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# Intensity noise of injection-locked lasers: Quantum theory using a linearized input-output method

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We derive analytical expressions for the quantum-noise spectra of an atomic laser using a linearized input-output method. We generalize the method to the problem of injection-locked lasers. We identify three distinct spectral noise regimes in the solution, unifying previous results. The approach offers insights into the physical processes and is especially suited for comparison with experiment. The quantum spectral noise properties of the pump laser and the injected laser appear explicitly in the solution. [S1050-2947(96)03710-9]

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## I. INTRODUCTION

The search for ever quieter light sources for applications in communications and high precision interferometry has fueled experimental interest in the noise properties of injection-locked lasers [1,2]. In Ref. [1] it was shown that the injection of a small signal into a pump noise suppressed semiconductor laser could suppress small longitudinal side modes while leaving the nonclassical noise characteristics of the free-running laser unaffected. In contrast Ref. [2] found that in Nd:YAG (YAG denotes yttrium aluminum garnet) lasers the free-running noise characteristics were suppressed and the output noise was an amplified version of the injected signal.

Although classical noise theories of injection locking have been around for some time [3], quantum-mechanical theories are more recent [4–6]. A quantum theory is needed when noise powers approach or go below the standard quantum-noise limit (QNL). In order to understand, motivate, and model experimental research into injection locking a fully quantum-mechanical theory which incorporates the spectral noise properties and detunings of the various input fields and unifies the apparently dissimilar results of Refs. [1] and [2] is required. In this paper we present such a theory. In a following paper [7] we obtain experimental results from miniature Nd:YAG lasers which are in good quantitative agreement with the theory.

Consider the schematic representation shown in Fig. 1. We wish to be able to model both the slave and the master laser explicitly. Alternatively we may wish to inject an empirically determined intensity spectrum as the master. We wish to be able to nominate spectra for the pump lasers, allowing for pump noise or pump noise suppression. We also wish to allow for detuning between the slave laser cavity mode and the injected master. A rigorous solution to this model can be obtained by using standard techniques [8] to write down a master equation for the system. Such a master equation would incorporate the quantum statistics of the pumps through quantized pump modes [9,10] and the coupling between master and slave laser via the cascaded system formalism [11]. Then, using (for example) the positive $P$ representation [12], amplitude or phase spectra for the linearized fluctuations can be calculated. Unfortunately the complexity of such an approach leads only to numerical results.

An alternative approach is to derive operator equations of motion for the lasers in which the various quantum-noise sources appear as zero-point input fields. These equations will be nonlinear. However, they can be solved by linearization around their stable steady-state values [13]. The amplitude or phase fluctuation spectrum of the output field can then be calculated in terms of the input fields using the input-output formalism [9]. This approach is similar to that employed for semiconductor lasers [14]. The advantages of this approach are its conceptual simplicity and the production of analytical solutions in terms of the various noise sources. The simplicity of the approach means that many details omitted from previous treatments, such as the full spectral noise properties of the injected field and the pump sources, can be included. In fact, once the linearized operator equations have been written down the solution is basically just algebraic, so quite complex systems can be solved. The separation of the solution in terms of the various noise sources allows for a greater physical understanding of the contributions of the different processes to the final spectrum. We have compared the results obtained using this method to numerical results obtained using the positive $P$ representation (as outlined in the preceding paragraph) and find them to be in excellent agreement.

In Sec. II we derive the quantum-mechanical Langevin operator equations of motion for a free-running laser. The equations are written in a form suited to linearization around the semiclassical solutions. In Sec. III the linearized equations for the quantum fluctuations are obtained and the amplitude and phase noise spectra are solved. The behavior of

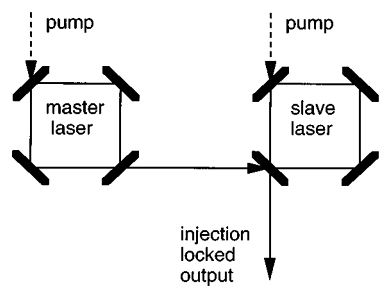


FIG. 1. Schematic representation of the injection-locked laser model.

 

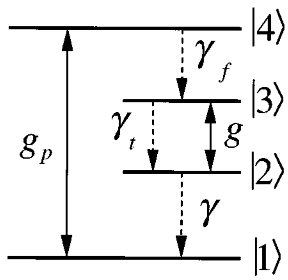


FIG. 2. Energy level scheme of the active atoms.

the solutions is discussed. In particular, the physical mechanism for frequency-dependent squeezing in open (undepleted pump level) lasers is explained. In Sec. IV the method is generalized to the case of injection-locked lasers. The amplitude noise spectrum is obtained and the solutions discussed. In particular, we find that there are three distinct frequency regimes with quite different noise properties. In Sec. V we summarize our results.

## II. OPERATOR EQUATIONS FOR THE FREE-RUNNING LASER

We consider $N$ four-level atoms (see Fig. 2) interacting with two optical ring cavity modes. The first mode, represented by the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively, is the lasing mode. It interacts with the active atoms via the resonant Jaynes-Cummings Hamiltonian

$$\hat{H}_{\mathrm{rev1}}=ihg(\hat{a}^\dagger\hat{\sigma}_{23}-\hat{a}\hat{\sigma}_{23}^{+}), \tag{1}$$

where carets indicate operators, $g$ is the dipole coupling strength between the atoms and the cavity, and $\hat{\sigma}_{ij}$ and $\hat{\sigma}_{ij}^{+}$ are the collective Hermitian conjugate atomic lowering and raising operators between the $i$th and $j$th levels. The field phase factors have been absorbed into the definition of the atomic operators. The second mode, represented by the annihilation and creation operators $\hat{b}$ and $\hat{b}^\dagger$, respectively, is the pump mode. It interacts with the active atoms via the resonant Jaynes-Cummings Hamiltonian

$$\hat{H}_{\mathrm{rev2}}=ihg_p(\hat{b}^\dagger\hat{\sigma}_{14}-\hat{b}\hat{\sigma}_{14}^{+}), \tag{2}$$

where $g_p$ is the dipole coupling strength between the atoms and the cavity.

We couple the atoms and cavities to reservoirs to describe the irreversible transitions of the system. Under the Markov approximation that the coupling constants between the system and the reservoirs are frequency independent the Hamiltonian for the irreversible processes is

$$\begin{aligned}\hat{H}_{\mathrm{irrev}}=ih\int_{-\infty}^{\infty}d\omega\Bigg\{&\left(\frac{2\kappa_m}{2\pi}\right)^{1/2}[\hat{A}(\omega)\hat{a}^\dagger-\hat{A}^\dagger(\omega)\hat{a}]\\&+\left(\frac{2\kappa_l}{2\pi}\right)^{1/2}[\hat{A}_l(\omega)\hat{a}^\dagger-\hat{A}_l^\dagger(\omega)\hat{a}]+\left(\frac{2\kappa_b}{2\pi}\right)^{1/2}\\&\times[\hat{B}(\omega)\hat{b}^\dagger-\hat{B}^\dagger(\omega)\hat{b}]+\left(\frac{\gamma_f}{2\pi}\right)^{1/2}\\&\times[\hat{C}_f(\omega)\hat{\sigma}_{34}^{+}-\hat{C}_f^\dagger(\omega)\hat{\sigma}_{34}]\\&+\left(\frac{\gamma_t}{2\pi}\right)^{1/2}[\hat{C}_t(\omega)\hat{\sigma}_{23}^{+}-\hat{C}_t^\dagger(\omega)\hat{\sigma}_{23}]\\&+\left(\frac{\gamma}{2\pi}\right)^{1/2}[\hat{C}(\omega)\hat{\sigma}_{12}^{+}-\hat{C}^\dagger(\omega)\hat{\sigma}_{12}]+\left(\frac{\gamma_P}{4\pi}\right)^{1/2}\\&\times[\hat{C}_P(\omega)-\hat{C}_P^\dagger(\omega)](\hat{\sigma}_3-\hat{\sigma}_2)\Bigg\}.\end{aligned} \tag{3}$$

Reservoir operators are indicated by capitals and $\hat{\sigma}_i$ is the collective population operator for the $i$th level. Included in our laser model are atomic spontaneous emission from level $|4\rangle$ to level $|3\rangle$, from level $|3\rangle$ to level $|2\rangle$, and from level $|2\rangle$ to level $|1\rangle$, at rates $\gamma_f$, $\gamma_t$, and $\gamma$, respectively. The rate of collisional or lattice induced phase decay of the lasing coherence is $\gamma_P$. The laser cavity damping rate due to the output mirror is $2\kappa_m$. The laser cavity damping rate due to other losses is $2\kappa_l$. The total laser cavity damping rate is $2\kappa=2\kappa_m+2\kappa_l$. The pump mode damping and pumping occurs through an input-output mirror with a rate $2\kappa_b$. In the following the pump mode, upper-pump level, and pump coherence will all be adiabatically eliminated. Hence the details of the pumping process dynamics are unimportant and are kept as simple as possible.

