1988

Caustics and virtual cathodes in electron beams

Evangelos A. Coutsias

Recommended Citation
Journal of Plasma Physics, 40(2): 369-384

J. Plasma Physics (1988), *vol.* 40, *part* 2, *pp.* 369–384
Printed in Great Britain

Caustics and virtual cathodes in electron beams

By EVANGELOS A. COUTSIAS

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.

(Received 22 April 1987)

A simplified model is discussed that captures the basic physics of the phenomenon of oscillatory virtual cathodes in electron beams. A monoenergetic non-relativistic one-dimensional electron beam is injected through a conducting grid into a semi-infinite drift space. Attraction from image charges (and, possibly, an adverse externally applied electric field) cause particle reflection and the formation of a caustic where the charge density has an integrable singularity. The steady-state solution of the Vlasov equation describing the flow is known from numerical simulation to be unstable, but analytical demonstration of this instability has proved intractable. Here we derive an integral-delay equation describing the time-dependent evolution of the electron beam under the assumption that the caustic accelerates much more slowly than the electrons in its neighbourhood and thus at most two streams are present at each point. Under this assumption we show that the charge singularity is $\sim |x-x_c|^{-\frac{1}{2}}$ in the presence of an external field, exactly what it would be for non-interacting particles, but in the absence of applied field it is weaker, $\sim (x-x_c)^{-\frac{1}{3}}$. Our methods can be used to estimate the charge singularity, and thus the collisionless 'shock conditions' for virtual cathodes in any geometry. The importance of delay effects for the onset of beam oscillations is demonstrated in an exactly solvable version of the model in which the interaction between the two streams is ignored. This solution, although unphysical, can provide a means for testing the performance of numerical schemes, which have difficulties in problems of this type due to the charge singularity.

1. Introduction

Virtual cathode oscillations were first encountered in the study of diode tubes (Birdsall & Bridges 1966). Their occurrence in all space-charge-limited devices (Miller 1982) makes them an essential feature of non-neutral plasma flows interacting with conducting walls. Current interest in this phenomenon is due mainly to applications of intense charged particle beams in inertial confinement fusion (Kuhn 1984) and microwave generation (Sullivan *et al.* 1984). For a more detailed discussion see Coutsias & Sullivan (1983) and references therein. Miller (1982) is an excellent introduction to the subject, providing all the necessary background material, while Whitham (1976) supplies the necessary mathematical techniques.

In this study we consider the simplest physical arrangement in which virtual cathode (VC) oscillations occur (figure 1): a non-relativistic monoenergetic electron beam is injected in the positive x-direction in a semi-infinite vacuum,

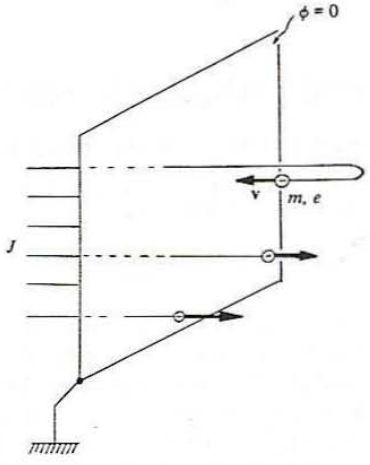

FIGURE 1. Schematic of model: a mononergetic electron beam is injected through an infinite planar grounded conducting grid from the left against an adverse electric field E.

through an infinite grounded conducting grid at $x = 0$. A constant electric field E_∞ is applied, repelling the electrons towards the grid. The injection velocity of the electrons is $u(0, t) = u_0$ and charge density $n(0, t) = n_0$. A very strong magnetic field in the x-direction, $B_x \approx \infty$, ensures that the beam stays one-dimensional. If we ignore radiation effects (which should be included if one wants to study microwave generation) and collisions (justified for typical beam densities of 10^{-13} cm^{-3}) the flow is described by the one-dimensional Vlasov equation (Davidson 1974)

$$\partial_t f + u\,\partial_x f - \frac{e}{m}(E_\infty + E_{\text{self}})\,\partial_u f = 0, \tag{1.1}$$

where $f = f(u, x, t)$ is the electron velocity distribution function, from which the electron charge density n is found as

$$n(x, t) = \int_{-\infty}^{\infty} f(u, x, t)\,du. \tag{1.2}$$

The self-consistent electric field satisfies

$$\partial_x(E_{\text{self}}) = \sum_i \frac{1}{\epsilon_0} n_i, \tag{1.3}$$

so that the total electric field is given by

$$E = E_\infty + \frac{1}{\epsilon_0}\sum_i \int_\infty^x n_i\,dx. \tag{1.4}$$

We assume a monoenergetic beam, so that the distribution function has the form

$$f(u, x, t) = \sum_i n_i(x, t)\,\delta[u - u_i(x, t)]. \tag{1.5}$$

The reason we consider a Vlasov equation instead of a fluid description lies in the fact that we expect the presence of multiple streams. This means that the functions n and u in (1.3)–(1.5) are expected to be multi-valued in general, and the summation in these expressions is over the various streams present at each point of the space.

Introducing the dimensionless variables

$$\left.\begin{aligned} E' &= \frac{E}{E_\infty}, \quad u' = \frac{u}{u_0}, \quad x' = \frac{e}{m}\frac{E_\infty}{u_0^2}x, \\ t' &= \frac{e}{m}\frac{E_\infty}{u_0}t, \quad n' = \frac{n}{|n_0|}, \quad f' = \frac{f}{|n_0|}, \end{aligned}\right\} \tag{1.6}$$

(1.1)–(1.4) become

$$\partial_{t'} f' + u' \partial_{x'} f' + E' \partial_{u'} f' = 0, \tag{1.7a}$$

$$f'(0, u', t') = -\delta(u'-1) \quad (u' > 0), \tag{1.7b}$$

$$E' = 1 + \lambda E_i \int_\infty^{x'} n_i' ds, \quad n' = \int_{u'} f' du, \tag{1.7c–d}$$

with

$$\lambda = \frac{u_0^2 |n_0|}{(e/m)\epsilon_0 E_\infty^2} \tag{1.8}$$

a dimensionless parameter measuring the ratio of self-consistent to external field. In the subsequent discussion we deal with (1.7*a–d*), dropping the primes for convenience.

