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Squeezed light from a coherently pumped four-level laser

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We calculate the amplitude squeezing in the output of a coherently pumped four-level laser and compare it with that from a similar incoherently pumped laser. We find that squeezing may be considerably enhanced by pumping with coherent light. The squeezing in both types of laser is explained by a simple statistical model.

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Recently Khazanov, Koganov and Gordov [1], Ralph and Savage [2], and Ritsch *et al.* [3] have shown that multilevel lasers are intrinsically less noisy than standard "two-level" lasers, under certain conditions. This can lead to sub-Poissonian photon statistics in the output light. The novel physics is the suppression of pump noise in the lasing levels by the level structure of the laser medium itself, rather than by an externally imposed regularization. Previously, sub-Poissonian output had only been predicted for lasers in which a regular pumping mechanism reduced the population fluctuations in the lasing levels [4]. Various schemes have been proposed to achieve this regular pumping [4–8]. Sub-Poissonian photon statistics have been measured in diode lasers with sub-shot-noise pump current [9–11].

We introduce the following terminology. "Pump cycle" refers to the sequence of levels involved in an electron moving from the lower lasing level to the upper lasing level, while "pump" refers to the actual energy absorbing transition. The basic requirement for sub-Poissonian output, without regular pumping, is a pump cycle containing at least two steps with approximately equal rates. One of these steps may be the pump itself. Any other rates must be faster. Standard laser models do not meet this requirement. The two-level laser has only one step in the pump cycle. Multilevel treatments routinely assume the pump rate is much slower than all other rates [12,13]. Many lasers operate in regimes in which the two-level approximation or the slow pump rate assumption is appropriate. However, we see no fundamental reason why lasers could not be built which operate in regimes meeting the preceding requirement for squeezing.

In this paper, we calculate the squeezing spectra of two types of four-level lasers modeled by master equations. One is pumped by another laser (coherently pumped), while the second is pumped by some other mechanism such as electrical discharge or thermal light (incoherently pumped). We explore their characteristics and show that greater squeezing can occur in the coherently pumped case. We discuss the physical mechanism of noise suppression using a simple model. We find excellent agreement between the predictions of the simple model and those of the full master equation.

The atomic level scheme we have analyzed is depicted in Fig. 1. It is a coherently ($\Gamma=0$) or incoherently ($E=0$) pumped four-level system. Our model consists of N of these four-level atoms interacting with a single optical ring cavity mode via the resonant Jaynes-Cummings Hamiltonian

$$\hat{H}_{\mathrm{JC}}=i\hbar g\sum_{\mu=1}^{N}(\hat{a}^{\dagger}\hat{\sigma}_{23\mu}^{-}-\hat{a}\hat{\sigma}_{23\mu}^{+}) , \tag{1}$$

where carets indicate operators, g is the dipole coupling strength between an atom and the cavity, μ labels the different atoms, $\hat{a}$ and $\hat{a}^{\dagger}$ are the cavity mode annihilation and creation operators, and $\hat{\sigma}_{ij\mu}^{-}$ and $\hat{\sigma}_{ij\mu}^{+}$ are the Hermitian conjugate lowering and raising operators between levels $|i\rangle$ and $|j\rangle$ for the μth atom. The field phase factors have been absorbed into the definition of the atomic operators. The coherent pump is described by a resonant classical field interacting with the atoms via the Hamiltonian

$$\hat{H}_{\mathrm{CP}}=i\hbar E\sum_{\mu=1}^{N}(\sigma_{14\mu}^{-}-\sigma_{14\mu}^{+}) . \tag{2}$$

Following standard techniques [12,14] we couple the atoms and cavity to reservoirs and derive a master equation for the reduced density operator $\hat{\rho}$ of the atoms and cavity. Included in our laser model are atomic spontaneous emission from level $|4\rangle$ to level $|3\rangle$, from level $|3\rangle$ to level $|2\rangle$, and from level $|2\rangle$ to level $|1\rangle$, at rates γ_{34}, γ_{23},

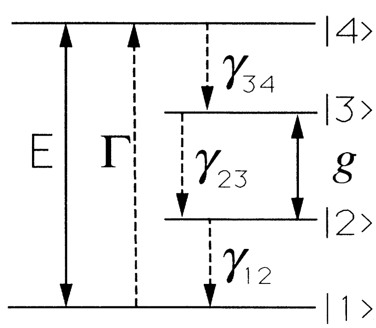

FIG. 1. Schematic diagram of the level structure considered. Pumping is either incoherent at rate Γ or coherent with field strength proportional to E.

