9-3-2010

Pell's Equation and nearly equilateral triangles

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PELL'S EQUATION
AND
NEARLY EQUILATERAL TRIANGLES

BY

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B.S., MATHEMATICS, JAMES MADISON UNIVERSITY, 2007

THESIS

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science
Mathematics

The University of New Mexico
Albuquerque, New Mexico

July 2010

ACKNOWLEDGEMENTS

I would like to acknowledge Dr. Alex Stone, my thesis committee chair for being the most understanding and patient master's thesis advisor I could have asked for! Thanks for helping me through each step of the way and for dealing with me when I get a little overly stressed!

I would also like to thank my committee members, Dr. Kristin Umland and Dr. Jens Lorenz for taking the time out of their busy schedules to work with me. I appreciate all of your helpful comments and encouraging words. Thank you for all of your efforts at helping me graduate!

I cannot ever do anything mathematical without thanking those who guided me on my path to get here. I thank all my prior math teachers, but especially Mr. Maier at Sandia High School and Dr. Laura Taalman and Dr. Elizabeth Brown at James Madison University. I would never have continued in my mathematics education with the wonderful enthusiasm, support and guidance from these wonderful instructors!

I would like to thank my parents for always pushing me academically. I know I need a little extra prodding sometimes, and thanks for always providing that. I also thank them for making it possible for me to study in higher education.

Finally, I would like to thank my wonderful husband, Preston. No one else has to deal with my breakdowns and stress as much as you do! I still cannot believe God has blessed me with such a patient, understanding, and encouraging partner! You are such an amazing man, and I promise to at least make an effort to calm down the stress level!

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ABSTRACT

In this paper, we seek a family of triangles that have integer side lengths and integer area. We observe that it is impossible to have such triangles that are equilateral. Then we study briefly the isosceles case. The majority of the paper concentrates on constructing this family of scalene triangles that we name Nearly Equilateral Triangles. These are triangles such that the side lengths are consecutive integers.

In the process of describing this family, we find a connection between Pell's equation and Nearly Equilateral Triangles. There is a brief introduction into Pell's Equation as well as a detailed description of their use in forming the family of triangles we seek. Finally, we see an interesting connection between the Pell's equation solutions and Archimedes' approximation of $\sqrt{3}$. We follow up this connection with another interesting connection to his approximation.

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Introduction

In this paper, we seek to construct a family of triangles that have integer sides as well as integer area. In the process of describing this family we will explore Heron's formula for the area of a triangle. This formula will lead to an in-depth use of Pell's equation in relation to the family of triangles. The study of Pell's equation will also lead to connections to Archimedes' approximation for $\sqrt{3}$. First, we will use Heron's formula to restrict the type of triangles we will study.

The formula attributed to Heron of Alexandria (ca: 75 AD) for the area of a triangle in terms of the lengths of its sides appears in Heron's Metrica. It is likely that the result was also known to Archimedes centuries earlier and possible Archimedes had a proof of this formula. The formula can be found in Burton's The History of Mathematics. It states that a triangle with sides of length x, y, and z has area A given by

$$A = \sqrt{\sigma(\sigma - x)(\sigma - y)(\sigma - z)}$$

where σ is the half perimeter:

$$\sigma = \frac{x + y + z}{2}$$

A derivation of the formula using elementary mathematics was given by Reuben Hersh in his article in Focus. His derivation states that if there is a formula for the area of a triangle in terms of side lengths then it could be of several forms including a quadratic or it could be of the form $A = k\sqrt{P(a,b,c)}$ where k is a constant and $P(a,b,c)$ is a symmetric, homogenous polynomial of degree four in a, b, c which are the side lengths of the triangle. Then we use the Factor Theorem and a degenerate triangle where $a + b = c$ or $a + c = b$ or $b + c = a$ and thus $A = 0$. Then if P is a polynomial in a, it must have roots at $b + c$, $b - c$ and $c - b$. Therefore, by the factor theorem it has factors $a - (b + c)$, $a - (b - c)$ and $a - (c - b)$. Also, since P is a quartic polynomial, it must have another factor and since it is symmetric, this factor must be $k_1(a + b + c)$ where k_1 is a constant. So thus far we have

$$A = \sqrt{k_1(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}$$

and we will use a triangle with known area to calculate the constant. Consider the triangle with lengths $a = 3$, $b = 4$, $c = 5$. This triangle is a right triangle so the area is $A = \frac{1}{2}(3)(4) = 6$. Plugging these values into our polynomial we have

$$A = \sqrt{k_1(3 + 4 + 5)(-3 + 4 + 5)(3 - 4 + 5)(3 + 4 - 5)} = \sqrt{k_1(576)}$$

which leads to

$$6 = \sqrt{k_1(576)} \tag{1}$$

$$36 = k_1(576) \tag{2}$$

$$k_1 = \frac{1}{16} \tag{3}$$

Therefore, we have the polynomial that was sought after and

$$A = \sqrt{\frac{1}{16}(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}$$

Now, substituting $a = x$, $b = y$, $c = z$, and $\sigma = \dfrac{x+y+z}{2}$ gives Heron's formula $A = \sqrt{\sigma(\sigma - x)(\sigma - y)(\sigma - z)}$.

This is only a derivation of the formula and not a proof. Possibly the first proof ever discovered of this formula is given in Heron's Diopatra, but a more modern proof utilizing trigonometric identities is provided here derived from the proof provided on "Heron's Formula" as cited.

Proof. Let a, b, c be the sides of a triangle with opposite angles A, B, C respectively. Then by the law of cosines

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

and since $\sin^2(x) + \cos^2(x) = 1$ we have

$$\begin{aligned} \sin(C) &= \sqrt{a - \cos^2(C)} && (4) \\ &= \sqrt{1 - \frac{a^2 + b^2 - c^2}{4a^2b^2}} && (5) \\ &= \sqrt{\frac{4a^2b^2 - (a^2 + b^2 - c^2)}{4a^2b^2}} && (6) \\ &= \frac{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)}}{2ab} && (7) \end{aligned}$$

Now let A_Δ be the area of the triangle. Also let σ be as defined in Heron's formula. By the law of sines, the altitude, h, of the triangle if the base is side a is given by $b\sin(C)$, thus,

$$A_\Delta = \frac{1}{2}bh \tag{8}$$

$$= \frac{1}{2}ab\sin(C) \tag{9}$$

$$= \frac{1}{4}\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2} \tag{10}$$

$$= \frac{1}{4}\sqrt{(2ab - (a^2+b^2-c^2)(2ab + (a^2+b^2-c^2)} \tag{11}$$

$$= \frac{1}{4}\sqrt{(c^2 - (a-b)^2)((a+b)^2 - c^2)} \tag{12}$$

$$= \frac{1}{4}\sqrt{(c-(a-b))(c+(a-b))(a+b-c)(a+b+c)} \tag{13}$$

$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{c+b-a}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{a+b-c}{2}\right)} \tag{14}$$

$$= \sqrt{\sigma(\sigma-a)(\sigma-b)(\sigma-c)} \tag{15}$$

□

It is important to note that a triangle with integer sides will have integer area if the product $\sigma(\sigma-x)(\sigma-y)(\sigma-z)$ is a perfect square. It is fairly simple to construct examples of this type of triangle. For instance, if $x = 17$, $y = 25$, and $z = 28$, the resulting area A is 210. However, the following theorem states one of the main restrictions on finding other triangles with integer area.

