Strategy-proof Choice of Acts: A Preliminary Study*

Yves Sprumont†

June 2, 2016

Abstract

We model social choices as acts mapping states of the world to (social) outcomes. A (social choice) rule assigns an act to every profile of subjective expected utility preferences over acts. A rule is strategy-proof if no agent ever has an incentive to misrepresent her beliefs about the world or her valuation of the outcomes; it is ex-post efficient if the act selected at any given preference profile picks a Pareto-efficient outcome in every state of the world. We show that every two-agent ex-post efficient and strategy-proof rule is a *top selection*: the chosen act picks the most preferred outcome of some (possibly different) agent in every state of the world. The states in which an agent's top outcome is selected cannot vary with the reported valuations of the outcomes but may change with the reported beliefs. We give a complete characterization of the ex-post efficient and strategy-proof rules in the two-agent, two-state case, and we identify a rich class of such rules in the two-agent case.

JEL Classification: D71.
Keywords: Social choice under uncertainty, strategy-proofness, subjective expected utility.

1 Introduction

We address the problem of designing incentive-compatible rules for making social choices under uncertainty. Following Savage (1954), we model such choices as acts mapping states of the world to outcomes, and we assume that agents compare acts according to the subjective expected utility they yield. Society chooses acts on the basis of the preferences of its members: a social choice rule asks agents to report full-fledged preferences over acts, and assigns an act to every preference profile. If individual preferences are private information, it is important that a rule be incentive-compatible. This paper focuses on the condition of strategy-proofness, which requires

*This paper evolved from numerous discussions with D. Mishra and A. Sen, and, later, with S. Horan. The author also benefited from comments by H. Moulin. Many thanks to all of them. The research reported in this paper was supported by a FRQSC grant.

†yves.sprumont@umontreal.ca, Département de Sciences Économiques and CIREQ, Université de Montréal, C.P. 6128, succursale Centre-ville, Montréal QC, H3C 3J7, Canada.

that reporting one's true preferences be a dominant strategy: no agent should ever have an incentive to misrepresent her beliefs about the world or her valuation of the outcomes. Because subjective expected utility preferences form a restricted domain, the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) does not apply. This raises the problem of describing the set of strategy-proof social choice rules.

To the best of our knowledge, this problem has not been studied. The related literature may be divided into three strands. The first strand is concerned with the problem of eliciting an agent's assessment of the likelihood of events in which she has no stake. The best known incentive-compatible elicitation procedures are Savage's (1971) proper scoring rules; see Gneiting and Raftery (2007) for a survey of the literature on the topic. Other procedures include de Finetti's (1974) promissory notes method and Karni's (2009) direct revelation mechanism. These methods do not elicit the agent's valuation of the outcomes and do not address the problem of choosing a social act based on individual preferences.

The second relevant strand studies strategy-proofness in the context of risk, that is, when society chooses lotteries rather than acts. The seminal contribution is due to Gibbard (1977), who analyzes social choice rules asking agents to report their preferences over sure outcomes only. Hylland (1980), Dutta, Peters and Sen (2007, 2008), and Nandeibam (2013) allow agents to report full-fledged von Neumann-Morgenstern preferences over lotteries. A central finding in this literature is that every strategy-proof and ex-post efficient rule is a random dictatorship. Ex-post efficiency requires that the chosen lottery attaches zero probability to every Pareto-dominated sure outcome. A random dictatorship selects each agent's most preferred outcome with a probability that does not depend on the reported preference profile.

Finally, let us mention that the issue of preference aggregation under uncertainty has received a good deal of attention: see Hylland and Zeckhauser (1979), Mongin (1995), Gilboa, Samet and Schmeidler (2004), and Gilboa, Samuelson and Schmeidler (2014), among others. This literature, which is normative in nature, is not concerned with the incentive-compatibility issue and is therefore only tangentially related to our work. It shows that utilitarian aggregation of preferences is problematic; it also questions the desirability of Pareto efficiency when individual beliefs differ, and proposes weakened versions of it.

In line with the literature on strategy-proofness under risk, we restrict attention to social choice rules that are ex-post efficient. Under uncertainty, ex-post efficiency means that the act selected at a given preference profile should recommend a Pareto-efficient outcome in every state of the world. The requirement does not imply (ex-ante) Pareto efficiency.

Our results are restricted to the two-agent case. Proposition 1 establishes that every two-agent strategy-proof and ex-post efficient social choice rule must be a *top selection*: at every preference profile, the chosen act must pick the most preferred outcome of some (possibly different) agent in every state of the world. The analog of random dictatorship consists in *exogenously* assigning each state to an agent and picking each agent's top outcome in the states assigned to her. A top selection need not be a random dictatorship because the states in which an agent's top outcome is selected may vary with the reported preference profile.

Proposition 2 states that, in order to guarantee strategy-proofness, the states in which an agent's top outcome is selected cannot vary with the reported valuations of the outcomes.

On the other hand, it turns out that the *beliefs* of the agents can be used to assign states to agents: the mechanism designer can exploit the differences in subjective probabilities so as to have each agent select the outcome in states that she finds relatively more likely. This can be done in at least two ways. A *dictatorial assignment rule* lets one agent select from an exogenous menu the event that she considers most likely; the social choice rule then picks that agent's top outcome in every state in that event, and the other agent's top outcome in the remaining states. Under a *consensual assignment rule,* two non-nested events are exogenously selected. The first is tentatively assigned to agent 1 and its complement is assigned to agent 2. However, if agent 1 reports that the second event is more likely than the first *and* agent 2 reports the opposite belief, they exchange events. The social choice rule picks an agent's reported top outcome in every state in the event that the consensual assignment rule has assigned to her. Proposition 3 states that, when there are only two possible states of the world, a two-agent social choice rule is strategy-proof and ex-post efficient if and only if it is a top selection generated by a dictatorial or consensual assignment rule.

When the state space contains more than two states, Proposition 4 shows how the basic rules just described can be combined to generate a rich family of fairly flexible and reasonably symmetric two-agent strategy-proof and ex-post efficient social choice rules.

A last and technical introductory remark is in order. The set of acts is a Cartesian product, and subjective expected utility preferences over acts are additively separable. It is known that when individual preferences over a product set of social alternatives are separable *and* form a suitably rich domain, strategy-proof social choice rules are products of strategy-proof "sub-rules" defined on the marginal profiles of preferences over the components of the social alternatives. Le Breton and Sen (1999) offer general theorems of this type; earlier papers proving variants of the result include Border and Jordan (1983), Barberà, Sonnenschein and Zhou (1991), and Barberà, Gul and Stacchetti (1993). This decomposition property does not hold in our setting. The reason is that subjective expected utility preferences do not form a rich domain. Le Breton and Sen's (1999) richness condition requires that for any collection of admissible preferences over the components of the social alternatives there exists a preference over the social alternatives which induces marginal preferences over components coinciding with the ones in that collection. Since in our setting all the state-contingent preferences over outcomes induced by a subjective expected utility preference over acts are the same, Le Breton and Sen's condition is violated. It is this lack of richness that allows one to define non-decomposable rules where beliefs affect the states where each agent's top outcome is selected.

