^s,$$

whenever $s \geq 0$ is no larger than the greatest integer $\leq \kappa - 1$. If κ is not an odd integer, then $s < s_c < \frac{n}{2} < k$, and $\kappa - 1 \geq k$, for all s under consideration in Theorem 4.0.3 this conclusion holds. If $\nu \ll 1$ and $1 \ll \tilde{t} \leq c|\ln \nu|^c$, (4.6) implies that

$$\|\phi^\nu(\tilde{t})\|_{H^s} \sim \tilde{t}^s. \tag{4.20}$$

This estimate indicates that when time progresses, the function $\phi^\nu(\tilde{t})$ transfers its energy to increasingly higher frequencies. We now exploit the supercriticality of s via the scaling parameter λ to create arbitrarily large H^s norms at arbitrarily small times. Applying (4.6), we have

$$[u^{(\nu,\lambda)}(\lambda^2\tilde{t})]^\wedge(\xi) = \lambda^{\frac{-4}{\kappa-1}}(\frac{\lambda}{\nu})^n[\phi^\nu(\tilde{t})]^\wedge(\frac{\lambda}{\nu}\xi).$$

By the change of varables $\eta := \frac{\lambda}{\nu}\xi$

$$\|u^{(\nu,\lambda)}(\lambda^2\tilde{t})\|^2_{H^s} \geq c\lambda^{\frac{-8}{\kappa-1}}(\frac{\lambda}{\nu})^n \int |[\phi^\nu(\tilde{t})]^\wedge(\eta)|^2(1 + |\frac{\nu}{\lambda}\eta|^2)^s d\eta.$$

Since $\frac{\lambda}{\nu} \leq 1$,

$$\int |[\phi^\nu(\tilde{t})]^\wedge(\eta)|^2(1 + |\frac{\nu}{\lambda}\eta|^2)^s d\eta \geq (\frac{\lambda}{\nu})^{-2s} \int_{|\eta|\geq 1} |[\phi^\nu(\tilde{t})]^\wedge(\eta)|^2|\eta|^{2s} d\eta$$

$$\geq (\frac{\lambda}{\nu})^{-2s}(c\|\phi^\nu(\tilde{t})\|^2_{H^s} - C\|\phi^\nu(\tilde{t})\|^2_{H^0}).$$

From (4.20), it is that $\|\phi^\nu(\tilde{t})\|_{H^0} \ll \|\phi^\nu(\tilde{t})\|_{H^s}$ for $\tilde{t} \gg 1$. Thus by (4.19) and (4.20)

$$\|u^{(\nu,\lambda)}(\lambda^2\tilde{t})\|_{H^s} \geq c\lambda^{\frac{-4}{\kappa-1}}(\frac{\lambda}{\nu})^{\frac{n}{2}-s}\|\phi^\nu(\tilde{t})\|_{H^s} \geq c\epsilon\tilde{t}^s.$$

Therefore for $\|u(t)\|_{H^s}$, when $\tilde{t} \approx c|\ln \nu|^c$, choose ν is small enough such that

$$c|\ln \nu|^c \gg \epsilon^{-\frac{2}{s}},$$

for $t = \lambda^2 \tilde{t}$, ν sufficiently small,

$$t \approx c|\ln \nu|^c \lambda^2 = C|\ln \nu|^c \nu^{2(\frac{n/2-s}{s_c-s})} \epsilon^{\frac{2}{s_c-s}} < \epsilon,$$

we have

$$\|u(t)\|_{H^s} \geq \epsilon^{-1}.$$

Theorem 4.0.3 follows. □

Chapter 5

Strichartz estimates for the beam equation on domains

5.1 Introduction

Recently, Strichartz estimates have been developed for nontrivial geometries. This chapter is primarily concerned with proving Strichartz estimates for the beam equation on Riemannian manifolds (Ω, g) with boundary, for compact manifolds and when Ω is exterior of a smooth, non-trapping obstacle in Euclidean space. Define Laplace-Beltrami operator on Riemannian manifolds (Ω, g):

$$\triangle : H_0^2(\Omega) \longrightarrow L^2(\Omega),$$

and

$$\triangle_g \phi = \frac{1}{\sqrt{\det g_{ij}}} \partial_i (g^{ij} \sqrt{\det g_{ij}} \partial_j \phi).$$