Theorem 1. *An equilateral triangle with integer sides cannot have integer area.*

Proof. Let z be the length of the sides of the equilateral triangle. By Heron's formula,

$$\sigma = \frac{x+y+z}{2} \tag{16}$$

$$= \frac{3z}{2} \tag{17}$$

and

$$A = \sqrt{\sigma(\sigma - z)(\sigma - z)(\sigma - z)} \tag{18}$$

$$= \sqrt{\sigma(\sigma - z)^3} \tag{19}$$

$$= \sqrt{\frac{3z}{2}(\frac{3z}{2} - z)^3} \tag{20}$$

$$= \sqrt{\frac{3z^4}{16}} \tag{21}$$

$$= \frac{\sqrt{3}z^2}{4} \tag{22}$$

So if z is any integer, the area A of the equilateral triangle will contain $\sqrt{3}$ and is, therefore, not an integer.

$\square$

On the other hand, there are many ways of constructing isosceles triangles with integer sides and integer area. One could, for example, consider a triangle with two sides of length z and choose z to be the hypotenuse of any right triangle. In fact, if z is a prime of the form $4k + 1$, the following theorem from Sierpinski's Elementary Theory of Numbers applies.

Theorem 2. *Any prime number z of the form $z = 4k + 1$ for some integer k is expressible uniquely as a sum of squares, $z = x^2 + y^2$.*

Proof. Let p be a prime number of the form $4k + 1$ and let $a = \left(\frac{p-1}{2}\right)!$. Now it can be shown that p of this form divides $\left[\left(\frac{p-1}{2}\right)!\right]^2 + 1$. So we have $p|a^2 + 1$ with a being relatively prime to p. Now by Thue's Theorem there exist two natural numbers x and y each $\leq \sqrt{p}$ such that for a suitable choice of $+$ or $-$ the number

$ax \pm y$ is divisible by p. Then it follows that the number $a^2x^2 - y^2 = (ax - y)(ax + y)$ is divisible by p.

It is also the case that $a^2x^2 + x^2 = (a^2 + 1)x^2$ is divisible by p since $p|(a^2 + 1)$. Consequently the number $x^2 + y^2 = a^2x^2 + x^2 - (a^2x^2 - y^2)$ is divisible by p. But, since x and y are natural numbers $\leq \sqrt{p}$ they must be strictly $< \sqrt{p}$, because p being a prime is not a square of a natural number. Thus $x^2 + y^2$ is a natural number > 1 and $< 2p$ and it is divisible by p so it must be equal to p. Therefore $p = x^2 + y^2$ for any prime number p of the form $4k + 1$. $\square$

Now if we are considering an integer z that is the hypotenuse of a right triangle then a few results concerning right triangles should be discussed first. Possibly one of the most well-known equations in mathematics is from the Pythagorean Theorem which states that if x and y are the lengths of the legs of a right triangle and z is the length of the hypotenuse, then $x^2+y^2 = z^2$. This equation is so commonly used that a shorthand notation is introduced. If three natural numbers satisfy this condition, we call it a Pythagorean triple and use the notation (x, y, z). A primitive Pythagorean triple adds the property that the three numbers x, y, and z have no common divisor other than 1. This leads directly to the fact that no two of the numbers x, y, and z can have a common divisor greater than one. That is to say, they are all relatively prime.

It can be easily shown that both x and y cannot be even. If they were, then x^2+y^2 would also be even, which would mean that z is even and then x, y, and z have the common factor 2 which contradicts the property of primitive triples. This leads to the following theorem as stated in Burton.

Theorem 3. *If (x, y, z) is a primitive Pythagorean triple, then one of the integers x and y is even and the other is odd.*

Proof. It was already discussed that both x and y cannot both be even, so it must now be shown that both cannot be odd.

Assume x and y are both odd. Then there exist integers n and m such that

$$x = 2n + 1 \text{ and } y = 2m + 1$$

Then since x, y, and z must satisfy $x^2 + y^2 = z^2$,

$$\begin{aligned} z^2 = x^2 + y^2 &= (2n+1)^2 + (2m+1)^2 && (23) \\ &= 4n^2 + 4n + 1 + 4m^2 + 4m + 1 && (24) \\ &= 4(n^2 + m^2 + n + m) + 2 && (25) \\ &= 4h + 2 && (26) \end{aligned}$$

for some integer h.

Since $z^2 = 4h + 2 = 2(2h + 1)$, z^2 must be an even number. Now, z must be even in order for z^2 to be even, but the square of any even number is divisible by 4. However, $z^2 = 2(2h+1)$ cannot have a factor of 4. This situation is impossible since z must be an integer. Therefore, one of x and y must be even and the other must be odd. $\square$

Now since in a primitive Pythagorean triple (x, y, z) exactly one of x and y must be an odd integer, for convention we will write our triples such that x is even and y is odd. This implies that z is odd as well because if not then $\gcd(x, z) \geq 2$ which violates the primitive condition. Then we have the following theorem adapted from Redmond's Number Theory; An Introduction.

Theorem 4. *Given a primitive Pythagorean triangle with side lengths x, y, z where x is even and y is odd, then there exist relatively prime integers a and b such that*

$$x = 2ab \tag{27}$$

$$y = a^2 - b^2 \tag{28}$$

$$z = a^2 + b^2 \tag{29}$$

where $a > b > 0$ and a and b are of opposite parity, i.e. one is even and the other is odd.

Proof. Let (x, y, z) be any primitive Pythagorean triple. Take x even, and both y and z odd. Then $z - y$ and $z + y$ are both even integers. Say $z - y = 2u$ and $z + y = 2v$. Now $x^2 + y^2 = z^2$ can be rewritten as

$$x^2 = z^2 - y^2 = (z - y)(z + y)$$

and dividing both sides by 4 leads to

$$\left(\frac{x}{2}\right)^2 = \left(\frac{z-y}{2}\right)\left(\frac{z+y}{2}\right) = uv$$

Now u and v must be relatively prime integers because if $\gcd(u, v) = d > 1$, then $d|(u-v)$ and $d|(u+v)$ which means $d|y$ and $d|z$, which violates the primitive condition.

We will use the fact that if the product of two relatively prime integers equals the square of an integer, then each of them is also a perfect square. Then we can conclude that u and v are perfect squares. So let

$$u = a^2 \text{ and } v = b^2$$

where a and b are positive integers. Then substituting these values of u and v we have

$$z = u + v = a^2 + b^2 \tag{30}$$

$$y = u - v = a^2 - b^2 \tag{31}$$

$$x^2 = 4uv = 4a^2b^2 \tag{32}$$

$$x = 2ab \tag{33}$$

Since any common divisor of a and b divides y and z, the relation $\gcd(y, z) = 1$ forces the relation $\gcd(a, b) = 1$. We must still show, however, that a and b have opposite parity. Now if a and b were both even this would cause both y and z to be even since the square of an even integer is even and the sum or difference of two even integers is also even. This is impossible because then $\gcd(y, z) \geq 2$ which violates the primitive condition. Similarly, if a and b were both odd, then their squares would each be odd as well. Since the sum or difference of two odd numbers is even, then again y and z would both be even which violates our condition that the triple be primitive. Hence, a and b have opposite parity.