2 Definitions

There is a finite set of agents $N = \{1, ..., n\}$ with $n \geq 2$, a finite set of states of the world $\Omega = \{\omega_1, ..., \omega_K\}$ with $K \geq 2$, and a finite set of outcomes $X = \{x_1, ..., x_M\}$ with $M \geq 3$. Subsets of Ω are called events. The set of acts is $\mathcal{F} := X^\Omega$. Agent i's preference ordering $\succcurlyeq_i$ over acts is assumed to be of the subjective expected utility type: there exist a valuation function $v_i : X \to \mathbb{R}$ and a subjective probability measure p_i on the set of events such that for all $f, f' \in \mathcal{F}$,

$$f \succsim_i f' \Leftrightarrow \sum_{\omega \in \Omega} p_i(\omega) v_i(f(\omega)) \geq \sum_{\omega \in \Omega} p_i(\omega) v_i(f'(\omega)),$$

where we write ω instead of $\{\omega\}$ to alleviate notation. Of course, since the set of acts is finite, neither the valuation function v_i nor the subjective probability measure p_i representing the preference ordering $\succcurlyeq_i$ are determined uniquely.

Throughout the paper, we assume that $\succcurlyeq_i$ is a linear ordering. Since the set of acts is finite, this is not an outrageous assumption. It implies that for any (p_i, v_i) representing $\succcurlyeq_i$, (i) v_i is injective and (ii) p_i is injective: for all $E, E' \subseteq \Omega$, $p_i(E) = p_i(E') \Rightarrow E = E'$. Because $p_i(\varnothing) = 0$, it follows from (ii) that $p_i(\omega) > 0$ for all $\omega \in \Omega$. We further assume, without loss of generality, that v_i is normalized: $\min_X v_i = 0 < \max_X v_i = 1$. We denote by $\mathcal{V}$ the set of normalized injective valuation functions v_i and by $\mathcal{P}$ the set of (necessarily positive) injective measures p_i.

A *(social choice) rule* is a function $\varphi : \mathcal{V}^N \times \mathcal{P}^N \to \mathcal{F}$. If $(v, p) \in \mathcal{V}^N \times \mathcal{P}^N$ and $\omega \in \Omega$, we denote by $\varphi(v, p; \omega)$ the outcome chosen by the act $\varphi(v, p)$ in state ω. We call $v = (v_1, ..., v_n) \in \mathcal{V}^N$ a *valuation profile* and $p = (p_1, ..., p_n) \in \mathcal{P}^N$ a *belief profile.* A rule assigns an act to each profile of valuations and beliefs. We emphasize that the chosen act is allowed to change when an agent's valuation function is replaced with another that induces the same ranking of the outcomes: no information about individual preferences over acts is *a priori* discarded. Note also that, in principle, our formulation allows a rule φ to choose different acts for profiles of valuations and beliefs (v, p) and (v', p') that represent the same profile of preferences $(\succcurlyeq_1, ..., \succcurlyeq_n)$. Of course, the requirement of strategy-proofness defined below will rule this out: in effect, a strategy-proof rule assigns an act to every profile of subjective expected utility preferences $(\succcurlyeq_1, ..., \succcurlyeq_n)$ over acts. With a slight abuse of terminology, we therefore call any $(v, p) \in \mathcal{V}^N \times \mathcal{P}^N$ a *preference profile.* We denote the set of social choice rules by $\Phi(N)$.

As usual, $v_{-i} \in \mathcal{V}^{N \setminus \{i\}}$ and $p_{-i} \in \mathcal{P}^{N \setminus \{i\}}$ denote the valuation and belief sub-profiles obtained by deleting v_i from v and p_i from p, respectively. A rule φ is *strategy-proof* if, for all $i \in N$, all

$(v_i, p_i), (v_i', p_i') \in \mathcal{V} \times \mathcal{P}$, and all $(v_{-i}, p_{-i}) \in \mathcal{V}^{N\setminus\{i\}} \times \mathcal{P}^{N\setminus\{i\}}$,

$$\sum_{\omega\in\Omega} p_i(\omega) v_i(\varphi(v, p; \omega)) \geq \sum_{\omega\in\Omega} p_i(\omega) v_i(\varphi((v_i', v_{-i}), (p_i', p_{-i}); \omega)).$$

This means that distorting one's preferences –by misrepresenting one's valuation function or one's beliefs– is never profitable. A weaker condition rules out misrepresentations of one's valuation function: φ is *misvaluation-proof* if, for all $i \in N$, all $v_i, v_i' \in \mathcal{V}$, all $v_{-i} \in \mathcal{V}^{N\setminus\{i\}}$, and all $p \in \mathcal{P}^N$,

$$\sum_{\omega\in\Omega} p_i(\omega) v_i(\varphi(v, p; \omega)) \geq \sum_{\omega\in\Omega} p_i(\omega) v_i(\varphi((v_i', v_{-i}), p; \omega)).$$

A rule φ is *ex-post efficient* if for all $(v, p) \in \mathcal{V}^N \times \mathcal{P}^N$ and all $\omega \in \Omega$, there is no $x \in X$ such that $v_i(x) > v_i(\varphi(v, p; \omega))$ for all $i \in N$. This requirement does not imply that the acts chosen by φ are (ex-ante Pareto) efficient at all preference profiles.

3 Results

Throughout this section we assume that $N = \{1, 2\}$. For any $v_i \in \mathcal{V}$, let $\tau(v_i)$ denote the unique maximizer (or *top*) of v_i in X. A rule $\varphi \in \Phi(\{1, 2\})$ is a *top selection* if $\varphi(v, p; \omega) \in \{\tau(v_1), \tau(v_2)\}$ for all $(v, p) \in \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}}$ and all $\omega \in \Omega$.

Proposition 1. *If a social choice rule $\varphi \in \Phi(\{1, 2\})$ is misvaluation-proof and ex-post efficient, then φ is a top selection.*

All proofs are in the Appendix.