In §2 we derive the steady-state solution and we see that it is a double-streaming state with electrons coming to rest at some distance $x = x_c$ from the grid and then being accelerated back to the grid. The charge density becomes infinite at x_c, but the singularity is integrable, of type $\sim (x_c - x)^{-\frac{1}{2}}$. In the limit of vanishing external field the singularity is weaker, $\sim (x_m - x)^{-\frac{1}{3}}$, but still integrable.

A Lagrangian description of the flow in terms of particle trajectories shows that in a space–time diagram the trajectories have an envelope at $x = x_c$, which we shall refer to as a caustic, using the terminology of geometrical optics (figure 2). Simulations show that this flow is numerically unstable and, at least in the case $E_\infty = 0$ of no imposed field, that there is no unique caustic for the time-dependent state (see figures 6–8 of Dunn & Ho 1963). Any effort to prove the instability of the steady state analytically must take account of the density singularity in perturbing around the steady state, and the fact that in time-dependent states the singularity moves and might break into several branches. However, it is possible to give a discussion for the case of perturbations of the steady state that result in modulations of the caustic preserving the double-streaming property. A sufficient condition for such a perturbation is that the acceleration of the caustic be much smaller than the acceleration of the particles passing through each of its points. This is because in this case (see figure 2) the particle trajectories in the (x, t) diagram will stay always on the grid side of the caustic.

Under this assumption, we formulate a double-streaming model of the flow with a gently modulated caustic. We close §2 with the derivation of an exact

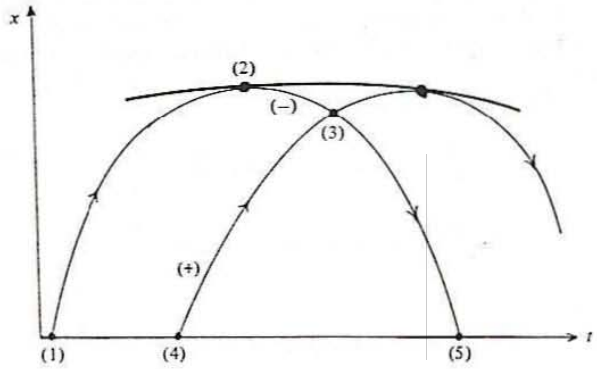

FIGURE 2. The beam envelope; the points shown have Eulerian co-ordinates (t, x) as follows: (1) $(s_1, 0)$; (2) $(s_1+T(s_1), \dot{\psi}(s_1+T(s_1)))$; (3) (t, x); (4) $(s_2 = S^+(t, s_1), 0)$; (5) $(s_1+R(s_1), 0)$.

system of equations that describe the time-dependent behaviour of the caustic. This system seems extremely difficult to solve in general, but it can be used to draw several conclusions.

In §3 we present a linearized analysis of this system for the special case of no externally imposed field ($\lambda \to \infty$). Since our main purpose is to show the difficulties involved in such analyses, we do not present the case for general λ. We conjecture that for small enough λ the steady state is stable, since in this limit the self-interaction that is responsible for destabilization becomes arbitrarily weak. We have not been able to find a rigorous proof of this assertion. We derive an integral equation that can be analysed, in principle, with the help of the computer to yield a 'dispersion relation' for the small-amplitude behaviour of double-streaming perturbations of the steady state.

In §4 we show how to derive the form of the charge singularity at the caustic. It turns out that the charge singularity on a weakly accelerating caustic is the same as for the steady state.

In §5, by ignoring the interaction between the two streams, we produce an exact solution of the integral equations found in §2. For λ sufficiently small there is a steady state that is stable, and we find its domain of attraction. There is also an unsteady oscillatory state that eventually leads to a breakdown of our assumptions. Above a critical value of λ ($= \frac{3}{2}$), the domain of attraction of the steady state vanishes and the oscillatory state disappears. The solution grows until a point where, again, the assumptions leading to our model break down. In any case, we obtain a time-dependent solution valid for a certain finite time.

2. The two-stream model

In this section we present a formulation based on a Lagrangian (characteristic) description that assumes the presence of two streams. The discussion of the steady-state solutions of (1.7) given at the end of the section will help to further clarify the reasons for this choice.

In Lagrangian coordinates each point in space is represented in terms of the particles that occupy it at a given time by

$$x = x(s, t), \qquad t = t, \tag{2.1}$$

where t is the physical time and s is the entry time of the particle passing by the position x at time t. Obviously, in the case of multiple streams, the transformation (2.1) is many-to-one. Throughout we shall assume that the shape of the trajectories in an (x, t) diagram is typified by figure 2, i.e. there is an envelope, $x = \psi(t)$, so that there are always two trajectories crossing every point with $x < \psi$ and for $x = \psi$ the trajectories are tangent to the envelope. We see that when two trajectories cross at some point $(x < \psi(t), t)$, say with $s = s_1$ and $s = s_2 > s_1$, then all the particles (sheaths) that entered the drift space between $t = s_1$ and $t = s_2$ are further from the grid than x. In the one-dimensional geometry we consider the self-consistent field on a particle at x is due only to the total charge further away from the grid than x (and its image) and is independent of distance. We may think of our particles as sheaths of charge.

We shall use the convention that a particle is on the ascending (+) phase of its motion before its encounter with the caustic and on the descending (−) phase after that encounter. The expressions $S^{(\mp)}(t, s)$ (figure 2) give the entry time of the descending (ascending) particle crossing the trajectory of the ascending (descending) particle that entered at time s and is at the point with Eulerian co-ordinates (x, t).

