

8-27-2009

Exploring the Mathematical Thinking of Bilingual Primary-Grade Students: CGI Problem Solving From Kindergarten Through 2nd Grade

Mary Elisabeth Marshall

Follow this and additional works at: https://digitalrepository.unm.edu/educ_llss_etds

Recommended Citation

Marshall, Mary Elisabeth. "Exploring the Mathematical Thinking of Bilingual Primary-Grade Students: CGI Problem Solving From Kindergarten Through 2nd Grade." (2009). https://digitalrepository.unm.edu/educ_llss_etds/24

This Dissertation is brought to you for free and open access by the Education ETDs at UNM Digital Repository. It has been accepted for inclusion in Language, Literacy, and Sociocultural Studies ETDs by an authorized administrator of UNM Digital Repository. For more information, please contact disc@unm.edu.

Mary Elisabeth Marshall

candidate

Language, Literacy and Sociocultural Studies

730/20/10000

This dissertation is approved, and it is acceptable in quality and form for publication:

Approved by the Dissertation Committee:

Lylia Celedon-Pattichia

, Chairperson

Heather Marshall

John David Galvan

Richard S. Ketchum

**EXPLORING THE MATHEMATICAL THINKING OF BILINGUAL
PRIMARY-GRADE STUDENTS: CGI PROBLEM SOLVING FROM
KINDERGARTEN THROUGH 2ND GRADE**

BY

MARY ELISABETH MARSHALL

B. A., Elementary and Secondary Education, College of Santa Fe, 1997
M. A., Education of At-Risk Youth, College of Santa Fe, 1999

DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy

Language, Literacy and Sociocultural Studies

The University of New Mexico
Albuquerque, New Mexico

August, 2009

©2009, Mary E. Marshall

DEDICATION

This work is dedicated to Gina, Yolanda, Gerardo and Omar, whose mathematical thinking is revealed in the following pages, and also to Ana, Brisa, Jenna and Dolores who participated in this study. All eight children sat through seemingly endless hours of interviews. They approached our questions with humor, gave us their best efforts, and forgave us for their frustrations. They willingly came back time and again, always with smiles and hugs. Over the years they became seasoned research participants who knew they were involved in something important even though they didn't understand the significance of the study for their lives and their community. Without their enthusiasm for mathematics problem solving, this project would not have been possible. They gave us their love.

I am also indebted to the parents and families of these children for supporting our effort and allowing their children to participate in this research. They knew our goal was educational opportunity for their children, but our formal qualitative methodology was not part of their life experience. They gave us their trust.

Finally, this work is dedicated to the teachers at La Joya Elementary School who supported our research effort. They cheerfully tolerated our interview schedules as we pulled students from their classes and disrupted their daily routines. They believed in us.

ACKNOWLEDGMENTS

A dissertation is too massive a project to be the work of only one person. I have had the great fortune in the last several years to be supported in this effort by wonderful personal and professional relationships.

First, I give my deepest thanks to my advisor, Sylvia Celedón-Pattichis and the CEMELA (Center for the Mathematics Education of Latinos/as) research team. Sylvia has been involved in this work at every step of the way, always supportive, never critical and willing to spend long hours making sure I would be successful with the research and the final writing project. Sandra Musanti, a former CEMELA post-doc, was at my side for many hours of interviewing with thoughtful comments and suggestions that made this work stronger in multiple ways. Richard Kitchen has been another critical presence in this effort. Rick carries the banner and leads the march in equity for Latina/o students in mathematics education. He inspired me on a moral as well as a professional level. The other CEMELA fellows, Barb, Edgar, Laura, Lisa and past fellows Alan and Berenice have been my greatest friends and supporters as I struggled toward this goal. Susan Metheny, our administrator, made CEMELA run like clockwork and we all owe a debt to her for every accomplishment. To all the affiliated professors and support staff for CEMELA at UNM, I offer you my deepest thanks.

Just like this dissertation, CEMELA is a broad effort. I am sincerely indebted to the National Science Foundation for their financial support and the vision that made CEMELA possible. To the CEMELA faculty and fellows at University of Arizona, UC Santa Cruz, and the University of Illinois at Chicago, I say a warm thank you and wish you all the best as you finish your research. I want to especially acknowledge Marta

Civil at the University of Arizona, who has dedicated countless hours of her time, enjoyed the benefits and bore the brunt of the frustrations of leading a large research grant. A special thanks also to Erin Turner, now at University of Arizona, who initiated the research in the kindergarten classroom that resulted in this project.

The staff and students at La Joya Elementary School deserve a special mention. They represent a community rich in resources and potential. The teachers are dedicated to the success of the students and support the families in their dreams of educational opportunity for their children. The children are delightful. One of our greatest pleasures in visiting the classrooms was the warm reception from students. We received as much as we gave in the relationships we built over time at La Joya.

Finally, I want to thank my very dear family and the special friends who have supported me, listened to my extended conversations, and most all of believed in me. I could not have done this work without the friends who helped me develop the ability to speak in Spanish and the family who stood behind me. To my friends, children Colin and Peter, and their dad Bob Marshall, thank you more than I can say for your love and encouragement.

**EXPLORING THE MATHEMATICAL THINKING OF BILINGUAL
PRIMARY-GRADE STUDENTS: CGI PROBLEM SOLVING FROM
KINDERGARTEN THROUGH 2ND GRADE**

BY

MARY E. MARSHALL

ABSTRACT OF DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy

Language, Literacy and Sociocultural Studies

The University of New Mexico
Albuquerque, New Mexico

August, 2009

**EXPLORING THE MATHEMATICAL THINKING OF BILINGUAL PRIMARY-
GRADE STUDENTS: CGI PROBLEM SOLVING FROM KINDERGARTEN
THROUGH 2ND GRADE**

by

Mary E. Marshall

B.A., Elementary and Secondary Education, College of Santa Fe, 1997

M.A., Education of At-Risk Youth, College of Santa Fe, 1999

Ph.D., Language, Literacy and Sociocultural Studies, University of New Mexico, 2009

ABSTRACT

This study explores the mathematical thinking of native Spanish-speaking, primary-grade Latina/o students learning in bilingual classrooms where the majority of their mathematics instruction has been in Spanish. Guided by sociocultural theory, which emphasizes the important connection between language and conceptual development (Mahn, 2008; Sfard, 2001; Van Oers, 2001; Zack & Graves, 2001), and the theory and methods of Cognitively Guided Instruction [CGI] (Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Carpenter, Fennema & Franke, 1994; Carpenter, Fennema & Franke, 1996; Carpenter, Fennema, Franke, Levi & Empson, 1999; Fennema, Franke, Carpenter & Carey, 1993; Turner, Celedón-Pattichis & Marshall, 2008), data were collected on students' developing abilities to solve CGI problems and explain their thinking about their solutions. An expanded notion of mathematical explanations and discourse was used in the analysis that goes beyond student language to include their gestures, the tools they

selected as problem solving aids, and their drawings and equations (Gee & Green, 1998; Moschkovich, 2002). This qualitative, longitudinal study is based on individual CGI interviews with four students over the course of their first three years in school and follows a grounded theory tradition (Creswell, 1998; Glaser & Strauss, 1980) to uncover themes in their mathematical thinking.

The overarching motivation for this research was one of equity where the broader methodology sought to encourage a high-quality mathematics learning environment for Spanish-speaking, Mexican immigrant students in bilingual classrooms (Secada & De La Cruz, 1996). Groundbreaking findings from this study add to the literature on how young students make sense of the numbers in mathematical word problems (Fuson, 1988). The findings demonstrate that students are making sense of the numbers in fundamentally different ways and carry major implications for CGI theory, mathematics teaching and learning, and sociocultural theory. Of particular interest to the field of bilingual education is the way students engage with CGI problems when the problems have been contextualized within students' native language and culture (Cummins, 2001; Secada & De La Cruz, 1996). Additionally, this study demonstrates how bilingual students use two languages, Spanish and English, to explain their mathematical thinking and describe their problem solving strategies.

TABLE OF CONTENTS

TABLE OF CONTENTS	x
LIST OF FIGURES.....	xiv
LIST OF TABLES	xv
CHAPTER 1. A Commitment to Equity	1
Introduction.....	1
Statement of the Problem.....	2
Significance of the Study.....	3
Research Questions	4
Background to the Research	4
Equity in Mathematics Education	5
Purpose of this Study.....	9
Organization of this Document and Overview of the Findings	10
CHAPTER 2. Exploring Systems of Meaning Through Cognitively Guided Instruction (CGI)	13
Introduction.....	13
Sociocultural Theory	18
The Role of Language in Sociocultural Theory.....	19
Systems of Meaning	21
Verbal thinking in the construction of systems of meaning.....	24
Qualitative Transformation – Language Acquisition	26
Qualitative Transformation – Introduction of Scientific Concepts.....	28
Mathematical Thinking.....	30
Competing Theories for Exploring Conceptual Mathematical Development	33
Constructivism	33
Social Constructivism/Emergent Theory.....	36
Thinking as Communication: Interactivity Analysis.....	40
Problem Solving in Mathematics Education	45
Cognitively Guided Instruction.....	48
A Bridge Between Informal and Formal Systems of Meaning.....	50
From Direct Modeling to More Efficient Problem Solving Strategies	50
CGI Problem Types.....	53
Joining and Separating.....	53
Multiplication and Division.....	55
Extending CGI to Base Ten Thinking and Multidigit Numbers.....	57
Teaching and Learning Mathematics in Spanish	60
Gaps in the Literature	63
CHAPTER 3. Methodology.....	65
Introduction.....	65
Background to the Research	66
Kindergarten.....	67
1 st and 2 nd grade.....	69
Research Questions	72
Mode of Inquiry and Philosophy.....	73

Research Participants and Site Description	75
The four final participants.....	78
Methods	79
Tools available for problem solving.....	80
Interview locations.....	80
Qualitative data collection.....	81
Quantitative analysis of the kindergarten and 1 st grade data.....	85
Problem solving strategy analysis for 1 st grade.....	86
Longitudinal Data Analysis	87
Stage 1: Choosing a subset of students.....	88
Stage 2: Choosing a subset of problems.....	89
Stage 3: Transcribing interview data.....	90
Stage 4: Profiling students.....	90
Stage 5: Comparing and contrasting students.....	92
Reflexivity During Data Collection and Analysis.....	93
Researcher Positionality	93
Trustworthiness	95
Triangulation of Data.....	95
Peer Debriefing	95
Transferability.....	96
Limitations of the Study	97
CHAPTER 4. Student Language Profiles.....	99
Introduction.....	99
Gina	102
Gerardo	107
Yolanda.....	111
Omar	115
CHAPTER 5. Student Problem-Solving Profiles	120
Introduction.....	120
Student Profile: Omar.....	122
Join problems.....	123
Compare problems.....	126
Part-Part-Whole problems.....	127
Multiplication problems.....	129
Partitive Division problems.....	131
Omar's problem solving summary.....	132
Student Profile: Yolanda.....	133
Join problems.....	133
Compare problems.....	136
Part-Part-Whole problems.....	138
Multiplication problems.....	140
Partitive Division problems.....	143
Yolanda's problem solving summary.....	144
Student Profile: Gerardo.....	145
Join problems.....	145
Compare problems.....	147

Part-Part-Whole problems.....	149
Multiplication problems.....	150
Partitive Division problems.....	155
Gerardo’s problem solving summary.....	155
Student Profile: Gina	156
Join problems.....	156
Compare problems.....	160
Part-Part-Whole problems.....	162
Multiplication problems.....	165
Partitive Division problems.....	168
Gina’s problem solving summary.....	169
CHAPTER 6. Making Meaning During CGI Problem Solving	170
Introduction.....	170
A Closer Examination of CGI Problem Solving Strategies.....	174
The role of student language to reveal mathematical thinking.....	176
Discourse variations among students.....	177
Individual student strategy patterns.....	178
Analysis of Omar’s CGI strategies.....	180
Analysis of Yolanda’s CGI strategies.....	183
Analysis of Gerardo’s CGI strategies.....	186
Analysis of Gina’s CGI strategies.....	188
Comparison of Student CGI Problem Solving: Kindergarten and 2 nd Grade.....	191
Multiplication (3, 6), Kindergarten: 3 bags of marbles, 6 in each bag.....	193
Omar’s solution.....	193
Yolanda’s solution.....	194
Gerardo’s solution.....	196
Gina’s solution.....	197
Student Approaches to Partitive Division (84, 4), 2 nd Grade.....	198
Omar’s solution.....	199
Gina’s solution.....	201
Yolanda’s solution.....	203
Gerardo’s solution.....	206
CHAPTER 7. Discussion and Implications.....	208
Discussion About Sequential Thinking.....	210
Discussion About Discrete Thinking.....	212
Implications of These Findings.....	214
For CGI theory:.....	214
For mathematics education:.....	216
For systems of meaning and native language learning environments:.....	219
For equity in mathematics education for Latina/Latino students:.....	220
Areas for Further Research.....	221
The Final Word.....	221
APPENDICES	223
Appendix A. System of Meaning and Planes of Verbal Thinking (Mahn, 2009).....	224
Appendix B. CGI Problem Types.....	225
Appendix C. Kindergarten Post-Assessment.....	226

Appendix D. 1 st Grade Pre-Assessment	227
Appendix E. 1 st Grade Post-Assessment	229
Appendix F. Materials for Problem Solving.....	230
Appendix G. Student Explanations	231
Appendix H. Examples of Intermediary Analysis	237
Appendix I. Longitudinal Tendencies	251
Appendix J. Problem Solving Profile Tables.....	259
Appendix K. Students' Problem Solving Tendencies	275
Appendix L. Students' Solutions to Partitive Division (84, 4).....	279
REFERENCES	292

LIST OF FIGURES

Figure 1. Planes of Verbal Thinking (Mahn, 2009).....	24
Figure 2. Channels of interaction during pairs problem solving (Sfard, 2001).....	42
Figure 3. Omar, Join Change Unknown (15, 25), 1 st grade.....	123
Figure 4. Omar, Join Start Unknown (7, 22), 2 nd grade.	125
Figure 5. Erin’s number line for Omar, Multiplication (3, 6), kindergarten.	129
Figure 6. Omar, Partitive Division (12, 6) and (24, 6), 1 st grade.....	131
Figure 7. Yolanda, Join Start Unknown (5, 15) and (10, 55), 1 st grade.	135
Figure 8. Yolanda, Part-Part-Whole (75, 50), 1 st grade.	139
Figure 9. Yolanda, Multiplication (15, 7), 2 nd grade.....	142
Figure 10. Gerardo, Compare Difference Unknown (9, 12), kindergarten.	148
Figure 11. Gerardo, Multiplication (7, 5), 1 st grade.....	152
Figure 12. Gerardo, Multiplication (8, 5), 2 nd grade.	153
Figure 13. Gerardo, Multiplication (15, 7), 2 nd grade.	154
Figure 14. Gina, Join Start Unknown (3, 5) and (5, 13), 1 st grade.....	158
Figure 15. Gina, Part-Part-Whole (13, 7), 1 st grade.	163
Figure 16. Gina, Part-Part-Whole (30, 20), 1 st grade.....	164
Figure 17. Gina, Multiplication (15, 7), 2 nd grade.	167
Figure 1. Planes of Verbal Thinking (Mahn, 2009).....	172
Figure 4. Omar, Join Start Unknown (7, 22), 2 nd grade.	182
Figure 18. Omar, Partitive Division (84, 4), 2 nd grade.....	200
Figure 19. Yolanda, Partitive Division (84, 4), 2 nd grade.....	205

LIST OF TABLES

Table 1. Interview Protocols K-2 nd Grade: CGI Problem Types and Numbers Used.....	83
Table 2. Comparison of Advanced Strategies to Correct Answers – 1 st Grade.....	86
Table 3. CGI Problem Types and Strategies, (based on Carpenter et al., 1999)	122
Table 4: Breakdown of All Students’ Strategies, Kindergarten – 2 nd Grade.....	178
Table 5: Omar’s Strategies, K-2 nd Grade	180
Table 6: Yolanda’s Strategies, K-2 nd Grade	183
Table 7: Gerardo’s Strategies, K-2 nd Grade.....	186
Table 8: Gina’s Strategies, K-2 nd Grade.....	188
Table 9: Sequential versus Discrete Thinking.....	192

CHAPTER 1. A Commitment to Equity

All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics.

(NCTM, 2000, p. 12)

Introduction

This qualitative study explores the developing mathematical thinking and conceptual development around Cognitively Guided Instruction [CGI] problem solving (Carpenter, Fennema, Franke, Levi & Empson, 1999) of four native Spanish-speaking students as they progress from kindergarten through 2nd grade. These students come from bilingual classrooms where the majority of their mathematics instruction is in Spanish. Of particular importance in this study is that the CGI problems used for individual student interviews arose from the context of students' native language and culture (Secada & De La Cruz, 1996), a culture based in a Mexican immigrant community where the majority of residents have a lower socioeconomic status. The research and analysis presented in this dissertation are based on the understanding from sociocultural theory that there is a direct connection among concept development, social interaction and language (Mahn, 2009; Vygotsky, 1987), and from CGI theory and mathematics education research that students' mathematical concept development is reflected in the approaches they take to solve complex problems and the ways they explain their mathematical thinking (Carpenter, Fennema & Franke, 1994; Hiebert and Carpenter, 1992; Lerman, 2001). Within the CGI framework, the specific strategies students use to solve a variety of problem types help teachers target areas of mathematical concept development and

understand how students are making sense of the number operations and relationships they encounter in the problems (Carpenter et al., 1999).

Statement of the Problem

Equity in mathematics education means that *all* students regardless of native language, culture, or socioeconomic class are entitled to engage in challenging classroom learning experiences that help them develop a deeper understanding of mathematics. This is stated in the Equity Principle of the *Principles and Standards for School Mathematics [PSSM]* (National Council of Teachers of Mathematics [NCTM], 2000). These activities should invite students to engage intellectually and push them toward excellence. All students need classroom environments that honor their diversity of language and culture and provide them with consistent support toward successful learning. Further, to give Spanish-speaking Mexican immigrant students, many from lower socioeconomic backgrounds, access to equity in mathematics education, students need opportunities to learn in their native language, Spanish (Trueba, 1999), and opportunities to solve problems based in familiar contexts (Secada & De La Cruz, 1996; Secada, 1989). Bilingual classrooms where young students learn cognitively demanding concepts in Spanish before transferring these to English have been found to provide the most successful learning environments overall (Cummins, 2001; Thomas & Collier, 2000). In addition, solving word problems beginning in kindergarten allows students to make sense of mathematics in familiar situations, make connections among ideas, and gives students the opportunity to engage in important mathematical processes, including verbal communication and justification of their thinking (Carpenter et al., 1999; NCTM, 2000; Turner, Celedón-Pattichis & Marshall, 2008). Finally, teaching practices that help

students develop the behavioral norms and discursive skills required in standards-based mathematics¹ curricula are vital if students from culturally, economically and linguistically diverse backgrounds are to be successful academically (Cobb & Yackel, 1996; Lubienski, 2002).

Significance of the Study

While some research has shown that indeed Spanish-speaking Latina/o students and students from such diverse communities can successfully engage in complex mathematical processes and that their problem-solving skills can improve through conceptually challenging activities (Kamii et al., 2005; Secada, 1991; Secada & De La Cruz, 1996; Turner et al., 2008; Villaseñor & Kepner, 1993), very little research has documented young Spanish-speaking students' mathematical problem-solving and discourse development over time within Spanish learning environments and the way these students explain their thinking in their native language (see Turner et al., 2008). This dissertation study provides a unique opportunity to explore bilingual students' longitudinal mathematical development and how they use Spanish, their first language, and English, their developing second language, in connection with problem solving. The study analyzes how young children approached CGI problem solving from kindergarten through 2nd grade and reveals important patterns and a new way of understanding how students make sense of the mathematics in word problems.

The students in this study demonstrated creativity, flexibility and insight in their problem solving. Their powerful thinking and the importance of the implications for

¹ The conceptual knowledge and process skills students need at each grade level are outlined in a collection of standards found in the PSSM document (NCTM, 2000).

mathematics education for all populations give value to the richness of resources found within the Mexican immigrant community. The study strengthens the notion that resources lie within all populations and that all students benefit from attention to language and culture in teaching and learning (González, Moll & Amanti, 2005).

Research Questions

The research questions for this study include the following:

- How do bilingual primary-grade students learning mathematics in Spanish language environments communicate their mathematical thinking during CGI problem solving over the course of three years?
- What does student communication in the form of the strategies they use to solve the problems, the tools and materials they choose as aids, and students' verbal explanations reveal about how they are making sense of the mathematics in the CGI problems?

Background to the Research

This investigation into student thinking began four years ago with Mexican immigrant students in an urban, bilingual elementary school in the Southwestern United States. During the 2005-2006 school year, students in one bilingual kindergarten classroom began to solve context-embedded word problems from their very first weeks in school based on CGI problem types (Carpenter et al., 1999). Hand in hand with the CGI problem solving activities was an expectation from the kindergarten teacher, Ms. Arenas, that the students would always discuss their mathematical thinking (see Turner et al., 2008 for a description of this research). Videotape data were collected during the year on

the teacher's implementation of CGI problem solving in the classroom and also on students' mathematical thinking and explanations during individual CGI-based interviews.

The dual emphasis on CGI problem solving and mathematical language development continued in two first grade bilingual classrooms during the following 2006-2007 school year. The research team collected data on individual student thinking with eight of the previous years' kindergarten students, and once again on the implementation of CGI, this time with the two first grade teachers. Next, the eight students from 1st grade moved into two 2nd grade bilingual classrooms where the research on student thinking around CGI problem solving continued. See the methodology chapter of this dissertation for a detailed description of the three years of the study and the eight participants involved.

Equity in Mathematics Education

My personal commitment to equity in education drives this research. Unfortunately, equal access to quality mathematics experiences is not a reality for many students who may live in poverty, come from immigrant communities, and/or speak a dialect or native language other than Standard English (Jordan, Kaplan, Oláh, & Locuniak, 2006; Kamii, Rummelsburg, & Kari, 2005; Khisty, 1997; Lubienski, 2000; Moschkovich, 2002; National Assessment of Educational Progress [NAEP], 2005; Ortiz-Franco, 1999; Pappas, Ginsburg, & Jiang, 2002; Secada, 1995). Most of these students wind up in classrooms that stress drill and practice at the expense of more conceptually challenging tasks (Kamii et al., 2005; Secada & De la Cruz, 1996).

Although the PSSM document (NCTM, 2000) clearly calls for equity, making mathematics accessible for every student has been an illusive goal (Allexsaht-Snider, 2001; Gutstein, 2003; Kitchen, 2003, 2004; Lubienski, 2002; Powell, 2004). Two decades ago, Secada (1989a, 1989b) caught the attention of the mathematics education research community when he challenged popular notions of equity as more aligned with the economic goals of mainstream white America than promoting the needs of minority students. First at the American Educational Research Association Annual meeting in 1988 and then in the Peabody Journal of Education, Secada laid out an agenda to move the dialogue about equitable mathematics education away from the periphery of research and bring conversations about social justice to the center of all discussions about mathematics teaching and learning.

Secada criticized the conversations about equal access to mathematics education as ultimately serving mainstream society, specifically saying, “It is in our *enlightened self-interest* to invest extra time and resources in order to ensure the adequate mathematics preparation of this country’s girls, non-White/non-Asian minorities, and children from lower SES backgrounds” (1989b, p. 26). Although equal opportunity in education provided the enlightened stream in his argument, the self-interest came from worries about international competitiveness, the decline in blue-collar labor, the technological demands of military and business, and an unemployable underclass that would create an economic drain on the U. S. economy.

In his attempt to turn the equity conversation in mathematics education away from economic self-interest, Secada proposed an agenda based on three main points for action:

- 1) examination of the social structures that determine who succeeds at what kind of

mathematics, 2) research into curriculum and instructional practices that open up access to all learners, and 3) exposure and restructuring of the dominant beliefs by the educational community about who can and should succeed in mathematics. Secada explicitly framed his argument in terms of social justice, where the basis of educational opportunity and access should lie in what is fair and just for each student, regardless of race, class, gender, language, ethnicity, or whether or not he or she aspires toward assimilation into mainstream culture.

Secada's three points for action converge in the mathematics classroom where the socially constructed labels of race, class and ethnicity play out in an environment designed for teaching and learning, and where the mathematical knowledge of certain cultural groups is privileged over other ways of knowing (Gutstein, 2003; Kozol, 2005; Trueba, 1999). Secada argued that deficit thinking about students who do not fit the middle-class, English-speaking model and/or come from poverty resulted in continual drill and practice and an emphasis on basic skills at the expense of tasks involving higher order thinking. He said that if schools expected these students to come up to a certain norm *before* having access to more advanced mathematical tasks then schools were "chasing a moving target" (Secada, 1989b, p. 39). He argued instead for "real contexts that reflect the lived realities of people who are members of equity groups...rich in the sorts of mathematics which can be drawn from them" (p. 49).

Today, two decades later, Secada's call for real contexts and critical thinking tasks that make mathematics accessible to a diverse population has not been fulfilled for all students. One of the most pervasive problems limiting all students' access to this type of mathematics education is deficit thinking among teachers and administrators about

students' ability to be successful (Alleksaht-Snider, 2001). Instead of drill and practice in the classrooms, or administrative decisions that place diverse students into lower mathematics tracks in secondary schools, Alleksaht-Snider calls for increased attention to classroom practices that specifically give students a sense of "belongingness" and specific tasks that invite students to "engage, bringing their own background experiences into the process of learning" (p. 97). Practitioner/researcher Gutstein (2003) has taken this idea and put the PSSMs' five process skills of problem-solving, communication, reasoning, representations, and connections (NCTM, 2000) at the heart of his curriculum with a Latina/o population to explore issues of social justice from a mathematics perspective within their own communities.

While the case of Gutstein above is successful in giving a specific population of Mexican immigrant secondary students the opportunity to learn standards-based mathematics through real contexts and critical thinking tasks, overall the challenges for educating Latina/o students remain high. Latina/o immigrant students find particular challenges in issues of language, class, and culture in U. S. schools (Khisty, 1997; Lubienski, 2002). The Pew Hispanic Center (2004) reports that 75% of all English language learners (ELLs) in U. S. classrooms speak Spanish at home and 35% of all ELLs are living in poverty. To add to the language issue, most Spanish-speaking immigrant students are being educated in English language classrooms (Thomas & Collier, 2002), limiting their ability to use their own linguistic resources to make sense of mathematical concepts (Trueba, 1999).

When educators believe that there should be one common language and culture for teaching and learning in America's schools, students from diverse linguistic and

cultural groups are at an immediate disadvantage (Cummins, 2000; Ortiz-Franco, 1999; Powell, 2004). Engaging in the processes of solving word problems and justifying their thinking in a developing second language challenges students' initial comprehension of problem situations and limits their ability to fully express their thinking (Khisty, 1997; Moschkovich, 2002). In addition, when norms of classroom behavior and learning are based on a model of middle class, white, English-speaking discourse and social interaction styles, immigrant students from Latino cultures and/or lower socioeconomic classes can be left without an access point (Lubienski, 2000, 2002). In addressing these norms of social interaction, Lubienski argues that the culture of the U. S. mainstream middle class, the culture that underscores most classroom dynamics, builds into its children a sense of efficacy, the ability to handle ambiguity, and aggressive discursive styles that encourage debate. Lubienski suggests these dynamics may not serve the needs of all students.

Purpose of this Study

The purpose of this study is to explore Latina/o student thinking when they have access to equitable mathematics practices that honor their native language and cultural experiences. Specifically, this study seeks to gain a greater understanding of children's mathematical thinking and how they make meaning during CGI problem solving. It adds to the scarce literature on teaching mathematics for understanding in Spanish/English bilingual contexts (Fuson, Smith & Lo Cicero, 1997; Secada, 1991; Secada & De La Cruz, 1996). More broadly, this study promotes equity in mathematics education for low-income, Spanish-speaking, Latina/o students (Khisty, 1997; Ortiz-Franco, 1999). My goal is to bring the issue of equity in mathematics education for Latina/o students to

the forefront of research and argue that all students have potential for success in mathematics. Children from every background have rich intellectual resources and unique and powerful ways of making sense of mathematics, and we have much to learn by exploring children's thinking in diverse settings. Regardless of cultural or linguistic diversity, all children have the right to high quality mathematics instruction that develops their critical thinking and language skills from the very beginning of their formal education (Secada, 1989a, 1989b, 1991, 1992, 1995).

Data collected in kindergarten showed that Spanish-speaking Mexican immigrant children were successful with challenging mathematics (Turner et al., 2008). In this longitudinal study, I focus on how four of these students, from kindergarten through 2nd grade tackled challenging CGI problems and the ways they talked about their thinking in relation to the strategies they used to solve the problems. I show that they have enormous potential for academic success through bilingual and standards-based mathematics education and underscore my belief that to deny these students the right to these rich learning experiences is to limit their educational opportunities and career possibilities.

Organization of this Document and Overview of the Findings

This dissertation combines solid qualitative research methodology with an invitation to the reader to connect with these children on a personal level. One of my goals in the following chapters is to paint a portrait of each child that provides the backdrop on which to understand his or her mathematical thinking. It is important to me that these portraits reveal the students' unique personalities and give glimpses into the remarkable abilities each one brings to the learning environment. All four children are highly motivated mathematics students who gave us their very best efforts as research

participants. These chapters present Latina/o children as people not just demographic descriptions. Their voices once revealed, I hope, make it impossible for anyone to read this dissertation without a commitment to these children's educational success.

Beginning with the traditional format of theory and methodology chapters (Chapter 2 and Chapter 3), I next paint portraits of the students in two profile chapters. The first of these chapters, Chapter 4, presents a narrative of each child. An important aspect of this chapter is each child's perceptions on Spanish and English and how she or he makes decisions to use one language or the other. The freedom to communicate in the language of their thinking (Mahn, 2009; Vygotsky, 1987) whether it is in Spanish or English removes ambiguity from their learning mathematics with understanding (Secada & De La Cruz, 1996) and strongly promotes the theme of equity for these children in mathematics education (Cummins, 2000; Thomas & Collier, 2002). The second profile chapter, 5, summarizes the students' CGI problem solving over the three years of the study. My intention behind including Chapter 5 in this dissertation is twofold: 1) to develop the problem solving personalities of each child, and 2) to present evidence that the powerful trends I uncover in my analysis of the data, including a new way of understanding children's thinking about the numbers in mathematics word problems are not selected to support my conclusions, but instead are consistently found in each child over time and therefore lead to these conclusions.

The dissertation continues with Chapter 6, my analysis of the trends I uncovered in the problem solving portraits of the students, and finally ends with Chapter 7 where I present my conclusions and specifically the implications I feel are important for CGI theory, mathematics education, sociocultural theory, and equity in mathematics for

Latina/o students. These trends show that the sense students make of the problems and the strategies they use to find solutions are based on the meaning they attach to the numbers contained in the problems. If students are thinking of the numbers as part of a sequential and ordered arrangement, their solution strategies and the ways they explain their thinking are fundamentally different from those of students who think of the numbers as representing discrete sets of objects that can be manipulated independent of arrangement. Because both sequential and discrete numerical meanings are valid (Fuson, 1988), two groundbreaking implications arise from this research: 1) As students develop mathematically, the sense they make of problems follows the trajectory more closely associated with the meaning they attach to the numbers and therefore students at the same grade level may be on very different learning trajectories, and 2) All students need the flexibility to shift between these two valid meanings if they are to successfully solve the range of problems contained within formal mathematics.

CHAPTER 2. Exploring Systems of Meaning Through Cognitively Guided

Instruction (CGI)

The CGI analysis differs from many other characterizations of students' thinking that focus on identifying students' misconceptions and errors. In CGI, the emphasis is on what children can do rather than on what they cannot do. This leads to a very different approach to dealing with errors than an approach in which the goal is to identify students' misconceptions in order to fix them. For CGI teachers the goal is to work back from errors to find out what valid conceptions students do have so that instruction can help students build on their existing knowledge.

(Carpenter, Fennema & Franke, 1996, p. 14)

Introduction

Equitable access to mathematics education means focusing on student strengths instead of weaknesses, beginning with what children *can* do, and using instruction to connect new concepts to children's existing knowledge. The goal of equity is explicitly stated by the National Council of Teachers of Mathematics (2000) as academic success for *all* students. Success in mathematics has far reaching consequences. Mathematics as a formal discipline is the basis for many careers in science, technology and business. Extending beyond its direct relevance for careers that rely on numerical calculations, mathematics also acts as a gatekeeper for access to higher education in general (Gutstein, 2003; Kitchen, 2007; Secada, 1995). In addition, mathematics is a valuable tool for understanding the complex world in which we live and for making sense of global economic and social policies (Gutstein; 2003). For linguistic and cultural minority students such as the Latina/o students in this study, mathematics knowledge is power (Secada & De La Cruz, 1996). A solid mathematics education will give them the power to seize the opportunity for higher education and the power to challenge inequitable

social policies that privilege the white middle class (Kozol, 2005) over their own immigrant communities.

One of the key ideas in equitable mathematics education is to give students access to high quality instruction in the classroom, including problems situated in real contexts (Secada, 1989a, 1989b; Secada & De La Cruz, 1996). Contextualized problem solving can begin as soon as students enter the primary grades and can help students build a solid conceptual foundation in mathematics (Carpenter et al., 1993; NCTM, 2000). CGI problem solving in the classroom targets students' emerging sense of number and helps them develop conceptual understanding of numeric operations (Carpenter et al., 1999). Contextualized word problems based in familiar situations give young students the opportunity to use what they know about the world to make sense of the formalized mathematical concepts embedded in these problems (Secada & De La Cruz, 1996). The research in this study is based in contextualized CGI word problems and the way young students in kindergarten, 1st and 2nd grade develop in their abilities to make sense of these problems and explain their thinking.

This study builds on two important theories in cognitive development that have not been previously unified, and explores the longitudinal development of young students' mathematical thinking and communication during problem solving. The foundational perspective comes from sociocultural theory. This perspective describes cognitive development as a refinement and expansion of the internal systems of meaning individuals construct in their constant interactions with the external systems of meaning they encounter in their social and cultural environments (Vygotsky, 1987). At the heart of sociocultural theory's notion of concept formation is the idea that language, including

gesture is the first mediator of meaning in a child's life and that early spontaneous concepts are structured and represented internally by the meaning attached to words. Since making meaning is also a thought process, meaning results from the unification of thinking and speaking. Language is a central symbol system and sign operation for conscious thought, and therefore mediates the interplay between the internal and external systems of meaning (Mahn, 2009; Vygotsky, 1987). The meanings symbolized by words are generalizations and each of these generalizations is a concept, so words and meanings are the conceptual medium of thinking and speaking. In this way words and their associated meanings are the building blocks of conscious thought that lead to greater degrees of conceptual understanding. How children explain their thinking during CGI problem solving in this study gives me a window into how they are making meaning of the mathematics in the problems.

The second framework used in this study is Cognitively Guided Instruction (CGI), a framework for understanding the mathematical thinking of young children during problem solving around number stories (Carpenter et al., 1999). CGI theory explains that children think about problems in specific ways, depending on the actions and relationships contained in the stories. A key idea of CGI is that children do not need to be shown specific strategies for solving these problems, but rather come to school with an intuitive ability to model the actions and relationships they encounter in a word problem when they can comprehend the situation of the story (Carpenter et al., 1996). Children do not need a formal understanding of number operations before they can begin solving word problems. With only a beginning understanding of counting and one-to-one correspondence, children can be successful problem solvers using what they already

know about the world to make sense of the numeric actions and relationships in the problems.

Early CGI research observed generalizations in the way children as young as five years old approach different kinds of problems, the strategies they use, and the way these strategies change and develop over time as the children incorporate a more formal understanding of numbers and operations (Carpenter et al., 1993). The CGI framework contains a clear classification of problem types and associated strategies. This framework acts as a guide for teachers to understand a young student's mathematical thinking based on the strategies the child uses to solve a problem. CGI also provides a structure to bring children forward in their mathematical understanding through increasingly complex problem types. Increasing the size of the numbers in CGI problems and varying the problem types help students develop more flexible thinking, deepen their sense of number, and connect problem solving with the mathematical concepts they are learning in the classroom. In short, CGI problem solving helps students learn mathematics with understanding (Hiebert & Carpenter, 1992).

The Cognitively Guided Instruction framework can be understood through a sociocultural lens. In particular, the informal knowledge children bring to the classroom can be conceptualized as an important aspect of the internal system of meaning that helps them tackle mathematical word problems. The idea of internal systems of meaning is fundamental to this study (Mahn, 2009; Vygotsky, 1987), and the nature and development of children's systems of meaning will be discussed in detail in a later section of this chapter. Because many of the children studied in early CGI research approached specific types of problems in similar ways and developed advanced strategies

following much the same continuum (Carpenter et al., 1993, 1994, 1996), the CGI framework allows for generalizations to be made about children's systems of meaning as they enter formal schooling. Continued CGI problem solving can be used to explore how children's interactions with formal mathematics influence their concept development. This process of concept development is precisely the interplay noted above when an individual's internal system of meaning incorporates new meaning and structure from the external environment. For this reason, CGI theory as seen through a sociocultural lens, helps teachers and researchers understand how children are building conceptual knowledge by targeting specific mathematical concepts.

In the sections below, I begin with a general discussion of sociocultural theory and then focus on verbal thinking (Vygotsky, 1987), a particular aspect of sociocultural theory that is not discussed widely in mathematics education research. I will show how the powerful notion of verbal thinking that explores the unification between thought and language to understand concept formation adds to the theoretical foundation for this study (Mahn, 2009; Vygotsky, 1987). The research participants received CGI problems through verbal input, their thinking to solve the problems was mediated by their internal sign systems including language, and they talked about their thinking using verbal output. While student communication in the form of verbal explanations is critical, I expand communication to include the gestures students make as they are explaining their thinking (Domínguez, 2005) and the tools and materials they choose to help them find problem solutions (Gee & Green, 1998; Moschkovich, 2002), arguing that these gestures, tools and symbols also carry meaning to mediate student thinking.

Following a discussion of sociocultural theory, I present three competing theories from mathematics education research that potentially have much to offer as ways to understand children's thinking, but fall short due to their incomplete understanding of how conceptual thinking and cognitive development result from the interplay between internal and external systems of meaning. Two theories from the point of view of individual cognitive development in mathematics education, constructivism and social constructivism are discussed. In addition, a socioculturally-based theory that focuses on the direct connection between language production and mathematics learning called thinking-as-communication (Sfard, 2001) is discussed. Although these three theories have great potential for exploring the mathematical thinking of the young students, I explain my reasons for accepting certain aspects and rejecting others.

Following the discussion of competing theories, I provide a justification for solving word problems to promote mathematics learning, particularly with young primary-grade students. Finally, I look closely at Cognitively Guided Instruction and show how CGI theory in practice provides the structure to explore students' informal systems of meaning as they begin to incorporate formal mathematical concepts.

Sociocultural Theory

Young children's social, cultural and linguistic contexts, and their informal experiences with numbers before they enter kindergarten and outside the mathematics classroom are important to this study. The sociocultural perspective on learning recognizes the strong relationship between children's internal cognitive development and the continual interactions they have with their social environment, mediated by language, as they attempt to make sense of their world (Vygotsky, 1987). In this relationship,

language not only carries meaning, but gives a symbolic structure through a unification with thinking to the dynamically developing internal system of meaning children construct about their world (Mahn, 2009; Vygotsky, 1987). Sociocultural theorists in mathematics education research believe language and social experience are major mediating factors in children's developing understanding about mathematical concepts and processes (Sfard, 2001; Van Oers, 2001; Zack & Graves, 2001). When children are actively engaged in learning mathematics with understanding (Hiebert & Carpenter, 1992), they are doing so by connecting, transforming, and expanding what they already know from their social, cultural and classroom experiences as they incorporate new knowledge. Hiebert and Carpenter (1992) describe children's growing mathematical understanding as a dynamic mental network where conceptual nodes are connected in a variety of strong and flexible ways, similar to a system of meaning (Mahn, 2009; Vygotsky, 1987). Learning mathematics with understanding is an important goal for all children (NCTM, 2000) and especially for Latina/o students (Secada & De La Cruz, 1996) to turn around negative statistical trends (NAEP, 2005; Otiz-Franco, 1999).

The Role of Language in Sociocultural Theory

This study explores students' mathematical thinking as they solve word problems, a conscious mental activity. Vygotsky (1987), the Russian psychologist on whose work sociocultural theory is based, attempted to understand conscious thinking by means of studying the mental processes involved in concept formation. He felt that the most accessible process to study consciousness was verbal thinking, a process that unifies thinking and language processes. Language and concepts are first encountered on the external, social plane, and it is through social interaction that these concepts are

internalized and transformed. The language associated with concepts is internalized through unification with thinking so that a Vygotskian approach to the relationship between thinking and speaking views child development as fundamentally social (John-Steiner & Mahn, 1996).

From the time of birth, children's understanding of the world is mediated through their social interactions. Caregivers help children associate specific objects and gestures with sounds. Children's babbling noises and actions are reinforced by reactions from their caregivers, and gradually the sounds in a baby's world take on meaning. Through trial and error with their caregivers, children begin to build a vocabulary of sounds/words about their world. As children grow, the meanings associated with words evolve from specific associations into generalizations. In this process of generalization, the word becomes a concept, taking on a meaning apart from any specific object as it moves inward to become a mental representation for that meaning and concept. In this way words become a medium for conceptual thinking (Vygotsky, 1987).

Social dialogue and communication are critical in the development of consciousness. Language mediates children's understanding of the world and conversation helps children make connections between words, meanings, and ideas about the world in which they live (Vygotsky, 1987). As consciousness develops, dialogues and meanings are internalized, transformed, and integrated to build children's personal understanding of the external world. Meaning lies at the interface between pure thought and language and in this way the meanings associated with words also structure thinking because of the unification of thought and language. Vygotsky called the unit of meaning

contained in both the thought and the word *znachenie slova*, the “vital and irreducible part of the whole” (1987, p. 46).

This unification of the separate processes of thinking and speaking results in the structure and hierarchy of generalizations that create an internal, dynamically growing sense of the world. Drawing on Vygotsky, Mahn (2009) calls this dynamic structure a *system* of meaning. The continual interplay between children’s growing internal systems of meaning and the external systems children encounter in their day-to-day experiences results in a modification and refinement of the internal structures and leads children toward a conceptual understanding of their world. The dynamic construction of increasingly more complex concepts and refined generalizations, symbolized by meaning, continues to be facilitated by communication about the world as children grow. Of the dynamics of concept formation Vygotsky (1987) said, “The relationship of thought to word is not a thing but a process, a movement back and forth from thought to word and from word to thought” (p. 250).

Systems of Meaning

The idea of systems of meaning (Mahn, 2009; Vygotsky, 1987) is central to this study. Children’s mathematical thinking will be explored from the perspective of the expanding internal system of meaning they are constructing about numbers and about the patterns and relationships they are discovering in our number system. Specifically, this research focuses on how children attempt to solve and explain word problems over a period of three years, from kindergarten through 2nd grade, during which time the external formal mathematics system the children encounter in school reorganizes,

transforms and expands their internal, informal and experiential system of meaning about mathematics.

Systems of meaning are a way to understand thinking processes. We move, interact, solve problems and live in this world based on the understanding we have of how the world works and our place in it. The sense we make of our world is constructed through the meaning we attach to objects, ideas and activities. These meanings are not randomly collected in our minds, but are structured, ordered, and connected to each other so that our overall interpretation of reality is a related complex of meanings (Mahn, 2008; Vygotsky, 1987). Because we are continually adapting to our environment and the changing conditions of our lives, including incorporating new knowledge, we are continually modifying our understanding of the world. How we understand the world is rooted in more than just structure, which implies a fixed anatomical configuration. For this reason, Mahn (2009) has chosen to refer to the evolving meanings we attached to our world as a *system* to reflect the dynamic nature of our continual interaction with our social and physical environments.

In his conception of a system of meaning, Mahn (2009) describes the continual interplay between the meaning encountered socially and externally and the meaning we construct internally in response to our interactions with our environment. Everyone lives surrounded by social, cultural and historical bodies of knowledge that influence us and help us develop our understanding of the world. We encounter these in school, in our cultural traditions, and in the knowledge we co-construct as society unfolds and our place in it evolves and changes. It is this combined knowledge, both fixed and changing, that grounds us culturally. In addition, there is the affective and interpreted knowledge that

comes from each individual's interactions and experiences in specific cultural and social situations. Vygotsky (1987) called this personal experience of the social environment and its consequences on our perspective *perezhivanie*, defined by Mahn (2009) to be "the way children perceive, emotionally experience, appropriate, internalize, and understand interactions in their social situations of development" (p. 18). Because of the dynamic relationship between the individual and the environment, the environment is in turn influenced by the experiences of the individual.

An example of a system of meaning is the sense I make of how to manage my money so that I meet my needs and also have something left over for pleasure. The foundations are in my early interactions with money and the examples set by my family. My cultural, social and historical environments define money for me, set its value, establish a system for exchanging money for goods and services, create living options, and determine the requirements for maintaining those living options such as paying rent and electrical bills. Activities where I spend money for pleasure come from a set of options common to my culture and social class. I can go to movies, meet friends at a restaurant, or join a social club; however, the choices I eventually make are influenced by my *perezhivanie*. What I choose is based on my personality and my experiences within my social environment. My system of meaning about money is connected in some way to every other aspect of my life and is continually being influenced by my surrounding environment and my own evolving sense of how I want to live my life. My financial sense of meaning is a dynamic and evolving construction of my consciousness and the driving force behind my thinking and decision-making about money.

The developing systems of meaning that guide my conscious thought are influenced by dialogues inside my head. As I think and plan about my life, I use these internal dialogues to direct my thinking, where words and their meanings are the substance of my thoughts. Meaning, or *znachenie slova*, as noted earlier is “the internal structure of the sign operation” (Vygotsky, 1997a, p.133), and it is the fundamental unit of both thought and word so that thinking and speaking are unified in a process called *verbal thinking* (Vygotsky, 1987).

Verbal thinking in the construction of systems of meaning.

Verbal thinking is a major process of our conscious mind and leads to conceptual development. Mahn (2009) has developed a schematic for verbal thinking that shows Vygotsky’s notion of the interaction between the external and internal planes of thought (see Figure 1 and also Appendix A).

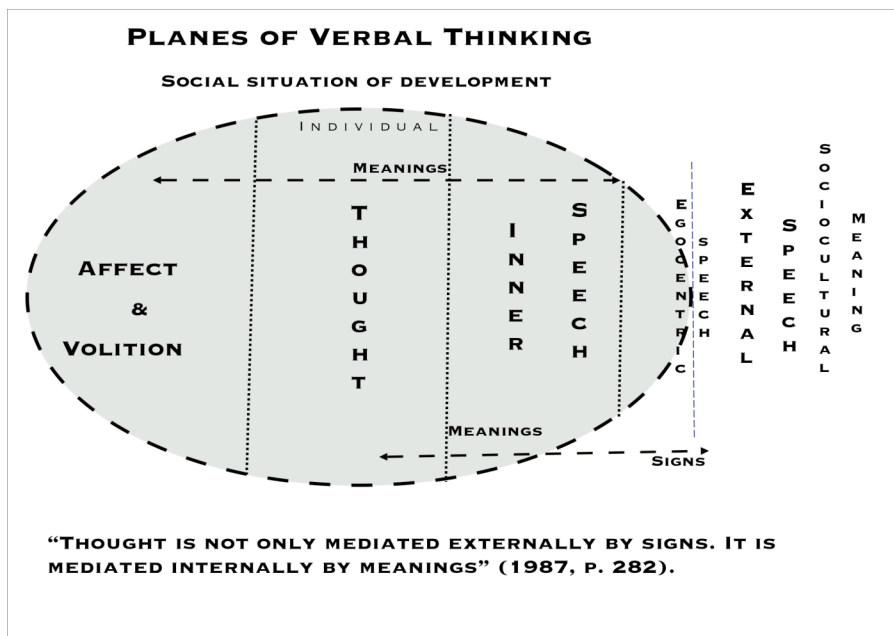


Figure 1. Planes of Verbal Thinking (Mahn, 2009).

Here external speech and sociocultural meanings move inward and are transformed to inner speech and inner meanings. Meanings, in this schematic, continue to move further inward toward pure thought and finally reach a plane Mahn calls affect and volition drawing on Vygotsky (1987) who noted that this final plane “includes our inclinations and needs, our interests and impulses, and our affect and emotion. The affective and volitional tendency stands behind thought” (p. 282). Meaning moves among the planes, belongs to all planes, but does not conflate with any. The idea that meaning is not one and the same as the word is critical. Meanings mediate thought, but they are not the thought. The importance of this idea has significance for bilingual education theories such as common underlying proficiency (Baker, 2006; Cummins, 1981), where conceptual meanings learned in one language transfer straight across to developing languages and do not need to be relearned for each language.

To understand the origins of verbal thinking, we explore children’s earliest interactions with their caregivers in their facial expressions, babbling, crying, and gestures (Vygotsky, 1987). The actions of very young children through pointing and other types of indication are the beginnings of representation and generalization through symbols. In other words, the action stands for something. The intention caregivers attach to children’s gestures lays the foundation for symbolic representation that is later expressed in words. To illustrate, at a very young age my son would point to the direction he wanted me to carry him. Before he could speak and while still a baby in my arms, he could communicate, “I want to go over there,” by simply pointing vigorously in the direction he wanted to go. Like all caregivers, I gave meaningful feedback to his

gestures through my words and my interpretation of the intention behind his action. I carried him “over there.”

Children begin the process of social communication through a sign system (gestures), and increasingly that sign system develops into language through continual interaction and interpretive feedback provided by caregivers (Vygotsky, 1987). As language and communication develop in the child, needs are expressed in words and the world starts to be organized around the generalizations contained in word meanings. The child can ask questions and receive answers through verbal communication. Language acquisition gives the child the ability to make sense of the world through verbal interactions.

Qualitative Transformation – Language Acquisition

Children begin to develop a conceptual understanding of the world through continual interaction with others in their social world. A qualitative transformation in conceptual development occurs when a child begins to acquire language. According to Vygotsky (1987), thought and speech as mental processes have distinct origins and distinct paths of development until about the age of two. Prior to this, thought is connected to discovery and exploration of the world. It is a process related to purpose and action (Vygotsky, 1987). On the other hand, early speech is a process of social connection that “has nothing to do with the development of thinking” (Vygotsky, 1986, p. 81). However, at about the age of two a revolution in thought occurs as language and thinking meet and unify around the meanings children are internalizing, as noted earlier.

The process of thought is qualitatively changed when thinking also becomes a verbal activity. Collections of meanings start to form systems that connect with each

other and with events. Each meaning that the child internalizes through language has aspects that are socially and historically constructed and connected to other meanings. In this way, a child creates a system of meaning based in language with roots in her social environment and connections to the historical development of her language community. This unification of the distinct processes of thinking and speaking form a new way of thinking that is qualitatively different from prelinguistic thought (Mahn, 2008; Vygotsky, 1987). Vygotsky felt verbal thinking was central to the development of consciousness. It is not the only form of thinking as Vygotsky noted, but it is the one judged by Vygotsky as the most accessible to study conceptual development because verbal thinking functions through the medium of language and can be explored through social communication. I used this theoretical perspective when I asked my research participants to tell me about their thinking in an attempt to explore their systems of meaning about mathematics.

As very young children acquire more and more language, verbal thinking helps them in problem solving; for example, how to reach a door handle that is too high, how to put on their clothes, or how to get the cookies they want when they can't reach them (John-Steiner & Mahn, 1996). Children talk themselves through these problems and their thinking processes out loud. Anyone who has lived with young children has observed this self-directed communication, what Piaget called egocentric speech (as cited in Vygotsky, 1987). Vygotsky disagreed that this egocentric speech later disappeared, as Piaget claimed, but instead moved inward to become inner speech and structure the thinking process. Mahn explains that, "it becomes internalized in the form of inner speech as part of the process of intermental/external functioning becoming intramental/internal functioning. He [Vygotsky] points out that in this internalization

process the function and structure of speech change bringing about a qualitative change in the system of meaning” (Mahn, 2009, p. 28). Children have experienced this qualitative change by the time they enter school, which marks the next critical period in conceptual development.

Qualitative Transformation – Introduction of Scientific Concepts

The second transformation in conceptual development happens when children enter school and are exposed to more formal and systematic ways of generalizing information (Mahn, 2009). Vygotsky (1987) called these formalized concepts located in school contexts *scientific concepts*. According to Vygotsky, there are two general types of concepts, spontaneous and scientific. Conceptual development for children begins with spontaneous concepts based in informal experiences. These spontaneous concepts form the foundational structure of children’s systems of meaning. Children’s conceptual development then organizes around the scientific concepts they encounter in school. Vygotsky visualized the spontaneous concepts developing upward toward the generalizations contained in scientific concepts (Mahn, 2009). Schooling introduces the formal scientific concepts that reach toward children’s foundational understanding. These scientific concepts begin with broader generalizations about the world then grow downward (in Vygotsky’s analogy) to meet, transform and incorporate the specific cases found in the spontaneous concepts reaching upward.

Ideally, classroom activities tap into children’s systems of spontaneous concepts and help children make connections between what they already know and the new ideas generalized in school’s scientific concepts (Van Oers, 2001). In this way, the scientific concepts expand children’s systems of meaning and help children to organize,

consolidate, refine and/or transform what they know and are learning about the world. Children's systems of meaning that link their informal and formal knowledge about numbers and help students make sense of mathematics in the classroom has been similarly described in mathematics education literature as the network of understanding (see above and in Hiebert & Carpenter, 1992). Whether children's mathematical conceptual knowledge and numeric processes are described as a system of meaning or a network of understanding, in both these descriptions concepts must be linked by multiple, strong and flexible connections if children are to successfully apply what they know about mathematics to solve new and challenging problems.

An example of a classroom activity linking spontaneous and scientific concepts is the contextualized problem solving in CGI (Carpenter et al., 1999). A series of multiplication type problems could ask kindergarteners the following. If there are three apple trees and each tree has four apples, how many apples are there in all? If there are six chickens and each chicken has two legs, how many legs are there in all? If there are two children and each has 10 fingers, how many fingers are there in all? These problems link the generalized concept of multiplication with specific problems based in children's own experiences that they can directly model through drawing pictures or by using counters to find the answer. The general concept of combining equally sized groups of objects to find how many in all (multiplication) will reach down to meet children's knowledge of how to find the answer for each specific case. The concept of multiplication gathers all similar problems under one category, and because the children understand how to solve each problem, the problems themselves stretch upward to this category and help children make sense of the concept of multiplication. Children's

thinking about the object world is refined and their existing knowledge is reconceptualized and generalized in this process (Van der Veer & Valsiner, 1991).

Mathematical Thinking

As a psychologist, Vygotsky believed that consciousness is composed of a broad range of thinking processes and that verbal thinking is only one of these. In his famous analogy he noted, “Consciousness is reflected in the word like the sun is reflected in a drop of water” (1987, p. 285). As examples of thinking, we can engage in verbal thinking, mathematical thinking, musical thinking, kinesthetic thinking, and visual-spatial thinking (Gardner, 1993; John-Steiner & Mahn, 1996), using other symbols and signs for mediation. Since the research presented in this dissertation is built upon CGI mathematics word problems that were presented to children orally, the children had to think both verbally and mathematically to find solutions to the problems. Their mathematical thinking was mediated by a sign system related to numeric representations such as numerals and number lines. Examples of these representations are found in my analysis.

Once again, meaning is critical to mathematical thinking. Just as words have been internalized as meanings in connection with thinking, so too are other representations children work with in the mathematics classroom such as the numerals “1”, “4”, or “10”, the number line that progresses around the walls of the classroom, and the chart of numbers from 1 to 100, arranged in rows of 10, that hangs on the easel. Meanings associated with these symbols develop in social interaction, similar to meanings associated with words, and can be complex and must be sorted out by children (Fuson, 1988). For example, the simple number 5 can represent either a cardinal

quantity, an ordinal position, or a measurement amount. Five can be the number of discrete objects in a group, it can be the fifth item in a sequence and represent a relative position, or it can be a measurement and represent a continuous quantity.

To add to the complexity, the meanings related to the symbols can shift during an activity, as when children are learning to count. They need to understand that the last word they use in the sequence of numbers assigned to objects represents the whole set of objects not just the last item, even though in the sense of one-to-one correspondence between object and number, it is the discrete 5th object that is associated with the word “five.” In her work with young children, Fuson (1988) notes that, “An important development throughout the age range...(age 2 to 8) is the increasing ability to shift among meanings and, finally, to integrate several of these meanings. Adults shift so easily and have such integrated meanings that it is difficult for adults even to comprehend how separate these meanings are for young children” (p. 5). This shifting from one meaning to another is facilitated in children through the multiple links they are constructing among concepts in their system of meaning about numbers and operations (Hiebert & Carpenter, 1992). The idea of multiple meanings attached to numbers becomes critical in the analysis of children’s thinking in this study.

A wide range of formal numerical and mathematical representations are part of the external system of meaning students encounter in school. Children are encouraged to appropriate these representations and create their own (NCTM, 2000). I have observed in my work with the participants in this study that in the constant interplay between the internal and external systems of meaning, students are appropriating new ways of thinking about numbers and their relationships. As children have more experience with

representations, the symbols and associated meanings move inward and become tools for their mathematical thinking. I propose that the numerals, the number line, the 100s chart and even algorithms such as carrying and borrowing are internalized and transformed to mediate thought.

This study explores students' mathematical thinking as they solve CGI word problems over the course of their first three years of formal schooling. It is also an exploration of their verbal thinking. The CGI problem solving interviews used to collect data reflected the interplay between external and internal systems of meaning where language mediated both the ways children made sense of the problems and children's own explanations of their thinking processes. Children used both verbal and mathematical thinking to solve the problems, revealing the complexity of mathematical sense making within a system of systems. Children's descriptions about how they were thinking should directly reflect the meaning they were attaching to the numbers in the problems and the verbal thinking processes they used to make sense of the problem situations (Mahn, 2009).