Following standard procedure [8,9] we define the input fields,

$$\begin{aligned}\delta\hat{A}(t)&=\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{A}_0(\omega),\\\delta\hat{A}_l(t)&=\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{A}_{l0}(\omega),\\\hat{B}(t)&=\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{B}_0(\omega),\\\delta\hat{C}_f(t)&=\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{C}_{f0}(\omega),\\\delta\hat{C}_t(t)&=\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{C}_{t0}(\omega),\\\delta\hat{C}(t)&=\frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{C}_0(\omega),\end{aligned} \tag{4}$$

$$\delta\hat{C}_P(t) = \frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega(t-t_0)}\hat{C}_{P0}(\omega),$$

where $\hat{A}_0,\ldots,\hat{C}_{P0}$ are the values of $\hat{A},\ldots,\hat{C}_P$ at some arbitrary initial time $t_0$. Zero-point fields are indicated by $\delta$'s. It can be shown that the dynamics of any system operator ($\hat{x}$) is given by the quantum Langevin equation,

$$\dot{\hat{x}} = -\sum_{i=1,2}\frac{1}{h}[\hat{x},H_{\mathrm{rev}\,i}] - \sum_{j=1,\ldots,7}\left\{[\hat{x},\hat{a}_j^\dagger]\left(\frac{\gamma_j}{2}\hat{a}_j + \sqrt{\gamma_j}\hat{A}_j\right) - \left(\frac{\gamma_j}{2}\hat{a}_j^\dagger + \sqrt{\gamma_j}\hat{A}_j^\dagger\right)[\hat{x},\hat{a}_j]\right\}, \tag{5}$$

where we have used the notation, for the system operators,

$$(\hat{a}_1,\hat{a}_2,\hat{a}_3,\hat{a}_4,\hat{a}_5,\hat{a}_6,\hat{a}_7) \equiv (\hat{a},\hat{a},\hat{b},\hat{\sigma}_{34},\hat{\sigma}_{23},\hat{\sigma}_{12},\hat{\sigma}_3-\hat{\sigma}_2),$$

for the input fields,

$$(\hat{A}_1,\hat{A}_2,\hat{A}_3,\hat{A}_4,\hat{A}_5,\hat{A}_6,\hat{A}_7) \equiv (\delta\hat{A},\delta\hat{A}_l,\hat{B},\delta\hat{C}_f,\delta\hat{C}_t,\delta\hat{C},\delta\hat{C}_P),$$

and for the coupling constants,

$$(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7) \equiv (2\kappa_m,2\kappa_l,2\kappa_b,\gamma_f,\gamma_t,\gamma,\gamma_{P/2}).$$

Using Eq. (5) we can write down the following operator equations of motion for the laser:

$$\dot{\hat{a}} = g\hat{\sigma}_{23} - \kappa\hat{a} + \sqrt{2\kappa_m}\delta\hat{A}_m + \sqrt{2\kappa_l}\delta\hat{A}_l,$$

$$\dot{\hat{b}} = g_p\hat{\sigma}_{14} - \kappa_b\hat{b} + \sqrt{2\kappa_b}\hat{B},$$

$$\dot{\hat{\sigma}}_{14} = g_p(\hat{\sigma}_4-\hat{\sigma}_1)\hat{b} - \frac{\gamma_f}{2}\hat{\sigma}_{14} - \sqrt{\gamma_f}(\hat{\sigma}_4-\hat{\sigma}_1)\delta\hat{C}_f,$$

$$\dot{\hat{\sigma}}_{23} = g(\hat{\sigma}_3-\hat{\sigma}_2)\hat{a} - \frac{1}{2}(2\gamma_P+\gamma_t+\gamma)\hat{\sigma}_{23} - \sqrt{2\gamma_P}\hat{\sigma}_{23}(\delta\hat{C}_P+\delta\hat{C}_P^\dagger) - \sqrt{\gamma_t}(\hat{\sigma}_3-\hat{\sigma}_2)\delta\hat{C}_t - \sqrt{\gamma}(\hat{\sigma}_2-\hat{\sigma}_1)\delta\hat{C},$$

$$\dot{\hat{\sigma}}_1 = g_p(\hat{\sigma}_{14}\hat{b}^\dagger + \hat{\sigma}_{14}^+\hat{b}) + \gamma\hat{\sigma}_2 - \sqrt{\gamma}(\hat{\sigma}_{12}\delta\hat{C}^\dagger + \hat{\sigma}_{12}^+\delta\hat{C}),$$

$$\dot{\hat{\sigma}}_2 = g(\hat{\sigma}_{23}\hat{a}^\dagger + \hat{\sigma}_{23}^+\hat{a}) + \gamma_t\hat{\sigma}_3 - \gamma\hat{\sigma}_2 + \sqrt{\gamma}(\hat{\sigma}_{12}\delta\hat{C}^\dagger + \hat{\sigma}_{12}^+\delta\hat{C}) - \sqrt{\gamma_t}(\hat{\sigma}_{23}\delta\hat{C}_t^\dagger + \hat{\sigma}_{23}^+\delta\hat{C}_t),$$

$$\dot{\hat{\sigma}}_3 = -g(\hat{\sigma}_{23}\hat{a}^\dagger + \hat{\sigma}_{23}^+\hat{a}) - \gamma_t\hat{\sigma}_3 + \gamma_f\hat{\sigma}_4 - \sqrt{\gamma_f}(\hat{\sigma}_{34}\delta\hat{C}_f^\dagger + \hat{\sigma}_{34}^+\delta\hat{C}_f) + \sqrt{\gamma_t}(\hat{\sigma}_{23}\delta\hat{C}_t^\dagger + \hat{\sigma}_{23}^+\delta\hat{C}_t),$$

$$\dot{\hat{\sigma}}_4 = -g_p(\hat{\sigma}_{14}\hat{b}^\dagger + \hat{\sigma}_{14}^+\hat{b}) - \gamma_f\hat{\sigma}_4 + \sqrt{\gamma_f}(\hat{\sigma}_{34}\delta\hat{C}_f^\dagger + \hat{\sigma}_{34}^+\delta\hat{C}_f). \tag{6}$$

As they stand Eqs. (6) are intractable. However, by making some approximations and using the operator relationships to rewrite certain terms we will obtain equations which may be solved using the standard linear approximation.

First we make the following assumptions.

(i) The pump cavity decays very rapidly ($\kappa_b$ very large). This is consistent with the normal experimental situation in which the pump mode is not resonant with the cavity.

(ii) The upper-pump level decays very rapidly ($\gamma_f$ very large). This is a desirable condition for efficient pumping and is normally satisfied.

(iii) Very little of the pump is absorbed by the active atoms.

(iv) The phase decay of the laser coherence is very rapid ($\gamma_P$ very large). This is a good assumption for most atomic lasers.

Using these assumptions we can adiabatically eliminate the equations for $\dot{\hat{b}}$, $\dot{\hat{\sigma}}_{14}$, $\dot{\hat{\sigma}}_{23}$, and $\dot{\hat{\sigma}}_4$.