Now we must show that any triple (x, y, z) satisfying the conditions above is in fact a Pythagorean triple. Take $x = 2ab$, $y = a^2 - b^2$, and $z = a^2 + b^2$ then the

Pythagorean identity holds:

$$x^2 + y^2 = (2ab)^2 + (a^2 - b^2)^2 \tag{34}$$

$$= 4a^2b^2 + a^4 - 2a^2b^2 + b^4 \tag{35}$$

$$= a^4 + 2a^2b^2 + b^2 \tag{36}$$

$$= (a^2 + b^2)^2 \tag{37}$$

$$= z^2 \tag{38}$$

Lastly, we must show that the triple (x, y, z) is primitive. Assume, seeking a contradiction, that x, y, and z have a common divisor $d > 1$. Consider any prime divisor p of d. Now since a and b have opposite parity, $a^2 + b^2 = z$ must be odd. Then we see that $p \neq 2$ since it must divide the odd integer z. Since $p|y$ and $p|z$, then we also have $p|(z + y)$ and $p|(z - y)$ which implies that $p|2a^2$ and $p|2b^2$. But then $p|a$ and $p|b$ which is impossible since $\gcd(a, b) = 1$. Therefore, $d = 1$ is the largest divisor and (x, y, z) as given in terms of a and b is a primitive Pythagorean triple. $\square$

Now we will construct a set of isosceles triangles that have integer sides as well as integer area. The two sides of the triangle that are equal we will call z. Using the previously stated theorem, if we take z to be a prime number of the form $4k + 1$ then we can write $z = a^2 + b^2$ where a and b are some particular integers. In order to use the additional theorems above, we will construct this isosceles triangle by putting two right triangles together with the hypotenuse that has length z outside. We will call the base of each right triangle y so the base of the isosceles triangle is $2y$. Then we will call the other side of the right triangle x. Then x is the height of the isosceles triangle. So we can use the theorems above and let $z = a^2 + b^2$. Then we can set $y = a^2 - b^2$ and $x = 2ab$. Since a and b are integers, it follows directly that x and y are integers as well. Then we will use the traditional equation for area, $A = \frac{1}{2}bh$ where b represents the base of the triangle and h represents the height. So we have

$$\begin{aligned} A &= \frac{1}{2}bh && (39) \\ &= \frac{1}{2}(2y)(x) && (40) \\ &= \frac{1}{2}(2(a^2 - b^2))(2ab) && (41) \\ &= 2ab(a^2 - b^2) && (42) \\ &= xy && (43) \end{aligned}$$

Since x and y must be integers, this shows that the area A of the isosceles triangle with sides $(2y, z, z)$ is also an integer. Similarly, for an isosceles triangle with sides $(2x, z, z)$ using the same identities for x and z above will have integer area. Therefore, for each prime number of the form $z = 4k + 1$ we can construct two isosceles triangles that have integer area. It should be noted that it is not a trivial fact that there are infinitely many primes of this form. The proof of this fact is in Redmond's *Number Theory*. Therefore, we have constructed an infinite set of isosceles triangles with integer sides and integer area.

Thus far we have shown that an equilateral triangle with integer sides cannot have integer area. We have also shown that isosceles triangles with integer side lengths and integer area can be formed. From here we will move on to the last type of triangles. We will study scalene triangles and see if there are any that can have integer area and integer sides. Since there are several different ways to create a scalene triangle, we will first define the particular type of scalene triangle that will be studied.

A **nearly equilateral triangle** or **NET** is a scalene triangle with consecutive integer sides. We will consider a class of NETs whose sides are represented by the consecutive integers $x-1$,x, $x+1$ and explore the possibility of choosing an integer x so that the area of the NET is also an integer. The most commonly studied example is the (3,4,5) primitive Pythagorean triangle. This scalene triangle also falls into the special category of right triangles so the area is very simply calculated to be $A = \frac{1}{2}(3)(4) = 6$. So there is at least one NET that has integer area. A few calculations will show that the (13,14,15) triangle is also a NET with integer area 84. However, this NET is not a Pythagorean triangle. So it is a slightly more interesting result that this NET has integer area since the area is not as simply calculated in this case. The next step is to search for other NETs with integer area. We will show that there is, in fact, an infinite collection of NETs with integer area, and that these can be found by obtaining solutions to a particular version of Pell's equation. Moreover, for each NET with integer area, the radius of the inscribed circle is also a positive integer; however, the radius of the circumscribed circle is never an integer. To get an idea of the first few NETs, we have included a Matlab program in Appendix A. This program calculates all the NETs up to an integer, n, which must be input by the user. This method only helps in getting an initial set of data. Our task of proving that the list of NETs is in fact infinite will require more complicated methods involving Pell's equation. First, we will explore some basic facts of Pell's equation.

Pell's Equation

The equation $r^2 - Ns^2 = 1$ is usually called Pell's equation. Here N is a parameter and we seek integer solutions for r and s. This equation has an interesting history. John Pell (1611-1685) was an English mathematician and clergyman. He made no actual contributions to the history of this equation. The equation is a Diophantine equation that was named Pell's equation after a mistaken historical reference made by Leonhard Euler. It is speculated that Euler actually meant to call the equation Fermat's equation. Fermat was the first to propose a challenge to the mathematicians in Europe to find integer solutions for Pell's equation with certain values for N. Although mathematicians Wallis and Brouncker found methods of solutions, it is believed that Fermat had the first solution. He undoubtedly knew there were infinitely many integer solutions when he posed the problem. However, Lagrange first published his solution to the problem in 1768. An algorithm that gives solutions to special cases of this equation can also be linked back to Indian mathematicians Bhaskara and Brahmagupta, ca. 600 AD, although they provided no proof of the efficiency or sufficiency of their procedure.

In 1759, Euler devised a procedure for finding the smallest integer solution for Pell's equation. The publication by Lagrange mentioned previously contained the first rigorous proof that the continued fraction expansion of N would provide all integer solutions to $r^2 - Ns^2 = 1$. These solutions (r_n, s_n) are given by the following formulas as stated in Gelfond's The Solution of Equations in Integers. For $n = 1, 2, 3....$

$$r_n = \frac{1}{2}[(r_1 + s_1\sqrt{N})^n + (r_1 - s_1\sqrt{N})^n]$$

$$s_n = \frac{1}{2\sqrt{N}}[(r_1 + s_1\sqrt{N})^n - (r_1 - s_1\sqrt{N})^n]$$

The pair (r_1, s_1) is a minimal solution in the sense that $(r_1 + s_1\sqrt{N}) \leq (r_i + s_i\sqrt{N})$ for all other solutions (r_i, s_i). It should be noted that there are solutions when r and s are either both positive or both negative. We will, however, only consider the case when both r and s are positive integers. There are also recursion relations for the values of r and s that have the form

$$r_{n+2} = 2r_1 r_{n+1} - r_n$$

$$s_{n+2} = 2r_1 s_{n+1} - s_n$$

for $n = 1, 2, 3, ...$

Consider the Pell's equation $r^2 - Ns^2 = 1$ where $\alpha = \sqrt{N}$ is an integer. In this particular form, the equation can be written in the form

$$r^2 - \alpha^2 s^2 = (r + \alpha s)(r - \alpha s) = 1$$

and since α is an integer and if r_0 and s_0 are integers satisfying the equation, the equations can be separated into the equations

$$r_0 + \alpha s_0 = 1, \; r_0 - \alpha s_0 = 1$$

or the equations

$$r_0 + \alpha s_0 = -1, \; r_0 - \alpha s_0 = -1$$

since the product of two integers may be equal to 1 only when they are both separately equal to 1 or both separately equal to -1. These two systems of two equations in two unknowns r_0 and s_0 have only the trivial solutions

$$r_0 = 1,\ s_0 = 0 \tag{44}$$

$$r_0 = -1,\ s_0 = 0 \tag{45}$$

So the Pell's equation with N equal to the square of an integer has only trivial solutions. Thus more interesting results come when $\sqrt{N}$ is irrational

In the next section we will show that one of the equations which we are interested in obtaining solutions to is $u^2 - 12v^2 = 144$. One possible approach which we will not use is to find all integer solutions to Pell's equation $r^2 - 12s^2 = 1$. We will, however, use this particular Pell's equation simply as an example to complete this section describing Pell's equation.