In the next section, I briefly discuss other theories from mathematics education research that focus on children's thinking. From here, I summarize the importance of problem solving in mathematics education and return to a more detailed discussion of Cognitively Guided Instruction (Carpenter et al., 1999), the theory that is used in conjunction with sociocultural theory to build the theoretical framework for this study. I close this chapter with a discussion of the connection between learning mathematics and learning in the native language and describe gaps in the literature on students' mathematical thinking.

Competing Theories for Exploring Conceptual Mathematical Development

Constructivism

Constructivism rose in prominence in the 1960s and 1970s and was influenced by Piaget's cognitive development psychology (Steffe & Kieren, 1994). Steffe and Kieren explained that constructivist thought challenged behaviorism, the idea that students simply learn behavioral objectives in mathematics education. From the constructivist perspective, learning is not a passive incorporation of skills and information, but an active construction of knowledge where "individuals construct their own reality through actions and reflections on actions" (p. 74), and where, according to Piaget, this knowledge is constructed through psychological interactions with the environment.

From a constructivist perspective, mathematics is learned as individuals participate in cognitively driven activities to make sense of concepts and ideas using their own psychological tools. In other words, knowledge is individually constructed as learners attempt to discover what works and what does not (Kamii, Rummelsburg & Kari, 2005). Radical constructivism extended this idea to focus solely on individual actions and reflections on those actions without consideration to the influences of the social environment. Researchers such as von Glasersfeld (as cited in Steff and Kieran, 1994) claimed that individuals do not strive to construct an external reality, but construct their own personal reality in learning. In fact, external reality is not a consideration. It is the individual's own mathematics knowledge he or she is building, not appropriating a system of mathematics already established.

Research conducted by Kamii et al. (2005), based on the constructivist ideas promoted by Piaget, seeks to show that children begin with physical knowledge about the

world and from this construct their logico-mathematical knowledge. Out of this knowledge comes children's sense and system of number. Starting with the assumption that children from low socioeconomic backgrounds come to school with reduced logico-mathematical knowledge, these researchers hypothesized that by providing logic games such as pick-up-sticks to first grade students, the children would build mental relationships as they developed physical strategies and thereby develop a greater sense of logic. With this newly developed sense of logic, children would gain the basis they needed for the logico-mathematical thinking critical for mathematics learning and the development of a sense of number. The researchers and teachers in this study actively engaged students in games that were fun for the children, but made them work hard mentally toward desired outcomes.

As the children played and developed their physical strategies, they were constructing mental relationships that led to logical and connected thinking (Kamii et al., 2005). The games that were chosen gave students immediate feedback on their choices, thus employing the two-part thrust of this constructivist approach, the action and the reflection on that action as described above. Post-tests among students with low physical knowledge and mathematic abilities at the beginning of the school year showed a great deal of success in students' abilities to apply logical thinking to find problem solutions.

It is interesting to note in the above study that the post-test problems used for evaluation were word problems, although there is no mention of the experiential knowledge children needed to understand the context of these problems. Four problems were given, which involved standing in a lunch line, serving soup and crackers in bowls, and sharing cookies and candies. Children in constructivist classrooms performed much

better in the sharing problems and slightly better in the other two contexts. Researchers concluded that these problems were a “test of the children’s logic” (Kamii et al., 2005, p. 45) without questioning why poor children could solve sharing problems better than a problem about serving soup and crackers in a bowl. They did not ask what other types of knowledge children need, beyond logical thinking, to comprehend and solve word problems. Additionally, researchers put students into groups to discuss problems, saying, “logico-mathematical knowledge is constructed by each child’s thinking, and this thinking is stimulated by the exchange of viewpoints among children” (Kamii et al., 2005, p. 49). They did not acknowledge the direct role of language and social communication in the development of this thinking.

Although constructivism acknowledges the mental activity of the learner and the importance of meaning making, it only focuses on an individual’s internal system of meaning and how actions and reflections by the individual develop that system of meaning. The above research by Kamii et al. (2005), while contributing valuable information to the literature on the importance of active student engagement in problem solving and meaning making, does not acknowledge that the social environment and language played a direct role in students’ knowledge systems nor the way that the social environment of the mathematics classroom and its formal system of meaning was in constant interplay with students’ own processes of meaning construction.

Because constructivism does not deal directly with the sign system that connects socially constructed meaning to internal meaning, it does not acknowledge the role that language plays in carrying meaning. Nor does constructivism account for the earliest experiences of young children learning about their environment from their caregivers,

young children's questions about their world, or how language carries meaning and forms the basis for children's developing generalizations about objects and processes.

Constructivism is concerned with only the development of an individual's internal system of meaning. It is not a sufficient theory to form a compatible relationship with sociocultural theory and verbal thinking, and is therefore rejected for this study.

Social Constructivism/Emergent Theory

Social constructivism arose as a competing theory to constructivism in that it acknowledges the importance of social interaction in mathematics learning (Cobb & Yackel, 1996; Steff & Kieran, 1994). Although there is an increased attention to socially constructed knowledge and how the individual learns in relation to the learning of the group, the link between the two systems of meaning, internal and external, is viewed as indirect. In the social constructivist view, the social learning environment sets the stage and provides the opportunity for learning, but it is still up to the individual to make sense of the mathematics and construct his or her own internal system of meaning (Cobb & Yackel, 1996).

Addressing a shortcoming they interpreted in sociocultural theory, Cobb and Yackel (1996) developed a nuanced social constructivist perspective they call emergent theory where they account for the differences in individual learning within social learning environments. They felt that sociocultural theory did not adequately explain why different children learning in the same classroom environment had such varying degrees of success. In their exploration of the development of sociomathematical norms, they examined what they call the indirect connection between social knowledge constructed through in-class discussions and individually student-constructed knowledge.

Cobb and Yackel's analytical framework follows the interactions between the social and individual planes during in-class mathematical problem solving and the development of norms for communication in this environment. They focus on the individual's reflection between what is happening on the social plane in the classroom and the personal, psychological plane. As classroom social norms are developed, individuals restructure their beliefs about themselves and their own efficacy in contributing to knowledge construction. As sociomathematical norms are developed on the collective plane, individuals restructure their beliefs and values about mathematics and what it means to contribute to mathematical practices. As classroom mathematical practice norms are developed, individuals engage in constructing meaning and incorporate and transform mathematical concepts in ways that make sense to them.

From the emergent perspective, the development of sociomathematical norms plays a key role in individual mathematics learning (Cobb & Yackel, 1996). These norms include what counts as a valid explanation or strategy, what is a more efficient strategy, what is faster, what is more sophisticated, and what connections can be made with other concepts to improve problem solving. In short, these norms establish better ways of solving problems. Engaging in discussions around these norms moves students forward in their conceptual understanding and problem solving abilities. As these norms are developed collectively in the mathematics classroom community, individuals restructure their thinking and strategic approaches to problem solving.

An example of analyzing the development of sociomathematical norms comes from a first grade classroom (McClain & Cobb, 2001). The McClain and Cobb research team had the dual goals of developing specific mathematical thinking and discourse along

with developing students' sense of agency and beliefs about themselves as mathematically competent. Social norms were already in place in this first grade classroom and laid the groundwork for the development of the more specific forms of mathematical communication, the sociomathematical norms. Students already knew that they had to communicate their thinking and their reasoning and listen respectfully to others' ideas. They were expected to indicate non-understanding and if they did not agree with another student's explanation, they had to explain why they did not find it acceptable.

With the social norms in place, the research team began with the development of what counts as a different solution and built on this idea to develop the rest of the sociomathematical norms. Researchers found that students could not distinguish between alternative solutions until the teacher made the differences explicit. For example, she said, "I don't mean just another way to count, but if you grouped them [the objects under investigation] in a different way, or you saw them in a different way, that's what will help us" (McClain & Cobb, 2001, p. 250). Once students had a concrete idea of what the teacher was expecting as a different solution, they were able to build on this idea and a shift occurred in students' thinking that allowed students to judge for themselves what was different. Once the idea of difference was established and "taken-as-shared" (p. 257), distinctions could be made among the solutions and students were ready to understand why some solutions were more sophisticated, more efficient, and easier than others. In each case, it was the teacher who first spoke in these terms and used them to guide her questioning, then students quickly developed the ability to use the terms on their own.

The social constructivist perspective makes important contributions to the literature on students learning mathematics in classroom communities and teaching formats that facilitate student learning through communication. Cobb and his co-researchers (Cobb & Yackel, 1996; McClain & Cobb, 2001) have worked extensively to improve student learning through a situated approach and have acknowledged the importance of recognizing and honoring different ways of knowing and different cultural contexts. However, the emergent view of an indirect connection between the learning of the group and the learning of the individual sets up an invalid dichotomy between the developing internal system of meaning of the individual and the external system of meaning encountered in the environment (Mahn, personal communication, 2008). It does not account for the dynamic and direct interplay between external and internal meaning and how the external is moving inward as both a sign system and structure. On its way in, the external system of meaning is transformed and transforms what is already present. This transformation is highly influenced by each students' *perezhivanie*, as noted earlier, i.e. their personal experience of the environment (Mahn, 2009; Vygotsky, 1987). Even though each student has a different personal experience of classroom learning resulting in differences in performance, this does not mean that the connection between the external social learning environment and the internal developing system of meaning is indirect for students.

Because of this dichotomy between social and individual systems of meaning, social constructivism does not provide the necessary theory compatible with sociocultural theory to understand the interplay between systems of meanings as students develop mathematical understanding over time. In the final competing theory described below, I

turn to a perspective from mathematics education that *does* make a direct connection between external and internal systems of meaning to connect thinking with verbal communication. It comes from work in sociocultural theory that focuses on communication in general and language specifically as keys to mathematical development.

Thinking as Communication: Interactivity Analysis

Sociocultural theory makes a direct connection between thought and language and acknowledges that concept development is mediated through social communication. Unlike the indirect connection made by social constructivists, the Bakhtinian notion of social communication associated with sociocultural theory, credited to Bakhtin who was a contemporary but not an associate of Vygotsky, asserts that through the utterances contained in dialogue around a new idea, concept, or problem, new knowledge is created for both interlocutors (Sfard, 2001). From this perspective, it is the act of communication and the negotiation of meaning on the social plane that creates the possibility for new insight to be gained.

To explore the development of knowledge through communication in mathematics learning, some researchers examine student dialogue during cooperative problem solving, considering this communication at the margin between social discourse and individual thinking (Kieran, 2001; Sfard, 2001). From this point of view, thinking is contained within the communication. This is an extremely rich area for analysis of the connection between thought and language, especially as new mathematical concepts are being explored and students are trying to make sense of mathematical ideas as they work with their peers. Justifying communication as the focus for analysis, Sfard says, “Once

thinking has been conceptualized as communicating, the dynamic, ever changing and extremely context-sensitive dimension of thinking comes to the fore” (p. 43).

Sfard (2001) has analyzed student dialogue as students struggle to make sense of new ideas and co-construct problem solutions. She has asked why joint construction of knowledge is so difficult and why many times students simply talk past each other. In her analysis, she argues against the differentiation of collective and individual knowledge construction and promotes the idea that thinking *is* communicating and can be analyzed as such through student interactions. Sfard uses an interactivity analysis to demonstrate graphically if and when learning is taking place on the individual plane and how this is influenced by the on-going social interaction and learning on the social plane. Her analysis shows that many times students can talk past each other, resulting in no learning for one or both of the interlocutors.

An example of Sfard’s graphic interactivity analysis of the discussion between two twelve-year-old boys (2001) Gur and Ari follows a personal channel for each boy and a social channel between the two. These channels are demonstrated in Figure 2.

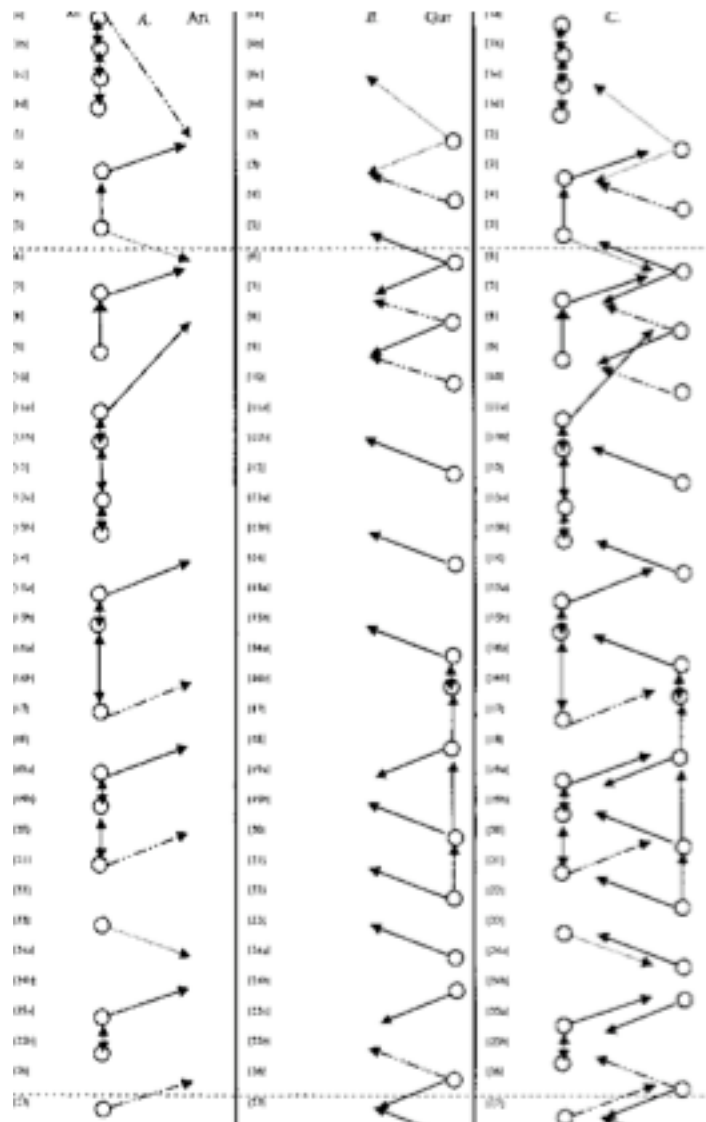


Figure 2. Channels of interaction during pairs problem solving (Sfard, 2001).

Within these three channels, there are both reactive and proactive arrows that show “whether interlocutors are really addressing and interpreting their partner’s questions and comments or, in fact, are concentrating on a ‘conversation with themselves’” (p. 41).

Figure 2 above, shows that in Ari’s channel on the left he is proactively working the problem, not in cooperation with Gur, but more to himself as he progresses through the reasoning. This is represented but the vertical arrows downward. He only reacts to Gur’s questions as necessary, represented by the arrows moving back and upward to the right.

Gur, in the middle channel continually asks Ari what he is doing, trying to follow Ari's reasoning, but not engaging in the reasoning himself. This is represented by the arrows moving upward and to the left. In the channel on the right, we see the combination of the two personal channels and how Ari's proactive reasoning dominates and drives the interaction.

An example of the power of this analysis scheme happens when Ari begins constructing knowledge on his individual plane, without attempting to make his thinking clearer to his partner Gur. Ari's personal channel shows the majority of the activity, while Gur's personal channel has very little activity. Gur tries to gain some understanding of Ari's train of thought through questioning, but gets only reactive responses. Ari's personal channel shows he is organizing and consolidating his thinking in his utterances to himself, but only gives lip service to Gur, "keen to protect his private channel from distractions" (p. 42).

Using a similar methodology to examine object-level, problem-based, and meta-level, relationship-based, discourses, Kieran's (2001) analysis of thirteen-year-old pairs during problem solving highlights the complexity of making mathematical discussions productive for both partners. She says, "bridging the individual and the social in mathematical problem solving can be extremely difficult to put into practice...making one's emergent thinking available to one's partner in such a way that the interaction be highly mathematically productive for both may be more of a challenge to learners than is suggested by the current mathematics education research literature" (p. 220).

As in Sfard's example above with Ari and Gur, the pair's discourse analyzed by Kieran (2001) shows that it is in the unfolding dynamics of the meta level utterances,

those comments that nurture the relationship between the two, that co-construction of knowledge occurs. In other words, it is in the proactive attempts of one partner to try to make his or her thinking transparent to the other, and the repeated questioning of the other partner in an effort to understand, that allows for learning for both to occur. Kieran demonstrates how the utterances that take place on the interactive plane are governed by individual needs confounded with the social norms of communication and are highly dependent on the norms established in the classroom, including the power relationship between the two partners. As an example, Keiran notes that when one student from a different pair, Sho, had insight into a problem solution “he spoke quietly, as if he did not want to lose his train of thought...which suggests he may have been distracted by the question, at the very moment he was trying to grab hold of a newly emerging idea” (p. 217).

Kieran’s method contains a second analytic component beyond interactivity analysis that explores how the partners were able to perform on similar tasks individually in a follow-up lesson. She found that individuals who could successfully work alone at a later time were those who had maintained a strong frequency of object level (problem focused) utterances during pairs work, either by directly talking to their partner or simply talking out loud to themselves. In this case, Sfard’s thinking as communicating is demonstrated as both social and personal communication, by talking through the thought process out loud in communication with others and/or in communication with self.

Although students’ conversations with each other, as described above, give a powerful window into each student’s developing thinking, conversations alone do not give a complete picture of how students are thinking mathematically. To describe

thinking as communicating is to conflate two distinct processes, albeit processes that are tightly interwoven (Mahn, 2009). In this notion, Sfard's thinking as communicating theory has diverged from a clear Vygotskian distinction between thinking and talking. Vygotsky (1986) described thought and speech as two intersecting circles, but was careful to explain that they are different processes. He said, "Verbal thought, however, does not by any means include all forms of thought or all forms of speech. There is a vast area of thought that has no direct relation to speech" (p. 88).

The assumptions I make for my study that there is pure thought not connected to a symbol system (See Mahn's graphic, Appendix A), there are systems of meaning not structured by language, and there are other forms of symbolic representation that mediate thought limits the thinking as communication theory as a way to understand the students' mathematical thinking in my research. In addition, because I am exploring the mathematical thinking of bilingual students, conflating a thought with a word to equal a meaning, contradicts a powerful theory from bilingual research mentioned earlier, the idea of a common underlying proficiency (Baker, 2006; Cummins, 1981). The essential idea in both common underlying proficiency and verbal thinking is that meaning mediates between thought and word, unites the two, but does not tie the word to thinking. Because meaning is at the interface between thought and language, concepts learned in one language can be transferred to another language. The symbol system can expand to include words in other languages without changing the internalized meaning.

Problem Solving in Mathematics Education

Solving complex word problems is a valuable component to the mathematics curriculum. Problem solving engages students in the work of actively constructing their

own knowledge according to constructivist theory (Kamii et al., 2005). Context-rich problems give individual students an opportunity to use what they know about the world to connect with formal mathematical concepts and processes and learn mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996). Social constructivists and socioculturalists argue that group problem solving gives all students the opportunity to broaden their perspectives, learn from each other, develop mathematical language and refine their own thinking (Cobb & Yackel, 1996; NCTM, 2000; Sfard, 2001; Zack & Graves, 2001). In CGI theory, authentic problem solving activities give teachers the opportunity to understand students' mathematical thinking, assess students' strengths, reveal gaps in student thinking, and move all students forward in their mathematical concept development (Fennema, Franke, Carpenter & Carey, 1993).

Children's first experiences with mathematics problem solving happen in out-of-school contexts. It is here where they are first exposed to the idea of numbers and operations on numbers. From a sociocultural perspective, what children know about solving problems in informal settings form the spontaneous concepts and informal systems of meaning that provide a foundation for later formal learning (John-Steiner & Mahn, 1996). As noted above in the section on sociocultural theory, context-embedded word problems link students' informal or spontaneous numerical knowledge of the world with the formal structure of school mathematics and allow for the interplay between these two systems of meaning (Carpenter et al., 1996; Cobb & Yackel, 1996; McClain & Cobb, 2001; Moschokovich, 2002; Turner et al., 2008).

Problem solving as a way to study children's thinking is widely supported across diverse research paradigms. All the mathematics education research studies outlined

above, including constructivist (Kamii et al., 2005), social constructivist (Cobb & Yackel, 1996), and the sociocultural and Bahktinian perspectives (Kieran 2001; Van Oers, 2001; Sfard, 2001; Zach & Graves, 2001) were built around problem solving activities. In addition, research in developmental psychology connects problem solving with higher order thinking including the ability to reflect on one's own approaches to finding solutions (Lerman, 2001).

Researchers in Finland exploring young students' mathematical abilities and rate of learning found a positive correlation between students' metacognitive skills entering school and their ability to solve word problems (Aunola, Leskinene, Lerkkanen & Nurmi, 2004). In a longitudinal study from kindergarten through second grade, Aunola et al. found that students with better metacognitive skills had more flexible and successful problem solving strategies and their rates of mathematical development increased more rapidly than students who had lesser abilities to examine their own thinking and approach problems flexibly. I interpret this research to imply, based on my understanding of the interplay between internal and external systems of meaning, that if flexible thinking and problem solving are linked then increased activities in problem solving with reflection can improve students' abilities to examine their own thinking and approach problem solving more successfully.

In the next section of this chapter, I discuss the CGI problem-solving framework (Carpenter et al., 1999). I have chosen CGI as a mathematics learning theory for this study because I believe it provides the generalizations necessary to examine children's systems of meaning. The CGI framework is built around basic scientific concepts for mathematics such as addition, subtraction, multiplication and division. CGI and

sociocultural theory taken together provide a powerful theoretical basis for understanding students' mathematical thinking. From sociocultural theory we have the notion that children's internal systems of meaning are being shaped by their exposure to the external, formal systems they encounter in the classroom (Mahn, 2009). CGI problem solving helps us target specific concepts in children's mathematical thinking and analyze how students are making sense of the mathematics (Carpenter et al., 1999). When we follow students' approaches to CGI problems from kindergarten to 2nd grade, we can observe how their informal system of meaning is being refined, transformed, and expanded in relation to these specific concepts as children incorporate new knowledge about mathematics into their CGI problem solving.

The developers of CGI acknowledge a connection with sociocultural theory and the powerful framework that CGI provides to understand student thinking. They note, "Our analysis of the development of children's mathematical thinking can be thought of as scientific knowledge, as defined by Vygotsky... that provides a basis for teachers to interpret, transform, and reframe their informal or spontaneous knowledge about students' mathematical thinking" (Carpenter et al., 1996, p. 5).

Cognitively Guided Instruction

Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993), Carpenter, Fennema, and Franke (1994, 1996), and Carpenter, Fennema, Franke, Levi, and Empson (1999) have done extensive research on young children's problem solving approaches and have developed a framework for teachers to help them understand how children initially think about problem solving and how children develop increasingly sophisticated problem solving strategies. This framework is called Cognitively Guided Instruction (CGI).

Building on the work of Fuson (1988, 1992), who studied children's approaches to solving addition and subtraction problems, and drawing on the general perspectives provided by both constructivist and sociocultural theories, Carpenter et al. (1996) concluded that even young children have a wide range of practical experiences and intuitive knowledge about problem solving they can use to find solutions to word problems. CGI researchers observed that children do not need to be shown specific solution strategies but can develop problem solving strategies on their own within socially supportive learning environments. They say, "Our major thesis is that children bring to school informal or intuitive knowledge of mathematics that can serve as the basis for developing much of the formal mathematics of the primary school curriculum" (Carpenter et al., 1996, p. 6).

The basic tenet of CGI is that students will "do what comes naturally" when it comes to problem solving (Carpenter et al., 1994, p. 3). When given contextualized word problems that reflect familiar, real-life situations, children will intuitively create a direct model of the actions and relationships of the numbers in the problems. Given opportunities to solve a variety of problems as well as the opportunity to share their thinking and strategies in social situations that connect the problems to formal mathematics, children develop a deeper sense of number and begin to make sense of the generalizations contained in formal mathematical concepts.

The strategies children use to solve CGI problem types are an important component of CGI theory and reflect the increasingly abstract ways children think about numbers (Carpenter et al., 1999). As children begin to incorporate formal mathematical ideas and relationships into their own ways of thinking, counting strategies dominate over

direct modeling strategies. As children add number facts to their mathematical knowledge, they begin to use facts to solve problems. Numbers facts can be used as a jumping off point to find solutions by deriving facts. As number sense grows, children apply trial and error strategies and begin to develop their own methodology and invented algorithms. Use of place value concepts with multidigit computations reflects student's developing understanding of the patterns and relationships in our number system.

A Bridge Between Informal and Formal Systems of Meaning

In the CGI framework, the more advanced strategies children use to solve problems are all “progressive abstractions” (Carpenter et al., 1996, p. 6) of earlier strategies beginning with the direct modeling children do in their first problem solving activities. When symbolic representations of numbers replace concrete manipulatives or the drawing of each object, the numeric symbols themselves become the objects of manipulation, directly extending and building on students' earlier models and strategies. When students are able to produce formal equations that represent their own problem solving solutions, a bridge has been made between students' informal system of meaning and the formal system of mathematical concepts and processes. In the following paragraphs, I give more details of the CGI framework, beginning with a discussion of the progression of strategies followed with examples of CGI problem types.

From Direct Modeling to More Efficient Problem Solving Strategies

CGI research found that students begin by directly modeling the actions and relationships contained in the CGI word problems (Carpenter et al., 1999). To directly model means to represent each object in the problem, reproduce the action or relationship

of the problem, and then count objects to find the solution. Children may use cubes, counters, their fingers, or drawing, but the key to direct modeling is that children show every object (Carpenter et al., 1993; Carpenter et al., 1999). Take a problem where the student is to find the total number of candies if she has six candies and her friend has eight candies. To directly model, she would count out a group of six cubes, count out another group of eight cubes, then put the two groups together and count how many cubes are in this combined group, 14.

As students' sense of number grows, they move toward more efficient counting strategies, which are extensions of the direct modeling described above. Carpenter et al. (1999) found that children will invent these increasingly efficient strategies for solving problems on their own and do not need to be shown. The more efficient counting strategy is an abstraction of direct modeling according to CGI theory. In a counting solution to the above problem, the child would recognize that it is not necessary to build each set. She would realize that she could treat the number six as an abstract object and the starting point, then count only the second group saying, "7, 8, 9, 10, 11, 12, 13, 14." Counting strategies can also include place-holder approaches. For example, in a problem where a student has four candles on a birthday cake and needs to find out how many more candles she needs to have seven on the cake, she could begin counting at four, count on to seven using three fingers to keep track of the count. She would then see that the three fingers in the counting on sequence represent the answer to how many more candles are needed.

As children begin to learn number facts, these facts become part of their solution strategies. Double facts, e.g. $5+5$ and $6+6$ are usually learned quite early (Carpenter et

al., 1996). Children will use this knowledge to derive solutions. In the problem above with six and eight candies, a derived fact strategy would be one where the student knows that six and six make 12, that eight is two more than six, so the answer is two more than 12, or 14. As children memorize more number facts and fact families, they can use this growing body of knowledge to recognize answers. A student who already knew that six plus eight is 14 would simply say the answer. It is important to recognize that even though students may know an answer or a more efficient strategy, for whatever reason they may sometimes choose to fall back to direct modeling or less efficient strategies (Carpenter et al, 1999).

Flexibility in problem solving strategies reflects a developing sense of number and results in students connecting a variety of concepts in their problem solving approaches. Students' flexibility indicates that they are learning mathematics with understanding because they are recognizing multiple connections and making generalizations (Hiebert and Carpenter, 1992). I have seen students demonstrate flexibility when they connect representations for money or time to problems that are not about these situations. Another indication of flexibility is when students recognize that operations can be reversed, i.e. addition is a strategy to solve a subtraction problem, or division can be approached through multiplication. In other words, an action can be undone with the inverse operation (Carpenter et al., 1999). This is particularly useful when solving problems where the starting value is unknown. Even when children do not have a clear strategy for solving a problem in mind, flexible thinking and their deepening sense of number help children narrow down the possibilities and approach trial and error more efficiently.

CGI Problem Types

Research in Cognitively Guided Instruction began with observations of children solving problems (Carpenter et al. 1993, 1994, 1996, 1999). From these observations, the developers of CGI classified problems into types depending on the ways children appeared to think about the problems and the generalizations they noted in children's approaches to finding solutions. CGI problem types include join, separate, compare, part-part-whole, multiplication, and division (See Appendix B for a detailed list of CGI problem types). The most basic types, introduced early in kindergarten, are join and separate problems with the result unknown, followed by part-part-whole and compare problems. Join problems put sets together, separate problems remove a set from the main group, compare problems match sets, and part-part-whole problems either combine the parts or take a part away from the whole to find the other part. An important distinction between these specific types is that join and separate problems involve *actions* on the numbers and compare and part-part-whole problems involve *relationships* between the numbers (Carpenter et al., 1999). CGI research observed that problems containing actions are much easier to solve than problems based on relationships because children can intuitively model the action of the problem.

Joining and Separating.

Whereas adults will generalize with addition or subtraction to solve a variety of problem situations, young children will think about these problem situations quite differently (Carpenter et al., 1993, 1999). For example, three problems where adults would all subtract are: 1) you had eight cookies and you ate three cookies. How many cookies are left? 2) You have \$3 to buy cookies that cost \$8. How many more dollars do

you need to buy the cookies? 3) You have \$3 and your friend has \$8. How many more dollars does your friend have? For all the problems adults will subtract the three from the eight to get five, but children who approach the solution through modeling, think of the problems in different ways. For problem 1) a student might start with eight cubes, take away three, and count what remains to see that five cubes are left. For problem 2) a student might start with three cubes, add cubes one at a time until there are eight cubes, and then count the added cubes, getting five dollars more. Finally, in problem 3) a student might make two rows of cubes, one containing three and the other eight, count the extra cubes in the longer row, and arrive at five dollars for an answer.

In the above examples, the three problems are all different CGI problem types because children solve them differently. The first problem is called a Separate Result Unknown because three cookies are being separated from the eight and how many are left is unknown. The second problem is a Join Change Unknown because the unknown quantity is how much money to add to \$3 to get \$8. The result is known, but not the change. The third problem is a Compare Difference Unknown because two known sets are being compared to discover the difference between the two.

Of the three example problems, the first is the easiest because it can be directly modeled from the actions in the story, and these actions follow one another in sequence. The second problem is more difficult because it requires a degree of preplanning. In this case, when the five cubes are added to the three cubes to finally arrive at eight altogether, the student needs to keep track of which are the original cubes and which have been added in order to distinguish the five from the original three. The third problem is also more difficult for children than the first because comparison of sets does not involve

action. In this problem, there are two static sets and the child's task is to figure out how to arrive at an answer using some type of action not explicitly given in the problem. A further breakdown of problem types and strategies will be discussed in the methodology chapter of this dissertation and student strategies will form the basis for the analysis of student thinking.

Multiplication and Division.

The multiplication and division problems used in this study involve grouping and partitioning of objects. While the CGI literature includes other problem types such as rate, price, multiplicative comparison and symmetrical arrays (Carpenter et al., 1999), they are outside the scope of this project. All the multiplication problems used in this study combine equally sized groups of objects to find a total number. All the division problems partition a certain quantity of objects into equally sized groups. The division problems further break down into two different CGI types. Measurement division problems have the total quantity of items as known and also how many of each item are in the groups, but the number of groups is unknown. An example of a measurement division problem is: You have 18 cookies and some bags. You want to put three cookies in each bag. How many bags do you need? In contrast to knowing how many items are in each group, partitive division problems state the total number of items, the number of groups, but not how many of each item are in each group. Using the same situation above, a partitive division problem would be: You have 18 cookies and six bags. How many cookies can you put in each bag so they all contain the same amount?

Multiplication problems are fairly straightforward for children to model directly. Even early in kindergarten, children can model the action by constructing equally sized

groups, putting these groups together and counting the total (Turner et al., 2008). Simple measurement and partitive division problems with small numbers can also be modeled directly. For a measurement division problem using the cookie example above, children may set out the 18 cubes, put them into groups of three and count how many groups they have made resulting in six. For partitive division, children may try to model the action by creating the correct number of groups, in the example above six, and doing a one by one distribution into each group until the cubes are used up, in this case resulting in three for each group.

Children develop more advanced strategies for multiplication and division problems by abstracting the numbers in much the same way as they do with addition and subtraction problems (Carpenter et al., 1999). However, the counting strategies children use tend to be based on some form of skip counting, which may take longer for children to develop. Children develop skip counting by 2s, 5s and 10s earlier than the other numbers because this is what they encounter in the classroom. This knowledge can be exploited and reinforced in CGI problems. For example, if you have seven bags of marbles with five marbles in each bag, children will realize fairly quickly that they can count by fives seven times to get 35, when they have been counting by fives as part of the regular classroom routine. This same strategy can be used for measurement division problems, but it is not as apparent to children as it is for multiplication problems because it essentially reverses the action. If there are 10 cookies and children can put two in each bag, they can count by twos until they reach 10 and see that they need five bags. Partitive division problems are the most difficult to solve with skip counting strategies because the number in each group is not known (Carpenter et al., 1999). For a direct model of a

partitive division problem, children will distribute the items equally into the groups. Distributing one by one is a common strategy, distributing in groups requires a certain amount of trial and error for young children. For partitive division, a more advanced strategy tends to combine counting with trial and error methods (Carpenter et al., 1999).

Extending CGI to Base Ten Thinking and Multidigit Numbers

CGI continues as a powerful tool for understanding student thinking as they move into base ten concepts and multidigit problem solving. It is important to introduce and reinforce these concepts in classrooms because children do not have intuitive knowledge of the base ten number system and place value. These concepts are social conventions that children cannot discover on their own (Carpenter et al., 1996; Kato, Kamii, Ozaki & Nagahiro, 2002). Some children may have emergent understanding of these concepts if they have been exposed to them in specific activities such as counting money. For all children, the key idea in learning base ten is that *groups* of 10, 100, 1000, etc. can be counted (Steff & Cobb, 1988). This idea is developed in CGI by problems that group items into tens and in working with manipulatives like base ten blocks that provide concrete representations for these groups. Multidigit number operations and problem solving strategies build on the ideas of base ten, place value and decomposing numbers into tens and ones.

The progression of strategies children use for solving CGI problems with multidigits are similar to those for single digits, only in this case as students develop their sense of number they model with tens and ones and begin to think of a group of ten as one object. It is important to note that children do not need to have a solid understanding of base ten before they engage in multidigit numbers operations (Carpenter et al., 1999).

CGI problems promote this understanding and give children the opportunity to manipulate groups of tens, and in the process discover more efficient strategies (Fuson, 1992, as cited in Carpenter et al., 1996). For example, children can do multiplication problems where they have five boxes of markers with 10 markers in each box. How many markers do they have in all (50)? Or, they have five boxes of markers with ten in each box and four single markers. How many markers do they have in all (54)? An example of a measurement division problem would be: You have 50 markers and want to put them in boxes with 10 in each box. How many boxes can you fill? Similarly in partitive division: You have 50 markers and five boxes. How many markers can you put in each box if they all contain the same amount?

As before when working with smaller numbers, each progression to a more advanced strategy in problems with larger numbers builds on previous strategies and reflects a greater degree of abstraction (Carpenter et al., 1999). When using base ten blocks, a direct modeling strategy to add 54 and 48 would involve building two groups with the correct numbers of rods (tens) and cubes (ones), then combining the groups and counting the combined amount (see base ten materials in Appendix F). A more advanced strategy for the same problem involves beginning with one of the numbers say 54, then counting on with four rods and eight cubes to find the answer. For example, start with 54 then count on by tens 64, 74, 84, 94 then finish by counting the eight ones, 95, 96, 97, 98, 99, 100, 101, 102. A further progression in strategy involves a more abstract approach. A student might start with 54, then without building the second number, mentally manipulate the tens and ones as objects, saying 54, then keeping track with four fingers for the 4 tens count up to 94, finally using the fingers again to count up by ones to 102.

It is important to note that when children move to advanced strategies as in the final case above, they carry out the operations on the symbolic numbers themselves instead of the concrete materials (Carpenter et al., 1996). The symbols that stand for the numbers become the objects of manipulation, and additionally the words that represent the numbers become objects of reflection (Steff, von Glasersfeld, Richards, & Cobb, 1983, as cited in Carpenter et al., 1996). It is precisely in this connection between numeric symbols and words that I make a strong connection to systems of meaning theory in the CGI. I see the integration of symbols with words expanding the sign operations as students' informal systems of meaning begin to incorporate and be restructured by the formal system of mathematics they encounter in the classroom. In solving CGI problems, children use both verbal and mathematical thinking to find their answers.

A further connection to sociocultural theory in CGI is the interplay between the internal reflection of the individual around problem solving and reflection on the social plane. Children's personal reflection as they describe their thinking and their strategies to the group helps them integrate words and symbols into their system of meaning. Carpenter et al. (1996) comment on this interplay among concrete object manipulation, numeric representations, symbolic manipulations and specifically language when students solve problems and discuss their thinking.

What we are proposing is that the manipulations of the blocks [cubes] become objects of reflection. At some point the numbers involved in counting the blocks also become objects of reflection so that students can operate on the numbers independently of the blocks. A key factor in this

process is the continuing discussion of alternative strategies. Students regularly are called on to articulate their solutions, to describe in words what they have done with the blocks. In order to be able to describe their strategies, they need to reflect on them, to decide how to report them verbally. Initially, the descriptions are of procedures that have already been carried out. Eventually, *the words that students use to describe their manipulations of blocks become the solutions themselves*. Thus, the verbal descriptions of modeling strategies provide a basis for connecting manipulations of tens blocks and invented algorithms using numbers only. (my italics, p. 13).

Teaching and Learning Mathematics in Spanish

Research described by Secada and De La Cruz (1996) in a 2nd grade bilingual classroom in Texas around CGI-type problem solving and student explanations emphasized the benefits of students having access to their native language, Spanish, to make sense of mathematics problems (Secada & De La Cruz, 1996). In their chapter on teaching mathematics to bilingual students, Secada and De La Cruz argue that the best approach to raise unacceptable Latina/o student proficiency scores in mathematics is by teaching mathematics with understanding (Hiebert & Carpenter, 1992). To learn mathematics with understanding, students must be able to make sense of the problems they encounter. Ambiguity in understanding is removed when students have access to their native language to comprehend problem contexts. When students have access to their native language to explain their thinking about problem solutions, they are free to be clear and precise (Secada & De La Cruz, 1996).

Clear explanations from the 2nd graders in their native language helped the teachers in the Texas study assess students' mathematical understanding and move students forward in their mathematical development. Secada and De La Cruz give specific Spanish language examples to emphasize that the 2nd graders were learning mathematics with understanding. In the examples, students were able to “put a new twist on an idea...relate and/or apply new ideas to the problem...point out something that is wrong...[and] solve a problem in a different way or give a new way of justifying an idea” (p. 294). Students' clear and precise explanations about their thinking would not have been possible if they had to use a developing second language (Cummins, 2001).

Other research has documented the benefits of developing both the native language, Spanish, and the second language, English for successful CGI problem solving. Secada (1991) found that bilingual 1st graders who had well developed linguistic skills in both languages performed better in CGI problem solving. The cognitive benefits of bilingualism increase as proficiency in both languages increases according to the thresholds theory (Bialystock, 2001; Cummins, 2000). Baker (2006) explains that when children reach a level where they have age-appropriate communicative competence in both languages they have reached a threshold where they may begin to have some cognitive advantages over monolingual children due to their increased metalinguistic awareness.

Before English Language Learners (ELLs) reach the threshold of communicative competence in English, however, they need the opportunity to develop concepts in their primary language if they are to keep pace academically (Cummins, 1981, 2000, 2001). As students develop more proficiency in English, concepts learned in the native language

transfer directly to the second language. Students' ability to transfer conceptual knowledge to a second language is understood by the theory of common underlying proficiency mentioned previously in this chapter (Cummins, 1981). Applying the theory to mathematics, Baker (2006) says,

Teaching a child to multiply numbers in Spanish or use a dictionary in English easily transfers to multiplication or dictionary use in the other language. A child does not have to be re-taught to multiply numbers in English. A mathematical concept can be easily and immediately used in English or Spanish if those languages are sufficiently well developed. (p. 169)

Common underlying proficiency theory supports systems of meaning and verbal thinking because of the separation of the processes of thinking and speaking (see Mahn's Planes of Verbal Thinking and Systems of Meaning diagrams, Appendix A). If a concept is incorporated into a system of meaning, expanding the symbols for that system to include words in a second language does not change the underlying structure of the system. Meaning has been internalized in a structure that connects directly with thought processes. It is meaning that mediates *between* thought and language, not a specific language that links meaning to thought (Mahn, 2009). Words in a language come to internally symbolize meaning, and this link is not lost when new words from another language take on similar meanings. The symbol system for representing meanings expands. In my understanding of the theory, just as the word/concept of two is expanded with the symbol "2" for monolingual children when they learn number representation, "dos" and "2" are further expanded when Spanish-speaking children learn the word

“two” in English. The concept of two-ness does not change with the addition of multiple linguistic representations.

The theories of language thresholds, common underlying proficiency, and systems of meaning further support the equitable practice of teaching Spanish-speaking children cognitively demanding subjects like mathematics in their native language while developing a second language (Thomas & Collier, 2002). When children are able to develop conceptual knowledge in their native language at the same time as they are learning English, they move forward academically without losing the ground they would have lost otherwise by struggling to incorporate new concepts in an unfamiliar language (Cummins, 2000). Learning formal concepts in an underdeveloped language throws up barriers between the informal concepts growing upward and the formal concepts growing downward.

Gaps in the Literature

Although there are strong connections between the CGI framework and sociocultural theory, combining the two theories to explore students’ mathematical thinking over the critical time period of their first three years of formal schooling has not been done. This study offers a unique opportunity to longitudinally explore bilingual students’ mathematical thinking using the CGI framework and analyze how students make meaning of the numeric actions and relationships in word problems over time.

Research that values the thinking of Spanish speaking students and promotes challenging mathematical problem solving in bilingual classrooms addresses equity for Latina/o students and adds to the scarce literature on mathematical development in Spanish language contexts. My research shows how students begin to understand CGI

problems and explain their thinking in their first language, Spanish, and developing language, English. It challenges the assumption from national test results (NCES, 2005) that linguistic minority students and students from lower socioeconomic backgrounds cannot be successful in advanced mathematics and must begin with drill and practice to learn their number facts *before* they engage in cognitively challenging activities.

CHAPTER 3. Methodology

Interviewers are listeners incarnate; machines can record, but only you can listen. At no time do you stop listening, because without the data your listening furnishes, you cannot make any of the decisions inherent in interviewing...The spontaneity and unpredictability of the interview exchange precludes planning most probes ahead of time; you must, accordingly, think and talk on your feet, one of those many interview-related skills that improves with practice.

(Glesne, 2006, p. 92)

Introduction

The qualitative study presented in this dissertation follows a descriptive and interpretative approach (Creswell, 1998) and culminates a three-year longitudinal exploration of individual CGI mathematics problem solving by native Spanish-speaking children who are learning in bilingual classrooms. The study covers the time period from kindergarten through 2nd grade where mathematics instruction for these children was delivered in Spanish. Extensive problem-solving interview data was collected on eight students using the CGI framework (Carpenter et al., 1999). Analysis of the data was based in the problem solving strategies described in the CGI framework, including the tools students used to help them find problem solutions. Analysis was expanded to include students' explanations of their thinking about the problems from the perspective of verbal thinking (Mahn, 2009; Vygotsky, 1987). The data were systematically coded to uncover trends and themes in the mathematical thinking of individual children over the three-year period. A description of four children's approaches to five specific CGI problem types over time and an explanation of individual student's unique ways of making sense of the mathematics in the CGI problems are the outcomes of this study.

My role as co-researcher for several related studies has brought me into close association with teachers and students at this site for the last four years and I have been both a participant and an observer during classroom mathematics lessons (see Celedón-Pattichis, Musanti & Marshall, in press; Musanti, Celedón-Pattichis & Marshall, 2009; Musanti, Marshall, Cebolla & Celedón-Pattichis, in press; Turner et al., 2008, 2009). I have gained the confidence of the research participants in this longitudinal study by being a supportive weekly presence in their classrooms, through extensive one-on-one interviews with them around CGI word problem solving, and in the many conversations I have had with them about their mathematical thinking. I have developed a warm relationship with the children in this study and admit to a bias that positions me as an advocate for their academic success. Throughout the three years I worked with these children, I became familiar with their learning styles and their personal ways of communicating and I had extensive conversations with my co-researchers about their mathematical thinking. These experiences gave me a unique opportunity to create an in-depth picture of their mathematical thinking and development over time.

Background to the Research

The work that serves as the background to this longitudinal study is with an educational research project supporting the mathematics education of Latino/a students called CEMELA². My specific work during the past four years as a CEMELA doctoral fellow has been in bilingual primary-grade classrooms at La Joya Elementary School³ working with Spanish-speaking students and their teachers around CGI problem solving.

² Center for the Mathematics Education of Latinos/as, NSF grant ESI-0424983

³ a pseudonym

During the first two years of my involvement at this site, I was part of a research team that worked with kindergarten and 1st grade teachers to develop word problems embedded in contextually familiar situations. This team explored both teacher professional development in mathematics and students' mathematical learning when solving word problems. The theoretical basis for the word problems came from CGI, a framework for understanding children's mathematics thinking (Carpenter et al., 1999).

In the typical CGI lessons we created collaboratively with teachers, kindergarten and 1st grade students were presented with familiar situations that had been mathematized. For example, "You go to the balloon fiesta and see 20 balloons. Then seven balloons fly away. Now how many do you see?" The CGI literature breaks problems such as this one down into specific types and describes common student strategies for solutions (see the theoretical framework in Chapter 2). Using blocks, counters, or pencil and paper, we encouraged the children to invent their own strategies for solving the problems. Key to the CGI approach is to have children explain their thinking and justify their solutions. Not only does this help students build connections between their own intuitive understanding of numbers and formal mathematics (NCTM, 2000), it gives teachers access to student thinking about the concepts contained within the problems (Carpenter, Fennema & Franke, 1994).

Kindergarten.

Our first CEMELA research project at La Joya Elementary was in one kindergarten classroom during the 2005-2006 school year. The teacher was part of a CGI training cohort that had been recruited when the CEMELA program first received funding during the 2004-2005 school year. The teacher is from Guatemala and a native

Spanish speaker. All the students in the classroom were native Spanish speakers from Mexican immigrant families and 90% of the instruction was in Spanish. All students understood and spoke Spanish, although two or three were dominant in English for their social communication. One student, Omar, had become dominant in English during his preschool experience and was relearning Spanish. Omar is one of the participants for this longitudinal study and he will be described later in this chapter.

The CEMELA research team for this kindergarten year was led by Dr. Erin Turner and Dr. Sylvia Celedón-Pattichis. I was the graduate student member most actively involved at this site. Beginning in October of 2005, the kindergarten teacher asked students to solve problems and explain their thinking even though many of them were still developing one-to-one correspondence and counting abilities up to ten. Initially, they used interlocking blocks to solve problems, and then later in the year moved to individual dry-erase boards and at times pencil and paper. The CEMELA research team conducted individual pre assessment interviews with eight of the students in this class in October 2005, and post assessment interviews with sixteen of the students in the class in May 2006 to record students' problem solving strategies, their answers, and their explanations about their thinking. The problems for the interviews were adapted by Dr. Erin Turner based on the study with kindergarten students by Carpenter et al. (1993). The problem types used in the post assessment interview are presented in Table 1 later in this chapter. (See Appendix C for the results of the kindergarten post assessment from May 2006).

Data analysis from the kindergarten interviews showed that all students were highly successful in solving a variety of problem types not usually introduced in

kindergarten, including multiplication and division problems (Turner et al., 2008). Students were able to use drawings to help them solve the problems and explain their thinking. They were even successful with problems that are traditionally hard to model involving relationships between numbers (Carpenter et al., 1994) such as part-part-whole and compare type problems (see a description of the findings in Turner et al., 2008).

1st and 2nd grade.

After a year of exploring kindergarten students' mathematical thinking through CGI problem solving, I wanted to continue the investigation with these same students into 1st grade. In addition, Dr. Sandra Musanti joined the CEMELA team in the summer of 2006, replacing Dr. Erin Turner. Sandra expressed an interest in continuing the research in professional development around CGI problem solving. In order to pursue our complementary research interests, Sylvia, Sandra and I contacted two 1st grade bilingual teachers who were interested in working with our team to explore CGI problem solving in their classrooms for the 2006-2007 school year. As a result of student placement decisions at La Joya Elementary, eight of the students from the bilingual kindergarten classroom were placed with these two 1st grade teachers. In the summer of 2006, I extended the research design on student learning to continue CGI problem solving for these eight students. The primary source of data for this study was based on individual student interviews around CGI problem solving, similar to the interviews conducted at the end of kindergarten. The problems presented in these interviews are presented in Table 1 later in this chapter.

The data collected throughout the 1st grade year showed students' continued success in problem solving and a more sophisticated use of language to explain their

mathematical thinking. Extensive individual interviews using all twelve of the CGI problem types were conducted with seven of the students in November 2006 and again in May of 2007. The eighth student, Dolores, chose not to participate in the May interviews. Her mathematical development was less advanced than the other students and she appeared to have limited confidence when asked to solve problems on her own.

Similar to kindergarten, the students in 1st grade showed success with challenging problem types, including addition and subtraction problems where the change or the starting number was unknown (Carpenter et al., 1999). These problems are difficult to solve for 1st graders because once again they are hard to directly model. Increasingly, the students moved from a reliance on direct modeling of the problems to more advanced strategies such as counting, recalled facts, derived facts, and trial and error. Their strategy use was an important focus of analysis for the 1st grade data because advanced strategies indicate a deepening sense of number according to Carpenter et al. (1999). Appendices D and E contain the quantitative results of the November 2006 and May 2007 individual interviews.

The same eight students interviewed in 1st grade were placed in two 2nd grade classrooms by the principal at La Joya Elementary so that we could conveniently continue with the CEMELA study the following year. I interviewed the students in September of their 2nd grade year and again in February. I used the same interview protocol for these first two interviews with a reduced set of CGI problem types. The choice of the four problems used came after I analyzed how students performed on the problems during 1st grade and also from discussions with Sandra about my goals for the interviews. Another consideration that led to a shortened interview was our accessibility

to the students and the amount of time we could take them out of the classroom during 2nd grade. Unlike the previous year when we worked closely in professional development with the 1st grade teachers, the two 2nd grade teachers were less involved with CGI problem solving professional development and had greater demands for student time.

For the first two 2nd grade interviews, I used four problem types: Compare, Multiplication, Part-Part-Whole, and Join Start Unknown. I chose Compare type problems because I noticed throughout our work in the 1st grade classrooms, children consistently had trouble understanding and solving these kinds of problems. Compare problems involve relationships among numbers and are more challenging than problems that involve actions (Carpenter et al., 1999). We noticed in our debriefings with the teachers that the concepts of more than and less than were challenging for many students to comprehend. Additionally, I chose to make the compared set unknown in the Compare problem instead of the difference unknown between sets to see if the research participants could understand the relationship between the numbers when only the smaller set and the difference between the two sets were known.

I chose one more relationship problem and two action problems to complete the interviews. I chose the Multiplication problem because I wanted to see if students would use an advanced skip counting strategy to find the answer instead of direct modeling this action-type problem. To this end, I created a problem about bags of marbles with five marbles in each bag to see if the students would count by fives to get an answer. Like Compare problems, Part-Part-Whole problems are about relationships instead of actions and are difficult to model. I chose Part-Part-Whole to balance action and relationship problems in the interviews having two of each type. Finally, for another action problem I

chose Join Start Unknown (JSU). This is the most difficult of the Join problems to model because students do not know the starting number. I found in the interviews in 1st grade that students were more challenged with JSU than Separate Start Unknown (SSU) problems. Students in 1st grade tended to just add the two numbers in the SSU problem together and so it was not clear if they were understanding the problem situation or not.

A final expanded interview with students was conducted in April of 2nd grade after my proposal for this longitudinal study was approved. Details of this interview follow in later sections of the chapter.

Research Questions

My interest in students' mathematical thinking and problem solving began in the bilingual kindergarten classroom. It continued the following two years in their 1st and 2nd grades. During 1st grade and the beginning of 2nd grade, I refined my understanding about how students approach CGI problems and my interest began to focus on the range of ways students communicate their mathematical thinking and what this communication reveals about the way they make sense of number problems. As I analyzed data from the 1st grade and reflected on how the students approached similar problems in kindergarten, I began to see connections among students' choice of strategies, problem-solving aids like number lines, and their verbal explanations, and how all of these components reveal aspects of students' mathematical thinking. I saw how verbal thinking (Vygotsky, 1987) is interwoven with mathematical thinking to help students make sense of the problems and also to make their mathematical thinking more transparent to themselves and their interviewers.

Because of the multi-year background I had with the student participants in this study and the volume of data the CEMELA team had collected about their problem solving, I realized I was in a perfect position to conduct longitudinal research into students' mathematical thinking over the course of the first three years of formal schooling. In addition, I was interested in how these students, now in 2nd grade and seasoned research participants, would approach the problems in more than one way if given the chance and what their opinions would be about why one method and/or tool might be better than another for a particular problem type. To explore students' mathematical thinking around CGI problems from a longitudinal perspective, my research questions ask:

1. How do bilingual primary-grade students learning mathematics in Spanish language environments communicate their mathematical thinking during CGI problem solving over the course of three years?
2. What does student communication in the form of the strategies they use to solve the problems, the tools and materials they choose as aids, and students' verbal explanations reveal about how they are making sense of the mathematics in the CGI problems?

Mode of Inquiry and Philosophy

I chose a qualitative descriptive and interpretive study based on individual problem-solving interviews from eight bilingual students learning mathematics in Spanish language classrooms as the mode of inquiry for this research. It most resembles the tradition of grounded theory (Creswell, 1998; Glaser & Strauss, 1967) in that there is a definite purpose for the study, i.e. the description of how students make meaning during

mathematical problem solving, and there is a rigorous coding scheme to develop a substantive theory about student thinking. This research builds on existing theory with the interpretation and analysis of data guided by the literature on CGI (Carpenter et al., 1999) and systems of meaning (Mahn, 2009; Vygotsky, 1987) as outlined in my theoretical framework.

In 1967 Glaser and Strauss presented a method for discovering theory within a qualitative research design. They called the result a *grounded* theory because of its intimate link to a body of data that has been systematically collected and analyzed. Glaser and Strauss explained that through a constant comparative analysis, “hypotheses and concepts...come from the data [and] are systematically worked out in relation to the data” (p. 6). During comparative analysis, conceptual categories emerge that lead to generalities based in facts and delimit the boundaries of the theory. Categories and generalities are continually verified against new data to discover similarities and differences and “bring out distinctive elements of the case” (p. 25). This process advances in four stages: 1) developing and refining categories in relation to the data, 2) integrating categories and developing generalizations, 3) delimiting the theory, and finally 4) writing the theory in relation to the research site and participants.

Grounded theory research relies on extensive data collection to gain as much information as possible about the situation being studied. In many cases this data comes from interviews (Creswell, 1998). Analysis on the data begins before the end of the study and the preliminary categories that emerge at this early stage are used to refine further data collection until a type of saturation occurs where ideally no more new information can be obtained related to the situation. In this way, data collection and analysis go back

and forth in grounded theory in an effort to continually refine the interpretation of the data toward a substantive theory. Because of the extensive amount of data obtained in grounded theory research, and the investigative purpose to allow theory to emerge from the data, it is important that analysis in grounded theory be as “scientific and objective” as possible (Creswell, 1998, p. 34).

The longitudinal study on students’ mathematical thinking described in this dissertation contains all the key elements for grounded theory. The groundbreaking results that led to the discovery of new theory are given added weight by the multi-year engagement with the research participants. Extensive interview data were collected. Eight students were interviewed six times, once in kindergarten, twice in 1st grade, and three times in 2nd grade. The numbers of problems for each interview ranged from four to 12. These interviews resulted in a large amount of data on students’ approaches to CGI word problems, the particular situation being studied. Comparative analysis on the data was on-going and was used to inform subsequent interviews. There was a theoretical purpose to the study, which was to make claims about student thinking and about how students make sense of the mathematics when they are solving CGI-type word problems. Finally, coding of the interview data proceeded in a rigorous, systematic, and as much as possible, objective manner given the interpretive nature of qualitative research. While coding, I tried to set aside any conclusions I had about how students were making sense of the problems so that categories could emerge from the data.

Research Participants and Site Description

I begin with a description of the site for this research to highlight the overarching theme of equitable mathematics education for Latina/o students. I argue that students

from sites such as this one deserve the opportunity to engage in rigorous mathematical tasks from their first days in kindergarten and to develop conceptual knowledge and mathematical language through contextualized problem solving. Furthermore, I argue that they are fully capable of performing at high levels in mathematics when given these opportunities (Secada, 1991; Secada & De La Cruz, 1996).

The research site for this study, La Joya Elementary, is located in a low-income neighborhood in a large urban area in the Southwestern United States. The surrounding neighborhood has a large Mexican immigrant population and 86% of the nearly 700 students speak Spanish as their first language. The school services 100% free or reduced breakfast and lunch to all students. La Joya Elementary promotes a maintenance bilingual program in grades K-5 with the goal of bilingualism and biliteracy for all Spanish-speaking students. The maintenance bilingual program begins with 90% Spanish and 10% English in kindergarten and ends with 50% Spanish and 50% English in 5th grade. The primary grade teachers who participated in this study introduced all mathematical concepts in Spanish and attempted to reinforce much of the material in English, depending on the time they had available. All of the student participants had at least some time during the day devoted to learning English as a second language (ESL).

Eight bilingual students were interviewed for this study over a period of three years. They were first interviewed during the 2005-2006 school year when they were in kindergarten, along with all students in this classroom. Results of these end-of-year interviews can be found in Appendix C. The following year these eight students were asked to participate in 1st grade because they were in either one of two classrooms with teachers who had an interest in professional development around CGI problem solving.