In particular when Ω is a subset of $\mathbb{R}^n$ we have $\triangle_g \phi = \sum_j^n \partial_{x_j}^2 \phi$. Here $H_0^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm $\|\phi\|_{H_0^2(\Omega)} = \sum_{|\alpha| \le 2} \|\partial^\alpha \phi\|_{L^2(\Omega)}$, it also defines the Sobolev space on domain Ω of order 2. The remaining Sobolev spaces $H^s(\Omega)$ can be defined by interpolation and duality. This is equivalent to defining them using the functional calculus $(I-\Delta_g)^{s/2}f \in L^2$. Especially, when Ω is compact manifold with boundary, we could define Sobolev space $H^s(\Omega)$ as (see [16]):

$$H^s(\Omega) = \{f \in L^2(\Omega) : \sum_{j=1}^{\infty}(1+\lambda_j^2)^s \|E_j(f)\|_{L^2}^2 < \infty\},$$

where the eigenvalues $0 < \lambda_j \nearrow \infty$, E_j are the corresponding eigenspaces.

We consider the beam equation

$$\begin{cases} \partial_t^2 u(t,x) + \triangle_g^2 u(t,x) = F(t,x) \\ u \mid_{t=0} = f(x) \\ \partial_t u \mid_{t=0} = g(x) \end{cases} \tag{5.1}$$

with Dirichlet boundary conditions

$$u(t,x)|_{x\in\partial\Omega} = 0, \qquad \triangle_g u(t,x)|_{x\in\partial\Omega} = 0,$$

where $\triangle_g$ denotes the Laplace-Beltrami operator on (Ω, g). The homogeneous beam equation (5.1) can formally be factorized as the following product

$$(\partial_t^2 + \triangle_g^2)u = (i\partial_t + \triangle_g)(-i\partial_t + \triangle_g)u,$$

which displays the relation with the Schrödinger equation. This suggests to recover Strichartz estimates for the beam equation from the ones for the Schrödinger equation.

The Strichartz estimates for the Schrödinger equation on domains is by now deeply studied. On a compact manifold, M. Blair, H. Smith and C. Sogge [1] improved on the current results for compact (Ω, g) where either $\partial\Omega \neq \varnothing$, (or $\partial\Omega = \varnothing$ and g Lipschitz), by showing that Strichartz estimates hold with a loss of $s = 4/3p$ derivatives. Consider the Schrödinger equation,

$$\begin{cases} i\partial_t u(t,x) + \triangle_g u(t,x) = 0 \\ u \mid_{t=0} = f(x) \end{cases} \tag{5.2}$$

with Dirichlet boundary conditions

$$u(t,x)|_{x\in\partial\Omega} = 0.$$

Definition 5.1.1. The exponent pair (p,q) is admissible if

$$2 \le p, q \le \infty, \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad (p,q,n) \neq (2,\infty,2).$$

We have the following:

Theorem 5.1.2. *Let (Ω, g) be a smooth compact Riemannian manifold with boundary. Then the following Strichartz estimates holds for any admissible pair (p,q)*

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|u\|_{H^{\frac{4}{3p}}(\Omega)}. \tag{5.3}$$

We now consider the case where Ω is the exterior of a smooth, non-trapping obstacle in Euclidean space, that is, $\Omega = \mathbb{R}^n \setminus \Theta$ for some compact set Θ with smooth boundary. Non-trapping means that every unit speed generalized bicharacteristic escapes each compact subset of Ω in finite time. O. Ivanovici [9] deduced classical Strichartz estimates for the Schrödinger equation outside a strictly convex obstacle. All Strichartz estimates are valid when Θ is strictly convex with the exception of endpoint estimates with $p = 2$. M. Blair, H. Smith and C. Sogge [2] proved scale invariant Strichartz estimates on domains exterior to a non-trapping obstacle for the Schrödinger equation as the following:

Theorem 5.1.3. *Let $\Omega = \mathbb{R}^n \setminus \Theta$ be a domain exterior to a compact nontrapping obstacle with smooth boundary, and $\triangle$ is the standard Laplace operator on Ω, subject to Dirichlet conditions. Suppose that $p > 2$ and $q < \infty$ satisfy*

$$\begin{cases} \frac{3}{p} + \frac{2}{q} \le 1, & n = 2 \\ \frac{1}{p} + \frac{1}{q} \le \frac{1}{2}, & n \ge 3. \end{cases} \tag{5.4}$$

Then for any solution $u = exp(it\triangle)f$ to the Schrödinger equation (5.2), the following estimates hold

$$\|u\|_{L^p([0,T];L^q(\Omega))} \leq C_T \|f\|_{H^s(\Omega)}, \tag{5.5}$$

provided that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s.$$

5.2 Strichartz estimates for the beam equation on compact domains

By the relation between the beam equation and Schrödinger equation, we deduce the Strichartz estimates of the beam equation as the following:

Theorem 5.2.1. *Let (Ω, g) be a smooth compact Riemannian manifold with boundary. If u is a solution to (5.1), then*

$$\|u\|_{L^p([-T,T];L^r(\Omega))} \lesssim \|f\|_{H^{\frac{4}{3p}}} + \|g\|_{H^{\frac{4}{3p}-2}} + \|F\|_{L^1([-T,T];H^{\frac{4}{3p}-2}(\Omega))}, \tag{5.6}$$

where

$$2 \leq p, r \leq \infty, \quad \frac{2}{p} + \frac{n}{r} = \frac{n}{2}, \quad n \geq 2, \quad (p, r, n) \neq (2, \infty, 2).$$

Proof. First of all, consider the solution of (5.1) with the Dirichlet boundary condition and $F = 0$,

$$v(t, x) = \cos(t\triangle_g)f + \frac{\sin(t\triangle_g)}{\triangle_g} g.$$

Since we have the simple but efficacious formulas

$$\cos(t\triangle_g) = \frac{e^{it\triangle_g} + e^{-it\triangle_g}}{2}, \quad \frac{\sin(t\triangle_g)}{\triangle_g} = \frac{e^{it\triangle_g} - e^{-it\triangle_g}}{2i\triangle_g}.$$

By Theorem 5.1.1, we have

$$\|\cos(t\triangle_g)f\|_{L^p([-T,T];L^r(\Omega))} \lesssim \|f\|_{H^{\frac{4}{3p}}}.$$

Now consider,

$$\|\frac{e^{it\triangle_g}}{\triangle_g}g\|_{L^p([-T,T];L^r(\Omega))}.$$

Let $h = (\triangle_g)^{-1}g$, because Ω is compact with boundary, denote the eigenvalues of $\sqrt{-\triangle_g}$ by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \cdots$, and the corresponding eigenspaces by $E_j, j = 1, 2, 3, \ldots n \ldots$, then

$$g = \sum_{j=1}^{\infty} E_j(g),$$

$$\triangle_g E_j(g) = -\lambda_j^2 E_j(g),$$

and

$$(\triangle_g)^{-1}g = \sum_{j=1}^{\infty} \lambda_j^{-2} E_j(g),$$

Since,

$$\|g\|^2_{H^{\frac{4}{3p}-2}} = \sum_{j=1}^{\infty}(1+\lambda_j^2)^{\frac{4}{3p}-2}\|E_j(g)\|^2_{L^2} < \infty.$$