Through systematic trial and error and using simple calculations, it can be shown that the minimal solution in the case of $r^2 - 12s^2 = 1$ is $r_1 = 7$ and $s_1 = 2$. Therefore, using the formulas given above for $n = 1, 2, 3, \ldots$ we have

$$r_n = \frac{1}{2}[(7 + 2\sqrt{12})^n + (7 - 2\sqrt{12})^n]$$
$$s_n = \frac{1}{2\sqrt{12}}[(6 + 2\sqrt{12})^n - (7 - 2\sqrt{12})^n]$$

These formulas simplify to

$$r_n = \frac{1}{2}[(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n]$$

$$s_n = \frac{1}{4\sqrt{3}}[(7 + 4\sqrt{3})^n - (7 - 4\sqrt{3})^n]$$

We can also use the recursion relations for $n = 1, 2, 3, ...$ given above in the case $r^2 - 12s^2 = 1$ to obtain

$$r_{n+2} = 14r_{n+1} - r_n$$

$$s_{n+2} = 14s_{n+1} - s_n$$

So clearly, if the minimal solution for a Pell's equation is an integer, all subsequent solutions will also be integers. This fact will be important in the following sections when applying Pell's equation to the construction of NETs.

Thus, we can tabulate the first six positive solutions to the Pell's equation $r^2 - 12s^2 = 1$, which is similar to the equation $r^2 - 12s^2 = 144$ that we will explore further in relation to NETs.

Table 1: Solutions to $r^2 - 12s^2 = 1$

n	r_n	s_n
1	7	2
2	97	28
3	1351	390
4	18817	5432
5	262087	75658
6	3650401	1053780

Nearly Equilateral Triangles

If $x-1, x, x+1$ are consecutive integer sides of a triangle, then as defined in the first section, $\sigma = \frac{3x}{2}$ and using Heron's formula for the area, A, of this triangle produces the following formula for A.

$$
\begin{aligned}
A &= \sqrt{\sigma\left(\sigma-(x-1)\right)\left(\sigma-x\right)\left(\sigma-(x+1)\right)} && (46)\\
&= \sqrt{\left(\frac{3x}{2}\right)\left(\frac{x+2}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x-2}{2}\right)} && (47)\\
&= \sqrt{\frac{3x^2}{16}(x^2-4)} && (48)\\
&= \frac{x}{4}\sqrt{3x^2-12} && (49)
\end{aligned}
$$

Now, if we assume that $3x^2-12$ is a perfect square and let $w^2 = 3x^2-12$, then

$$
\begin{aligned}
3x^2-12-w^2=0 \implies x &= \frac{0 \pm \sqrt{0-4(3)(-12w^2)}}{6} && (50)\\
&= \frac{\sqrt{144+12w^2}}{6} && (51)
\end{aligned}
$$

and we can ignore the negative because x is a side length which has domain $[0,\infty)$. We can also now write that $A = \frac{x\cdot w}{4}$.

Now if $12w^2+144$ is a perfect square, say $y^2 = 12v^2+144$, then $y^2-12w^2 = 144$. This equation is the variant of Pell's equation mentioned in the previous section.

However, we showed formulas to construct all solutions to $r^2 - Ns^2 = 1$. The recursion relation for the solutions that was given is also for $r^2 - Ns^2 = 1$ which is not the form of the Pell type equation $y^2 - 12w^2 = 144$ that will produce NETs. So we must evaluate this equation and see if there is an equation of the form $r^2 - Ns^2 = 1$ that will give the same results. Then we can apply the formulas in section 2 and construct NETS. We will now make another attempt to relate the length of the side of a triangle to the area with a Pell equation that has the form $r^2 - Ns^2 = 1$, and we will find a much more fruitful method.

Theorem 5. *The Pell equation $r^2 - 3s^2 = 1$ relates the side length of an NET to its area.*

Proof. First, consider again that

$$A = \frac{x}{4}\sqrt{3x^2 - 12}$$

Then let

$$v^2 = 3x^2 - 12 = 3(x^2 - 4)$$

and set

$$3u = v$$

Then

$$v^2 = (3u)^2 = 9u^2 \tag{52}$$

$$3x^2 - 12 = 9u^2 \tag{53}$$

$$x^2 - 4 = 3u^2 \tag{54}$$

Now if u were odd, say $u = 2h+1$ then $3u^2 = 3(2h+1)^2 = 12h^2+12h+3$. This then means that $x^2 - 4 = 12h^2 + 12h + 3$ and so $x^2 = 12h^2 + 12h + 7$ which is odd. Then x must be odd since only the square of odd numbers is odd. However, if x is odd then $A = \frac{x}{4}\sqrt{3x^2 - 12}$ would not be an integer. This is because $\sqrt{3x^2 - 12}$ would be odd (if it happens to be an integer) and then $x\sqrt{3x^2 - 12}$ would be odd as well since the product of two odd numbers is itself an odd number. But then $A = \frac{x}{4}\sqrt{3x^2 - 12}$ cannot be an integer since 4 cannot divide an odd number without remainder. Hence, both u and x must be even and we can write $u = 2t$ and $x = 2s$. Then we have

$$3u^2 = x^2 - 4 \tag{55}$$

$$x^2 - 3u^2 = 4 \tag{56}$$

$$(2s)^2 - 3(2t)^2 = 4 \tag{57}$$

$$4s^2 - 3(4t^2) = 4 \tag{58}$$

$$s^2 - 3t^2 = 1 \tag{59}$$

Thus, we have the Pell equation $s^2 - 3t^2 = 1$ that relates one of the sides, x, of a NET to its area. □

Now, if positive integer solutions s and t can be obtained, then $x = 2s$ will be a positive integer. Then we have the following sides of the respective triangle:

$$(x-1) = 2s - 1 \tag{60}$$

$$x = 2s \tag{61}$$

$$(x+1) = 2s + 1 \tag{62}$$

Since $\sigma = \dfrac{x+y+z}{2}$ we have

$$\sigma = \frac{(x-1)+x+(x+1)}{2} \tag{63}$$
$$= \frac{(2s-1)+2s+(2s+1)}{2} \tag{64}$$
$$= \frac{6s}{2} \tag{65}$$
$$= 3s \tag{66}$$

Then we obtain the following:

$$\sigma - (x-1) = 3s - (2s-1) = s+1 \tag{67}$$

$$\sigma - x = 3s - 2s = s \tag{68}$$

$$\sigma - (x+1) = 3s - (2s+1) = s-1 \tag{69}$$

Now since $s^2 - 3t^2 = 1$ we have

$$-3t^2 = -s^2 + 1 \tag{70}$$

$$3t^2 = s^2 - 1 \tag{71}$$

This leads to the equation for the area using Heron's formula.

$$A = \sqrt{\sigma\left(\sigma-(x-1)\right)\left(\sigma-x\right)\left(\sigma-(x+1)\right)} \tag{72}$$

$$= \sqrt{3s\left(s+1\right)\left(s\right)\left(s-1\right)} \tag{73}$$

$$= \sqrt{3s^2(s+1)(s-1)} \tag{74}$$

$$= s\sqrt{3(s^2-1)} \tag{75}$$

$$= s\sqrt{3(3t^2)} \tag{76}$$

$$= 3st \tag{77}$$

Hence the area, A, of the NET will be an integer provided s and t are integers. However, since s and t are the solutions to the Pell equation $s^2 - 3t^2 = 1$, we are only interested in integer solutions and thus we will have a set of NETs with integer sides and integer area if we find solutions to the Pell equation in question. We will look at these values in the next section.

Next, we will discuss the relationships between s and t given above to the radius of the inscribed circle r and the radius of the circumscribed circle R of a NET. The formulas for r and R can be found in *Standard Mathematical Tables and Formulae; 30th Edition*.

For a triangle with side lengths x, y, and z and for the semiperimeter σ as defined previously, the radius of the inscribed circle is given by

$$r = \sqrt{\frac{(\sigma - x)(\sigma - y)(\sigma - z)}{\sigma}} \tag{78}$$

$$= \sqrt{\frac{\sigma(\sigma - x)(\sigma - y)(\sigma - z)}{\sigma^2}} \tag{79}$$

$$= \left(\sqrt{\frac{1}{\sigma^2}}\right)\sqrt{\sigma(\sigma - x)(\sigma - y)(\sigma - z)} \tag{80}$$

$$= \frac{A}{\sigma} \tag{81}$$

So for the triangle with side lengths $x - 1$, x, and $x + 1$ that we consider in this section, r will be given as follows:

$$r = \frac{A}{\sigma} \tag{82}$$

$$= \frac{3st}{3s} \tag{83}$$

$$= t \tag{84}$$

Therefore, the radius,r, of the inscribed circle of the NETs will also be an integer since we will only take t to be an integer in the construction.