Student placement in these 1st grade classrooms was based on administrative decisions at the school and not the purposeful selection of students for the study. However, the results of this placement determined the eight students from the original kindergarten classroom who would be followed for the three-year time period. Of these eight students, six are girls and two are boys. The principal at La Joya Elementary, in an effort to support our CEMELA research, purposefully placed these same eight students in two 2nd grade classrooms for the 2007-2008 school year with teachers interested in our work.

The eight students were Ana, Brisa, Gina, Jenna, Yolanda, Dolores, Gerardo and Omar. Of the eight, four students additionally participated in pre assessment interviews in October of their kindergarten year, Ana, Jenna, Gerardo and Omar. All eight students speak Spanish as their first language and come from Mexican immigrant families. They have varying degrees of access to English outside of school, and there is diversity in the economic levels of the homes and the formal education of their parents. Some families are bilingual and the students speak a mixture of Spanish and English at home and at school. Other students come from monolingual Spanish speaking home environments. All students are bilingual, i.e. they use two languages for authentic purposes (Baker, 2006), although some are more advanced in their English development than others. Chapter 4 contains the language profiles of the four students analyzed in this study. Some of the students come from low-income families with parents who have not completed high school, while others come from middle-income families and have parents with more formal education. In this way, the eight students reflect the demographics of La Joya Elementary School.

The four final participants.

Only four of the original eight students were chosen for the analysis phase of this research due to the extensive amount of data the team had collected over the three years. The four chosen were two girls, Yolanda and Gina, and the two boys, Omar and Gerardo. They were chosen to balance the research between boys and girls and because each of these four students demonstrated confidence in problem solving. In my opinion, they all were able to think outside the box, so to speak, and to move away from prescribed classroom methods for problem solving. I believe their approaches to the problems may have been closer to their own conceptual understanding of the number situations than the other students. Of the four girls that were not chosen, two were high-level students; however, I noticed that they tended to follow traditional, teacher-prescribed approaches for problem solving and so I chose not to include them. The two other girls demonstrated some challenges in their mathematical understanding, and therefore I believe access to their thinking about the problems might have been clouded by other factors. In the case of Dolores for example, her 2nd grade teacher told me she suspected an audio processing challenge in her overall learning.

Language also played a role in my selection of the four students as I attempted to balance their linguistic strengths. While Yolanda is Spanish dominant, Omar is English dominant, and Gerardo and Gina appear to use both languages more or less comfortably. Gerardo prefers to use English and has had this preference since 1st grade. Gina made a transition to English usage in conversations with me in the middle of 2nd grade, but uses more Spanish for problem solving. Gerardo and Gina are also quite verbal and like to explain their thinking. Omar and Yolanda are less verbal, although Yolanda appears to

have greater access to her own thinking than Omar. All students are capable and successful problem solvers who enjoyed the CGI interviews and felt comfortable in front of the camera. I chose these four academically successful students to emphasize that Spanish-speaking, Latina/o students from Mexican immigrant families are capable of engaging in high-level mathematical tasks when given the opportunity.

Methods

As appropriate to a grounded theory study, data for this project came from extensive interviews with the research participants. All interviews were videotaped and there were at least two adults present at each session. One adult was the interviewer, one operated the video camera, and if there was a third adult, she took notes. One of the adults present was always a native Spanish speaker. I operated the video camera for Erin's kindergarten interviews with Omar, Yolanda, and Gerardo. Sylvia conducted the kindergarten interview with Gina while another graduate student videotaped. Beginning in students' 1st grade, I was present for all the interviews. In November of 1st grade, I conducted the interviews that were in English and videotaped the others. By the end of 1st grade, and for all three 2nd grade interviews, I was the interviewer. Either Sylvia or Sandra was present for these interviews to operate the camera, to correct any mistakes I made in Spanish and to help clarify the problems for students if necessary. They also helped me interpret comments by students if I was not sure of what the students had said in Spanish.

Tools available for problem solving.

Students always had a choice of tools to help them in solving the problems. In kindergarten, students had paper and markers for drawing and connecting cubes to use as counters. Connecting cubes are brightly colored, large, one-inch cubes that connect together linearly. Children can either use them as counters individually, or put them together in various sized groups, like fives and tens. In 1st grade students again had paper and markers, the connecting cubes, and base ten blocks to aid in their solution strategies. Base ten blocks aid in place value groupings. They come in sets of individual cubes, rods with ten sections the same size as the cubes, flats which are 10x10 squares with 100 sections, and large cubes which are 10x10x10 representing 1000 individual cubes. In 2nd grade students had pencil and paper again, one inch tiles for the September interview, and they were given the choice of base ten blocks, 100s charts, and number lines to use. The 100s charts have numbers up to 100 with each row a group of 10 so as children go from left to right the numbers increase by one and from top to bottom the numbers increase by tens. Examples of all materials are in Appendix F. At each grade level, the tools made available to the students during the interviews reflected the typical tools they used in the classroom. I did not give the students counters in 2nd grade because I wanted to push them to more advanced strategies rather than directly modeling with the counters.

Interview locations.

The interviews were always conducted outside of the classroom in the quietest places we could locate. At times, the location was not ideal. However, we were forced to use whatever space the school had available and to adapt ourselves to the unfolding dynamics of the school day. In kindergarten we had the luxury of using an adjoining

classroom that happened to be free part of the day. Our interview sites changed frequently in 1st grade and for the first two 2nd grade interviews. At times we could still use the empty kindergarten classroom, but very often it was in use with various testing and evaluation procedures. We conducted some interviews in a small office area that adjoined one of the 1st grade classrooms, but there were interruptions when teachers and aides needed to retrieve supplies from the room. We tried to use a kitchen area once, but the noise quality was very poor so we gave up on this location.

Finally, for the interviews at the end of 2nd grade, we gained the cooperation of the librarian and she let us use a corner of the library. There was no other space at La Joya Elementary at this time, except the hallways. We were grateful to the librarian for giving us a space for approximately 16 hours of interviewing, but this location was not without challenges. Students came into the library to borrow books and at times they were quite noisy. The librarian conducted classes for students and sometimes turned off the lights to show slides. Once when there were electrical problems at the school, we had workmen in the background while we interviewed. The students, all seasoned research participants by this time, carried on almost seamlessly during these shifting conditions.

Qualitative data collection.

The qualitative data for this study came from individual, video taped interviews with the students. These interviews were more like conversations where we explored students' thinking around the CGI problems rather than formal interviews. The first interview data for this longitudinal study were collected at the end of the students' kindergarten year. Students were interviewed again in November and May of 1st grade. The study continued into students' 2nd grade year with two similar interviews conducted

in September and February, and a final interview conducted in April. See Table 1 on the following page for the problem types and numbers used in each of the interviews.

Table 1. Interview Protocols K-2nd Grade: CGI Problem Types and Numbers Used

Kinder - End	1 st Grade – Nov.	1 st Grade - May	2 nd Grade – Sept.	2 nd Grade – Feb.	2 nd Grade – April
JRU (6,6)	JRU (16,8)	JRU (16,8)			
SRU (13,5)	SRU (19,11)	SRU (19,11)			
JCU (11,7)	JCU (25,15) (11,7)	JCU (25,15)			JCU (88, 46)
	SCU (10,8) (20,12)	SCU (12,5) (30,15)			SCU (70,39)
CDU (12,9)	CDU (21,13) (16,10)	CDU (13,2) (31,29)	CRU (13,6)	CRU (13,6)	CCU (53,35)
MULT (3,6)	MULT (4,7)	MULT (4,7)	MULT (8,5)	MULT (8,5)	MULT (15,7)
PartDiv (15,5)	PartDiv (24,6) (24,4) (18,3) (18,2)	PartDiv (12,6) (24,6)			PartDiv (84,4)
MeasDiv (10,2)	MeasDiv (18,3)	MeasDiv (18,3) (40,10)			
Multi (2,4,3)	Multi (3,4,5)	Multi (3,4,5)			
PPW (10,6)	PPW (13,7)	PPW (30,20) (75,50)	PPW (24,12)	PPW (24,12)	PPW (100,65)
	JSU (5,13) (3,5)	JSU (5,13)	JSU (7,22)	JSU (7,22)	JSU (40,112)
	SSU (3,6)	SSU (3,6)			SSU (95,25)
9 problems	12 problems	12 problems	4 problems	4 problems	8 problems

The abbreviations below refer to CGI problem types. The problems with an * were used for analysis in this study.

- JRU: Join Result Unknown
- SRU: Separate Result Unknown
- *JCU: Join Change Unknown
- SCU: Separate Change Unknown
- *CDU: Compare Difference Unknown
- *CRU: Compare Referent Set Unknown
- *CCU: Compare Compared Set Unknown
- *MULT: Multiplication
- *PartDiv: Partitive Division
- MeasDiv: Measurement Division
- Multi: Multi-step
- *PPW: Part-Part-Whole
- *JSU: Join Start Unknown
- SSU: Separate Start Unknown

The dynamics between interviewers and students shifted as the study progressed from kindergarten to 2nd grade. The purpose of the kindergarten interview was to discover if students could solve the problems and what strategies they would use without scaffolding or help from the interviewer. However, during the actual interviews we found it difficult not to give students some guidance if they were confused. For the interviews in 1st and 2nd grades, the CEMELA team made the conscious decision that we did not want a child to leave an interview session feeling unsuccessful, so we provided more scaffolding than in kindergarten. This took the form of suggestions that the child draw a model, use a different tool, or we guided student approaches with questions related to the story. For example, if a student was modeling a partitive division problem (24, 4) with cubes and she was making three groups instead of four, and if she could not self-correct after we repeated the story, we would ask her to show us the group of cubes for one or two of the friends. If students self-corrected with a small amount of scaffolding, we counted their answers as correct. If we had to guide them all the way toward a correct answer, we counted their solution as incorrect. If they did not need or want guidance, but came up with an answer that was incorrect, we would ask them to retell the story and/or explain their thinking to see if they could self-correct. When students self-corrected, we counted the answer as correct.

In addition to the varying amounts of scaffolding given to students, sometimes the numbers in the problems changed. The reasons for changing the numbers varied. If a student was having difficulty understanding or working with a particular set of numbers, we gave the student smaller numbers. If the student was solving problems quickly, we

decided to give them more challenge so made the numbers larger. Sometimes, the first students interviewed would consistently have trouble with a number pair, and so for the rest of the interviews we would make the numbers smaller. This was the case of the Partitive Division (24, 6) in November of 1st grade, as seen in the table above, where we eventually used four different pairs of numbers because the problem proved challenging for the students. The research team always debriefed after each interview, and the numbers could change based on these conversations. This happened in the case of the compare problem in November of 1st grade. Our original numbers were (21, 13), but we changed them to (16, 10) to test students' base ten thinking.

Quantitative analysis of the kindergarten and 1st grade data.

Quantitative analysis on student interview data have been done since the study began in kindergarten (see Appendix C) as a way to explore trends in students' problem solving. In kindergarten, we wanted to know how many valid strategies students used and how many of the valid strategies were advanced strategies according to the Carpenter et al. (1999) definitions. We also wanted to know how many problems each student solved correctly. For each problem on the kindergarten protocol, we wanted to know what percentage of students in the class approached the problem with a valid strategy and what percentage of students solved the problem correctly. For the two interviews in 1st grade (see Appendices D and E), I calculated the above percentages for each student and in addition I calculated students' percentages of advanced strategies. I did not create similar files for the first two 2nd grade interviews, but rather incorporated all the 2nd grade results into the tables created for the longitudinal analysis.

Problem solving strategy analysis for 1st grade.

Throughout the longitudinal study, there has been attention to students' uses of advanced strategies. These strategies, as described in the theoretical framework, are defined when students move away from directly modeling and manipulating objects in the problem and begin to manipulate the sequence of numbers as objects to find solutions. Because the use of advanced strategies indicates a deepening sense of number and conceptual development according to Carpenter et al. (1994), the use of strategies beyond direct modeling should be an important marker of mathematical thinking. To explore students' transformation from direct modeling to advanced strategies in 1st grade, I created the table below that compared the November and May interviews for each student with percentages of advanced strategies and correct answers.

Table 2. Comparison of Advanced Strategies to Correct Answers – 1st Grade

Student	Advanced Strategies Nov. 2006	Correct Answers Nov. 2006	Advanced Strategies May 2007	Correct Answers May 2007
Brisa	0%	92%	33%	89%
Ana	0%	86%	22%	96%
Yolanda	71%	71%	75%	92%
Omar	64%	86%	81%	86%
Jenna	0%	63%	36%	86%
Gina	36%	93%	43%	100%
Gerardo	31%	81%	44%	90%
Averages	29%	82%	48%	91%

Seeing movement of all students toward the advanced strategies, I designed the protocols used in the 2nd grade interviews to push students toward the advanced strategies. I selected larger numbers that would be less convenient to directly model and number combinations that would facilitate skip counting and base ten thinking.

Although the research team chose to analyze the student interview data from kindergarten and 1st grade quantitatively, it is important to note that these numbers do not represent clear-cut distinctions in student strategies in every case. At times, students clearly made a direct model of a problem or clearly applied an advanced strategy. However, frequently students combined direct modeling and advanced strategies. For example, they would model each object, but use a counting strategy to find the answer. Or they would find the answer first with a counting strategy and then make a direct model as they explained their thinking. Several of the students drew a direct model in the 1st grade interviews in an attempt to do their very best work for our interview sessions, in my opinion. They were trying to give us what they believed we wanted. Whether they actually needed the model to help them solve the problem was a judgment call we had to make during the analysis. In many cases, classifying a strategy required a degree of judgment.

Longitudinal Data Analysis

The analysis of students' mathematical thinking over three years of the study focuses on the CGI strategies they use to solve problems, the tools and aids they choose to help them find their solutions, and the way they explain their thinking and problem solving. I began my analysis with a careful examination of all the data, followed by a narrowing of the focus to a subset of students and problems. Once I made these decisions, I classified students' problem solving strategies using the CGI framework. With this initial classification of strategies in place, I used a refinement of coding methodology to uncover themes. I began with open coding then axial and finally selective coding on student strategies, their explanations, and the tools they used to help them

solve the problems (Creswell, 1998). My constant reflection on my own thoughts, developing ideas, and conversations with other researchers over the course of a year helped me uncover these themes, and also kept my focus open to alternative perspectives. Below is a description of the stages I followed for analysis.

Stage 1: Choosing a subset of students.

The first and most lengthy stage of data analysis for this longitudinal study was the important task of narrowing down an extensive body of interview data to a subset of the eight students and a subset of the problem types. With six interviews over three years, I was left with approximately 50 problems for each student resulting in a possible 400 problems. Even though Dolores did not participate in each interview, I still had many different examples of students solving CGI problems. My first task was to examine each interview for each student and make notes that highlighted interesting approaches and language, indicating problems I would like to analyze further. At this point in the analysis, I made the careful decision to choose as my subset of students those who were engaged and successful CGI problem solvers and expressed a positive attitude toward mathematics. These students, I felt, would give me the clearest picture of their mathematical thinking and promote the equity issue in this dissertation.

Using the above criteria, I was able to eliminate Dolores from this study because of her general struggle with mathematics. While Brisa was an excellent student with strong language skills, her ability to comprehend number relationships was still emerging. Even in 2nd grade she was not able to solve Compare type problems and so she was eliminated. Jenna also struggled with Compare problems until the final 2nd grade interview where she appeared to make a shift in comprehension and mathematical

development. Ana and Jenna were successful and engaged problem solvers and I very much wanted to include them in the final analysis, but in the end rejected them for one specific reason. They seemed to be students concerned with solving problems the “right” way, i.e. the teacher’s way. Their teachers evaluated their problem solving in a similar manner confirming my suspicion. Finally, I chose Yolanda and Gina for their apparent focus on approaching the problems with little regard to a “correct” method. As luck would have it, both Gerardo and Omar, the only boys, also had the ability to think outside the box. This gave me four students for my final analysis.

Stage 2: Choosing a subset of problems.

More or less simultaneously with note taking about students, I kept notes on interesting problems and I knew that I could not possibly analyze all problem types. Looking at Table 1 of the interview protocols from kindergarten through the end of 2nd grade, I realized that I had the most consistent data on Compare, Multiplication, Part-Part-Whole, and Join Start Unknown problems. Choosing these four problems would give me a balance between action problems and relationship problems. Compare and Part-Part-Whole problems involve relationships and Multiplication and Join problems involve actions. Because Join Change Unknown problems are part of the Join class, I decided to include these as well. After carefully looking at the interviews at the end of 2nd grade, the Partitive Division problem reflected the most striking differences in student approaches. For this reason I decided that it would be valuable to include as a cross-student comparison in the analysis.

Throughout stages 1 and 2, I knew that I had to keep CGI strategies uppermost in my mind as the starting point for analysis for each of the four students and the five

different problem classes. As I prepared to analyze each student's thinking, I returned to the CGI literature and created a table of strategies for the problem types that I chose. This became my point of reference as I started to examine the data in depth. I built Table 3, which appears in Chapter 5, from the information in Carpenter et al. (1999). This table lists the common direct modeling and advanced strategies for each of the problem types, and also includes my general comments about each problem.

Stage 3: Transcribing interview data.

My next step as I moved closer to analyzing student thinking was to transcribe each of the five problem classes for all six interviews for each of the four students. Complete transcriptions were possible for all but one of the interviews, May of 1st grade. Unfortunately, we had audio problems with our camera and only recorded the first twenty minutes of sound for each student. However, we kept careful notes during the interviews about students' strategies and choices of tools, and these notes included some direct student quotations. In spite of this problem, I was still left with a large amount of data. Counting five interviews, four students, and five problems, I transcribed approximately 100 problems. To get a better feel of the language in student explanations, I created a students' explanation file (see Appendix G) where I listed the best explanations for each student from all the interviews.

Stage 4: Profiling students.

My next step in the analysis was to examine how each student solved each problem class over time. To do this I organized tables for each problem class with the following headings: Grade and time of interview, Questions and clarifications, Strategies

and answers, Student explanations, and My comments. At the end of this process I had five tables for each student, or twenty tables in all (see Appendix I for examples of these tables). Each table ended with a summary of tendencies I saw in each student's approach to the problem class. Since these tables were only focused on one student at a time, I also wanted to compare students by problem class so I created a Longitudinal Tendencies Table (see Appendix J), where for each problem class, I listed side-by-side the summaries I had written for each student for that particular problem.

To explicitly write about each student's approach to each problem class over three years was still too much information for this longitudinal study. Sylvia and I decided the readers would become lost in the details of students' problem solving. Instead, I decided to create a profile for each student where I would summarize their problem solving over time only including the highlights. As I developed the individual student profiles, the analysis for each student came together in the following way. Before a written summary of each problem class, I created a table for that student and problem to guide my writing. These tables are presented in Appendix K with the following headings, where the strategy column specifically lists the CGI strategy that students used, based on the strategies table above: Interview (grade & time), Problem description, and Student's strategy & Result.

The last row in each table in Appendix K lists the number of problems, the number of problems directly modeled and the number of problems solved with advanced strategies, the number of problems correct and the number incorrect. I counted a strategy as direct modeling if the child represented each object, whether by drawing, using cubes, tally marks, or base ten blocks where each section of a ten rod represents an object. If the child used a number line or the 100s chart I counted this as an advanced strategy because

he or she was working with the number sequence itself. The written summaries of students' problem solving profiles are contained in Chapter 5. Concise lists of student problem solving tendencies for all five problems including any patterns seen in their use of CGI defined strategies are in Appendix L. These tendencies are not the same as the CGI problem solving strategies. They include the preferred strategies, and also include personality characteristics, language usage, and the mathematical disposition of each child.

Stage 5: Comparing and contrasting students.

The final analysis stage compared students' problem solving strategies and the overall tendencies that were uncovered in their profiles. My attention returned to the use of CGI strategies, as explained in Chapter 6, and included the tools students chose to help them solve problems, and the way they explained their thinking about how they reached their answers. At this stage, themes are consolidated and claims are made about how each child was making sense of the CGI problems. To support these claims, two additional problems are used as examples that contain large sections of transcription data. The first problem is taken from kindergarten and is the multiplication type where the context is three bags of marbles with six marbles in each bag. The second problem is the partitive division problem from the end of 2nd grade where the problem is to share 84 pencils equally among four children. For both of these problems, a comparison is made in how the four students solved the problem, their strategies, their tools, their insights and challenges, and their explanations of their mathematical thinking.

Reflexivity During Data Collection and Analysis

This longitudinal study gave me three years of experience with the students and the opportunity to get to know them well, both individually during the CGI interviews and as students in their classrooms. From kindergarten through 2nd grade, I was involved in CEMELA related research that took me into their classrooms on a weekly basis. I was involved in professional development around CGI problem solving with all of their teachers. The professional development research team, including teachers, met on a weekly basis to do problem solving during mathematics lessons and then to have conversations about students' thinking. This work gave me many opportunities to observe the research participants in the classroom setting.

Outside the classroom, I have had extensive conversations with the CEMELA researcher team who supported me in this project as we refined our thinking about the CGI framework, talked about how the students were responding to the problem types, discussed how these particular research participants were making sense of the problems in the interviews, and how we should modify the problems as the students matured. My extensive contact with these students and the many opportunities I have had to discuss their thinking has given me a unique and rich perspective on which to build my analysis and lends credibility to my findings.

Researcher Positionality

My interest in working with Mexican immigrant students began with my involvement in the above kindergarten CGI research project during the 2005-2006 school year. As I observed students' mathematical development throughout the year and helped videotape individual student's CGI problem solving interviews, I was impressed at the

progress students were making in mathematics and the sophisticated ways they approached the problems to find solutions. Their mathematical performance was on par with their middle-class, English-speaking peers (Carpenter et al., 1993; Turner et al., 2008) and the language they used to talk about their thinking was clear and insightful. These students presented a different picture of Latino achievement in mathematics than found in national statistics (NAEP, 2005). Even though most of these students came from lower socio-economic backgrounds and all were members of families with linguistic and cultural models different from those of the white, English-speaking, U. S. middle-class, it was clear that these students had the conceptual and linguistic foundations they needed to be successful mathematics learners.

I acknowledge that I am an outsider to the social, cultural and linguistic community of the students. I have been an elementary teacher who has focused on mathematics and has experience in the classroom with students from grades one to six, but I have never taught in a bilingual classroom. Further, I am a Spanish language learner who is dominant in English, my native language. However, I have worked with these student participants since the beginning of their kindergarten year, have met their parents on numerous occasions, and have been involved in all the interviews with the students through this time period. While my Spanish is not equivalent to my English, I am able to communicate with the students and ask probing questions about their thinking in the Spanish of the classroom. Furthermore, I have developed a warm rapport with these students, have learned much Spanish from them, and feel they are comfortable with my level of Spanish communication.

Trustworthiness

The trustworthiness of this study is enhanced by my prolonged engagement at the research site, as explained above, the experience I have had in designing and conducting the CGI interviews, and the member checking I conducted with other members of the research team (Erlandson, Harris, Skipper, & Allen, 1993). In addition to myself, this team has included three university researchers, Dr. Sylvia-Celedón Pattichis, Dr. Sandra Musanti and Dr. Erin Turner. Erin participated in the kindergarten year of the study and Sandra began her involvement the summer before the participants entered 1st grade. Sandra continued with the project through the 2nd grade interviews and took the lead in the CGI professional development efforts with the teachers. Sylvia, the chair of my dissertation committee, has been involved at all stages of this research.

Triangulation of Data

Triangulation of data comes from comparison of interview data beginning with kindergarten then 1st and 2nd grade data where students solved the same problem types with increasingly larger numbers. There is sufficient quantity of data, as appropriate for a grounded theory methodology, to approach the point of saturation.

Peer Debriefing

As a member of the CEMELA research team that supports the implementation of CGI problem solving at La Joya Elementary, I have been involved in extensive conversations with teachers and fellow researchers where the mathematical thinking of the research participants has been part of the broader discussions. Further, either Sandra or Sylvia videotaped, took field notes, or conducted interviews during the 1st grade year.

They also videotaped and/or took notes during all of the interviews I conducted in 2nd grade. As mentioned previously, after each interview we always debriefed about the participants' thinking and made decisions together about how to proceed for the next interviews.

The most important area of trustworthiness in this study falls in the area of language. I am a second language Spanish speaker, but have conducted interviews with the students in Spanish and have analyzed their explanations in Spanish to uncover their thinking. My two Spanish-speaking research colleagues have been invaluable in providing language support at all phases of the research. They have clarified my Spanish for the students when necessary during the interviews and they were also able to clarify student responses when I was unsure what the students were saying. They have provided valuable input and feedback on all stages of analysis for this dissertation study, especially checking my language transcriptions.

Transferability

Since this is a qualitative exploration of four students' mathematical thinking, the trends and conclusions from the analysis are not generally transferable to other contexts. However, since these young Latina/o research participants have successfully solved challenging CGI mathematics problems in the primary grades, there are implications for problem solving success in mathematics with diverse populations. There are also implications for teaching and learning with Latina/o students in bilingual classrooms. Findings from this study will contribute to the knowledge of teaching and learning mathematical concepts through CGI problem solving in primary grade bilingual settings

and will further the literature on how young children make sense of mathematical word problems.

Limitations of the Study

Thinking is a complex process and even my own thinking as an adult researcher is at times difficult to explain. This exploration of students' mathematical thinking through CGI problem solving attempts to get at their thinking, but recognizes that what the children said, the strategies they used to solve problems, and the tools they chose only provide me with a window into their thinking processes. It is possible that since students were enculturated into certain mathematical practices of their classroom, that they believed they must solve problems in a certain way, or express their thinking in a way that matched the culture of the classroom or preferences of their teacher, rather than risk doing problems in a way that made more sense to them. The research team had conversations with participating teachers about students who attempted to solve problems in the teacher's way rather than their own way. Students' interpretation of adult expectations must be kept in mind by any study that attempts to explore their conceptual understanding through problem solving.

Further limitations of this study come from my unfamiliarity with the culture of my research participants. Although I have some familiarity with the community in which the school is located, I am an outsider to that culture and do not know the unfolding dynamics of life within the community. I do not know the patterns of regular discourse within families that have played such a vital role in students' intellectual and linguistic development (John-Stein & Mahn, 1996; Mahn, 2009). These limitations have been

addressed throughout the study by regular conversations with the students' teachers and the other researchers.

CHAPTER 4. Student Language Profiles

In the background we hear the librarian reading a story, “The sky is falling. The sky is falling.”

“I know that story,” says Gina.

“¿Te gusta la historia? (Do you like the story?)” I ask.

“La maestra la lee en la clase... (The teacher reads it in the class...)”

Gina begins.

“Sobre el cielo, (About the sky,)” I continue, referring to the story about Chicken Little.

“Pues, [la lee] El Pollito Chiquito en español, pero ella la dice en inglés allí, (Well, [the teacher reads] Chicken Little in Spanish, but she [the librarian] says it in English there,)” comments Gina, knowing the story from the classroom in Spanish, but now recognizing the same story in English.

(Gina and Mary, April 2008)

Introduction

An important part of the methodology in this study was to base the interviews in students' native language, Spanish, but let the students choose at every step of the way the language they preferred both for the interview questions and their own explanations. These students were learning mathematics in the classroom in Spanish, and the decision of the research team to give them the freedom to access both Spanish and English for problem solving freed them to make sense of the mathematics without challenges to their comprehension (Cummins, 2001; Mahn, 2009; Secada & De La Cruz, 1996; Trueba, 1999). This study underscores the importance of children having access to their native language for learning conceptually demanding subjects such as mathematics (Cummins, 1981; Thomas & Collier, 2002).

The theory of underlying proficiency mentioned in Chapter 2 states that concepts are not tied to a specific language, but can easily be transferred to another language when students have developed sufficient skills in that other language (Baker, 2006; Cummins,

1981). The introductory exchange with Gina above gives strong evidence for the validity of this theory. Gina learned the story of Chicken Little in Spanish, but she recognized the same story in English because she had reached a level of English usage that allowed her to access that knowledge. The same is true for the mathematics concepts explored in this study. When the children entered kindergarten, they began building their mathematical conceptual knowledge mediated by the Spanish language. As some of them developed proficiency in English, they were able to mediate their expanding knowledge in English as well as Spanish. For the children who had reached a threshold of comprehension in English (Cummins, 1982), their underlying system of mathematical meaning did not change. The complex system of systems of meaning (Mahn, 2009; Vygotsky, 1987) that influenced both their verbal and mathematical thinking expanded to include their developing system of meaning in L2 (the second language), English, now interacting with an existing system of meaning in L1 (the first language), Spanish. L2 joined with L1 to inform students' verbal thinking, which in turn integrated with their mathematical thinking to help them make sense of the problems. In my opinion, this is what it means to have the power of two languages to make sense of learning.

In this chapter, I profile the children and give their thoughts about Spanish and English in their lives and how they think about one language or the other as it relates to their learning. My purpose for this chapter is to give the reader a broad understanding of these children as individuals and to show in students' own words how they believe the two languages interact with their learning. We start to see the *perezhivanie* of each child, or how she or he perceives, experiences and internalizes the social environments of the home and the classroom (Mahn, 2009; Vygotsky, 1987). The following narratives are

based on the conversations Sandra, Sylvia and I had with the students following the final CGI interviews in April of 2nd grade. It is important to note that the quotations of student language in this chapter and throughout the rest of the dissertation are verbatim, including any grammatical mistakes students made in Spanish or English. Sandra and/or Sylvia, both native Spanish speakers, were present during all these conversations to assist in the questioning and validate the exchanges.

All four of the students in this study are members of Spanish-speaking, Mexican immigrant families. They have lived the majority of their lives in the urban, working class and poor neighborhoods that feed into La Joya Elementary where most of the neighbors also speak Spanish. I begin the narratives with Gina who was the most comfortable of the four students in discussing her use of two languages. Next, I present Gerardo who explains why he prefers to use English and why he is uncomfortable using Spanish. Yolanda's story follows. She is Spanish dominant and explains why she needs to learn in Spanish. She is even more uncomfortable than Gerardo with the language questions. Finally, in contrast to Yolanda, I present Omar who is English dominant. He tries to avoid the questions about language altogether. For three of the four children, it will become clear in what they say that the topic of language preference is difficult for them to discuss. Social implications of language use and reminders of their minority status surround these children (Delpit, 1988). While the politics of English dominance in the U.S. is beyond the scope of this study, it is important to understand that there is complexity in the children's language choices.

Gina

Gina did not really warm up to me until 2nd grade. Maybe this had something to do with her ability to use English, or maybe it was a matter of maturity. For whatever reason, something clicked in 2nd grade and our relationships changed from more formal and standoffish to warm and playful. At this same time she went from hesitant use of English to near fluent-sounding ability. She could even joke around in English and use popular expressions and mannerisms common among young people. For example, once when we were talking about something that surprised her she said, “Didn’t see that coming!”

She is a tall, slender girl with long curly, dark hair. Her mother works at La Joya as an educational assistant and the two are like twins in appearance. To see them together is to see two attractive girls with the same abundance of curly hair and bright smiles. I know Gina has an older brother who is a successful high school student, that the two children live with both parents and that there is extended family in the area.

Gina is a mature and precocious student in every subject area. She told me during one of the interviews that her 2nd grade teacher recommended her for the gifted program. She can be a little impatient in the classroom with activities that seem too simple for her, and also with other students who she might feel are a little less intelligent than she is. She likes to work slowly and methodically by herself to reach a problem solution. She is not the gregarious “teacher helper” type, although she is agreeable and respectful.

Gina’s playful nature came out when we interviewed her about her language choices. Initially, she feigned reluctance, wondering why I would want to know about her two languages. Then she decided to have fun with the questions. For example, when I

asked her when she used English she said, “Oooh, that’s a hard question. Actually, no. I use English mostly, when I’m singing,” and she laughed because this is not really true. She admitted that she used English when she was teaching it to her six-year-old cousin. From previous conversations I have had with her, I also know that her brother uses English. Her close relationship with him may be the reason that her own English is so good.

When I asked her when she used Spanish in her life, she replied “Spanish? In my home...to get food,” and she laughed at her own joke. Then she added more seriously that she used Spanish to talk to her parents because they do not speak much English. She realized that the conversations at home moved from Spanish to a combination of languages when she said, “Spanish...spinglish,” and laughed at her made up word. Elaborating she explained, “But I like to talk to them in, in...Spanish. Because my dad talk really weird in English.” I asked her if she could give me an example of what she considered really weird in her dad’s use of English. “O.K.,” she explained, “cuz, he talk like this, and he reads like this too. Can I read, can I have something, that,” pointing to my paper, “to read? O.K. He talks like this: yooouuu haaaad...” she dragged out the words, “and he makes me read.”

I wanted to know if he read like that in Spanish as well, and Gina said, “Both, cuz he had to learn by himself.” This simple statement gave me a window into the world of her family. Like many others families in the La Joya neighborhood, they probably came from a rural community in northern Mexico where access to education was limited. Her father took the initiative to teach himself to read, although I have no idea if he was a child or an adult. It is clear he is passing his value of reading on to Gina. With her mother

working at the school, her brother doing well in high school, and Gina headed for the gifted program, a portrait of a family from a humble background with high academic aspirations for their children emerges.

Next, our conversation moved onto language preferences in school. I asked her when she used English and when she used Spanish, but she could only say that it depended on if the activity was in English or Spanish. Sandra was present at the interview and persisted with the questions to see if she could get more details. Gina just said either language was the same for her and it depended on the context. We suspected that she was stronger in Spanish, but did not say anything. Sandra and I had noticed that when problems became a little more challenging, Gina always switched to Spanish, so we felt that she was still dominant in her native language.

After the discussion about languages in her home, I asked her about school and specifically about numbers and mathematics. I wanted to see how she thought about problem solving and if her verbal and/or mathematical thinking during problem solving were tied to one language or the other. When I asked about counting and language, she just looked at me with an expression of disbelief and said, “I just talk. In my mind I talk numbers.”

“But how do they sound?” asked Sandra.

Gina continued to look surprised and said emphatically as though Sandra and I were a little slow on the uptake, “Uh...numbers, only numbers, it doesn’t talk!”

Now I realized we were approaching Gina’s mathematical thinking apart from verbal thinking, at least a separation in her way of understanding, so I wanted to know if she could explain the symbol system she used for numeric thinking. She stood up as

though she was now the teacher and said, “O.K., let me see. Pretend it’s one hundred,” and she picked up the 100s chart. “Ok. I’m just going to give you an example here. Pretend this is my mind.” She waved her hand over the 100s chart. “And if I want to do it in here…” she pointed to her head. “And if I want to do it in here, it just has to be one, zero here,” she said as she pointed to the top right corner of the 100s chart above the number 10. “Pretend this is my mind. It’s white and black too.” She closed her eyes and held her hands to her head. “I’m like that too. O.K., but with my eyes open. First I think of the number that’s on the board,” she said shifting the conversation to another symbol for thinking. She went on to explain that her mind is like a blank piece of paper where she can write an algorithm and then solve it, exactly like she would do in the classroom.

Now, curious about the actual thinking process she was describing, I asked her, “Oh, so you can imagine putting the plus sign?”

“First I put the line,” she explained, drawing an imaginary line across her forehead, laughing. “Then I put the plus sign, then I put the numbers.” She pretended to write on her forehead with her finger. “O.K. Like five with five. It’s ten, so I put zero and a one,” as though she were writing the zero in the one place of the answer and carrying the ten. She did this on her forehead with her finger, keeping her eyes closed. “Yeah, like that. It’s how I do it.”

“And then the answer, you see it down below the line?”

“Yes. It just pops out right now.” She held her hands to her head making circles with her fingers, and then she exploded them saying, “Cheeww.”

Sandra tried to direct the conversation back to the idea of words in Gina’s mind, hoping that she would be able to tell us if she counted silently in Spanish or English.

Gina, however, continued to insist that it was not a matter of words and gave us a strong indication of what she believes is the separation between mathematical thinking and verbal thinking. She said, “Numbers. I just see numbers in my mind. My mind doesn’t talk to me.” On second thought, however, she decided that in fact her mind did talk to her at times. She has an active imagination and was really enjoying the conversation at this point. She knew we were asking her about her two languages and trying to get at how she thinks so she explained that her mind was divided into two parts, an English side and a Spanish side.

“Oh, so your mind is like in half? You see one side then the other side?” I asked.

“Yeah.” She held her arms out in front of herself, pretending to divide herself in half. “Ok. If I want it in Spanish,” she pointed to the left, “I go that way. If I want it in English, I go that way,” and she pointed to the right. “Now, right now, I’m talking in...English, so I go that way,” and she pointed to the right again.

“Now speak in Spanish,” I asked.

“¿Por qué? (*Why?*), she asked in Spanish.

“Porque quiero ver si haya una diferencia, si puedo ver una diferencia. (*Because I want to see if there is a difference, if I can see a difference.*)”

“Um, O.K.” She put her hands to her head and with an expression of concentration on her face said, “I can’t take off my skin.” Then she mimed taking off her skin and we laughed. She continued with her description, saying, “It’s just a red line right here,” and she drew an imaginary line down from her forehead with her fingers. “I can’t really see it.” As the conversation digressed into the subject of passwords, I realized this was more of Gina’s playful side coming forward and I suspected she wanted to hang

around with us rather than go back to the classroom. Our obligations to the teacher and our need to continue the interviews with the other students trumped our desire to chat with Gina so we ended the interview.

Gerardo

Gerardo is one of the youngest students in his class. His birthday is officially listed with the school as July of 2000, which means he was only one month into his fifth year when he began kindergarten in August of 2005. He was selected as one of the original eight pre-assessed students by his kindergarten teacher, Ms. Arenas, because of his “average” ability with numbers. What we discovered over the course of three years however, is a boy who is far from average with remarkable insight and creativity. An added bonus is that he loves to explain his thinking.

Gerardo is naturally charming with an infectious grin that changes to a broad smile with ease. He has always been friendly and outgoing in the classroom and during the interviews. He is enthusiastic about problem solving and willing to tackle difficult problems. He has been very successful at reaching correct solutions and has shown great perseverance. Even when he is having difficulty reaching a solution, or does not really understand a problem, he keeps explaining what he is doing or what he thinks he is doing with the same perseverance. It is as though eventually, with enough explanation and effort, the solution will become clear to him. Gerardo reminds me of Sfard’s (2001) theory of thinking as communicating. While he is talking, he is thinking.

Gerardo’s parents live apart and he has an older brother. His mother and Omar’s mother are sisters, making the two boys cousins. His choice of language for the CGI interviews has been English since 1st grade, and even in kindergarten I observed he was

using English socially with his friends. In our conversation about language choices, he revealed his preference for English and in an uncharacteristically serious tone, his insecurity with his own ability to talk in Spanish. In his body language as much as his exact words, he showed how uncomfortable he was with his own Spanish.

I began the conversation by asking him when he liked to use English and when he liked to use Spanish. He answered quickly, “I like to use English at full time.” Not sure what he meant by this, I asked him again and he said that he liked to use English all the time. His reason was simple, “Because that’s the thing that I know more,” he said. He explained that he uses English with his dad at home, but that he uses Spanish with his mother. I have had conversations with his mother in English about my research and she has expressed interest in reading the dissertation. I believe she may be a fairly balanced bilingual speaker. Gerardo’s older brother probably speaks English with him, so I suspect his home environment is tipped toward English usage. Gerardo indicated this as well when he spoke of his home language, saying, “I think English because he’s the one that always...cuz everybody in my family, everybody, me...everybody in my family could talk English.”

Next we talked about language in school. Like all the children in this study, Gerardo has been in bilingual classes where all new content material was introduced in Spanish. He said that the classroom language was mostly Spanish, but that he uses English with his friends. When I asked him about his preference, he began to be uncomfortable. “What do you like to hear? If someone...when do you like to hear someone speak English and when do you like to hear someone speak Spanish? Or do you?” He bounced his pencil on the table and avoided eye contact.

“Like, when?” he hedged, then said, “That’s a hard one,” and he repeated more loudly, “That’s a hard one.”

“Are there some things, some subjects you like to hear in English and some subjects in Spanish?” I asked.

“I don’t like Spanish,” he answered with a surprisingly negative tone. “Because. I don’t like talking in Spanish,” he said while looking down. “Because sometimes I say stuff wrong.” We asked him if he understood better in one language than the other, but he felt he understood equally well in both languages. However, we noticed during the interviews, especially at the end of 2nd grade when I made an effort to start the interview in Spanish, that his comprehension of the problem was better in English. Finally, I asked if he made a conscious choice to use one or the other language at different times, but he replied, “I just talk. I just say what I want to say...”

“So you don’t really think about what language you’re using?” I asked again. He shook his head. “If someone speaks to you in Spanish, do you sometimes speak in...?” I began.

“English?” he asks. “Yeah.”

“English. And in your mind, when you’re solving a problem, when you’re talking to yourself in your mind, what language do you use?”

He said with certainty, “English.” In this response I don’t know if he really *was* thinking exclusively in English or just *wanted* to think exclusively in English. For whatever reason, he was showing during this interview that he valued English over Spanish for his own usage.

Sandra wanted to pursue an earlier point about his discomfort with his mistakes in Spanish so she asked, “Gerardo, you said that sometimes you speak wrong in Spanish. So how, how do you know that you speak wrong?”

He looked down, seemed ashamed or embarrassed, and started bouncing the pencil between his fingers again. “Because, people laugh at me,” he said.

Sandra continued, “People laugh at you? Well, we think you speak really good. A language, and Spanish is my first language, so...” she began, but Gerardo jumped into the conversation again.

“English was my first language,” he said, once again giving English priority in his own usage. From conversations I have had with his kindergarten teacher and because his mother speaks to him in Spanish, I doubt that English was his first language, although he may have had simultaneous exposure to both languages. His family is from Mexico and they visit Mexico regularly, so Spanish plays a major role in his life.

When we asked him if he thought two languages were better than one, he agreed. But here again, he gave value to English over Spanish saying, “Because, every time I go to Mexico and I go to my friends...um I go to my cousin’s house, I go to his school sometimes, um, they always get really, they always get really happy that I know English, I know English...’ah, de verdad? (...ah, really?)’ They get really excited.”

“But you can talk to them in Spanish too, don’t you?” Sandra asked.

In his response we saw that value for English follows Gerardo across the border and reinforces his preference. He explained, “They tell me to talk to them in English, even though they don’t understand what I say.”

Yolanda

Yolanda is a dominant Spanish speaker who began to use more English during 3rd grade after the CGI interviews were concluded for this longitudinal study. She was born in July of 2000, making her one of the youngest students in her class, similar to Gerardo. She was not one of the original eight pre-assessed students in kindergarten. However, during the post-assessment interviews that year, it was clear she had a powerful way of thinking about numbers and her strategies set her apart from the rest of the students as she demonstrated a strength and confidence in her own abilities for problem solving.

Yolanda is a quiet and studious girl, highly motivated academically. Like all the children in this study, she is attractive with long, dark hair and a beautiful face. She arrives at school with her hair and clothing neat as a pin, ready to learn. Yolanda has an intensity and seriousness about her that is not present in the other students. Rarely does she joke around with us, although she is capable of light-hearted behavior. She is determined to be successful in school and spends long hours on homework with the support of her parents. She has an older brother who is an honor student and who was with the same kindergarten teacher as Yolanda. I have met Yolanda's parents and have a friendly relationship with her and her family.

Although I know Yolanda's family well, her Spanish has been the most difficult for me to understand. She speaks quietly and perhaps since they come from central Mexico, their dialect is more difficult for me than the Spanish spoken in northern Mexico. Fortunately, Yolanda enjoys the CGI interviews and works with me to help me understand what she is saying. She is less verbal than either Gerardo or Gina and explanations of her thinking tend to be concise and to the point. When I asked her about

English and Spanish in her life she showed a definite reluctance to talk about language. Possibly she was embarrassed about her ability to speak in English, which is still emerging even though she has lived all but three months of her life in the United States.

I began the questions in the same way as with all the students, wondering when she used Spanish and when she used English in her life. Yolanda responded in an uncharacteristically emotional manner, saying “No me gusta...hablando en inglés. Pues le entiendo poquito, pero en español sí le entiendo mucho. (*I don't like...talking in English. Well, I understand it a bit, but in Spanish I do understand it a lot.*)” When I asked if she ever used English with anyone outside of school, she said, “Con mi hermano, Memo. (*With my brother Memo.*)” I know her brother Guillermo speaks very good English and picked it up early. His parents said he had an English-speaking friend in 2nd grade and at that time developed the ability to use English comfortably in social settings.

Sandra wanted to get a little more specific information about this context for English usage so asked, “¿Hablas con tu hermano? (*Do you speak with your brother?*)”

Yolanda only nodded and said, “A veces. (*Sometimes.*)”

Sandra continued, “¿Cuándo, por ejemplo? (*When for example?*)” Yolanda did not respond. “¿Habla sobre la escuela o sobre la televisión? (*Do you talk about school or television?*)” Still there was no response from Yolanda. “¿No te acuerdas? (*You don't remember?*)” Yolanda shook her head. Either she would not or could not elaborate on any English usage outside the school.

I switched contexts to the school environment to see if I could get more response. “En la escuela, ¿cuándo usas inglés y cuando usas español? (*In the school, when do you use English and when do you use Spanish?*)”

“Cuando estoy en centro de ESL. (*When I am in the ESL center.*)” She continued, “Cuando la maestra está hablando en...(*When the teacher is talking in...*)” and her voice faded away. After a few seconds she continued, “O cuando está poniendo algo de ESL a toda la clase, tengo que hablar puro inglés. (*Or when she is giving something about ESL to the whole class, I have to speak all English.*)”

I decided to ask her about her preference, although I knew it was Spanish.

“¿Cuándo prefieres oír inglés y cuándo prefieres oír otra persona hablar en español? (*When do you prefer to hear English and when do you prefer to hear another person speak in Spanish?*)” She did not respond. “¿Hay, hay algún tiempo que prefieres oír otra persona hablar en inglés? (*Are there sometimes that you prefer to hear somebody talk in English?*)” She shook her head no, but I pressed on, “¿Qué piensas sobre tu habilidad hablar en inglés? (*What do you think about your ability to talk in English?*)” She just looked at me. “¿Hablas en inglés un poco? (*Do you talk in English a little?*)”

“A little,” she responded in English. I noticed she was starting to get more uncomfortable with these questions and was playing with the eraser on the end of her pencil, not making eye contact with me. Sandra then asked about when Yolanda heard English in the classroom and Yolanda returned to the subject of the teacher. “Ella casi a veces usa palabras en español y unas en inglés. Casi los estamos hablando todo el tiempo porque usa palabras en inglés y en español cuando ya se me hace eso, (*Many times she uses words in Spanish and in English. We are almost always talking [this way] because she uses words in English and Spanish it seems to me now,*)” she finally explained, referring to the almost continual use of both languages we observed her teacher use in the classroom.

Sandra asked her if the continual back and forth between the two languages helped her learn English. Yolanda demonstrated little enthusiasm about this method and said, “Poquito, me ayuda más el ESL...cuando estamos en La Joya. (*A little, ESL helps me more...when we are at school.*)” I asked if ESL time was conducted in both languages or just English. Yolanda hesitated then said, “Se me hace la mitad inglés y la mitad español. (*It seems to me half in English and half in Spanish.*)”

Wondering if she listened to both languages or just the Spanish, I asked, “Y ¿cuándo haya una mitad en inglés y una mitad en español, estás escuchando los dos o solo las palabras en español? (*And when there is half in English and half in Spanish, are you listening to the two or just the words in Spanish?*)”

“Los dos, (*The two,*)” she began, and then added, “pero, casi no lo escucho porque como está...ella solamente habla inglés cuando nos está diciendo algo en inglés...(*But, I almost don't listen because like it is...she only speaks English when she is telling us something in English.*)” Yolanda seemed to be saying she was still struggling to understand in English in 2nd grade. I know her kindergarten teacher used 10% English, and that her 1st grade teacher did not use much more than that and depended on subjects like P.E. to provide the English in students' day. It could be that for Yolanda as a serious academic, the only topics that really mattered were being said in Spanish.

Finally, I wanted to know what she thought helped a person learn English. What made a difference for her brother when he was learning? I asked, “¿Este forma te ayuda aprender inglés, o cuál es la...en tu opinión Yolanda, cuál es la manera mejor de aprender otro idioma? Para tí inglés, para mí, español. ¿Cuál es la manera para aprender mejor? Porque, Memo habla mucho inglés. ¿Qué pasó con Memo para aprender? (*This ways*

helps you learn English, or which is the...in your opinion Yolanda, which is the best way to learn another language? For you English, for me Spanish. Which is the way to learn best? Because, Memo speaks a lot of English. What happened with Memo to learn?) ”

She didn't know why her brother was able to learn English well and just considered it the way he was. For her part, she valued the help she got from her father who knew a little English. She said, “Es que Memo casi siempre sali. Yo tengo que, mi papá sabe poquito inglés, y mi papá me ayuda. Yo lo leo el ESL y mi papá, si tengo errores me los corrije. *(It is that Memo almost always was this way. I have to, my father knows a little English, and my father helps me. I read it in ESL and my father, if I have mistakes he corrects me.)*”

Yolanda had nothing more to say about language even though we asked her if she knew a song or liked to watch T.V. in English. Her attachment to Spanish was in sharp contrast to Gerardo. Whereas Gerardo valued English over Spanish and this preference was continually reinforced through his experiences in both the U. S. and Mexico, Yolanda appeared to be struggling to maintain her native language. Although she is an assertive mathematics student and her academics overall are high, when it comes to English she is tentative. I have seen her venture into English usage only in the most secure occasions, say a few words, and then quickly retreat to Spanish.

Omar

Omar's school records indicate he was born in September of 1999, making him almost six when he entered kindergarten. He was one of the original pre-assessed students in kindergarten and was chosen by his teacher because of his high ability to work with numbers. Over the course of this study, he has always preferred to use English

in the CGI interviews and he was the only student interviewed in English in kindergarten. Omar and Gerardo are cousins so there is probably some overlap in where they use English in their lives.

The little I know of Omar's language history presents a picture of a student caught between Spanish and English. According to his kindergarten teacher, Ms. Arenas, he was in a special preschool program conducted exclusively in English for some type challenge, but I am not sure what that was. Although his first language was Spanish, the program contributed to his English dominance, again according to Ms. Arenas. He has consistently shown ability for mental mathematics, but has difficulty expressing his thoughts verbally and in writing.

Without a doubt Omar was the most difficult child to interview. This was true for every interview and every year, although by 2nd grade his focus had greatly improved. In kindergarten and 1st grade he would continually get out of his chair, be distracted by the tools, try to avoid the questions and control the interview with his own questions. On the other hand, he was sweet and loving and there was nothing he liked more than coming up to all members the research team to give us hugs when we visited his classroom. He cares about us deeply and the feeling is mutual. Even though there were times in 1st grade when we all seriously considered dropping him from the study, his incredible ability in mathematical thinking and his problem solving strategies changed our minds. By 2nd grade, we could see how much he enjoyed solving challenging mathematics problems in his head.

Omar is ambivalent about the roles of Spanish and English in his life and the topic of language in general is not comfortable for him. He is much more at home in the world

of numbers and mathematics. Academically, language is not an area of strength for him and over the years he has consistently had difficulty putting his thoughts into words.

When I asked him what he preferred to speak, he said English, but then could not answer why. When I asked him if he had the opportunity to speak Spanish outside of school and with whom, he hedged, saying, “Um, my mom, my dad, my mom, my dad, my brother. When? Uh, my cousin, my brother, my cousin, and...” He did add that his family prefers to speak Spanish.

“They like to speak Spanish?” I asked, and he nodded. “So do your mom or your dad speak to you in Spanish and then you answer back in Spanish? Or do you answer in English?”

“Both.”

“Or, what happens,” I persisted.

“I answer in English.”

“Uh huh. Do they usually start in English or in Spanish?”

“Spanish,” he said, and now I feel like we might be getting somewhere.

“Uh huh. Do you think you understand English and Spanish both the same?” He nodded. “And what about speaking? Do you think you can speak English and Spanish both the same?” He nodded again, but I knew this was not true both from our own experience in interviewing him and the anecdotal information from his teachers that he is English dominant. Even though he thinks he can speak both languages equally well, he said he prefers English. I asked him if he preferred one over the other and he started avoiding the subject again. He asked, “The one or the other, um, other?” with a note of

silliness in his expression. I asked him again if it was one or the other, and he simply answered first “other” and then “one.”

Moving on I asked, “What do you like to hear your teacher use? What do you understand better? Or do you understand both the same? When do you like to hear your teacher speak English? When do you like to hear your teacher use English?” He made a circle with the fingers of his right hand and put one of his left fingers through it. Not understanding the gesture I asked, “Is it a zero?” He moved his left finger to the side of the circle. “Aaaall the time,” he answered, stretching out the /a/ sound. When he liked to hear Spanish was, “At the end of the day. Cuz then I go home,” he said avoiding a direct answer.

Changing the context again I asked, “I’ve noticed that you’ve gone up to the board a lot and explained your thinking and shown the other students how to do problems. Do you like doing that?” He just nodded. “And when you do this, you use Spanish. Why do you choose Spanish?” I asked, remembering that during this current school year his teacher had encouraged him to explain his thinking in mathematics in front of the class and he seemed to enjoy sharing his mathematical thinking.

He answered, “Because, everyone doesn’t know English.”

“Oh, ah ha, and how do you feel about the Spanish that you use? How do you feel about yourself when you explain in Spanish?” I asked, glad that I finally got a straight answer.

But he was grabbing some base ten blocks and looking at them instead of me. “I don’t feel anything,” he responded.

Sandra decided to ask how he thought he learned English because he said his first language was Spanish. On further examination, however, he said that he couldn't remember a time when he didn't know English.

Changing the subject to the language of his thinking, I asked, "When you think about problems, or, do you ever think in Spanish? Do you sometimes think in Spanish and sometimes think in English, or what language do you think in?"

"Both," he responded, while absent-mindedly stacking cubes. We could get no more information than this out of him.

Changing the theme for the last time, we questioned Omar about the value of knowing two languages. He agreed there is value, but when asked about his preference of a classroom for next year he said he either wanted to learn Chinese or be in a bilingual English-Navajo classroom. We pointed out that this was not a choice for him, but when we gave him a chance to elaborate on the value of knowing two languages he remarked, "Cuz, um, then I could understand everybody...except Chinese people." Sticking to this theme of Chinese and succeeding in avoiding anymore discussion of Spanish learning, he concluded by saying, "If I, if I learn better," he pauses, "when I learn, um, Chinese, I'm going to call for, um for, to Chinese people, and um if I have enough money I could go to China and I could order Chinese food."

CHAPTER 5. Student Problem-Solving Profiles

The thesis of CGI is that children enter school with a great deal of informal or intuitive knowledge of mathematics that can serve as the basis for developing understanding of the mathematics of the primary school curriculum.

(Carpenter, et al., 1999, p. 4)

Introduction

This chapter continues the students' profiles, this time as mathematics problem solvers. In the following paragraphs, I present an overview of each student's approaches to solving the five different CGI problem types from kindergarten through 2nd grade and include analysis on their problem-solving strategies. The sections are arranged by student and begin with Join problems, followed by Compare problem types, Part-Part-Whole, Multiplication, and finally Partitive Division problems. I highlight each student's approaches in kindergarten, 1st and 2nd grades with the intention of introducing the reader to the tendencies each child showed toward CGI problem solving. These tendencies go beyond problem solving strategies to reveal the children's personalities and dispositions toward mathematics, in other words, their *perezhivanie* or how they are individually perceiving, experiencing and internalizing the formal, mathematics learning environment (Mahn, 2009; Vygotsky, 1987). I build on the narratives from the previous chapter to give a more complete picture of each student. Appendix L contains a detailed list of each student's problem solving tendencies.

My analysis of student thinking began by relating students' problem-solving strategies to the CGI Problem Types and Strategies Table 3 (see below). This table was created from information in Carpenter et al. (1999), based on CGI research, and describes

common strategies children use for each of the five problem types explored in this study. Strategies are divided into two classes in this table, direct modeling and advanced strategies. As will be further explained in Chapter 6, Carpenter et al. (1993, 1994, 1996, 1999) make an important distinction between these two classes, saying that the movement from direct modeling to counting and other advanced strategies is an important marker in children's mathematical development.

For each problem, I paid attention to when students made a direct model of the problem and when they did not, and used this distinction as the first stage in my analysis of their thinking. Recall that a direct model is when students have represented all the objects in the problem. In addition to the CGI strategies described in Table 3, the students in this study also used a direct modeling strategy I call "decomposition." In this approach, they created the whole amount under investigation then broke it into parts to find their answer. I describe this strategy in more detail when it occurs in the profiles. Complete tables of students' problem solving strategies for each problem over all three years are presented in Appendix K.