Therefore, by the Theorem 5.1.1 again. we get,

$$\|\sin(t\triangle_g)h\|_{L^p([-T,T];L^r(\Omega))} \lesssim \|h\|_{H^{\frac{4}{3p}}} = \|(\triangle_g)^{-1}g\|_{H^{\frac{4}{3p}}} = \|g\|_{H^{\frac{4}{3p}-2}}.$$

Let w be the solution of (5.1) with vanishing initial data,

$$w(t,\cdot) = \int_0^t \frac{\sin((t-s)\triangle_g)}{\triangle_g} F(s)ds.$$

By Minkowski inequality, we have the inhomogeneous Strichartz estimates

$$\left\|\int_0^t \frac{e^{i(t-s)\triangle_g}}{\triangle_g}F(s)ds\right\|_{L^p([-T,T];L^r(\Omega))} \lesssim ||F||_{L^1([-T,T];H^{\frac{4}{3p}-2}(\Omega))}.$$

Therefore for the solution of (5.1) $u = v + w$ we have

$$\|u\|_{L^p([-T,T];L^r(\Omega))} \lesssim \|f\|_{H^{\frac{4}{3p}}} + \|g\|_{H^{\frac{4}{3p}-2}} + \|F\|_{L^1([-T,T];H^{\frac{4}{3p}-2}(\Omega))}.$$

□

5.3 Strichartz estimates on domains exterior to a compact non-trapping obstacle with smooth boundary

Let $\Omega = \mathbb{R}^n \setminus \Theta$ be the domain exterior to a compact nontrapping obstacle with smooth boundary. For the Strichartz estimates of the beam equation on this kind of domain we have the following:

Theorem 5.3.1. *Let $\Omega = \mathbb{R}^n \setminus \Theta$ be the domain exterior to a compact nontrapping obstacle with smooth boundary, and $\triangle$ the standard Laplace operator on Ω, subject to Dirichlet conditions. Suppose*

$$\begin{cases} \frac{3}{p} + \frac{n}{q} \leq \frac{n}{2}, n = 2, \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, n \geq 3, \end{cases}$$

and

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s.$$

Then for any solution of beam equation (5.1) with Dirichlet boundary conditions, the following estimates hold

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-2}} + \|F\|_{L^1([-T,T];H^{s-2}(\Omega))}. \tag{5.7}$$

We begin the proof with the following Bernstein-type lemma on domains.

Lemma 5.3.2. *Let $\beta \in C^\infty$, supp$(\beta) \subseteq (\frac{1}{4}, 4)$, and define $\beta_j(\xi) = \beta(2^{-2j}\xi)$, then*

$$\|\beta_j u\|_{L^r} \lesssim 2^{jn(\frac{1}{q}-\frac{1}{r})}\|u\|_{L^q},$$

where, $r \geq q$. Furthermore,

$$\|\triangle \beta_j u\|_{L^q} \approx 2^{2j}\|\beta_j u\|_{L^q}.$$

Proof. Set

$$\beta(-2^{-2j}\triangle) = \int e^{-2^{-2j}(1-it)\triangle}\psi(t)dt,$$

where $\psi(t)$ is a Schwartz function. By Corollary 3.2.8 and Theorem 3.4.8 of E.B. Davies [6] the complexified heat kernel of the Dirichlet Laplacian satisfies

$$\begin{aligned} K(2^{-2j}(1-it), x, y) &\lesssim (2^{-2j})^{-\frac{n}{2}} \exp\{-Re(\frac{b|x-y|^2}{(1+\epsilon)2^{-2j}(1-it)})\} \\ &\lesssim 2^{jn}(1+\frac{2^{2j}|x-y|^2}{1+t^2})^{-N}, \end{aligned}$$

then

$$\begin{aligned} K_{\beta_j}(x,y) &= \int K(2^{-2j}(1-it), x, y)\psi(t)dt \\ &\lesssim \int |K(2^{-2j}(1-it), x, y)||\psi(t)|dt \\ &\lesssim 2^{jn}\int_\Omega \left(1+\frac{2^{2j}|x-y|^2}{1+t^2}\right)^{-N} |\psi(t)|dt. \end{aligned}$$