Secondly we use the operator relationships $\hat{\sigma}_{ij}\hat{\sigma}_{ij}^+ = \hat{\sigma}_i$, $\hat{\sigma}_{ij}^+\hat{\sigma}_{ij} = \hat{\sigma}_j$, and $\hat{\sigma}_{ij}\hat{\sigma}_{ij} = \hat{\sigma}_{ij}^+\hat{\sigma}_{ij}^+ = 0$ to rewrite coherence operators (that appear in the noise terms) in terms of population operators. For example,

$$(\hat{\sigma}_{23}+\hat{\sigma}_{23}^+)^2 = (\hat{\sigma}_3+\hat{\sigma}_2) \rightarrow (\hat{\sigma}_{23}+\hat{\sigma}_{23}^+) = \sqrt{\hat{\sigma}_3+\hat{\sigma}_2}.$$

This second step must be made to ensure that the linearized equations feature the standard semiclassical invariance under scaling by atomic number (see Sec. II). We obtain the following simpler equations of motion:

$$\begin{aligned}
\dot{\hat{n}} &= \tilde{G}(\hat{\sigma}_3-\hat{\sigma}_2)\hat{n} - 2\kappa\hat{n} + \sqrt{2\kappa_m}(\hat{a}^\dagger\delta\hat{A}_m + \hat{a}\,\delta\hat{A}_m^\dagger) + \sqrt{2\kappa_l}(\hat{a}^\dagger\delta\hat{A}_l + \hat{a}\,\delta\hat{A}_l^\dagger) - \sqrt{\tilde{G}\hat{n}(\hat{\sigma}_3+\hat{\sigma}_2)}(\delta\hat{C}_P + \delta\hat{C}_P^\dagger),\\
\dot{\hat{\sigma}}_1 &= -\Theta\hat{\sigma}_1\hat{B}^\dagger\hat{B} + \gamma\hat{\sigma}_2 - \sqrt{\gamma(\hat{\sigma}_2\delta\hat{C}\,\delta\hat{C}^\dagger + \hat{\sigma}_1\delta\hat{C}^\dagger\delta\hat{C})} - \sqrt{\Theta\hat{\sigma}_1(1-\Theta\hat{\sigma}_1)}\hat{B}(\delta\hat{C}_f^\dagger + \delta\hat{C}_f),\\
\dot{\hat{\sigma}}_2 &= \tilde{G}(\hat{\sigma}_3-\hat{\sigma}_2)\hat{n} + \gamma_t\hat{\sigma}_3 - \gamma\hat{\sigma}_2 + \sqrt{\gamma(\hat{\sigma}_2\delta\hat{C}\,\delta\hat{C}^\dagger + \hat{\sigma}_1\delta\hat{C}^\dagger\delta\hat{C})} - \sqrt{\gamma_t(\hat{\sigma}_3\delta\hat{C}_t\delta\hat{C}_t^\dagger + \hat{\sigma}_2\delta\hat{C}_t^\dagger\delta\hat{C}_t)} - \sqrt{\tilde{G}\hat{n}(\hat{\sigma}_3+\hat{\sigma}_2)}\times(\delta\hat{C}_P + \delta\hat{C}_P^\dagger),\\
\dot{\hat{\sigma}}_3 &= -\tilde{G}(\hat{\sigma}_3-\hat{\sigma}_2)\hat{n} - \gamma_t\hat{\sigma}_3 + \Theta\hat{\sigma}_1\hat{B}^\dagger\hat{B} + \sqrt{\Theta\hat{\sigma}_1(1-\Theta\hat{\sigma}_1)}\hat{B}(\delta\hat{C}_f^\dagger + \delta\hat{C}_f) - \sqrt{\gamma_t(\hat{\sigma}_3\delta\hat{C}_t\delta\hat{C}_t^\dagger + \hat{\sigma}_2\delta\hat{C}_t^\dagger\delta\hat{C}_t)} + \sqrt{\tilde{G}\hat{n}(\hat{\sigma}_3+\hat{\sigma}_2)}\times(\delta\hat{C}_P + \delta\hat{C}_P^\dagger),
\end{aligned} \tag{7}$$

where

$$\tilde{G} = \frac{2g^2}{\gamma_p},$$

$$\Theta = \frac{8g_p^2}{\gamma_f\kappa_b}, \tag{8}$$

and the photon number operator is given by $\hat{n} = \hat{a}^\dagger\hat{a}$.

The semiclassical equations of motion are obtained by ignoring the zero-point fields (indicated by $\delta$'s). In the next section we will linearize Eqs. (7) about the semiclassical steady state and hence solve for the noise spectra.

## III. SPECTRUM OF THE FREE-RUNNING LASER

### A. Solving for the free-running spectrum

To solve for the full quantum mechanics we proceed by assuming we can write the solutions in the scaled linear form

$$\frac{\hat{a}(t)}{\sqrt{N}}=\alpha+\delta\hat{a}(t),\quad \frac{\hat{\sigma}_i(t)}{N}=J_i+\delta\hat{\sigma}_i(t),\quad \frac{\hat{B}(t)}{\sqrt{N}}=B+\delta\hat{B}(t), \tag{9}$$

where $\alpha$ and $J_i$ are the stable semiclassical steady-state solutions to Eq. (7) for the amplitude per root atom and population of level $i$ per atom, respectively. $B$ is the coherent amplitude per root atom of the pump mode. Without loss of generality we can take $\alpha$ to be real. The quantum fluctuations, $\delta\hat{a}$, $\delta\hat{B}$, and $\delta\hat{\sigma}_i$, are considered small perturbations to the steady state. By substituting Eq. (9) into Eq. (7) and retaining only linear terms in the fluctuations we obtain the following linearized equations of motion for the scaled quantum fluctuations:

$$\begin{aligned}
\delta\dot{\hat{X}}_a&=G(\delta\hat{\sigma}_3-\delta\hat{\sigma}_2)\alpha+\sqrt{2\kappa_m}\delta\hat{X}_{Am}+\sqrt{2\kappa_l}\delta\hat{X}_{Al}\\
&\quad-\sqrt{G(J_3+J_2)}\delta\hat{X}_{cp},\\
\delta\dot{\hat{\sigma}}_1&=-\Gamma\delta\hat{\sigma}_1+\gamma\delta\hat{\sigma}_2-\sqrt{\Gamma J_1}\delta\hat{X}_B\\
&\quad-\sqrt{\gamma(J_2\delta\tilde{\hat{C}}\delta\tilde{\hat{C}}^\dagger+J_1\delta\tilde{\hat{C}}^\dagger\delta\tilde{\hat{C}})},\\
\delta\dot{\hat{\sigma}}_2&=G(\delta\hat{\sigma}_3-\delta\hat{\sigma}_2)\alpha^2+G(J_3-J_2)\alpha\delta\hat{X}_a+\gamma_t\delta\hat{\sigma}_3-\gamma\delta\hat{\sigma}_2\\
&\quad+\sqrt{\gamma(J_2\delta\tilde{\hat{C}}\delta\tilde{\hat{C}}^\dagger+J_1\delta\tilde{\hat{C}}^\dagger\delta\tilde{\hat{C}})}\\
&\quad-\sqrt{\gamma_t(J_3\delta\tilde{\hat{C}}_t\delta\tilde{\hat{C}}_t^\dagger+J_2\delta\tilde{\hat{C}}_t^\dagger\delta\tilde{\hat{C}}_t)}\\
&\quad-\sqrt{G(J_3+J_2)}\alpha\delta\hat{X}_{Cp},\\
\delta\dot{\hat{\sigma}}_3&=-G(\delta\hat{\sigma}_3-\delta\hat{\sigma}_2)\alpha^2-G(J_3-J_2)\alpha\delta\hat{X}_a-\gamma_t\delta\hat{\sigma}_3\\
&\quad+\Gamma\delta\hat{\sigma}_1+\sqrt{\Gamma J_1}\delta\hat{X}_B+\sqrt{\gamma_t(J_3\delta\tilde{\hat{C}}_t\delta\tilde{\hat{C}}_t^\dagger+J_2\delta\tilde{\hat{C}}_t^\dagger\delta\tilde{\hat{C}}_t)}\\
&\quad+\sqrt{G(J_3+J_2)}\alpha\delta\hat{X}_{Cp},
\end{aligned} \tag{10}$$

where the quadrature amplitude fluctuations of the fields are defined by

$$\delta\hat{X}_a=\delta\hat{a}+\delta\hat{a}^\dagger,\quad \delta\hat{X}_{Am}=\delta\tilde{\hat{A}}_m+\delta\tilde{\hat{A}}_m^\dagger,\ldots,$$

$$\delta\hat{X}_B=\sqrt{\Gamma J_1\eta}(\delta\hat{B}+\delta\hat{B}^\dagger)+\sqrt{\Gamma J_1(1-\eta)}(\delta\hat{C}_f+\delta\hat{C}_f^\dagger).$$

The tildes on the input field operators indicate they have been scaled by the root of the atomic number. The rate $G$ is proportional to the stimulated emission cross section $\sigma_s$ via

$$G=\frac{2g^2N}{\gamma_p}=\sigma_s\rho c,$$

where $\rho$ is the atomic density and $c$ is the speed of light in the medium. We have defined the incoherent pump rate via $\Gamma=\Theta|B|^2N$ and $\delta\hat{X}_B$ is the quadrature amplitude fluctuation of the absorbed pump field where the fraction absorbed is $\eta\ll1$. (Note the requirement that $\eta$ be small can be relaxed provided there is little depletion of the atomic ground state.) An important property of the semiclassical equations is their invariance under scaling by atomic number (provided the atomic density is constant). Notice (from the definitions of the operators, the semiclassical values, and of $G$) that Eqs. (10) still display this invariance. This indicates that the noise operators have been correctly rewritten (see Sec. II) in a form compatible with the linearization assumption [Eq. (9)].