Table 3. CGI Problem Types and Strategies, (based on Carpenter et al., 1999)

Problem Types and Examples	Direct Modeling Strategies	Advanced Strategies**	Comments
Join Change Unknown Omar wants to buy a toy car that costs \$11. He only has \$7. How many more dollars does Omar need to buy the car?	Joining To A set of 7 objects is constructed. Objects are added to this set until there is a total of 11 objects. The answer is found by counting the number of objects added.	Counting On To A forward counting sequence starts from 7 and continues until 11 is reached. The answer is the number of counting words in the sequence.	<i>This is an action problem. Strong consistency in children's observed strategies.</i>
Join Start Unknown Daniela had some candies. Then her friend gave her 5 more and now she has 13. How many candies did Daniela have to start?	Trial and Error A set of objects is constructed. A set of 5 objects is added to the set, and the resulting set is counted. If the final count is 13, then the number of objects in the initial set is the answer. If not, a different initial set is tried.	Trial and Error	<i>This is an action problem, but it is hard to directly model because children do not know the starting value. Children were observed to use trial and error.</i>
Compare Gerardo has 12 pencils. His cousin Omar has 9 pencils. How many more pencils does Gerardo have than Omar.	Matching A set of 12 objects and a set of 9 objects are matched 1-to-1 until one set is used up. The answer is the number of unmatched objects remaining in the larger set.	(no common strategy)	<i>This is a relationship problem. A direct modeling strategy of matching was consistently observed, but no consistency was seen in counting strategies.</i>
Part-Part-Whole Gina has 10 balloons. Six of the balloons are blue and the rest are red. How many balloons are red?	(no common strategy)	Counting On To (as above) Counting Down A backward counting sequence is initiated at 10 and goes for six counts. The last number in the counting sequence is the answer.	<i>This is a relationship problem. Children employed a variety of ways to directly model, counting strategies showed more consistency when used.</i>
Multiplication Yolanda has three bags of marbles. There are seven marbles in each bag. How many marbles does Yolanda have altogether?	Grouping Make 3 groups with 7 objects in each group. Count all the objects to find the answer.	Skip Counting (in specific cases) If numbers in each group are easily skip counted, children will count by these and may keep track with their fingers, e.g. 5s 10s.	<i>This is an action problem. There is high consistency in how children directly model it, but counting strategies are observed when the size of the groups is convenient for children to skip count.</i>
Partitive Division David has 15 marbles. He wants to share them with 3 friends so that each friend gets the same amount. How many does he give to each friend?	Partitive Divide 15 objects into 3 groups with the same number of objects in each group. Count the objects in one group to find the answer.	(no common strategy)	<i>This is an action problem. Counting strategies are difficult because children do not know the size of the groups. They tend to use trial and error to figure out what to count by.</i>
** For multidigit problems children will invent strategies that decompose numbers into tens and ones, increment in steps to reach a five or a ten, and/or compensate by solving with an easier number and then adjust their answer when they have reached a solution.			

Student Profile: Omar

The following sections profile Omar's problem solving from kindergarten through 2nd grade on the five problem types used for analysis in this study. Omar is considered high in mathematics by his teachers, actively engages in problem solving, has the ability

to work with large numbers in his head, but he has difficulty explaining his thinking. Omar is English dominant and used almost no Spanish in the CGI interviews.

Join problems.

Join problems highlight Omar's ability in using mental counting strategies to find solutions to CGI problems. At the end of kindergarten he used a Counting On To strategy to find the change between 7 and 11. He looked up at the number line on the wall of the room, started at 7 and counted up to 11, getting the four counts as his answer. Omar showed in kindergarten that he was using the number line as an aid to problem solving (Ernest, 1985).

In 1st grade, Omar continued to count on to find solutions for Join problems and began to apply base ten thinking. He solved the change between 15 and 25 by Counting On from 15 to 25 by ones after setting up the problem in a drawing. Figure 3 shows how he went from 15 to 25 by drawing seven circles below the box that contained the target number and three more to the right and above to have a change of 10 altogether.

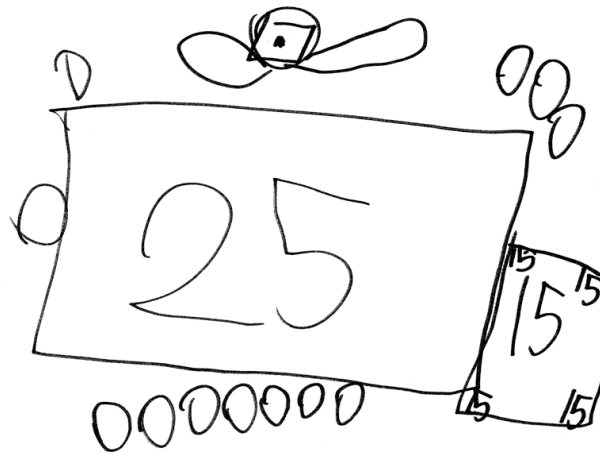


Figure 3. Omar, Join Change Unknown (15, 25), 1st grade.

Later in the year, Omar solved the change between 25 and 45 by Counting On To using two groups of ten. He easily solved the simpler Join Start Unknown problem in mid-year, but had more difficulty when the numbers increased. For the situation where he had an unknown number of candies and then a friend gave him three more resulting in five altogether, he recognized that $2 + 3 = 5$. He said the answer was two. Omar did not know how to apply a similar equation or a counting strategy to the larger numbers (5, 13) during the same interview, and had to use a trial and error drawing model. He used his model with a Joining To strategy to find the solution of eight. He drew five lines then counted up until he had 13 lines. By the end of 1st grade, Omar was comfortable with Join Start Unknown problems and applied equations to both sets of numbers we gave him. For (5, 15), he wrote “ $? + 5 = 15$ ” and said the answer was ten. For (10, 34), he used the same strategy and wrote, “ $? + 10 = 34$ ” and gave 24 as his answer.

In the 2nd grade Join Start Unknown problems, Omar described how he imagined solutions as number lines in his head. The problem for the first two interviews was to begin with an unknown quantity of candies, add 7 more candies to end with 22 candies and then tell how many candies you had to start. In September, he solved the problem by looking at the wall in the interview room where there was a sequence of numbers up to 20. He said, “Since it was twenty, it was up to twenty, I went in my imagination and I imagined that paper, there was two more papers.” Then he said, “I counted down...seven,” and he got 15 as his answer. This gives us explicit evidence that Omar is using a number line representation as a symbol for thinking (Mahn, 2009). For the same problem in February, he drew two simultaneous number lines on a piece of paper, one

going up from 1 to 7 as the other went down from 22 to 15. Figure 4 shows the two columns of numbers. Omar added the correct equation above after he found the answer.

Figure 4. Omar, Join Start Unknown (7, 22), 2nd grade.

He could not explain why this made sense to him, saying, “I just drew these and I got the answer.” Attaching an ordinal meaning to the number line representation in these examples and then using it for thinking (Ernest, 1985) indicates the possibility that Omar has incorporated the number line as a symbol into his system of meaning (Mahn, 2009).

At the end of 2nd grade, I gave Omar a very hard Join Start Unknown problem (340, 1012), which he requested and wanted to solve in his head. He was to begin with an unknown quantity, add 340 with a result of 1012. The question involved finding the beginning number. Omar’s thinking was transparent as he thought out loud to solve the problem, saying, “Um, I had seven hundred and...hm...I had...I had six hundred...six hundred sixty...um...sixty...sixty twelve, um, I mean, sixty, no, I mean seventy, two candies.” He appeared to have switched the addends, begin with 340, then Count On To to get 1012. He said seven hundred at first as if he added seven hundred to three hundred to get one thousand. Then he remembered that he would make another hundred with the tens and ones so he backed down to six hundred. Next, he took the forty in 340 and

Counted On To to the next hundred, which was 60, and then added 12 more to get 72. In his mind he had counted on from 340 to 1012, first by the hundreds and then by the tens and ones together to get his answer of 672. Omar used the sequencing of numbers to find the answer, decomposed numbers into place values, and possibly again used the number line as a tool for thinking (Mahn, 2009).

Compare problems.

Throughout all interviews on Compare problems, Omar never used a Matching direct modeling strategy, which is noted as a common beginning strategy in the CGI Strategies Table 3. In each case he abstracted the objects in the problem to sequences of numbers and used this sequencing to find his answers (Carpenter et al., 1999). In kindergarten, Omar used a Counting On To strategy and the number line running along the wall in the interview room to compare the numbers 12 and 9. He counted on from 9 to 12 and got 3 as an answer. He made sense of the comparison as a sequence of numbers and not as two sets by working with an ordinal meaning for the numbers (Fuson, 1988). Omar used this same Counting On To strategy in 1st grade to compare 29 and 31, getting an answer of 2 as he counted in his head from 29 to 31.

In the first two 2nd grade interviews, we made the difference between the two compared sets known and the quantity of one of the sets unknown. The problem was to find the quantity of the second set when the first set contains 13 objects and the second set contains 6 more than the first. Omar solved the problem by counting up in his mind by ones from 13 to 19, again using a Counting On To strategy. On the second interview with the same problem, Omar decomposed the 13 into tens and ones, added the three and six ones in his head to get nine then recomposed this answer with the ten to get 19. He

said, “three and three is six, but three and three and three is nine,” showing evidence that he decomposed the numbers into more than just tens and ones. He also decomposed the ones in a way that made sense to him as three groups of three (Mahn, 2009).

For the interview at the end of 2nd grade, I changed the unknown set in the comparison to the smaller number and worded the problem in such a way as to not make it an obvious subtraction problem. Omar was to find the number of markers his friend had if Omar himself had 53 markers and these 53 markers were 36 more than the number his friend had. Omar counted both forward and backward simultaneously in his mind between 53 and 36 to find his answer. He explained, “I knew, um, it was, two plus three is five [going up 20 by tens from 36 to 56], and he had thirty less, thirty six less, and I...I just guessed.” In this explanation he described how he thought about the tens as two plus three is five, but he was not able to describe how he removed an additional three numbers from 20, the difference between 56 and 53. Omar had difficulty explaining his thinking after the fact, but in his partial explanation we get a window into the ordinal meaning he is using to make sense of the problem (Fuson, 1988).

Part-Part-Whole problems.

Omar used counting and number sequences on the Part-Part-Whole problems, approaching the problems with similar strategies to those he used for Join and Compare problems. Omar’s fundamental approaches to problems did not change, even as the problem types themselves changed between actions and relationships. The Part-Part-Whole problem in kindergarten was one of the few problems that challenged Omar over the course of three years. He tried to apply a counting strategy to find the number of red balloons in a problem about ten balloons where six were blue and the rest red. Rather

than building a direct model, he started counting up on a number line from one, but suddenly said “ten” as his answer. He appeared not to comprehend the problem situation, even though Erin asked it in both English and Spanish.

By the time of the 1st grade interviews, Omar understood the part-part-whole context. He used one of his few direct models in November to solve Part-Part-Whole (7, 13) by drawing seven balloons then counting on from 7 to 13 using a Joining To direct modeling strategy as seen in the CGI Strategies Table 3 (See Omar’s tables in Appendix K for the problem wording). He solved a Part-Part-Whole (30, 20) at the end of the year by simply answering “ten”. In the same interview, he solved the (75, 50) balloon problem by writing “Q Q Q” for three quarters and then crossing off two, which left him with one Q, and 25 as his answer. Omar showed he was learning mathematics with understanding by flexibility applying a money representation to solve a problem about balloons (Hiebert & Carpenter, 1992).

In 2nd grade, Omar solved all the Part-Part-Whole problems in his head. His explanation for solving (12, 24) in September was that he did it in his head, but gave no details. In February he explained his Counting On To strategy, saying, “Cause, cause two plus two is four [the ones], and plus...and ten plus ten is, is twenty [the tens], and two plus two is four.” He decomposed the problem into tens and ones to Count On To from 12 to 24. For the final interview, I gave Omar a much harder problem (805, 565), after he requested larger numbers and was confident he could do the problem in his head. He used Counting On To mentally, getting first 440 and then 340, when the answer should have been 240. He appeared to have worked with the hundreds first in his mind going up 300 counts from 500 to 800 and then adding 40 to 65 to get 105 of the total 805 balloons.

His answer was 340 because he did not account for the 100 he gained when he added 40 to 65. When he did the problem a second way on paper using the borrowing algorithm, he saw that the answer was actually 240. He explained his mistake by referencing the borrowing in the hundreds place of the algorithm, but this did not shake his confidence in his preference for Counting On To mentally.

Multiplication problems.

Multiplication problems continued to highlight Omar’s strengths in counting, and he was only challenged once with this problem type over the three years. In kindergarten we see his preference for the number line as a tool to aid problem solving when he tackled the problem situation of three bags of marbles with six marbles in each bag. He looked at the number line on the wall and said, “I know. I know now how many marbles that is.” He continued to look at the wall, nodded his head as though he was counting silently and then wrote “ $3 \times 6 = 18$ ” on a piece of paper. He explained, “I umm, I umm, I just counted, six, six, six.” Because he kept getting up from his seat to look at the number line, Erin, the interviewer, decided to draw a section of the number line for him on paper. (See Figure 5 below.) This problem is explained more fully in the next chapter.

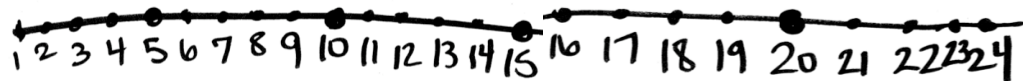


Figure 5. Erin’s number line for Omar, Multiplication (3, 6), kindergarten.

Omar made one of his few direct models of the Multiplication (4, 7) in November of 1st grade. Even though he modeled the problem, he tried to find the answer by skip counting the four groups of seven. His counting was off by one and he said 27 instead of

28 for his answer. He confidently stayed with his answer of 27 rather than examining his direct model and self-correcting. He had more confidence in his mental count than in his model. This is an example where a student who uses more advanced strategies has fallen back to direct modeling (Carpenter et al., 1999). However, in Omar's case he did not fall back successfully. I say more about this in Chapter 6. At the end of 1st grade, Omar skip counted mentally by fives to find the answer to the (7, 5) problem, confidently returning to a counting strategy and mental mathematics to find the total number of marbles in seven bags with five in each bag.

Omar showed flexibility in his multiplication problem solving in 2nd grade. At the beginning of the year for the problem with eight bags and five marbles in each bag, he combined skip counting by twos, fives and tens to find the answer. He used eight tiles, said that each tile represented five marbles, but then counted two tiles at a time by tens to get forty. For the final interview with 15 bags of marbles and 7 marbles in each bag, he transposed the equation saying, "you mean fifteen times seven?" instead of seven 15 times as reflected in the structure of the story. Then he used his knowledge of the clock to skip count by 15s seven times. He tried to do this mentally, but got confused after 60 minutes and used the 100s chart to find the correct answer. Omar creatively applied the cyclical sequence of time rather than the linear sequence of the number line to find his answer. The ability to apply a specific type of representation to make sense of an unrelated problem context shows that Omar is learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Partitive Division problems.

This problem type was the most difficult for Omar to solve over the course of the three years and he did not apply a counting strategy until the end of 1st grade (See Appendix K). Carpenter et al. (1999) note that Partitive Division problems do not lend themselves to counting strategies because the size of the groups is not known. In kindergarten, Omar drew a direct model and applied trial and error to divide 15 into three equal groups. Omar also used a direct modeling approach to divide 18 into three groups in mid-year of 1st grade. For this problem he drew 18 dots and tried to solve the problem putting five in each group. When he saw he had three dots left over for his trial of five, he knew he could put one more in each group to make it six. He said, “And then I figured out it was six cause it was even three more.” He had combined trial and error with a Partitive strategy to distribute the remaining three dots. At the end of 1st grade, he divided 12 into six groups easily and 24 into six groups with more difficulty by linearly repeating subtraction. Figure 6 shows how he wrote the series of subtractions, first for (12, 6) and later for (24, 6).

The image shows handwritten mathematical work. On the left, the numbers 18 and 12 are written vertically. To the right, there are three rows of subtraction problems:

- The first row shows a series of subtractions: -2 , -2 , -2 , -2 , -2 , -2 . Below this is the number 4.
- The second row shows a series of subtractions: -2 , -2 , -2 , -2 , -2 , -2 . This entire row is crossed out with a large diagonal line.
- The third row shows a series of subtractions: -4 , -4 , -4 , -4 , -4 , -4 .

Figure 6. Omar, Partitive Division (12, 6) and (24, 6), 1st grade.

He knew that six groups of two made 12, but he needed scaffolding to discover that he could take away four six times from 24 to solve the more difficult problem. The 18 and

the series of subtractions of three that are crossed out in the above figure are part of the scaffolding for the interview. Omar did not directly model any of these problems, but worked with the sequence of numbers from the larger number down to zero.

We did not ask Partitive Division problems for the first two 2nd grade interviews. The problem yielded very interesting results for the interview at the end of the year, which I explain later in Chapter 6.

Omar's problem solving summary.

In the profile above we see a strong tendency for Omar to use Counting On To strategies and the number line as a tool for counting and explaining (Ernest, 1985). It is possible that Omar had the number line representation internalized as a symbol for thinking when he manipulated large numbers in his mind (Mahn, 2009; Vygotsky, 1987). As he matured, he could decompose numbers, impressively manipulate the parts of large numbers in his head, and then recompose them to get an answer. Although he struggled to explain his thinking, we saw in his limited explanations and his thinking out loud during problem solving, that he was counting up and down sequentially to get most of his answers. Omar flexibly combined tools and representations to show that he is beginning to learn mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Omar is an impressive problem solver, but does not have ready access to his thinking and it is difficult for him to reflect on what he has done. He will stay with a wrong answer that he has calculated mentally rather than self-correct with a direct model. It is not easy for him to recognize when his thinking has been inaccurate. It is interesting that he cannot recall how he was thinking even as he does impressive mental calculations.

The only problem in three years where he did not have a valid strategy for the solution was the Part-Part-Whole (10, 6) balloon problem in kindergarten.

Student Profile: Yolanda

Next, I present a profile of Yolanda's problem solving over the three years.

Yolanda is a motivated mathematics student and actively engages in problem solving.

She shows a motivation to solve problems quickly, she uses mental calculations when she can, and she applied operations and algorithms on paper early in 1st grade as a strategy to find problem solutions. She has the ability to explain her thinking, and when she does, her explanations are concise and to the point. She is Spanish dominant and used no English in the CGI interviews.

Join problems.

Yolanda moved quickly from direct modeling of Join problems to apply counting strategies. In kindergarten, Yolanda had a unique solution to the change between 7 and 11. She directly modeled the problem, but not with a Joining To strategy as noted in the CGI Strategies Table 3. Instead, she used what I like to call a decomposition type strategy with her fingers. Students begin with all the objects represented in the model, and then they identify one part to reveal the other part as their answer. This is similar to a CGI strategy called Separating From where students solve subtraction problems by removing a set from the whole amount, but in this case a set is not removed. Students simply mark the division between the sets to find their answer. To solve this problem, Yolanda spread out her ten fingers on the table, placed a marker next to the little finger of one hand as though it were an 11th finger, kept seven fingers extended (the known) and

folded down the remaining three fingers next to the marker to find the answer of four (three fingers and the marker represent the change). She explained how she used the marker as an 11th finger, knew she had seven fingers extended, and then saw that four was her answer. By using the sequence of her fingers and one marker to model this problem instead of discrete cubes, Yolanda may have demonstrated a tendency toward ordinal rather than cardinal thinking about the numbers in the problem (Fuson, 1988).

In 1st grade, Yolanda used counting strategies and recalled facts for the Join Change Unknown problems. She said, “Cinco (*Five*),” for the change between 15 and 25 in the mid-year interview. Even though she began by counting out 15 cubes, she did not continue with the direct model to find the solution, but calculated the answer mentally. She was confident with her answer of five and did not choose to check it with a direct model. Although she made a mistake by saying 5 instead of 10, I think it is likely she recognized a relationship between 15 and 25. Here is another example of a student, like Omar above, who fell back to a direct modeling strategy (Carpenter et al., 1999), but the direct model did not help her successfully solve the problem. On a simpler comparison between 7 and 11 during this same interview, Yolanda used Counting On To from 7 to 11 four times. She said, “Lo hice y conté uno, dos, tres, cuatro. (*I did it and I counted one, two, three, four.*)” At the end of 1st grade, Yolanda solved the change between 25, 45 by skip counting on by fives four times to get 20. She explained, “Porque los conté no más como venticinco, treinta, treinta y cinco, cuarenta, cuarenta y cinco. (*Because I counted them, just like 25, 30, 35, 40, 45.*)” For these problems, Yolanda was making sense of the situation by attaching an ordinal meaning to the numbers and then counting (Fuson, 1988).

The Join Start Unknown problems in 1st grade were initially difficult for Yolanda, but by the end of the year she was comfortable with the problem structure. When Yolanda first heard the Join Start Unknown problem in mid-year of 1st grade, where she had an unknown number of candies, was given three more and then had five, she approached it flexibly thinking of the problem as both an addition and subtraction situation. She said two as the answer immediately and explained her thinking that two plus three are five. She explained further that if she took three away from five she would have two. When we gave her the same problem with larger numbers (5, 13) during the same interview, she got confused and needed scaffolding to find the correct answer. At the end of 1st grade, Yolanda wrote equations and filled in the first addend to find the answers. For Join Start Unknown (5, 15), she wrote “10 + 5 = 15.” For (10, 55) during the same interview, she wrote “45 + 10 = 55.” She appeared to have good comprehension of the problem structure by the end of 1st grade. Figure 7 shows how Yolanda wrote equations that matched the Start Unknown structure.

$$\boxed{10} + \textcircled{5} = 15$$

$$\overset{\triangle}{45} + 10 = 55$$

Figure 7. Yolanda, Join Start Unknown (5, 15) and (10, 55), 1st grade.

She showed she was learning mathematics with understanding by applying formal equations and a sense of base ten to match the structure of the problems (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

By 2nd grade, Yolanda was comfortable with Join Start Unknown problems. At the beginning of the year, she solved for the unknown starting number that adds to 7 to get 22 by lining up two rods and two cubes from the base ten blocks. She used a decomposition strategy to build the whole amount and then count what was left after she identified the seven sections part on her cube model. It is interesting to note that she used a discrete tool, the base ten blocks (Van Wageningen, Flora & Walker, 1976), in a sequential way, by lining up the rods and cubes. This may indicate that a linear model more closely matched how she was making sense of the problem (Mahn, 2009). At mid-year of 2nd grade, Yolanda quickly and confidently applied the standard borrowing algorithm to the same (7, 22) problem and subtracted 7 from 22. She used the same borrowing algorithm for the final Join Start Unknown (40, 112) problem, taking 40 away from 112 to reach the correct answer. Although I asked her if she could use base ten blocks to solve the problem initially, she expressed confidence in the algorithmic approach and was determined to find her answer first with an algorithm before using another tool. Yolanda's preference for the algorithmic approach may indicate she had incorporated this representation into her system of meaning for problem solving (Mahn, 2009).

Compare problems.

Yolanda first solved the compare problem in kindergarten with direct modeling, but on all later interviews she used counting strategies. She compared 9 and 12 in kindergarten by drawing 12 lines, drawing a vertical marker line after the 9th line and counting the rest. She once again applied a decomposition strategy to model the whole amount, identify the known part and then find the unknown part. She said, "Porque son tres más y puse una línea para que... porque éstos son de Gina y los demás míos."

(Because there are three more and I put a line so that...because these are Gina's and the rest are mine.)"

By 1st grade, Yolanda was no longer direct modeling compare problems. In November for a comparison of the number 10 and 16, she started counting up quickly on her fingers using Counting On To before the interviewer, Sandra, even finished the problem description, getting six for her answer. At the end of 1st grade I gave her two compare problems. The first was (13, 2). She responded with 11, again before I finished the question. When asked how she knew, she said, "La pensé en mi mente, le quité dos. (*I thought it in my mind, I took away two.*)" Here she mentally applied subtraction or counted backward to find the answer. Since she solved this problem so quickly, I increased the numbers to (31, 29). After clarifying the numbers, she quickly answered "dos (*two*)". She explained how she used a Counting On To strategy to go from 29 to 31, putting first one finger on the table and then another. Yolanda showed flexibility in her thinking as she counted down in one case and up in the other to find her answers (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996). She attached an ordinal meaning to the numbers in both problems (Fuson, 1988).

In 2nd grade, I changed the compare problems to have the difference between the sets known and one of the sets unknown. The compare problem in September asked for the quantity of the larger set when the first set was 13 and the second set 6 more than the first. When Yolanda heard the first part of the problem, once again she answered before I finished, saying "diecinueve (*nineteen*). " She had counted on with her fingers from 13 six times very quickly while listening to the problem. To explain why she used her fingers she said, "Porque no más cuento con mis dedos porque ya no, ya no puedo

marchar inside [sic] con la mente...Porque en las vacaciones no estudiaba mucho, y tengo que estudiar más. (*Because I just count with my fingers because now, now I can't function inside my mind...Because during the vacation I didn't study much, and I have to study more.*)” She showed her motivation in mathematics in this explanation and how she valued doing problems quickly in her mind, reflecting a mathematical disposition she may have developed in the classroom (McClain & Cobb, 2001).

During the rest of 2nd grade, Yolanda moved away from the reliance on her fingers as aids in mental math. At mid-year she did the same problem mentally, referring again to her mind saying, “Mi mente lo tenía grabado, y entonces, dije que trece más seis es igual a diecinueve. (*My mind had it recorded, and then, I said that 13 plus 6 is equal to 19.*)” For the end-of-the-year compare problem of 53 in one set where this set was 36 more than the second set, she chose to use an algorithm and subtracted to find the smaller set. She initially thought it was 53 and 33 and so subtracted tens quickly on paper to get 20. After she was told the first set was 36 more, she used the borrowing algorithm and answered 17. She showed her preference again for applying algorithms in this problem and did not have trouble comprehending the rather complicated situation that a person had 53 markers and this was 36 more than her friend. How many markers did her friend have?

Part-Part-Whole problems.

Yolanda applied part-part-whole modeling in kindergarten and 1st grade, but by the end of 1st grade she was using advanced strategies for Part-Part-Whole problems. In kindergarten, Yolanda showed the ability to decompose a number into parts to discover how many balloons are red if six out of ten balloons are blue. Once again using her

fingers, she spread out all ten fingers and said *cuatro (four)*. Looking at her fingers to explain, she said simply, “Porque seis son azules y cuatro son rojos. (*Because six are blue and four are red.*)” She made the whole amount, 10, with her fingers, then divided it into a set of six and the rest to get her answer. She used the sequence of fingers again instead of discrete cubes to model the problem.

In mid-year of 1st grade, Yolanda used a similar strategy to solve Part-Part-Whole (13, 7). She counted out 13 cubes then moved away 7 cubes and counted the rest to get 6. By the end of 1st grade she was able to use a recalled fact and base ten thinking to solve (30, 20). She said, “Porque veinte más diez es igual a treinta. (*Because 20 plus 10 is equal to 30.*)” When I gave her a harder pair of numbers, (75, 50) in the same interview, she initially began by counting up from 50 to 75 by twos using a Counting On To strategy. Figure 8 shows how Yolanda wrote the sequence of numbers, adding one to the end to go from 74 to 75, but she did not immediately arrive at 25 as an answer.

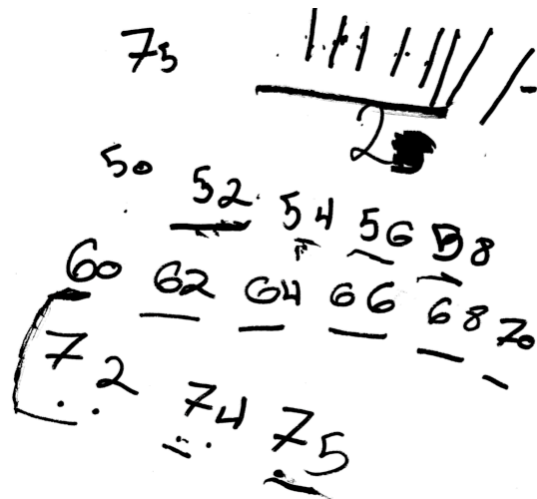


Figure 8. Yolanda, Part-Part-Whole (75, 50), 1st grade.

After writing the numbers, she thought of a better way so she switched to counting by fives to get the correct answer of 25. She was able to self-correct her counting strategy

with another way of counting. This is different from a case described below in the summary of Multiplication problems where Yolanda was not able to self-correct when she used direct modeling and may indicate that ordinal rather than cardinal numeric meanings facilitated Yolanda's problem solving (Fuson, 1988).

In 2nd grade, Yolanda showed she was learning mathematics with understanding by making connections between concepts (Hiebert & Carpenter, 1992). She was able to relate the Part-Part-Whole (24, 12) problem in February to what she knew about telling time and fractions to answer correctly. She used the idea of fractions in her explanation saying, "Y entonces...hay veinticuatro...yo, me dije hay veinticuatro horas, y la mitad de veinticuatro horas son doce y entonces lo hice con los globos, entonces dije que la mitad de veinticuatro son doce. (*And then...there are 24...I, I said to myself there are 24 hours, and half of 24 is 12 and then I did it with the balloons, then I said that half of 24 is 12.*)" In this explanation there is evidence of her verbal thinking when she says to herself that there are 24 hours in a day (Mahn, 2009; Vygotsky, 1987). For the final interview Part-Part-Whole problem of (65, 100), Yolanda chose the 100s chart and quickly found the answer by starting at 65, counting down three rows of ten to 95, then five ones to 100 to get 35. She applied a sequential tool, the 100s chart, to find the answer to a problem about two discrete sets of balloons, showing flexible understanding (Hiebert & Carpenter, 1992) and that she could shift between cardinal and ordinal meanings attached to numbers (Fuson, 1988).

Multiplication problems.

A tendency to apply counting strategies and the rewards and challenges Yolanda had with this approach is apparent in the Multiplication problem type over the three

years. For the kindergarten problem about three bags of marbles with six marbles in each bag, Yolanda knew the problem was three groups of six, but chose to transpose the three sixes into six threes. Then she imagined the six threes lined up vertically on a paper and counted these mentally to get 17. Erin encouraged her to draw a model, but this did not help her self-correct. Instead she adjusted the model of 18 lines by erasing one to match her mental calculation. This problem is explained in detail in Chapter 6. Yolanda had a similar challenge in November of 1st grade. She modeled four bags of marbles with seven in each bag using cubes and then counted the cubes to get 27. Similar to kindergarten, she had confidence in her original answer and the direct model did not help her self-correct. She chose to fall back to direct modeling (Carpenter et al., 1999), but like Omar, she did not fall back successfully.

At the end of 1st grade and beginning of 2nd grade, Yolanda showed her facility with skip counting by fives and tens and her developing base ten thinking. She explained her solution to seven bags of marbles with five in each bag by saying, “Porque le puse cinco, diez, quince, veinte, veinticinco, treinta, treinta y cinco, (*Because I put five, ten, fifteen, twenty, twenty five, thirty, thirty five.*)” as she tapped the table with alternating hands for each count. When given the additional multiplication problem of seven groups of ten and six more single marbles during the same interview, she applied base ten thinking to get 70 and then added the six ones, all done mentally and said, “No más le puse siete, siete y sabía que era setenta. Y luego le puse otro seis, y era setenta y seis. (*I just put seven, seven and I knew it was seventy. And then I put another six and it was seventy six.*)” The Multiplication (8, 5) problem in 2nd grade was trivial for her. She quickly counted up by fives eight times mentally to get 40 for an answer. By explaining

her thinking in terms of counting sequences instead of referring to the objects in the story, Yolanda showed that she had abstracted the numbers in the problem to a sequence (Carpenter et al., 1999) and had attached an ordinal meaning to them (Fuson, 1988).

Yolanda was the only student among the four in this study who found the right solution completely on her own for the final multiplication problem of 15 bags of marbles with seven in each bag. In her creative approach to solving this problem, she used a counting strategy with the 100s chart and created her own list of multiples of seven to keep track of the sum as she counted each new group. Figure 9 shows her work.

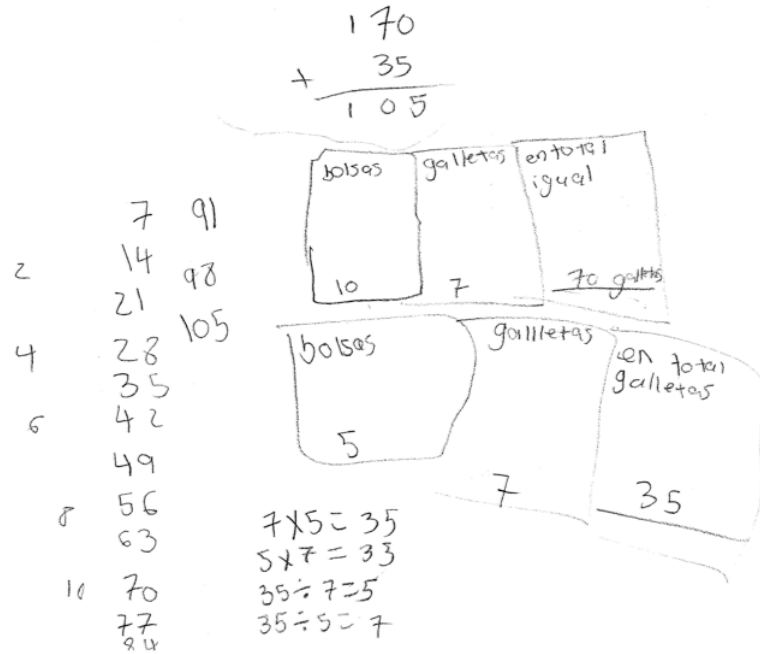


Figure 9. Yolanda, Multiplication (15, 7), 2nd grade.

As she was counting up on the 100s chart to get the multiples of seven, she made a column of these multiples on a piece of paper and kept track of how many multiples she had written counting these by twos in a second column until she had 14. Then she counted one more group of seven. She worked confidently and quickly, never flagging once she had her strategy in mind. She showed a high level of motivation and

engagement as she counted forward and organized the counting sequence. She was able to use the 100s chart as a guide and was also able to expand it when she needed to go to 105 on her last count of seven. Her strategy to solve this problem was to use three sequences of numbers, the numbers on the 100s chart, the multiples of seven, and the multiples of twos to keep track of the groups up to 15. The columns of numbers she created are found in the left hand part of Figure 9 above. Her drawings on the right reflect additional questions about this problem not analyzed in this study. Yolanda's creative approach showed she was learning mathematics with understanding (Secada & De La Cruz, 1996) and also making sense of the problem with an ordinal meaning for the numbers (Fuson, 1988).

Partitive Division problems.

Although Yolanda showed a tendency to move quickly to counting strategies for as many problems as she could, she was challenged to apply counting to Partitive Division and fell back to direct modeling (Carpenter et al., 1999). As mentioned earlier, there are no clear counting strategies for Partitive Division problems (Carpenter et al., 1999). Yolanda modeled the partitioning of 15 into three parts in kindergarten by drawing lines and using trial and error. In November of 1st grade, she partitioned 18 in half and explained that she knew the number fact of nine plus nine is 18. When we gave her the harder problem to partition 18 into three equal groups, she attempted trial and error with her fingers then changed to cubes. Once again she was not successful falling back to direct modeling and was unable to distinguish between how many groups, the known, and how many in each group, the unknown. This problem was unusually difficult for her and she solved it incorrectly like a measurement division with three in six groups.

As an interesting aside, she correctly solved the measurement division problem with the same numbers (18, 3) later in the interview. At the end of 1st grade she partitioned by 12 and 24 into six equal groups by drawing lines and using trial and error to reach correct solutions. How she solved the problem during the final interview is saved for Chapter 6.

Yolanda's problem solving summary.

Yolanda used counting strategies like Omar, but also used decomposition direct modeling strategies in kindergarten and the beginning of 1st grade. Her decomposition models were more often with fingers or drawing lines, instead of using discrete objects like cubes and may have indicated a move toward ordinal and sequential thinking (Fuson, 1988). She showed the ability to think of number problems flexibly from both the point of view of counting sequences and as parts of a whole. This flexibility showed she was learning mathematics with understanding (Hiebert & Carpenter, 1992). For the majority of problems she showed good problem comprehension and remembered the numbers well. She also applied addition and subtraction operations, algorithms and equations early to find problem solutions. In fact, frequently she used more than one operation in her solutions and explanations, showing additional mental flexibility and mathematical understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Overall, Yolanda seemed to have incorporated classroom mathematical norms described by Cobb and Yackel (1996) to solve problems quickly and efficiently by using advanced CGI strategies. She was a risk-taker with confidence in her own mental counting capabilities and liked to push herself toward more advanced strategies. She had less confidence in her direct models and twice they did not help her self-correct when she fell back to this approach. Because of the consistent confidence she showed in counting

and her move away from direct modeling over the years, it is probable she tried to make sense of the CGI problems using an ordinal meaning for the numbers (Fuson, 1988).

Student Profile: Gerardo

Gerardo is a determined problem solver with a charming and easy-going manner. He usually started his problem solutions by drawing and showed confidence and persistence in trial and error approaches. He gave extensive explanations of his thinking, many times referring specifically to his model, and used more English than Spanish.

Join problems.

The Join problems highlighted Gerardo's ability to use a Joining To strategy (see Table 3) and trial and error with direct modeling to find correct answers. He also showed clarity in his explanations early in kindergarten. To find the change between 7 and 11 in kindergarten, Gerardo modeled the problem by Joining To, counting out 7 cubes and then adding more cubes until he had 11. He counted how many more he added (4) then explained, "Cuando decía siete los conté en estos [pointing to the 7 cubes] y luego y luego me sobraron estos [pointing to the 4 he added] (*When you said seven I counted them in these [the seven] and then and then I had these extra [the four]*)." He made sense of the problem with a cardinal meaning using distinct groups of cubes (Fuson, 1988).

Gerardo drew a correct Joining To model in 1st grade to find the change between 15 and 25, but got lost in his drawing and could not say the answer. He drew 15 lines and another 10 lines, but could not identify the 10 lines he joined to the 15. When I changed the numbers in this same problem to (7, 11), he solved the problem easily with Joining To

and drawing. At the end of 1st grade he solved the change between 9 and 18 by counting and even said the words, “*counting on*” in his explanation. He explained, “I started counting on with my fingers. I had nine dollars [he points to the ‘9’ he has written on his paper] and it costs eighteen dollars [he points to the ‘18’ he has written]. Ten, eleven, twelve, thirteen, fourteen [he holds up five fingers, one at a time, from his left hand as he counts] fifteen, sixteen, seventeen, eighteen [he holds up four more fingers, one at a time, from his right hand]. Five plus four is nine [and he shows the 5 and the 4 fingers].”

Gerardo was using his fingers to keep track of the count. This count gave him his answer.

When Gerardo encountered the Join Start Unknown (5, 13) problem in the middle of 1st grade, he showed remarkable perseverance. He knew he had some candies and then received 5 more to have 13 in all, but he did not know how to model the problem with his drawing. He tried to approach it with the same Joining To strategy he had used successfully with the change unknown problems, but this did not work for him because he did not know where to start. Finally, through continued drawing and trial and error, and on his fourth attempt, he discovered that he could start by drawing five lines and then Join To until he got to 13. By repeatedly counting the initial five with what he was adding, he found the answer to be eight. In his explanation he said, “I needed to *start* with *these* [pointing to the group of five lines],” emphasizing that he had to start with the known number even though it was not the beginning of the problem.

In 2nd grade, Gerardo still modeled to answer Join problems. He used drawing and trial and error to solve Join Start Unknown (7, 22) in September where he was to begin with an unknown amount of candies, receive seven more, and then finish with 22 candies. He began with an estimation of 11 and gave a lengthy explanation of his

thinking as he solved the problem. Here is an excerpt from that explanation. “When I heard that [22]...I told myself to stop at eleven [7 initial lines and 11 more are now 18 lines] so I could count...to see if it was twenty two and then once I knew that it wasn’t twenty two...I did three more.” He got an answer of 14, the 11 lines of his estimate and three more. His answer was off by one because he made a mistake in counting not because he failed to understand the problem or his strategy.

Gerardo subtracted objects to solve the same problem in February, giving a very clear explanation. He said, “I was thinking if I had twenty two altogether, then I should start taking away the seven that he gave me so I could start where I had.” He did not count back, but removed seven objects from the 22 in a cardinal approach (Fuson, 1988). For the final interview problem beginning with an unknown quantity of candies, adding 40, and ending with 112, he modeled it with base ten blocks instead of drawing. Gerardo used 11 base ten rods and two cubes to build 112 then removed the four rods that made 40. He saw he had 7 rods and two cubes left so his answer was 72.

Compare problems.

Compare problems highlight Gerardo’s ability to model sets of objects. In kindergarten, Gerardo used a Matching strategy to compare 9 and 12. He drew two rows of circles then made a 1-to-1 correspondence between the rows and saw that there were three extra. He knew the difference was three because he said, “Puedo ver aquí, (*I can see here,*)” as he drew a line around the last three circles on the right of the top row that stuck out from the line of circles below (see Figure 10 below).

Handwritten work for a compare problem. The top line shows the equation $12 - 9 + 3 = 12$. Below it are two rows of circles representing sets. The first row has 12 circles, and the second row has 9 circles. A vertical line is drawn between the 9th and 10th circles in the first row, and a horizontal line is drawn below the 9th circle in the second row. The bottom line shows the equation $12 + 9 = 21$.

Figure 10. Gerardo, Compare Difference Unknown (9, 12), kindergarten.

Gerardo directly modeled all the compare problems as two sets in 1st grade and solved them all correctly. He compared 13 and 21 by Matching again, and explained that he counted the difference between the two using “my inside my voice.” [sic] With this statement he gave me a window into his awareness of his verbal thinking (Mahn, 2009; Vygotsky, 1987). When Gerardo compared 2 and 13 at the end of 1st grade, he still modeled the two sets, but this time compared them side-by-side instead of matching them one under the other. He combined his direct model of the sets with subtraction to find the answer and he explained that if you take 2 away from 13 you have 11. This strategy shows that Gerardo is thinking of the numbers in the problem cardinally as two distinct sets and also ordinally where two counts back from one number gives the other number (Fuson, 1988).

In 2nd grade, Gerardo used both direct modeling and advanced strategies to solve the compare problems, indicating he may be shifting between ordinal and cardinal meaning for the numbers (Fuson, 1988). In September, he found the quantity of the larger amount in the problem about 13 toy cars in one set and 6 more in the second set by again modeling both sets. He drew a row of 13 circles and below this he drew another row of

13 circles then added six more circles to the second row to get 19. Here he applied discrete thinking to the problem (Van Wageningen et al., 1976). In February, he solved the same problem in his head. He explained the answer in terms of the larger number, 19, passing the smaller number, 13, by six. In this solution he appeared to use ordinal thinking (Fuson, 1988).

In the interview at the end of 2nd grade, Gerardo used the 100s chart to find the smaller set when he knew one set was 53 and the other 36 less. He started at 53 on the chart and counted back by ones 36 times to land on 17 as his answer. When asked if he could count faster on the chart, he says he could “take out the threes” and he began counting at 50. Then he counted down by tens saying “thirteen, twenty three, thirty three” and finally counted back by ones three times to 17. He did this by decomposing 36 into three, thirty and three. He was able to use a sequential tool to find the answer to a problem about sets (Ernest, 1985). Gerardo showed flexibility in thinking both cardinally and from an ordinal perspective in 2nd grade and using a combination of problem solving strategies (Fuson, 1988). This flexibility showed he was learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Part-Part-Whole problems.

Gerardo used direct modeling for Part-Part-Whole problems throughout the three years of the study. In kindergarten, he had an interesting approach to the problem about 10 balloons, where six are blue and the rest red. He began by drawing all ten balloons in the story. Even though he knew that there were six blue balloons, he started coloring balloons red, one at a time, and counting those not colored, until he had six not colored, the number he knew were blue. This left him with four that were colored red. He thought

about the whole group of balloons, then approached the problem from the unknown and worked toward the known, flexibly manipulating the parts of the whole. This is what I describe as a decomposition direct modeling strategy, as I mentioned above in Yolanda's profile. In mid-year of 1st grade, Gerardo used the Joining To strategy on the two (7, 13) and (10, 16) balloon problems. Gerardo drew the smaller number of balloons, then added balloons until he had the total. At the end of the year he used base ten thinking to solve (30, 20) saying, "thirty minus twenty is ten". When I gave him a harder problem in the same interview, (75, 50), he counted up by tens first, realized this was not the best strategy and then was able to self-correct his own counting.

Gerardo shifted between direct modeling and counting for the 2nd grade Part-Part-Whole problems. He used the Joining To strategy on the (12, 24) problem in September. He drew the known quantity balloons, 12, and continued drawing until he had 24 to find the unknown quantity. In the second interview for 2nd grade, he did the same problem mentally, describing his verbal thinking (Mahn, 2009; Vygotsky, 1987) as, "What I, when I was starting at twelve and starting and counting like one and one, and then once I got to twenty four I started...I started thinking again, I started counting again to see if it's right and it was." Gerardo solved the final Part-Part-Whole problem (100, 65) quickly and correctly with a borrowing algorithm on paper.

Multiplication problems.

Gerardo preferred to directly model multiplication problems for the three years of the study. In kindergarten, Gerardo directly modeled three bags of marbles with six in each bag using a Grouping strategy and referred to the context of the story in his explanation. He counted the total to get his answer then said, "Estaba contando como mi

imagina, (*I was counting like my imagination,*)” then he counted out loud in Spanish, “1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.” For this kindergarten equation, he wrote, “ $3 + 6 + 6 + 6 = 18$,” beginning with a three by mistake. He struggled with the words until he was happy with his explanation saying, “tres más tres...más seis más seis más seis es igual a dieciocho... Porque...porque hay tres bolsas y solo puse así... Porque hay en seis, hay seis canicas adentro, (*3 plus 3...plus 6 plus 6 plus 6 is equal to 18...Because...because there are 3 bags and I only put them like this...Because there are in 6, there are 6 marbles inside.*)” Gerardo’s reference to the objects in the story showed he had not abstracted the numbers to a counting sequence (Carpenter et al., 1999) but was thinking about them cardinally (Fuson, 1988).

Gerardo continued to use direct modeling for multiplication in 1st grade. He drew four bags with seven marbles in each bag for the mid-year problem then had an interesting explanation that showed his metacognition and verbal thinking (Mahn, 2009; Vygotsky, 1987). He said, “I started counting from, I started counting from, I started counting from inside...Like from my head [and he puts his hands to his forehead]...counting from seven [pauses to count up silently] fourteen [pauses and counts again] twenty one [counts silently] twenty eight.” At the end of 1st grade for the Multiplication problem (7, 5), Gerardo again drew each bag, but then he wrote the numbers by fives on the bags, 5, 10, 15, 20, 25, 30, 35, and counted by fives to find the answer, 35. Figure 11 shows his drawing with both modeling and counting strategies.



Figure 11. Gerardo, Multiplication (7, 5), 1st grade.

He said, “Um, every time when, um, when it’s a five, I go five and five.” This type of skip counting is a common strategy noted in the CGI Strategies Table 3 for multiplication problems. Here Gerardo has combined modeling with a counting strategy to show that he is thinking about the numbers both cardinally and ordinally (Fuson, 1988).

Gerardo shifted between direct modeling and skip counting again to solve Multiplication problems in 2nd grade (Fuson, 1988). At the beginning of the year, he drew a direct model of the eight bags for the (8, 5) problem and took the time to put the numbers one to five in each (see Figure 12).

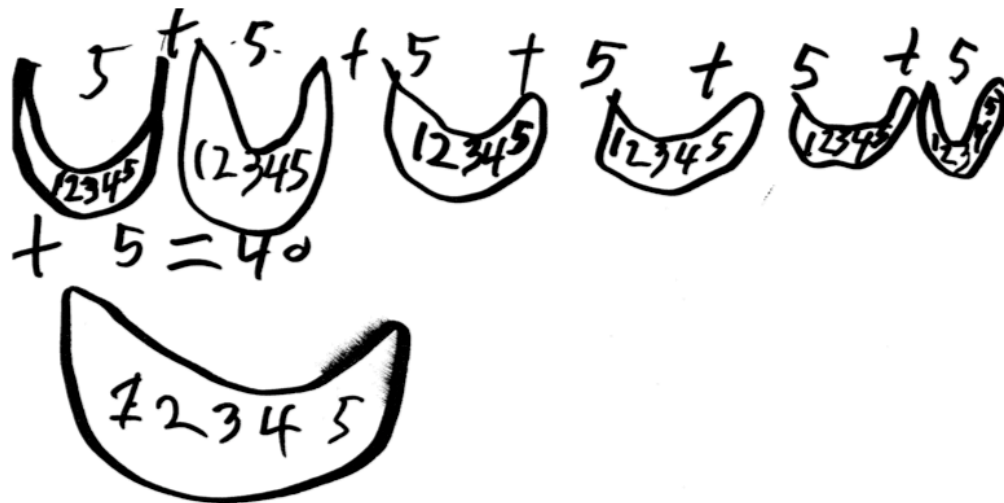


Figure 12. Gerardo, Multiplication (8, 5), 2nd grade.

He counted by fives to find his answer, 40, saying “Because there’s five in each one...I needed to...I wanted to count in five and five because there were five in each bag.” For the same problem later in the year, he used his fingers to keep track of the eight counts of five. He explained, “When I was going like this [and he puts out his fingers] when I was putting my fingers out, I started counting five and five.” This was a pure counting strategy (Carpenter et al., 1999).

Gerardo chose to directly model the final problem with 15 bags and 7 marbles in each. He was one of the earliest interviewees and I let him continue with the model as seen below in Figure 13. The same is true for Gina. After these early interviews, I decided that students were going to consistently directly model so I asked the remaining students, including Omar and Yolanda, to solve the problem another way. When Gerardo’s drawing was finished for the Multiplication problem, he tried to count by sevens to find the answer.

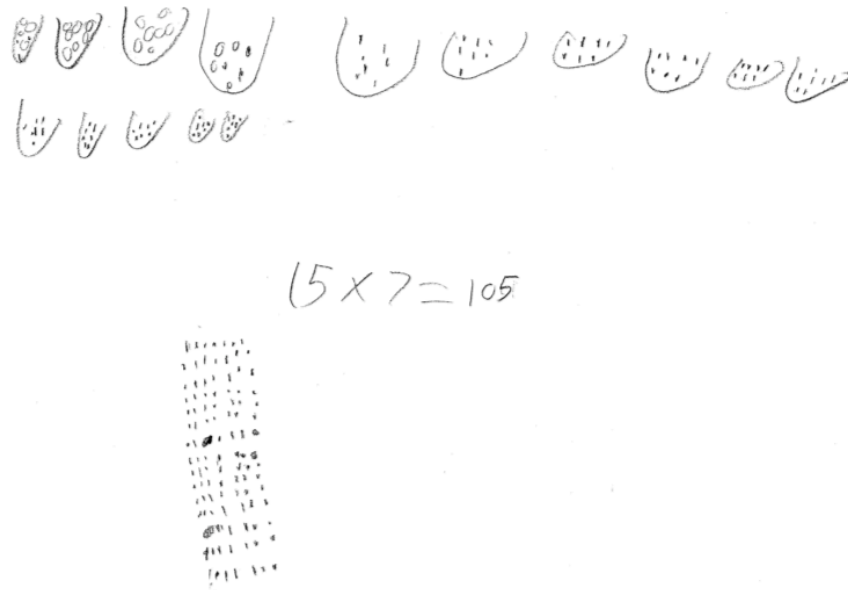


Figure 13. Gerardo, Multiplication (15, 7), 2nd grade.

An excerpt from his explanation shows his knowledge of tens and ones. “I did the bags, put the marbles, and then I started counting em, on one, but to make it faster, every time I started in, like, like, if I’m like in thirty I just go to thirty seven, and do the other ones. If I get, it’s like, it’s like, uh, fifty, I could just do fifty seven the next time.” He did not know the multiples of seven so he had to count by ones for the larger numbers and was off by one in his answer. However, he was anxious to show that he knew how to write the equation for the problem, and put “ $15 \times 7 = 104$ ”. He later changed his answer to 105 after solving the problem with a matrix (see Figure 13 above). Gerardo communicated a clear understanding of the concept of multiplication in this problem through his actions, words and equation (Moschkovich, 2002).

Partitive Division problems.

In kindergarten and 1st grade, Gerardo directly modeled all the Partitive Division problems. He divided 15 lines into three equal groups in kindergarten by trial and error. He used trial and error partitioning again in 1st grade to divide 24 into four equal groups, but he appeared to mix up the number of groups with the number of objects in each group and solved the problem incorrectly like a Measurement Division where the number in each group is known. At the end of 1st grade, he partitioned 12 into six equal groups by drawing and trial and error. He called his answer of two a “lucky guess.” This was one of the few times when he was not verbose in his explanation. When I asked him to partition 24 into six equal groups to challenge him a bit in this same interview, he grouped the 24 squares in fours right away. Once again he called his thinking a, “lucky guess,” and added, “I know these things,” matter-of-factly. How he approached this problem type for the final interview in 2nd grade will be discussed in Chapter 6.

Gerardo’s problem solving summary.

We saw in the above descriptions that Gerardo has a good ability to directly model problems, preferred to draw the problem situations, and used more direct modeling than counting strategies to find solutions. He explained his thinking clearly and had the additional ability to reflect on his verbal thinking and problem solving (Mahn, 2009; Vygotsky, 1987). His explanations showed a comprehension of the actions and relationships in the problems, he tied what he said to the structure of the problem, and he talked about the voice inside his head. He was persistent and confident in using trial and error even when he did not have a clear idea how to proceed. Gerardo showed flexibility in shifting between cardinal and ordinal meaning of numbers (Fuson, 1988) because he

could explain his thinking in terms of the objects of the story and/or in terms of counting sequences. He could create a model of the whole problem situation, represent the problem as sets, but then use a linear strategy like skip counting to find the answer. I believe this flexibility was why he was successful at trial and error; he could shift gears, so to speak, looking at the whole problem as made up of parts or as a linear progression toward the whole. This flexibility showed he was learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Student Profile: Gina

Gina was a methodical problem solver who liked to take her time in finding the solutions and was very successful over the three years of this study. She made models using concrete manipulatives when they were available, or at times used drawing. She gave detailed explanations of her thinking and recounted her own questioning and answering thinking processes. She was playful during problem solving and liked to engage in discussions on various unrelated topics during the interviews. By the end of 2nd grade, she was comfortable in both Spanish and English. She transitioned to using English with me in February of 2nd grade.

Join problems.

Gina approached Join problems over the three years of the study using direct models and creating decomposition type representations for most problem situations. She found the change between 7 and 11 in kindergarten with the same decomposition strategy I described above in Yolanda and Gerardo's profiles. Gina began her solution with a drawing of the total number, 11 circles, then removed the part she knew, 7, and counted

the rest, 4. Her answer was the number remaining that she could see after she drew 11 circles and crossed out the *first* seven circles. Crossing out the first seven of the circles of the drawing and not the last seven is important, I believe in this case. With her direct model she represented the order of the action of the problem. She explained her thinking very clearly for someone so young. “Es que mi hermano quería comprar un avión y costó...no más tenía siete. Entonces le puse las crucecitas para saber cuántos hay y los conté para que supiera. *(It is that my brother wanted to buy a plane that cost...only he had seven. Then they [I] put the little crosses to know how many there are and I counted them so I would know.)*” As an insight into her own thinking at this young age, she said initially on hearing the problem that she usually needed to have the numbers repeated. “Es que necesito los veces...necesito repitan. *(It is that I need sometimes...I need them to repeat.)*” Over the years I have noticed that she was right and usually does need numbers repeated.

Gina continued direct modeling Join problems in 1st grade using Joining To strategies and part-part-whole thinking. She found the change between 15 and 25 by separating 25 tally marks into two sets. At the end of 1st grade, she found the change between 9 and 18 by building the number 18 with a group of ten connected cubes and another group of eight connected cubes. Then she removed one cube from the ten and added it to the eight to have nine and nine, so her answer was nine. She explained, “Porque él tiene nueve, como si le ponen uno con los demás es diez y le van a quedar ocho, y como ya tienen nueve y le quité uno y si puse aquí, y ahora le quedan nueve, *(Because he had nine, because if they put one with the others it is ten and that leaves eight, and like they have [one row of cubes] nine and I took one off and put it here, now*

they are left with nine [the other row of cubes].)” When Gina was given a similar problem with bigger numbers in the same interview, (45, 25), she again built two distinct groups of cubes to represent each number using connected groups of ten and five cubes. She then compared the smaller set (2 tens and 5 cubes) with what was in the larger set (4 tens and 5 cubes), and removed 2 tens and 5 cubes from the 45, leaving her 2 tens, or 20 as her answer. By building two distinct groups to represent each number, she showed that she was using a cardinal meaning for the numbers in the problem (Fuson, 1988).

When Gina first heard the Join Start Unknown problem with smaller numbers (3, 5) in mid-year of 1st grade, she solved it quickly by drawing “||+ |||= 5”, as seen below in Figure 14.

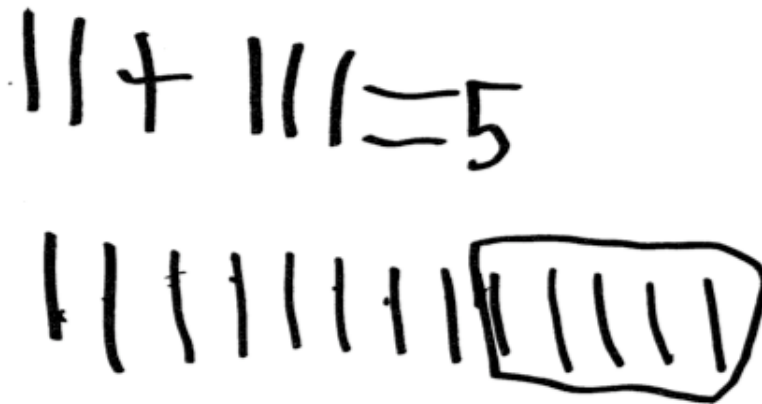


Figure 14. Gina, Join Start Unknown (3, 5) and (5, 13), 1st grade.