Proposition 1 says that misvaluation-proofness and ex-post efficiency forbid choosing acts that select "compromise outcomes". Suppose that $X = \{a, b, c\}$ and consider a preference profile (v, p) such that $v_1(a) = v_2(c) = 1$, $v_1(b) = v_2(b) = .99$, $v_1(c) = v_2(a) = 0$, and $p_1(b) = p_2(b) = .99$. By Proposition 1, the natural compromise b cannot be picked in any state of the world at this profile. The only admissible form of compromise consists in allowing different agents to choose the final outcome in different states of the world. An obvious corollary is that no two-agent misvaluation-proof rule is (ex-ante Pareto) efficient.

Proposition 1 implies that, if a two-agent social choice rule φ is strategy-proof and ex-post efficient, the state space must be partitioned into an event where agent 1 dictates the outcome and a complementary event where agent 2 does: there exists a function $\sigma : \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}} \to 2^\Omega$ such that

$$\varphi(v, p; \omega) = \begin{cases} \tau(v_1) \text{ if } \omega \in \sigma(v, p), \\ \tau(v_2) \text{ if } \omega \in \complement\sigma(v, p), \end{cases}$$

where $\complement E$ denotes the complement of event E in Ω. The function σ is essentially unique: the event $\sigma(v, p)$ is uniquely defined at every profile (v, p) where $\tau(v_1) \neq \tau(v_2)$; it is indeterminate if and only if $\tau(v_1) = \tau(v_2)$. Proposition 1, however, is not a characterization result: it does not spell out the restrictions that strategy-proofness implies on the function σ associated with φ. Our first step in that direction is recorded in Proposition 2 below. It asserts that the valuation profile cannot be used to partition the state space: the chosen partition may only depend upon the belief profile p.[1] Call a function $s : \mathcal{P}^{\{1,2\}} \to 2^{\Omega}$ an *(Ω-) assignment rule.*

Proposition 2. *If a social choice rule $\varphi \in \Phi(\{1,2\})$ is strategy-proof and ex-post efficient, then there exists a unique Ω-assignment rule s such that, for all $(v,p) \in \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}}$ and all $\omega \in \Omega$,*

$$\varphi(v,p;\omega) = \begin{cases} \tau(v_1) \ \textit{if} \ \omega \in s(p), \\ \tau(v_2) \ \textit{if} \ \omega \in \complement s(p). \end{cases} \tag{1}$$

When (1) holds, we say that s is associated with, or generates, φ. We are now left with the task of identifying the restrictions that strategy-proofness of a social choice rule φ implies on its associated assignment rule s. With a slight abuse of terminology, let us call $s : \mathcal{P}^{\{1,2\}} \to 2^{\Omega}$ *strategy-proof* if misrepresenting one's belief never allows one to obtain an event that one judges more likely:

$$\begin{aligned} p_1(s(p_1,p_2)) &\geq p_1(s(p_1',p_2)) \text{ for all } p_1, p_1', p_2 \in \mathcal{P}, \\ p_2(\complement s(p_1,p_2)) &\geq p_2(\complement s(p_1,p_2')) \text{ for all } p_1, p_2, p_2' \in \mathcal{P}. \end{aligned}$$

If φ is strategy-proof, its associated assignment rule s must also be strategy-proof.[2] Conversely, if an Ω-assignment rule s is strategy-proof, it is clear that the two-agent top-selection rule φ it generates is strategy-proof.[3]

There are two basic types of strategy-proof Ω-assignment rules. The first type uses the beliefs of (at most) one agent. That agent receives from an exogenous menu the event that she reports to be the most likely; the complement of that event is then assigned to the other agent. Formally, an Ω-assignment rule s is *dictatorial* if there exists a nonempty collection $\mathcal{E}$ of non-nested events such that $s(p)$ maximizes p_1 over $\mathcal{E}$ for all $p \in \mathcal{P}^{\{1,2\}}$ (in which case agent 1 is called a dictator) or $\complement s(p)$ maximizes p_2 over $\{\complement E : E \in \mathcal{E}\}$ for all $p \in \mathcal{P}^{\{1,2\}}$ (in which case agent 2 is called a dictator). The range of such an assignment rule is $\mathcal{E}$; its size may be as large as the maximal number of

[1] More precisely: the assignment $\sigma(v,p)$ cannot vary with v at any profile (v,p) where $\tau(v_1) \neq \tau(v_2)$. If $\tau(v_1) = \tau(v_2)$, the assignment $\sigma(v,p)$ could be affected by a change in v that leaves $\tau(v_1)$, $\tau(v_2)$ unchanged. But since $\tau(v_1) = \tau(v_2)$, this is immaterial: σ can always be replaced with an assignment function that is constant in v and generates the same social choice rule φ.

[2] The problem of describing the strategy-proof assignment rules is mathematically equivalent to the problem of describing the stratgy-proof procedures for allocating strictly desirable indivisible objects to agents with additive preferences over sets of such objects. Some such procedures have been studied in the literature (see, e.g., Pápai (2007)) but no general characterization is known.

[3] This converse statement does not extend to more than two agents.

non-nested events.[4] As usual, dictatorship is understood to hold on the range of the rule. If $\mathcal{E}$ contains a single event, then s is constant and both agents are (trivial) dictators.

The second basic type of strategy-proof Ω-assignment rule uses the beliefs of *both* agents. Two non-nested events are exogenously selected from the state space. The first event is assigned to agent 1 and its complement is assigned to agent 2 *unless* agent 1 reports that she finds the second event more likely than the first *and* agent 2 reports the opposite belief. In that case the second event is assigned to agent 1 and its complement is assigned to agent 2. We call such rules consensual assignment rules. Formally, s is *consensual (with default B)* if there exist two non-nested events $A, B \subseteq \Omega$ such that for all $(p_1, p_2) \in \mathcal{P}^{\{1,2\}}$ we have $s(p_1, p_2) = A$ if $p_1(A) > p_1(B)$ and $p_2(\complement A) > p_2(\complement B)$, and $s(p_1, p_2) = B$ otherwise. The range of a consensual Ω-assignment rule is of size two.

Our next result is a complete characterization of the two-agent strategy-proof and ex-post efficient social choice rules for the particular case where the state space is of size two.

Proposition 3. *Suppose $\Omega = \{\omega_1, \omega_2\}$. A social choice rule $\varphi \in \Phi(\{1,2\})$ is strategy-proof and ex-post efficient if and only if there exists a dictatorial or consensual Ω-assignment rule $s : \mathcal{P}^{\{1,2\}} \to 2^\Omega$ such that, for all $(v,p) \in \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}}$ and all $\omega \in \Omega$,*

$$\varphi(v, p; \omega) = \begin{cases} \tau(v_1) \text{ if } \omega \in s(p), \\ \tau(v_2) \text{ if } \omega \in \complement s(p). \end{cases}$$

The important point is that the mechanism designer is allowed to use the agents' beliefs to determine in which state their top outcome is selected. This leads to possible Pareto improvements with respect to the less sophisticated rules where the states in which each agent's top outcome is selected are fixed exogenously. For instance, consider the social choice rule φ^s generated by the consensual assignment rule $s(p_1, p_2) = \{\omega_2\}$ if $p_1(\omega_2) > p_1(\omega_1)$ and $p_2(\omega_2) < p_2(\omega_1)$, and $s(p_1, p_2) = \{\omega_1\}$ otherwise: an agent's top outcome is selected in the event bearing her name unless both agents prefer to swap their "endowment events". This social choice rule Pareto-dominates the rule $\varphi(v, p; \omega_i) = t(v_i)$ where an agent's top outcome is always selected in the event bearing her name.