The boundary condition at the output mirror is [9]

$$\hat{A}_{\rm out}=\sqrt{2\kappa_m}\hat{a}-\delta\hat{A}_m, \tag{11}$$

where $\hat{A}_{\rm out}$ is the laser output field. In terms of the amplitude quadrature fluctuations

$$\delta\hat{X}_{\rm out}=\sqrt{2\kappa_m}\delta\hat{X}_a-\delta\hat{X}_{Am}. \tag{12}$$

In frequency space Eq. (12) can be used to eliminate the internal fields from Eq. (10) and hence obtain the following expression for the output amplitude quadrature fluctuations in terms of the input field fluctuations:

$$\begin{aligned}
\delta X_{\rm out}&=\{[2\kappa_m-i\omega+F_1(\omega)]\delta X_{Am}+\sqrt{2\kappa_m}\sqrt{\Gamma J_1}F_2(\omega)\delta X_B\\
&\quad+\sqrt{2\kappa_m}F_3(\omega)\sqrt{\gamma(J_2\delta\tilde{C}\delta\tilde{C}^\dagger+J_1\delta\tilde{C}^\dagger\delta\tilde{C})}\\
&\quad+\sqrt{2\kappa_m}F_4(\omega)\sqrt{\gamma_t(J_3\delta\tilde{C}_t\delta\tilde{C}_t^\dagger+J_2\delta\tilde{C}_t^\dagger\delta\tilde{C}_t)}\\
&\quad+\sqrt{2\kappa_m}\sqrt{G(J_3+J_2)}[1-F_4(\omega)]\delta X_{Cp}\\
&\quad+2\sqrt{\kappa_m\kappa_l}\delta X_{Al}\}/\{i\omega-F_1(\omega)\},
\end{aligned} \tag{13}$$

where the absence of the circumflex indicates Fourier transforms and the functions $F_i(\omega)$ are defined in the Appendix. Hence the amplitude noise spectrum of the output field, $V_{\rm out}=\langle\delta X_{\rm out},\delta X_{\rm out}\rangle N$, is given by

$$\begin{aligned}
V_{\rm out}(\omega)&=\{[2\kappa_m-i\omega+F_1(\omega)]^2+2\kappa_m\Gamma J_1[F_2(\omega)]^2V_p(\omega)\\
&\quad+2\kappa_m\gamma J_2[F_3(\omega)]^2+2\kappa_m\gamma_tJ_3[F_4(\omega)]^2\\
&\quad+2\kappa_mG(J_3+J_2)\\
&\quad\times[1-F_4(\omega)]^2+4\kappa_m\kappa_l\}/[i\omega-F_1(\omega)]^2,
\end{aligned} \tag{14}$$

where square brackets indicate absolute squares, $V_p$ is the amplitude noise spectrum of the absorbed pump, and we have used the fact that the fields associated with spontaneous emission and phase decay noise are vacuum fields. The physical origin of each of the noise terms in Eq. (14) is clear by comparison with Eq. (13).

We may also obtain the phase noise spectrum of the free-running laser from Eq. (10) by solving for the phase quadrature fluctuations; $\delta\hat{X}_a^-=\delta\hat{a}-\delta\hat{a}^\dagger$. The phase noise spectrum is then given by

$$V_{\rm out}^-(\omega)=\{[2\kappa_m-i\omega]^2+2\kappa_mGJ_3+2\kappa_mGJ_2+4\kappa_m\kappa_l\}/\omega^2. \tag{15}$$

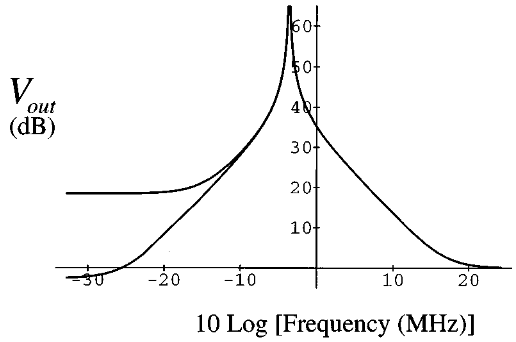


FIG. 3. Intensity-noise spectrum of the free-running laser [Eq. (16)] with classical pump noise; $V_p=100$ (upper trace) and nonclassical pump noise suppression; $V_p=0.1$ (lower trace). Other parameters are $G=1.5\times10^{12}$ $\mathrm{s}^{-1}$, $\Gamma=6.0$ $\mathrm{s}^{-1}$, $\gamma_t=4.6\times10^3$ $\mathrm{s}^{-1}$, $\kappa_m=8.0\times10^7$ $\mathrm{s}^{-1}$, and $\kappa_l=5.2\times10^7$ $\mathrm{s}^{-1}$. The QNL is at 0 dB. The frequency scale is in logarithms to base 10.

### B. Intensity-noise characteristics of the free-running laser

Equation (14) is equivalent to an intensity-noise power spectrum normalized to the quantum-noise limit. We illustrate the strength of our approach by showing that our result unifies previous results in a physically transparent manner. We consider first the limit in which the decay rate out of the lower lasing level is the most rapid in the problem, i.e., $\gamma\gg\gamma_t,\Gamma,\kappa,G\alpha^2$. This is typical of gas and solid-state lasers operating close to threshold (within approximately ten times). In this limit we get the following expression for the spectrum:

$$\begin{aligned}V_{\mathrm{out}}(\omega)=1+\{&(2\kappa_m)^2[\omega^2+(G\alpha^2+\gamma_t+\Gamma)^2]\\&-8\kappa_m\kappa G\alpha^2(G\alpha^2+\gamma_t+\Gamma)\\&+2\kappa_m G^2\alpha^2[\Gamma J_1 V_p(\omega)+\gamma_t J_3]+2\kappa_m G\\&\times[(\gamma_t+\Gamma)^2+\omega^2](J_3+J_2)+4\kappa_m\kappa_l\\&\times[(G\alpha^2+\gamma_t+\Gamma)^2+\omega^2]\}/\{(\omega_r^2-\omega^2)^2\\&+\omega^2(\gamma_L)^2\},\end{aligned}\tag{16}$$

where

$$\omega_r=\sqrt{2G\alpha^2\kappa}\tag{17}$$

and

$$\gamma_L=G\alpha^2+\gamma_t+\Gamma.$$

There is a resonance in the spectrum at frequency $\omega_r$. If the resonance is underdamped ($\omega_r>\gamma_L$) an oscillation is produced known as the resonant relaxation oscillation [15] (RRO). Below this frequency the pump noise and the quantum noise due to spontaneous emission and phase decay of the coherence appear in the spectrum. Above the RRO these noises roll off due to the filtering effect of the cavity. In the limit of high frequencies the spectrum tends to 1, indicating the laser is quantum-noise limited at high frequencies. Figure 3 illustrates these features for parameters typical of solid-state lasers. Comparison of Eq. (16) with experimental spectra from miniature Nd:YAG ring lasers produces excellent quantitative agreement [7]. Sufficiently far above threshold the low frequency region is dominated by noise from the pump source. If the pump spectrum is below the quantum-noise level ($V_p<1$) this noise suppression can be transferred to the laser spectrum producing squeezing [14]. This effect has been observed in diode lasers with regularized pump currents [16] and is also illustrated in Fig. 3. This is the only nonclassical effect possible under the approximation of Eq. (16). The squeezing shown in Fig. 3 is limited by the inclusion of intracavity losses. Also, because the laser is not far above threshold, there is significant obscuring noise from the RRO.