We define $T(s)$ to be the time it takes for the particle entering at $t = s$ to reach the caustic, so that (see figure 2)

$$x(s+T(s), s) = \psi(s+T(s)), \tag{2.2}$$

and $R(s)$ to be the time it takes for the particle entering at $t = s$ to return to the grid:

$$x(s+R(s), s) = x(s, s) = 0. \tag{2.3}$$

For convenience, we define

$$0 \leqslant \Delta^{\pm}(t, s) = \pm(S^{\pm}(t, s) - s). \tag{2.4}$$

In these co-ordinates we also have that

$$x_t(s, s) = 1 \tag{2.5}$$

is the entry velocity; $\qquad x_t(s+T(s), s) = \psi_t(s+T(s)) \tag{2.6}$

guarantees that particle trajectories are tangent to the caustic; and

$$x_s(s+T(s), s) = 0 \tag{2.7}$$

is the geometric condition that the trajectories have an envelope. In the last two expressions no distinction is made as to whether we speak of ascending or descending trajectories. First derivatives of the trajectories are continuous on the caustic, but self-interaction introduces discontinuities in higher derivatives.

With these definitions, the equation describing particle trajectories (the equation for the characteristics of (1.7)) becomes

$$\frac{\partial^2 x}{\partial t^2} = -1 - \lambda \Delta^{\mp}(t, s), \tag{2.8}$$

where the top sign $(-)$ is taken if $s \leqslant t < s+T(s)$ (ascending phase) and the bottom sign $(+)$ is taken if $s+T(s) < t \leqslant s+R(s)$ (descending phase). This equation simply expresses the fact that the collective force on a sheath at position x is proportional to the total number of sheaths whose distance from the grid is greater than x (figure 2).

Integrating (2.8) subject to the conditions (2.2)–(2.7), we find for the particle trajectories the expressions

$$x(t, s) = -\tfrac{1}{2}(t-s)^2 + (t-s) - \lambda \int_s^t (t-\xi)\, \Delta^-(\xi, s)\, d\xi \quad \text{for } s \leqslant t < s+T(s), \tag{2.9}$$

$$x(t, s) = -\tfrac{1}{2}[t-s-T(s)]^2 + \psi_t(s+T(s))\,[t-s-T(s)] + \psi(s+T(s)) - \lambda \int_{s+T(s)}^t (t-\xi)\, \Delta^+(\xi, s)\, d\xi \quad \text{for } s+T(s) < t \leqslant s+R(s). \tag{2.10}$$

These expressions, when supplemented by the compatibility conditions

$$x(t, s) = \begin{cases} x(t, s-\Delta^-(t, s)) & (s \leqslant t < s+T(s)), \quad (2.11) \\ x(t, s+\Delta^+(t, s)) & (s+T(s) < T \leqslant s+R(s)) \quad (2.12) \end{cases}$$

that must be satisfied by crossing trajectories, and the obvious relations

$$\Delta^-(s+T(s), s) = 0 = \Delta^+(s+T(s), s), \tag{2.13}$$

$$\Delta^+(s+R(s), s) = R(s), \tag{2.14}$$

lead to two coupled integral equations for Δ^+ and Δ^-.

Since these are very complicated they will not be presented here. We shall only mention that, in principle, (2.2), (2.3), (2.6), (2.7) and (2.9)–(2.14) provide a closed system of integral–delay equations for $\psi(t)$ that might be studied to provide us with the behaviour of the caustic as the external field is varied. We close this section with a discussion of steady-state solutions of this system.

For steady solutions, symmetry arguments show that ψ, T and $R = 2T$ are constant and that x and $\Delta^{\pm}$ are functions of $z = t-s$. It is easy to see that

$$\Delta^+(t, s) = 2(z-T) \quad (T \leqslant z \leqslant 2T),$$

$$\Delta^-(t, s) = 2(T-z) \quad (0 \leqslant z \leqslant T),$$

so that

$$x(z) = \begin{cases} \tfrac{1}{3}\lambda z^3 - (\lambda T + \tfrac{1}{2})\, z^2 + z & (0 \leqslant z \leqslant T), \quad (2.15a) \\ -\tfrac{1}{3}\lambda (z-T)^3 - \tfrac{1}{2}(z-T)^2 + \psi & (T \leqslant z \leqslant 2T). \quad (2.15b) \end{cases}$$

Applying (2.6) to (2.15a), we find

$$-\lambda T^2 - T + 1 = 0, \tag{2.16}$$

so that

$$T = \frac{-1+(1+4\lambda)^{\frac{1}{2}}}{2\lambda}, \tag{2.17}$$

and (2.2) in (2.15a) gives

$$\psi = \tfrac{1}{6}T(T+2). \tag{2.18}$$

Utilizing (2.16) and (2.18) in (2.15a, b), we find

$$x(z) = \pm\tfrac{1}{3}\lambda(z-T)^3 - \tfrac{1}{2}(z-T)^2 + \psi, \tag{2.19}$$

with $(+)$ if $z \leqslant T$ and $(-)$ if $z \geqslant T$, so that the third derivative of x is discontinuous at the caustic if $\lambda \neq 0$.

The natural question now is whether the steady state found above for $0 \leqslant \lambda < \infty$ is stable for all λ, as it obviously is if $\lambda = 0$. In the next section we formulate (after appropriate rescaling) the problem of linearized stability of the steady state for the case $\lambda \to \infty$ (vanishing external field). This will help to demonstrate the difficulties that must be surmounted in studying the stability of the steady solution (17)–(19) for any λ.