and γ_{12}, respectively. Note that γ_{23} is the rate of spontaneous decay into modes other than the laser mode. Incoherent pumping from level $|1\rangle$ to level $|4\rangle$ occurs at rate Γ, and the cavity damping rate is 2κ. The resulting interaction picture master equation is

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho} &= \frac{1}{i\hbar}[\hat{H}_{\rm JC},\hat{\rho}]+\frac{1}{i\hbar}[\hat{H}_{\rm CP},\hat{\rho}] \\ &\quad +\tfrac{1}{2}(\Gamma L^{\dagger}_{14}+\gamma_{34}L_{34}+\gamma_{12}L_{12}+\gamma_{23}L_{23})\hat{\rho} \\ &\quad +\kappa(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a}) , \\ L_{ij}\hat{\rho} &= \sum_{\mu=1}^{N}(2\hat{\sigma}^{-}_{ij\mu}\hat{\rho}\hat{\sigma}^{+}_{ij\mu}-\hat{\sigma}^{+}_{ij\mu}\hat{\sigma}^{-}_{ij\mu}\hat{\rho}-\hat{\rho}\hat{\sigma}^{+}_{ij\mu}\hat{\sigma}^{-}_{ij,u}) , \\ L^{\dagger}_{ij\mu}\hat{\rho} &= \sum_{\mu=1}^{N}(2\hat{\sigma}^{+}_{ij\mu}\hat{\rho}\hat{\sigma}^{-}_{ij\mu}-\hat{\sigma}^{-}_{ij\mu}\hat{\sigma}^{+}_{ij\mu}\hat{\rho}-\hat{\rho}\hat{\sigma}^{-}_{ij\mu}\hat{\sigma}^{+}_{ij\mu}) .\end{aligned} \tag{3}$$

This operator master equation is equivalent to a c-number Fokker-Planck equation for the positive P representation of Drummond and Gardiner [15]. A correspondence is defined between c numbers and operators

$$\begin{aligned}&\alpha^{\dagger}\leftrightarrow\hat{a}^{\dagger},\quad \alpha\leftrightarrow\hat{a} , \\ &J^{+}_{ij}\leftrightarrow\hat{J}^{+}_{ij}\equiv\sum_{\mu=1}^{N}\hat{\sigma}^{+}_{ij\mu}e^{i\mathbf{k}\cdot\mathbf{r}_{\mu}} , \\ &J_{i}\leftrightarrow\hat{J}_{i}\equiv\sum_{\mu=1}^{N}\hat{\sigma}_{i\mu},\quad J_{ij}\leftrightarrow\hat{J}_{ij}\equiv\sum_{\mu=1}^{N}\hat{\sigma}^{-}_{ij\mu}e^{-i\mathbf{k}\cdot\mathbf{r}_{\mu}} ,\end{aligned} \tag{4}$$

by introducing the normally ordered characteristic function

$$\begin{aligned}\hat{\chi} &= \exp(i\lambda_{17}\hat{J}^{+}_{24})\exp(i\lambda_{16}\hat{J}^{+}_{12})\exp(i\lambda_{15}\hat{J}^{+}_{34})\exp(i\lambda_{14}\hat{J}^{+}_{13})\exp(i\lambda_{13}\hat{J}^{+}_{23})\exp(i\lambda_{12}\hat{J}^{+}_{14}) \\ &\quad\times\exp(i\lambda_{11}\hat{J}_{4})\exp(i\lambda_{10}\hat{J}_{3})\exp(i\lambda_{9}\hat{J}_{2})\exp(i\lambda_{8}\hat{J}_{14})\exp(i\lambda_{7}\hat{J}_{23})\exp(i\lambda_{6}\hat{J}_{13}) \\ &\quad\times\exp(i\lambda_{5}\hat{J}_{34})\exp(i\lambda_{4}\hat{J}_{12})\exp(i\lambda_{3}\hat{J}_{24})\exp(i\lambda_{2}\hat{a}^{\dagger})\exp(i\lambda_{1}\hat{a}) .\end{aligned} \tag{5}$$

We define a P representation in seventeen-dimensional complex phase space by

$$\begin{aligned}&P(\boldsymbol{\alpha})=\int\partial^{2}\lambda_{1}\cdots\partial^{2}\lambda_{17}e^{-i\lambda\cdot\alpha}\mathrm{Tr}(\hat{\chi}\hat{\rho}) , \\ &\boldsymbol{\lambda}=(\lambda_{1}\ldots\lambda_{17}), \\ &\boldsymbol{\alpha}=(\alpha_{1},\alpha_{2},\ldots,\alpha_{17})=(\alpha,\alpha^{\dagger},J_{24},\ldots,J^{+}_{12},J^{+}_{24}) .\end{aligned} \tag{6}$$