Nonclassical effects are optimized in the absence of losses and far above threshold. Hence to study these effects we abandon the approximation of Eq. (16) and adopt instead the approximation that losses are negligible, i.e., $\gamma_t\cong\kappa_l\cong0$ and the stimulated emission rate is the most rapid in the problem, i.e., $G\alpha^2\gg\Gamma,\kappa,\gamma$. In this limit Eq. (14) becomes

$$\begin{aligned}V_{\mathrm{out}}=&\frac{\omega^2}{\omega^2+(2\kappa)^2}\\&+\frac{(2\kappa)^2(\omega^2+\gamma^2)}{[2\kappa(\gamma+2\Gamma)-2\omega^2]^2+\omega^2(4\kappa+\gamma+2\Gamma)^2}V_p\\&+\frac{(2\kappa)^2[\omega^2+(2\Gamma)^2]}{[2\kappa(\gamma+2\Gamma)-2\omega^2]^2+\omega^2(4\kappa+\gamma+2\Gamma)^2}.\end{aligned}\tag{18}$$

This equation unifies all the nonclassical effects previously noted for this type of laser [14,17,18]. The only noise terms still significant are those due to the vacuum field input at the mirror (first term), noise from the pump source (second term), and spontaneous emission noise due to the decay from the lower lasing level (third term). Consider first the situation in which the decay from the lower lasing level is much faster than the cavity decay rate and the pump rate, i.e., $\gamma\gg\Gamma,\kappa$. Equation (18) reduces to

$$V_{\mathrm{out}}=\frac{\omega^2}{\omega^2+(2\kappa)^2}+\frac{(2\kappa)^2}{\omega^2+(2\kappa)^2}V_p.\tag{18a}$$

The contribution from the lower lasing level decay is negligible. If the pump is quantum-noise limited, i.e., $V_p=1$, the first and second terms add up to 1, for all frequencies. This is the standard result that lasers are quantum-noise limited well above threshold and is the limiting result for Eq. (16), well above threshold. We see that the quantum noise arises from the pump at low frequencies and the vacuum noise at high frequencies [14]. As we observed for Eq. (16) if the pump is squeezed then squeezing appears in the output spectrum at low frequencies. In this case the roll off of squeezing with frequency is a Lorentzian with a half-width of the cavity decay rate. Consider next the situation in which both the pump and lower lasing level decay rate are much faster than the cavity decay rate, i.e., $\gamma,\Gamma\gg\kappa$. Equation (18) reduces to

$$\begin{aligned}V_{\mathrm{out}}=&\frac{\omega^2}{\omega^2+(2\kappa)^2}+\frac{(2\kappa)^2\gamma^2}{[(2\kappa)^2+\omega^2](\gamma+2\Gamma)^2}V_p\\&+\frac{(2\kappa)^2(2\Gamma)^2}{[(2\kappa)^2+\omega^2](\gamma+2\Gamma)^2}.\end{aligned}\tag{18b}$$

Now both the pump and lower lasing level decay noise contribute to the output noise. In fact the division of the noise between these two independent processes reduces the total noise contributed [19]. This leads to 50% squeezing in the output at low frequencies when the pump and decay rates are "matched" ($\gamma=2\Gamma$). Once again the roll off of squeezing with frequency is Lorentzian.

The division of the low frequency noise between the pump and the lower lasing level decay noise can also occur in a frequency-dependent way, even if the pump rate is much slower than the decay rate. Consider the situation in which the cavity decay rate is faster than the lower level decay rate but both are much faster than the pump rate, i.e., $\kappa>\gamma\gg\Gamma$. Now Eq. (18) reduces to

$$V_{\text{out}}=\frac{\omega^2}{\omega^2+(2\kappa)^2}+\frac{(2\kappa)^2(\omega^2+\gamma^2)}{(2\kappa\gamma-2\omega^2)^2+\omega^2(4\kappa+\gamma)^2}V_p+\frac{(2\kappa)^2[\omega^2+(2\Gamma)^2]}{(2\kappa\gamma-2\omega^2)^2+\omega^2(4\kappa+\gamma)^2}. \tag{18c}$$

Under this condition the laser can be considered an open system, i.e., the ground level is an undepleted reservoir [as for Eqs. (16) and (18a)]. The resonance in the denominator of Eq. (18c) at

$$\omega_s=\sqrt{\kappa\gamma} \tag{19}$$

is now inside the cavity linewidth. At frequencies close to the resonance, noise is divided between contributions from the pump and the lower level decay rate in much the same way as when the rates are matched. In the limit that $\kappa\gg\gamma$, 50% squeezing is produced at frequencies close to $\omega_s$. The physical mechanism behind this effect is the storage of electrons in the lasing levels for times long compared to the cavity decay time. The subsequent noise filtering effect produces the noise suppression.

In this section we have shown how the three-level laser equations can be solved for their fluctuation spectrum so as to produce analytical solutions containing more detail in a physically clear way than previous methods. In the next section we generalize the approach to the problem of injection-locked lasers.

## IV. SPECTRUM OF THE INJECTION-LOCKED LASER

### A. Solving for the injection-locked spectrum

We now consider the effect of an injected field. The lasing transition of the $N$ atoms now interacts with two optical ring cavity modes via the Jaynes-Cummings Hamiltonian

$$\hat{H}=ihg(\hat{a}_i^\dagger\hat{\sigma}_{23i}+\hat{a}_f^\dagger\hat{\sigma}_{23f}-\hat{a}_i\hat{\sigma}_{23i}^+-\hat{a}_f\hat{\sigma}_{23f}^+). \tag{20}$$

The mode $\hat{a}_i$ is resonant with the input field from the master but may be detuned from the slave laser cavity resonance. The mode $\hat{a}_f$ is the resonant free-running mode of the slave. Each mode induces an independent dipole interaction with the lasing levels modeled by lowering and raising operators for each mode (also labeled $i$ and $f$). We have assumed that for the range of frequencies considered the dipole coupling strength for the two modes is the same. Following the procedure of Sec. II, the operator equations of motion can be written

$$\begin{aligned}
\dot{\hat{a}}_f&=\frac{G}{2}(\hat{\sigma}_3-\hat{\sigma}_2)\hat{a}_f-\kappa\hat{a}_f\\
&\quad+\sqrt{2\kappa_m}\delta\hat{A}_m+\sqrt{2\kappa_l}\delta\hat{A}_l-\sqrt{G}\hat{a}_f\hat{\sigma}_{23}(\delta\hat{C}_{Pf}+\delta\hat{C}_{Pf}^\dagger),\\
\dot{\hat{a}}_i&=\frac{G}{2}(\hat{\sigma}_3-\hat{\sigma}_2)\hat{a}_i-(\kappa+i\Delta)\hat{a}_i+\sqrt{2\kappa_m}\hat{A}+\sqrt{2\kappa_l}\delta\hat{A}_l\\
&\quad-\sqrt{G}\hat{a}_i\hat{\sigma}_{23}(\delta\hat{C}_{Pi}+\delta\hat{C}_{Pi}^\dagger),\\
\dot{\hat{\sigma}}_1&=-\Theta\hat{\sigma}_1\hat{B}^\dagger\hat{B}+\gamma\hat{\sigma}_2-\sqrt{\gamma(\hat{\sigma}_2\delta\hat{C}\delta\hat{C}^\dagger+\hat{\sigma}_1\delta\hat{C}^\dagger\delta\hat{C})}\\
&\quad-\sqrt{\Theta\hat{\sigma}_1(1-\Theta\hat{\sigma}_1)}\hat{B}(\delta\hat{C}_f^\dagger+\delta\hat{C}_f),
\end{aligned} \tag{21}$$

$$\begin{aligned}
\dot{\hat{\sigma}}_2&=G(\hat{\sigma}_3-\hat{\sigma}_2)(\hat{n}_f+\hat{n}_i)+\gamma_t\hat{\sigma}_3-\gamma\hat{\sigma}_2\\
&\quad+\sqrt{\gamma(\hat{\sigma}_2\delta\hat{C}\delta\hat{C}^\dagger+\hat{\sigma}_1\delta\hat{C}^\dagger\delta\hat{C})}\\
&\quad-\sqrt{\gamma_t(\hat{\sigma}_3\delta\hat{C}_t\delta\hat{C}_t^\dagger+\hat{\sigma}_2\delta\hat{C}_t^\dagger\delta\hat{C}_t)}-\sqrt{G\hat{n}_f(\hat{\sigma}_3+\hat{\sigma}_2)}\\
&\quad\times(\delta\hat{C}_{Pf}+\delta\hat{C}_{Pf}^\dagger)-\sqrt{G\hat{n}_i(\hat{\sigma}_3+\hat{\sigma}_2)}(\delta\hat{C}_{Pi}+\delta\hat{C}_{Pi}^\dagger),\\
\dot{\hat{\sigma}}_3&=-G(\hat{\sigma}_3-\hat{\sigma}_2)(\hat{n}_f+\hat{n}_i)-\gamma_t\hat{\sigma}_3+\Theta\hat{\sigma}_1\hat{B}^\dagger\hat{B}\\
&\quad-\sqrt{\gamma_t(\hat{\sigma}_3\delta\hat{C}_t\delta\hat{C}_t^\dagger+\hat{\sigma}_2\delta\hat{C}_t^\dagger\delta\hat{C}_t)}\\
&\quad+\sqrt{\Theta\hat{\sigma}_1(1-\Theta\hat{\sigma}_1)}\hat{B}(\delta\hat{C}_f^\dagger+\delta\hat{C}_f)+\sqrt{G\hat{n}_f(\hat{\sigma}_3+\hat{\sigma}_2)}\\
&\quad\times(\delta\hat{C}_{Pf}+\delta\hat{C}_{Pf}^\dagger)+\sqrt{G\hat{n}_i(\hat{\sigma}_3+\hat{\sigma}_2)}(\delta\hat{C}_{Pi}+\delta\hat{C}_{Pi}^\dagger),
\end{aligned}$$

where the definitions are the same as in Sec. II A. The subscripts $i$ and $f$ are included when fields pertain to the injected or free modes, respectively. The input field from the master laser is labeled $\hat{A}$. The detuning between the input field and the slave cavity is $\Delta$.