Her equation represented the action of the problem, but was also a direct model using lines to represent each object. She appeared to have recognized the relationship immediately among the numbers with a recalled fact. When she next solved Join Start Unknown to find the beginning number when 5 is added and the result is 13, she drew 13 lines, identified the *last* five, drew a box around these, counted the beginning amount and said eight for her answer (see Figure 14 above). It is significant that she identified the last

five instead of the first five, once again representing the progression of action in her model and approaching the problem as parts of a whole. At the end of 1st grade, she used a recalled fact to know that the number you add to five to get 15 is 10. Then she used cubes and a decomposition strategy to find the answer to Join Start Unknown (10, 34) during the same interview. She built 34 with cubes, removed 10, and got 24 as the solution. Within the space of one interview she used two different approaches to the same problem type, recalled fact and modeling, to demonstrate that she is learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Gina showed increasing flexibility in her direct modeling in 2nd grade. She solved Join Start Unknown (7, 22) by drawing 22 circles in four rows of five each and one row of two, crossed off the last seven (the last row of five and the row of two) and saw that there were 15 circles for her answer because she was left with the first three rows of five. She solved the same problem later in the year by reversing her direct modeling strategy. She drew seven lines, enclosed them in a circle then used a Joining To strategy until she reached 22. This direct modeling strategy is common for Join Change Unknown problems, see Table 3, but she applied it to Join Start Unknown by reversing the addends. It is possible with her ability to think about numbers discretely as parts and wholes (Fuson, 1988; Van Wageningen et al., 1976), she had no trouble at this stage in her mathematical development reversing the order of the parts to easily find the answer.

Gina first used a subtraction algorithm at the end of 2nd grade to find the answer to how many candies she had before her friend gave her 40, if the gift resulted in 112 candies. Next, she modeled the problem with base ten blocks, using 11 rods and two

cubes. When I asked her if the model in front of her was the beginning or the end of the story, she replied that it was the end. Then she said she could make it the beginning if I wanted and she removed four rods from the 11 rods. In this specific answer and in the models she used to solve previous join type problems, especially when she crossed off lines at the beginning, middle or end of her models, she demonstrated a conceptual understanding of the action of join, a differentiation between the change as unknown and the start as unknown, and a flexibility in thinking about the numbers in these problems. All of these approaches show that she is learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

Compare problems.

Gina used direct modeling in her approaches to Compare problems in kindergarten and 1st grade. In kindergarten she compared 9 and 12 with a decomposition strategy. She drew 12 lines, the larger set, in two rows of six lines. Then she asked again about the numbers in the problem and crossed out the six lines on top and three below, nine altogether, to leave her with three lines. She began with the whole as the larger part and removed the smaller part to see what was left. Gina used Matching to find solutions to compare problems in 1st grade. She matched 10 and 16 with a twist. She used cubes to build 16 in two groups of eight then she built 10 in two groups of five. She compared a 5 and an 8, and another 5 and 8, saw that there was a difference of three for each pair. She combined the two partial answers to get her final answer of six. At the end of 1st grade, Gina compared both the (2, 13) problem and the (29, 31) problem by building and lining up two distinct sets of cubes and counting the difference. She was clearly thinking cardinally about these problems as sets (Fuson, 1988; Van Wagenen et al., 1976).

Gina used counting strategies for compare problems in the first two 2nd grade interviews. This could have been due to the structure where the problem was to find the larger set when the smaller set and the relationships between the two were known. She knew that the smaller set was 13 and the larger set was 6 more than the smaller. She simply Counted On from 13 six times in the first interview. She explained, “O.K., como...tenía trece...y luego, um, dije, a ver, le voy a contar 6 más a trece, y luego fui contando en uno en uno, y luego dije, hasta seis. Y luego puse mis dedos así [she puts the 5 fingers of her left hand and thumb of her right hand on the table], y luego fui contando. Y ya supe la respuesta. (O.K., like...he had 13...and then, um I said, let’s see, I am going to count six more to 13, and then I was counting by ones, and then I said, until six. And then I put my fingers like this [she puts the 5 fingers of her left hand and thumb of her right hand on the table] and then I was counting. And I just knew the answer.)” Her explanation describes her verbal thinking to solve the problem (Mahn, 2009; Vygotsky, 1987). She used the same counting strategy for the second interview of 2nd grade, making sense of the problem with an ordinal meaning (Fuson, 1988) by abstracting the objects to number sequences (Carpenter et al., 1999).

In the final interview, Gina directly modeled to find the amount in the smaller set when she knew it was 36 less than the larger set of 53. She used base ten blocks, took five rods and three cubes, removed three rods and then counted down six on one of the rods, initially to get 14 because she forgot about the three cubes she had set aside for the 53. When she remembered the three cubes, she moved her finger back up the rod she was marking and changed her answer to 17 to compensate for the three cubes. The direct model with base ten blocks helped her self-correct. Further, these compare problems with

the compared set or the referent set unknown did not challenge her. She simply either added or subtracted to get the answer.

Part-Part-Whole problems.

Gina directly modeled all of the Part-Part-Whole problems over the course of three years. This may be due to the problem structure, which reflected her tendency to use decomposition strategies and the way she attached meaning to the numbers in the CGI word problems and used discrete thinking (Fuson, 1988; Mahn, 2009). In kindergarten, Gina directly modeled the (10, 6) balloon problem by first drawing ten lines and then partitioning off six. She counted the rest to get her answer. In mid-year of 1st grade, Gina solved (13, 7) problem about balloons by first connecting seven green cubes then adding more red and pink cubes until she had first 14 and then 13. This was a Joining To strategy where she built one part and then added the other part until she had the total. She also wanted to draw a model of the balloons shaped as hearts (see Figure 15).



Figure 15. Gina, Part-Part-Whole (13, 7), 1st grade.

Gina’s explanation of how she self corrected her answer in the above problem when she was modeling with cubes is interesting. She said, “Pues, conté estos, (*Well, I counted these,*)” as she grabbed the six red and pink cubes, “y luego de estos, (*and then these,*)” and she grabbed the seven green cubes. “Y luego, me dije esto, (*and then I said this to myself,*)” she put the extra pink cube that she had earlier with the group of six reds and pinks. “Hice a ver, deja contarlos, uno, dos, tres, cuatro, cinco, seis, siete, (*I did it to see, let’s count them, one, two, three, four, five, six, seven*) ” she said aloud pointing to the red and pink cubes as she counted. She then continued counting the green cubes, “ocho, nueve, diez, once, doce, trece, (*eight, nine, ten, eleven, twelve, thirteen,*)” and she stopped with her finger on the second to last cube. “Y mejor quité este de rosa, (*And it is better to get rid of this pink,*)” and she removed this last pink cube. This lengthy explanation describes her internal dialogue and her verbal thinking to find the correct solution (Mahn, 2009; Vygotsky, 1987).

At the end of 1st grade Gina solved the Part-Part-Whole (30, 20) problem with a direct model and decomposition strategy rather than base ten thinking. She drew 30 lines then circled the first 20. She counted the rest to find the answer of 10. Figure 16 shows her drawing, with some of the lines almost off the paper on the top right, and the dots that she made when she counted the 10 lines not circled. It is not clear why Gina chose to draw a model of this problem when she knew how to count by tens as shown below in the multiplication problem for the same interview.

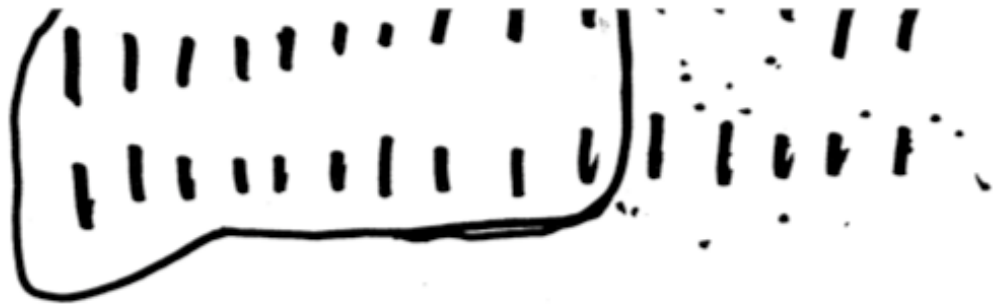


Figure 16. Gina, Part-Part-Whole (30, 20), 1st grade.

Gina used direct modeling to find solutions for all the Part-Part-Whole problems in 2nd grade as well. In September for the (24, 12) problem, she used two rods and four cubes from the base ten blocks. To find the solution she lined them up, counted along one rod and two sections of the next rod, marked the place with her finger, counted the rest of this rod and the four cubes to get 12. Gina solved the same problem in February by partitioning 24 tally marks into a group of 12 and then counting the rest, a similar strategy to the one she used in September. She had difficulty remembering the numbers in February, which happened frequently with her.

For the final interview problem of (100, 65), she started with the 100s chart, but made a mistake in counting between 65 and 100. Then she put a cube on each number to

mark the space, but still came up with two answers 35 and 40 and was not able to count confidently between the two numbers. She used base ten blocks to resolve the situation by taking 10 rods and removing six rods and five cubes. To show the removal of the five cubes she did not trade a rod for cubes, but instead stacked the five cubes on top of the seventh rod, then counted the remaining sections together with the three rods to get 35 as an answer. It is interesting that she put aside the 100s chart, a sequential tool she rarely chose to use. She consistently showed that she was more comfortable with direct modeling tools like cubes and base ten blocks instead of sequential tools like the 100s chart, possibly because these tools reflected the way she was making sense of the problems (Mahn, 2009) from a cardinal perspective (Fuson, 1988).

Multiplication problems.

Gina preferred directly modeling all the Multiplication problems. She approached the problem in kindergarten with confidence. She did a direct model by drawing three bags, six circles in each bag, and counted while she drew them to get 18 for an answer. Gina continued directly modeling problems in 1st grade even as she incorporated counting strategies. In mid-year of 1st grade for the (4, 7) problem, she drew four rectangles and used the dots she puts inside like tally marks, four dots with a line through them then two more below so there were seven dots in all. However, she counted by ones to get 28 as an answer, relying on a direct model instead of a counting strategy. At the end of 1st grade, the problem, (7, 5) was designed to see if students could just apply skip counting to find the answer. Gina seemed to enjoy using the connecting blocks and directly modeling this problem in seven groups of five cubes, but then she used skip counting by fives to get her answer. When I made the problem a little harder, seven bags with ten

marbles in each and six single marbles, she built a model with seven groups of 10 cubes and six single cubes. She counted by tens and then added the ones to get 76. I did not think at the time she needed these cubes to find the answer, but that she was enjoying using the cubes while doing the mathematics. However, it is possible she was shifting between cardinal and ordinal meanings of the numbers and her strategy reflected this shift in thinking (Fuson, 1988).

At the beginning of 2nd grade, Gina wanted to directly model the (8, 5) problem with tiles, but I asked her to do it a faster way, so she skip counted by fives eight times to find the answer. Her explanation described her verbal thinking during problem solving (Mahn, 2009; Vygotsky, 1987). “Luego dije, a ver [she taps her cheek again with the forefinger of the right hand] voy, voy a hacer tres de mis dedos de la, de una mano y cinco de la aquí [she puts down 3 fingers of her right hand and all the fingers of her left hand on the table] y así estoy a contando de cinco en cinco y para, usando los dedos. *(Then I said, let’s see [she taps her cheek] I am going, I am going to do 3 of my fingers of the, of one hand and five of the one here [she puts fingers on the table] and in this way I am counting by fives for, using the fingers.)*” For the same problem later in the year, she drew the (8, 5) model with eight circles and five dots in each then said quickly it was 80, but self-corrected after looking at her model. She said in English, “No, it’s not eighty, it’s forty...I just look at it... Yeah, and I know that two fives are ten. This is ten [she draws a circle around two bags] and that ten [circles two more bags] and that ten, [again] and ten [underlining the last two bags]. So ten, twenty, thirty, forty.” She looked at the whole set of eight bags and consolidated them into four groups of ten. This was a part-part-whole approach as she manipulated the parts into tens and then used skip counting to find the

answer. Unlike Omar and Yolanda above, the direct model helped Gina correct her solution.

Finally, for the Multiplication problem of $(15, 7)$, Gina began with the base ten blocks, but quickly gave up when they did not lend themselves to the direct model she was trying to build with seven single cubes in 15 groups. She went to paper and drew out all the 15 bags with lines inside for the marbles in tally marks. Figure 17 shows her final drawing. Note that two of the original seven lines in each bag have been partially erased.

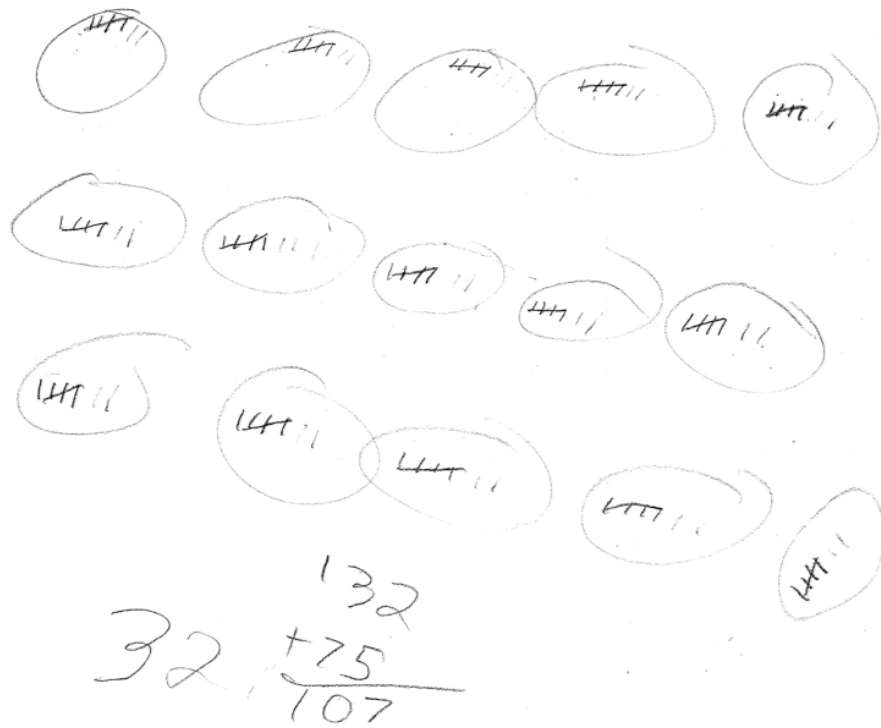


Figure 17. Gina, Multiplication $(15, 7)$, 2nd grade.

As I explained above in Gerardo's profile, I let her continue because hers was one of the earlier interviews. It was only later I asked some of the other students to find a method other than direct modeling. Gina's method of counting the marbles in her model was interesting and showed how she combines direct modeling and counting strategies. She first counted the two extra lines in each bag, by twos, then erased these, counted the rest

by fives, then added the two together. Her answer was off by two because she miscounted. She said after counting the twos, “OK treinta y dos [she writes ‘32’ on the paper] menos...sé éstos...los...que están sueltos, entonces éstos a los voy a borrar [she begins to erase the two dots in each bag to leave just the group of five tallies] para, para que sea de cinco en cinco...así...para los...para que los conté, y luego los puedo sumar aquí. *O.K., 32 minus...I know these...the...these are single, then I am going to erase these so, so that it is by fives...it in this way...so that... so that I counted these, and then I can add them here.*”) It is interesting that she felt she needed to erase the two dots so that she could count the groups of five tallies in each bag. It may have been important to her that the direct model dynamically reflected how she was making sense of the problem, and in turn reflected her system of meaning about the numbers (Mahn, 2009).

Partitive Division problems.

Gina used direct modeling with trial and error to solve Partitive Division problems. In kindergarten, she drew 15 lines and divided them into three equal groups of five. For the (18, 3) problem at mid-year of 1st grade, Gina used the same approach, drawing the 18 lines and then trial and error to make three equal groups. She used seven as one of her trials, then decided to try five, saying mostly to herself, “Espera...es decir, es mejor...cinco...(*Wait...that is to say...it is better...five...*)” Here she demonstrated how she talked herself through the trial and error strategy using language to mediate her thinking (Mahn, 2009; Vygotsky, 1987). Finally, she tried six successfully. She explained, “Dije, seis en cada una o no? Deja contar éstas. Uno, dos, tres, cuatro, cinco, seis [the 2nd group of 6]. Ah sé. Uno, dos, tres, cuatro, cinco, seis [she counts the 3rd group]. Bueno, sí. (*I said, six in each one or not? Let me count these. 1, 2, 3, 4, 5, 6 [the*

2nd group]. *Ah, I know. 1, 2, 3, 4, 5, 6, [the 3rd group]. Good, yes.*” She reproduced in her explanation her self-questioning and her internal dialogue about her strategy.

At the end of 1st grade she solved (12, 6) easily by grouping 12 lines into twos. She says, “Es fácil de dos en dos. Pues son 6. (*It’s easy by 2s. Well, there are six.*)” When she is asked the harder problem, (24, 6), she begins with connecting blocks in two groups of ten and 4 singles, but she discards them and goes back to her direct modeling drawing strategy. She makes tally marks, uses trial and error, and quickly puts four in each group. Her approach to the problem to the final problem will be discussed in the analysis chapter.

Gina’s problem solving summary.

In almost every case, Gina drew a direct model or used some type of manipulative to find solutions. The use of these manipulatives appeared to be fun for her, and she used them to build a model of the problem even when I thought she could have solved it with a more advanced strategy. Gina liked to talk about her thinking and had the ability to recreate her internal dialogue by retelling the questions she asked herself and her answers. Her explanations reflected her verbal thinking (Mahn, 2009; Vygotsky, 1987). She approached problems flexibly with good comprehension and showed creative aspects in her direct modeling. She combined direct modeling with counting strategies, shifted between ordinal and cardinal thinking (Fuson, 1988), and used skip counting and recalled facts to show that she was learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

CHAPTER 6. Making Meaning During CGI Problem Solving

*“Let me make this very clear, there is no one concept of number.”
(Karen Fuson, NCTM annual meeting presentation, April 2009)*

Introduction

In Chapter 2, I presented the theory of systems of meaning in sociocultural theory (Mahn, 2009; Vygotsky, 1987) and built this longitudinal study on the idea that learning is a dynamic process where both children and adults incorporate new information into what they already know to restructure, refine and expand their knowledge about the world. Children learning mathematics in the early grades must build bridges between their informal experiences with numbers and the formal generalizations, conventions, concepts and abstractions presented in the classroom. These scientific concepts from formal mathematics restructure students’ existing spontaneous concepts (Vygotsky, 1987) to refine the system of meaning they are building about numbers (Mahn, 2009).

Children make sense of mathematical problem solving based on the system of meaning they are constructing about numbers and operations. Successful mathematics education helps children construct robust systems of meaning, or as Hiebert and Carpenter (1992) describe, flexible networks of mathematical understanding around the concepts and processes of formal mathematics. In this chapter, I consolidate the analysis of students’ mathematical thinking during CGI problem solving over a three-year period and explain how I believe these students made sense of the problems. I demonstrate as Karen Fuson noted above, that the four children in this study are using more than one concept of number to approach CGI problems. When children make sense of mathematics, they attach meaning to concepts, develop symbols to mediate their thinking

and organize these concepts and symbols into a system of meaning (Mahn, 2009). In this way, how the children made sense of the problems to reach successful solutions were an indication of the strength and validity of the systems of meaning they were constructing about numbers and operations and if they were learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

The research questions for this study guided my analysis when I explored: 1) how students communicate their mathematical thinking, and 2) what students' communication reveals about how they are making sense of the mathematics in the CGI problems (Moschkovich, 2002; Mahn, 2009). Students communicate their mathematical thinking through their problem solving strategies (Carpenter et al., 1999), which include the tools they use as aids to their problem solving, and also in the verbal descriptions of their thinking (Forman et al., 2001; Kieran, 2001; NCTM, 2000; Sfard, 2001). Because students' external speech is directly connected to their inner speech and verbal thinking processes, and these processes are directly connected to children's thought processes, children's communication gives a direct link to how they are making sense of the CGI problems.

I use an analogy of windows to describe how students' communication gives me access to their systems of meaning. I imagine the connection as a series of windows, each going deeper into students' conscious thought. I use Mahn's (2009) Planes of Verbal Thinking diagram to begin in the external environment and move inward to students' consciousness in the following way.

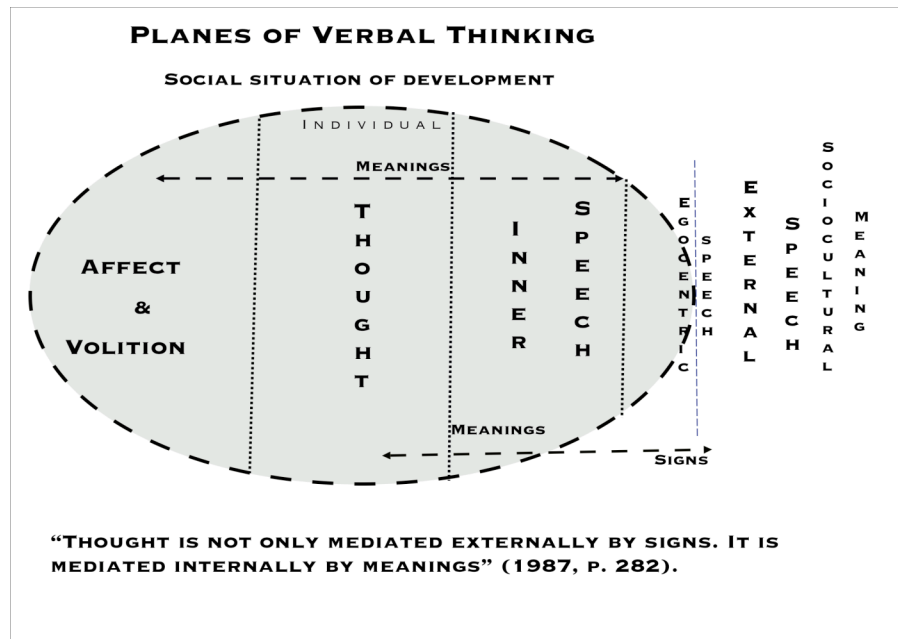


Figure 1. Planes of Verbal Thinking (Mahn, 2009).

1. The plane of external speech and sociocultural meaning: Student communication during CGI problem solving includes their strategies, the tools they choose to help them, and how they explain and justify their solutions.
2. The plane of inner speech: Students' explanations give a window into how they thought about the problems, whether they counted along a sequence of numbers, manipulated the objects, and/or engaged in internal dialogues.
3. The plane of meaning: From students' own descriptions of their problem-solving approaches and verbal thinking, we have a window into how they made sense of the numbers in the problems.
4. The plane of thought: How students made sense of the numbers in the problems reflects the system of meaning they are constructing about mathematics.

Viewed schematically:

communication \Leftrightarrow *verbal thinking* \Leftrightarrow *meaning making* \Leftrightarrow *system of meaning*

I have used bidirectional arrows in this schematic representation because sociocultural theory describes learning and meaning making as the continuous interplay between internal consciousness and the external environment (Mahn, 2008, 2009).

In the following sections, I elaborate on student thinking and present the following grounded theory:

The way children approach CGI problem solutions reflects the way they make sense of the numbers in the problems. Children make sense of the numbers in different ways and in turn their problem-solving strategies differ, depending on if they are attaching a cardinal or an ordinal meaning to the numbers (Fuson, 1988).

I argue that primarily two of these four students made sense of the number actions and relationships in the CGI problems by thinking of the numbers ordinally and the two other students primarily thought of the numbers cardinally. I call the first two students *sequential thinkers* because they tended to use counting strategies on number sequences. They used an ordinal perspective. I call the second two students *discrete thinkers* because they tended to manipulate discrete sets of objects to find their answers. They used a cardinal perspective. The data below show that these patterns in students' mathematical thinking were visible in kindergarten and followed the students through the interviews at the end of 2nd grade.

My argument unfolds in two parts. In the first part, I discuss problem solving strategies from CGI theory and present strategy tables from my analysis for each student by problem type and also by grade level. As part of the strategies section, I include examples of students' explanations, using their own language to highlight their thinking. Students' verbal descriptions of their thinking directly connect to their internal speech and verbal thinking as outlined above, so that a great deal about how students were

making sense of the mathematics is revealed in their language (Mahn, 2009; Vygotsky, 1987).

In the second section of this chapter, I begin with a summary of the characteristics of the sequential thinkers, Omar and Yolanda, and the discrete thinkers, Gina and Gerardo, based on my analysis of their problem solving strategies. The summary is followed by a description of each of the four students solving two specific problems, drawing attention to how the characteristics played out in their problem solving. I describe how each of the four students solved the Multiplication problem in kindergarten then how they each solved the Partitive Division problem at the end of 2nd grade.

A Closer Examination of CGI Problem Solving Strategies

CGI strategies were my basis for analyzing student thinking and I started with the classification and description of strategies developed by Carpenter et al. (1993, 1994, 1996, 1999) within the CGI framework. Two important classes of strategies in CGI theory are direct modeling and counting. Although mentioned in the theoretical framework for this study and in the students' problem solving profiles, it is important that I present the definition once again in the words of Carpenter et al. (1999) before I continue.

Direct Modeling is distinguished by the child's explicit physical representation of each quantity in a problem and the action or relationship involving those quantities before counting the resulting set. In using a Counting strategy, a child essentially recognizes that it is not necessary to actually construct and count sets. The answer can be figured out by focusing on the counting sequence itself. (p. 22)

Essentially, a child directly models a problem when he or she represents each object in the number story. In other words, the objects and their associated numbers are the focus of how the child makes sense of the problem. When a child uses a counting strategy, he or she is not focused on the objects themselves but on the numbers that represent the objects and their positions in a sequence of numbers that can be counted (Fuson, 1988). With a counting strategy, the child makes sense of the problem through an abstraction of the concrete situation to a number sequence. Counting strategies are considered advanced, along with recalled and derived facts and algorithmic approaches like carrying in addition and borrowing in subtraction (Carpenter et al., 1999). Carpenter et al. (1999) note,

Counting strategies are abstractions of the corresponding Direct Modeling strategies they [children] used previously...Gradually over a period of time children replace concrete Direct Modeling strategies with more efficient Counting strategies, and the use of Counting strategies is *an important marker* in the development of number concepts. (p. 28)

I have added the italics in the above quote to make a point. If the difference between a direct model and a counting approach represents a marker from concrete to abstract mathematical thinking, then this distinction gives me important information on the system of meaning students are using to solve these CGI problems. Do they think about each object in the story and manipulate groups of objects independently, or do they abstract the objects to numbers in a sequence and operate within that sequence to find their answers? Using this distinction, I chose to classify the strategies the four students used in this study by categorizing their approaches to each problem as either a direct

model or an advanced strategy. I labeled their strategy a direct model if they represented each object in the problem even if they used a counting strategy within their model to find the answer. I labeled their strategy advanced if they did not represent each object, but used counting, recalled fact, derived fact, a number operation like addition or subtraction, or some type of algorithm like carrying or borrowing to find their answer.

The role of student language to reveal mathematical thinking.

According to Vygotsky (1987), social communication through language gives us one of the best avenues to study conscious thought. The informal mathematical thinking of the bilingual students in this study began early in their lives when they connected meaning to words in Spanish. When these children entered kindergarten, the formal classroom mathematics taught in Spanish bridged their informal systems of meaning to the formal mathematical systems encountered in the classroom and they were able to solve CGI problems successfully (Appendix C; Carpenter et al., 1996; Turner et al., 2008). New mathematical concepts, context-based problem-solving activities and discussions in Spanish supported these native Spanish-speaking students to categorize, organize, and generalize their informal knowledge toward the generalizations of formal, scientific mathematical concepts without language obstacles (Thomas & Collier, 2002). In the analogy used by Vygotsky (1987) to describe the connection between spontaneous and scientific concepts, teaching mathematics in Spanish to these young children whose first language was Spanish had the power to directly link their informal concepts growing upward with the formal concepts growing downward.

CGI interview problems used in this longitudinal study were created in both Spanish and English and were based in familiar contexts so that the children could fully

understand the problem situations. The conversations during the interviews were in the child's language of choice and children had the option to use either Spanish or English to explain their thinking. In order for students to develop accurate solution strategies, they had to make sense of the mathematics in the problems. The problem situations used in this study drew on students' native language and cultural contexts, thus giving the bilingual students in this study the best opportunities to comprehend the problems (Secada & De La Cruz, 1996), employ both verbal and mathematical thinking (Mahn, 2008, Vygotsky, 1987), and then discuss their solutions in the language of their thinking (Sfard, 2001). During the three years of this study, students developed academic language in Spanish at the same time as they were beginning to incorporate the use of English. By the end of the study, several students were comfortable explaining their thinking in English because they had transferred the concepts learned in their native language to their second language (Cummins, 1981, 2001; Thomas & Collier, 2002).

Discourse variations among students.

Reflecting on their thinking during problem solving was not equally easy for all the students. Gerardo and Gina had the ability to reflect on their thinking, loved to talk about their thinking, talked out loud while problem solving, and Gina frequently retold the internal dialogue she had during problem solving. Yolanda preferred to work silently while problem solving, but when she explained her thinking it was clear and concise. She knew when she had solved the problem in her head and could explain it this way. Omar was more challenged to explain what he had done until he reached 2nd grade. However, when he thought out loud, especially during the final interviews, the words reflected his thinking processes. For Omar, thinking out loud at this time was communicating (Sfard,

2001). Additionally, because Gerardo and Gina were direct modelers, they had the advantage of their models in front of them to aid in their explanations (Carpenter et al., 1999). Because Omar and Yolanda liked to solve problems mentally, they did not have a model as an aid for their explanation, which may have made it more difficult for them to reflect on their thinking.

Individual student strategy patterns.

The table below shows a breakdown of student strategies for the five problems examined in this study. Students do not have exactly the same number of problems because in some cases, as explained in the methodology chapter, if students could solve a simpler problem quickly, and if time permitted and they were willing, I gave them a more complex problem by increasing the number size or difference between numbers (Carpenter et al., 1999; Secada & De La Cruz, 1996). Time constraints from the classroom and student dispositions for a particular interview day always played a role in these decisions. However, all students were given all five problem types. As described in Chapter 3, the interviews contained additional problem types, but these are not discussed in this study and are not reflected in the data below.

Table 4: Breakdown of All Students’ Strategies, Kindergarten – 2nd Grade

Student	Number of Problems	Number Correct	Direct Model and Combinations of DM and Counting	Advanced Strategies: Counting, Recalled Facts, Add., Sub., Carrying or Borrowing
Omar	37	33 (89%)	5 (14%)	32 (86%)
Yolanda	39	33 (85%)	10 (26%)	29 (74%)
Gerardo	35	31 (94%)	24 (69%)	11 (31%)
Gina	36	35 (97%)	30 (83%)	6 (17%)

This table shows that Omar and Yolanda preferred to use advanced strategies 86% and 74% of the time, respectively. They did not reach as many correct solutions as the other two students, but were still quite successful problem solvers. Gerardo and Gina used direct modeling 69% and 83% of the time, respectively. They were more successful in finding correct solutions to the problem. From the perspective of correct answers, it could be argued that the CGI problems lend themselves to direct modeling as they are based in contextually rich story problems about objects (Carpenter et al., 1993, 1999), which gives an advantage to the direct modelers in my opinion. One could also argue that Omar and Yolanda pushed themselves toward more advanced approaches and therefore took more risks, my opinion once again. Correct solutions aside, the table above presents a clear distinction between the strategies Omar and Yolanda used to solve CGI problems over time versus the strategies Gerardo and Gina used.

Below, I further break down the above strategies information into tables for each student to show how this information plays out by problem type and by grade. “Num” in the tables stands for the number of problems, “DM” stands for direct model, “ADV” stands for an advanced strategy, and “#C” gives the number of correct problems for this problem type or grade. The left hand columns of the tables show the problem types, how many of each were given over three years and whether the problem was solved with a direct model or an advanced strategy. The right hand columns of the tables take the same problems and separate them by interview, progressing from kindergarten through 2nd grade and once again classify the problem solutions into a direct model or an advanced strategy. Because there were five problem types and six interviews, there is a blank row

at the bottom left of the tables. The analysis of each student's problem solving strategies follows the tables. For complete problem descriptions and solutions see Appendix K.

Analysis of Omar's CGI strategies.

Table 5: Omar's Strategies, K-2nd Grade

Type	Num	DM	ADV	#C	Grade	Num	DM	ADV	#C
Join	11	2	9	11	Kinder	6	1	5	5
Compare	7	0	7	7	1st-Mid	8	4	4	7
PPW	7	1	6	5	1st-End	10	0	10	10
PartDiv	6	1	5	6	2nd-Beg	4	0	4	4
Mult	6	1	5	4	2nd-Mid	4	0	4	3
					2nd-End	5	0	5	4
Totals and %	37	5 14%	32 86%	33 89%	Totals and %	37	5 14%	32 86%	33 89%

In the above table for Omar, we see that he has overwhelmingly chosen to use advanced strategies for all problem types and for all grades. Only in mid year of 1st grade did he show any reliance on direct modeling. Furthermore, it is interesting that of his five direct models, two he solved incorrectly, the Part-Part-Whole (10, 6) in kindergarten and the Multiplication (4, 7), from the middle of 1st grade. As described in his profile in the previous chapter, Omar was not as successful direct modeling as counting and only fell back to direct modeling when he could not find a counting strategy (Carpenter et al., 1999). In the table above, Omar shows an accelerated tendency to move away from direct modeling to advanced strategies (Carpenter et al., 1999) and by the end of 1st grade, he did not use direct modeling for any further problems.

Three examples of Omar's verbal explanations highlight how he abstracted the objects in the problems to a sequence of numbers and then counted to find his answers (Carpenter et al., 1999; Fuson, 1988). His explanations give evidence that he is using an ordinal meaning for the numbers (Fuson, 1988). The first problem was the compare problem in 2nd grade with the larger set unknown, (13, and 6 more). The question asked how many objects were in the second set. Omar said of his answer, "nineteen...I thought in my mind. I just thought of it...I thought...I mean I count...thirteen, and then fourteen, fifteen, sixteen, seventeen, eighteen and nineteen." Omar began with the number 13 in a sequence then counted on six times to get his answer. The words for the number sequence mediated his thinking (Mahn, 2009; Vygotsky, 1987).

The second problem is also from 2nd grade where Omar solved the Join Start Unknown (7, 22) problem about candies. The question was how many did he have to start if he was given seven and wound up with 22? He said, "I'm counting up," as he wrote the numbers from 1 to 7 on a piece of paper, "and counting down," he said of the other sequence as he wrote 21 to 15, "...til I get seven right here...I counted up till seven, and then I counted down until I got fifteen." Omar used a number sequence for his explanation and also wrote the sequences on a piece of paper. As a reference, in Figure 4 below I present his work that I also included in his profile in Chapter 5.

$$\begin{array}{r}
 22 - 7 = 15 \\
 21 \\
 20 \\
 19 \\
 18 \\
 17 \\
 16 \\
 15
 \end{array}$$

Figure 4. Omar, Join Start Unknown (7, 22), 2nd grade.

It is probable that he used number line representations in his mind to mediate his thinking and that he has incorporated this sequential representation of numbers into his system of meaning to make sense of join problems (Mahn, 2009).

Finally, during the interview at the end of 2nd grade Omar related the multiplication of 15 bags of marbles with seven marbles each bag to the representation of a clock. He transposed the problem to seven groups of 15 and explained the relationship to the clock by saying, “Cause, cause, it’s like the clock...cause it has a quarter and a quarter of the clock, and then another quarter, and then another.” Omar had internalized the concept of time advancing in groups of 15 minutes and he used skip counting by 15 seven times because he knew that on a clock face the sequence was in four parts, 15, 30, 45 and then 60 minutes. In his explanation, Omar reveals that he has abstracted the problem away from a focus on marbles, taken the sequence from one to 60 of progressing time, and used this symbol to mediate his thinking and help him solve the multiplication problem.

Analysis of Yolanda's CGI strategies.

Table 6: Yolanda's Strategies, K-2nd Grade

Type	Num	DM	ADV	#C	Grade	Num	DM	ADV	#C
Join	12	2	10	10	Kinder	5	4	1	4
Compare	7	1	6	7	1st-Mid	9	3	6	5
PPW	7	2	5	6	1st-End	12	2	10	12
PartDiv	6	4	2	5	2nd-Beg	4	1	3	3
Mult	7	1	6	5	2nd-Mid	4	0	4	4
					2nd-End	5	0	5	5
Totals and %	39	10 26%	29 74%	33 85%	Totals and %	39	10 26%	29 74%	33 85%

Like Omar, Yolanda showed a strong tendency to use counting and advanced strategies, which increased rapidly as she matured. It was only in kindergarten that she favored direct modeling. By 1st grade she preferred counting and using applications of facts and operations, as noted in her profile in the previous chapter. It was also in 1st grade that Yolanda began demonstrating a high degree of motivation in mathematics, showing increasing confidence in her own problem solving approaches and a desire to solve problems quickly and efficiently. She had internalized the sociomathematical norms of her classroom (McClain & Cobbb, 2001), a clear demonstration of the effects of *perezhivanie* (Mahn, 2009; Vygotsky, 1987) or how she has perceived and internalized the external environment. For all the problems in this study, she only showed a preference for direct modeling on one of the problem types, Partitive Division, possibly

because “It is much more difficult to use strategies involving counting or adding to solve Partitive Division problems” (Carpenter et al., 1999, p. 41).

Interestingly, two of Yolanda’s ten direct models did not help her reach correct solutions, the Multiplication (4, 7) and the Partitive Division (18, 3) problems in 1st grade as described in her profile. In both of these cases the direct models did not help her conceptualize and solve these problems and it appears Yolanda was not able to shift from an ordinal meaning of counting a sequence to the cardinal meaning of discrete groups (Fuson, 1988). I believe that if Yolanda had made meaning of the numbers discretely, the discrete models in front of her would have helped her. Instead, I suggest there was a mismatch between her sequential thinking and her direct model.

Three of Yolanda’s explanations highlight her sequential thinking to solve the CGI problems. In kindergarten she directly modeled the Join Change Unknown problem where she had \$7 to buy a toy that cost \$11. The question was how many more dollars did she need to buy the toy. As described in the profile, she used her fingers and a marker to model the problem. She said, “Pensé y luego conté, conté con mis dedos...Conté, me faltaban cuatro, porque me conté el marcador como así, uno, dos, tres, cuatro, porque no tengo once dedos...Puse el marcador aquí, como estos son once. Con este marcador son once, y vi que faltaba cuatro. (*I thought and then I counted, I counted with my fingers...I counted, I needed four, because I counted the marker like this, one, two, three, four, because I don’t have eleven fingers...I put the marker here, like these are eleven. With this marker there are eleven, and I saw that I needed four.*)” I argue that she made sense of this problem using a sequence of numbers represented both externally and in her mind by her fingers (Mahn, 2009). She saw that she needed four more in the

sequence of numbers to get from seven to 11. Her explanation shows that her focus was on the counting sequence from 1 to 11 rather than on a discrete amount of seven dollars and 11 dollars.

In the middle of 1st grade for the Join Start Unknown (3, 5) problem, Yolanda conceptualized the problem flexibly in relation to the addition and subtraction operations on the sequence of numbers (see Yolanda's profile in Chapter 5 for the problem description). She explained her answer of two candies as the starting amount by saying, "Porque tres más dos son cinco y luego quité tres, tres menos, y luego eran dos más. (*Because 3 plus 2 is 5 and then I removed 3, 3 minus, and then there were two more.*)" In Yolanda's explanation we see her using a number sequence to mediate her thinking about this problem. Her thinking forward and backward along this number sequence is similar to Omar's solution of the Join Start Unknown (7, 22) problem above where he drew two simultaneous number sequences.

Finally, instead of solving the Compare Difference Unknown (31, 29) problem about toy cars at the end of 1st grade by matching two discrete sets, Yolanda counted on from 29 to find the difference. She explained, "Porque no... porque no más le puse... [my cousin] tenía veintinueve y le puse uno más y eran treinta y otro más eran treinta y uno. (*Because, because I just put... [my cousin] had 29 and I put one more and they were 30 and another one and they were 31.*)" To compare these two numbers, she thought of the two sets ordinally as one sequence of numbers where she only needed to count on from the smaller number to reach the larger number (Fuson, 1988). Her explanation reflects her thinking in numbers when she says she put one more with 29 to get 30 and another one to get 31 (Mahn, 2009).

Analysis of Gerardo's CGI strategies.

Table 7: Gerardo's Strategies, K-2nd Grade

Type	Num	DM	ADV	#C	Grade	Num	DM	ADV	#C
Join	9	6	3	7	Kinder	5	5	0	5
Compare	7	5	2	7	1st-Mid	9	9	0	7
PPW	8	4	4	8	1st-End	8	3	5	8
PartDiv	5	5	0	4	2nd-Beg	4	4	0	3
Mult	6	4	2	5	2nd-Mid	4	0	4	4
					2nd-End	5	3	2	4
Totals and %	35	24 69%	11 31%	31 89%	Totals and %	35	24 69%	11 31%	31 89%

Gerardo shows a different pattern of strategies from Omar and Yolanda. Now we see a tendency to continue direct modeling into 2nd grade. We know from his profile in Chapter 5 that he began to incorporate counting and advanced strategies with his direct models rather than replacing direct models with counting as described in CGI theory (Carpenter et al., 1999). We also know from Gerardo's profile that he shifted between direct modeling and counting and we see this shift in the table above. Of all the students in this study, the data show that Gerardo is the most flexible in his use of direct modeling and advanced strategies and can shift between sequential and discrete thinking (Fuson, 1988). Although he prefers directly modeling sets of objects, he is able to use counting strategies when the problem structure lends itself to this type of thinking and he can also combine the two strategies to find his solutions. This indicates he is beginning to make the "shift among meaning" described by Fuson (1988, p. 5) between cardinal and ordinal

number meanings, and is also demonstrating the flexibility needed to learn mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

In his explanations to problem solutions, Gerardo revealed that he tends to make sense of the problems using discrete thinking and a cardinal meaning for the numbers primarily (Fuson, 1988). He explained how he compared two toy dinosaurs with 13 toy dinosaurs in 1st by grade by saying, “Cuz if, um, if, if my cousin Alan, that normal, that’s normal, has two, and Monkey Boy has...thirteen, and Alan has two...and there’s thirteen and you take away two more, it’s going to be eleven.” He continued, referring to the objects in the story, saying, “Alan had two dinosaurs, Monkey Boy had thirteen, and if you take away two you have to count em...” Gerardo referred to his drawing of a direct model of the two sets of dinosaurs and explained his thinking as removing the amount in one set (2) from the other (13).

For the compare problem in 2nd grade of 13 toy cars in one set and 6 more in another, Gerardo showed he was still thinking about the objects as sets rather than as a sequence of numbers. He used a recalled fact to find the answer, but then he explained, “Because if I have thirteen, plus the ones that Omar has, he has the nineteen. Cause he already passed my number with the six that he had.” In his explanation, I can almost see the two discrete sets of cars lined up with Omar’s six extra cars out in front of Gerardo’s because of his use of the Matching strategy in the past. He did not abstract the numbers in the problem to a counting sequence, but compared two sets of objects (Carpenter et al., 1999).

Finally, Gerardo gives an interesting explanation for his thinking about the Join Start Unknown problem in the final interview. I initially misspoke in Spanish, saying

“cien doce” instead of “ciento doce” for the number 112, and he understood the problem to be JSU (40, 100) instead of JSU (40, 112). He mentally solved the situation of an unknown number plus 40 more to give 100 in all by a manipulation of the groups of tens in the problem. He said, “Because, if my, if I get a hundred at last, and I just take the forty away and give it back to my friend, I’ll have sixty candies. But if I keep having those, I already know that I have sixty and forty together.” Here he is flexibly adding and removing a set of 40 candies to 60 candies to get 100 candies. I believe he thought about two discrete sets of candies while at the same time used base ten thinking to manipulate the groups of ten (Fuson et al., 1997).

Analysis of Gina’s CGI strategies.

Table 8: Gina’s Strategies, K-2nd Grade

Type	Num	DM	ADV	#C	Grade	Num	DM	ADV	#C
Join	11	8	3	11	Kinder	5	5	0	5
Compare	7	5	2	7	1st-Mid	7	6	1	7
PPW	6	6	0	6	1st-End	11	10	1	11
PartDiv	5	5	0	5	2nd-Beg	4	2	2	4
Mult	7	6	1	6	2nd-Mid	4	3	1	4
					2nd-End	5	4	1	4
Totals and %	36	30 83%	6 17%	35 97%	Totals and %	36	30 83%	6 17%	35 97%

Just as Omar shows a strong preference for sequential counting strategies, Gina strongly prefers to make discrete models of the problem situations. She seems to enjoy the time spent modeling and carefully approaches each problem purposefully. Unlike

Yolanda, Gina does not approach problems in a way that focuses on the most efficient and quickest route to a solution. These two girls have incorporated different perspectives on what counts as the best way to approach a problem. These perspectives illustrate how *perezhivanie* (Mahn, 2009) interacts with the sociomathematical norms of the classroom (McClain & Cobb, 2001). Recall that the concept of *perezhivanie* explains how individuals perceive, experience, internalize and influence the surrounding sociocultural environment. Like all children in this study, Gina is on her own unique trajectory of mathematical development. She does not show a progressive abstraction of direct modeling to counting strategies as she goes from kindergarten to 2nd grade (Carpenter et al., 1999). However, this does not reflect a student who is challenged academically. She is a mature student who has been recommended for the gifted and talented program at her school. She is a very successful CGI problem solver as seen in the table above. The only problem she solved incorrectly over the course of three years was the Multiplication (15, 7) at the end of 2nd grade where she needed to find the number of marbles in 15 bags if each bag had seven marbles inside. She miscounted and was off by two.

Gina's language demonstrates how she makes sense with discrete sets and thinks in terms of parts and wholes. In kindergarten she approached the Part-Part-Whole (10, 6) problem about balloons by drawing 10 lines. Explaining her solution, she says, "Como (*Because,*)" "ya sabía...ya sé cuantos son seis, y luego puse la rayita, (*I already knew, I already know how many are six, and then I put a little line,*)" "y luego me quedaron cuatro. (*and then I was left with four.*)" She explains that she knew there were six in one group, and then she removed this set of lines from her total to find the remaining set. She thought about the problem discretely as dividing the total into two distinct groups.

In the middle of 1st grade, Gina modeled the Join Start Unknown (5, 13) problem again by creating the whole amount and removing a set to leave the rest as her answer. She said, “Pues, um...hay...le puse 13 y luego conté 5, y le puse...ay...una cajita [around the five]...y luego conté los demás. (*Well, um...there are...I put 13 and then counted five, and it put...ay...a little box [around the 5]...and then I counted the rest.*)” In her solution she operated on the discrete sets of objects rather than the counting sequence, and therefore made sense of the problem based in the cardinal meaning of the numbers (Fuson, 1988).

Finally, in the Join Start Unknown (40, 112) problem, Gina explained her choice of materials to find the solution and showed she understood the structure of the problem. In the problem she had some candies and then her friend gave her 40 more resulting in 112 candies altogether. She said, “O.K., first I didn’t know what I had, but my friend give me forty then I had one hundred and twelve. So, I had to do it on paper [an algorithm] because you wouldn’t have as much of these [the small cubes from the base ten blocks set] and I would get lost in that [the 100s chart].” She used an algorithm because there were not enough blocks, and she felt she would get lost in the 100s chart, a sequential tool. Her rejections of the 100s chart may indicate that she thought of the numbers ordinally (Fuson, 1988). When she finally built 112 with base ten rods and cubes to solve the problem in a different way, I asked her, “Is this the beginning of the story or the end of the story?” She said, “The end, but I can make it the beginning by taking off forty.” She was able to conceptualize both the problem beginning and ending structures by adding or removing the set of 40. Once again her explanation reflects her discrete thinking about the problem (Fuson, 1988).

Comparison of Student CGI Problem Solving: Kindergarten and 2nd Grade

In the previous analyses, the students showed a tendency to use two different ways of making sense of the number situations in the CGI problems, either ordinally using sequences of numbers, or cardinally using sets of discrete objects (Fuson, 1988). Two of the children, Omar and Yolanda, tended to explain their thinking and chose strategies in terms of counting sequences. The other two children, Gina and Gerardo, tended to explain their thinking in terms of the objects in the problems and made direct models of their solutions. Although all the students had instances when they approached problems from the other perspective, which indicates a continuum between the perspectives instead of a dichotomy, the trends above were present in kindergarten and followed the students through 2nd grade. Omar showed the strongest evidence of sequential thinking, Gina showed the strongest evidence of discrete thinking, and Yolanda and Gerardo were closer to the middle of the continuum. Yolanda did more direct modeling than Omar and used decomposition strategies, which indicated she was also capable of thinking discretely. Gerardo showed evidence that he was incorporating sequential thinking into his discrete perspective and was able to shift between meanings depending on the problem structure. Below is a side-by-side comparison of the problem solving tendencies of the two types of thinking.

Table 9: Sequential versus Discrete Thinking

<p>Omar and Yolanda: The <i>sequential thinkers</i> who tend to use a System of Meaning based on an <i>ordinal</i> interpretation of the numbers.</p>	<p>Gina and Gerardo: The <i>discrete thinkers</i> who tend to use a System of Meaning based on a <i>cardinal</i> interpretation of the numbers.</p>
<ul style="list-style-type: none"> • CGI strategies were mostly counting, number operations and mental math. They fell back to direct modeling when they could not find a counting strategy. • Yolanda frequently applied operations and algorithms. Omar decomposed and recomposed numbers in his head. • They began with a number and worked sequentially up and/or down to find the answer. Even when they were comparing sets of objects, they operated on the number sequences rather than on the sets of objects. • They were unsure of their direct models. Did not show confidence in their answers when they did model, and models did not help them self-correct. • They had confidence in their mental math and counting strategies, remembered the numbers well in the problems, and rarely needed problems repeated. • Omar had trouble explaining his thinking, but did talk out loud while problem solving. Yolanda gave concise explanations of her approaches and thinking. 	<ul style="list-style-type: none"> • CGI strategy was usually direct modeling. They tended to start with a direct model of the problem even when they applied a counting strategy to find their answer. • Gerardo tended to begin with a drawing. Gina drew or chose connecting cubes or base ten blocks when they were available. • They began by modeling the whole amount in the problem and then breaking it into parts if possible. They operated on the sets of objects. In comparison problems they built both sets. • They showed confidence in their models and the models helped them self-correct. They were good at manipulating their models. Gerardo was especially good at trial and error and had confidence in this approach. • They did not remember the numbers well in the problems and needed problems repeated. • Both gave detailed explanations of their thinking and were able to reflect on their own thought processes. Both showed metacognition.

In the remainder of this chapter I describe two problems that highlight these different ways of thinking. The first problem is the Multiplication (3, 6) from kindergarten. I have chosen this problem because it lends itself well to direct modeling

with a CGI Grouping strategy (Carpenter et al., 1999) and also to counting strategies since the equal groups of the problem can be counted sequentially. The second problem is the Partitive Division (84, 4) from the end of 2nd grade. This problem *does not* lend itself to counting strategies as noted by Carpenter et al. (1999) because the size of the groups is not known so it provides a good balance to the multiplication problem.

Multiplication (3, 6), Kindergarten: 3 bags of marbles, 6 in each bag

Omar's solution.

Erin, the interviewer, began by reading the problem to Omar. “Omar you have three bags of marbles, three little bags of marbles. And there are six marbles in each of your bags. How many marbles do you have altogether?” Omar wrote “3” on a blank piece of paper, but then stopped and looked up. Erin asked if he wanted to hear the problem again, but he said no. She suggested he draw or use cubes, and he even started to reach for cubes, but then he put them back. After thinking silently for a few seconds he said, “Wait a minute, I know. I know now.” He crossed out the “3” on his paper and wrote “3 x 6”. Erin asked Omar why he wrote that and he said, “It just popped up,” but he could explain no further.

As Erin was reading the problem again to Omar, he began to study the number line running along the wall of the room and then turned to his paper and wrote “= 18” after the “3 x 6.” He could not explain how he found this answer so Erin drew a section of the number line on two pieces of paper and put these in front of him to help with his explanation. Initially, she left out the number 23 on the number line, but Omar pointed this out to her and insisted that the number line be drawn correctly. When he was

satisfied, he gave Erin his explanation. He started by pointing to number one on the number line and counting forward. He said in English, “I just counted...1, 2, 3, 4, 5, 6...1, 2, 3, 4, 5, 6...1, 2, 3, 4, 5, 6,” landing on 6, 12 and 18 after each group of six counts.

Omar’s use of the number line as a tool to aid his thinking, his solution as a sequence of counts along the number line, and his abstraction of the objects in the problem to a sequential progression of equal quantities along the line all reflect that he is making sense of the problem sequentially. He thought about using cubes to build a model, but rejected them for the number line on the wall. I believe this was because the number line more closely matched the system of meaning he had about solving this problem (Mahn, 2009). It is interesting that he immediately saw there was a missing number in the number line Erin drew for him, number 23, further evidence of his ordinal thinking (Fuson, 1988).

Yolanda’s solution.

Erin interviewed Yolanda in Spanish, beginning with the problem, “Tú tienes tres bolsitas de canicas. En cada bolsita Yolanda tiene seis canicas. ¿Cuántas canicas tiene Yolanda en total? (*You have three bags of marbles. In each bag Yolanda has six marbles. How many marbles does Yolanda have in all?*)” Yolanda began by counting to herself on her fingers. She counted the right hand first and then the left hand twice, and then a third time. Apparently not satisfied, she put two fingers on a blank piece of paper as though she was imagining numbers arranged in a column. She finally said, “Diecisiete? (*Seventeen?*)” For an explanation she said, “tres, y tres... y tres y tres y tres y tres, (*three*

and three...and three and three and three and three,” six times in all, and continued, “Y cuento en mi mente eran diecisiete. *(And I count in my mind, there were seventeen.)*”

Because the answer should have been 18 instead of 17, Erin suggested to Yolanda that she draw a picture of her solution to be sure. Yolanda drew three circles, each with six lines inside, apparently counting to herself while she drew. Under the circles she wrote “ $6 + 6 + 6 = 17$ ”. Erin asked her to count again to make sure and when she did, arriving at 18 in her count, Yolanda said, “Uh oh, diecisiete. Tengo que quitar uno. *(Uh oh, seventeen. I have to get rid of one.)*” Then she erased the last line from the third circle in her drawing. When Erin asked which answer Yolanda thought was correct, 17 or 18, Yolanda chose 17.

It is not clear whether Yolanda was making sense of the problem as a counting sequence or as a set of six discrete groups of three, but her confidence in the count she obtained mentally over the direct model she drew shows that the direct model does not reflect how she thought about the problem. She did not talk about the objects in her explanation, but about the series of threes she counted, which may indicate she is moving toward sequential thinking. I believe she was trying to move away from a focus on the objects to an abstraction of the numbers (Carpenter et al., 1999), and may have been in the process of shifting between the cardinality of the answer she saw on paper to a more ordinal answer obtained from her sequential counting (Fuson, 1988). In any case, since the direct model she drew did not represent the way she was thinking about the problem, I believe she chose to change the model to match her mental image.

Gerardo's solution.

Once again Erin was the interviewer and she asked Gerardo the question in Spanish. Gerardo started to draw the bags and marbles on a blank piece of paper even before Erin finished the problem description, counting to himself and moving only his lips. He finished his drawing of three bags with six dots in each bag, and then counted the lines all again silently, indicating the bags with his head as he counted. Finally, he wrote “18” on the paper. Erin asked him what he did and he said, “Estaba contando como mi imagina [sic]. (*I was counting like my imagination.*)” She asked him to show her how he counted, which he did, pointing to the lines in the bags on his paper and counting aloud from one to 18 in Spanish.

When Erin asked Gerardo how he could write the problem with numbers, he wrote, “ $3 + 6 + 6 + 6 = 18$ ” and read this as, “Tres más tres...más seis más seis más seis es igual a dieciocho. (*3 plus 3...plus 6 plus 6 plus 6 is equal to 18.*)” Erin asked him, “Y ¿Cómo supiste ponerlo así? (*And how did you know to put it this way?*),” and he responded, “Porque...porque hay tres bolsas y solo puse así. (*Because...because there are three bags and I just put it that way.*)” She asked him why he added the sixes and he said, “Porque hay en seis, hay seis canicas adentro. (*Because there are in six, there are six marbles inside.*)”

Gerardo's approach shows a difference in his thinking from Omar and Yolanda and a focus on the objects in this multiplication problem. Even though all four students were in the same kindergarten classroom and had the same experiences throughout the school year, this problem shows they were not all making sense of the mathematics in the same way. Their sum total of personal experiences inside and outside of the classroom

learning environment reflected their *perezhivanie* (Mahn, 2009), or how they were individually internalizing and developing mathematical knowledge in relation to the CGI problems.

When Gerardo heard the multiplication problem, he confidently jumped right into drawing the three bags with the six marbles inside. Then he counted all the marbles to find his answer. In his explanation, he referred to the bags and the six marbles inside each and made a direct connection between the equation he wrote and the objects in the story. Gerardo's focus was on the three sets of objects in the problem rather than on a number sequence and showed discrete thinking (Fuson, 1988). His explanation further reinforced his focus on the objects and he explicitly described his verbal thinking when he said he counted with his imagination (Mahn, 2009).

Gina's solution.

Finally, I turn to Gina's solution. Sylvia was the interviewer and began, "Sara tiene tres bolsitas de canicas. Hay seis canicas en cada bolsita. (*Sarah has three bags of marbles. There are six marbles in each bag.*)" Gina began to draw the first bag before Sylvia asked the question, "¿Cuántas canicas tiene Sara en total? (*How many marbles does Sarah have in all?*)" When Sylvia was done with the problem description, Gina drew six circles in the bag. Then she asked two more times how many bags there were in all and Sylvia clarified each time by re-reading the problem. After Gina was clear on how many bags, she drew all three in the same way, drawing the marbles inside in pairs with one under the other working from left to right inside the bags. Then she wrote "18" on her paper. Sylvia asked how she knew and Gina said, "Porque puedes contar cuando los estás haciendo. (*Because you can count when you are doing them.*)"

When Sylvia asked Gina to explain what she did, Gina said, “Como, ya sabía que eran seis [she indicates first bag] entonces puse seis y luego éstas eran doce [indicates 2nd bag] y luego éstas [indicates 3rd bag] eran dieciocho, y ahora cuando conté todos [indicates all the bags] ya sabía que eran dieciocho. (*Because, I already knew that there were six [she indicates the 1st bag] then I put six and then these were twelve [indicates the 2nd bag] and then these [indicates the 3rd bag] were eighteen, and now when I counted all of them [indicates all the bags] I already knew that there were eighteen.*)”

Next Sylvia asked her to write the problem with numbers. Gina wrote, “3”, said , “Tres cajitas, (*Three little boxes,*)”, hesitated, wrote “x,” and finally wrote “6 = 18” to have altogether “3 x 6 = 18” on her paper. She read her equation as, “Tres veces seis es igual a dieciocho. (*Three times six is equal to eighteen.*)”

Like Gerardo, Gina confidently began with a direct model of each of the bags and the marbles inside them. We see in her approach a strong emphasis on the objects in the problem. She explained how she counted to find her answer, first a set of six marbles, then another set of six to make 12 marbles, and finally a third set of six marbles to make 18 in all. She made sense of the problem by counting up three discrete sets of six to get her answer, thinking cardinally about each quantity to find the total (Fuson, 1988). She also explained how she thought verbally, counting the numbers as she was drawing her model (Mahn, 2009).