Hence

$$\frac{\partial}{\partial t}P(\boldsymbol{\alpha})=\int\partial^{2}\lambda_{1}\cdots\partial^{2}\lambda_{17}e^{-i\lambda\cdot\alpha}\mathrm{Tr}\left[\hat{\chi}\frac{\partial\hat{\rho}}{\partial t}\right] . \tag{7}$$

Equation (7) can be expanded using Eqs. (3) and (5) to obtain a partial differential equation containing derivatives of infinite order in the atomic populations. Adopting the following standard scaling of the variables and dipole coupling constant with the number of atoms:

$$\alpha=\tilde{\alpha}N^{1/2},\quad J_{ij}=\tilde{J}_{ij}N,\quad J_{i}=\tilde{J}_{i}N,\quad g=\tilde{g}N^{-1/2} ,$$

where a tilde denotes a scaled quantity, reveals that the terms containing derivatives of higher than second order are negligible in the limit of many atoms [6,14,15]. This scaling is appropriate well above threshold [6,13].

Truncating higher-order derivatives we obtain the Fokker-Planck equation

$$\frac{\partial}{\partial t}P(\tilde{\boldsymbol{\alpha}})=\left[-\frac{\partial}{\partial\alpha_{\xi}}A_{\xi}(\tilde{\boldsymbol{\alpha}})+\frac{1}{2}\frac{\partial}{\partial\alpha_{\zeta}}\frac{\partial}{\partial\alpha_{\xi}}D_{\zeta\xi}(\tilde{\boldsymbol{\alpha}})\right]P(\tilde{\boldsymbol{\alpha}}) . \tag{8}$$

Equation (8) is equivalent to a system of stochastic differential equations [16]. These equations contain deterministic terms obtained from the drift vector $A_{\xi}(\tilde{\boldsymbol{\alpha}})$ as well as δ-correlated noise terms with zero mean. The noise terms arise from the diffusion matrix $D_{\zeta\xi}(\tilde{\boldsymbol{\alpha}})$ and describe the quantum fluctuations. If the noise terms are ignored we obtain the following semiclassical equations of motion:

$$\begin{aligned}\dot{\tilde{\alpha}}&=\tilde{g}\tilde{J}_{23}-\kappa\tilde{\alpha}, \\ \dot{\tilde{J}}_{12}&=E\tilde{J}^{+}_{24}+\tilde{g}\tilde{\alpha}^{\dagger}\tilde{J}_{13}-\tfrac{1}{2}(\Gamma+\gamma_{12})\tilde{J}_{12} , \\ \dot{\tilde{J}}_{13}&=E\tilde{J}^{+}_{34}-\tilde{g}\tilde{J}_{12}\tilde{\alpha}-\tfrac{1}{2}(\Gamma+\gamma_{23})\tilde{J}_{13} , \\ \dot{\tilde{J}}_{14}&=E(\tilde{J}_{4}-\tilde{J}_{1})-\tfrac{1}{2}(\Gamma+\gamma_{34})\tilde{J}_{14} , \\ \dot{\tilde{J}}_{23}&=\tilde{g}(\tilde{J}_{3}-\tilde{J}_{2})\tilde{\alpha}-\tfrac{1}{2}(\gamma_{23}+\gamma_{12})\tilde{J}_{23} , \\ \dot{\tilde{J}}_{34}&=-E\tilde{J}^{+}_{13}-\tilde{g}\tilde{J}_{24}\tilde{\alpha}^{\dagger}-\tfrac{1}{2}(\gamma_{23}+\gamma_{34})\tilde{J}_{34} , \\ \dot{\tilde{J}}_{2}&=\tilde{g}(\tilde{J}_{23}\tilde{\alpha}^{\dagger}+\tilde{J}^{+}_{23}\tilde{\alpha})+\gamma_{23}\tilde{J}_{3}-\gamma_{12}\tilde{J}_{2} , \\ \dot{\tilde{J}}_{3}&=-\tilde{g}(\tilde{J}_{23}\tilde{\alpha}^{\dagger}+\tilde{J}^{+}_{23}\tilde{\alpha})+\gamma_{34}\tilde{J}_{4}-\gamma_{23}\tilde{J}_{3} , \\ \dot{\tilde{J}}_{4}&=-E(\tilde{J}_{14}+\tilde{J}^{+}_{14})+\Gamma\tilde{J}_{1}-\gamma_{34}\tilde{J}_{4} .\end{aligned} \tag{9}$$

Setting the derivatives in Eq. (9) to zero enables us to solve for the semiclassical steady state. In Fig. 2 we have plotted the semiclassical steady-state cavity photon number per atom versus the pumping rate for the coherent and incoherent cases.