Let us now return to state spaces of arbitrary size. Dictatorial Ω-assignment rules are fairly flexible (in the sense that their range may be large) but they are exceedingly asymmetric. Consensual rules are more symmetric (as they use both agents' beliefs) but they are extremely rigid (since their range contains only two events). When there are more than two states, these two basic types of rules can be combined to produce more balanced procedures. For each event Ω_t in an exogenously specified partition of Ω, the mechanism designer may use a different dictatorial or consensual Ω_t-assignment rule to assign the states belonging to that event. Because subjective

[4] By a theorem of Sperner (1928), this number is $\binom{K}{\lfloor K/2 \rfloor}$.

expected utility preferences are separable, the resulting social choice rule will be strategy-proof.

Formally, fix a partition $\{\Omega_1, ..., \Omega_T\}$ of Ω. For $t = 1, ..., T$, let $\mathcal{P}_t$ denote the set of injective probability measures on 2^{Ω_t}. If $p \in \mathcal{P}^{\{1,2\}}$ and $i \in \{1, 2\}$, define $p_{i,t} \in \mathcal{P}_t$ by

$$p_{i,t}(E) = \frac{p_i(E)}{p_i(\Omega_t)} \text{ for all } E \in 2^{\Omega_t}$$

and $p_t = (p_{1,t}, p_{2,t})$: this is the profile of conditional probability measures on 2^{Ω_t} generated by p. An Ω_t-assignment rule is a function $s_t : (\mathcal{P}_t)^{\{1,2\}} \to 2^{\Omega_t}$. Extending our earlier terminology, we call s_t *dictatorial* if there exists a nonempty collection $\mathcal{E}_t$ of non-nested subsets of Ω_t such that $s_t(p_t)$ maximizes $p_{1,t}$ over $\mathcal{E}_t$ for all $p_t \in (\mathcal{P}_t)^{\{1,2\}}$ or $\complement_t s_t(p_t)$ maximizes $p_{2,t}$ over $\{\complement_t E : E \in \mathcal{E}_t\}$ for all $p_t \in (\mathcal{P}_t)^{\{1,2\}}$ (where $\complement_t E := \Omega_t \setminus E$). We call s_t *consensual (with default B)* if there exist two non-nested sets $A, B \subseteq \Omega_t$ such that for all $p_t \in (\mathcal{P}_t)^{\{1,2\}}$ we have $s_t(p_t) = A$ if $p_{1,t}(A) > p_{1,t}(B)$ and $p_{2,t}(\complement_t A) > p_{2,t}(\complement_t B)$, and $s_t(p_t) = B$ otherwise.

Proposition 4. *Let $\{\Omega_1, ..., \Omega_T\}$ be a partition of Ω. For each $t = 1, ..., T$, let $s_t : (\mathcal{P}_t)^{\{1,2\}} \to 2^{\Omega_t}$ be a dictatorial or consensual Ω_t-assignment rule. For all $(v, p) \in \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}}$ and all $\omega \in \Omega$, let*

$$\varphi(v, p; \omega) = \begin{cases} \tau(v_1) \text{ if } \omega \in \cup_{t=1}^{T} s_t(p_t), \\ \tau(v_2) \text{ otherwise.} \end{cases}$$

The social choice rule $\varphi \in \Phi(\{1, 2\})$ so defined is strategy-proof and ex-post efficient.

The proof follows directly from Proposition 3 and the fact that subjective expected utility preferences are additively separable; we therefore omit it.

To illustrate the richness of the class identified in Proposition 4 and the flexibility of some of the rules it contains, we describe two examples for the case of 4 states of the world. It will be convenient to write p_{ik} instead of $p_i(\omega_k)$ and $\varphi(v, p; \omega) = i$ instead of $\varphi(v, p; \omega) = \tau(v_i)$.

Example 1. The exogenous partition of the state space is $\{\Omega_1, \Omega_2\} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. The Ω_1-assignment rule s_1 has agent 1 dictate over $\mathcal{E}_1 = \{\{\omega_1\}, \{\omega_2\}\}$ and the Ω_2-assignment rule s_2 has agent 2 dictate over $\mathcal{E}_2 = \{\{\omega_3\}, \{\omega_4\}\}$. The resulting social choice rule φ is shown in the table below.

	$p_{11} > p_{12}$	$p_{11} < p_{12}$
$p_{23} < p_{24}$	$(1, 2, 1, 2)$	$(2, 1, 1, 2)$
$p_{23} > p_{24}$	$(1, 2, 2, 1)$	$(2, 1, 2, 1)$

The first cell means that $(\varphi(v, p; \omega_1), \varphi(v, p; \omega_2), \varphi(v, p; \omega_3), \varphi(v, p; \omega_4)) = (\tau(v_1), \tau(v_2), \tau(v_1), \tau(v_2))$ whenever $p_1(\omega_1) > p_1(\omega_2)$ and $p_2(\omega_3) < p_2(\omega_4)$.

Example 2. The exogenous partition of the state space is $\{\Omega_1, \Omega_2\} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. The Ω_1-assignment rule s_1 is consensual with default $\{\omega_1\}$, the Ω_2-assignment rule s_2 has agent 2 dictate over $\mathcal{E}_2 = \{\{\omega_3\}, \{\omega_4\}\}$. The resulting social choice rule φ is shown below; its range is the same as in the previous example but the rule uses more information.

	$p_{11} > p_{12}$	$p_{11} < p_{12}$
$p_{21} < p_{22}$ and $p_{23} < p_{24}$	$(1,2,1,2)$	$(1,2,1,2)$
$p_{21} < p_{22}$ and $p_{23} > p_{24}$	$(1,2,2,1)$	$(1,2,2,1)$
$p_{21} > p_{22}$ and $p_{23} < p_{24}$	$(1,2,1,2)$	$(2,1,1,2)$
$p_{21} > p_{22}$ and $p_{23} > p_{24}$	$(1,2,2,1)$	$(2,1,2,1)$

4 Concluding comments

Our results are obviously a very small step towards a complete characterization; we hope they will stimulate more research on what appears to be a difficult problem.