Once again the semiclassical equations of motion are obtained by ignoring the zero-point fields. We assume that the input field is real. Notice that this forces the locked mode to be complex for all but zero detunings. An examination of the behavior of the semiclassical steady state as a function of the detuning between the input field and the cavity resonance reveals the standard injection locking behavior [4]. At large detunings there is very little buildup of the input field in the laser cavity and hence negligible interaction with the active atoms. The output of the laser is just the geometric addition of the master and slave fields. As the detuning is reduced the injected field intracavity intensity increases and starts to rob gain from the free-running mode, which begins to drop in intensity. Eventually, when the gain to loss balance of the injected mode equals that of the free-running mode, the free-running mode is extinguished. This occurs when

$$\Delta_l=\frac{\sqrt{2\kappa_m}A}{|\alpha_f|}. \tag{22}$$

The detuning $\Delta_l$ is called the locking range and $\alpha_f$ is the semiclassical steady-state value of the free-running laser per root atom in the absence of an injected field. $A$ is the semiclassical amplitude of the injected field per root atom. Note that the input fields are traveling wave fields thus $|A|^2$ is the photon flux of the master field in units of photons per atom per second. For detunings less than the locking range the free-running mode does not oscillate and the slave laser acts as an optical amplifier for the injected field. Linearization and spectral analysis of the fluctuations proceeds as in Sec. III A. An additional complication, arising from the detuning of the locked mode, is the coupling of the amplitude quadrature of the output field with the phase quadrature of the internal and injected fields. The amplitude fluctuation spectrum is given by

$$V_{\rm out}(\omega)=\left\{\left[2\kappa_m-i\omega+F_1(\omega)-\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}-\frac{2\kappa_m A_I^2}{|\alpha_i|^2\left(i\omega+\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}\right)}\right]^2 V_{\rm in}(\omega)\right.$$
$$+2\kappa_m\Gamma J_1[F_2(\omega)]^2V_p(\omega)+2\kappa_m\gamma J_2[F_3(\omega)]^2+2\kappa_m\gamma_t J_3[F_4(\omega)]^2+2\kappa_m G(J_3+J_2)$$
$$\times\left([1-F_4(\omega)]^2+\left[\frac{\sqrt{2\kappa_m}A_I}{|\alpha_i|\left(i\omega+\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}\right)}\right]^2\right)+(2\kappa_m)^2\left[\frac{\sqrt{2\kappa_m}A_I}{|\alpha_i|\left(i\omega+\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}\right)}\right]^2 V_{\rm in}^-(\omega)$$
$$\left.+4\kappa_m\kappa_l\left(1+\left|\frac{\sqrt{2\kappa_m}A_I}{|\alpha_i|\left(i\omega+\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}\right)}\right|^2\right)\right\}\Bigg/\left[i\omega-F_1(\omega)+\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}+\frac{2\kappa_m A_I^2}{|\alpha_i|^2\left(i\omega+\frac{\sqrt{2\kappa_m}A_R}{|\alpha_i|}\right)}\right]^2, \tag{23}$$

where the definitions are as for Eq. (14) (see also the Appendix) with the substitution $\alpha\rightarrow|\alpha_i|$. The amplitude fluctuation spectrum of the injected master is denoted $V_{\rm in}$ while the phase fluctuation spectrum of the injected master is denoted $V_{\rm in}^-$. Also we have used

$$A_I=\frac{\Delta|\alpha_i|}{\sqrt{2\kappa_m}}$$

and

$$A_R=\sqrt{A^2-A_I^2}.$$

The master can be modeled explicitly by using Eqs. (14) and (15) as the input amplitude and phase spectra, respectively. Alternatively idealized or empirically determined spectra can be used.

### B. Intensity-noise behavior of the injection-locked laser

We will now examine the properties of Eq. (23). First we consider the limit of the lower lasing level decay rate being very rapid [as for Eq. (16)]. We limit our discussion, initially, to the case in which the injected field is resonant with the slave cavity. We get the expression

$$V_{\rm out}=V_{\rm in}+\left\{V_{\rm in}(2\kappa_m)^2[\omega^2+(G\alpha_i^2+\gamma_t+\Gamma)^2]\right.$$
$$-V_{\rm in}4\kappa_m\left(G^2\alpha_i^2(J_3-J_2)+\frac{\sqrt{2\kappa_m}A_i}{\alpha_i}(G\alpha_i^2+\gamma_t+\Gamma)\right)$$
$$\times(G\alpha_i^2+\gamma_t+\Gamma)-V_{\rm in}4\kappa_m\omega^2\frac{\sqrt{2\kappa_m}A_i}{\alpha_i}$$
$$+2\kappa_m G^2\alpha_i^2(\Gamma J_1V_p+\gamma_t J_3)+2\kappa_m G[(\gamma_t+\Gamma)^2+\omega^2]$$
$$\times(J_3+J_2)+4\kappa_m\kappa_l$$
$$\left.\times[(G\alpha_i^2+\gamma_t+\Gamma)^2+\omega^2]\right\}\Big/\{(\omega_{rl}^2-\omega^2)^2+\omega^2\gamma_{Ll}^2\}, \tag{24}$$

where

$$\omega_{rl}=\left(G^2\alpha_i^2(J_3-J_2)+\frac{\sqrt{2\kappa_m}A_i}{\alpha_i}(G\alpha_i^2+\gamma_t+\Gamma)\right)^{1/2}$$

and

$$\gamma_{Ll}=G\alpha_i^2+\gamma_t+\Gamma+\frac{\sqrt{2\kappa_m}A_i}{\alpha_i}.$$

Note that $\alpha_i$ is real for zero detunings. We can identify three distinct frequency regimes in Eq. (24). For simplicity we assume that the locking range and cold cavity linewidth are much larger than the stimulated emission rate, pump rate, and spontaneous emission rate, i.e.,

$$\Delta_l\approx\kappa\gg G\alpha_i^2,\Gamma,\gamma_t. \tag{25}$$

This is a good approximation for most solid-state lasers.

*(i) Amplification regime.* The resonance that appeared in the free-running case [see Eq. (16)] is still present in the

denominator for the locked case at roughly the same frequency, i.e., $\omega_{rl} \approx \omega_r$. However, the damping term is now much larger due to the presence of the injected field. In fact the damping term is now roughly equivalent to the locking range, i.e., $\gamma_{Ll} \approx \Delta_l$. For sufficiently large values of the injected field such that $\gamma_{Ll} > \omega_{rl}$ (or roughly such that the locking range is larger than the free-running RRO frequency) the resonance is overdamped and does not produce an oscillation. Instead in the frequency region close to the resonance the injected fluctuations are amplified. If we simplify the spectrum [Eq. (24)] using Eq. (25) and assume that we are close to the resonance, i.e.,

$$\omega \approx \omega_{rl}, \tag{26}$$

we obtain

$$V_{\rm out} = V_{\rm in}\left(1 + \frac{2\kappa_m \alpha_i^2}{A^2} - \frac{2\sqrt{2\kappa_m}\alpha_i}{A}\right) + \frac{G(J_3+J_2)\alpha_i^2}{A^2}$$
$$= V_{\rm in}H + (H-1) + \frac{4\kappa_l \alpha_i^2}{A^2}, \tag{27}$$

where $H = (2\kappa_m \alpha_i - A)^2/A^2$ is the semiclassical intensity amplification factor [see Eq. (11)]. In deriving Eq. (27) we have also used the fact that in the presence of rapid decay from the lower lasing level, $J_3 \gg J_2$, to simplify the second term. In the limit of negligible losses ($\kappa_l = 0$), Eq. (27) reduces to the standard result for an ideal linear optical amplifier [20]. The injected fluctuations are amplified by the semiclassical amplification factor [2] (first term). In addition quantum fluctuations introduced by the phase decay of the lasing coherence [14] (second term) and the intracavity losses (third term) are also amplified. Note that other noise features of the free-running slave, such as pump noise, are suppressed. The width of the amplification regime depends on how strongly the resonance is damped and hence is inversely related to the locking range.