We now assume that the quantum fluctuations are sufficiently small that we can treat them as linear perturbations. We write the full solutions in the form

$$\tilde{\alpha}=\tilde{\alpha}_{0}+\delta\tilde{\alpha},\quad \tilde{J}_{ij}=\tilde{J}_{ij;0}+\delta\tilde{J}_{ij},\quad \tilde{J}_{i}=\tilde{J}_{i;0}+\delta\tilde{J}_{i} ,$$

where $\tilde{\alpha}_{0}$, $\tilde{J}_{ij;0}$ and $\tilde{J}_{i;0}$ are stable steady-state solutions to Eq. (9). Note that the phases of the complex solutions are undetermined; without loss of generality we take them to be real. Fixing this phase limits the time for which our approximation of small quantum fluctuations is valid. This is because the laser phase diffuses away from its initial phase. Hence the spectral quantities we shall calculate are not reliable for frequencies less than the laser linewidth $\Delta\omega \approx \kappa/\alpha_0^* \alpha_0$, which is very small.

The following linearized stochastic differential equation describes to first order the fluctuations in the field and atomic variables:

$$\frac{d}{dt}\delta\tilde{\boldsymbol{\alpha}} = -A_0 \delta\tilde{\boldsymbol{\alpha}} + \mathbf{F}(t) , \tag{10}$$

where

$$\delta\tilde{\boldsymbol{\alpha}} = (\delta\tilde{\alpha}, \delta\tilde{\alpha}^\dagger, \ldots, \delta\tilde{J}_{24}^{+}) ,$$

$$A_{\zeta,\xi;0} = -(\partial/\partial\tilde{\alpha}_\xi)\dot{\tilde{\alpha}}_\zeta|_{\tilde{\boldsymbol{\alpha}}=\tilde{\boldsymbol{\alpha}}_0} ,$$

and $\dot{\tilde{\alpha}}_\zeta$ is the right-hand side of the corresponding Eq. (9). The noise correlations are given by

$$\langle F_\zeta(t) F_\xi(t') \rangle = D_{\zeta,\xi;0}\delta(t-t') ,$$

$$D_{\zeta,\xi;0} \equiv D_{\zeta,\xi}|_{\tilde{\boldsymbol{\alpha}}=\tilde{\boldsymbol{\alpha}}_0} .$$

The nonzero terms of the linearized, symmetric, diffusion matrix $D_{i,j;0}$ are

$$D_{9,9;0} = -2\tilde{g}\tilde{J}_{23;0}\tilde{\alpha}_0 + \gamma_{12}\tilde{J}_{2;0} + \gamma_{23}\tilde{J}_{3;0} ,$$

$$D_{10,10;0} = -2\tilde{g}\tilde{J}_{23;0}\tilde{\alpha}_0 + \gamma_{34}\tilde{J}_{4;0} + \gamma_{23}\tilde{J}_{3;0} ,$$

$$D_{5,16;0} = D_{4,15;0} = D_{3,14;0} = D_{6,17;0} = E\tilde{J}_{23;0} ,$$

$$D_{5,14;0} = D_{6,15;0} = E(\tilde{J}_{3;0} - \tilde{J}_{1;0}) + \gamma_{23}\tilde{J}_{14;0} ,$$

$$D_{4,16;0} = -2E\tilde{J}_{14;0} + \Gamma\tilde{J}_{2;0} + \gamma_{23}\tilde{J}_{3;0} ,$$

$$D_{6,14;0} = -2E\tilde{J}_{14;0} + (\Gamma + \gamma_{23})\tilde{J}_{3;0} ,$$

$$D_{12,12;0} = 2E\tilde{J}_{14;0} ,$$

$$D_{8,8;0} = 2E\tilde{J}_{14;0} + \gamma_{34}\tilde{J}_{4;0} ,$$

$$D_{11,11;0} = -2E\tilde{J}_{14;0} + \Gamma\tilde{J}_{1;0} + \gamma_{34}\tilde{J}_{4;0} ,$$