Finally, we will consider the radius of the circumscribed circle of the NET. This value is given by

$$R = \frac{abc}{4A}$$

where x, y, and z are the side lengths and A is the area of the given triangle. So for the NET in which the side lengths are $x = x - 1$, $y = x$ and $z = x - 1$ and applying $x = 2s$ and $A = 3st$ we have the following

$$\begin{aligned} R &= \frac{xyz}{4A} && (85) \\ &= \frac{(x-1)(x)(x+1)}{4A} && (86) \\ &= \frac{(2s-1)(2s)(2s+1)}{12st} && (87) \\ &= \frac{4s^2-1}{6t} && (88) \end{aligned}$$

A simple analysis of the relationships we have developed between s and t and the NETs will show that the conditions for x, A, and r to be integers follow directly from s and t being integers. However, it is also clear that in order for R to be an integer, much more complicated divisibility conditions must be satisfied. In fact, R is never an integer. Using the relationship $s^2 = 3t^2 - 1$ we have the following

$$\begin{aligned} R &= \frac{4s^2-1}{6t} && (89) \\ &= \frac{4(3t^2-1)-1}{6t} && (90) \\ &= \frac{12t^2-5}{6t} && (91) \\ &= 2t - \frac{5}{6t} && (92) \end{aligned}$$

Now, clearly $2t$ is an integer since we will require t to be an integer. However, $\dfrac{5}{6t}$ will never be an integer if t is any positive integer. Thus, it is impossible for $R = 2t - \dfrac{5}{6t}$ to be an integer while also satisfying the conditions on t.

The previous section included an analysis of Pell equations in general. We will use the formulas described in that section and apply them to find integer solutions to the particular Pell equation $s^2 - 3t^2 = 1$. These integer solutions lead directly to solutions for x, A, r, and R for NETs. In the next section, we construct NETs in exactly this manner and tabulate the first several NETs.

Construction of Nearly Equilateral Triangles

In this section, we now seek positive integer solutions to the equation $s^2 - 3t^2 = 1$. Educated and systematic trial and error calculations will give the minimal solution $(s_1, t_1) = (2, 1)$. Now we can apply the formulas given in the Pell's Equation section to construct all solutions. For $n = 1, 2, 3, \ldots$ we have

$$s_n = \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n)\right]$$

$$t_n = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$$

Recursion relations for (s_n, t_n) also exist. From the Pell's Equation section we have the following:

$$s_{n+2} = 4s_{n+1} - s_n$$

$$t_{n+2} = 4t_{n+1} - t_n$$

These formulas will give all positive integer solutions to $s^2 - 3t^2 = 1$ which gives rise to all solutions to the set of NETs. In the previous section we found that if we have positive integers s_n and t_n then x_n, A_n and r_n are also integers. Using the following formulas we can construct the NETs we have sought.

$$x_n = 2s_n \tag{93}$$

$$A_n = 3s_nt_n \tag{94}$$

$$r_n = t_n \tag{95}$$

However, as discussed in the previous section, the radius of the circumscribed circle, R, cannot be said to be an integer in general even though s_n and t_n are since the following formula does not directly calculate as an integer as the other formulas do.

$$R_n = \frac{4s_n^2 - 1}{6t_n}$$

This relationship between the integers s_n and t_n is much more complicated, and we will see in the table that R is clearly not an integer in general.

The first eight positive solutions to $s^2 - 3t^2 = 1$ as well as the corresponding values of $x-1$, x, $x+1$, σ, A, r and R appear in the table below. Only the values for R have rounded decimal values. Although the values for R when $n = 7$ and $n = 8$ appear to be integers, this is only due to a lack of decimal places. They are not, in fact, integers.

Table 2: Solutions to NETs

n	s_n	t_n	$x_n - 1$	x_n	$x_n + 1$	σ_n	A_n	r_n	R_n
1	2	1	3	4	5	6	6	1	2.5
2	7	4	13	14	15	21	84	4	8.125
3	26	15	51	52	53	78	1170	15	30.033
4	97	56	193	194	195	291	16296	56	112.009
5	362	209	723	724	725	1086	226974	209	435.502
6	1351	780	2701	2702	2703	4053	3161340	780	1560.001
7	5042	2911	10083	10084	10085	15126	44031786	2911	5822.00
8	18817	10864	37633	37634	37635	56451	613283664	10864	21728.00

An interesting result should be noted from this table. The recursion relationships for σ_n, x_n, and r_n are all the same as the recursion relations for u_n and v_n. So

$$u_n = 4u_{n-1} - u_{n-2} \tag{96}$$

$$v_n = 4v_{n-1} - v_{n-2} \tag{97}$$

$$\sigma_n = 4\sigma_{n-1} - \sigma_{n-2} \tag{98}$$

$$x_n = 4x_{n-1} - x_{n-2} \tag{99}$$

$$r_n = 4r_{n-1} - r_{n-2} \tag{100}$$

There is also a recursion relation for A_n that can be seen in the table. It is as follows:

$$A_n = 14A_{n-1} - A_{n-2}$$

We can use the recursion relation for x_n to find an explicit formula that gives x_n for any n. We start by rewriting the relation as

$$y_n - 4y_{n-1} + y_{n-2} = 0$$

Then define the following function

$$K = 1 - 4x + x^2$$

Using the quadratic formula to find solutions of K we can write

$$K = [x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]$$

Now define the infinite sum

$$G = a + bx + y_2 x^2 + \ldots + y_n x^n + \ldots$$

where $y_0 = a$ and $y_1 = b$ are the initial solutions for the recursion relation. The first scalene triangle made up of three consecutive integers is the degenerate triangle $(1, 2, 3)$, we will leave this one out since its area is 0 and thus it is uninteresting. So the first triangle we will actually consider and the first one that appears in the table above is the $(3, 4, 5)$ triangle. Since we have chosen x_n to be the even integer in the middle of the three consecutive ones, $a = 4$. The next NET is $(13, 14, 15)$ and thus

$b = 14$. Then,

$$KG = (1 - 4x + x^2)(a + bx + y_2x^2 + y_3x^3 + \ldots + y_nx^n + \ldots) \tag{101}$$

$$= a - (-4a + b)x + (a - 4b + y_2)x^2 + (b - 4y_2 + y_3)x^3 \ldots \tag{102}$$

$$+ (y_n - 4y_{n-1} + y_{n-2})x^n + \ldots \tag{103}$$

$$= a + [-4a + b]x + [a - 4b + y_2]x^2 + (y_1 - 4y_2 + y_3)x^3 \ldots \tag{104}$$

$$+ [y_n - 4y_{n-1} + y_{n-2}]x^n + \ldots \tag{105}$$

$$= a + [b - 4a]x + [y_2 - 4y_1 + y_0]x^2 + (y_3 - 4y_2 + y_1)x^3 \ldots \tag{106}$$

$$+ [y_n - 4y_{n-1} + y_{n-2}]x^n + \ldots \tag{107}$$

Then since $y_n - 4y_{n-1} + y_{n-2} = 0$,

$$KG = a + [b - 4a]x + 0x^2 + 0x^3 + \ldots + 0x_n + \ldots = a + [b - 4a]x$$

And then since $K = [x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]$ we have

$$G = \frac{a + (b - 4a)x}{[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]}$$

and since $a = 4$ and $b = 14$,

$$G = \frac{4 - 2x}{[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]}$$

Now we will use partial fractions to write G as two separate fractions. We start with

$$G = \frac{4 - 2x}{[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]} = \frac{A}{[x - (2 + \sqrt{3})]} + \frac{B}{[x - (2 - \sqrt{3})]}$$