So

$$\|\beta_j u\|_{L^r} = \left\{\int_\Omega \left|\int_\Omega K_{\beta_j}(x,y)u(y)dy\right|^r dx\right\}^{\frac{1}{r}}.$$

Since

$$\sup_x \left(\int_\Omega \left|K_{\beta_j}(x,y)dy\right|^k\right)^{\frac{1}{k}} \lesssim \left(\int_\Omega 2^{jnk}\left(\int_\Omega \left(1+\frac{2^{2j}|x-y|^2}{1+t^2}\right)^{-N}\psi(t)dt\right)^k dy\right)^{\frac{1}{k}},$$

by Minkowski inequality for integrals,

$$\sup_x \left(\int_\Omega \left| K_{\beta_j}(x,y) dy \right|^k \right)^{\frac{1}{k}} \lesssim 2^{jn(1-\frac{1}{k})} \int_\Omega |\psi(t)|(1+t^2)^{\frac{n}{2k}} dt \lesssim 2^{jn(1-\frac{1}{k})}.$$

Since $\psi(t)$ is Schwartz function and the same bound holds with the roles of x and y reversed, by Young's inequality,

$$\|\beta_j u\|_{L^r} \lesssim 2^{jn(1-\frac{1}{k})} \|u_j\|_{L^q}, \text{where,} \quad \frac{1}{k} = 1 - (\frac{1}{q} - \frac{1}{r}).$$

Then we have

$$\|\beta_j u\|_{L^r} \lesssim 2^{jn(\frac{1}{q}-\frac{1}{r})} \|u\|_{L^q}.$$

To see the final claim, apply the same argument to the function $\xi^{-2}\beta(\xi)$. □

Proof of Theorem 5.3.1. The solution to the beam equation can be written in the form

$$u(t,\cdot) = \cos(t\triangle)f + \frac{\sin(t\triangle)}{\triangle} g + \int_0^t \frac{\sin((t-s)\triangle)}{\triangle} F(s) ds.$$

We still set $u = v + w$, where

$$v(t,x) = \cos(t\triangle)f + \frac{\sin(t\triangle)}{\triangle} g,$$

$$w(t,\cdot) = \int_0^t \frac{\sin((t-s)\triangle)}{\triangle} F(s) ds.$$

By Euler's formula

$$\cos(t\triangle) = \frac{e^{it\triangle} + e^{-it\triangle}}{2}, \qquad \frac{\sin(t\triangle)}{\triangle} = \frac{e^{it\triangle} - e^{-it\triangle}}{2i\triangle},$$

which shows that the estimates for $\cos(t\triangle)$ follow directly from the ones for the Schrödinger equation. By Theorem 5.1.3,

$$\|\cos(t\triangle)f\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|f\|_{H^s},$$

whereas estimates on $\frac{\sin(t\triangle)}{\triangle}$ can be obtained by the propagator $\frac{e^{it\triangle}}{\triangle}$. By taking a Littlewood-Paley decomposition of g in the x variable with respect to the spectrum of $\triangle$, we have,

$$\frac{\sin(t\triangle)}{\triangle}g = \frac{\sin(t\triangle)}{\triangle}\overline{\beta}(\triangle)g + \sum_{j=1}^{\infty}\frac{\sin(t\triangle)}{\triangle}\beta_j(\triangle)g, \tag{5.8}$$

where $\overline{\beta}(\cdot) = 1 - \sum_{j=1}^{\infty}\beta(2^{-2j}\cdot)$, $\sum_{j=1}^{\infty}\beta(2^{-2j}\triangle) = 1$ for $s \geq 2$, and β is supported by $s \in [\frac{1}{2}, 2]$. Consider

$$\|\triangle^{-1}e^{it\triangle}g_j\|_{L^p([-T,T];L^q(\Omega))}.$$

For fixed time t, by the Lemma 5.3.2,

$$\|\triangle^{-1}e^{it\triangle}g_j\|_{L^q(\Omega)} \approx 2^{-2j}\|e^{it\triangle}g_j\|_{L^q(\Omega)}.$$