$$D_{3,16;0} = D_{4,17;0} = E(\tilde{J}_{2;0} - \tilde{J}_{1;0}) ,$$

$$D_{13,13;0} = D_{7,7;0} = 2\tilde{g}\tilde{J}_{23;0}\tilde{\alpha}_0 ,$$

$$D_{10,9;0} = 2\tilde{g}\tilde{J}_{23;0}\tilde{\alpha}_0 - \gamma_{23}\tilde{J}_{3;0} ,$$

$$D_{10,11;0} = -\gamma_{34}\tilde{J}_{4;0} ,$$

$$D_{3,17;0} = \gamma_{12}\tilde{J}_{4;0} + \Gamma\tilde{J}_{1;0} ,$$

$$D_{7,13;0} = \gamma_{12}\tilde{J}_{3;0} + \gamma_{23}\tilde{J}_{4;0} ,$$

$$D_{9,7;0} = D_{9,13;0} = \gamma_{12}\tilde{J}_{23;0} ,$$

$$D_{5,15;0} = \gamma_{23}\tilde{J}_{4;0} + \Gamma\tilde{J}_{1;0} ,$$

$$D_{6,16;0} = D_{4,14;0} = \Gamma\tilde{J}_{23;0} ,$$

$$D_{8,12;0} = \Gamma(\tilde{J}_{1;0} + \tilde{J}_{4;0}) .$$

The amplitude squeezing spectrum is defined by [17]

$$V(X,\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\langle \hat{X}(t+\tau), \hat{X}(t) \rangle d\tau , \tag{11}$$

where $\hat{X}(t) = \hat{a}_{\text{out}}(t) + \hat{a}_{\text{out}}^\dagger(t)$ is the quadrature phase amplitude of the cavity output field and $\langle \hat{c}, \hat{b} \rangle = \langle \hat{c}\hat{b} \rangle - \langle \hat{c} \rangle \langle \hat{b} \rangle$. The amplitude squeezing spectrum is given by [17]

$$V(X,\omega) = 1 + 2\kappa[S_{12}(\omega) + S_{21}(\omega) + S_{11}(\omega) + S_{22}(\omega)] , \tag{12}$$

where the spectral matrix $S(\omega)$ is given by

$$S(\omega) = (A_0 - i\omega I)^{-1} D_0 (A_0^T + i\omega I)^{-1} , \tag{13}$$

and the laser cavity is assumed to be one sided. In Fig. 3 we plot the amplitude spectral variance at the zero-frequency local minimum (or maximum) as a function of pump rate for the coherent and incoherent cases. The full spectra are approximately Lorentzians (in the region shown) with linewidths corresponding to that of the laser cavity. Closer to threshold non-Lorentzian spectra are found.

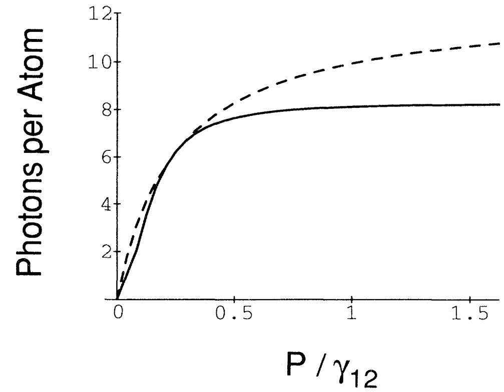

FIG. 2. Semiclassical cavity photon number per atom $\tilde{\alpha}_0^* \tilde{\alpha}_0$ vs pump rate for the incoherently pumped (dashed line) and coherently pumped (solid line) lasers. For the incoherently pumped case $P = \Gamma$ and for the coherently pumped case $P = E$. Parameters in units of γ_{12} are $\gamma_{23} = 10^{-6}$, $\gamma_{34} = 0.5$, $\tilde{g} = 1$, $\kappa = 0.01$.