In the two-agent case, we conjecture that all ex-post efficient strategy-proof rules are of the type described in Proposition 4.

The n-agent case is wide open. We do not know whether Proposition 1 generalizes. An n-agent ex-post efficient and strategy-proof rule can be shown to be a top selection in the particular case where it is restricted to use only the agents' beliefs and their preference orderings over outcomes: this is the analog of Gibbard's (1977) assumption in the setting of lotteries. Rules of the variety described in Proposition 4 can be defined; they are strategy-proof if, for each component of the underlying partition of the state space, the corresponding assignment rule involves only two agents.

5 Appendix

5.1 Proof of Proposition 1

Fix a misvaluation-proof and ex-post efficient rule $\varphi \in \Phi(\{1,2\})$ and a belief profile $p = (p_1, p_2) \in \mathcal{P}^{\{1,2\}}$. For any $w \in \mathcal{V}^{\{1,2\}}$ and $A \subseteq X$, define $\Omega_A(w) = \{\omega \in \Omega \mid \varphi(w,p;\omega) \in A\}$. This is the set of states for which the act chosen at the profile (w,p) picks an outcome in A. We write $\Omega_a(w)$, $\Omega_{ab}(w)$, $\Omega_{abc}(w)$ instead of $\Omega_{\{a\}}(w)$, $\Omega_{\{a,b\}}(w)$, $\Omega_{\{a,b,c\}}(w)$.

Lemma 1. *Let $a, b, c \in X$ be three distinct outcomes.*

(i) If $(v_1, v_2), (w_1, w_2) \in \mathcal{V}^{\{1,2\}}$ are such that

$$\begin{aligned} 1 &= v_1(a),\ 1 = v_2(b) > v_2(a) > v_2(x) \text{ for all } x \in X \setminus \{a,b\}, \\ 1 &= w_1(c),\ 1 = w_2(b) > w_2(c) > w_2(x) \text{ for all } x \in X \setminus \{b,c\}, \end{aligned}$$

then $\Omega_b(v_1, v_2) = \Omega_b(w_1, w_2)$.

(ii) If $(v_1, v_2), (w_1, w_2) \in \mathcal{V}^{\{1,2\}}$ *are such that*

$$\begin{aligned} 1 &= v_2(a),\ 1 = v_1(b) > v_1(a) > v_1(x) \text{ for all } x \in X \setminus \{a,b\}, \\ 1 &= w_2(c),\ 1 = w_1(b) > w_1(c) > w_1(x) \text{ for all } x \in X \setminus \{b,c\}, \end{aligned}$$

then $\Omega_b(v_1, v_2) = \Omega_b(w_1, w_2)$.

Proof. We only prove statement (i); up to a permutation of the agents, the proof of statement (ii) is identical. Fix $(v_1, v_2), (w_1, w_2) \in \mathcal{V}^{\{1,2\}}$ satisfying the premises of statement (i). If $0 < \varepsilon_1, \varepsilon_2, \delta_1, \delta_2 < 1$, construct $(v_1^{\varepsilon_1}, v_2^{\varepsilon_2}), (w_1^{\delta_1}, w_2^{\delta_2}) \in \mathcal{V}^{\{1,2\}}$ such that

$$\begin{aligned} 1 &= v_1^{\varepsilon_1}(a) > 1 - \varepsilon_1 = v_1^{\varepsilon_1}(c) > v_1^{\varepsilon_1}(x) > 0 = v_1^{\varepsilon_1}(b) \text{ for all } x \in X \setminus \{a,b,c\}, \\ 1 &= v_2^{\varepsilon_2}(b) > \varepsilon_2 = v_2^{\varepsilon_2}(a) > v_2^{\varepsilon_2}(x) \text{ for all } x \in X \setminus \{a,b\}, \\ 1 &= w_1^{\delta_1}(c) > 1 - \delta_1 = w_1^{\delta_1}(a) > w_1^{\delta_1}(x) > 0 = w_1^{\delta_1}(b) \text{ for all } x \in X \setminus \{a,b,c\}, \\ 1 &= w_2^{\delta_2}(b) > \delta_2 = w_2^{\delta_2}(c) > w_2^{\delta_2}(x) \text{ for all } x \in X \setminus \{b,c\}. \end{aligned}$$

Step 1. By ex-post efficiency,

$$\Omega_{ab}(v_1, v_2) = \Omega_{ab}(v_1^{\varepsilon_1}, v_2^{\varepsilon_2}) = \Omega_{bc}(w_1, w_2) = \Omega_{bc}(w_1^{\delta_1}, w_2^{\delta_2}) = \Omega \tag{2}$$

and

$$\Omega_{abc}(w_1^{\delta_1}, v_2^{\varepsilon_2}) = \Omega. \tag{3}$$

for all $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 \in (0,1)$.

Step 2. By misvaluation-proofness and (2),

$$\begin{aligned} \Omega_x(v_1, v_2) &= \Omega_x(v_1^{\varepsilon_1}, v_2^{\varepsilon_2}) \text{ for } x = a, b \text{ and all } \varepsilon_1, \varepsilon_2 \in (0,1), \\ \Omega_x(w_1, w_2) &= \Omega_x(w_1^{\delta_1}, w_2^{\delta_2}) \text{ for } x = b, c \text{ and all } \delta_1, \delta_2 \in (0,1). \end{aligned}$$

Step 3. Because Ω is finite and p_1, p_2 are injective, we have

$$\min_{\substack{E, E' \subseteq \Omega: \\ E \neq E'}} |p_i(E) - p_i(E')| =: \alpha_i > 0 \text{ for } i = 1, 2. \tag{4}$$

We claim that

$$\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2}) = \Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}) \text{ whenever } 0 < \delta_1 < \alpha_1, \tag{5}$$

and

$$\Omega_b(w_1^{\alpha_1}, v_2^{\varepsilon_2}) = \Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}) \text{ whenever } 0 < \varepsilon_2 < \alpha_2. \tag{6}$$

Suppose not.

Case 1: Statement (5) is false.

Since p_1 is injective, (i) there exists $\delta_1 \in (0, \alpha_1)$ such that $p_1(\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2})) > p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))$ or (ii) there exists $\delta_1 \in (0, \alpha_1)$ such that $p_1(\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2})) < p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))$.