Then

$$4 - 2x = A[x - (2 - \sqrt{3})] + B[x - (2 + \sqrt{3})]$$

so we have

$$-2 = A + B \implies A = -2 - B$$

and

$$4 = -2A + \sqrt{3}A - 2B - \sqrt{3}B \tag{108}$$

$$\implies 4 = -2(-2 - B) + \sqrt{3}(-2 - B) - 2B - \sqrt{3}B \tag{109}$$

$$\implies 0 = -2\sqrt{3} - 2\sqrt{3}B \tag{110}$$

$$\implies B = -1 \tag{111}$$

$$\implies A = -1 \tag{112}$$

Thus

$$G = -\frac{1}{[x - (2 + \sqrt{3})]} - \frac{1}{[x - (2 - \sqrt{3})]}$$

Now we will use geometric series to find the explicit formula for y_n. The basic property of a geometric series is for $r \in \mathbb{R}$ and $|r| < 1$ we have

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

So we'll start by writing

$$G = -\frac{1}{[x - (2 + \sqrt{3})]} - \frac{1}{[x - (2 - \sqrt{3})]} = A + B$$

and then rewrite A as follows

$$A = -\frac{1}{[x-(2+\sqrt{3})]} \tag{113}$$

$$= \frac{-1}{(2+\sqrt{3})\left[\frac{x}{2+\sqrt{3}}-1\right]} \tag{114}$$

$$= \frac{1}{(2+\sqrt{3})\left[1-\frac{x}{2+\sqrt{3}}\right]} \tag{115}$$

$$= \frac{1}{2+\sqrt{3}}\left[1+\frac{x}{2+\sqrt{3}}+\frac{x^2}{(2+\sqrt{3})^2}+\ldots+\frac{x^n}{(2+\sqrt{3})^n}+\ldots\right] \tag{116}$$

$$= \frac{1}{2+\sqrt{3}}+\left(\frac{1}{(2+\sqrt{3})^2}\right)x+\left(\frac{1}{(2+\sqrt{3})^3}\right)x^2+\ldots+\left(\frac{1}{(2+\sqrt{3})^{n+1}}\right)x^n+\ldots \tag{117}$$

Similarly, we can write B as follows

$$B = -\frac{1}{[x-(2-\sqrt{3})]} \tag{118}$$

$$= \frac{-1}{(2-\sqrt{3})\left[\frac{x}{2-\sqrt{3}}-1\right]} \tag{119}$$

$$= \frac{1}{(2-\sqrt{3})\left[1-\frac{x}{2-\sqrt{3}}\right]} \tag{120}$$

$$= \frac{1}{2-\sqrt{3}}\left[1+\frac{x}{2-\sqrt{3}}+\frac{x^2}{(2-\sqrt{3})^2}+\ldots+\frac{x^n}{(2-\sqrt{3})^n}+\ldots\right] \tag{121}$$

$$= \frac{1}{2-\sqrt{3}}+\left(\frac{1}{(2-\sqrt{3})^2}\right)x+\left(\frac{1}{(2-\sqrt{3})^3}\right)x^2+\ldots+\left(\frac{1}{(2-\sqrt{3})^{n+1}}\right)x^n+\ldots \tag{122}$$

Notice that the geometric series property mentioned above is used to get equations

116 and 121. Since $A+B=G$ we have

$$G=\left[\frac{1}{2+\sqrt{3}}+\left(\frac{1}{(2+\sqrt{3})^2}\right)x+\left(\frac{1}{(2+\sqrt{3})^3}\right)x^2+\ldots+\left(\frac{1}{(2+\sqrt{3})^{n+1}}\right)x^n+\ldots\right] \tag{123}$$

$$+\left[\frac{1}{2-\sqrt{3}}+\left(\frac{1}{(2-\sqrt{3})^2}\right)x+\left(\frac{1}{(2-\sqrt{3})^3}\right)x^2+\ldots+\left(\frac{1}{(2-\sqrt{3})^{n+1}}\right)x^n+\ldots\right] \tag{124}$$

$$=\left[\frac{1}{2+\sqrt{3}}+\frac{1}{2-\sqrt{3}}\right]+\left[\frac{1}{(2+\sqrt{3})^2}+\frac{1}{(2-\sqrt{3})^2}\right]x \tag{125}$$

$$+\left[\frac{1}{(2+\sqrt{3})^3}+\frac{1}{(2-\sqrt{3})^3}\right]x^2+\ldots+\left[\frac{1}{(2+\sqrt{3})^{n+1}}+\frac{1}{(2-\sqrt{3})^{n+1}}\right]x^n+\ldots \tag{126}$$

Now recall that we defined the infinite sum

$$G=a+bx+y_2x^2+\ldots+y_nx^n+\ldots=y_0+y_1x+y_2x^2+\ldots+y_nx^n+\ldots$$

So it follows that

$$y_n=\frac{1}{(2+\sqrt{3})^{n+1}}+\frac{1}{(2-\sqrt{3})^{n+1}}$$

and since

$$\frac{1}{2+\sqrt{3}}\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)=2-\sqrt{3}$$

and

$$\frac{1}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)=2+\sqrt{3}$$

we have the simple explicit formula

$$y_n=(2-\sqrt{3})^{n+1}+(2+\sqrt{3})^{n+1}$$

This formula will give x_n, which is the even integer in the middle of the NET triple $(x-1,x,x+1)$.

Further Pell's Equation Applications

In the previous section, we saw how solutions to Pell's equation were useful in finding an infinite set of triangles with integer sides and integer area. We will discuss two more applications of Pell's equation here. The first involves Archimedes famous approximations of $\sqrt{3}$ and the second is the determination of nearly isosceles Pythagorean triangles.

Finding approximate values of $\sqrt{N}$ where N is not a perfect square has been a task that mathematicians have studied for centuries. One of Archimedes' approximations for $\sqrt{3}$ is one that shows up in most history of mathematics books. He estimated that

$$\frac{223}{71} < \pi < \frac{22}{7}$$

by inscribing and circumscribing regular polygons of sides 6, 12, 24, 48, and 96 in a circle. He describes his process in his *Measurement of a Circle*. In his process, these estimates of $\sqrt{3}$ were certainly necessary. However, his estimate of

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

is much closer and yet he does not explain how he came to this estimate in any of his writings. These are the bounds which we will be considering in relation to Pell's equations. Consider the following Pell equations

$$x^2 - 3y^2 = 1 \text{ and } r^2 - 3s^2 = -2$$

We will start by tabulating the first seven solutions to each equation.

Table 3: Solutions to $x^2 - 3y^2 = 1$ and $r^2 - 3s^2 = -2$

n	x_n	y_n	r_n	s_n
1	2	1	1	1
2	7	4	5	3
3	26	15	19	11
4	97	56	71	41
5	362	209	265	153
6	1352	780	989	571
7	5042	2911	3691	2131

Since $x^2 - 3y^2 = 1 > 0$ we can write

$$x^2 - 3y^2 > 0 \tag{127}$$

$$x^2 > 3y^2 \tag{128}$$

$$\sqrt{x^2} > \sqrt{3y^2} \tag{129}$$

$$x_n > \sqrt{3}y_n \tag{130}$$

$$\frac{x_n}{y_n} > \sqrt{3} \tag{131}$$

and so thus an upper bound for $\sqrt{3}$ comes from the ratio of the solutions to the Pell equation. To get the bound that Archimedes had used, look at $n = 6$. This gives the upper bound

$$\frac{x_n}{y_n} = \frac{1351}{780} > \sqrt{3}$$

Similarly, the Archimedean lower bound comes from a "Pell-like equation" $r^2 - 3s^2 = -2$. Since

$r^2 - 3s^2 = -2 < 0$ we can write

$$r^2 - 3s^2 < 0 \tag{132}$$

$$r^2 < 3s^2 \tag{133}$$

$$\sqrt{r^2} < \sqrt{3s^2} \tag{134}$$

$$r_n < \sqrt{3}s_n \tag{135}$$

$$\frac{r_n}{s_n} < \sqrt{3} \tag{136}$$

and so we have a lower bound on $\sqrt{3}$. If we take $n = 5$ we will have the bound that Archimedes stated. Thus, using the Pell equation and the "Pell-like" equation we have the following

$$\frac{265}{153} = \frac{r_n}{s_n} < \sqrt{3} < \frac{x_n}{y_n} = \frac{1351}{780}$$

Now it seems as though it would make sense to further this argument and possibly get more accurate lower bounds by looking at the "Pell-like" equation $x^2 - 3y^2 = -1$. However, the following proof from Gelfond's The Solution of Equations in Integers shows why this is not possible.