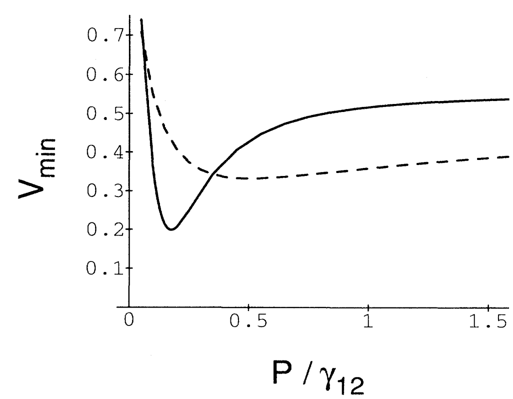

FIG. 3. The amplitude squeezing spectral variance $V_{\min} = V(X, \omega = 10^{-4}\gamma_{12})$ vs pump rate for the incoherently pumped (dashed line) and coherently pumped (solid line) lasers (the variance is undefined at exactly $\omega = 0$ due to phase diffusion). For the incoherently pumped case $P = \Gamma$ and for the coherently pumped case $P = E$. A coherent state has $V(X,\omega) = 1$, while perfect squeezing has $V(X,\omega) = 0$. Parameters are the same as for Fig 2.

Parameters have been chosen to show maximum squeezing. $V_{\min}=0$ is perfect squeezing and $V_{\min}=1$ is the coherent-state spectral variance. The incoherently pumped case has a minimum spectral variance of 0.33 when $\Gamma=\gamma_{34}=0.5\gamma_{12}$. The coherently pumped case has a minimum of 0.2 when $\sqrt{8}E=\gamma_{34}=0.5\gamma_{12}$. The best squeezing is found in the coherent case, but the incoherent case has better squeezing for larger pump rates.

The amplitude squeezing spectrum from a laser can be observed by direct photodetection, as has been done in the laser diode squeezing experiments [9–11]. As long as the squeezing is small the classical amplitude of the laser light α_0 acts like an in-phase local oscillator for the quantum fluctuations, and the amplitude squeezing spectrum is equivalent to the intensity fluctuation spectrum. This follows from a calculation of the intensity fluctuation spectrum using the approximation $\hat{a}^\dagger=\alpha_0+\delta\hat{a}^\dagger$, α_0 real:

$$\begin{aligned}\langle \hat{n}_\tau,\hat{n}\rangle &= \langle(\alpha_0+\delta\hat{a}_\tau^\dagger)(\alpha_0+\delta\hat{a}_t),(\alpha_0+\delta\hat{a}^\dagger)(\alpha_0+\delta\hat{a})\rangle\\ &= \langle(\alpha_0^2+\alpha_0\delta\hat{X}_\tau+\delta\hat{a}_\tau^\dagger\delta\hat{a}_\tau),(\alpha_0^2+\alpha_0\delta\hat{X}+\delta\hat{a}^\dagger\delta\hat{a})\rangle\\ &= \alpha_0^2\langle\delta\hat{X}_\tau,\delta\hat{X}\rangle+O(\alpha_0)\ ,\end{aligned} \tag{14}$$

where we have used a subscript τ to indicate quantities to be evaluated at time $t+\tau$.

The three-level limit can be approached by adjusting the spontaneous decay rates. In the incoherent case, if γ_{34} is made much faster than γ_{12}, we move towards the three-level results of our previous paper [2], where the minimum spectral variance was 0.5. However, if γ_{34} is made much slower than γ_{12}, then it is the bottom level that is effectively removed giving the model by Khazanov, Koganov, and Gordov [1]. The minimum variance is again 0.5. The coherent case is quite different. If a fast decay precedes the coherent transition ($\gamma_{12}\gg\gamma_{34}$), then the minimum variance increases slightly to 0.25. If, however, the fast decay follows the coherent pump ($\gamma_{34}\gg\gamma_{12}$), then the minimum variance increases significantly to 0.5. If the coherent pump is to have an advantage it must immediately precede a decay that is as slow or slower than all other decays in the pump cycle.

In order to understand the physics of these lasers we will now present a simple model which produces results in agreement with those of the complete model just discussed. First we relate the variance in the time the pump cycle takes to place an electron in the upper lasing level of an individual atom to the photon number variance of the output. We assume the laser is well above threshold and has a strong enough dipole coupling that the lasing transition time can be considered to have zero variance [8]. Also we assume there is no spontaneous emission out of the upper lasing level, so every time the pump cycle places an electron in the upper lasing level a laser photon is emitted. The mean number of photons $\bar{n}$ leaving the cavity in a time T, which is long compared to the cavity lifetime, is thus given by $\bar{n}=N(T/\bar{t})$ where $\bar{t}$ is the mean time it takes for the electron to arrive in the upper lasing level. The standard deviations in $\bar{n}$ and $\bar{t}$ are hence related by

$$\left[\frac{\Delta\bar{n}}{\bar{n}}\right]^2=\left[\frac{\Delta\bar{t}}{\bar{t}}\right]^2 , \tag{15}$$