Suppose (i) holds. Fix agent 2's valuation at $v_2^{\alpha_2}$. Let agent 1's true valuation be $w_1^{\delta_1}$. Reporting truthfully gives a utility not higher than $1 - p_1(\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2}))$ while reporting $w_1^{\alpha_1}$ yields at least $[1 - p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))]\,(1 - \delta_1)$. The utility gain from misrepresenting is at least

$$
\begin{aligned}
& [1 - p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))]\,(1 - \delta_1) - \left[1 - p_1(\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2}))\right] \\
= \;& \left[p_1(\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2})) - p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))\right] - [1 - p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))]\,\delta_1 \\
\geq \;& p_1(\Omega_b(w_1^{\delta_1}, v_2^{\alpha_2})) - p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) - \delta_1 \\
\geq \;& \alpha_1 - \delta_1 \\
> \;& 0,
\end{aligned}
$$

contradicting misvaluation-proofness.

If (ii) holds, a similar contradiction is obtained when agent 1 has valuation $w_1^{\alpha_1}$ and reports $w_1^{\delta_1}$.

Case 2: Statement (6) is false.

Interchanging the roles of the agents and using the fact that p_2 is injective, a completely symmetric argument delivers again a contradiction to misvaluation-proofness.

Step 4. We claim that $\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}) = \Omega_b(v_1, v_2)$.

Let agent 2's reported valuation be $v_2^{\alpha_2}$; this is fixed throughout the proof of Step 4.

Suppose agent 1's true valuation is $v_1^{\varepsilon_1}$, where $0 < \varepsilon_1 < 1$. By (2) in Step 1,

$$\Omega_{ab}(v_1^{\varepsilon_1}, v_2^{\alpha_2}) = \Omega$$

and by Step 2,

$$\Omega_x(v_1^{\varepsilon_1}, v_2^{\alpha_2}) = \Omega_x(v_1, v_2) \text{ for } x = a, b.$$

Truthfully reporting $v_1^{\varepsilon_1}$ yields the utility $p_1(\Omega_a(v_1^{\varepsilon_1}, v_2^{\alpha_2})) = p_1(\Omega_a(v_1, v_2))$. Reporting $w_1^{\alpha_1}$ yields $p_1(\Omega_a(w_1^{\alpha_1}, v_2^{\alpha_2}))+ p_1(\Omega_c(w_1^{\alpha_1}, v_2^{\alpha_2}))(1 - \varepsilon_1)$. Misvaluation-proofness thus requires $p_1(\Omega_a(v_1, v_2)) \geq p_1(\Omega_a(w_1^{\alpha_1}, v_2^{\alpha_2}))+ p_1(\Omega_c(w_1^{\alpha_1}, v_2^{\alpha_2}))(1 - \varepsilon_1)$. None of the three events in this inequality changes with ε_1. Therefore, letting $\varepsilon_1 \to 0$ yields $p_1(\Omega_a(v_1, v_2)) \geq p_1(\Omega_{ac}(w_1^{\alpha_1}, v_2^{\alpha_2}))$. Since by Step 1 $\Omega_{ab}(v_1, v_2) = \Omega = \Omega_{abc}(w_1^{\alpha_1}, v_2^{\alpha_2})$, we get

$$p_1(\Omega_b(v_1, v_2)) \leq p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})). \tag{7}$$

Next, suppose agent 1's true valuation is $w_1^{\alpha_1}$. Truth-telling yields a utility of at most $1 -$

$p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}))$. Reporting $v_1^{\varepsilon_1}$, $0 < \varepsilon_1 < 1$, gives $(1-\alpha_1)p_1(\Omega_a(v_1^{\varepsilon_1}, v_2^{\alpha_2})) = (1-\alpha_1)\left[1 - p_1(\Omega_b(v_1^{\varepsilon_1}, v_2^{\alpha_2}))\right] = (1-\alpha_1)\left[1 - p_1(\Omega_b(v_1, v_2))\right]$. Misvaluation-proofness therefore requires $1 - p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) \geq (1-\alpha_1)\left[1 - p_1(\Omega_b(v_1, v_2))\right]$, that is,

$$p_1(\Omega_b(v_1, v_2)) \geq \frac{p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) - \alpha_1}{1-\alpha_1} > p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) - \alpha_1. \tag{8}$$

Combining inequalities (7) and (8) gives

$$p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) - \alpha_1 < p_1(\Omega_b(v_1, v_2)) \leq p_1(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})).$$

By definition of α_1, this means that $\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}) = \Omega_b(v_1, v_2)$.

Step 5. We claim that $\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}) = \Omega_b(w_1, w_2)$.

Let agent 1's reported valuation be fixed at $w_1^{\alpha_1}$.

Suppose agent 2's true valuation is $w_2^{\delta_2}$, where $0 < \delta_2 < 1$. By Step 2,

$$\Omega_x(w_1^{\alpha_1}, w_2^{\delta_2}) = \Omega_x(w_1, w_2) \text{ for } x = b, c, \tag{9}$$

hence by Step 1,

$$\Omega_{bc}(w_1^{\alpha_1}, w_2^{\delta_2}) = \Omega. \tag{10}$$

Truthfully reporting $w_2^{\delta_2}$ yields the utility $p_2(\Omega_b(w_1^{\alpha_1}, w_2^{\delta_2})) + p_2(\Omega_c(w_1^{\alpha_1}, w_2^{\delta_2}))\delta_2 = p_2(\Omega_b(w_1, w_2)) + p_2(\Omega_c(w_1, w_2))\delta_2$ because of (9). Reporting $v_2^{\alpha_2}$ yields at least $p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) + p_2(\Omega_c(w_1^{\alpha_1}, v_2^{\alpha_2}))\delta_2$. Misvaluation-proofness thus requires $p_2(\Omega_b(w_1, w_2)) + p_2(\Omega_c(w_1, w_2))\delta_2 \geq p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) + p_2(\Omega_c(w_1^{\alpha_1}, v_2^{\alpha_2}))\delta_2$. None of the four events in this inequality changes with δ_2. Therefore, letting $\delta_2 \to 0$ yields

$$p_2(\Omega_b(w_1, w_2)) \geq p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})). \tag{11}$$

Next, suppose agent 2's true valuation is $v_2^{\varepsilon_2}$. Truth-telling yields a utility of at most $p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) + [1 - p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})]\alpha_2$. Reporting $w_2^{\alpha_2}$ gives at least $p_2(\Omega_b(w_1^{\alpha_1}, w_2^{\alpha_2})) = p_2(\Omega_b(w_1, w_2))$, where the equality holds by Step 2. Misvaluation-proofness thus requires $p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) + [1 - p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})]\alpha_2 \geq p_2(\Omega_b(w_1, w_2))$. Because none of the three events in this inequality varies with ε_2, letting $\varepsilon_2 \to 0$ gives

$$p_2(\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2})) \geq p_2(\Omega_b(w_1, w_2)). \tag{12}$$

Inequalities (11), (12) and the fact that p_2 is injective give $\Omega_b(w_1^{\alpha_1}, v_2^{\alpha_2}) = \Omega_b(w_1, w_2)$.