3. Linearized stability

We now investigate some of the stability properties of the two-streaming solutions discussed in §2. Specifically, we wish to examine the case most commonly considered in practice, in which there is no external electric field. For this, we need to consider the limit $\lambda \to \infty$. We define

$$\mu = \lambda^{-\frac{1}{2}} = \frac{E_\infty}{u_0}\left(\frac{\epsilon_0 e}{|n_0| m}\right)^{\frac{1}{2}}, \tag{3.1}$$

and we introduce new dimensionless variables x'', t'', s'', T'', etc., related to the dimensionless variables defined in §1 by a scaling by μ, i.e.

$$x'' = \frac{1}{\mu}x', \quad t'' = \frac{1}{\mu}t', \quad s'' = \frac{1}{\mu}s', \quad T'' = \frac{1}{\mu}T', \quad \text{etc.}, \tag{3.2}$$

where, to avoid confusion, we have reinstated the primes in the dimensionless variables defined by (1.6). This scaling amounts to a transformation

$$E'' = \frac{1}{u_0}\left(\frac{\epsilon_0 e}{|n_0| m}\right)^{\frac{1}{2}} E,$$

$$x'' = \frac{1}{u_0}\left(\frac{|n_0| e}{\epsilon_0 m}\right)^{\frac{1}{2}} x, \qquad t'' = \left(\frac{|n_0| e}{\epsilon_0 m}\right)^{\frac{1}{2}} t$$

$$n'' = \frac{1}{|n_0|} n, \qquad u_0'' = \frac{1}{u_0} u,$$

where t scales by the plasma frequency. In the new variables, (2.8)–(2.10) become (again the primes are conveniently dropped)

$$\frac{\partial^2 x}{\partial t^2} = -\mu - \Delta^{\mp}(t, s), \tag{3.3}$$

integrating to

$$x(t,s) = -\tfrac{1}{2}\mu(t-s)^2 + (t-s) - \int_s^t (t-\xi)\,\Delta^-(\xi,s)\,d\xi \tag{3.4}$$

$$\text{for } s \leqslant t < s+T(s), \tag{3.5}$$

$$x(t,s) = -\frac{\mu}{2}(t-s-T(s))^2 + \psi_t(s+T(s))\,[t-s-T(s)] + \psi(s+T(s)) - \int_{s+T(s)}^t (t-\xi)\,\Delta^+(\xi,s)\,d\xi \tag{3.6}$$

$$\text{for } s+T(s) < t \leqslant s+R(s). \tag{3.7}$$

The steady state in this scaling is given by

$$T = \tfrac{1}{2}[-\mu + (4+\mu^2)^{\frac{1}{2}}], \quad \psi = \tfrac{1}{6}T(2+\mu T), \tag{3.6a, b}$$

$$x(z) = \pm\tfrac{1}{3}(z-T)^3 - \tfrac{1}{2}\mu(z-T)^2 + \psi, \tag{3.6c}$$

while, as before,

$$\Delta^\pm(z) = \pm 2(z-T) \tag{3.6d}$$

((+) if $T \leqslant z \leqslant 2T$, (−) if $0 \leqslant z \leqslant T$). To study the (linearized) stability of the steady state given by (3.6*a–d*), we limit our discussion to the case $\mu = 0$. We let

$$T(s) = 1 + \epsilon T\, e^{ws}, \quad R(s) = 2 + \epsilon R\, e^{ws}, \tag{3.7a, b}$$

$$\psi(s+1) = \tfrac{1}{3} + \epsilon\Psi\, e^{ws}, \quad \Delta^\pm(t,s) = \pm 2(z-1-\epsilon T\, e^{ws}) + \epsilon\phi(z;w)\, e^{ws}, \tag{3.7c, d}$$

where T, Ψ and R are (unknown) constants, $z = t-s$, and $\epsilon \ll 1$ gives the amplitude of the perturbations. Substituting in (3.4) and (3.5), we get (to $O(\epsilon)$)

$$x = \pm\tfrac{1}{3}(z-1)^3 + \tfrac{1}{3} + \epsilon e^{ws}\begin{cases} -z^2 T - \int_0^z (z-\zeta)\,\phi(\zeta;w)\,d\zeta & (0 \leqslant z \leqslant 1), \\ (z-1)^2 T + [(z-1)w+1]\,\psi - \int_1^z (z-\zeta)\,\phi(\zeta;w)\,d\zeta & (1 \leqslant z \leqslant 2), \end{cases} \tag{3.8}$$

while (2.2) and (2.3) with (2.6), (2.13) and (2.14) give respectively (also to $O(\epsilon)$)

$$\Psi = -T - \int_0^1 (1-\zeta)\,\phi(\zeta;w)\,d\zeta, \quad R = -2T - \int_0^2 (2-\zeta)\,\phi(\zeta;w)\,d\zeta, \tag{3.9a, b}$$

$$\phi(1;w) = 0, \quad R = 2T - \phi(2;w). \tag{3.9c, d}$$

A final relation is found by considering

$$\underline{s - S^-(s,s) = \Delta^-(s,s) = R(S^-).}$$

Now $\Delta^-(s,s) = 2 + \epsilon 2T\, e^{ws} + \epsilon\phi(0,w)\, e^{ws}$, so that

$$S^-(s,s) = s - 2 - \epsilon[2T + \phi(0;w)]\, e^{ws},$$

$$R(S^-) = 2 + \epsilon R\, e^{w(s-2)} + O(\epsilon^2),$$

giving, finally,

$$\phi(0) = R\, e^{-2w} - 2T. \tag{3.9c}$$

We now utilize the above expressions in the conditions (2.11) and (2.12) for crossing trajectories. We obtain the two integral equations

$$(z-1)^2\phi+(z^2-4z+2)\,T-\int_0^z(z-\zeta)\,\phi(\zeta)\,d\zeta$$
$$=e^{2(z-1)}\left\{(z-1)^2T+[(1-z)\,w+1]\,\psi-\int_1^{2-z}(2-z-\zeta)\,\phi(\zeta)\,d\zeta\right\}\quad\text{for } 0\leqslant z\leqslant 1, \tag{3.10a}$$

$$(z-1)^2\phi-(z-1)^2T+[(z-1)\,w+1]\,\psi-\int_1^z(z-\zeta)\,\phi(\zeta)\,d\zeta$$
$$=e^{2(z-1)}\left[-(2-z)^2-\int_0^{2-z}(2-z-\zeta)\,\phi(\zeta)\,d\zeta\right]\quad\text{for } 1\leqslant z\leqslant 2. \tag{3.10b}$$