where we have assumed the number of atoms N is constant over the time T. The standard deviation in the mean of $\bar{n}$ arrival times $\Delta\bar{t}$ is related to the standard deviation of one arrival time Δt by

$$\Delta\bar{t}=\frac{\Delta t}{\sqrt{\bar{n}}} . \tag{16}$$

Using this in Eq. (15) the spectral variance at zero frequency $V_{\min}$ is given by

$$V_{\min}\approx\frac{\Delta\bar{n}^2}{\bar{n}}=\frac{\Delta t^2}{(\bar{t})^2} , \tag{17}$$

where the approximate equality holds if the squeezing is not too large. Now we relate Δt and $\bar{t}$ to atomic parameters.

In a three-level laser the pump cycle has two steps: a spontaneous decay from the lower lasing level at rate γ and a pump from the ground state to the upper lasing level at rate Γ. In the absence of the lasing transition the mean arrival time in the upper lasing level would be $1/\gamma+1/\Gamma$. With the lasing transition present the electron oscillates between the upper and lower lasing levels. It is only available to decay out of the lower lasing level a fraction of the time. If the dipole coupling is strong the populations of the upper and lower lasing levels will be approximately equal above threshold. This implies the electron is only available to decay out of the lower lasing level for half the time and the *effective* decay rate is $\gamma'=0.5\gamma$. Thus $\bar{t}=2/\gamma+1/\Gamma$. Using the standard rules for adding independent noises we obtain $\Delta t=[(2/\gamma)^2+(1/\Gamma)^2]^{1/2}$ and thus from Eq. (17)

$$V_{\min}\approx\frac{(2/\gamma)^2+(1/\Gamma)^2}{(2/\gamma+1/\Gamma)^2} . \tag{18}$$

$V_{\min}$ depends, for a given γ, on the pump rate. A minimum value of 0.5 (50% squeezing) occurs when $\Gamma=0.5\gamma$ in agreement with the results of our previous paper [2]. This result is readily generalized to multilevel atoms. For an $(r+3)$-level laser

$$V_{\min}\approx\frac{(2/\gamma_L)^2+(1/\gamma_1)^2+\cdots+(1/\gamma_r)^2+(1/\Gamma)^2}{(2/\gamma_L+1/\gamma_1+\cdots+1/\gamma_r+1/\Gamma)^2} , \tag{19}$$

where γ_L is the decay rate out of the lower lasing level, Γ is the pump rate, and $\gamma_1\cdots\gamma_r$ are the rates of the other steps in the pump cycle. The rates are matched for optimum noise reduction when $\Gamma=\gamma_1=\cdots=\gamma_r=0.5\gamma_L$. The minimum value of $V_{\min}$ is then $1/(r+2)$. This result for matched rates was reported by Ritsch *et al.* [3]. The multistep pump cycle is "quieter" than the single-step pump due to the independence of the noise introduced in each step [2].

Agreement between the preceding simple model and that based on the master equation is excellent. A graph of Eq. (19) with $r=1$ is identical to the incoherent plot on

Fig. 3. We noted earlier that a slow pump rate leads to Poissonian statistics. As $\Gamma \rightarrow 0$ in Eq. (19) $V_{\min} \rightarrow 1$. Conversely the terms associated with very fast rates tend to zero and become unimportant. Figure 3 is plotted for a scaled dipole coupling strength of $\tilde{g} = \gamma_{12}$. This is consistent with the assumptions of our simple model. The assumption that the effective decay is Poissonian breaks down with smaller $\tilde{g}$, as the variance of the laser transition itself becomes important. For $\tilde{g} < 0.15\gamma_{12}$ the squeezing is destroyed.

The above discussion concerned incoherently pumped lasers. If the pump cycle contains a coherent step we must solve for its dynamics explicitly. The pump cycle of our coherently pumped four-level laser has three steps. The middle step is coherently driven. The first step has an *effective* decay rate of $0.5\gamma_{12}$ and the third step has decay rate γ_{34}. A master equation for the pump cycle gives the following system of linear differential equations:

$$\begin{aligned} &\dot{\sigma}_{2c} = -0.5\gamma_{12}\sigma_{2c} \ , \quad \dot{\sigma}_{1c} = 2E\sigma_{14c} + 0.5\gamma_{12}\sigma_{2c} \ , \\ &\dot{\sigma}_{4c} = -2E\sigma_{14c} - \gamma_{34}\sigma_{4c}, \quad \dot{\sigma}_{3c} = \gamma_{34}\sigma_{4c} \ , \\ &\dot{\sigma}_{14c} = (\sigma_{4c} - \sigma_{1c})E - 0.5\gamma_{34}\sigma_{14c} \ . \end{aligned} \tag{20}$$