Student Approaches to Partitive Division (84, 4), 2nd Grade

Next, I present students’ approaches to a different type of problem two years later that asks students to divide 84 pencils among four of their friends, so that each friend gets the same number of pencils. If Omar and Yolanda think about the numbers in this CGI

problem as contained within a sequence that arrives at 84, how will they use their sequential understanding to solve this problem? If Gina and Gerardo think about the numbers in this problem as contained within distinct groups that combine to make 84, how will they apply this discrete thinking to solve the problem? Due to the length of these problem solving interview sessions, I provide summaries of the strategies for each student below and include the full descriptions of students' approaches and the dialogue in Appendix N.

Omar's solution.

Solving this problem was a challenging experience for Omar. He had a good mental estimate to begin with, twenty, but then surprisingly he could not figure out how to share the four additional pencils. His thinking around his initial estimate was transparent. He said, "I thought about the numbers by two and two...two, four, six, and eight." Here he describes how he thought of the tens as units, counted them as eight, divided them into four groups of two each, and counted by twos. He knew that the two were actually two tens so his answer made 20. He showed good base ten thinking (Fuson, 1997) and made sense of the problem with a sequential progression of 20s (Fuson, 1988). When I asked him to use a tool to show me his answer, he chose the number line, correctly counting down by 20 from 84 and winding up at four, reinforcing his sequential approach. But then he admitted that he had not shared all the pencils with his answer of 20 and did not know what the answer would be if he did share all 84 pencils.

We moved on to base ten blocks and Omar began to grow weary. The frustration he felt as part of his *perezhivanie* (Mahn, 2009) was affecting his problem solving ability.

However, he continued to struggle to find a sequential approach. He asked if he could just show and not tell the answer. He represented the problem as eight rods and four cubes, but did not see how he could partition these into four equal groups, although he tried several combinations. He apparently could not shift from thinking sequentially to thinking discretely (Fuson, 1988). He gave up on partitioning and tried to make sequential groups of 14 by lining up the rods and cubes. His choice of 14 was not clear, but it is possible that he was thinking about the four ones in 84. However, building a series of 14s with base ten blocks was not a useful approach I decided, and hoping that he would see what appeared to me to be an obvious solution of partitioning the eight rods and four cubes into four equal groups, I tried to encourage him in that direction. I was not effective. Omar gave up on the base ten blocks and fell back to direct modeling with trial and error on paper using 17 as a possible answer. In Figure 18 below, we see he was able to model and count four groups of 17 accurately, but his good counting skills did not move him any closer to the answer, and he gave up on his direct model as well.



Figure 18. Omar, Partitive Division (84, 4), 2nd grade.

As noted in kindergarten with Yolanda, if Omar had had a system of meaning that incorporated a solid discrete sense of number, then he should have been able to shift to

discrete thinking with the base ten blocks or the trial and error method (Fuson, 1988). I believe Omar could not make this shift because his discrete sense of number was not as well developed as his sequential sense, if at all.

Finally, on his fifth and final method using the 100s chart, Omar selected 19 as a starting point and began to count up, apparently using a trial and error strategy again. Through scaffolding and encouragement, he was able to arrive at the correct answer of 21. The only tool that facilitated Omar's thinking about this problem and helped him find an answer was a sequential tool, the 100s chart. I believe this is further indication that his system of meaning about numbers is rooted in a sequential representation like the number line (Mahn, 2009). It is possible that Omar's strong tendency to think sequentially about the numbers in the problem prevented him from seeing that he could partition 84 discrete objects into four equal groups quite easily by partitioning the tens first and then the ones.

Gina's solution.

Now I would like to switch from the most sequential thinker, Omar, to the most discrete thinker, Gina. If Omar makes sense of number problems from the perspective of a sequential progression of numbers and uses tools that are sequential, I argue that Gina makes sense of number problems by thinking about the numbers as contained in discrete groups that can be combined and separated (Fuson, 1988). She uses tools that facilitate her discrete thinking like drawing, cubes and base ten blocks.

Gina chose base ten blocks to solve this problem and commented that she thought it would be "tricky". To her surprise she solved it easily saying "none tricky," taking the entire amount represented by eight rods and four cubes and just moving them into four equal groups with two rods and one cube in each group. She gave us a window into her

internal dialogue and verbal thinking when she explained her distribution. She said, “I was thinking four first, but it was going to be two, so then I thought there’s no way three, but there is way two, so...I thought two and two? Yeah.” As evidence of her discrete thinking about the number of pencils as a quantity that could be separated (Fuson, 1988), she described her partitioning by saying, “Right, there are four [rods]...put one in each for they could be...equally.”

She chose to also solve the problem on the 100s chart, but her thinking was not particularly clear and she got mixed up in her counting even though she knew the answer to the problem. It could be that this sequential tool did not match the way she was thinking discretely about the problem (Mahn, 2009). It appeared she was trying to use the same approach on the 100s chart that she used with the base ten blocks, separating the numbers into distinct groups, but the 100s chart does not lend itself to the clear separation of sets. Finally, Gina chose to just divide 80 instead of 84 possibly because the 100s chart is presented in rows of ten and she could think about these rows discretely more easily because they are visually distinct.

When I asked if she could do the problem on paper, Gina began with a direct model using a partitive strategy then gave up because of the amount of time she thought it would take her to distribute all 84 lines by ones. It is not clear why she did not think of distributing the objects in groups of 10 like she did with the base ten blocks. This may indicate she is still focused on the discrete objects in the story rather than abstracting them to groups of ten, or she cannot abstract them to groups of ten without an aid like the rods in the base ten blocks. When pushed for an equation, Gina said she couldn’t write one for this problem, indicating she could not think of a way to solve or represent this

problem with number operations. She liked the base ten blocks tool because she could manipulate the discrete groups of ten and explained, “they’re already made!...They’re just tens.” As a discrete thinker, I believe she preferred the base ten blocks tool that allowed her the flexibility to separate the quantity into distinct groups of objects because this reflected the way she made sense of the problem.

In spite of Gina’s ease with solving this Partitive Division problem, it is interesting that she could not think of a way to represent the problem with a number operation. Perhaps similar to Omar, whose strong tendency to think sequentially prevented him from taking a partitioning perspective to find the answer, Gina’s strong tendency to think discretely prevented her from seeing how this problem could be represented in a sequential manner. She may not have had a well-developed sequential sense of number in her system of meaning and for this reason could not shift easily between meanings (Fuson, 1988; Mahn, 2009).

Yolanda’s solution.

The next student I present is Yolanda, a sequential thinker like Omar, who has a strong tendency to think in terms of number operations and algorithmic approaches to problem solving. Instead of using the number line like Omar, Yolanda frequently applied her sequential thinking to addition and subtraction algorithms. This tendency came out in her eventual successful solution to the Partitive Division problem. Whereas Gina could not think of a way to represent the problem with an equation, adding a series of numbers was the only method that eventually worked for Yolanda.

Yolanda struggled to find a solution to this problem. She is a very motivated problem solver who has incorporated the sociomathematical norms of solving problems

efficiently (McClain & Cobb, 2001), and I think she was frustrated with herself for not finding a quick strategy at the beginning. Like Omar, the emotional component of Yolanda's *perezhivanie* (Mahn, 2009) affected her ability to solve this problem. When she tried to draw a direct model, she proceeded unsurely and her strategy for partitioning the 84 pencils was not clear. She appeared to be uncomfortable and confused with the partitive strategy of her drawing. (See the narrative in Appendix N for her emotional reaction to this problem.) Once again I ask: If she had abstracted direct modeling to counting strategies as described in the CGI literature (Carpenter et al., 1999), why couldn't she fall back to a direct model successfully?

Yolanda's confidence as a mathematics problem solver seemed to be undermined right from the beginning as she selected and rejected several tools. Her recall of the numbers and situation in the problem were good, but she explained, "Pero no sé como hacerlo. (*But I don't know how to do it.*)" Yolanda's counting ability and base ten number sense were apparent in her choice of 20 as an estimate when she tried the 100s chart (Fuson et al., 1997), but it appeared that separating all 84 into four equal groups challenged her thinking in the same way Omar was challenged. I believe that neither student could shift from sequential to discrete thinking to partition 84 into four equal groups. Yolanda was very close to the answer at two different times when she was trying to model the problem with base ten blocks, but she could not see it, and gave up before finishing the distribution.

It was a credit to Yolanda's persistence that at the height of her frustration she rallied. She got an idea and went with it, returning to her confidence in the algorithmic approach and especially her ability to apply number operations. She took the inverse of

the division problem, multiplication, and further reduced this approach to an addition of four equal groups. She showed flexibility and an understanding of the connections between operations in her strategy (Hiebert & Carpenter, 1992). Now comfortable with a strategy and a renewed focus, she started with an estimate of 16. With the answer too small, she tried adding four 18s, then four 19s. Suddenly, she realized the answer was four groups of 21. Her calculations are presented in Figure 19 along with her early attempt to directly model the problem.

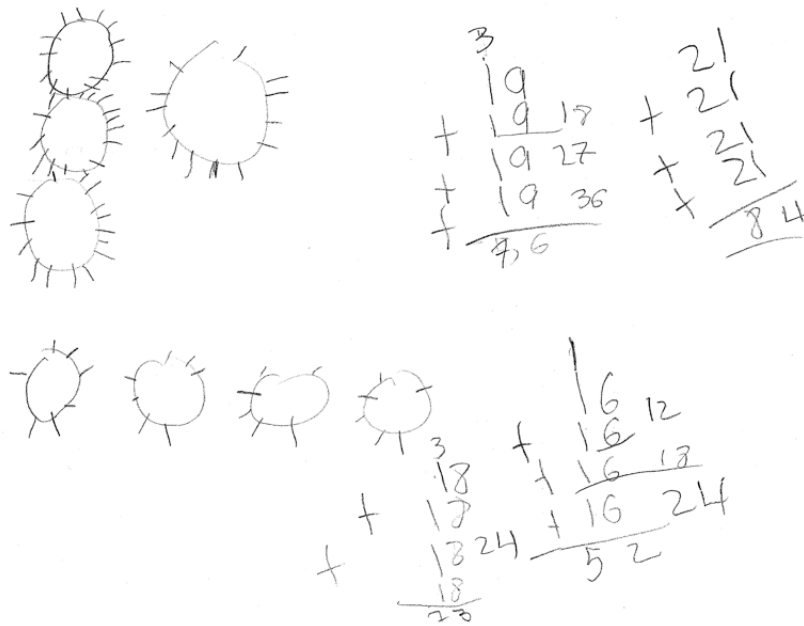


Figure 19. Yolanda, Partitive Division (84, 4), 2nd grade.

Yolanda had to struggle to find a method that reflected the way she made sense of the CGI problems. Finally, she settled on a successful counting strategy with addition. Yolanda’s successful strategy for solving this problem as a sequential thinker is exactly the method that Gina could not apply as a discrete thinker. Recall that Gina said this problem could not be expressed as an equation. Similarly, what was immediately obvious to Gina as a discrete thinker, breaking the 84 into four equal groups, was not at all

obvious to Yolanda. Neither student could shift easily between meanings (Fuson, 1988) to help them approach problems in different ways.

Gerardo's solution.

As described earlier, Gerardo appears to be approaching some flexibility between discrete and sequential thinking and the ability to shift between the two (Fuson, 1988). He is a strong direct modeler and tends to think about the numbers in problems as discrete sets. However, the data show that he is incorporating more counting strategies with his direct models as he matures. He is using recalled facts and number operations (Carpenter et al., 1999), and at times shows that he can apply multiple types of thinking to one problem.

Gerardo's discrete thinking dominates in his approach to this problem. It is interesting that the only tool that Omar could use to find a successful answer to the problem, the 100s chart, was the tool that challenged Gerardo. Try as he might, Gerardo could not uncover the counting sequence he wanted on the chart. He tried to think of the answer as sequential groups, and his first guess of 20 was very good as were the guesses of Omar and Yolanda. When Gerardo saw that his guess of 20 was not going to come out exactly, he tried other numbers, first 30 and then 15. The trial and error attempts challenged him to keep track of the numbers, although he tried very hard, by marking the count with his fingers. Because he was thinking discretely and he was trying to make sense of the problem by thinking of distinct groups, I believe the sequential tool did not match his system of meaning (Mahn, 2009; Vygotsky, 1987) and therefore did not facilitate his thinking.

Similar to Yolanda, Gerardo's characteristic persistence was impressive for this problem even though he could not reach the solution by his first method. For these two students, we see that *perezhivanie* (Mahn, 2009) also includes positive aspects that help children be successful learners. However, when I directed Gerardo toward base ten blocks, he saw the answer quickly because I believe at this point he was able to use discrete thinking and approach the problem with a decompositional discrete strategy (Fuson, 1988). He was able to look at the rods and cubes as discrete objects and partition them out into four equal groups, first the rods and then the cubes (See Appendix N for a complete description of how he solved this problem).

Because he had tried so hard at the beginning of the interview and was unsuccessful on the 100s chart, I asked him to go back to the chart and see if he could show the answer once he knew what it was. His explanation for how he could apply this same strategy to the 100s chart was clear, when combined with his gestures (Moschkovich, 2002; Domínguez, 2005). He did essentially the same thing as he did with the rods, remove rows two at a time from 80 then partition the four ones. This explanation showed that he could use a sequential tool, with some prior knowledge of the answer, in a discrete way, which indicates he was beginning to think flexibly using both cardinal and ordinal meanings (Fuson, 1988).

CHAPTER 7. Discussion and Implications

When investigating whether students understand the mathematics they are performing, we often ask students to explain their reasons for doing what they do. We look for evidence in their explanations that they have connected the pieces of knowledge that support their performance. Students' explanations are their theories of how things work. Asking students to verbalize their theories allows us to interact with them about their thinking.

(Hiebert & Carpenter, 1992, p. 92)

This research examines young students' CGI problem solving in an attempt to understand how they are making sense of the mathematics in the problems. Their explanations about their problem solving strategies were a key component in my analysis based on the connections among external speech, internal speech, thought processes and students' systems of meaning from sociocultural theory (Mahn, 2009; Vygotsky, 1987). The importance of how students describe their thinking is also emphasized in the mathematics education literature. As Hiebert and Carpenter (1992) note above, students can *verbalize their theories* of how things work. We gain more information about students' theories when we expand their language to consider their problem solving approaches and the tools they choose to aid their thinking (Gee & Green, 1998). We also gain a more nuanced understanding of student thinking through CGI problem solving because the CGI framework helps us target specific mathematical concepts (Carpenter et al., 1993, 1994, 1996, 1999).

As I examined and re-examined the four students in this study and their problem-solving strategies over the period from kindergarten through 2nd grade, two distinct ways of making sense of the mathematics in CGI problems stood out. Two of the children tended to think about the numbers ordinally and two of the children tended to think about

the numbers cardinally (Fuson, 1988). Yolanda and Omar appeared to approach the numbers as linear sequences as early as kindergarten. Both Omar and Yolanda generally used counting strategies to solve CGI problems (Carpenter et al., 1999), made mental calculations frequently, quickly reached problem solutions, and looked for ways to apply formal operations and algorithms. When they compared two discrete sets of objects, they abstracted the numbers to a sequence and then counted either up or down to find their answers (Carpenter et al., 1999). Neither student chose direct modeling as his or her first approach and they only used this method when they did not have an idea about a counting or advanced strategy. For Omar and Yolanda, it was difficult to fall back to a direct modeling strategy.

Gina and Gerardo, on the other hand, generally approached problems from a discrete perspective and described their thinking in relation to the objects in the CGI problems not as counting sequences. They both frequently drew direct models of the problems and represented the problem situations by partitioning the whole quantity into parts. In the case of compare problems, Gina and Gerardo built both sets of numbers even when they used a counting strategy to compare the numbers and find their solutions (Carpenter et al., 1999). Gina used connecting cubes when they were available to build direct models, but then was able at times to find the answer to the problems in her head. In the same way, Gerardo frequently drew sets of numbers then showed he could find the answer mentally by counting or a recalled fact.

The tools students chose to help them solve the problems, or the tools I directed them toward if they were unsure of a strategy played a major role in revealing how these students were thinking about the problems. If the tools were based on a sequential

ordering of numbers, like the number line or 100s chart, and if the students were attaching an ordinal meaning to the numbers in the problem, then the tools helped them make sense of the problems in a sequential way and find the correct answers. The same is true for tools that facilitated discrete thinking. Connecting cubes and base ten blocks helped students make sense of problems when they were assigning meaning to the numbers from a cardinal perspective as sets of objects. When students were thinking about the numbers in the problems from one perspective, but trying to use a tool that facilitated the other numeric meaning, for example trying to manipulate discrete sets using a sequential 100s chart, then the tool worked against their sense-making efforts.

Discussion About Sequential Thinking

I believe Omar and Yolanda made sense of the CGI problems by abstracting the numbers to a linear sequence and using some form of this mental representation to move forward and backward either by ones or by skip counting. Tools that facilitated their thinking were sequential, continuous and facilitated counting strategies (Ernest, 1985). Of the tools used in this longitudinal study, the number line was the most popular for Omar, and he gave evidence that he had internalized it as a symbol for his thinking (Mahn, 2009). Yolanda frequently chose the 100s chart and quickly counted up and down on the chart with ease. The 100s chart is sequential because it represents a linear progression of numbers when read left to right and top to bottom. The 100s chart particularly aids skip counting with its arrangement because patterns in number progression are easily encountered. Even though the number line and 100s chart can be physically divided into sets by cutting them apart, they are still fundamentally sequential and subsets made from them are sequential as well. It is possible to use base ten blocks

sequentially to skip count, but they do not intrinsically contain a sequence of numbers like the above two tools. To promote sequential thinking in students, I believe the best tools must facilitate ordering and progression in a linear way.

CGI literature promotes children abstracting the numbers in the problems away from the objects and toward counting sequences, and considers counting strategies more advanced than direct modeling (Carpenter et al., 1993, 1994, 1996 1999). Would sequential thinkers be more advanced mathematically than discrete thinkers? The curriculum used at La Joya Elementary School promotes counting strategies and the children in this study spent a great deal of time developing counting by ones, twos, fives and tens in kindergarten, 1st grade, and 2nd grades. When numbers are seen as being part of a sequential and regular progression, patterns emerge. Numbers in our base ten system progress by ones, but they also progress in groups of tens, hundreds and thousands. Sequential thinking is critical if students are to develop a sense of number patterns and relationships necessary for advanced mathematics including algebra (Fuson, 1988; Fuson et al., 1996; Jordan et al., 2006; Kamii et al., 2005). However, discrete thinking is a critical component of formal mathematics as well (Van Wagenen et al., 1976).

Does sequential thinking about numbers provide a sufficient foundation on which to build a robust system of meaning and mathematical understanding? A sequential ordering of numbers does not necessarily reflect the informal experiences children have with numbers outside of school. I propose that learning to count numbers in order to develop sequential knowledge, like going from 1 to 100, may not be as culturally universal or applicable to children's lives as actually counting objects to determine how many there are, a cardinal task. Children need to be able to connect their informal

experience with formal mathematics to learn mathematics with understanding (Hiebert & Carpenter, 1992; John-Steiner & Mahn, 1996). When formal mathematics in the early grades focuses a great deal of students' time on developing counting strategies, does this focus rob sequential thinkers of the development they need in taking a discrete perspective?

Discussion About Discrete Thinking

I believe Gina and Gerardo made sense of the numbers in the CGI problems by thinking of collections of objects that they could manipulate to find their answers. Tools that facilitated their discrete thinking allowed them to join and separate the objects into groups and included connecting cubes, counters, and base ten blocks (Van Wageningen et al., 1976). Connecting cubes had the advantage of representing the whole amount in the problem situation when they were physically connected and representing the parts of the whole when they were disconnected into sets. Counters facilitated partitioning a whole amount into equal groups with a flexibility that was completely independent of any counting sequence. I believe this is an important characteristic for solving Partitive Division problem situations where the counting sequence is not known. The children's drawings also facilitated discrete grouping because they could devise and manipulate their own representations for the different parts defined in a problem. Number lines and 100s charts challenged the discrete thinkers because their sequential nature was still present when students tried to separate them into parts.

Is discrete thinking less abstract than sequential thinking? If discrete thinkers prefer direct modeling of CGI problems, as is suggested by the data in this study, then CGI theory would suggest that these discrete thinkers are still concrete and should be

encouraged to move toward more advanced strategies like counting and number operations (Carpenter et al., 1993, 1994, 1996, 1999). However, I argue there is great value in direct modeling. While discrete thinking is not synonymous with direct modeling, the direct modeling strategies used by these two students revealed a great deal about how they were making sense of the problems. Gina and Gerardo were successful CGI problem solvers who began to incorporate counting strategies while maintaining their preference for direct modeling. Based on the success of Gerardo and Gina's problem solving in this study, their ability to apply counting strategies to their direct models, and their ability to explain their thinking, I am reluctant to say that they were less "advanced" mathematically than Omar and Yolanda or are less capable of abstract thinking.

If linear thinkers have abstracted the relationship between objects and associated numbers to the number positions in a sequence, is there a way that discrete thinkers abstract number relationships? How would discrete thinkers organize numbers in their minds if not sequentially? Is it possible that discrete thinkers have a conception of numbers that is hierarchical, i.e. they start with the whole collection and decompose it into the parts? Could they think discretely first and *then* sequentially? Fuson (1988) states that children need to develop the ability to shift between thinking ordinally and thinking cardinally about numbers. How they accomplish this task and how they build a system of meaning about numbers that connects sequential and discrete thinking is an area for further research.

Implications of These Findings

For CGI theory:

The development of students' mathematical thinking is complex. The findings from this study show that students can have fundamentally different ways of making sense of CGI problems as early as kindergarten and the meanings they attached to numbers continue to influence their mathematical problem solving throughout the primary grades (Fuson, 1988; Mahn, 2009). CGI theory generalizes the idea that students move from direct modeling to counting strategies, that students' counting strategies are abstractions of their direct models, and that students tend to leave behind direct modeling as they develop the ability to focus on the number sequences as objects of manipulation instead of the discrete objects in a problem (Carpenter et al., 1993, 1994, 1999). The analysis presented in this research refines the theoretical perspective of CGI that all students develop mathematical problem solving along this type of trajectory. The grounded theory discovered in this research (Glaser & Strauss, 1967) shows that students' mathematical problem solving is influenced by more than one meaning attached to the numbers in problems (Fuson, 1988) and that students advance in their mathematical development along different trajectories.

The four students profiled in this study refine our thinking about the way children make sense of the mathematics during CGI problem solving. Two of the students, Gerardo and Gina, are powerful and successful problem solvers who approached CGI problems from a discrete perspective. They did not leave behind direct modeling in their approaches to the problems in 2nd grade, but continued to focus on the objects in the stories to make sense of even complicated problems. Although discrete thinking is not

synonymous with direct modeling, their affinity for direct modeling revealed their understanding of the numbers in the problems as representing discrete collections of objects that they could draw. The strategies Gina and Gerardo used and the explanations they gave of their problem solving demonstrate that they made sense of the problems primarily by discovering the part-part-whole relationships among the objects, and then manipulating the parts and whole to find their answers. Their explanations usually included mention of the objects, which further indicates that they were making sense of the problems by focusing on the objects (Mahn, 2009).

Just as Gina and Gerardo approached the problems from a discrete perspective, Yolanda and Omar used a sequential perspective for their approaches at every opportunity. The evidence from this study suggests that Yolanda and Omar may not have abstracted direct modeling to counting and advanced strategies as was found by the Carpenter et al. research (1993, 1994, 1996 1999), but arrived at their understanding of numbers in a different way. If they had followed this trajectory, I argue they would have been able to fall back successfully to direct modeling approaches, but this was not the case. From the time of kindergarten with Omar and November of 1st grade with Yolanda, these two students demonstrated more focus on the sequences of numbers than the objects in the stories. In their explanations of their thinking about the problems, they referred to the numbers and how they counted rather than the objects in the problems, indicating their thinking was about the numbers and not the objects.

I argue that students should be able to fall back to direct modeling and that being able to create a concrete representation or diagram is an important component in students' mathematics learning (NCTM, 2000). I further argue that if students cannot

create a model of the problem, this indicates a lack of complete understanding of the problem situation and prevents them from approaching the problem with sufficient flexibility. Students' ability to directly model a CGI problem *as well as* demonstrate counting and advanced strategies gives teachers valuable information about whether students can approach problems from more than one perspective. With this knowledge, teachers would be able to help students develop both cardinal and ordinal meanings more fully and promote learning mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz, 1996).

For mathematics education:

The implications for CGI theory in this study extend to mathematics education for all children. Flexibility in problem solving and helping students develop the power behind multiple ways of thinking are important goals for all students (Hiebert & Carpenter, 1992; NCTM, 2000) and even more so for Spanish-speaking Latina/o children who need the access to academic opportunity that mathematics offers (Secada & De La Cruz, 1996). Children also need to be able to use their unique talents to find approaches to problems that make sense to them and help them build bridges between their informal conceptual knowledge and the formal concepts they encounter in the classroom (Lerman, 2001; Vygotsky, 1987).

All four of the students in this study brought unique talents to CGI problem solving. Gina and Omar were very impressive problem solvers and applied their powerful ways of making sense of the mathematics in the CGI problems in sophisticated ways. Omar demonstrated an amazing ability to manipulate numbers in his head and the power of a deep understanding of the patterns and relationships contained in number sequences.

Gina revealed the power and clarity of discrete thinking when she immediately recognized that she could divide a large number, 84, into four equal groups using base ten blocks. Gerardo incorporated increasing flexibility in his thinking (Fuson, 1988) and Yolanda showed she had incorporated the sociomathematical norms important for academic success (McClain & Cobb, 2001). Gerardo demonstrated he could apply two types of thinking simultaneously. As Yolanda matured, she took more risks and constantly pushed herself to more efficient and sophisticated approaches.

Flexibility in both sequential and discrete types of thinking needs to be encouraged and further research could explore how students' mental flexibility expands as they are exposed to increasingly abstract concepts and more variety in problem types. Omar, with his strong sequential thinking, does impressive mental calculations. Gina has an amazing ability to take a discrete perspective to understand complicated problems. Omar and Gina should be encouraged to develop these remarkable abilities, while at the same time encouraged to stretch into other ways of thinking. Important further research could explore how Gina and Omar build on their strengths and develop mental flexibility as they move into the upper grades.

The most powerful mathematical development comes when students can think about problems from both sequential and discrete perspectives (Fuson, 1988). When they can shift between meanings, they have learned mathematics with understanding (Hiebert & Carpenter, 1992; Secada & De La Cruz) and they have built robust systems of meaning that have multiple strong connections among meanings and representations (Mahn, 2009). To gain this flexibility, children need problem types that promote their sequential thinking and also problems that draw on their ability to think discretely and use

decomposition strategies to break numbers down into parts of a whole. Lessons that focus on counting skills and sequential thinking are part of the standard elementary curriculum. I suggest that an increased emphasis be placed on Partitive Division and Part-Part-Whole type CGI problems to help students develop the flexibility they need in discrete thinking for advanced mathematics.

In conjunction with a focus on developing both sequential and discrete thinking, mathematics educators need to recognize the important role that tools play in facilitating children's thinking. Not all tools help children solve all types of problems, as seen in this research. If the problem type and the meaning children attach to the numbers in the problem match the tool they are using, then the tool helps them find a solution. On the other hand, if the tool does not match the meaning children are attaching to the numbers and/or does not help students make sense of the problem using a more appropriate meaning, then the tool interferes with the sense making process. A clear example of this occurred when students tried to use base ten blocks, a discrete tool, to solve the Multiplication (15, 7) problem, which was better approached from a sequential perspective. Another example occurred when students tried to use the 100s chart, a sequential tool, to solve Partitive Division (84, 4), a problem more easily approached from the discrete perspective with base ten blocks. When teachers recognize that certain tools are better for certain types of problems, they are more able to direct children toward appropriate tools that help the children develop the flexibility in thinking important for mathematical development.

For systems of meaning and native language learning environments:

This study examined the strategies students used to solve CGI problems and provided a rich opportunity to explore the sense children were making of the mathematics in the problems. Combining the strategy with students' own explanations of how they approached the problems gave us a window into their thinking that would not have been possible if we only focused on the equations they wrote and/or their correct answers to the problems. Their explanations reflected how they were thinking verbally during problem solving (Mahn, 2009; Vygotsky, 1987). Their verbal thinking was in turn connected to how students made sense of the problems and further illuminated the system of meanings students were constructing about mathematics. This window would have been obscured if the students were forced to make sense in a language other than the language of their thinking (Cummins, 2001; Mahn, 2009; Thomas & Collier, 2002). The words that mediated their thoughts became part of the external expression of their thinking (Mahn, 2009; Vygotsky, 1987). There was no conflict between the language they used for learning mathematics and the language they used for thinking. The four students in this study had the power of two languages to make sense of the mathematics in the CGI problems (Cummins, 2000; Khisty, 1997; Moschkovich, 2002).

I believe the findings in this study show that students have access to their own thinking and can explain this thinking even at a young age when given the opportunity to frame their explanations around specific, contextualized problem solving situations and the actions they used to find answers. Students' explanations revealed their theories of how the mathematics worked in the problems (Hiebert & Carpenter, 1992). The young children in this study gave us a window into their thought processes and described how

they thought in their own words. They used either English or Spanish, whichever language mediated their thinking (Mahn, 2009; Vygotsky, 1987). They described their internal dialogues and gave us evidence that they used both mathematical thinking *and* verbal thinking to solve these CGI problems (Vygotsky, 1987).

For equity in mathematics education for Latina/Latino students:

When given the opportunity to solve challenging mathematics problems and share with us how they made sense of the problem situations, these four students showed a richness of thinking and sophistication of language that moved us closer to understanding children's mathematical development. It is clear that these four students have great potential for academic achievement in mathematics and beyond, and that they should be given access to quality educational opportunities (Secada, 1989a, 1989b, 1991, 1992, 1995). Unfortunately, their academic futures are not secure (Kitchen, 2003). We must not forget that they are Latina/o students, their first language is Spanish, and they come from a Mexican immigrant community. The statistics for academic achievement are not in their favor (NAEP, 2005). Where would these four children be if they had been in primary classes where their only access to mathematical knowledge was through drill and practice (Kamii et al., 2005)? Where would they be if they had to try to make sense of challenging mathematical concepts in an unfamiliar language (Cummins, 2001)?

A major goal for this dissertation has been to promote equity in mathematics education for Latina/o students. It is my hope that by presenting the study through the words and actions of Omar, Yolanda, Gerardo and Gina, everyone who reads these words comes to know the children, feels a strong attachment for them, and cares about the quality of the education they will receive in the future. I hope these children are no

longer just a statistic, but are now the face of potential within the Latino immigrant community. I want my readers to become advocates for all Latina/o children so that these children may have equitable access to high quality mathematics education.

Areas for Further Research

My next step in researching the mathematical thinking of young children is to explore more deeply natural number concepts in the mathematics education literature (Ernest, 1985; Fuson, 1988; Van Wageningen et al., 1976). I admit that I need to expand my understanding of how young children develop the multiple meanings of cardinality, ordinality and measurement. I would like to design research that builds on this dissertation with CGI problem solving and use CGI problem types with the specific goal of promoting flexible understanding in children where they can take multiple perspectives to find sophisticated solutions to problems. I would also like to explore how various tools such as the number line help students develop meaning and how these tools mediate thinking. Questions for further study along this line include what tools children might be internalizing to mediate their cardinal thinking just as the number line mediates ordinal thinking in Omar.

The Final Word

In closing, I give Gerardo the final word.

“O.K., great,” I say to him at the end of the final interview after he has solved the Partitive Division (84, 4) problem. “Is there, have I asked you what’s fun, what you find fun? Oh, you were telling me about ejes, ejes? (*axes, axes?*)” He nods. “Are other kinds of problems fun? Was this problem kind of fun or not?”

He nods and says, “Yeah.”

“Why was it fun? When did it get to be fun? Cuz it was frustrating...”

“At last when I figured it out with these,” and he holds some base ten rods.

“When you figured it out? It was more fun. O.K., do you...”

“I don’t know why,” he continues spontaneously, “but sometimes, every time I see like an easy way, I always have a song in my mind.”

“What? Do you have a song right now?” I ask, thinking he might since he seems very happy to have solved the problem after so much initial frustration.

“No.”

“Oh, but you have kind of a song in your mind?” I press.

“Yeah,” he agrees.

“So do you feel good with a song in your mind?”

Smiling broadly he says, “Every time like I have like a good answer to put on, I, a song always comes into my head.”

I want Gerardo to continue to have a song in his mind as he advances in mathematics education. This can only come through equitable learning opportunities that recognize the potential in every child and open the doors that make academic achievement a reality in each of their lives.

APPENDICES

Appendix A. System of Meaning and Planes of Verbal Thinking (Mahn, 2001)

Appendix B. CGI Problem Types

Appendix C. Kindergarten Post-Assessment

Appendix D. 1st Grade Pre-Assessment

Appendix E. 1st Grade Post-Assessment

Appendix F. Materials for Problem Solving

Appendix G. Student Explanations

Appendix H. Examples of Intermediary Analysis

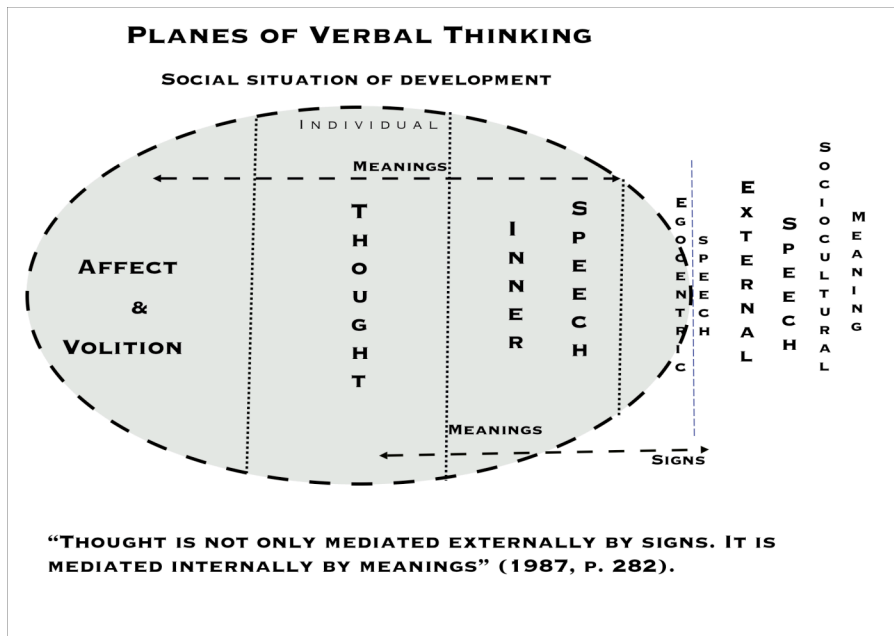
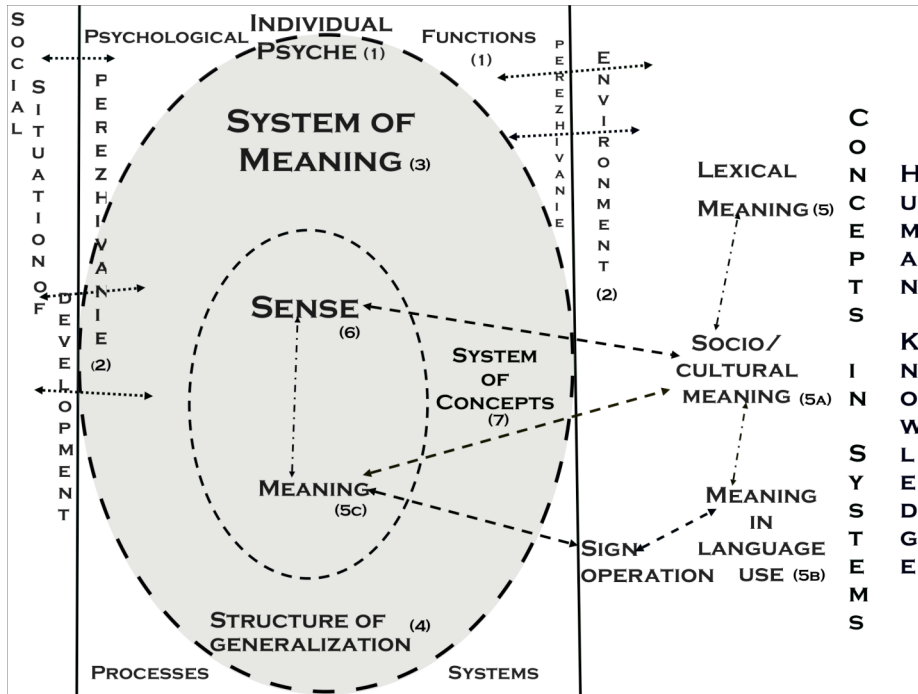
Appendix I. Longitudinal Tendencies

Appendix J. Problem Solving Profile Tables

Appendix K. Students' Problem Solving Tendencies

Appendix L. Students' Solutions to Partitive Division (84, 4)

Appendix A. System of Meaning and Planes of Verbal Thinking (Mahn, 2009)



Appendix B. CGI Problem Types

Join Result Unknown (JRU): Julio has 16 candies. His sister gives him 8 more. How many candies does Julio have in all?

Join Change Unknown (JCU): Karla wants to buy a toy plane that costs 25 dollars. She has 15 dollars. How many more dollars does Karla need so that she can buy the plane?

Join Start Unknown (JSU): Alex had some candies. David gave him 5 more candies. Now he has 13 candies. How many candies did Alex have to start?

Separate Result Unknown (SRU): Paco had 19 cookies. He ate 11 of them. How many cookies does Paco have left?

Separate Change Unknown (SCU): Connie had 20 marbles. She gave some to Juan. Now she has 12 marbles left. How many marbles did Connie give to Juan?

Separate Start Unknown (SSU): María had some pencils. She gave 3 to Jennifer. Now she has 6 left. How many pencils did María have to start?

Multiplication (MULT): Sara has 4 bags of marbles. There are 7 marbles in each bag. How many marbles does Sara have altogether?

Compare Difference Unknown (CDU): Fernando has 21 toy cars. Anabel has 13 toy cars. How many more toy cars does Fernando have than Anabel?

Compare Quantity Unknown: Juan has 5 marbles. Connie has 8 more than Juan. How many marbles does Connie have?

Compare Referent Unknown: Connie has 13 marbles. She has 5 more marbles than Juan. How many marbles does Juan have?

Partitive Division (PART DIV): Jorge had 24 marbles. He shared the marbles with 6 friends so that each friend got the same number of marbles. How many marbles did each friend get?

Part-Part-Whole (PPW): Hector has 35 balloons. 20 balloons are blue and the rest are red. How many balloons are red?

Measurement Division (MEAS DIV): Alan had 18 cookies and some little bags. He put 3 cookies in each bag to give to his friends. How many bags did he put cookies in?

Multi-step (MULTI): Javier has 3 bags of candy. There are 4 candies in each bag. Javier eats 5 of the candies. How many candies are left?

Appendix C. Kindergarten Post-Assessment

Post Test Summary: Ms. Arenas

For each

Problem Type

Note: Valid (V) or Invalid (I) / Correct (C) or Incorrect (I)

Note Strategy: IF VALID -- Direct Model (DM), Counting (C), Derived Fact (DF), Recalled Fact (RF), None Evident (NE), or Unclear (U).

IF strategy is INVALID -- Partial Direct Modeling (PD), Partial Counting (PC), Unrelated Direct Modeling (UD), Unrelated Counting (UC), None Evident (NE), Unrelated (U)

Child	JRU	SRU	JCU	MULT	COMP	PD	PPW	MD	Multi-S	MD-R	#V	#C	#AS
n=16													
Ana	V/C/C	V/C/C	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	10	10	2
Alan	V/C/DM	V/C/DM	V/I/DM	V/C/DM	V/C/DM	I/I/PD	I/I/PD	V/C/DM	V/C/DM	I/I/PD	7	6	0
Brisa	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/I/DM	10	9	0
Donna	V/I/DM	V/C/DM	V/C/DM	V/C/DM	I/I/PD	I/I/PD	I/I/PD	I/I/PD	I/I/PD	I/I/UD	4	3	0
Daniel	V/C/C	V/C/DM	V/C/DM	V/I/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/I/DM	10	8	1
Dolores	V/C/DM	V/I/DM	I/I/PD	V/C/DM	I/I/PD	I/I/PD	I/I/UD	I/I/PD	V/C/DM	V/C/DM	5	4	0
Eddy	V/C/DM	V/C/DM	V/C/DM	V/C/DM	I/I/UD	V/C/DM	V/C/DM	V/C/DM	I/I/UD	V/C/DM	8	8	0
Gabby	V/C/DM	V/C/DM	V/C/DM	I/I/NE	I/I/UD	I/I/PD	I/I/UD	I/I/NE	I/I/UD	I/I/NE	3	3	0
Gerardo	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	I/I/PD	I/I/UD	8	8	0
Gina	V/C/C	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/I/DM	10	9	1
Jesus	V/C/DM	V/C/DM	V/I/C	V/C/DM	V/I/DM	V/C/DM	V/I/DM	I/I/PD	V/C/DM	I/I/PD	8	5	0
Jenna	V/C/C	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	10	10	1
Maria	V/C/DM	V/C/DM	V/I/DM	V/C/DM	V/I/DM	V/C/DM	V/C/DM	V/C/DM	V/I/DM	V/C/DM	10	7	0
Porfi	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	I/I/PD	V/C/DM	I/I/UD	8	8	0
Omar	V/I/DM	V/C/DM	V/C/C	V/C/C	V/C/C	V/C/DM	I/I/NE	V/C/DM	I/I/PD	V/C/DM	8	7	3
Yolanda	V/C/C	V/C/C	V/C/C	V/I/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	10	9	3
#valid	16	16	15	15	12	12	11	11	11	10			
#correct	14	15	12	13	10	12	10	11	10	7			
#adv strat	5	2	3	1	1	0	0	0	0	0			
% Valid	100	100	93.75	93.75	75	75	68.75	68.75	68.75	62.5			
% Correct	87.5	93.75	75	81.25	62.5	75	62.5	68.75	62.5	43.75			
Averages											8.1	7.13	

Appendix D. 1st Grade Pre-Assessment

Primary Study - 1st Grade Year - November 2006 PRE ASSESSMENT SUMMARY **DRAFT**

Mary Marshall and Sandra Musanti

PRE ASSESSMENT SUMMARY - Yellow box indicates advanced strategy

Child numbers	JRU	SRU	JCU	JCU-E	SCU	SCU-H	MULT	CDU-E	CDU	CDU-H
Brisa	V/C/DM	V/C/DM	V/C/DM		V/C/DM	V/C/DM	V/C/DM		V/C/DM	
Ana	V/C/DM	V/I/DM	V/C/DM		V/C/DM	V/I/DM	V/C/DM			V/C/DM
Yolando	V/C/C	V/C/C	V/I/DF	V/C/C	V/C/DM		V/I/DC			V/C/C
Omar	V/C/DM	V/C/DM	V/C/C			V/I/DM	V/I/DC			V/C/DM
Jenna	V/C/DM	V/C/DM	V/C/DM		V/C/DM	V/C/DM	V/C/DM	V/I/DM	V/I/DM	
Gina	V/C/DM	V/C/DM	V/C/DRF		V/C/DRF	V/I/DRF	V/C/DM			V/C/DM
Gerardo	V/C/DM	V/C/DM	V/I/DM	V/C/DC	V/C/DM	V/I/DM	V/C/DM		V/C/DM	V/C/DM

#Val		7/7	7/7	7/7	2/2	6/6	6/6	7/7	1/1	3/3	5/5
#Cor		7/7	6/7	5/7	2/2	6/6	2/6	5/7	0/1	2/3	5/5
#Adv		1/7	1/7	3/7	2/2	1/6	1/6	2/7	0/1	0/3	2/5
% Val		1.00	100	100		100	100	100		100	100
% Corr		1.00	0.86	0.71		100	0.33	0.71		0.66	100
% Adv		0.14	0.14	0.43		0.17	0.17	0.29		0.0	0.40

PartDiv	PartDI	PartDiv	PPW*	PPW-E	PPW	Mdiv-E	Mdiv	Multi	JSU-E	JSU	SSU	#Valid	#Correct
18,2	18,3	24,4	24,6	10,6	13,7	10,2	18,3	3,4,5	3,5	5,13	3,6		
	V/C/DM				V/C/DM		V/C/DM	V/C/DM		V/C/DM	I/I/DM	12/13	12/13
		V/C/DM			V/C/DM		V/C/DM	V/C/DM		V/C/DM	V/C/DM	14/14	12/14
	V/C/RF	I/I/DTE			V/C/DM		V/C/DM	V/C/DM	V/C/RF	V/I/RF	V/C/DM	14/15	10/14
	V/C/RF	V/C/DTE			V/C/DRF		V/C/DM	V/C/DRF	V/C/RF	V/C/DM	V/C/RF	14/14	12/14
	I/I/DM				V/C/DM	V/C/DM	I/I/DM	V/C/DM	I/I/DM	I/I/DM	V/C/DM	12/16	10/16
		V/C/DTE			V/C/DRF		V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	14/14	13/14
			V/I/DTE	V/C/DC	V/C/DC		V/C/DM	V/C/DM		V/C/DTE	V/C/DM	16/16	13/16

2/3	3/4	1/1	1/1	1/1	7/7	1/1	6/7	7/7	3/4	6/7	6/7		
2/3	3/4	1/1	0/1	1/1	7/7	1/1	6/7	7/7	3/4	5/7	6/7		
2/3	3/4	0/1	1/1	1/1	3/7	0/1	0/7	1/7	2/4	2/7	1/7		
	0.75	100			100		0.86	100		0.86	0.86		
	0.75	100			100		0.86	100		0.71	0.86		
	0.75				0.43		0.0	0.14	0.50	0.29	0.14		

Average

I or Invalid (I) / Correct (C) or Incorrect (I)
IF VALID -- Direct Model (DM), Counting (C), Derived Fact (DF), Recalled Fact (RF), None Evident (NE), or Unclear
NVALID -- Partial Direct Modeling (PD), Partial Counting (PC), Unrelated Direct Modeling (UD), Unrelated Counting

#Adv Strat	% Adv	%Valid	%Cor	
0/13	0.00	0.92	0.92	Brisa
0/14	0.00	1.00	0.86	Ana
10/14	0.71	0.93	0.71	Yolanda
9/14	0.64	1.00	0.86	Omar
0/16	0.00	0.75	0.63	Jenna
5/14	0.36	1.00	0.93	Gina
5/16	0.31	1.00	0.81	Gerardo

% Adv	Average %Valid	Average %Correct
0.29	0.94	0.82

Added 8/5/07: **DC for a VALID combination of direct modeling and counting.**
 Added 8/5/07: **DTE for a VALID combination of direct modeling and trial and error.**
 Added 8/23/07: **BTT for VALID base ten thinking
 Added 8/23/07: **CRF for VALID combination of counting and recalled fact.
 Added 8/23/07: **DRF for VALID combination of direct modeling and recalled fact.

yellow is advanced strategy

Appendix E. 1st Grade Post-Assessment

Primary Study - 1st Grade Year - May 2007 POST ASSESSMENT SUMMARY **DRAFT**

Mary Marshall and Sandra Musanti

POST ASSESSMENT SUMMARY - Yellow box indicates advanced strategy

Child numbers	JRU	JRU-H	SRU	SRU-H	SRU-SC	JCU	JCU-H	SCU	SCU-H	MULT
	16,8	24,12	19,11	35,15	45,20	18,9	45, 25	12,5	30,15	7,5
Brisa	V/I/DM	skip	V/C/DM	V/C/DM		V/C/DM	skip	V/C/DM	skip	V/C/DC
Ana	V/C/C	V/C/DM	V/C/DM	V/C/DM		V/C/DM	V/C/DM	V/C/DM	skip	V/I/DM
Yolando	V/C/C	skip	V/I/DM	V/C/C		V/C/RF	V/C/C	V/C/C	skip	V/C/C
Omar	skip	V/C/C	skip	V/I/RF		skip	V/C/C	V/C/DM	V/I/DF	V/C/C
Jenna	V/C/DM	skip	V/C/DM	skip		V/I/DM	skip	V/C/DM	skip	V/C/DC
Gina	V/C/DM	V/C/C	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/RF	V/C/DM	skip	V/C/DC
Gerardo	V/C/DM	skip	V/C/DM	skip		V/C/DM	skip	V/C/DM	skip	V/C/C

#Val	6/6	3/3	6/6	5/5	1/1	6/6	4/4	7/7	1/1	7/7
#Cor	5/6	3/3	5/6	4/5	1/1	5/6	4/4	7/7	0/1	6/7
#Adv	2/6	2/3	0/6	2/5	0/1	5/6	3/4	1/7	1/1	6/7
% Val	100	100	100	100		100	100	100		100
% Corr	0.83	100	0.83	0.80		0.83	100	100		0.86

MULT_H	CDU	CDU-H	PartDIV	PartDiv-H	PPW	PPW-SC*	PPW-H	Mdiv	MD-SC*	MD-H	MD-SC*	Multi	
7,10	+6	13,2	31,29	12,6	24, 6	30,20	20,15	75,50	18,3	25,5	40,10	55,10	3,10,5
V/C/DC	V/C/DM	I/I/DM	V/C/DM	V/C/DTE	V/C/DC		skip	V/C/DM		V/C/DM		V/C/DC	
V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM	V/C/DM		V/C/DM	V/C/DM		V/C/DF		V/C/DM	
V/C/RF	V/C/C	V/C/C	V/C/DM	V/C/DM	V/C/RF		V/C/DF	V/C/DM		V/C/DF		V/C/DM	
skip	I/I/DM	V/C/C	V/C/DF	V/C/DF	V/C/RF		V/C/C	V/C/DM		V/C/DM	V/C/DF	V/C/RF	
V/C/C	V/I/DM	skip	V/C/DM	V/C/DM	I/I/DM	V/C/DM	skip	V/C/DM	V/C/DM	V/C/DF		V/C/DM	
V/C/DC	V/C/DM	V/C/DF	V/C/DM	V/C/DM	V/C/DM		skip	V/C/DM		V/C/DM		V/C/C	
skip	V/C/DM	skip	V/C/DM	V/C/DM	V/C/RF		V/C/C	V/C/DM		V/C/DM		V/C/DM	

5/5	6/7	4/5	7/7	7/7	6/7	1/1	4/4	7/7	1/1	7/7	1/1	7/7
5/5	6/7	4/5	7/7	7/7	6/7	1/1	4/4	7/7	1/1	7/7		7/7
4/5	6/7	3/5	1/7	2/7	4/7	0/1	3/4	0/7	0/1	3/7	1/1	3/7
100	0.86	0.80	100	100	0.86		100	100		100		100
100	0.86	0.80	100	100	0.86		100	100		100		100

Multi	SC*	Multi	SC*	Multi-H	JSU	JSU-SC*	JSU-H	SSU	SSU-H	#Valid	#Correct	#Adv Strat
3,5,5	5,10,5	10, 10, 10			5, 15	5,10	10,34	8,10	10,24			
		V/C/DC	V/C/DM				skip	V/C/DM	skip	17/18	16/18	6/18
		V/C/DM	V/C/RF				V/C/RF	V/C/DM	V/C/DM	23/23	22/23	5/23
		V/C/DF	I/I/DF	V/C/RF			V/C/RF	V/C/RF	V/C/RF	23/24	22/24	18/24
		V/C/DF	V/C/DF				V/C/DF	V/C/DF	V/C/DF	20/21	18/21	17/21
V/C/RF	V/C/DM	V/C/RF	V/C/RF				V/C/DM	V/C/RF	V/C/DF	21/22	19/22	8/22
		V/C/RF	V/C/RF				V/C/DF	V/C/RF	V/C/DM	23/23	23/23	10/23
		V/C/RF	V/C/RF				V/C/DF	V/C/RF	V/C/RF	18/18	18/18	8/18

1/1	1/1	7/7	6/7	1/1	6/6	7/7	6/6
1/1	1/1	7/7	6/7	1/1	6/6	7/7	6/6
1/1	0/1	6/7	6/7	1/1	5/6	5/7	4/6
		100	0.86		100	100	100
		100	0.86		100	100	100

% Adv Strat	%Valid	%Cor
0.33	0.94	0.89
0.22	1.00	0.96
0.75	0.96	0.92
0.81	0.95	0.86
0.36	0.95	0.86
0.43	1.00	1.00
0.44	1.00	1.00

Brisa
Ana
Yolanda
Omar
Jenna
Gina
Gerardo

Average %Valid Average %Correct
0.97 0.91

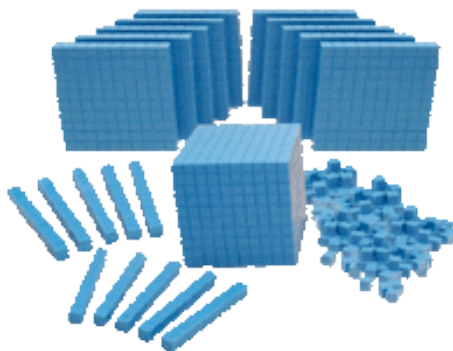
Note: Yellow indicates advanced strategy.

Appendix F. Materials for Problem Solving

Connecting Cubes



Base Ten Blocks



100s Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Appendix G. Student Explanations

Omar

Kinder, MULT (3, 6): “Wait a minute. I know. I know now.” “It just popped up.” “I know. I know now how many marbles that is.” “I umm, I umm, I just counted, six, six, six.” {On the number line} “1, 2, 3, 4, 5, 6...1, 2, 3, 4, 5, 6...1, 2, 3, 4, 5, 6.”

2nd Grade, Beginning: Compare (13, 6 more): “nineteen.” “I thought in my mind.” “I just thought of it...I thought...I mean I count...thirteen, and then fourteen, fifteen, sixteen, seventeen, eighteen and nineteen.”

2nd Grade Beginning: JSU (7, 22), “Fifteen.” “I counted from there,” and he points to the wall in front of him. “Since it was twenty, it was up to twenty, I went in my imagination and I imagined that paper, there was two more papers.” “I counted down,” “Because I counted down,” “Like this,” he counts, “one, two,” then he mouths some numbers, finishing with, “seven.”

2nd Grade, Middle: Compare (13, 6 more): the answer “popped out of my head.”, “three and three is six, but three and three and three is nine,”

2nd Grade Middle: PPW (12, 24) “Cause, cause two plus two is four [the ones], and plus...and ten plus ten is, is twenty [the tens], and two plus two is four.”

2nd Grade Middle: JSU (7, 22) “I’m counting up,” and he points to the increasing numbers from one to five, “and counting down,” he points to the decreasing numbers from 21 to 17. He continues, “...til I get seven right here,” and he indicates the column with increasing numbers. “I counted up till seven, and then I counted down until I got fifteen.” “Cause, it’s easier?” “I don’t know why it’s easy.” “I just drew these,” “and I got the answer.”

2nd Grade Dissertation: Compare (53, 36 less): “I knew, um, it was, two plus three is five [going up 20 with the tens from 36 to 56], and he had thirty less, thirty six less, and I, I just guessed.”

2nd Grade Dissertation: Mult (15, 7) “you mean fifteen times seven?” “Cause, cause, it’s like the clock...cause it has a quarter and a quarter of the clock, and then another quarter, and then another...”

2nd Grade Dissertation: JSU (340, 1012) “Um, I had seven hundred and...hm...I had...I had six hundred...six hundred sixty...um...sixty...sixty twelve, um, I mean, sixty, no, I mean seventy, two candies.” “Cause,” “when...when...when um, when I thought about it, when I was thinking some more,” “I, um, thought about it, and then I knew the answer.”

Yolanda

Kinder JCU (7, 11) “Pensé y luego conté, conté con mis dedos...Conté, me faltaban cuatro, porque me conté el marcador como así, uno, dos, tres, cuatro, porque no tengo once dedos...Puse el marcador aquí, como estos son once. Con este marcador son once, y vi que faltaba cuatro. (*I thought and then I counted, I counted with my fingers...I counted,*

I lacked four, because I counted the marker like this, one, two, three, four, because I don't have eleven fingers...I put the marker here, like these are eleven. With this marker there are eleven, and I saw that I lacked four.)"