Each of (3.10*a*, *b*) gives a relation between values of ϕ for $0\leqslant z\leqslant 1$ and $1\leqslant z\leqslant 2$. A possible strategy would be to express $\phi(z)$ for $1\leqslant z\leqslant 2$ in terms of $\phi(z)$ for $0\leqslant z\leqslant 1$ using (3.10*a*), then substitute into (3.10*b*) to get, eventually, a fourth-order ordinary differential equation for $\phi(z)$, $0\leqslant z\leqslant 1$. One then needs to determine for which values of w this equation can be solved, subject to the conditions (3.9*a*–*e*) relating the constants appearing in the equation to integrals of its solution. If a solution can be found corresponding to $\mathrm{Re}\,w>0$ then instability follows, but failure to find such solutions would be inconclusive since the steady state might still be unstable to perturbations that do not preserve a smooth envelope.

4. The charge singularity

We now examine the structure of the solutions of (2.8) in the neighbourhood of the caustic. Since the behaviour is different when the externally applied field vanishes, it is appropriate to study the scaled version of §3, so all the formulae from §2 will be used in this scaling.

Referring to figure 3, we define local co-ordinates by

$$\sigma=t-s-T(s),\quad \tau=s+T(s). \tag{4.1a, b}$$

Utilizing (2.13), we assume that in the neighbourhood of the caustic (i.e. for small σ), the delay terms in (2.8) can be approximated by

$$\Delta^{\pm}(t,s)=A_{\pm}(\tau)|\sigma|^{\alpha}+o(|\sigma|^{\alpha}). \tag{4.2}$$

Substituting (4.2) in (2.9) and (2.10), we find, after some manipulation, that near the caustic the particle trajectories have the form

$$x(\tau,\sigma)=\psi(\tau)+\sigma\psi'(\tau)-\tfrac{1}{2}\mu\sigma^2-\frac{A_{\pm}(\tau)}{(\alpha+1)\,(\alpha+2)}|\sigma|^{\alpha+2}+o(|s|^{\alpha+2}). \tag{4.3}$$

To determine the exponent α, we utilize the compatibility conditions (2.11) or (2.12). Referring again to figure 3, where the caustic co-ordinates are defined,

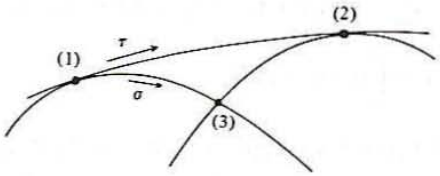

FIGURE 3. Caustic co-ordinates: (1) $\chi = \psi(t) = \psi(s_0+T(s_0))$, $\sigma_0 = 0$, $\tau_0 = s_0+T(s_0)$: (2) $x = \psi(s_1+T(s_1))$, $\sigma_1 = 0$, $\tau_1 = s_1+T(s_1)$; (3) $\sigma = t-s_0-T(s_0)$, $\tau = \tau_0 = s_0+T(s_0)$ are caustic co-ordinates of point (t, s) found from point (1), while $\sigma = t-s_1-T(s_1)$, $\tau = \tau_1 = s_1+T(s_1)$ are found from point (2). Here $s_1 = s_0+\Delta^+(t, s_0)$, or $s_0 = s_1+\Delta^-(t, s_1)$.

we see that the point (x, t) in physical space has two distinct descriptions in caustic co-ordinates:

$$(x, t) \sim (x(t, s), t) \sim (\sigma_0 = t-s-T(s), \quad \tau_0 = s+T(s)), \tag{4.4a}$$

$$(x, t) \sim (x(t, s+\Delta^{\pm}(t, s)), t) \sim (\sigma_1 = t-[s+\Delta^{\pm}(t, s)]-T(s+\Delta^{\pm}(t, s)),$$
$$\tau_1 = s+\Delta^{\pm}(t, s)+T(s+\Delta^{\pm}(t, s))). \tag{4.4b}$$

Nearness to the caustic is ensured by demanding that $\tau_1-\tau_0$, σ_0, σ_1 all be small in absolute value, while the sign $+(-)$ is used according to $\tau_1-\tau_0 > 0(<0)$, in which case $\sigma_0 > 0(<0)$ and $\sigma_1 < 0(>0)$.

We assume ϕ'' and T'' to be small compared with the quantities considered (where primes denote derivatives with respect to the argument), in accordance with our assumption that the caustic accelerates much slower than the particles crossing it. In a subsequent paper we shall discuss the effect on the singularity of highly curved and cusped caustics, and utilize our conclusions in formulating the bifurcation of the oscillating virtual cathode state on finite diodes.