*(ii) High frequency regime.* If we move to frequencies much higher than the resonance the amplification rolls off until at frequencies

$$\omega \gg \left(\frac{V_{\rm in}\kappa_m^2 + 4\kappa\kappa_m}{V_{\rm in}}\right)^{1/2} \tag{28}$$

we find

$$V_{\rm out} \cong V_{\rm in}. \tag{29}$$

Physically we are outside the laser cavity linewidth, hence high frequency fluctuations inside the cavity are suppressed while high frequency fluctuations on the injected field are simply reflected off the front mirror.

*(iii) Low frequency regime.* If we move to frequencies lower than the resonance we also get a roll off of the amplification of the fluctuations of the injected field. The corner frequency for this roll off is given by

$$\omega_c = \frac{\omega_{rl}^2}{\gamma_{Ll}} \tag{30}$$

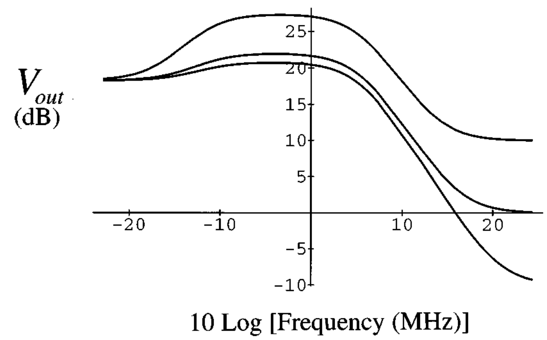


FIG. 4. Intensity-noise spectrum of the injection-locked laser for zero detuning [Eq. (24)] with classical pump noise; $V_p = 100$. The master spectrum is $V_{\rm in} = 10$ (upper trace), $V_{\rm in} = 1$ (middle trace), and $V_{\rm in} = 0.1$ (lower trace). The injected field flux is $A^2 = 0.07\ {\rm s}^{-1}$. Other parameters are the same as in Fig. 3.

(or roughly the square of the free-running RRO frequency divided by the locking range). In addition noise sources associated with the free-running dynamics of the slave laser roll on. If we add the extra condition that we are well above threshold ($G\alpha_l^2 \gg \Gamma \gg \gamma_t$) and consider frequencies very close to dc ($\omega \approx 0$) then Eq. (24) becomes

$$V_{\rm out} \cong \frac{V_{\rm in}A^2 + V_p \Gamma J_1}{(\sqrt{2\kappa_m}\alpha_i - A)^2}$$
$$\cong \frac{P_{\rm in}V_{\rm in} + P_{\rm free}V_f}{P_{\rm in} + P_{\rm free}}, \tag{31}$$

where we have used the fact that far above threshold, with small losses ($\kappa_m \gg \kappa_l$), energy conservation demands $\Gamma J_1 \cong 2\kappa_m \alpha^2 \propto P_{\rm free}$, where $P_{\rm free}$ is the free-running output power for the same pump rate, and also $(\sqrt{2\kappa_m}\alpha_i - A)^2 \propto P_{\rm out} \cong P_{\rm in} + P_{\rm free}$. Under these same assumptions the spectrum of the free-running slave [Eq. (14)] is just determined by the pump noise, i.e., $V_f = V_p$ (see Sec. III B). Hence at low frequencies the output spectrum is just the independent addition of the fluctuations of the injected and free-running modes and is analogous to the spectrum when the lasers are not locked.

Given the preceding discussion we may ask what the normalized spectrum of a large, noisy laser locked to a small quieter laser would be. In Fig. 4 we show the spectrum of a slave laser (whose free-running spectrum is the same as for Fig. 3, with classical pump noise), locked to a quantum-noise limited master, a master with white noise 10 dB above the quantum-noise limit, and a master with squeezed white noise 10 dB below the quantum-noise limit. The intensity amplification factor between the master and the locked slave is $\sim$40 times. At low frequencies the slave pump noise dominates [Eq. (31)]. The noise due to the master rolls on with increasing frequency until we reach the amplification regime where the noise of the master is amplified by the intensity amplification factor plus a quantum-noise penalty [Eq. (27)]. Finally, at frequencies well above the cavity linewidth, we see just the quantum noise of the master [Eq. (29)]. We compare the predictions of Eq. (24) with results obtained using miniature Nd:YAG ring lasers in Ref. [7] and find good quantitative agreement.

### C. Nonclassical effects

We now consider applications of the injection-locked laser for production and manipulation of squeezed light. In the amplification regime the amplified dipole fluctuations [second term in Eq. (27)] mask nonclassical effects for amplification factors of 2 or more. However, in the low and high frequency regimes nonclassical effects can be preserved.

*(i) High frequency regime*. According to Eq. (29) in the high frequency regime the output spectrum of the injection-locked laser is identical to that of the master. If the input light from the master is squeezed then the output from the slave will also be squeezed. Thus the intensity of a bright squeezed light source can be increased without destroying the squeezing [21]. This is illustrated by the spectrum for the squeezed master in Fig. 4. Notice that the signal resolution of the master (the separation between traces for different master noise levels) is reduced in the amplification regime but is preserved in the high frequency regime.

*(ii) Low frequency regime*. If the slave laser is well above threshold Eq. (31) tells us that at low frequencies the slave laser spectrum is an independent combination of fluctuations from the master laser and from the slave laser pump. If the master field is small compared to the output of the cavity ($A_i \ll \sqrt{2\kappa_m}\alpha_i$) the pump fluctuations will dominate. Therefore for a laser with a regularized pump (see Sec. III B) squeezing will still be seen with a small injected signal. In diode lasers pumped with a regularized current the presence of weakly lasing longitudinal side modes can degrade the squeezing produced. An injected field will suppress these side modes. In this way injection locking has been used to improve the squeezing obtained from semiconductor lasers [1]. In fact if we consider the situation in which the stimulated emission rate is very rapid and losses are negligible [as for Eq. (18)] we find

$$V_{\rm out}=\frac{\omega^2+\dfrac{2\kappa A^2}{\alpha_i^2}}{\omega^2+\left(2\kappa-\dfrac{\sqrt{2\kappa}A}{\alpha_i}\right)^2}V_{\rm in}+\frac{(2\kappa)\left(2\kappa-\dfrac{2\sqrt{2\kappa}A}{\alpha_i}\right)(\omega^2+\gamma^2)}{\left[\left(2\kappa-\dfrac{\sqrt{2\kappa}A}{\alpha_i}\right)(\gamma+2\Gamma)-2\omega^2\right]^2+\omega^2\left[2\left(2\kappa-\dfrac{\sqrt{2\kappa}A}{\alpha_i}\right)+\gamma+2\Gamma\right]^2}V_p$$
$$+\frac{(2\kappa)\left(2\kappa-\dfrac{2\sqrt{2\kappa}A}{\alpha_i}\right)[\omega^2+(2\Gamma)^2]}{\left[\left(2\kappa-\dfrac{\sqrt{2\kappa}A}{\alpha_i}\right)(\gamma+2\Gamma)-2\omega^2\right]^2+\omega^2\left[2\left(2\kappa-\dfrac{\sqrt{2\kappa}A}{\alpha_i}\right)+\gamma+2\Gamma\right]^2}. \tag{32}$$

Provided the master field is small compared to the output of the cavity and the slave spectrum is quantum-noise limited then the output spectrum is virtually unchanged from that of the free-running case for all frequencies, hence all squeezing mechanisms are unaffected. Note that in the limit of being very far above threshold there is no amplification regime.