Kinder Compare (12, 9) "Porque son tres más y puse una línea para que... porque estos son de Gina y los demás míos. *(Because there are three more and I put a line so that...because these are Gina's and the rest are mine.)"*

Kinder MULT (3, 6) "Diecisiete? *(Seventeen?)*" "Es como así, *(It's like this,)*" "tres, y tres... y tres y tres y tres y tres, *(three and three...and three and three and three and three,*" "Porque, *(Because,)*" "tres y tres son seis. *(three and three are six.)*" "Y luego puse aquí, *(And then I put here,)*" "tres y tres y tres y tres y tres. *(three and three and three and three and three and three)*" "Y cuento en mi mente eran diecisiete. *(And I count in my mind and there were seventeen.)"*

1st Grade Middle JSU (3, 5) "Porque tres más dos son cinco y luego quité tres, tres menos, y luego eran dos más. *Because 3 plus 2 is 5 and then I removed 3, 3 minus, and then there were two more.)"*

1st Grade End: Compare (13, 2) "La pensé en mi mente, le quité dos. *(I thought it in my mind, I took away two.)"*

1st Grade End: Compare (31, 29) "Porque no... porque no más le puse... [my cousin] tenía veintinueve y le puse uno más y eran treinta y otro más eran treinta y uno, *(Because, because I just put... [my cousin] had 29 and I put one more and they were 30 and another one and they were 31,)*"

1st Grade End MULT (7, 10 and 6 more): "Ya lo sé es setenta y seis, *(I already know it is seventy six,)*" "No más le puse siete, siete y sabía que era setenta. Y luego le puse otro seis, y era setenta y seis. *(I just put seven, seven and I knew it was seventy. And then I put another six and it was seventy six.)"*

2nd Grade Beginning: Compare (13, 6 more) "Porque no más cuento con mis dedos porque ya no, ya no puedo marchar inside con la mente... Porque en las vacaciones no estudiaba mucho, y tengo que estudiar más. *(Because I just count with my fingers because now, now I can't function inside my mind... Because during the vacation I didn't study much, and I have to study more.)"*

2nd Grade Middle Compare (13, 6 more): "Mi mente lo tenía grabado, y entonces, dije que trece más seis es igual a diecinueve. *(My mind had it recorded, and then, I said that 13 plus 6 is equal to 19.)"*

2nd Grade Middle PPW (12, 24): "Y entonces... hay veinticuatro... yo, me dije hay veinticuatro horas, y la mitad de veinticuatro horas son doce y entonces lo hice con los globos, entonces dije que la mitad de veinticuatro son doce. *(And then... there are 24... I, I said to myself there are 24 hours, and half of 24 is 12 and then I did it with the balloons, then I said that half of 24 is 12.)"*

Gerardo

Kinder MULT (3, 6): "Estaba contando como mi (?) imagina, *(I was counting like my imagination,)*" "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18," "tres más tres... más seis más seis más seis es igual a dieciocho... Porque... porque hay tres bolsas y solo puse

asi... Porque hay en seis, hay seis canicas adentro, (*3 plus 3...plus 6 plus 6 plus 6 is equal to 18...Because...because there are 3 bags and I only put them like this...Because there are in 6, there are 6 marbles inside.*)”

1st Grade Mid MULT (4, 7): “I started counting from, I started counting from, I started counting from inside...Like from my head [and he puts his hands to his forehead]...counting from seven [pauses to count up silently] fourteen [pauses and counts again] twenty one [counts silently] twenty eight.”

1st Grade Mid Compare (21, 13): “Eight.” “Cuz I started counting em...” “My inside my voice.” “one, two, three, four, five...” then starts again, “one, two, three, four, five, six, seven, eight.”

1st Grade End Compare (13, 2): “Cuz if, um,” “if, if my cousin Alan, that normal, that’s normal, has two, and Monkey Boy has...thirteen, and Alan has two...and there’s thirteen and you take away two more, it’s going to be eleven.” “Alan had two dinosaurs, Monkey Boy had thirteen, and if you take away two you have to count em...” “How many more question?” “If he had more than Alan, then he had more, um he had more dinosaurs than him,” pointing to the word Alan on the paper, “than Alan and he has eleven more than he has,”

2nd Grade Beginning JSU (7, 22): “When I heard that [22]...I told myself to stop at 11 so I could count...to see if it was 22 and then once I knew that it wasn’t 22...I did 3 more.”

2nd Grade Beginning Compare (13, 6 more): “He has, he has twenty,” “...and he needs six more to get twenty like his friend so they could be even.” “I counted with my...Once I got to thirteen, I counted with my fingers six.”

2nd Grade Middle JSU (7, 22): “I was thinking if I had 22 altogether, then I should start taking away the seven that he gave me so I could start where I had.”

2nd Grade Middle PPW (12, 24): “What I, when I was starting at twelve and starting and counting like one and one, and then once I got to twenty four I started...I started thinking again, I started counting again to see if it’s right and it was.”

2nd Grade Middle Compare (13, 6 more): “Because if I have thirteen,” “plus the ones that Omar has,” “he has the nineteen.” “Cause he already passed my number,” “with the six,” “that he had,”

2nd Grade Dissertation JSU (40, 100) ****misunderstood problem numbers but had a good explanation****. “Because, if my, if I get a hundred at last, and I just take the forty away and give it back to my friend, I’ll have sixty candies. But if I keep having those, I already know that I have sixty and forty together.”

2nd Grade Dissertation MULT (15, 7): “I did the bags, put the marbles, and then I started counting em, on one, but to make it faster, every time I started in, like, like, if I’m like in thirty I just go to thirty seven, and do the other ones. If I get, it’s like, it’s like, uh, fifty, I could just do fifty seven the next time.”

2nd Grade Dissertation Compare (53, 36 less): “He has seventeen!” “Cuz I started right here,” “Right here counting and I counted all the way.” “I counted in my mind in ones and ones, and once I got here,” once I already got here, it was thirty five, starting with one, this, it was thirty...[six]”

Gina

Kinder JCU (7, 11): “Es que mi hermano quería comprar un avion y costó...no más tenía siete. Entonces le pusen los crucecitas para saber cuántos hay y los conté para que supiera. (*It is that my brother wanted to buy a plane that cost...only he had seven. Then they [I] put the little crosses to know how many there are and I counted them so I would know.*)”

Kinder MULT (3, 6): “Como, ya sabía que eran seis [indicates the 1st bag] entonces puse seis y luego estos eran doce [indicates the 2nd bag] y luego estos [indicates 3rd bag] eran dieciocho, y ahora cuando conté todos [indicates all the bags] ya sabía que eran dieciocho. (*It's like I already knew there were six [indicates the 1st bag] then I put six and then these were twelve [indicates the 2nd bag] and then these [indicates 3rd bag] were eighteen, and now when I counted all [indicates all the bags] I already knew there were eighteen.*)”

Kinder PPW (10, 6): “Como (*Because,*)” “ya sabía...ya sé cuantos son seis, y luego puse la rayita, (*I already knew, I already know how many are six, and then I put a little line.*)” “y luego me quedaron cuatro, (*and then I was left with four.*)”

1st Grade Middle Partitive Division (18, 3): “Dije, seis en cada una o no? Deja a contar estas. Uno, dos, tres, cuatro, cinco, seis [the 2nd group of 6]. Ah sé. Uno, dos, tres, cuatro, cinco, seis [she counts the 3rd group]. Bueno, sí. (*I said, six in each one or not? Let me count these. 1, 2, 3, 4, 5, 6 [the 2nd group]. Ah, I know. 1, 2, 3, 4, 5, 6, [the 3rd group]. Good, yes.*)”

1st Grade Middle JSU (5, 13): “Pues, um...hay...le puse 13 y luego conté 5, y le puse...ay...una cajita...y luego conté los demas. (*Well, um...there are...I put 13 and then counted five, and it put...ay...a little box...and then I counted the rest.*)”

1st Grade Middle PPW (13, 7): {part of explanation} “Y luego, me di este, (*and then I said this to myself,*)” “Hice a ver, deja contarlos, uno, dos, tres, cuatro, cinco, seis, siete, (*I did it to see, let's count them, one, two, three, four, five, six, seven*)”

1st Grade End JCU (9, 18): “Porque el tiene nueve, como si le ponen uno con los demas es diez y le van a quedar ocho, y como ya tienen nueve y le quité uno y si puse aquí, y ahora le quedan nueve, (*Because he had nine, because if they put one with the others it is ten and that leaves eight, and like they have [one row of cubes] nine and I took one off and put it here, now they are left with nine [the other row of cubes].*)”

2nd Grade Beginning MULT (8, 5): “Luego dije, a ver [she taps her cheek again with the forefinger of the right hand] voy, voy a hacer tres de mis dedos de la, de una mano y cinco de la aquí [she puts down 3 fingers of her right hand and all the fingers of her left hand on the table] y así estoy a contando de cinco en cinco y para, usando los dedos. (*Then I said, let's see [she taps her cheek] I am going, I am going to do of my fingers of the, of one hand and five of the one here [she puts fingers on the table] and in this way I am counting by fives for, using the fingers.*)”

2nd Grade Beginning Compare (13, 6 more): “Ok, como...tenía trece...y luego, um, dije, a ver, le voy a contar 6 más a trece, y luego fui contando en uno en uno, y luego dije, hasta seis. Y luego puse mis dedos así [she puts the 5 fingers of her left hand and thumb of her right hand on the table], y luego fui contando. Y ya supe la respuesta. (*Ok, like...he*

had 13...and then, um I said, let's see, I am going to count six more to 13, and then I was counting by ones, and then I said, until six. And then I put my fingers like this [comment above] and then I was counting. And I just knew the answer.)”

2nd Grade Middle Compare (13, 6 more): “OK...first...eran trece, trece de mi prima.

Then sumé seis más y me hace que como seis más tres es nueve. Entonces puse diecinueve. (Ok...first...there were thirteen, thirteen of my cousin. Then I added 6 more and it seems to me that six plus three is nine. Then I put nineteen.)”

2nd Grade Dissertation JSU (40, 112): “Ok, first I didn't know what I had, but my friend give me forty. Then I had one hundred and twelve. So, I had to do it on paper because you wouldn't have as much of these [she points to the little cubes from the base ten blocks] and I would get lost in that [the chart] and these are too little [the number line].” {starting with 112 in rods} “Is this the beginning of the story or the end of the story?” “The end, but I can make it the beginning by taking off forty,”

Partitive Division Problem (84, 4) 2nd Grade: Language

Omar

“Twenty,” “Uh, cuz I,” “Um, I...I...In my mind? I, I...in my mind?” “I, I thought about the numbers by two and two,” “Two four six,” “and eight.” “And here's four, here's four,” {he is pointing to 4 groups 2 cubes each} “and then, and then I thought about it,” “and it was eight?” {marks number line at 84, 64, 44, 24} “Here's twenty.” “I didn't share all the pencils,”

{Now with base ten blocks} “There's ten.” “fourteen.” “here's another fourteen.”

“another fourteen.” “Can I start over?” “Seventeen?”

{Finally on the 100s chart} “I was going like this,” “Twenty, twenty one,”

Yolanda

“veinte (twenty)”, {on the 100s chart, then} “diecinueve...no entiendo. (nineteen...I don't understand.)” “Sí, pero no sé como hacerla. (Yes, but I don't know how to do it.)”

{After a try with base ten rods she develops her own method of addition} “Veintiuno,

(Twenty one,)” “Porque, (Because,)” “Veinte más veinte más veinte más veinte es

sesenta, y luego veintiuno... (20 plus 20 plus 20 plus 20 is 60 and then 21...)” “Veinte

veinte...20 mas 20 mas 20 mas 20?” “Dos cuatro seis ocho, (two four six eight,)” “Es

ochenta, (It's eighty,)” “Es ochenta y cuatro, (It is eighty four,)”

{Back to the base ten rods} “Veintiuno, veintiuno, veintiuno, veintiuno, (Twenty one,

twenty one, twenty one,)” “Es igual a ochenta y cuatro. (Is equal to eighty four.)” “Ya está

facil porque yo lo sé, (Now it's easy because I know it,)” {of the base ten blocks} “Esta

sea mas facil para mi pensar bien, (This is the easiest for me to think well,)” “porque los

demas tendria que ser [inaudible] ver la respuesta...pero no sé cual es mas facil. (because the others you have to be [inaudible] to see the answer...but I don't know which is the easiest.)

Se me hace que es este, (It seems to me it is this,)” pointing to the paper,

“porque solamente escribe los numeros alli y te sale la respuesta pero necesitas caber

todo, (because you just write the numbers there and you get the answer but you need to cover the whole thing,)”

Gerardo

{Begins on the 100s chart} "...ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty...I'm thinking that if I," "have eighty four and I take away twenty," "I might have...[inaudible] ok, so that's nine. I forgot what I was ..."
{Winds up with two} "How about thirty?" "I wonder if I should try..."
{Moves to base ten blocks} "One two three four five...six seven...eight," "One two three four." "Ok, I'm just going to move these four away," "So let me take out...I'm going to give...I have four friends, here," "I'm going to give ten to one," "ten to one, ten to one, and ten to another." "Ten to one, ten to one, ten to one, and ten to another," "and then I give one to one," "and one." "Ok, so they have," "and I give all of them one cube," "And then I don't get to have any." "Is twenty one,"

Gina

"Ah, this is going to be a little bit tricky,"
"None...tricky!" "I just did it,"
"Ok," "It was eight here. It's easy."
"Ok, estos los dejé por allí, (*Ok, these, I left these over there,*)" she says and moves the cubes aside. "I was thinking four first, but it was," "going to be two, so then I thought there's no way three, but there is way two, so" "I thought two and two? Yeah,"
"These?" she asks touching the 4 cubes. I agree. "Right, there are four," "put one in each for they could be," "equally."
"Eran ochenta y (*It was eighty and*)...oh yeah, I can't do it on paper." "I don't know how to do it on paper," "Don't like to. Well...I do but not in this problem."
"Yeah, but, con este, (*Yeah, but with this,*)" "No se me parece conmigo. No más en otros problemas, como minus, (*It doesn't seem to me that I can. Only in other problems, like minus,*)" {Can't write with addition or subtraction} "Si, porque no más son dos numeros, que tienen que repartir, (*Yes, because there are only two numbers, and they have to share,*)"
{Why she prefers base ten blocks for this problem} "I think this one in that problem."
"cause it's easy, cause they're already made!" "Look," "They're just tens."

Appendix H. Examples of Intermediary Analysis

Table H1. Omar – Join Change/Start Unknown

Grade	Questions/Clarifications	Strategies and Answer	Explanations
<p>Kinder (PRE) - JCU (11, 7) Alan esta poniendo velas en un pastel de cumpleaños para ti. Ahora, hay cuatro velas en el paste. Cuantas velas mas tiene que poner Alan para que haya siete velas en total?</p>	<p>When Sylvia mentions <i>his</i> birthday cake and for <i>him</i> this gets his attention and he looks at her quickly and smiles.</p> <p>Erin asks if he had 4 and then put 3 more how many would he have. At first he says 6, she asks him to be sure, he counts, then says 7.</p>	<p>He says “tres” right away without hardly – or even any apparent – thinking.</p> <p>He is playing with a string of unifix cubes connected.</p> <p>When Sylvia doesn’t respond he ventures diez?</p> <p>He is using the cubes to connect and disconnect groups of 3 and 4.</p>	<p>Closed questions: S: Y cuantos tenías aqui? O: cuatro S: ...tu supiste que tres nada mas? O: Tres, nada mas</p> <p>He does not explain the answer, but shows by putting together and taking apart groups of 3 and 4.</p> <p>E: Tenía [Alan] cuatro, y quería... O: ((jumps in)) Poner siete! ((and he puts the 3 and 4 groups of cubes together again))</p>
<p><i>Comments: Omar shows in October of Kinder that he has an instinctive feel for the join change unknown action. He knew right away, instinctively, that the four plus some more would give him seven. Somehow he went up from 4 to 7 in his head very fast and got 3. He could not explain. He shows clear understanding of the problem context.</i></p>			
<p>Kinder (POST)- JCU (11, 7) You want to buy a toy plane that costs 11 dollars, but you only have 7 dollars. How many</p>	<p>What? Erin repeats several times. It is not clear at what point he begins to understand. He does not ask questions.</p>	<p>Several attempts at equations. Probably not serious. At one point he says “3” Finally: I just umm..((looks around)) ***he may be looking at the number line on the wall.*** (then he says) 4. On paper: ((he finishes his equation “7+4=11”))</p>	<p>For 3: “It just popped out.” Finally he indicates he used the number line and started at 7. Then I ((he gets up and goes to his right. The camera moves to show O standing under the number line and pointing upward.)) 1, 2, 3, 4...11. ((Showing how he counted up from 7 to 11)).</p>

more dollars do you need...			
<i>Comments: Quite a bit of effort to get him to focus. When he does he uses the number line on the wall to start at 7 and count up to 11, getting 4. Very linear approach.</i>			
<p>1st Pre – JCU (25, 15) Gerardo wants to by a toy plane, but this toy plane is pretty expensive. It costs 25 dollars ((O begins to draw a rectangle)) and he only have 15 dollars. How many more dollars does Gerardo need ((O completes the rectangle)) to buy the plane?</p>	<p>He remembers the numbers.</p>	<p>So, he wants to by a toy plane that costs 25 dollars ((O writes a big “25” in the middle of the rectangle)), but he only has... He only <i>has</i> 15 (he knew this) ((O writes 15 to the right side of the rectangle)) How many more dollars does he need? ((O encloses the 15 in a box, then writes “15” in each of the 4 corners.)) He needs...((he draws 3 circles in the upper right hand corner of the paper, then 7 circles under the rectangle. He has draw 10 circles in all.)) 10. ((he has was moving with the marker in it as though he was counting the circles but the camera was not focused on his face.))</p>	<p>Because I counted them ((O gets another piece of paper))</p> <p>I counted like this ((He is uncovering a marker)) I counted like this ((He begins to draw circles to the left side of the paper.))</p> <p>I knew it was circles that could, um, could make it.</p> <p>16, 17, 18, 19 ((I is pointing to circles while he counts)) 20, 21, 22, 23, 24, 25 [This is not to help him, because he could do this confidently without – don’t know why I did this.]</p>
<i>Comments: He just counted up from 15 to 25. Did not appear to use base ten thinking. In his model he wrote 15 and then counted on. He remembered the numbers and did not ask questions.</i>			
<p>1st Pre – JSU (5, 13) OK. Let’s put you in too. Gerardo has some candy, and then you</p>	<p>He does not understand the first time: hum? 2nd reading: How many did I give him?</p> <p>I try smaller numbers 3 and 5: It was 2.</p> <p>You gave him 5 ((O</p>	<p>Much scaffolding for the larger nums. He goes to pencil paper. Draws 5 circles.</p>	<p>For smaller (3, 5): Because 2 plus 3 is 5 ((show with his fingers))</p> <p>I just guessed. He knows he has 13 circles, I ask how many he crossed off. I crossed off 5</p>

<p>give him 5 more candies and he ((O is writing words on a paper)) now has 13 candies. How many did Gerardo have to start with?</p>	<p>draws more circles. Now he has 5 circles below the original 5. Then he puts three more on the top of the paper.)) I like your picture. ((O crosses out the 3 on top and two more below.)) What's your answer? umm, 8.</p>		
<p><i>Comments: Omar does not comprehend this problem except with very small numbers that he can model on his fingers. It seems like he finally solved the problems by thinking of 5 plus something to get 13, using the same thinking as with the smaller numbers. Then he drew circles until he got the 13. Amazingly challenging for him.</i></p>			
<p>1st Post – JCU (90, 45) then (45, 25) The Playstation 3 costs 90 dollars and you have 45 dollars. How many more dollars do you need to buy the Playstation 3? Smaller: You want to buy this playstation that costs 45 dollars, but you have 25 right now. How many more</p>	<p>Huh? O: 65 He doesn't want to explain, asks for smaller nums. ((he sits quietly for a few seconds, then writes "2")) O: How much...does it cost? O: And how much do I have?</p>	<p>He is not focusing, but does not want the numbers smaller initially. : ((writes a 0 so now he had "20" on his paper))</p>	<p>O: 35 ((holds up a finger)) and then 45 ((holds up another finger. He was thinking and not distracted)).</p>

dollars do you need?			
<p><i>Comments: He is distracted initially by writing equations like 1000×1000</i></p> <p><i>He solved the smaller problem by adding ten to 25 to get 35 and then 10 to 35 to get 45. This is once again a linear progression (forward).</i></p>			
<p>2nd Pre – JSU (22, 7) This is about Alex, and Alex had some candies. We don't know how many ((O is still keeping his hands busy with the tiles and rods, but appears to be listening)). Remember, you can draw or use cubes or whatever. And then his friend David gave him 7 more candies, and now he has 22. How many did Alex have to start with? JSU (22, 7) ans 15</p>	<p>1st time: O: huh?...what? O: We...huh? Tell me again? He wants to hear the whole problem again.</p>	<p>O: ((after only 3 or 4 seconds staring in silence...)) 15.</p>	<p>O: I counted from there ((and his points toward the wall in front of him)). Since it was 20, it was up to 20, I went in my imagination and I imagined that paper, there was 2 more papers. I: So you imagined a 21 and a 22. O: I counted down. He says he counted down by 7 to get 15.</p>
<p><i>Comments: There is some number representation on the wall in front of him that goes up to 20. He imagines that it goes to 22 then he counts down 7 times to get 15. Still a linear</i></p>			

<p><i>conceptualization, but now he able to go down on the number line. He is showing more flexibility about moving up and down on the number line in his head. He is also imagining and extending the number line. This imagination shows that he is incorporating this symbol for thinking.</i></p>			
<p>2nd Mid - JSU (22, 7) You had some candies, and then Gerardo gave you 7 more candies, and after that you had 22 candies. How many did you have to start?</p>	<p>I ask him to retell: O: I, uh, I had some candies. Gerardo gave me 7. Altogether I have 22. O: How many candies do I have after. O: before, before</p>	<p>O: Let me do it with my fingers. O: Actually, a picture ((he reaches for paper)) O: Oh yeh, hm ((he pauses, then starts to write, "22 - 7 =". Then below he writes 21, and below 20, then next to the 21, next to the 20, 2. He pauses then writes "19 3" the "18 4" and "17 5"))</p>	<p>O: I'm counting up ((he point to the increasing numbers 1 to 5)) and counting down ((he points to the 21 to 17)) O: till I get 7 right here ((indicating the column from 1 to 5. He slowly writes "16 6" then "15)) 15! O: I, uh, counted, this up ((the 1 to 6)) til I got 7. O: I counted up til 7, and then I counted down until I got 15.</p>
<p><i>Comments: He did it this way because he said it was "easier" but he did not know why it was easier. Now we see him use that flexibility of going up and down on the number line in he same problem.</i></p>			
<p>2nd Diss. - JSU (340, 1012) This is about a huge number of candies. You have, you're in Candyland and you have a pile of candies in front of you and then your friend gives you three hundred and forty</p>	<p>O: Three hundred and, how much? Three hundred?</p>	<p>O: um ((he is looking off, lips move slightly, fingers move slightly)) um...I had seven hundred and...hm...I had...((obviously concentrating deeply)) I had six hundred...six hundred sixty...um...sixty...sixty twelve, um, I mean, sixty, no, I mean seventy, two candies. *****<i>You can just see his mind working. He seems to be going up from 300 to a 1000 first and gets 700, then he realizes the tens and ones are going to give him another hundred so</i></p>	<p>O: Cause...when...when...when um, when I thought about it, when I was thinking some more ((he is looking at the cubes not at me)) I, um, thought out it, and then I knew the answer ((he looks up at me))</p>

<p>more candies, and now you have a thousand and twelve candies ((he is just listening))</p>		<p><i>he backs down to 600. Then he works on the tens and ones together. He goes from 40 to the next hundred and gets sixty, then adds on the 12. He says sixty twelve but corrects himself to seventy two. *****</i></p>	
<p><i>Comments: His thinking is revealed as he is solving the problem and talking out loud. He cannot go back and recreate in words what he was doing. Now he is doing amazing work with a number line inside his head, manipulating large numbers by decomposing and recomposing them.</i></p>			
<p>Omar's Comments: He likes to solve problems in his mind. His responses are from the PPW problems. O: Mind ((he says right away)) O: Cause...I don't know ((he is still playing with the cubes)) He agrees to both easier and faster. I: We can do another problem. Was this one [the PPW(805, 565] fun with the big numbers? ((he nods more enthusiastically)) <i>**Omar does like manipulating big numbers in his head and getting correct answers. These big numbers do not frustrate him but grab his attention and focus**</i></p>			
<p>My final comments: Omar started in kindergarten actively using the number line on the wall to solve problems. During one interview problem Erin made a number line for him on paper (multiplication I think) so he could show her how he solved it. Then he went from moving forward to moving backward and moving both ways at the same time. We see his use of the number line in his head for the final problem with large numbers. He expresses a preference for using his mind.</p>			

Table H2. Gina – Part-Part-Whole

Grade	Questions/Clarifications	Strategies and Answer	Explanations
<p>Kinder- Part-Part-Whole (10, 6) Tu tienes 10 globos. 6 globos son azules... y los demas son rojos.... Cuantos globos son rojos?</p>		<p>((G starts to draws lines on her paper with a red marker)) ((She draws them in pairs, two above, two below, then one more above and below)) ((she draws 4 more lines alternating top and bottom rows. Now she has two rows of 5 lines to make 10. ((she makes a vertical line dividing 6 and 4 lines on the paper)) G: ah! ((this we only hear, but then she writes “4” on her paper))</p>	<p>G: Como ((she circles the 6 lines with her finger)) ya sabía... ya sé cuantos son 6, y luego puse la rajita ((indicating her vertical line with her finger)) y luego me quedaron 4 ((she circles the 4 lines with her finger))</p>
<p><i>Comments: This is definitely partitive thinking. She drew all and divided into parts based on what she knew, the 6. Then she recognized the other 4.</i></p>			
<p>1st Pre – PART-PART-WHOLE (13, 7) Gina, esta preparando por su compelanos y tenía 13 globos, y 7 de</p>	<p>G: Deja divertales ((propobably referring to the party. I says “que bueno”)) Cuantos tenía yo? I: 13 en total, y 7 de esos globos son azules, y el resto rojos.</p>	<p>G: uh huh ((she gets paper in front of herself and uncaps a red marker)) She begins to work with cubes, playing with the</p>	<p>G: Pues, conté estos ((she grabs the 6 red and pink cubes)) y luego de estos ((she grabs the 7 green cubes)) She describes how she self-</p>

<p>los globos eran azules ((G starts to draw a red balloon in a heart shape with a tail)) y el resto de los globos eran rojos ((G draws another balloon, then she looks up)) Cuantos globos rojos tiene Gina?</p>	<p>O: oh ((she now has 3 red balloons)) I: Queremos saber cuantos rojos tenías. G: Pues....((she stops drawing and pauses)) ohhh....hm ((stares off to the right and appears to be thinking)))....hay.....3, no mas estos? ((she points to the 3 balloons she has drawn)) I: 13 G: 13 tenías. Ah por eso estaba pensada que difícil, no?</p>	<p>colors. Below is he beginning. G: ((thinks, caps the marker, whispers something inaudible like “esta bien”, and begins to count out cubes. She gets 2 brown and a yellow, moves them aside, then starts to get red cubes. She says faintly to herself, but the camera is not on her face...)) mejor los rojos como ((she gets 4 reds)) y mejor los verdes para...((she starts getting green cubes)) Finally she goes back to her paper to draw the answer. This problem was easy for her and she played around with the materials, I think. G: Seis ((she says with confidence))</p>	<p>corrected. G: y luego, me di este ((she puts the extra pink one that she had before with the reds and pinks)) hice a ver, dejar contarlos, 1, 2, 3, 4, 5, 6, 7 ((she does this aloud pointing to the cubes, then continues with the green cubes)) 8, 9, 10, 11, 12, 13 ((she stops with her finger on the 2nd to last cube)) y lo mejor quite este de rosa ((she removes the last pink cube))</p>
<p><i>Comments: This problem was easy for her. She decided to do it with cubes because she likes cubes. She focused on getting the colors right. In her description she explains how she self-corrected.</i></p>			
<p>1st Post –</p>	<p>NO SOUND ON TAPE!!</p>		
<p>2nd Pre</p>		<p>G: ((in a</p>	

<p>PART-PART-WHOLE (24, 12) Herman tiene 24 globos ((G starts to get out tiles)) 12 de los globos de Herman son azules ((she has 2 base 10 rods)) y los demas son rojos. Cuantos son rojos? Cuantos son rojos?</p>		<p>whisper)) 24 ((she puts them back and reaches for a different tool)) ((she sets down the two rods and gets 4 small cubes. She said something inaudible)) ((she begins to count to herself along one of the rods with her finger, then she puts two fingers of her right hand together on the end of the other rod and used the finger of her left hand to count the sections)) G: ((she continues to count the rest of this one rod, then counts the 4 single cubes and says...)) doce. **She must used the first rod as 10 and the first two sections of the 2nd rod as 12. Then she counts the rest of the 24 she has built with 2 rods and 4 singles**</p>	<p>G: Ya sé que estas ((holding up one of the rods)) una decena G: Verdad? Luego, como...((she pauses holding the rod)) como...10...y luego le cuentas 2 son doce... G: ((takes the rod and puts it to the right of the 2 she is holding with her fingers so the two rods are lines up right to left with the 10 first and then the 2 sections to make 12)) como estos. Luego, conté los demas ((she draws the finger of her left hand along the rest of the second rod and then turn to grab the 4 single cubes)) G: Hay...uh...8 ((she just looks and does not count the sections)) G: Y...9 ((put a cube in front of the rods)) 10 ((moves another cube)) 11 ((the 3rd cube)) luego 12 ((moves the 4th cube)).</p>
<p><i>Comments: She used the rods as a linear tool. She made 24 then counted out 12, held the place with her finger and counted the rest. Even though she said it was a decena, she didn't use this to solve the problem.</i></p>			

<p>2nd Mid – PART-PART-WHOLE (24, 12) I: OK, tienes 24 globos... Y doce son rojos y los demas son azules. Cuantos son azules?</p>	<p>G: I have 24 balloons and.. ah ((she sighs and stops, starts to speak and then stops and looks at me)) I: do you remember how many were red? G: no ((she shakes her head)) I: doce G: OK, yeh I: and? Los demas son azules. G: I need to do it. I: and la pregunta es? G: I don't know the question ((she shakes her head)) REPEAT She is distracted. REPEAT IN ENGLISH</p>	<p>After English: G: OK ((to herself)) 24 ((she draws tally marks up to 24. She draws a line under two groups of 5s and then two more. She counts the rest, not forgetting to count the crossing line in the split group)) 12</p>	<p>G: ((she leans back and says softly)) I hate this problem G: Because...you make me answer questions...oh yeh, I could have done it by myself. I just put this one here, and two and yeh, ten and two ((indicating loosely the last tally marks)) G: I could have put this ((showing how she could have crossed the last group of 4 tallys)) and this would be this ((at the beginning of these just have 2 and not the cross that makes three)) and I know that five...</p>
<p><i>Comments: She is very contentious. Does not want to repeat the prob to begin and does not want to explain her thinking at the end.</i></p> <p><i>She makes tally marks up to 24. This is the same as earlier, count the first part and then count what is rest.</i></p>			
<p>2nd Diss. - PART-PART-WHOLE (100, 65) Tu tienes cien globos. Sesenta y cinco globos son azules y los demas son rojos. ¿Cuántos</p>		<p>G: ((she choses the 100s chart)) Son aquí y aquí ((she circles 100 and 65 on the chart)) Ya sé, como se, como se cual...vas a hacer, cual...porque</p>	

<p>globos son rojos? Dime el problema por favor</p>		<p>esto estaba rato ((points to 100)) un poco y esto tambien ((points to 65)) y me da todos estos ((we see her point to the numbers in between. The camera is focused on the paper)) G: ah, voy a contarlos, ok? G: ((in a whisper while she points to 75 and 85 and 95)) cinco diez quince veinte. Ok, now I see it ((she starts again. First she points to 65, then 70, 75, 80, 85, 90, 95, 100. We hear...)) cinco, diez, quince, veinte, veinticinco, treinta, ((treinta y cinco, cuarenta are silent)) cuarenta. <i>**she has counted 65 as 5 which put her off by 5***</i> [She checks] G: Cuarenta o...treinta y cinco...? Espera ((we see the same action with the pencil pointing to each 5 between 65 and 100) ah, treinta y cinco, entonces...o cuarenta? ((the</p>	
---	--	---	--

		<p>camera moves back and we see the questioning look on her face. She shakes her head and says something inaudible))</p> <p>G: es que necesito una ficha ((she gets two cubes to use as markers on her paper)) para que me ayuden un poquito. Uno ((she starts to count by ones from 65, pointing with her pencil)) ah, mejor de dos en dos ((she starts counting again along the numbers of the chart, moving her pencil, but in her head without whispering or moving her lips. She stops at 77 and seems puzzled, says “uh?”))</p> <p>dieciocho, veinte, veintidos, veinticuatro, veintiseis, veintiocho, sesenta ((she keeps counting more to herself in the 60s, then stops at 96 when she says sesenta y ocho. She starts</p>	
--	--	---	--

		with the 70s)) setenta y dos?	
<p><i>Comments: She tried the problem with the chart first. She got off by 5 by counting the 65 in her count, giving her 40, then she got 35 on her next.</i></p> <p><i>The she does it by 2s and really gets confused. She did not see the obvious on the chart which was to use the tens.</i></p>			
<p>2nd Diss. – Other ways.</p>	<p>Blocks: (without a correct ans) G; Let me try, let me try. I'm going to get the five ready ((she gets 5 cubes and sets them to her right. Then she gets some rods, counts them, gets some more, and says...)) there's more than a hundred! G: ((she pauses)) look ((she counts and moves the rods one at a time)) one two three four five six seven eight nine ten G: These are tens ((she grabs a rod)) so it means that...it's a hundred. So, it's, if this is a hundred, well then that could be another hundred ((she points to the rest of the blocks)), so it's two thousand I think <i>**attempt at estimation. She is serious but may be mixing up hundreds and thousands**</i> G: ((she gathers rods together, 6)) sixty five ((she pulls in the cubes)) OK, estos ((she gets one of the 4 other rods and starts stacking the cubes on top, we hear "cuatro" "cinco" and see that she has stacked the 5 on 5 sections of the rod. She counts the free sections)) one two three four five)) ok, thirty ((she gather the 3 extra rods, points to the free sections and says...)) thirty five</p>	<p>With paper:</p>	<p>The chart:</p>

<p><i>Comments: With the blocks she feels very secure. She makes the 100. What she does next is very clever. She gets the 5 blocks she set aside initially, then she moves away 6 rods. She sets the 5 cubes on one of the rods to indicate that these are being used, then she counts the rest and gets 65.</i></p>			
<p>Gina's comments/opinions: Cual manera es mas seguro, piensas es mas segura la respuesta? G: ((she points to the blocks)) G: ((the camera zooms into the blocks)) Porque con estos, asi no me juro(i), porque aqui ((the chart)) como lo mas poquito, asi no se ve y si cuentan dos en dos no me saliera la respuesta. Pero salen muchas respuesta aqui ((still the chart)) pero si lo hago en papel estoy gastando papel. Si hago en estos ((the number line)) hay demasiado chiquitos los numeros, y si hago en estos ((the blocks)) esta perfecto.</p> <p>She likes paper but is concerned about wasting it.</p>			
<p>My final comments: She started in kinder with very partitive thinking. Then in 1st and second she used a linear method of starting with what she knew and counting up. The diss problem was very interesting for her. In her first attempt with the chart she got messed up in the counting. She was trying to use it linearly and got off by 5. She didn't use it as tens. Then when she got the blocks she was able to partition again, stacking 5 cubes on top of a rod and counting the rest.</p>			

Appendix I. Longitudinal Tendencies

	Gerardo	Gina	Omar	Yolanda
Compare	<p>General: Gerardo seemed in first grade post to go from thinking in terms of sets to relating the sets to an operation which is linear. He went from set thinking to the operation which is linear. In contrast, Y and O went from thinking linearly about compare right to the algorithm. They did get confused with the larger numbers that did not fit into their linear formula, before they were able to make the conversation to using the larger numbers with carrying and borrowing.</p> <p><i>Can you use</i></p>	<p>Gina seems to be able to move flexibly between linear and partitive thinking without a strong preference for either. She is very connected to her thinking and able to think about her thinking in words. She does get confused when she tries to apply a linear approach using the base ten blocks. She has a clear preference for solving problems in the fastest and most efficient way.</p>	<p>He did not like the blocks as a method to solve the final problem. He started to solve it this way but then went to paper. It was showing the ones that stopped him. From kindergarten he has had linear thinking. He used the number line on the wall, then went to mental math. In first grade we saw that he was decomposing numbers and thinking about them as tens and ones. This came out in the join/change/start/unknown problems as well. Like Yolanda he is using the chart in a linear way and using the tens to move quickly forward and backward. He explains in terms of addition and subtraction.</p>	<p>She never really had any problem with compare problems. In kinder it was more a representation and sorting out how she was going to do it. Even in kinder it was clear she was thinking of a linear progression of numbers. In first grade she had the number line internalized and used it for mental calculations. When the numbers got bigger in second she used the algorithm. She was able to show the problem with the chart easily. The blocks forced her to use partitive strategies and she got a little confused on counting the answer, off by one. In her</p>

	<p><i>the chart to think in sets? Can you use the chart for partitioning? Blocks are partitioning – can you use the blocks linearly? O and Y did, or at least they tried.</i></p> <p>Gerardo moved from set thinking to algorithmic thinking. For the final diss problem he just subtracted on the chart. Could it be that the progression from set to set and algorithm to finally just algorithm demonstrates an understanding of what you need to do to find an answer in comparison? He gave great and coherent explanations.</p>			<p>partitive thinking she tried to still make it two linear sets. She does use base ten thinking to move forward and backward faster, jumping essentially up or back by tens.</p>
JCU/JSU	<p>Gerardo has no problem understanding this type of problem. For JCU he</p>	<p>Gina has a good sense of partitive thinking. She can see the problem</p>	<p>Omar started in kindergarten actively using the number line on the wall to solve problems. During one interview problem Erin</p>	<p>Yolanda is very similar to Omar in her linear thinking. She can keep numbers in her</p>

	<p>always knew that you have to begin with what you know and go up to what you had. When he went to JSU he began with trial and error to solve it the same way, start with some number and go up until you reach what you have. In 2nd for JSU he just realized that you could take away what they gave you to give you what you started with. Once again he is focused on what did you have to start as the key to finding the answer. His language develops. In 1st he got confused with the explanation, even though his strategy was good and he had the answer in his</p>	<p>as a part-part-whole where she has to find the other part. She either started with the whole and removed the part, then counted the rest, or started with the part and added until she had the whole, then counted what she added. She showed flexibility in doing this and even articulated this when asked if it was the beginning or the end of the problem. She said it was the end when she had the whole in front of her, but said it could be the beginning, she just had to remove the part.</p>	<p>made a number line for him on paper (multiplication I think) so he could show her how he solved it. Then he went from moving forward to moving backward and moving both ways at the same time. We see his use of the number line in his head for the final problem with large numbers. He expresses a preference for using his mind.</p>	<p>head and move up and down. Instead of the number line she uses her fingers as the symbolic representation. Initially she could solve the JSU with 3 and 5, but when it was 5 and 13 that was difficult. She takes risks and makes mistakes sometimes, but tries to apply more efficient strategies naturally, like thinking in 5s and 10s. She can connect the algorithm for subtracting to these problems and is most comfortable with this. She took the chart and extended it without problem, viewing as a snapshot without fixed end points. She shows how the chart can be extended easily and has recognized the pattern.</p>
--	--	---	---	---

	model. He was able to solve the same kind of problem at this time with smaller numbers.			
Multiplication	Gerardo: This is very easy for him to directly model. What he does is relates it to multiplication and tries to skip count. He does this for the 5s questions and then tries to do it for the diss with 7. His alternative is a matrix, but this does not really do anything more than show another way to directly model. However, with the choice of a matrix he has show a more global understanding of the multiplication. It is another way to show 15 groups of 7 that is not	Gina: This is the problem with the proof of common underlying proficiency and Chicken Little. Like Gerardo, she used direct modeling in drawing and then counted by 2s and then by 5s adding them together to get the answer. She could not model the problem with blocks, numbers too big, nor could she write an equation. This is an interesting problem for both Gina and Gerardo. It	Omar has not only skip counted like the other students, but has used multiplication to combine skip countings of 2s, 5s, and 10s, not always getting the answer correct. For the diss problem he used the idea of 15, 7 times instead of 7, 15 times like the problem structure. Then he related it to the clock. He didn't get the answer right the first time with the chart. He tried to keep track of the 7 times with his fingers, but going 15 each time, using ten and over 5, threw him off and he landed on 85. He explained with the clock idea, but then had to go to the chart and with some scaffolding with the clock he was able to come up with the right answer. **This is a very interesting series of problems for Omar.*** Good explanation for diss prob with clock ref.	Interesting thing with Yolanda, and we have seen this before, she has confidence in her answer and will make the model match the answer (see kinder). ***This is not cheating in the eyes of a young child, this is making things match.*** The progression of her thinking from kinder to Diss is a strong tendency to use skip counting, except in 1 st where she used cubes to get 4 groups of 7 and was off by one. She knows multiplication is counting/putting together equal sized groups. Her diss example was a good developed strategy. Very interesting

	linear. He always explains well.	is easy to do a model and lends itself to the idea of skip counting. Good explanations in 2 nd grade.		explanation in kinder. 1 st grade post is good also.
Part Division	I think Gerardo tends to be more linear like Yolanda and Omar, but he also has a flexibility to his thinking that comes from not such a strong tendency nor internalized number line. He is very verbal and likes to focus on the details. He does not remember the numbers as well as Y and O. His approach to solve the diss prob was using the chart, but in a linear way. He is between Yolanda and Gina in the way he thinks. This is not the most	Gina: All the data we have for Gina on this type of problem shows strong partitive thinking. She starts with the whole group then tries trial and error to find the answer, dividing the whole group into various parts and checking if they are equal. When she does the diss problem with btbs, the tool facilitates the way she is thinking. This is very clear. She treats the chart like	Omar: **Note: I think for thinkers like Yolanda and Omar, comprehending the problem is difficult. They seem to be able to solve the smaller numbers with trial and error and a linear approach once they understand it. Could it be that this type of problem challenges their linear representation of numbers, where adding and subtracting problems do not? *** In kinder Omar showed a strong preference for the number line. Then in 1 st grade pre he used linear trial and error. We have no data for 1 st post and did not interview with this question in 2 nd until the dissertation. For the diss he was able to estimate, but placing the last 4 so he had an exact answer was tough. He did not get the final answer, 21, until his last attempt with the chart and some scaffolding.	Partitive division is hard for Yolanda because she is such a strong linear thinker with a strong sense of number, but this number sense progresses in a linear fashion which makes partitive thinking hard. She has a strong tendency for algorithms and uses them as strategies to solve problems. At the beginning of 1 st she could not find the answer, but she understood the concept by the end. When she had to move beyond direct modeling in the diss she was incredibly frustrated. This, like Omar, is

	<p>interesting case for Gerardo.</p>	<p>rods with groups of tens. **Note that we saw Yolanda and Omar treat the chart like a number line.** When she gets to paper she cannot think of another method except one by one division. She cannot apply an equation to this problem because for equations she is thinking of adding and subtracting two numbers, I think. Her explanation of her thinking and why the blocks are better is excellent.</p>	<p>the downside of a very strong numeric thinker. Because our numbers progress linearly, kids with a natural sense of number are very comfortable manipulating numbers linearly in their mind, but thinking of sets and grouping appear to be much more difficult. The strategy that she finally found to solve the dissertation problem was by a series of trial and error additions. She started with 16, which was a good guess, then went upward. It is interesting that Gina, who so quickly solved this problem with BTBs could not think of how to create an equation to represent it. Yolanda, on the other hand, used the equation as the tool to</p>
--	--------------------------------------	--	--

				<p>solve. **Perhaps the goal of teachers is to help students think in both ways. Here is a good example of where both types of thinking are important. Put Yolanda and Gina's understanding together for the ideal.***</p>
<p>Part-Part-Whole</p>	<p>Gerardo showed remarkable consistency in his strategy from K through 2nd. He starts with what he know, then counts up to the full amount, keeping track of how many times he is counting on. He only got confused when he tried to explain his algorithm using blocks. He used the chart like a number line. ***good point – chart used linearly like a</p>	<p>Gina started in kinder with very partitive thinking. Then in 1st and second she used a linear method of strating with what she knew and counting up. The diss problem was very interesting for her. In her first attempt with the chart she got messed up in the counting. She was trying to use it linearly</p>	<p>Omar begins in kinder with not knowing how to comprehend the problem. In first grade the problem was simple so he just added on. In second he recalled a fact and then in mid he broke it into tens. He demonstrates his ability to break/decompose problems for the diss. He makes a mistake but finds the error in the algorithm. There is a better example of his decomposing number in one of the oher problems where he says “sixty twelve” and then “seventy two.”</p>	<p>In kinder Yolanda solved the problem very quickly with partitive thinking. **This is a very intereting problem for Yolanda. All the way through she has shown interesting thinking and it has inspired creativity. She related it to the clock in Mid 2nd, made a mistake in pre 2nd. She was added and challenged by the manipulatives in diss. She has shown both partitive and</p>

	<p>number line, but more efficient because of the skipping by 5s and 10s.***</p> <p>Even though part-part-whole is a relationship problem, it is a linear relationship. At least it seems like children conceive of it that way, unlike partitive division. Measurement division is more linear because they know how many in each group. Partitive division means breaking down/apart a whole set. Partitive division does not lend itself to linear thinking apparently in children.</p>	<p>and got off by 5. She didn't use it as tens. Then when she got the blocks she was able to partition again, stacking 5 cubes on top of a rod and counting the rest.</p>		<p>linear thinking in these problems.***</p>
--	--	---	--	--

Appendix J. Problem Solving Profile Tables

Table J1. Omar, Problem Class: Join

Interview	Problem Description	Omar's Strategy
Kinder – Beginning	JCU (7, 4) Alan is putting candles on a birthday cake. Now there are 4 candles and he wants to put 7. How many more candles does Alan need to put on the cake to have 7?	Joining To with cubes from 4 to 7. Direct Modeling/Correct
Kindergarten - End	JCU (11, 7) You want to buy a toy plane that costs \$11. You only have \$7. How many more dollars do you need to be able to buy the plane?	Counting On To using number line on the wall. Counting On To/Correct
1st Grade - Mid	JCU (25, 15) Gerardo wants to buy a toy plane that costs \$25, but he only has \$15. How many more dollars does Gerardo need to be able to buy the plane?	Counting On To by drawing 15 circles then adding more until he has 25. Counting On To/Correct
1 st Grade - End	JCU (45, 25) You go to the store and you want to buy a Playstation game that costs \$45, and you only have \$25. How many more dollars do you need to buy that game?	Counting on by 10s from 25 to 45. Counting by tens/Correct
1 st Grade – Mid	JSU (3, 5) Gerardo had some candies and then you gave you 3 more, and then he had 5 candies. How many candies did Gerardo have to start with?	Recalled fact, $2 + 3 = 5$ Recalled fact/Correct
1 st Grade - Mid	JSU (5, 13) Same as above.	Direct modeling with scaffolding and trial and error. Direct Modeling/Correct
1 st Grade - End	JSU (5, 15) Alex had some candies. David gave him 5 more and now he has 15. How many candies did Alex have to start with?	Recalled fact, $10 + 5 = 15$. Knows the answer is 10. Recalled fact/Correct
1 st Grade – End	JSU (10, 34) Same as above.	Recalled fact or base ten thinking. $24 + 10 = 34$ Base ten thinking/Correct
2 nd Grade - Beginning	JSU (7, 22) Alex had some candies, we don't know how many, and then his friend David gave him 7 more and now he has 22 candies. How many	Counting down from 22 seven times with aid of number line representation on wall. Counting/Correct

	did Alex have to start with?	
2 nd Grade - Mid	JSU (7, 22) You had some candies and then Omar gave you 7 more, and now you have 22. How many did you have to start with?	Simultaneous number line counting up and down. Counting/Correct
2 nd Grade - Dissertation	JSU (340, 1012) You had some candies. Later a friend gives you 4 more and now you have 112. How many candies did you have before your friend gave you the 40?	Mental calculation decomposing and recomposing numbers. Counting/Correct
11 problems, 2 direct modeling, 9 advanced strategies, 11 correct, 0 incorrect		

Table J2: Omar, Problem Class: Compare

Interview	Problem Description	Omar's Strategy & Result
Kindergarten - End	Compare Difference Unknown (12, 9) You have 12 toy cars. David has 9 toy cars. How many more toy cars do you have than David?	Counting On To using a number line. Starts at 9 and counts to 12. Answer is 3. Counting On To/Correct
1 st Grade – Mid	Compare Difference Unknown (16, 10) Andres has 16 toy cars and Gerardo had 10. Who has the most? How many more?	Possible base ten thinking. Knows 16 is 6 more than 10. Base ten thinking/Correct
1st Grade – End	Compare Difference Unknown (13, 2). You have 13 pieces of moon sand and Gerardo has 2. How has the most? How many more?	Added, $2 + 11 = 13$, answer is 11. Adding/Correct
1 st Grade - End	Compare Difference Unknown (31, 29) Same as above.	At first is confused between 2 and 3. Then uses Counting On To and gets 2. Counting On To/Correct
2 nd Grade - Beginning	Compared Set Unknown (13, 6) Jose has 13 toy cars and Pedro has 6 more toy cars than Jose. How many toy cards does Pedro have?	Mental counting on. Counting/Correct
2 nd Grade - Mid	Compared Set Unknown (13, 6) You have 13 toy cars and Gerardo has 6 more toy cars than you. How many more toy cars does Gerardo have than you?	Counting on mentally, decomposes and works with ones, 3 plus 6 is 9. Answer 19. Counting/Correct
2 nd Grade - Dissertation	Referent Unknown (53, 36) You have 53 markers, and you have 36 more markers than your friend. How many markers does your friend have?	Mental calculation counting up and down, likely tens first going up then adjusts the ones down. Answer 17. Counting/Correct
7 problems, 0 direct modeling, 7 advanced strategies, 7 correct		

Table J3: Omar, Problem Class: Part-Part-Whole

Interview	Problem Description	Omar's Strategy
Kindergarten - End	PPW (10, 6) You have 10 balloons. Six of your balloons are blue and the rest are red. How many balloons are red?	He could not solve the problem. Says 10. Tries to use number line. Counting/Incorrect
1st Grade - Mid	PPW (13, 7) You have 13 balloons. 7 are blue, and the rest are red. How many are red?	Joining To by drawing all balloons, appears to add 3 to 7 to get 10 and another 3 to get 13. Knows 3 + 3 is 6. Direct Modeling/Correct
1 st Grade - End	PPW (30, 20) You have 30 balloons. 20 are blue and the rest are red. How many are red?	Recalled fact and base ten thinking. $30 = 20 + 10$. Recalled fact/Correct
1 st Grade - End	PPW (75, 50) You have 75 balloons. 50 are blue and the rest are red. How many are red?	Uses money notation, Q for quarter. Three Qs and crosses off two. Leaves 25. Counting by 25/Correct
2 nd Grade - Beginning	PPW (24, 12) You have 24 balloons and 12 of the balloons are blue, and the rest are red. How many balloons are red?	Possible recalled fact. Answer is 12. Recalled fact/Correct
2 nd Grade - Mid	PPW (24, 12) You have 24 balloons. Twelve of your balloons are white and the rest are black. How many are black?	Base ten thinking and number decomposition, $12 + 12 = 24$. Addition/Correct
2 nd Grade - Dissertation	PPW (805, 565) You have 805 balloons. 565 of your balloons are blue and the rest are red. How many balloons are red?	Number decomposition, mentally adding on by parts and adjusting to get 340. Off by 100, corrects on paper by borrowing. Counting/Incorrect
7 problems, 1 direct modeling, 6 advanced strategies, 5 correct, 2 incorrect		

Table J4: Omar, Problem Class: Multiplication

Interview	Problem Description	Omar's Strategy
Kindergarten - End	MULT (3, 6) You have 3 bags of marbles. There are 6 marbles in each bag. How many marbles do you have in all?	Counting up six on the number line three times. Counting/Correct
1st Grade - Mid	MULT (4, 7) What if you had 4 bags of marbles, and there are 7 marbles in each bag. How many marbles would have altogether?	Grouping by drawing circles. Tries to count by 7s. Answer is 27. Direct Modeling/Incorrect

1 st Grade - End	MULT (7, 5) You have 7 bags of marbles. There are 5 marbles in each bag. How many marbles do you have altogether?	Skips counting by 5s. Keeps track with his fingers. Counting by 5s/Correct
2 nd Grade - Beginning	MULTI (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Skips counting using 2s, 5s and 10s using tiles. Skip counting/Correct
2 nd Grade - Mid	MULT (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Skip counting by 5s, combines two to make 10s, gets confused, answer is 35. Counting by 5s/Incorrect
2 nd Grade - Dissertation	MULT (15, 7) You have 15 bags of marbles and there are 7 in each bag. How many do you have altogether?	Changes to 7 groups of 15 and thinks of the clock and quarter of an hour. Gets correct answer by extending the 100s chart. Counting/Correct
6 problems, 1 direct modeling, 5 advanced strategies, 4 correct, 2 incorrect		

Table J5. Omar, Problem Class: Partitive Division

Interview	Problem Description	Omar's Strategy
Kindergarten - End	PartDiv (15, 3) Omar, you have 15 marbles and you are going to give the marbles to 3 friends so that each friend gets the same amount. How many does each friend get?	Trial and error on a number line to get 5. Counting/Correct
1st Grade - Mid	PartDiv (18, 2) You have 18 pieces of candy and you want to share them with 2 friends so that each friend gets the same amount. How many does each friend get?	Possibly recalled fact, $9 + 9 = 18$. Recalled Fact/Correct
1st Grade - Mid	PartDiv (18, 3) Same problem with 3 friends.	Trial and error with 18 dots to get 6. Direct Modeling/Correct
1 st Grade – End	PartDiv (12, 6) You have 12 marbles and you want to share with 6 friends so that each friend gets the same amount. How many does each friend get?	Possibly recalled fact plus use of repeated subtraction of 2. Subtraction/Correct
1 st Grade – End	PartDiv (24, 6) Same as above.	Repeated subtraction of 4. Subtraction/Correct
2 nd Grade - Dissertation	PartDiv (84, 4) You have 84 pencils and you want to give them to four friends so that each friend gets the same amount. How many does each	<i>Strategy described in a later section.</i> Counting/Correct

friend get?
6 problems, 1 direct modeling, 5 advanced strategies, 6 correct, 0 incorrect

Table J6. Yolanda, Problem Class: Join

Interview	Problem Description	Yolanda's Strategy
Kindergarten – End	JCU (11, 7) Gina wants to buy a toy plane that costs \$11. She only has \$7. How many more dollars does she need to be able to buy the plane?	Direct model with 10 fingers plus a marker. Separates 11 into two parts, 7 and the answer, 4. Direct Model/Correct
1st Grade – Mid	JCU (25, 15) Gerardo wants to buy a toy plane that costs \$25, but he only has \$15. How many more dollars does Gerardo need to be able to buy the plane?	Mental, possibly skip counting on by 5s or 10s, gets confused and answers 5. Counting/Incorrect
1 st Grade – Mid	JCU (11, 7) Same as above.	Counting On To from 7 to 11 four times. Counting/Correct
1 st Grade – End	JCU (18, 9) You want to buy a doll that costs \$18, and you only have \$9. How many more dollars do you need to buy the doll?	Recalled Fact Recalled fact/Correct
1 st Grade – End	JCU (45, 25) Same as above.	Counting On To with skip counting by 5s. Counting by 5s/Correct
1 st Grade – Mid	JSU (3, 5) Jenna had some candies and then Dolores gave her 3 more, and then she had 5 candies. How many candies did Jennifer have to start?	Knows the answer is 2 because $2 + 3 = 5$. Recalled fact/Correct
1 st Grade – Mid	JSU (5, 13) Same as above.	Appeared to be confused, thinking the change was still 3. Says " $10 + 3 = 13$ ". Addition/Incorrect
1 st Grade – End	JSU (5, 15) Alex had some candies. David gave him 5 more and now he has 15. How many candies did Alex have to start with?	No audio data but fills in 1 st addend, writing $10 + 5 = 15$ Recalled fact/ Correct
1 st Grade – End	JSU (10, 55) Same as above.	No audio data, writes $45 + 10 = 55$. Base ten thinking/Correct
2 nd Grade – Beginning	JSU (7, 22) Alex had some candies, we don't know how many, and then his friend David gave him 7 more and now he has 22 candies. How many did Alex have to start with?	Built the whole, 22, with rods, marked 7 as one part and the counted the rest. Direct Modeling/Correct

2 nd Grade – Mid	JSU (7, 22) You had some candies and then your brother gave you 7 more, and now you have 22. How many did you have to start with?	Subtraction with standard borrowing algorithm. Algorithm/Correct
2 nd Grade - Dissertation	JSU (40, 112) You had some candies. Later a friend gives you 4 more and now you have 112. How many candies did you have before your friend gave you the 40?	Subtraction with standard borrowing algorithm. Algorithm/Correct
12 problems, 2 direct models, 10 advanced strategies, 10 correct, 2 incorrect		

Table J7. Yolanda, Problem Class: Compare

Interview	Problem Description	Yolanda's Strategy
Kindergarten - End	Compare Difference Unknown (12, 9) You have 12 toy cars. Gina has 9 toy cars. How many more toy cars do you have than Gina?	Joining To, draws 9 lines then adds 3 to get 12. Direct Model/Correct
1 st Grade – Mid	Compare Difference Unknown (16, 10) Ana has 16 dolls and Dolores has 10. Who has the most? How many more?	Counting On To from 10 to 16. Counting On To/Correct
1 st Grade - End	Compare Difference Unknown (13, 2) Your brother has 13 toy cars and your cousin has 2. Who has the most? How many more?	Subtraction, $13 - 2 = 11$ Subtraction/Correct
1 st Grade – End	Compare Difference Unknown (31, 29) Same as above.	Counting On To from 29 to 31. Counting On To/Correct
2 nd Grade - Beginning	Compared Set Unknown (13, 6) Jose has 13 toy cars and Pedro has 6 more toy cars than Jose. How many toy cards does Pedro have?	Counts on from 13 six times to get 19. Counting/Correct
2 nd Grade - Mid	Compared Set Unknown (13, 6) Your neighbor has 13 Brats dolls and you have 6 more dolls than she does. How many Brats dolls do you have?	Recalled Fact. Recalled fact/Correct
2 nd Grade - Dissertation	Referent Unknown (53, 36) You have 53 markers, and you have 36 more markers than your friend. How many markers does your friend have?	Subtraction with borrowing algorithm. Algorithm/Correct
7 problems, 1 direct models, 6 advanced strategies, 7 correct, 0 incorrect		