Neglecting terms of order ψ'', T'' and higher, we have

$$\tau_1 \approx s+T(s)+(1+T')\,A_{\pm}(\tau_0)|\sigma_0|^2+\ldots, \tag{4.5a}$$

$$\sigma_1 \approx \sigma_0-(1+T')\,A_{\pm}(\tau_0)|\sigma_0|^2+\ldots. \tag{4.5b}$$

Equating the expressions (4.4a, b), where we substitute τ_1, σ_1 from (4.5a, b), we find that

$$\alpha = 1, \quad A_{\pm} = \frac{2}{1+T'},$$

so that

$$\Delta^{\pm}(t, s) = \frac{2}{1+T'}|\sigma|, \tag{4.6}$$

$$x(t, s) = \psi(\tau)+\psi_t(\tau)\,\sigma-\tfrac{1}{2}\mu\sigma^2-\frac{1}{3(1+T')}|\sigma|^3+o(|\sigma|^3). \tag{4.7}$$

Having the form of the trajectories near the caustic, we can estimate the charge singularity. On every continuous stream, the continuity equation

$$n_t+(nu)_x = 0 \tag{8}$$

leads to

$$n_t+un_x = -nu_x.$$

In characteristic co-ordinates. this is

$$\left(\frac{\partial n}{\partial t}\right)_s = -n\left(\frac{\partial u}{\partial x}\right)_t = -n\frac{(\partial u/\partial s)_t}{(\partial x/\partial s)_t}, \tag{4.9}$$

or, since

$$u = \left(\frac{\partial x}{\partial t}\right)_s, \tag{4.10}$$

we have

$$\frac{1}{n}\left(\frac{\partial n}{\partial t}\right)_s = -\frac{\partial^2 x/\partial t\,\partial s}{\partial x/\partial s},$$

so that

$$\frac{\partial}{\partial t}\ln\left|n\frac{\partial x}{\partial s}\right| = 0,$$

or

$$n\frac{\partial x}{\partial s} = \text{constant} \tag{4.11}$$

on each trajectory.

Matching the charge density in trajectory tubes at the caustic and at the entry point, we find that the constant in (4.11) is -1) for the ascending and $+1$ for the descending phase, so that

$$n = -\frac{1}{|x_s|}. \tag{4.12}$$

Computing the density of trajectories x_s from (7),

$$\begin{aligned} x_s &= (1+T')(x_\tau - x_\sigma) \\ &= -\tfrac{1}{2}\mu(1+T')\sigma - \tfrac{2}{3}\sigma^2 \operatorname{sgn}\sigma + o(\sigma^2), \end{aligned} \tag{4.13}$$

giving

$$n = -|\tfrac{1}{2}\mu(1+T')\sigma + \tfrac{2}{3}\sigma^2 \operatorname{sgn}\sigma + o(\sigma^2)|^{-1}. \tag{4.14}$$

Now,

$$\begin{aligned} x(t,s) - \psi(t) &\sim x(\tau,\sigma) - \psi(\tau) - \psi'(\tau)\sigma \\ &= -\tfrac{1}{2}\mu\sigma^2 - \frac{1}{3(1+T')}|\sigma|^3 + o(|\sigma|^3) \end{aligned} \tag{4.15}$$

relates σ to the distance from the caustic. We distinguish two possibilities for the behaviour of n near the caustic:

if $\mu \neq 0$, then

$$\begin{aligned} n &= -|\tfrac{1}{2}\mu(1+T')\sigma + o(\sigma)|^{-1} \\ &\approx -\frac{1}{1+T'}|\tfrac{1}{2}\mu(x-\psi)|^{-\frac{1}{2}}; \end{aligned} \tag{4.16}$$

if $\mu = 0$ then

$$\begin{aligned} n &= -\left|\frac{\lambda}{3(1+T')}\sigma^3 + o(|\sigma|^3)\right|^{-1} \\ &\approx -\left|\tfrac{1}{2}\mu(1+T')\left|\frac{3(1+T')}{\lambda}(x-\psi)\right|^{\frac{1}{3}}\right|^{-1} - \frac{2}{3^{\frac{1}{3}}(1+T')^{\frac{1}{3}}}|x-\psi|^{-\frac{1}{3}}. \end{aligned} \tag{4.17}$$

Since T' is assumed small, these estimates are comparable to the results for the steady state. We see that if there is an adverse field present ($\mu \neq 0$), the singularity is what we would have at a caustic in a system of non-interacting

particles (Arnold 1984). When $\mu = 0$ the self-interaction pulls particles apart at the caustic, weakening the singularity.

The above arguments are easily extended to other cases, and we report on the results elsewhere.

5. Delay effects

In the previous sections we have discussed the dynamics of the two-stream model and have seen that its linearized stability is governed by a fairly complicated system of integral equations that could be recast as a fourth-order ordinary differential equation. The mathematical analysis of this problem is further complicated by the fact that some of the coefficients in this equation actually depend on global properties of the solutions. It is also clear that in more general cases, where a larger number of separate streams are present, one can in principle apply similar ideas to produce equations of order equal to twice the number of streams. This seems to be a difficulty inherent in such problems whose resolution requires a combination of the geometrical ideas presented here and especially designed numerical schemes.

In this section we demonstrate some of the properties of the equations derived in §2 by deriving an exact solution subject to the following assumption: we suppose that the electrons are removed from the drift space as soon as they reach the caustic. Clearly, this is an unphysical situation (it requires a sink placed at the caustic), which cannot be realized experimentally. However, the discussion will offer some insight into the behaviour of the true, current-conserving solution, and especially into the importance of delay effects and interaction with the conducting walls in setting up the VC oscillations. This exact time-dependent solution can also provide a means of testing the performance of numerical codes, which owing to the charge singularity, might have problems in the time-dependent regime.

Following figure 4, we define $S_0(t)$ to be the entry time of the particle reaching the caustic at time t. Then, in the scaling of §2, the equation of motion of the sheath entering at time s is given by

$$\frac{\partial^2 x(t,s)}{\partial t^2} = -1 - \lambda[s - S_0(t)] \tag{5.1}$$

(since we ignore particles after their encounter with the caustic).