Increased squeezing can also be produced through a single mode effect [6]. If the slave laser is not far enough above threshold ($G\alpha_i^2 \geq \Gamma$) then squeezing from regularized pumping is reduced by the presence of dipole fluctuations and the RRO. Notice, however, that if the injected field is increased then the internal mode amplitude ($\alpha_i$) will also increase such that for a sufficient injected field $G\alpha_i^2 \gg \Gamma$ can be satisfied and the dipole fluctuations and RRO are suppressed. On the other hand, as the injected field gets larger it contributes more of its own fluctuations to the spectrum via Eq. (31). Nevertheless, if a quantum-noise limited master is used, squeezing from all three mechanisms discussed in Sec. III B can be produced at lower slave pump powers using injection locking. At pump powers at which squeezing is already optimized in the free-running case, injection locking can only reduce the amount of squeezing in a single mode situation by adding its own noise. The effect is illustrated in Fig. 5 by comparing the low frequency squeezing of the free-running laser (same parameters as Fig. 3, with pump noise suppression) with that of the locked laser. As noted in Sec. III B, the free-running laser is too close to threshold to show good squeezing. The injection of a large signal from a quantum-noise limited master significantly improves the frequency range of the squeezing.

### D. The effect of detuning

From Eq. (23) we find that a detuning between the input field and the slave laser cavity affects the output spectrum in three distinct ways. First the detuning introduces a frequency-dependent rotation of the noise quadrature of the master field which is amplified. Hence phase noise from the master appears in the output intensity spectrum of the slave. The rotation of the noise quadrature is most pronounced at lower frequencies and high detunings.

Secondly the additional terms in the denominator change the position of the corner frequency [Eq. (30)] marking the roll off of pump noise and the beginning of the amplification regime. The effect is to increase the damping term in the denominator and hence reduce the corner frequency leading to a lower frequency roll off of pump noise and a wider amplification regime. The combination of this effect with the rotation of the noise quadrature means that for detunings close to the edge of the locking range the amplification regime extends to low frequencies where almost pure phase

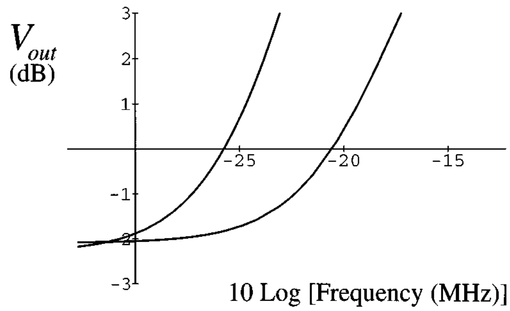


FIG. 5. Intensity-noise spectrum of the free-running laser [upper trace, Eq. (16)] and the injection-locked laser [lower trace, Eq. (24)] with nonclassical pump noise suppression; $V_p=0.1$. The master laser is quantum-noise limited; $V_{\rm in}=1$. The injected field flux is $A^2=0.42\ {\rm s}^{-1}$. Other parameters are the same as in Fig. 3.

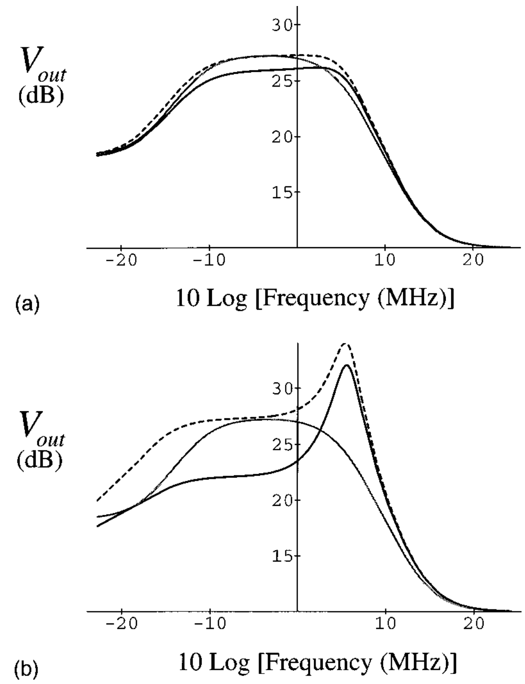


FIG. 6. Intensity-noise spectra of the injection-locked laser with detuning $\Delta=1.8$ MHz (a) and $\Delta=3.4$ MHz (b). The master noise is $V_{\rm in}=10$ and $V_{\rm in}^{-}=10$ (dashed trace and gray trace) or $V_{\rm in}^{-}=1$ (solid trace). The gray traces show zero detuning. All other parameters are the same as in Fig. 4.

noise from the master is amplified.

Thirdly, a second resonance appears in the denominator at approximately the detuning frequency once the detuning becomes larger than roughly half the locking range. This leads to an enhancement of all noise sources at frequencies close to the detuning frequency. The origin of this noise source is beating between transient photons at the free-running frequency and the strong photon field at the injected frequency.

We illustrate the various effects of detuning in Fig. 6. Figure 6(a) shows the effect of a moderate detuning. The dashed line is for a master with equally noisy amplitude and phase quadratures, while the solid line is for a master with amplitude noise but quantum-noise limited phase noise. The gap between these curves is a measure of the rotation of the amplified quadrature of the master. For comparison the zero detuning case is included (gray trace). Some quadrature rotation and broadening of the amplification regime can be observed. Figure 6(b) is for a detuning close to the edge of the locking range. Now strong quadrature rotation and broadening of the amplification regime can be observed. In addition the resonance at the detuning frequency is seen.

Although we will not elaborate here it is possible to extend the theory so as to examine the noise spectra of the two independent modes outside the locking range. The spectra move smoothly from the locked to unlocked case. The amplification of the injected mode noise drops with the amplification of its intensity as the detuning is made larger. While the beat note grows rapidly as a steady state, free-running intensity is established in the cavity. The spectrum of the free-running mode behaves in an analogous way to a laser being pumped further and further above threshold as the increasing detuning increases the gain available. Hence, for example, the RRO frequency will increase with the square root of the free-running intensity as per Eq. (17). At sufficiently high detunings the two spectra become completely independent.

## V. CONCLUSION

We have shown that the linear input-output method can be used to solve for the quantum fluctuations of an atomic laser. The method is straightforward in its solution and leads to analytical results for the full spectral properties of the output field. The separation of the solution into the various noise contributions allows greater physical insight into the noise processes and also allows for the inclusion of the full spectral properties of the pump field and any other input fields.

We have demonstrated the approach by solving for the amplitude quadrature spectrum of a four-level laser and describing its classical and nonclassical behavior. In particular, we have shown that the physical mechanism for nonzero frequency squeezing in open lasers is a noise filtering effect due to the storage of atoms in the lasing levels. At nonzero frequencies this distributes the noise between the pump and the lower lasing level decay in much the same way as in a rate matched laser.

We have applied the method to injection-locked lasers and derived solutions with more detail than those previously presented. We have investigated the behavior of the solutions and identified three distinct frequency regimes in the noise spectrum. At low frequencies the intensity fluctuations are just the geometric addition of the fluctuations from the master (scaled by the master intensity) and the slave (scaled by the slave intensity). This is almost the same as the situation when the master has a large detuning from the slave cavity resonance (much larger than the locking range). At middle frequencies the master fluctuations are amplified in the same way as by a linear optical amplifier. Slave noise sources are suppressed. At high frequencies the output is just the master fluctuations scaled by the slave intensity. We show that the boundaries of these regimes are determined by the amplification ratio between the master and the slave, the output intensity of the free-running slave, and the detuning between the master and the slave.

We have noted qualitatively the relationship between our results and those of experiments. The apparent discrepancy between Refs. [1] and [2] is shown to be because the results of Ref. [1] were obtained in the low frequency regime while those of Ref. [2] were obtained in the amplification regime. In Ref. [7] we make a quantitative comparison between theory and experiment.

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## APPENDIX

This appendix contains the definitions of the functions that appear in Eqs. (13), (14), and (24).

$$\begin{aligned}
F_1(\omega)&=\frac{G^2\alpha^2(J_3-J_2)(2i\omega+2\Gamma+\gamma)}{(i\omega+\Gamma)(i\omega+\gamma+2G\alpha^2+\gamma_t)+\gamma(G\alpha^2+\gamma_t)},\\
F_2(\omega)&=\frac{G\alpha(i\omega+\gamma-\gamma_t)}{(i\omega+\Gamma)(i\omega+\gamma+2G\alpha^2+\gamma_t)+\gamma(G\alpha^2+\gamma_t)},\\
F_3(\omega)&=\frac{G\alpha(i\omega+2\Gamma+\gamma_t)}{(i\omega+\Gamma)(i\omega+\gamma+2G\alpha^2+\gamma_t)+\gamma(G\alpha^2+\gamma_t)},,\\
F_4(\omega)&=\frac{G\alpha(2i\omega+2\Gamma+\gamma)}{(i\omega+\Gamma)(i\omega+\gamma+2G\alpha^2+\gamma_t)+\gamma(G\alpha^2+\gamma_2)}.
\end{aligned}\tag{A1}$$


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