Table J8: Yolanda, Problem Class: Partitive Division

Interview	Problem Description	Yolanda's Strategy
Kindergarten	PartDiv (15, 3) Yolanda, you have 15	Partitive Direct Model, using

- End	marbles and you are going to give the marbles to 3 friends so that each friend gets the same amount. How many does each friend get?	trial and Error with drawing. Direct Modeling/Correct
1st Grade - Mid	PartDiv (18, 2) You have 18 marbles and you want to share them with a friend so that you each get the same amount. How many does each friend get?	Recalled Fact, $9 + 9 = 18$ Recalled fact/Correct
1 st Grade – Mid	PartDiv (18, 3) Same as above sharing with 3 friends.	Trial and Error, cubes then fingers with 3 in each group. Tried to approach as measurement division. Direct Modeling/Incorrect
1 st Grade – End	PartDiv (12, 6) You have 12 marbles and you want to share with 6 friends so that each friend gets the same amount. How many does each friend get?	Trial and Error, same strategy for both, draws circles then finds 6 equal groups. Direct Modeling/Correct
1 st Grade – End	PartDiv (24, 6) Same as above.	Trial and error as above. Direct Modeling/Correct
2 nd Grade - Dissertation	PartDiv (84, 4) You have 84 pencils and you want to give them to four friends so that each friend gets the same amount. How many does each friend get?	<i>Described in detail later.</i> Addition/Correct
6 problems, 4 direct models, 2 advanced strategies, 5 correct, 1 incorrect		

Table J9. Yolanda, Problem Class: Part-Part-Whole

Interview	Problem Description	Yolanda's Strategy
Kindergarten - End	PPW (10, 6) You have 10 balloons. Six of your balloons are blue and the rest are red. How many balloons are red?	Used 10 fingers as the whole, viewed 6 as one part, which left 4 as the other part. Direct Modeling/Correct
1st Grade - Mid	PPW (13, 7) You have 13 balloons. 7 are blue, and the rest are red. How many are red?	Partitions a group of 13 blocks into 7, and the rest, which is 6. Direct Modeling/Correct
1 st Grade - End	PPW (30, 20) You have 30 balloons. 20 are blue and the rest are red. How many are red?	Recalled fact and base ten thinking, $20 + 10 = 30$. Recalled fact/Correct
1 st Grade – End	PPW (75, 50) Same as above.	Counting On to by 2s. 1 st attempt miscounts then 2 nd attempt counting by 5s is correct. Counting by 5s/Correct

2 nd Grade - Beginning	PPW (24, 12) You have 24 balloons and 12 of the balloons are purple, and the rest are blue. How many balloons are blue?	Counting On To from 12 to 20. Miscounts answer as 7. Counting/Incorrect
2 nd Grade - Mid	PPW (24, 12) You have 24 balloons. Twelve of your balloons are blue and the rest are green. How many are green?	Relates to 24 hours and the clock. Says half of 24 is 12. Fractions/Correct
2 nd Grade - Dissertation	PPW (100, 65) You have 100 balloons. 65 of your balloons are blue and the rest are red. How many balloons are red?	Counting On To by 10s and 1s. Counting/Correct
7 problems, 2 direct models, 5 advanced strategies, 6 correct, 1 incorrect		

Table J10. Yolanda, Problem Class: Multiplication

Interview	Problem Description	Yolanda's Strategy
Kindergarten - End	MULT (3, 6) You have 3 bags of marbles. There are 6 marbles in each bag. How many marbles do you have in all?	Counting with fingers 1 st then direct model drawing. Miscounts to 17. Counting/Incorrect
1st Grade - Mid	MULT (4, 7) What if you had 4 bags of marbles, and there are 7 marbles in each bag. How many marbles would have altogether?	Grouping: Uses blocks, but groups are not clear, Miscounts to 27. Direct Modeling/Incorrect
1 st Grade - End	MULT (7, 5) You have 7 bags of marbles. There are 5 marbles in each bag. How many marbles do you have altogether?	Counting On To by 5s using fingers. Counting/Correct
1 st Grade – End	MULT(7, 10, and 6 more) You have 7 bags of marbles with 10 in each bag. And you have 6 single crayons. How many crayons do you have?	Mental math using base ten. 7 groups of 10 plus 6 more. Base ten thinking/Correct
2 nd Grade - Beginning	MULTI (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Counting On To by 5s using fingers. Countings by 5s/Correct
2 nd Grade - Mid	MULT (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Mental skip counting. Counting by 5s/Correct
2 nd Grade - Dissertation	MULT (15, 7) You have 15 bags of marbles and there are 7 in each bag. How many do you have altogether?	Counting aided by 100s chart. Counting/Correct
7 problems, 1 direct models, 6 advanced strategies, 5 correct, 2 incorrect		

Table J11. Gerardo, Problem Class: Join

Interview	Problem Description	Gerardo's Strategy
Kindergarten - End	JCU (11, 7) Pepe wants to buy a toy plane that costs \$11. Pepe only has \$7. How many more dollars does Pepe need to be able to buy the plane?	Joining To, builds 7 cubes then adds on to he has 11, his answer is 4. Direct Modeling/Correct
1 st Grade – Mid	JCU (25, 15) You want to buy a toy plane that costs \$25, but you only have \$15. How many more dollars do you need to be able to buy the plane?	Direct modeling with lines, cannot see answer in his model, says 25. Direct Modeling/Incorrect
1st Grade - Mid	JCU (11, 7) Same as above.	Joining To with lines for (11, 7) to get 4. Direct Modeling/Correct
1 st Grade - End	JCU (18, 9) You go to the store and you want to buy a Sponge Bob game that costs \$18, and you only have \$9. How many more dollars do you need to buy that game?	Counting On To and recalled fact. Counting On To/Correct
1 st Grade - Mid	JSU (5, 13) You had some candies and then Omar gave you 5 more, and then you had 13 candies. How many did you have to start with?	Trial and error with drawing to eventually solve using a Direct Model. Direct Modeling/Correct
1 st Grade - End	JSU (5, 15) Alex had some candies. Then David gave him 5 more, and now he has 15 candies. How many candies did Alex have to start with?	Recalled Fact, 5 and 10 more are 15. Recalled fact/Correct
2 nd Grade - Beginning	JSU (7, 22) Alex had some candies, we don't know how many, and then his friend David gave him 7 more and now he has 22 candies. How many did Alex have to start with?	Estimation and drawing with Trial and Error, answer is 14. Direct Modeling/Incorrect
2 nd Grade - Mid	JSU (7, 22) You had some candies and then Omar gave you 7 more, and now you have 22. How many did you have to start with?	Counting down 7 times from 22. Counting/Correct
2 nd Grade - Dissertation	JSU (40, 112) You had some candies. Later a friend gives you 4 more and now you have 112. How many candies did you have before your friend gave you the 40?	Models with base ten blocks, removes 4 rods to get 72. Direct Modeling/Correct
9 problems, 6 direct models, 3 advanced strategies, 7 correct, 2 incorrect		

Table J12. Gerardo, Problem Class: Part-Part-Whole

Interview	Problem Description	Gerardo's Strategy
Kindergarten - End	PPW (10, 6) You have 10 balloons. Six of your balloons are blue and the rest are red. How many balloons are red?	Direct model, draws 10 circles, colors circles red until he is left with 6. Answer is 4. Direct Modeling/Correct
1st Grade - Mid	PPW (13, 7) You see 13 great big balloons go up and 7 are blue, and the rest are red. How many of those great big balloons are red?	Direct model, draws the known blue balloons and adds red until he has the final number. Direct Modeling/Correct
1 st Grade – Mid	PPW (10, 6) Same as above.	Same strategy as above with drawing. Direct Modeling/Correct
1 st Grade - End	PPW (30, 20) You have 30 balloons. 20 balloons are blue and the rest are red. How many balloons are red?	Recalled fact, $30 - 20 = 10$ Recalled fact/Correct
1 st Grade – End	PPW (75, 50) Same as above.	Counting On To, starting with tens, from 50 to 75. Writes $50 + 25 = 75$. Counting by tens/Correct
2 nd Grade - Beginning	PPW (24, 12) You have 24 balloons and 12 of the balloons are red, and the rest are black. How many balloons are black?	Joining To by drawing from 12 to 24. Direct Modeling/Correct
2 nd Grade - Mid	PPW (24, 12) You have 24 balloons. Twelve of your balloons are green and the rest are red. How many are red?	Counting On To using mental math. Counting On To/Correct
2 nd Grade - Dissertation	PPW (100, 65) You have 100 balloons. Sixty five of your balloons are blue and the rest are red. How many balloons are red?	Borrowing algorithm on paper. Algorithm/Correct
8 problems, 4 direct models, 4 advanced strategies, 8 correct, 0 incorrect		

Table J13. Gerardo, Problem Class: Multiplication

Interview	Problem Description	Gerardo's Strategy
Kindergarten - End	MULT (3, 6) Ana has 3 bags of marbles. There are 6 in each bag. How many marbles does Ana have in all?	Direct model using Grouping, draws 3 bags with 6 dots in each. Answer is 18. Direct Modeling/Correct
1st Grade - Mid	MULT (4, 7) Alex has 4 bags of marbles. There are 7 marbles in each bag. How many marbles does Alex have in all?	Direct model using Grouping, draws bags and marbles, but tries to count by 7s. Direct Modeling/Correct

1 st Grade - End	MULT (7, 5) Monkey Boy has 7 bags of marbles. There are 5 marbles in each bag. How many marbles does Monkey Boy have in all?	Skip counting by 5s. Skip counting by 5s/Correct
2 nd Grade - Beginning	MULTI (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Makes a complete model, but counts by 5s to find the answer. Direct Modeling/Correct
2 nd Grade - Mid	MULT (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Skip counting by 5s using fingers to keep track of the 8. Skip Counting by 5s/Correct
2 nd Grade - Dissertation	MULT (15, 7) You have 15 bags of marbles and there are 7 in each bag. How many do you have altogether?	Direct model with drawing, tries to count by 7s, off by one to get 104. Direct Modeling/Incorrect
6 problems, 4 direct models, 2 advanced strategies, 5 correct, 1 incorrect		

Table J14. Gerardo, Problem Class: Partitive Division

Interview	Problem Description	Gerardo's Strategy
Kindergarten - End	PartDiv (15, 3) Gerardo, you have 15 marbles and you are going to give the marbles to 3 friends so that each friend gets the same amount. How many does each friend get?	Trial and error, draws 15 circles and puts in 3 equal groups of 5. Direct Modeling/Correct
1st Grade - Mid	PartDiv (24, 4) You have 24 toy dinosaurs and you want to give share them with 6 friends so that each friend gets the same amount. How many does each friend get?	Drawing, solves as though measurement division with 4 in each group. Direct Modeling/Incorrect
1 st Grade - End	PartDiv (12, 6) You had 12 marbles. You want to share the marbles with 6 friends so that each one gets the same amount. How many marbles does each friend get?	Direct model with trial and error using drawing. Direct Modeling/Correct
1 st Grade – End	PartDiv (24, 6) Same as above.	Trial and error with drawing. Direct Modeling/Correct
2 nd Grade - End	PartDiv (84, 4) You have 84 pencils and you want to give them to four friends so that each friend gets the same amount. How many does each friend get?	<i>Described in a later section.</i> Direct Modeling/Correct
5 problems, 5 direct models, 0 advanced strategies, 4 correct, 1 incorrect		

Table J15. Gerardo, Problem Class: Compare

Interview	Problem Description	Gerardo's Strategy
Kindergarten - End	Compare Difference Unknown (12, 9) Gerardo has 12 toy cars. Alex has 9 toy cars. How many more toy cars does Gerardo have than Alex?	Matching, draws a row of 12 circles and a row of 9 circles. Answer is 3. Direct Modeling/Correct
1st Grade - Mid	Compare Difference Unknown (21, 13) You have 21 dinosaurs and your friend David has 13. Who has the most? How many more?	Matching, draws two rows of circles and counts extras. Direct Modeling/Correct
1 st Grade - Mid	Compare Difference Unknown (16, 10) You had 16 toy dinosaurs and your friend David had 10. If you had 16 and your friend had 10, can you think in your mid how many more you would have?	Matching, draws two rows of lines and counts extras. Direct Modeling/Correct
1 st Grade - End	Compare Difference Unknown (13, 2) Monkey Boy has 13 dinosaurs and his brother is not very interested in toy dinosaurs so he only has 2. How many more dinosaurs does Monkey Boy have than his brother?	Direct model and subtraction/difference between numbers. Direct Modeling/Correct
2 nd Grade - Beginning	Compared Set Unknown (13, 6) Jose has 13 toy cars and Pedro has 6 more toy cars than Jose. How many toy cards does Pedro have?	Drawing then counting on from 13 six times. Direct Modeling/Correct
2 nd Grade - Mid	Compared Set Unknown (13, 6) You have 13 Playstations and Omar has 6 more Playstations than you. How many more Playstations does Omar have than you?	Recalled fact, $13 + 6 = 19$. Recalled fact/Correct
2 nd Grade - Dissertation	Referent Unknown (53, 36) You have 53 markers, and you have 36 more markers than your friend. How many markers does your friend have?	Used 100s chart to count backwards by ones to 17. Counting/Correct
7 problems, 5 direct models, 2 advanced strategies, 7 correct, 0 incorrect		

Table J16. Gina, Problem Class: Join

Interview	Problem Description	Gina's Strategy
Kindergarten - End	JCU (11, 7) Herman wants to buy a toy plane that costs \$11. He only has \$7. How many more dollars does he need to be able to buy the plane?	Direct model of part-part-whole strategy. Draws 11 lines, crosses out 7 to leave 4. Direct Modeling/Correct
1st Grade -	JCU (25, 15) Leslie went to the toy	Joining To with Direct Model

Mid	store to buy a toy plane that costs \$25, but she only has \$15. How many more dollars does Leslie need to be able to buy the plane?	using tally marks. Direct Modeling/Correct
1 st Grade - End	JCU (18, 9) Karla wants to buy a toy plane that costs \$18. She has \$9. How many more dollars does Karla need sothat she can buy the plane?	Direct model of connected cubes 10 and 8, removes a cube from 10, gives to 8 to make 9 and 9. Direct Modeling/Correct
1 st Grade – End	JCU (45, 25) Similar to above.	Makes 2 groups of cubes, 45 and 25, compares and removes 25 from 45 to get 20. Direct Modeling/Correct
1 st Grade - Mid	JSU (3, 5) Gerardo had some candies and then you gave you 3 more, and then he had 5 candies. How many candies did Gerardo have to start with?	(3, 5) solved by recalled fact. Recalled fact/Correct
1 st Grade – Mid	JSU (5, 15) Same as above.	Solved with part-part-whole model, draws 13, crosses off 5 to leave 8. Direct Modeling/Corret
1 st Grade - End	JSU (5, 15) Alex had some candies. David gave him 5 more and now he has 15. How many candies did Alex have to start with?	Solved by recalled fact. Recalled fact/Correct
1 st Grade – End	JSU (10, 34) Same as above.	Direct model, builds 34 with cubes and removes 10. Direct Modeling/Correct
2 nd Grade - Beginning	JSU (7, 22) Alex had some candies, we don't know how many, and then his friend David gave him 7 more and now he has 22 candies. How many did Alex have to start with?	Part-part-whole strategy, draws 22 circles, crosses out 7 from the end and counts the rest. Direct Modeling/Correct
2 nd Grade - Mid	JSU (7, 22) You had some candies and then Herman gave you 7 more, and now you have 22. How many did you have to start with?	Joining To, draws 7 lines then draws more and counts until she reaches 22. Answer is 15. Direct Modeling/Correct
2 nd Grade - Dissertation	JSU (40, 112) You had some candies. Later a friend gives you 40 more and now you have 112. How many candies did you have before your friend gave you the 40?	Borrowing algorithm on paper. Algorithm/Correct
11 problems, 8 direct models, 3 advanced strategies, 11 correct, 0 incorrect		

Table J17. Gina, Problem Class: Compare

Interview	Problem Description	Gina's Strategy
Kindergarten - End	Compare Difference Unknown (12, 9) Fernando has 12 toy cars. Anabel has 9 toy cars. How many more toy cars do you have than David?	Part-part-whole, draws 12 lines and crosses out 9 to get 3. Direct Modeling/Correct
1 st Grade – Mid	Compare Difference Unknown (16, 10) Briana has 16 toy cars and Leslie has 10. How many more little cars does Briana?	Matching twice, uses to cubes to make two 8s and two 5s, combines two 3s to get 6. Direct Modeling/Correct
1 st Grade - End	Compare Difference Unknown (13, 2) Fernando has 13 toy cars. Anabel has 2 toy cars. How many more toy cars dos Fernando have than Anabel?	Matching, solved by matching rows of cubes. Direct Modeling/Correct
1 st Grade – End	Compare Difference Unknown (31, 29) Same as above.	Matching, solved by matching rows of cubes. Direct Modeling/Correct
2 nd Grade - Beginning	Compared Set Unknown (13, 6) Jose has 13 toy cars and Pedro has 6 more toy cars than Jose. How many toy cards does Pedro have?	Counting On. She counts up from 13 six times to get 19. Counting On/Correct
2 nd Grade - Mid	Compared Set Unknown (13, 6) Your cousin has 13 toy cats, and you have 6 more than your cousin. How many do you have?	Adds 6 to 13 to get 19. Addition/Correct
2 nd Grade - Dissertation	Referent Unknown (53, 36) You have 53 markers, and you have 36 more markers than your friend. How many markers does your friend have?	Uses base ten blocks to removed 36 from 53 without exchanging. Direct Modeling/Correct
7 problems, 5 direct models, 2 advanced strategies, 7 correct, 0 incorrect		

Table J18. Gina, Problem Class: Multiplication

Interview	Problem Description	Gina's Strategy
Kindergarten - End	MULT (3, 6) Sara has 3 bags of marbles. There are 6 marbles in each bag. How many marbles does Sara have in all?	Direct model by Grouping, draws 3 bags with 6 circles in each. Direct Modeling/Correct
1st Grade - Mid	MULT (4, 7) Gina has 4 bags of marbles, and there are 7 marbles in each bag. How many marbles does Gina have altogether?	Direct model by Grouping, draws 4 bags with 7 tally marks in each. Direct Modeling/Correct
1 st Grade - End	MULT (7, 5) Sara has 7 bags of marbles. There are 5 marbles in each	Direct model (7, 5) with cubes, then skip counting by 5s to find

	bag. How many marbles does Sara have altogether?	the answer. Direct Modeling/Correct
1 st Grade - End	MULT (7, 10, and 6 more) Same as above.	Direct model (7, 10) with cubes, counts by 10s and adds 6. Direct Modeling/Correct
2 nd Grade - Beginning	MULTI (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	<i>I did not let her directly model the problems.</i> Skip counting by 5s eight times. Counting by 5s/Correct
2 nd Grade - Mid	MULT (8, 5) You have 8 bags of marbles and in each bag there are 5. How many marbles do you have altogether?	Direct model as activity, but can solve with skip counting by 5s and grouping in 10s. Direct Modeling/Correct
2 nd Grade - Dissertation	MULT (15, 7) You have 15 bags of marbles and there are 7 in each bag. How many do you have altogether?	Direct model and skip counting, drawing the bags then counting by 2s first then 5s to get partial sums, adds these. Off by 2. Direct Modeling/Incorrect
7 problems, 6 direct models, 1 advanced strategies, 6 correct, 1 incorrect		

Table J19. Gina, Problem Class: Partitive Division

Interview	Problem Description	Gina's Strategy
Kindergarten - End	PartDiv (15, 3) Jorge has 15. He wants to give the marbles to 3 friends so that each friend gets the same amount. How many does each friend get?	Direct model with drawing, uses trial and error to divide 15 into three equal groups. Direct Modeling/Correct
1st Grade - Mid	PartDiv (18, 3) Marian has 18 pieces marbles and she wants to share them with herself and 2 friends so that each friend gets the same amount. How many does each friend get?	Direct model with drawing, uses trial and error to divide 18 into three equal groups. Direct Modeling/Correct
1 st Grade – End	PartDiv (12, 6) Jorge had 12 marbles. He shared the marbles with 6 friends so that each friend got the same amount. How many did each friend get?	Direct model for (12, 6), circles groups of 2. Direct Modeling/Correct
1 st Grade – End	PartDiv (24, 6) Same as above.	Direct model with tally marks for (24, 6). Trial and error to find answer of 4. Direct Modeling/Correct
2 nd Grade - Dissertation	PartDiv (84, 4) You have 84 pencils and you want to give them to four friends so that each friend gets the same amount. How many does each	<i>Described in later section.</i> Direct Modeling/Correct

	friend get?	
5 problems, 5 direct models, 0 advanced strategies, 5 correct, 0 incorrect		

Table J20. Gina, Problem Class: Part-Part-Whole

Interview	Problem Description	Gina's Strategy
Kindergarten - End	PPW (10, 6) You have 10 balloons. Six of your balloons are blue and the rest are red. How many balloons are red?	Direct model with 10 lines, divides into 6 and 4. Direct Modeling/Correct
1st Grade - Mid	PPW (13, 7) Gina, you are getting ready for your birthday and you have 13 balloons. 7 are blue, and the rest are red. How many are red?	Direct model with Joining To, uses cubes to make 7 and then another 6. Direct Modeling/Correct
1 st Grade - End	PPW (30, 20) Hector has 30 balloons. 20 are blue and the rest are red. How many are red?	Direct model with tally marks. Divides 30 into 20 and 10. Direct Modeling/Correct
2 nd Grade - Beginning	PPW (24, 12) Herman has 24 balloons and 12 of the balloons are blue, and the rest are red. How many balloons are red?	Direct model with base ten rods. Builds 24, counts to 12 then counts the rest. Direct Modeling/Correct
2 nd Grade - Mid	PPW (24, 12) You have 24 balloons. Twelve of your balloons are white and the rest are black. How many are black?	Direct model with tally marks. Draws 24 then divides into 12 and 12. Direct Modeling/Correct
2 nd Grade - Dissertation	PPW (100, 65) You have 100 balloons. 65 of your balloons are blue and the rest are red. How many balloons are red?	1 st on 100s chart counting, then BTBs to find answer. Direct model for correct answer. Direct Modeling/Correct
6 problems, 6 direct models, 0 advanced strategies, 6 correct, 0 incorrect		

Appendix K. Students' Problem Solving Tendencies

Omar's Tendencies

Strategies:

Prefers to solve problems mentally.
Strong tendency toward counting on and counting down strategies.
Can decompose numbers, isolate ones, tens and/or hundreds for the calculation, then recompose for the answer.
Skip counts and groups by 2s, 5s, and 10s.
Connects with other forms of representation like the clock in groups of 15 minutes or money representation.

Tools:

Number line as a tool used very early on.
Addition and subtraction algorithms, many are done mentally.
100s chart for skip counting in 10s.

Language:

Struggles to put his thinking into words.
Rarely can reflect on what he has done, has difficulty explaining how he was thinking about a problem.
Better able to talk about his thinking in 2nd grade.
Can remember numbers well, even when they are quite large.

Transparency of thinking:

His thinking is much more transparent to the interviewer than to Omar because of his frequent use of counting strategies and the number line.
His thinking is rarely transparent to himself.

Personal characteristics:

Motivated in mathematics, seems to have an innate ability to work with numbers.
He has confidence in his answers and likes to work with large numbers.
Rarely uses direct modeling and these models do not help him self correct.
He is difficult to interview and easily distracted.
When interested and motivated he shows intense focus on mental problem solving.
Drawing and/or using concrete manipulatives tend to distract him.

Challenging Problems:

Kinder PPW (10, 6) unsuccessfully tried an invalid counting strategy
1st grade mid JSU (5, 13) needed scaffolding and direct model to comprehend
1st grade mid MULT (4, 7) miscounts direct model to get 27
2nd grade mid MULT (8, 5) miscounts by 5s to get 35

Yolanda's Tendencies

Strategies:

- Counting on and counting down.
- Solving problems mentally.
- Flexibly works forwards and backwards with numbers and combines counting and part-part-whole strategies.
- Uses skip counting, thinking in groups of fives and tens, and base ten thinking.

Tools:

- Prefers to use fingers as aids.
- Skip counting on the 100s chart.
- Uses addition and subtraction to find solutions and applies carrying and borrowing algorithms.
- Applies number facts.

Language:

- Can succinctly explain her thinking and her strategy, especially when she describes her mental math.
- Remembers the numbers in the problems well.
- Refers to the state of her mind, how she counts and calculates mentally.

Transparency of thinking:

- When she is counting and relying on mental calculations and strategies her thinking is clear.
- Her direct modeling strategies are not always clear.

Personal characteristics:

- Comprehends problems quickly and prefers mental math.
- Thinks “outside the box” and is creative in her solutions.
- Transitioned to number operations and algorithmic approaches early.
- Motivated math student who likes to anticipate a question, calculate it mentally, and answer before the interviewer is finished asking the question.
- Shows confidence with mental calculations and counting.
- Less confidence with direct models, direct models do not help her self-correct.

Challenging Problems:

- Kinder MULT (3, 6) miscounts in mind to 17 and changes the model to match
- 1st grade mid – MULT (4, 7) miscounts direct model with cubes to get 27
- 1st grade mid – JCU (25, 15) using mental math thinks answer is 5 not 10
- 1st grade mid – JSU (5, 13) misunderstands the problem or was confused by (3, 5)
- 1st grade mid – PartDiv (18, 3) put 3 in each group instead of making groups of 3
- 2nd grade beginning – PPW (24, 12) miscounts up to 20 to get 7

Gerardo's Tendencies

Strategies:

- Direct modeling with a drawing continues beyond kindergarten.
- Flexibly works with part-part-whole representations.
- Creates sets to compare.
- Trial and error in direct modeling.
- Makes flexible combinations of direct models with counting strategies such as skip counting.

Tools:

- Preference for drawing.
- 100s chart for counting by ones, fives and tens
- Paper, pencil algorithms.

Language:

- Many explanations are tied closely to the structure of the problem.
- Enjoys expressing his thinking in words, gives lengthy explanations, reflects on his thinking and his actions.
- Refers to the voice inside his head.
- Does not tend to remember the specific numbers in the problems.

Transparency of thinking:

- His thinking is usually transparent, especially since he has the ability to go back and explain what he was doing and thinking. Even when he struggles to find solutions, his models reveal his thinking.
- Good access to his own thinking.

Personal characteristics:

- Persistent and confident, especially when using trial and error strategy.
- Shows attention to details of the problems.
- Shows he can combine linear and holistic thinking in problem solving.
- Positive attitude toward problem solving, rarely shows frustration, is almost always agreeable.

Challenging Problems:

- 1st grade mid JCU (25, 15) could model correctly but not see the answer
- 1st grade mid PartDiv (24, 4) puts in groups of 4 like measurement division

Gina's Tendencies

Strategies:

Strong tendency to directly model problems in part-part-whole type representations.

Can flexibly work with parts of the whole to facilitate problem solving.

Likes to draw or build direct model with manipulatives.

Combines directly modeling with counting strategies.

Tools:

Draws models of problem situations, uses tally marks.

Uses manipulatives like base ten blocks and connecting cubes, building groups of ten.

Language:

Good access to her own thinking and the ability to explain while she is solving problems and what she has done afterwards.

Metacognitive self-correcting abilities and the ability to retell the internal dialogue that goes on in her head while problem solving.

Does not remember the numbers in the problems well.

Transparency of thinking:

Because she has the ability to describe her thinking in detail and likes to draw or build models of the problems, the transparency of her thinking is good.

She has good access to her own thinking process.

Personal characteristics:

Confident problem solver who is playful and at times mischievous.

Enjoys problem solving and likes to add a creative aspect to her models.

She can remember problem contexts and shows clear understanding of actions and relationships, but it is difficult for her to remember the specific numbers.

Challenging Problems:

2nd grade dissertation MULT (15, 7) directly models, but miscounts by 2

Appendix L. Students' Solutions to Partitive Division (84, 4)

Omar's Solution

I began by giving Omar the problem and then asked him to retell the problem situation to me. He thought for a moment then responded, "and there's, uh, twenty, and there's four [inaudible] friends." To explain his answer of 20 he said, "I, I thought about the numbers by two and two...Two four six, and eight...And then, and then I thought about it and it was eight."

Since Omar estimated the problem answer using 80 pencils and not 84, I asked him if he could show me the problem a different way and share all the pencils. He chose the number line, placed his finger close to eighty-four and said, "Here's twenty." Then he moved his finger down along the line toward zero. Then he took a pencil and circled the numbers he was pointing to which were 84, 64, 44, 24 and 4, and he answered 20 again. When I asked him if he shared all the pencils, he said no, and added he didn't know how to share them all.

Omar next tried base ten blocks with eight rods and four cubes to solve the problem. He moved the cubes and rods around the table, but his approach was unclear. Finally, he moved two rods away from the group of eight, put one back with the group and said of the first rod, "There's ten." He put the four cubes with it and said, "fourteen." He moved a second rod with the first and marked the fourth section on a third rod near these and said, "Here's another fourteen." Then he moved this third rod with his finger still marking the place near the second rod. He got a fourth rod, counted the remainder of his marked third rod (6), placed another finger to mark the eighth section of this fourth

rod and says, “another fourteen.” With the apparent intention of building a series of 14s, he reached for another rod from the group.

I interrupted him and asked, “And why fourteen?”

He looked up and said, “Huh?”

“Why fourteen? Why are you using the number fourteen?” He smiled at me and gathered the rods together, apparently giving up on this strategy. I continued, “Do you know why you are using fourteen?” He shook his head while looking down.

“Can I start over?” he asked. He got a pencil and started tapping his head. I asked what he was thinking, and he responded with “seventeen.” Next he took a piece of blank paper and started a one by one distribution of lines into four different circled areas on the paper. He kept adding lines, one by one, inside the circles. When he had what appeared to be 17 lines in each circle, he counted all the lines one by one.

“And? How many?” I wanted to know.

“It’s just sixty eight,” he said, which meant he had 17 lines in each of four groups. “Is that...how many lines do you have in each one?” I asked to clarify.

“Seventeen,” he responded. Then he returned to adding lines inside the circles, falling back to a direct modeling strategy. Since he appeared to be counting the lines by ones, I asked if he could count them faster. At this request he just counted more quickly, but still counted by ones.

Shortly, Omar gave up on this method of direct modeling and reached for the 100s chart. He found 84 on the chart and put a finger on it. Then he pointed to 19 and started working down the chart and over to his left, skipping from row to row by tens. At first it was not clear where he was pointing or how he was working through this strategy.

Finally, he pointed to 21, and I asked what he was thinking. Omar responded by tapping the 21 on the chart, said nothing, tapped the 43, went back to pointing at the 21, then moved to the 42, back to the 21, then down two rows and over one square back to 42. He did this a second time and pointed to 51 and then to 62, and once again back to 21. He seemed uncertain of his strategy.

“What are you doing?” I asked, “Taking twenty one and then another twenty one? What do you get when you have 21 and another 21?” I was trying to scaffold and support his earlier try of 21. Without saying anything, Omar moved his finger again from 21 to 41 and then 42. Then he moved from 42 down two rows and over one square to 63, then down 2 rows and over one square to 84. I asked him what he was doing again and he said, “I was going like this,” and he showed me by moving down two rows and over one square. Finally, he seemed to have a clear strategy and his focus and confidence returned. He was able to now point at 21, 42, 63, and finally 84 and he knew the answer was 21 pencils for each friend.

Gina’s Solution

During the interview, I presented the problem to her in Spanish, “Tú tienes ochenta y cuatro lápices, ochenta y cuatro lápices, (*You have eighty-four pencils, eighty-four pencils,*)” she said “whoa” in feigned surprise at the quantity, and I continued, “Y los quieres repartir entre cuatro amigas. (*And you want to share them among four friends.*)” At this, she immediately reached for the base ten rods, but I said, “Escucha primero por favor, (*Listen first, please,*)” and she stopped. “Para que cada amiga reciba la misma cantidad de lápices. ¿Cuántos lápices recibe cada amiga? (*So that each friend gets the same amount of pencils. How many pencils does each friend receive?*)” After I was done

telling her the problem, she got eight rods and four cubes then asked for clarification of the numbers in the problem.

“Ah, this is going to be a little bit tricky,” she predicted, speaking in English. However, she moved eight rods and four cubes into four groups of two very quickly then said, “None...tricky!” I asked her what she did, and she said, “I just did it. O.K.,” and she moved the eight rods back together. “It was eight here. It’s easy,” she added. Then she moved the four cubes together. She explained, “O.K., estos los deajo por allí, (*O.K., these, I leave these over there,*)” and she moved the cubes aside. “I was thinking four [rods] first, but it was,” and she moved two rods away from the group, “going to be two, so then I thought there’s no way three, but there is way two, so” and she moved the rods into four groups of two again, “I thought two and two? Yeah.” When I asked what she did with the four single cubes, she said, “These?” touching the 4 cubes. “Right, there are four,” she said while holding them in her hand. Then she placed one next to each group of two rods saying, “put one in each for they could be,” and she waved a hand over all the groups, “equally,” she emphasized, finishing with a flourish.

When I asked her to do the problem in another way, she chose the 100s chart. “Oh boy,” she said to herself. “Ochenta y cuatro, (*Eighty four,*)” and put a cube on 84. Then she decided to move the cube to 80, explaining, “porque, no más que estarán ochenta. (*because, just so that there will be eighty.*)” She covered up the 90s row on the chart with her arm and said, “Entonces, puse tan cinco en cinco, van a ser ocho...si no, entonces, pongo así, (*Then, I put as by fives, they are going to be eight...if not, then I put like this,*)” and she put the cube on the 80 again, and then put cubes on 60, 40, and 20, adding, “para cada una. (*for each one.*)” I asked her what the cubes on the numbers represented and

she explained, “O.K. Estos representan dos, (*Ok, these represent two,*)” and she put fingers on the two cubes at 80 and 60, possibly meaning the amount of 20 pencils for two friends. She continued, “Estos..como, no, (*These...like, no,*)” now speaking more to herself than to me. “Si le quité los cuatro, más vale que, (*If I got rid of the four, it is better that,*)” she said as she studied the four cubes. “Estos dos, cuatro, seis, (*These two, four, six,*)” and she appeared to be counting by groups of 10 as she moved from one cube to another. “Oh, no... uno, dos, cuatro...ah...(*Oh no...one, two, four...ah.*)” She immediately looked up at me and smiled at her miscount. “Uno, dos, cuatro? Uno, dos, tres, cuatro. Entonces, si está bien, no me lo pongo cuatro más que son estos trickitos. (*One, two, four? One, two, three, four. Then, if this is good, I don't put more than these four little tricksters.*)” She laughed and indicated the vertical line of tens on the chart. I was a little confused at what she was trying to do, so I asked her again how many pencils for each friend. She said, “Oh...venti...uno, (*Oh...twenty...one,*)” as she held up her little finger.

I was curious if she could write an equation for the problem so I asked her if she could do it on paper. “I don't know how to do it on paper,” she replied then added, “Don't like to. Well...I do but not in this problem.” However, she suddenly got the idea of direct modeling on paper by drawing lines in four circles. After a minute or two she gave up on this idea saying it would take her “a million years” to draw in all the lines. When I asked her explicitly about an equation for this problem, she didn't think it was possible, saying, “No se me parece conmigo. No más en otros problemas, como minus, (*It doesn't seem to me that I can. Only in other problems, like minus,*)” She didn't think she could

use addition or subtraction, “Porque no más son dos numeros, que tienen que repartir. *(Because there are only two numbers, and they have to share.)*”

Gina had an interesting response when I asked her to choose the best tool to solve this problem. She pointed to the base ten blocks, saying, “I think this one...I think this one in that problem.” Then she got some rods and added, “cause it’s easy, cause they’re already made! “Look,” she continued and held the rods toward me. “They’re just tens.”

Yolanda’s solution

I began, “Tú tienes ochenta y cuatro lápices y los quieres repartir entre cuatro amigas para que cada amiga reciba la misma cantidad de lápices. ¿Cuántos lápices recibe cada amiga? *(You have eighty four pencils and you want to share them among four friends to that each friend receives the same amount of pencils. How many pencils does each friend get?)*” After I finished the problem description, Yolanda started to get a piece of paper, but changed her mind. Then she got the number line and said, “Casi no lo puedo ver, *(I almost can’t see it,)*” and rejected it. She looked at the 100s chart, but rejected it, and finally picked up a piece of blank paper and said something inaudible, ending with “un dibujo *(a drawing.)*”

Yolanda drew four circles on the paper then started to put lines on each one, counting to herself by ones. When she reached *vienticinco (twenty five)*, I stopped her because I believed at the time that she could easily find the answer by direct modeling one by one and I wanted to see what else she could do besides this type of direct modeling. She started to show some frustration at my stopping her, said, “hmm” and picked up the 100s chart. She began by pointing to the tens column, said, “veinte

(*twenty*)”, pointed to 20, then said with a slight whine in her voice, “diecinueve...no entiendo. (*nineteen...I don’t understand.*)”

Yolanda was on the right track with the 100s chart, and it looked like she was going to think in tens, maybe in twenties, and 20 would have been a good estimate. However, she decided to go back to her drawing, but rejected the drawing again, went to the small number line, pointed to a couple of places, maybe 84, said something inaudible, and rejected the number line again. Back to the 100s chart, she pointed to 84 then put her finger on 72, went up two rows of tens before making a whining noise of frustration.

“Recuerdas la historia? (*Do you remember the story?*)” I ask.

“Sí, pero no sé como hacerla. (*Yes, but I don’t know how to do it.*)” She turned back to her drawing with the four circles and put four more lines on each, one at a time. I decide not to stop her direct modeling approach this time. She continued drawing lines on the circles one at a time, counting softly to herself. Sometimes she added two lines to one circle. There did not seem to be a clear system to her direct model although she proceeded fairly methodically. At one point she stopped and counted around one circle then added another line. All of a sudden, seemingly very frustrated, she said, “no sé! (*I don’t know!*)” but it was not clear why she was getting so frustrated with her direct modeling strategy.

I suggested another method, and she decided to try the base ten rods. Feeling bad about her frustration I said, “Discúlpeme Yolanda por este difícil...(*Excuse me Yolanda for this difficult...*)” but she was intently counting the rods until she had eight rods altogether. “Y ¿cuántos lápices en total? ¿Cuántos tienes? (*And how many pencils in all? How many do you have?*)” Still she did not respond, concentrating on the rods and added

four cubes. She moved two rods away from the group of eight, and then she moved all the rods so she had three groups, in 3, 3 and 2. She moved them again so she had four groups of two. At this point she almost had the answer in front of herself and she just needed to partition the four cubes. Unfortunately, she didn't see how close she was. She looked at the four groups of two rods and moved the rods back into the 3, 3, and 2 arrangement.

“¿Recuerdas los números en la historia? (*Do you remember the numbers in the story?*)” I asked her again.

“Sí, (*Yes,*)” she said, “que yo tengo ochenta, ochenta... (*that I have eighty, eighty...*)” studying the rods.

“Ochenta y...cuatro, (*Eighty...four,*)” I tell her.

She continued the sentence, saying, “lápices...cuatro lápices. (*pencils...four pencils.*)” Then she picked up four cubes. “Y había cuatro niñas y cuánto le dio a cada niña, (*And there were four girls and how many did I give to each girl,*)” she added with some impatience. Her frustration was obvious. She put a cube with each of four rods then gave up. Once again, she was very close to the answer, but the frustration seemed to have blocked her thinking. “Yo no sé, (*I don't know,*)” she whined and gathered the rods together.

“¿Dónde están las cuatro niñas aquí? (*Where are the four girls here?*) I asked as I indicated the space in front of her, hoping she would see the answer with a little scaffolding. “Dónde están las cuatro? (*Where are the four?*)”

“No sé, (*I don't know*)” she whined as she put a rod with the four cubes. Then suddenly she got an idea, turned away from the rods and started adding four 16s

vertically on paper. “Seis, doce, dieciocho...*(Six, twelve, eighteen...)*” she said and she pointed to the 6s while she added. While she worked, her level of frustration at first inhibited her ability to add and she made a couple of mistakes in the process, which was unusual for her. I wanted to help her see her mistakes through questioning, but she paid no attention to me. She moved on to another place in her paper and was intently focused on adding four 18s. These didn’t add up to 84 so she started again adding four 19s and got an answer of 76. I noted out loud that she was getting close to 84. Without responding she repeated the process with four 21s. She added them quickly, counted the ones column by ones and the 2s in the tens column by twos then wrote 84 on her paper and said, “Veintiuno, *(Twenty one,)*” with confidence in her voice once again.

Once she knew the answer and recovered her confidence, I wanted to see if she could find the solution by partitioning the base ten blocks. She got eight rods and four cubes, said, “Veinte, veintiuno, *(Twenty, twenty one,)*” and set aside two of the rods and a cube. “Veintiuno, *(Twenty one,)*” and she set aside another two rods and a cube. Then she made the other two groups of 21 silently. “Veintiuno, *(Twenty one,)* she laughed, now lighthearted. “Veintiuno, veintiuno, veintiuno, veintiuno, *(Twenty one, twenty one, twenty one, twenty one,)*” and pointed to each group, “Es igual a ochenta y cuatro. *(Is equal to eighty four.)*” At the end of the problem, I asked her what she thought about this base ten block solution. Her opinion of this block solution was, “Ya está fácil porque yo lo sé. *(Now it’s easy because I know it.)*”

When asked what she thought was the best method to solve this problem, she said, “Esta sea más fácil para mí pensar bien, *(This is the easiest for me to think well,)*” indicating the blocks, “porque los demás tendría que ser [inaudible] ver la

respuesta...pero no sé cual es más fácil. (*because the others you would have to be [inaudible] to see the answer...but I don't know which is the easiest.*) Se me hace que es este, (*It seems to me it is this,*)” pointing to her paper, “porque solamente escribe los numeros allí y te sale la respuesta pero necesitas cubrir todo, (*because you just write the numbers there and you get the answer, but you need to cover the whole thing,*)” she said as she made circles above the paper with her pencil.

Gerardo's solution

I began the problem in Spanish, but switched to English because he seemed a little distracted. I said, “Ah, so you have eighty four pencils, and you want to share them, you want to give them to four friends so that each friend has the same amount.” He prepared to write on the paper while I was speaking, then he began by writing 84 on the piece of paper, but he decided to use the 100s chart instead.

Gerardo started counting silently while he pointed to the 100s chart. When I asked him to do it aloud, he began, “...ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty...I'm thinking that if I,” and he held the place with his finger, “that I have eighty four and I take away twenty,” his finger was on 44, “I might have...[inaudible] ok, so that's nine. I forgot what I was ...” Then he started again at 83. “One, two, three,” and he counted back to 24 out loud, stopping at 60. “Ahh,” he said with some frustration in his voice. Starting again at 83 he said, “One, two, three,” and he counted up to 20 as he kept track of the groups of twenty with his fingers and pointed with the pencil to the numbers on the chart. He counted up to 20 four times, but wound up at 2 on the 100s chart. He paused with a look of frustration on his face and then appeared to give up on the 100s chart.

“Can you think of another way maybe?” I asked encouragingly.

“Hmm. Ten? No.” He put his head down on the table on his crossed arms, but suddenly he picked his head up again with renewed interest. “How about thirty?” he asked. He went back to the 100s chart at 83 and started counting backwards again. He counted aloud from one to 30 going backwards and gets to 53. He did this two more times, but got confused. All the while he was counting in English. He made two more attempts to count by 30, and finally when he arrived at ten on the chart, he realized that thirty wouldn’t work as an answer. He put his head down on his hand on the table, once again frustrated. But then once again he rallied. He started again from 83, counted aloud backwards to 56, and then from 55 he went back by tens to 35, 25, 15, and 5. But once again he was stopped by frustration and slapped his hand to his head. Still, he was able to rally and said, “I wonder if I should try...” then started counting again. Counting very carefully, he went backwards two times by 30 counts and marked each group with a finger. Then he continued counting until he reached 20, which landed him on 3 instead of 4. He counted 2 and 1 in a mumble. He mumbled something else to himself, started counting, and appeared to be trying 15 again. Unfortunately, right at this moment Omar stopped by the interview table in the library. I nudged Omar away, but Gerardo was distracted, skipped a line and counted to 20 the third time instead of 15. Realizing this, he put his hand to his head again and said “Ahhhh!”

I finally suggested he try the base ten blocks. Gerardo counted them out twice in English, “One, two, three, four, five...six, seven...eight,” and then counted out the cubes in a whisper, “One, two, three, four.” More loudly he said, “O.K. I’m just going to move these four away,” and he moved the four cubes away from the rods. “So let me take

out...I'm going to give...I have four friends, here," and he touched a place on the table with a rod in his hand. "I'm going to give ten to one," he moved a rod, "ten to one, ten to one, and ten to another." He had moved out four rods then continued, "Ten to one, ten to one, ten to one, and ten to another," At this point he had moved all eight rods into four groups of two. Next he said, "and then I give one to one," distributing the single cubes, "and one." He started rearranging the rods and cubes and said, "O.K., so they have," as he clearly put two rods in each of four groups, "and I give all of them one cube." He put a cube with each group of two rods, gave me a look of triumph, smiled then said, "And then I don't get to have any."

"Right. So the answer is?" I asked him.

"Is twenty one," he said confidently.

I asked if he could do the problem on the 100s chart now that he knew the answer. To do this he said, "Just take off these four," and he put four fingers on the numbers 81 to 84 on the chart, "use these," he pointed to the tens column, "once they're all finished use these four again," and he pointed to the top row. "Just put these aside," he covered 81 to 84 with his fingers again, "then I'm going to have eighty" and he drew his finger along the tens column. "Let me just take this...take two, two, two and two," he said as he used two fingers to show removing two rows of ten at a time. With his gestures and his words he demonstrated removing the four ones and distributing the eighty pencils in four groups two rows each. Finally, he distributed the four ones, saying, "Then I'm gonna, since I'm gonna have these [the four ones]...It's gonna put em here," he puts his hand down on the table, "then I'm gonna, like, and then I'm gonna go here, I have, and then I'm gonna go

like if I have extras then I'll give em one of the extras that I had, “ and he moved his hand to demonstrate giving the four cubes to four people.

REFERENCES

- Alleksaht-Snider, M., & Hart, L. E. (2001). "Mathematics for all": How do we get there? *Theory Into Practice*, 40(2), 93-101.
- Aunola, K., Leskinen, E., Lerkkanen, M-K., & Nurmi, J-E. (2004). Developmental dynamics of math performance from preschool to grade 2. *Journal of Educational Psychology*, 96(4), 699-713.
- Bialystok, E. (2001). *Bilingualism in development: Language, literacy and cognition*. Cambridge: Cambridge University Press.
- Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education*, 24(5), 427-440.
- Carpenter, T., Fennema, E., & Franke, M. L. (1994). *Cognitively Guided Instruction: Children's thinking about whole numbers*. Madison, WI: Wisconsin Center for Education Research.
- Carpenter, T., Fennema, E., & Franke, M. (1996). Cognitively Guided Instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3-20
- Carpenter, T., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann/NCTM.
- Celedón-Pattichis, S., Musanti, S., & Marshall, M. (in press). Bilingual teachers' reflections on students' native language and culture to teach mathematics. Manuscript to appear in M. Foote (Ed.), *Mathematics teaching and learning in K-12: Equity and professional development*. New York, NY: Palgrave Mcmillan. Peer reviewed chapter.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of development research. *Educational Psychologist*, 31, 175-190.
- Cummins, J. (1981). The role of primary language development in promoting educational success for language minority students. In California State Department of Education (Ed.) *Schooling and language minority students: A theoretical framework* (pp. 3-49). Los Angeles: California State Department of Education.
- Cummins, J. (2000). *Language, power and pedagogy: Bilingual children in the crossfire*. Clevedon: Multilingual Matters.

- Cummins, J. (2001). Empowering minority students: A framework for intervention. *Harvard Educational Review*, 71(4), 649-675.
- Delpit, L. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Educational Review*, 58, 280-298.
- Domínguez, H. (2005). Bilingual students' articulation and gesticulation of mathematical knowledge during problem solving. *Bilingual Research Journal*, 29(2), 269-293. Retrieved 6/29/09 from <http://brj.asu.edu>.
- Eisenhart, M. & Borko, H. (1993). Contributions from cognitive psychology. In (Authors) *Designing classroom research: Themes, issues, and struggles* (pp. 23-40). Boston: Allyn & Bacon.
- Ernest, P. (1985). The number line as a teaching aid. *Educational Studies in Mathematics*, 16(4), 411-424.
- Forman, E., & Ansell, E. (2001). The multiple voices of a mathematics classroom community. *Educational Studies in Mathematics*, 46(1&3), 115-142.
- Forman, E., McCormick, D., & Donato, R. (1998). Learning what counts as a mathematical explanation. *Linguistics and Education*, 9(4), 313-339.
- Fennema, E., Franke, M., Carpenter, T., & Carey, D. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal*, 30(3), 555-583.
- Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer-Verlag.
- Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed), *Handbook of research on mathematics teaching and learning* (pp. 243-271). New York: Macmillan.
- Fuson, K., Smith, S., & Lo Cicero, A. (1997). Supporting Latino first graders' ten-structured thinking in urban classrooms. *Journal for Research in Mathematics Education*, 28(6), 738-766.
- Garner, H. (1993). *Frames of mind: The theory of multiple intelligences*. New York: Basic Books.
- Gee, J. P. & Green, J. L. (1998). Discourse analysis, learning, and social practice: A methodological study. *Review of Research in Education*, 23, 119-169.
- Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

- Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education*, 34(1), 37-73.
- Hiebert, J. & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). Reston, VA: National Council of Teachers of Mathematics.
- John-Steiner, V., & Mahn, H. (1996). Sociocultural approaches to learning and development: A Vygotskian framework. *Educational Psychologist*, 31(4), 191-206.
- Jordan, N.C., Kaplan, D., Oláh, L. N., and Locuniak, M. N. (2006). Number sense growth in kindergarten: A longitudinal investigation of children at risk for mathematics difficulties. *Child Development*, January/February 2006, 77(1), 153-175.
- Kamii, C., Kirkland, L., & Lewis, B. (2001). Representation and abstraction in young children's numerical reasoning. In A. Cuoco & F. Curcio (Eds.) *The Roles of Representation in School Mathematics: 2001 Yearbook* (pp. 24-34). Reston, VA: NCTM.
- Kamii, C., Rummelsburg, J., & Kari, A. (2005). Teaching arithmetic to low-performing, low-SES first graders. *Journal of Mathematical Behavior*, 24(1), 39-50.
- Kato, Y., Kamii, C., Ozaki, K. & Nagahiro, M. (2002). Young children's representations of groups of objects: The relationship between abstraction and representation. *Journal for Research in Mathematics Education*, 33(1), 30-45.
- Khisty, L. L. (1997). Making mathematics accessible to Latino students: Rethinking instructional practices. In J. Trentacosta & M. Kenney (Eds.), *Multicultural and gender equity in the mathematics classroom: The gift of diversity, 97th yearbook of the National Council of Teachers of Mathematics* (pp. 94-101). Reston, VA: NCTM.
- Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *Educational Studies in Mathematics*, 46(1&3), 187-228.
- Kitchen, R. S. (2003). Getting real about mathematics education reform in high poverty communities. *For the Learning of Mathematics*, 23(3), 16-22.
- Kitchen, R. S. (2004). Challenges associated with developing discursive classrooms in high-poverty, rural schools. *Mathematics Teacher*, 97(1), 28-31.

- Kitchen, R. S. (2007). An overview of schooling in high poverty communities. In R. S. Kitchen, J. DePree, S. Celedón-Pattichis, & J. Brinkerhoff, *Mathematics education at highly effective schools that serve the poor: Strategies for change* (pp. 1-19). Mahway, NJ: Lawrence Erlbaum Associates.
- Kozol, J. (2005). Still separate, still unequal: America's educational apartheid. *Harper's Magazine*, 311(1864), 41-54.
- Lerman, S. (2001). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics*, 46(1&3), 87-113.
- Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education*, 31(4), 454-482.
- Lubienski, S. T. (2002). Research, reform, and equity in U.S. mathematics education. *Mathematical Thinking and Learning*, 4(2,3), 103-125.
- Mahn, H. (2008). A dialogic approach to teaching L2 writing. In J. P. Lantolf & M. E. Poehner (Eds.), *Sociocultural theory and the teaching of second languages* (pp. 115-138). London: Equinox.
- Mahn, H. (2009). *Vygotsky's analysis of the unit "System of Meaning"*. Paper presented at American Educational Research Association, April, 2009. San Diego, CA.
- McClain, K. and Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, 32(3), 236-266.
- Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, 4(2&3), 289-212.
- Musanti, S., Marshall, M., Ceballos, K., & Celedón-Pattichis, S. (in press). Situating mathematics professional development: A bilingual teacher and researchers' collaboration. Manuscript to appear in J. Moschkovich & K. Téllez (Eds.), *Latinos and mathematics: Research on learning and teaching in classrooms and communities*. Charlotte, NC: Information Age Publishing.
- National Assessment of Educational Progress. (2005). *The Nation's Report Card: Mathematics 2005*. Retrieved on-line October 15, 2006 at: <http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2006453>
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

- National Center for Education Statistics (2005). *The Nation's Report Card Mathematics 2005*, NCEES 2006-453, August. Washington, D.C.: NCEES.
- Ortiz-Franco, L. (1999). Latinos, income, and mathematics achievement: Beating the odds. In L. Ortiz-Franco, N. G. Hernandez & Y. de la Cruz (Eds.), *Changing the faces of mathematics: Perspectives on Latinos* (pp. 13-20). Reston, VA: National Council of Teachers of Mathematics.
- Pappas, S., Ginsburg, H. , & Jiang, M. (2002). SES differences in young children's metacognition in the context of mathematical problem solving. *Cognitive Development*, 18(3), 431-450.
- Pew Hispanic Center (2004). Hispanic school achievement: Catching up requires running faster than white youth. *Fact Sheet*. Retrieved 1/10/08 from <http://www.pewhispanic.org>.
- Powell, A. (2004). The diversity backlash and the mathematical agency of students of color. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 37-54).
- Secada, W. (1989a). Agenda setting, enlightened self-interest, and equity in mathematics education. *Peabody Journal of Education*, 66(2), 22-56.
- Secada, W. (1989b). Editor's introduction: Needed: An agenda for equity in mathematics education. *Peabody Journal of Education*, 66(2), 1-5.
- Secada, W. (1991). Degree of bilingualism and arithmetic problem solving in Hispanic first graders. *The Elementary School Journal*, 92(2), 213-231.
- Secada, W. (1992). Race, ethnicity, social class, language, and achievement in mathematics, in: D. A. Grows (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 623-660). New York: Macmillan.
- Secada, W. (1995). Social and critical dimensions for equity in mathematics education. In W. Secada, E. Fennema & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 146-164). New York, NY: Cambridge University Press.
- Secada, W. G & De La Cruz, Y. (1996). Teaching mathematics for understanding to bilingual students. In J. L. Flores (Ed.) *Children of la frontera: Bilingual efforts to serve Mexican migrant and immigrant students*. (ERIC Document Reproduction Service No. ED393646). Retrieved from ERIC database.

- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46(1&3), 13-57.
- Steffe, L. P. & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education*, 25, 711-733.
- Thomas, W., & Collier, V. (2002). A national study of school effectiveness for language minority students' long-term academic achievement. Center for Research on Education, Diversity, and Excellence (CREDE). (ERIC Document Reproductive Service No. ED 475048). Retrieved from ERIC database.
- Trueba, E. T. (1999). Critical ethnography and a Vygotskian pedagogy of hope: The empowerment of Mexican immigrant children. *Qualitative Studies in Education*, 12(6), 591-614.
- Turner, E., Celedón-Pattichis, S., & Marshall, M. E. (2008). Cultural and linguistic resources to promote problem solving and mathematical discourse among Hispanic kindergarten students. In R. Kitchen & E. Silver (Eds.), *Promoting high participation and success in mathematics by Hispanic students: Examining opportunities and probing promising practices* [A Research Monograph of TODOS: Mathematics for ALL], 1, 19-42. Washington D.C.: National Education Association Press.
- Van der Veer, R. & Valsiner, J. (1991). The universe of words: Vygotsky's view on concept-formation. In *understanding Vygotsky: A quest for synthesis* (pp. 256-283). Cambridge: Blackwell.
- Van Oers, B. (2001). Educational forms of initiation in mathematical culture. *Educational Studies in Mathematics*, 46(1&3), 59-85.
- Van Wageningen, R. K., Flora, J. A. & Walker, A. A. (1976). The introduction of mathematics through measurement or through set theory: A comparison. *Journal for Research in Mathematics Education*, 7(5), 229-307.
- Villaseñor, A. Jr. & Kepner, H. S. Jr. (1993). Arithmetic from a problem-solving perspective: An urban implementation. *Journal for Research in Mathematics Education*, 24(1), 62-69.
- Vygotsky, L. S. (1986). *Thought and language*. (A. Kozulin, Ed.). Cambridge: MIT Press.
- Vygotsky, L. S. (1987). *Problems of general psychology. The collected works of L. S. Vygotsky: Vol. 1*. Including the volume *Thinking and Speech*. New York, NY: Plenum.

Zack, V. & Graves, B. (2001). Making mathematical meaning through dialogue: "Once you think of it, the z minus three seems pretty weird." *Educational Studies in Mathematics*, 46(1&3), 229-271.