Integrating twice and imposing the condition that $\partial z/\partial t = 1$ at $t = s$ (i.e. at $x = 0$), we find

$$\frac{\partial x}{\partial t} = -(t-s) + 1 - \lambda s(t-s) + \lambda \int_s^t S_0(\xi)\, d\xi, \tag{5.2}$$

$$x(t,s) = -\tfrac{1}{2}(t-s)^2 + t - s - \tfrac{1}{2}\lambda s(t-s)^2 + \lambda \int_s^t (t-\xi)\, S_0(\xi)\, d\xi \tag{5.3}$$

The condition that the trajectory reaches the caustic at $t = s + T(s)$ implies

$$\frac{\partial x}{\partial s}(s+T(s),\, s) = 0 = [t - s - 1 - \tfrac{1}{2}\lambda(t-s)^2 - \lambda s(t-s) - (t-s)\, S_0(s)]_{t-s+T(s)},$$

or

$$\tfrac{1}{2}\lambda T^2(s) - [1 + \lambda T(S_0(s))]\, T(s) + 1 = 0. \tag{5.4}$$

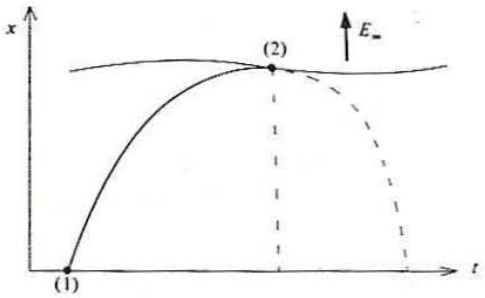

FIGURE 4. Basic configuration for 1-stream model: (t, x) co-ordinates of points shown are (1) $(s_0(t), 0)$; (2) $(t, \psi(t)) = (s_0(t)+T(s_0(t)), \psi(t))$.

Since, by definition, we have that

$$S_0(s)+T(S_0(s)) = s, \tag{5.5}$$

we can rewrite (5.4) using S_0 as our independent variable and solving for $T(s)$ in terms of $T(S_0(s))$ as

$$\lambda T(s) = \lambda T(S_0+T(S_0)) = 1+\lambda T(S_0)-\{[1+\lambda T(S_0)]^2-2\lambda\}^{\frac{1}{2}}, \tag{5.6}$$

where the $(-)$ sign was taken since it corresponds to the bounded solution, $T(s) \equiv 1$, of (5.4) as $\lambda \to 0$. We can view (5.6) as a double map by defining

$$S_n = S_{n-1}+T_{n-1}, \tag{5.7a}$$

$$\lambda T_n = 1+\lambda T_{n-1}-[(1+\lambda T_{n-1})^2-2\lambda]^{\frac{1}{2}}, \tag{5.7b}$$

where we set

$$T_n = T(S_n). \tag{5.7c}$$

Since (5.7*b*) is independent of S_n, we see that it can be used to define a sequence of iterates, which in turn can serve to give the times at which they occur through (5.7*a*). Let

$$g_n = 1+\lambda T_n. \tag{5.8}$$

We must now study the map

$$g_{n+1} = f(g_n;\lambda) = 1+g_n-(g_n^2-2\lambda)^{\frac{1}{2}} \tag{5.9}$$

The steady solution of (5.1) is found in terms of the fixed point, g_*, *of* (5.9). *We have*

$$g_* = f(g_*;\lambda) = 1+g_*-(g_*^2-2\lambda)^{\frac{1}{2}},$$

from which

$$g_* = (2\lambda+1)^{\frac{1}{2}}. \tag{5.10}$$

This implies

$$T = \frac{1}{\lambda}(2\lambda+1)^{\frac{1}{2}}-1]. \tag{5.11}$$

Substituting $S_0'(t) = t-T$ into (5.3), we get the expression for the steady trajectories in characteristic co-ordinates. Clearly, the map is defined for $g > (2\lambda)^{\frac{1}{2}}$. If for some value of g_n we get $g_{n+1} = f(g_n) < (2\lambda)^{\frac{1}{2}}$ then this signifies a breakdown of our model. Also, since

$$xf'(g) = 1-\frac{g}{(g^2-2\lambda)^{\frac{1}{2}}} < 0 \tag{5.12}$$

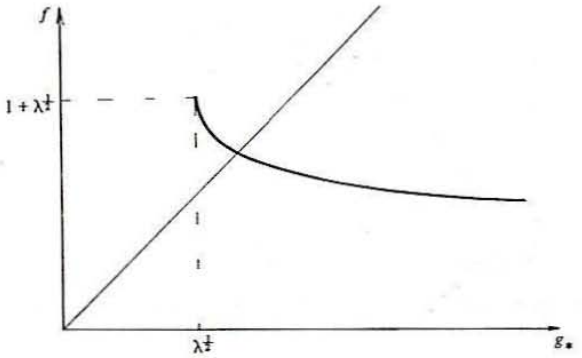

FIGURE 5. The function $f(g_*;\lambda)$ for $\lambda = 1$.

for $g > (2\lambda)^{\frac{1}{2}}$ (with $f'(g) \to -\infty$ as $g \to (2\lambda)^{\frac{1}{2}}$), we see that, for all g, $f(g) < 1+(2\lambda)^{\frac{1}{2}} = f((2\lambda)^{\frac{1}{2}})$. The fixed point $g_A = (1+2\lambda)^{\frac{1}{2}}$ lies between these two values. Stability of this steady state (which it is not necessary to give here explicitly) depends on the stability of the fixed point g_* of the map (5.10). Following the standard treatment (Thompson & Stewart 1986), we see that g_* is stable if

$$\left|\frac{\partial f}{\partial g}(g^*;\lambda)\right| < 1 \tag{5.13}$$

and unstable if that expression becomes greater than 1. Since

$$\left|\frac{\partial f}{\partial g}\right|_{g_*} = \left|1-\frac{g}{(g^2-2\lambda)^{\frac{1}{2}}}\right|_{g_*} = |1-(2\lambda+1)^{\frac{1}{2}}|, \tag{5.14}$$

we find that $$\left|\frac{\partial f}{\partial g}(g^*;\lambda)\right| \lessgtr 1,$$

according to whether $$\lambda \lessgtr \tfrac{3}{2}. \tag{5.15}$$

Therefore $\lambda = \frac{3}{2}$ is the critical ratio of external versus self-consistent field for this model. If $\lambda < \frac{3}{2}$ (external field stronger) then the steady state is stable, while for $\lambda > \frac{3}{2}$ (external field weaker) the steady state is destabilized.