Theorem 6. *The equation $x^2 - 3y^2 = -1$ is not solvable in integers x and y.*

Proof. Note that since any odd number a may be written in the form $a = 2N + 1$

where N is an integer we have the following

$$a^2 = (2N+1)^2 \tag{137}$$

$$= 4N^2 + 4N + 1 \tag{138}$$

$$= 4N(N+1) + 1 \tag{139}$$

$$= 8M + 1 \tag{140}$$

where M is an integer. This last line comes from the fact that either N or $N+1$ must be even so there is a factor of 2 in the term $4N(N+1)$ thus resulting in some integer $8M$. Thus note that the square of any odd integer under division by 8 always results in a remainder of 1.

So if $[x_0, y_0]$ is a solution of $x^2 - 3y^2 = -1$, then x_0 and y_0 must not both be even or both odd. If x_0 and y_0 were both even then x_0^2 and y_0^2 are also both even. So then $x_0^2 - 3y_0^2$ must be even and cannot be equal to -1. Similarly, if x_0 and y_0 are both odd then x_0^2 and y_0^2 are both odd but $x_0^2 - 3y_0^2$ again must be even and still cannot be equal to -1. Thus, x_0 and y_0 cannot be of the same parity. The next option is to consider x_0 and y_0 to have opposite parity.

First consider x_0 to be odd and y_0 to be even. Then x_0^2 would give a remainder of 1 upon division by 4, but $-3y_0^2$ would be divisible by 4. Therefore, $x_0^2 - 3y_0^2$ would give a remainder of 1 upon division by 4. This is impossible since the right hand side of the equation under division by 4 gives a remainder of -1 or 3 but not a remainder of 1.

Finally, we will consider x_0 to be even and y_0 to be odd. Then x_0^2 is divisible by

4 and as shown above

$$-3y^2 = -3(8M+1) \tag{141}$$

$$= -24M - 3 \tag{142}$$

$$= 4(-6M-1)+1 \tag{143}$$

and thus the left hand side of $x_0^2 - 3y_0^2 = -1$ will again have a remainder of 1 in division by 4 and this is impossible as stated before. Therefore, there do not exist integers x_0 and y_0 which satisfy $x_0^2 - 3y_0^2 = -1$ and thus this "Pell-like" equation cannot be used to calculate a better lower estimate for $\sqrt{3}$. $\square$

Although Pell's equation was not named as such until much after Archimedes made this claim on the bounds, we do not know whether Archimedes had ever studied similar equations. Since Archimedes provided no proof there is no definitive way to know how he came upon this bound. It is certainly possible he used some type of "Pell-like" equation.

We will now use arithmetic manipulation to show another possibility as to how Archimedes might have come upon these bounds. Consider first the upper bound of $\frac{1351}{780}$. We will start with $26 - \frac{1}{52}$ and manipulate this quantity as follows.

$$26 - \frac{1}{52} = \sqrt{(26 - \frac{1}{52})^2} \tag{144}$$

$$= \sqrt{26^2 - 2\left(\frac{26}{52}\right) + \left(\frac{1}{52}\right)^2} \tag{145}$$

$$= \sqrt{26^2 - 1 + \left(\frac{1}{52}\right)^2} \tag{146}$$

$$> \sqrt{26^2 - 1} \tag{147}$$

Then we have

$$\frac{1}{15}\left(26-\frac{1}{52}\right)=\frac{26}{15}-\frac{1}{(15)(52)}=\frac{1351}{780}$$

and from the previous inequality statement

$$\frac{1}{15}\left(26-\frac{1}{52}\right)>\frac{1}{15}\sqrt{26^2-1}$$

thus we have

$$\frac{1351}{780}=\frac{1}{15}\left(26-\frac{1}{52}\right)>\frac{1}{15}\sqrt{26^2-1}=\sqrt{\frac{26^2-1}{15^2}}=\sqrt{3}$$

which is the upper bound on $\sqrt{3}$ that we sought.

Now similarly, we will find the lower bound $\frac{265}{153}$ by arithmetic manipulation using $26-\frac{1}{51}$. The manipulation is as follows:

$$\begin{aligned}
\frac{265}{153} &= \frac{1}{15}\left(26-\frac{1}{51}\right) && (148)\\
&= \frac{1}{15}\sqrt{\left(26-\frac{1}{51}\right)^2} && (149)\\
&= \frac{1}{15}\sqrt{26^2-\frac{52}{51}+\frac{1}{2601}} && (150)\\
&= \frac{1}{15}\sqrt{26^2-1-\frac{1}{51}+\frac{1}{2601}} && (151)\\
&= \frac{1}{15}\sqrt{26^2-1-\frac{2550}{132651}} && (152)\\
&< \frac{1}{15}\sqrt{26^2-1} && (153)\\
&= \sqrt{3} && (154)
\end{aligned}$$

and thus we have secured a lower bound on $\sqrt{3}$ using only simple arithmetic. Again, we do not know if this is the exact way that Archimedes came upon the bounds, but this method would have been attainable using the mathematics of his day.

The second application of Pell's equation is the determination of nearly isosceles Pythagorean triangles with integer sides $(x, x+1, x+k)$. The problem here is to find integers x and k for which $x^2 + (x+1)^2 = (x+k)^2$. Using a similar method as used in the previous sections, it can be shown that solutions for these triangles can be obtained from the Pell equation $u^2 - 2v^2 = 1$. The first five solutions for these triangles are tabulated here.