Steps 4 and 5 together establish that $\Omega_b(v_1, v_2) = \Omega_b(w_1, w_2)$.■

Lemma 2. *Let $a, b, c \in X$ be three distinct outcomes. If $(v_1, v_2), (w_1, w_2) \in \mathcal{V}^{\{1,2\}}$ are such that*

$$\begin{aligned} 1 &= w_1(c),\ 1 = v_1(a) > v_1(c) > v_1(x) \text{ for all } x \in X \setminus \{a, c\}, \\ 1 &= w_2(c),\ 1 = v_2(b) > v_2(c) > v_2(x) \text{ for all } x \in X \setminus \{b, c\}, \end{aligned}$$

then $\Omega_c(v_1, w_2) \cap \Omega_c(w_1, v_2) = \varnothing$.

Proof. Let $(v_1, v_2), (w_1, w_2) \in \mathcal{V}^{\{1,2\}}$ satisfy the assumptions of the lemma. Let $(\widetilde{v}_1, \widetilde{v}_2) \in \mathcal{V}^{\{1,2\}}$ be such that

$$\begin{aligned} 1 &= \widetilde{v}_1(a) > \widetilde{v}_1(b) > \widetilde{v}_1(x) \text{ for all } x \in X \setminus \{a, b\}, \\ 1 &= \widetilde{v}_2(b) > \widetilde{v}_2(a) > \widetilde{v}_2(x) \text{ for all } x \in X \setminus \{a, b\}. \end{aligned}$$

By statement (i) in Lemma 1, $\Omega_b(w_1, v_2) = \Omega_b(\widetilde{v}_1, \widetilde{v}_2)$. Since by ex-post efficiency $\Omega_{bc}(w_1, v_2) = \Omega_{ab}(\widetilde{v}_1, \widetilde{v}_2) = \Omega$, it follows that

$$\Omega_c(w_1, v_2) = \Omega_a(\widetilde{v}_1, \widetilde{v}_2).$$

By statement (ii) in Lemma 1, $\Omega_a(v_1, w_2) = \Omega_a(\widetilde{v}_1, \widetilde{v}_2)$. Since by ex-post efficiency $\Omega_{ac}(v_1, w_2) = \Omega_{ab}(\widetilde{v}_1, \widetilde{v}_2) = \Omega$, it follows that

$$\Omega_c(v_1, w_2) = \Omega_b(\widetilde{v}_1, \widetilde{v}_2),$$

and therefore $\Omega_c(v_1, w_2) \cap \Omega_c(w_1, v_2) = \varnothing$.■

Proof of Proposition 1. For any $v = (v_1, v_2) \in \mathcal{V}^{\{1,2\}}$ such that $\tau(v_1) = \tau(v_2)$, ex-post efficiency directly implies that $\varphi(v, p; \omega) = \tau(v_1) = \tau(v_2)$ for all $\omega \in \Omega$. From now on, fix two arbitrary distinct outcomes a, b, let $\mathcal{V}(a) = \{v_1 \in \mathcal{V} \mid \tau(v_1) = a\}$ and $\mathcal{V}(b) = \{v_2 \in \mathcal{V} \mid \tau(v_2) = b\}$. We will show that $\Omega_{ab}(v) = \Omega$ for all $v \in \mathcal{V}(a) \times \mathcal{V}(b)$.

Fix $c \in X \setminus \{a, b\}$. Let $\mathcal{V}^*(acb) \subseteq \mathcal{V}(a)$ be the set of all valuation functions v_1 such that

$$1 = v_1(a) > v_1(c) > 1 - \alpha_1 > v_1(x) > 0 = v_1(b) \text{ for all } x \in X \setminus \{a, b, c\}, \tag{13}$$

and let $\mathcal{V}^*(bca) \subseteq \mathcal{V}(b)$ be the set of all valuation functions v_2 such that

$$1 = v_2(b) > v_2(c) > 1 - \alpha_2 > v_2(x) > 0 = v_1(a) \text{ for all } x \in X \setminus \{a, b, c\}. \tag{14}$$

Step 1. We show that there exist Ω_a^*, Ω_b^*, $\Omega_c^* \subseteq \Omega$ such that

$$\Omega_x(v) = \Omega_x^* \text{ for all } v \in \mathcal{V}^*(acb) \times \mathcal{V}^*(bca) \text{ and } x = a, b, c. \tag{15}$$

Suppose not. Without loss of generality, assume $\Omega_x(v_1, v_2) \neq \Omega_x(v_1', v_2)$ for some $v_1, v_1' \in$

$\mathcal{V}^*(acb)$, some $v_2 \in \mathcal{V}^*(bca)$, and some $x \in \{a, b, c\}$; the other possible cases are similar. Because v_1 and v_1' rank a, b, c in the same order and $\Omega_{abc}(v_1, v_2) = \Omega_{abc}(v_1', v_2) = \Omega$ by ex-post efficiency, misvaluation-proofness implies that $\Omega_x(v_1, v_2) \neq \Omega_x(v_1', v_2)$ for all $x \in \{a, b, c\}$.

Let $v_1(c) = 1 - \delta_1$, $v_1'(c) = 1 - \delta_1'$, where by assumption $0 < \delta_1, \delta_1' < \alpha_1$. Suppose, without loss of generality, that $p_1(\Omega_b(v_1, v_2)) > p_1(\Omega_b(v_1', v_2))$ (the case $p_1(\Omega_b(v_1, v_2)) < p_1(\Omega_b(v_1', v_2)$ is treated symmetrically). By definition of α_1, $p_1(\Omega_b(v_1, v_2)) - p_1(\Omega_b(v_1', v_2)) \geq \alpha_1$. Fix agent 2's valuation at v_2. If agent 1's valuation is v_1, reporting truthfully gives a utility not higher than $1 - p_1(\Omega_b(v_1, v_2))$ while reporting v_1' yields at least $[1 - p_1(\Omega_b(v_1', v_2))]\,(1 - \delta_1)$. The utility gain from misrepresenting is at least

$$
\begin{aligned}
& [1 - p_1(\Omega_b(v_1', v_2))]\,(1 - \delta_1) - [1 - p_1(\Omega_b(v_1, v_2))] \\
= \; & [p_1(\Omega_b(v_1, v_2)) - p_1(\Omega_b(v_1', v_2))] - [1 - p_1(\Omega_b(v_1', v_2))]\,\delta_1 \\
\geq \; & p_1(\Omega_b(v_1, v_2)) - p_1(\Omega_b(v_1', v_2)) - \delta_1 \\
\geq \; & \alpha_1 - \delta_1 \\
> \; & 0,
\end{aligned}
$$

contradicting misvaluation-proofness. This proves Step 1.