We now show how the solution (5.3) of our model can be constructed from the map (5.9), from $\lambda \leqslant \frac{3}{2}$. The idea is to start with an arbitrary value of T_0 at $S_0 = 0$; then $g_0 = 1+\lambda T_0$ and $T_1 = [f(g_0)-1]/\lambda$, $S_1 = T_1$. Referring to figure 5, we can arbitrarily (but in a smooth fashion) assign values for T in the interval $S_0 = 0 \leqslant S \leqslant S_1$.

Further care ensures that $T(s)$ stays smooth after the first iteration of the map, which will ensure that it stays smooth after further iterations. It is not hard to show that (5.9) maps the interval

$$(2\lambda)^{\frac{1}{2}} < 2-(1-\tfrac{2}{3}\lambda)^{\frac{1}{2}} \leqslant g \leqslant 2+(1-\tfrac{2}{3}\lambda)^{\frac{1}{2}} < 1+(2\lambda)^{\frac{1}{2}} \tag{5.16}$$

into itself, so that choosing initial conditions in this interval guarantees that iterates stay in the domain of definition of the map.

To get a smooth function $T(s)$ for all s, we take $T_0 = [(1+2\lambda)^{\frac{1}{2}}-1]/\lambda$, so that $T_1 = T_0$, and we make sure that $T(s)$, for s near zero, is chosen so that it is

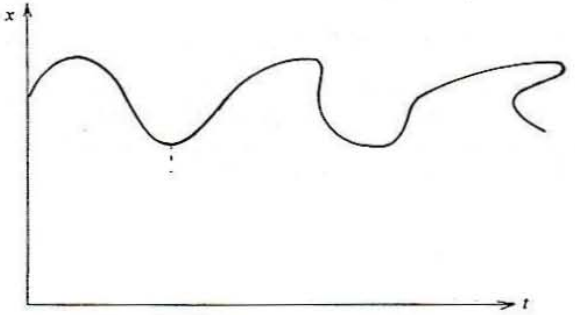

FIGURE 6. Evolution of the envelope for $\lambda = 1$ with a profile that lies between the values corresponding to $T_\pm = (1 \pm \sqrt{\frac{1}{3}})$. The distance between successive maxima or minima is 2, while the distance between points of inflection is $T = T_0 = \sqrt{3}-1$. Since $2T_0 < 2$, the profile steepens and breaks.

smooth around $s = T_0$. Then an initial profile for $T(s)$, $0 \leqslant s \leqslant T_0$ defines, by iteration, a profile $T(s)$ for all time. Since the fixed point is stable, the profile eventually decays to the steady-state value T_0. In any case, knowledge of this history of the profile allows us to fully determine the particle trajectories in characteristic co-ordinates, given by (5.3). A different situation occurs if T assumes either of the values $[1 \pm (1-\frac{2}{3}\lambda)^{\frac{1}{2}}]/\lambda$ in the initial interval. Since the map alternates between these values (they are unstable fixed points of order 2), it follows that in this case the profile does not decay to the value T_0. Instead, since the point with value $T(s) = T_0$ repeats with period $T_0 = [(1+2\lambda)^{\frac{1}{2}}-1]/\lambda$, while the points $T_\pm = [1 \pm (1-\frac{2}{3}\lambda)^{\frac{1}{2}}]/\lambda$ repeat with combined period $2/\lambda > 2T_0$, the profile will steepen and eventually break. This situation is reminiscent of breaking waves, and it leads to multi-valued $T(s)$ (figure 6). At this point the assumptions inherent in this model are no longer valid.

Thus, in effect, the interval (5.16) gives the domain of attraction of the steady state. At $\lambda = \frac{3}{2}$ this interval shrinks to the fixed point, which thus loses stability. Small perturbations about $T(s) = T_0$ will eventually grow, until either they fall outside the domain of definition of the map, or the profile breaks. Again, our assumptions are no longer valid at this point, and we cannot expect that the destabilization of the steady state leads to a steady oscillation with a smooth caustic.

The author wishes to acknowledge conversation with D. J. Sullivan and N. Roderick. The work was supported in part by AFOSR under Grant AFOSR-82-0277.

REFERENCES

ARNOLD, V. I. 1984 *Catastrophe Theory*. Springer.

BIRDSALL, C. K. & BRIDGES, W. B. 1966 *Electron Dynamics of Diode Regions*. Academic Press.

COUTSIAS, E. A. & SULLIVAN, D. J. 1983 *Phys. Rev.* A **27**, 1535.

DAVIDSON, R. C. 1974 *Theory of Nonneutral Plasmas*. Benjamin.

DUNN, D. A. & HO, I. T. 1963 *AIAA J.* **1**, 2770.

KUHN, S. 1984 *Phys. Fluids*, **27**, 1821.

MILLER, R. B. 1982 *An Introduction to the Physics of Intense Charged Particle Beams.* Plenum.

SULLIVAN, D. J., VOSS, D. E., ADLER, R. J., BOLLEN, W. M., JACKSON, R. H. & COUTSIAS, E. A. 1984 *Mission Research Corporation Report* AMRC-R-614.

THOMPSON, J. M. T. & STEWART, H. B. 1986 *Nonlinear Dynamics and Chaos.* Wiley.

WHITHAM, G. B. 1976 *Linear and Nonlinear Waves.* Wiley.