σ_{ic} is the population expectation value for the ith level of the atom. σ_{14c} is the expectation value of the coherence between the coherently pumped levels. For simplicity we assume the incoherent rates are matched, i.e., $0.5\gamma_{12} = \gamma_{34} = \gamma$. Solving the differential equations we find the probability of an electron being in the upper lasing level at time t if originally in the lower lasing level is

$$\sigma_{3c} = 1 + e^{-\gamma t} + [32E^2 - \gamma^2(e^{\Omega t} + e^{-\Omega t})]\frac{e^{-\gamma t/2}}{4\Omega^2} \ . \tag{21}$$

where $\Omega = (\gamma^2 - 16E^2)^{1/2}/2 \neq 0$. Ω becomes imaginary when $4E > \gamma$ and the fourth term becomes oscillatory with frequency $|\Omega|$. The mean arrival time and arrival time variance of the pump cycle are given by

$$\bar{t} = \int_0^\infty t\frac{\partial}{\partial t}\sigma_{3c}dt, \quad \Delta t^2 = \sum_{0^\infty} t^2\frac{\partial}{\partial t}\sigma_{3c}dt - \bar{t}^2 \ . \tag{22}$$

These integrals may be evaluated and $V_{\min}$ calculated for various values of E. Once again there is good agreement between the simple and rigorous models. The minimum value is found when $|\Omega|$, equals $\gamma/2$.

As E becomes large so does $|\Omega|$, producing rapid oscillations in the coherently pumped levels. As for the lasing transition only half the population is available for spontaneous decay out of level $|4\rangle$. The effective value of γ_{34} is halved and is no longer matched with γ_{12}. This is why the coherent pump squeezing is not as good as with the incoherent pump for high pump rates. If a fast decay precedes the coherent transition, then the pump tends to a two-step cycle (coherent$\rightarrow$incoherent). The removal of

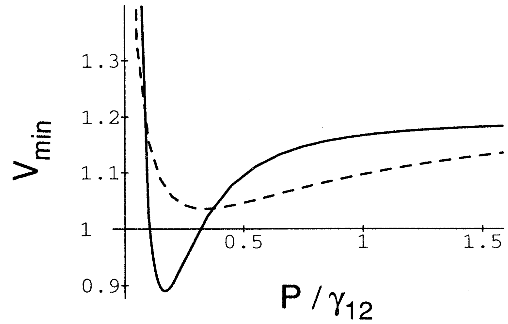

FIG. 4. The amplitude-squeezing spectral variance $V_{\min} = V(X, \omega = 10^{-4}\gamma_{12})$ vs pump rate for the incoherently pumped (dashed line) and coherently pumped (solid line) lasers. For the incoherently pumped case $P = \Gamma$ and for the coherently pumped case $P = E$. Parameters in units of γ_{12} are the same as for Fig. 2 except $\gamma_{23} = 0.3$.

a level increases $V_{\min}$ slightly to 0.25. If, however, the fast decay follows the coherent pump, then the pump becomes overdamped and the combination starts to behave like one incoherent step. Hence we move towards the incoherent three-level case and $V_{\min}$ increases to 0.5.

So far we have ignored noise sources other than those produced by the pump cycle. Technical noise may arise in various ways from the pump or the cavity. This must be brought as close as possible to the quantum limit if squeezing is to be observed. Also, spontaneous emission from the upper lasing level out of the cavity adds noise to the output light. In Fig. 4 we show the effect of introducing such noise $\gamma_{23} = 0.3\gamma_{12}$ on the two models. The coherent is more robust, showing some squeezing when the incoherent case has become exclusively super-Poissonian. It is interesting to note that the characteristic dip due to matched rates is still present when the photon statistics remain super-Poissonian. This suggests a classical analog of the squeezing effect may be observable in non-quantum-noise limited lasers. Notice also that the positioning of the dip gives a measure of the rate of the dominant decay in the pump cycle.

In summary, we have shown that a coherently pumped multilevel laser with matched rates can produce significantly better squeezing than an equivalent incoherently pumped laser. With a simple model we explained the mechanism of the noise suppression in both the coherently and incoherently pumped lasers. The simple model gives accurate predictions in the regime of interest. Finally, we note that the pump noise suppression mechanism may be operative even in non-quantum-noise-limited lasers.

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