Since

$$\|e^{it\triangle}g_j\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|g_j\|_{H^s}.$$

Then

$$2^{-2j}\|e^{it\triangle}g_j\|_{L^q(\Omega)} \lesssim \|g_j\|_{H^{s-2}}.$$

Then by the Littlewood-Paley squarefunction estimate (see Theorem 0.2.10 of [20]) and Sobolev embedding for the first term of (5.8) (see the proof of Theorem 2.2.1),

$$\|\frac{\sin(t\triangle)}{\triangle}g\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|g\|_{H^{s-2}}.$$

Combining with the estimates of $\cos(t\triangle)$ we have

$$\|v\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-2}}. \tag{5.9}$$

For w, by Minkowski inequality we have

$$\left\|\int_0^t \frac{e^{i(t-s)\triangle_g}}{\triangle_g}F(s)ds\right\|_{L^p([-T,T];L^q(\Omega))} \lesssim ||F||_{L^1([-T,T];H^{s-2}(\Omega))}.$$

Combining with (5.9), we have the estimate (5.7) $\square$

Similarly, when Θ is strictly convex, all Strichartz estimates are valid for the Schrödinger equation. By the methods above, we have the following:

Corollary 5.3.3. *Let $\Omega = \mathbb{R}^n \backslash \Theta$, where Θ is compact with smooth boundary. Suppose that $n \geq 2$ and $\partial\Omega$ is strictly geodesically concave throughout. Assume the pair (p, q) satisfies the scaling condition:*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s,$$

Then for the solution of the beam equation (5.1) with Dirichlet boundary conditions the following estimates hold

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-2}} + \|F\|_{L^1([-T,T];H^{s-2}(\Omega))}.$$

Chapter 6

Future Work

6.1 Future directions for the nonlinear beam equation

(1) I will continue to investigate the ill-posedness of the nonlinear beam equation in focusing case with initial data in $\dot{H}^s \times \dot{H}^{s-2}$ or $H^s \times H^{s-2}$. I had found the small dispersion analysis doesn't work in this case. For the wave equation in the focusing case, H. Lindblad and C. D. Sogge [15] proved that blow-up and ill-posedness can be obtained via the ODE method. They showed that truncating the initial data in space yields compactly supported solutions which blow up in finite time by virtue of the finite speed of propagation. Then, they transformed these blowup solutions using the scaling symmetry to establish blowup in arbitrarily short time when $s < s_c$. Because the beam equations don't satisfy finite speed of propagation, it becomes diffcult to use the ODE method. So I will try to discuss if almost finite speed propagation could be established for (1.1) with initial data in $\dot{H}^s \times \dot{H}^{s-2}$ or $H^s \times H^{s-2}$, or try the contradiction method to discuss this case.

(2) I will continue to consider the Strichartz estimates and establish local and global

properties of the solutions in low regularity Sobolev space for the nonlinear beam equation with the following type

$$\partial_t^2 u(t,x) + \triangle^2 u(t,x) + mu = \omega |u|^{\kappa-1} u, \quad \text{with} \quad m > 0.$$

6.2 Future directions for the nonlinear beam equations on domains

S. Levandosky and W. Strauss [13] derived an analogue of Morawetz' radial identity for the nonlinear beam equation. It follows that all solutions decay to zero in a certain sense as $t \to \infty$. By these results, J. E. Lin [14] showed that the local energy of solution is integrable in time and the local L^2 norm of the solution approaches zero as $t \to \infty$ for a nonlinear beam equation with the Euclidean spatial dimension > 5. I will investigate the local energy; that is, the norm in $H^2(\Omega) \times L^2(\Omega)$ for any exterior domain $\Omega \in \mathbb{R}^n$, is integrable and tends to zero as $t \to \infty$.

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