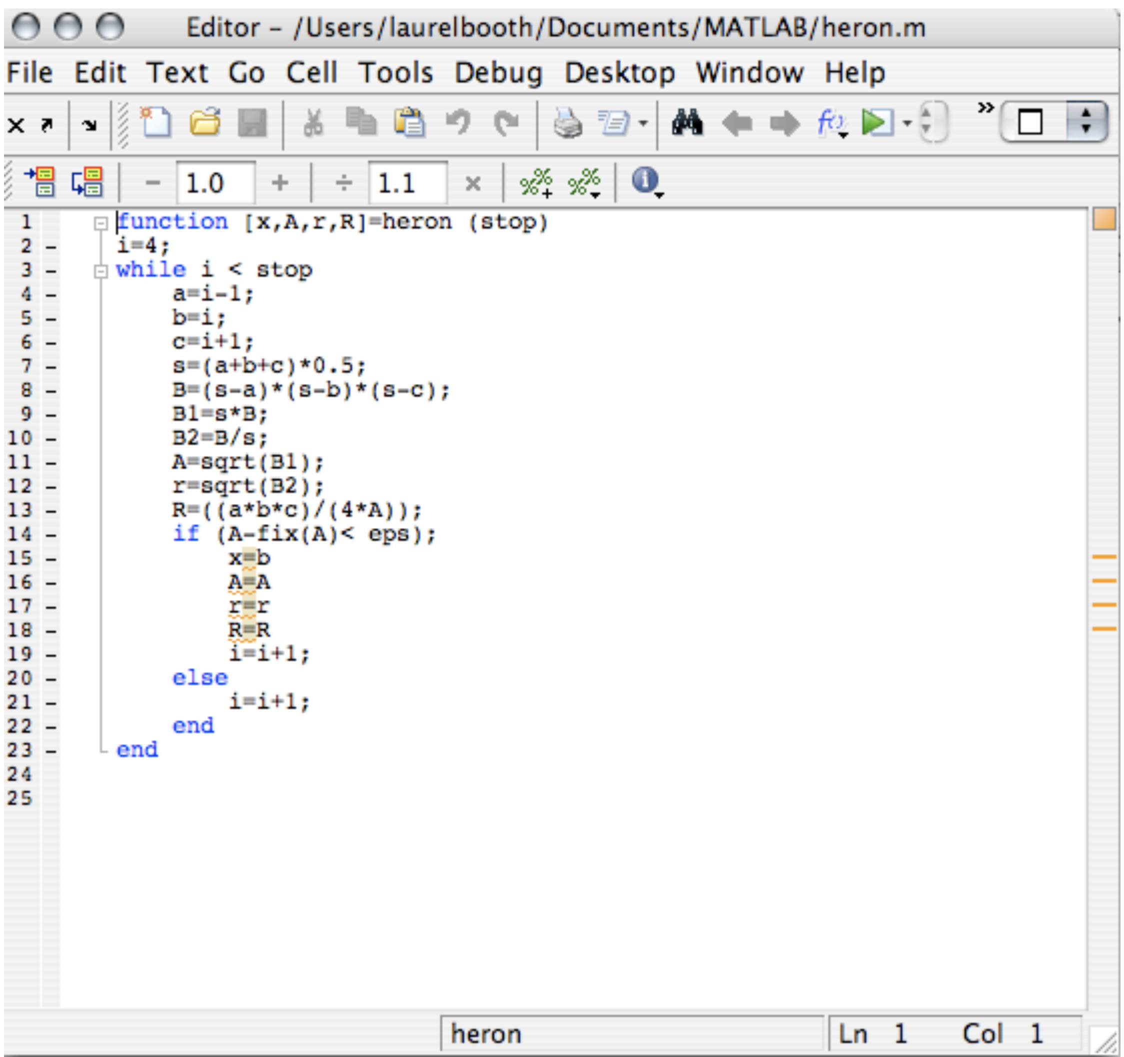
Table 4: Solutions to Nearly Isosceles Pythagorean Triangles

n	u_n	v_n	k	x	$x+1$	$x+k$
1	3	2	2	3	4	5
2	17	12	9	20	21	29
3	99	70	50	119	120	169
4	577	408	289	696	697	985
5	3363	2378	1682	4059	4060	5741

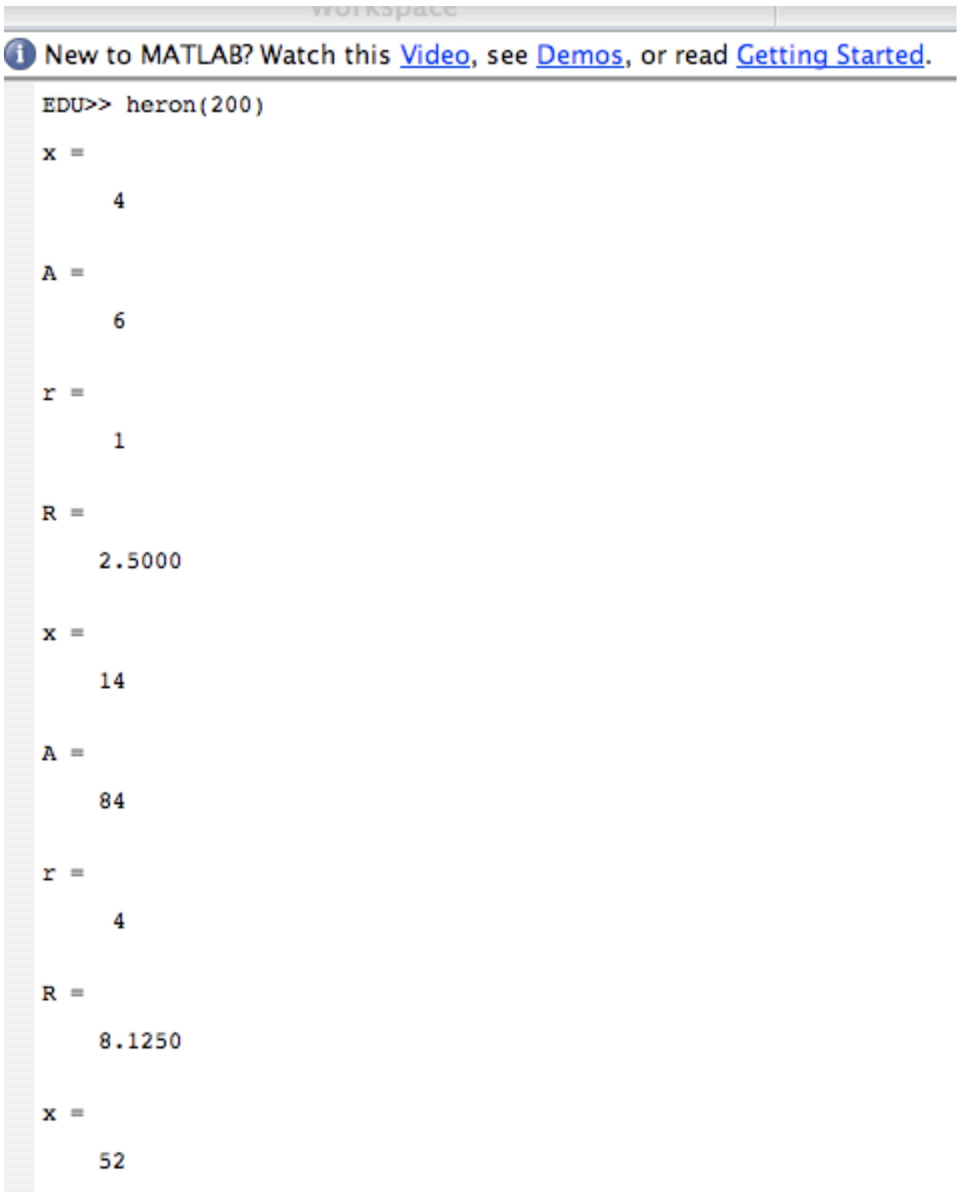
It is possible to obtain generalizations of nearly isosceles triangles to higher dimensions. For example, in dimension three we seek triangles of the forms $(x, x, x+1, x+k)$ and $(x, x+1, x+1, x+l)$. It can be shown that there are infinitely many triangles of these types. Two examples of the first are $6^2+6^2+7^2 = 11^2$ and $88^2+88^2+89^2 = 153^2$. Two examples of the second are $1^2+2^2+2^2 = 3^2$ and $23^2+24^2+24^2 = 41^2$. These examples can be constructed in a manner similar to the one used in the previous sections for two dimensions. The Pell equation that is relevant in this case is $x^2 - 3y^2 = 4$. For any dimensions $N > 3$, the relevant Pell equation is $x^2 - Ny^2 = (N-1)^2$ but solutions can only be obtained for values of N not a perfect square. Note that a minimum solution of $x^2 - Ny^2 = (N-1)^2$ is $x = N+1$ and $y = 2$. These higher dimension situations will not be discussed further here.

Appendix A: Matlab Program

This is a screen shot of the function file for the program in Matlab that will output the center side length for the first n NETs.



The output for this program gives the first few solutions of NETs. In the previous sections we verified the following output using Pell's equation. The first few output are as follows:



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