Step 2. By ex-post efficiency, $\Omega^*_{abc} = \Omega$. We show that $\Omega^*_{ab} = \Omega$, that is, $\Omega_{ab}(v) = \Omega$ for all $v \in \mathcal{V}^*(acb) \times \mathcal{V}^*(bca)$.

Let $v = (v_1, v_2) \in \mathcal{V}^*(acb) \times \mathcal{V}^*(bca)$. Let $v_1(c) = 1 - \delta_1$ and $v_2(c) = 1 - \delta_2$, where by assumption $0 < \delta_1 < \alpha_1$ and $0 < \delta_2 < \alpha_2$.

Step 2.1. Let $w_1 \in \mathcal{V}$ be such that $1 = w_1(c) > w_1(a) > w_1(x)$ for all $x \in X \setminus \{a, c\}$. We claim that

$$\Omega_c(w_1, v_2) = \Omega^*_{ac}. \tag{16}$$

Fix agent 2's valuation at v_2. For $0 < \varepsilon < 1$, let $w_1^\varepsilon \in \mathcal{V}$ be such that

$$1 = w_1^\varepsilon(c) > 1 - \varepsilon = w_1^\varepsilon(a) > w_1^\varepsilon(x) > 0 = w_1^\varepsilon(b) \text{ for all } x \in X \setminus \{a, b, c\} .$$

By ex-post efficiency, $\Omega_{bc}(w_1^\varepsilon, v_2) = \Omega_{bc}(w_1, v_2) = \Omega$ whenever $0 < \varepsilon < 1$. Using these equalities, misvaluation-proofness directly implies that $\Omega_x(w_1^\varepsilon, v_2) = \Omega_x(w_1, v_2)$ for $x = b, c$.

If agent 1's valuation is w_1^ε, preventing her from reporting v_1 requires $p_1(\Omega_c(w_1, v_2)) \geq p_1(\Omega_c(v_1, v_2)) + p_1(\Omega_a(v_1, v_2))(1 - \varepsilon) = p_1(\Omega^*_c) + p_1(\Omega^*_a)(1 - \varepsilon)$. Letting $\varepsilon \to 0$ gives

$$p_1(\Omega_c(w_1, v_2)) \geq p_1(\Omega^*_{ac}). \tag{17}$$

If agent 1's valuation is v_1, truth-telling yields a utility of $p_1(\Omega^*_a) + p_1(\Omega^*_c)(1 - \delta_1)$ while reporting

w_1 gives at least $p_1(\Omega_c(w_1, v_2))(1 - \delta_1)$. Applying misvaluation-proofness and letting $\delta_1 \to 0$ gives

$$p_1(\Omega^*_{ac}) \geq p_1(\Omega_c(w_1, v_2)). \tag{18}$$

Since p_1 is injective, (17) and (18) imply (16).

Step 2.2. Let $w_2 \in \mathcal{V}$ be such that $1 = w_2(c) > w_2(b) > w_2(x)$ for all $x \in X \setminus \{b, c\}$. We claim that

$$\Omega_c(v_1, w_2) = \Omega^*_{bc}. \tag{19}$$

Up to a permutation of players 1 and 2 and a permutation of outcomes a and b, the proof is identical to that of (16) and therefore omitted.

Step 2.3. From (16) and (19) we obtain $\Omega^*_c \subseteq \Omega^*_{ac} \cap \Omega^*_{bc} = \Omega_c(w_1, v_2) \cap \Omega_c(v_1, w_2)$. By Lemma 2, $\Omega_c(w_1, v_2) \cap \Omega_c(v_1, w_2) = \varnothing$. Therefore $\Omega^*_c = \varnothing$, hence $\Omega^*_{ab} = \Omega$.

Step 3. We show that $\Omega_{ab}(v) = \Omega$ for all $v \in \mathcal{V}^*(acb) \times \mathcal{V}(b)$ and for all $v \in \mathcal{V}(a) \times \mathcal{V}^*(bca)$.

Let $v = (v_1, v_2) \in \mathcal{V}^*(acb) \times \mathcal{V}(b)$; the case $v \in \mathcal{V}(a) \times \mathcal{V}^*(bca)$ is similar. Let $v_1(c) = 1 - \delta_1$, $0 < \delta_1 < \alpha_1$. Let C denote the set of outcomes other than a and b which are ex-post efficient at v. If $C = \varnothing$, ex-post efficiency directly implies $\Omega_{ab}(v) = \Omega$. Assume from now on that $C \neq \varnothing$. Define $\max\{v_2(x) \mid x \in C\} = 1 - k$ and note that $v_2(a) < 1 - k < 1$.

For $0 < \delta_2 < \alpha_2$, let $u_2^{\delta_2} \in \mathcal{V}$ be such that

$$1 = u_2^{\delta_2}(b) > u_2^{\delta_2}(c) \geq u_2^{\delta_2}(x) \geq 1 - \delta_2 > 0 = u_2^{\delta_2}(a) \text{ for all } x \in C.$$

Since $(v_1, u_2^{\delta_2}) \in \mathcal{V}^*(acb) \times \mathcal{V}^*(bca)$, Steps 1 and 2 imply $\Omega_x(v_1, u_2^{\delta_2}) = \Omega^*_x$ for $x = a, b$ and $\Omega_{ab}(v_1, u_2^{\delta_2}) = \Omega^*_{ab} = \Omega$.

Fix agent 1's valuation at v_1. If agent 2's valuation is $u_2^{\delta_2}$, truth-telling yields $p_2(\Omega^*_b)$ while reporting v_2 gives at least $p_2(\Omega_b(v_1, v_2))+ p_2(\Omega_C(v_1, v_2))(1 - \delta_2)$. Applying misvaluation-proofness and letting $\delta_2 \to 0$ gives

$$p_2(\Omega^*_b) \geq p_2(\Omega_{b\cup C}(v_1, v_2)). \tag{20}$$

If agent 2's valuation is v_2, preventing her from reporting $u_2^{\delta_2}$, $0 < \delta_2 < \alpha_2$, requires

$$\begin{aligned} & p_2(\Omega_b(v_1, v_2)) + p_2(\Omega_C(v_1, v_2))(1 - k) + [1 - p_2(\Omega_{b\cup C}(v_1, v_2))]\, v_2(a) \\ \geq\ & p_2(\Omega^*_b) + [1 - p_2(\Omega^*_b)]\, v_2(a). \end{aligned} \tag{21}$$

Because $v_2(a) < 1-k$, (21) implies $p_2(\Omega_{b\cup C}(v_1, v_2)) \geq p_2(\Omega^*_b)$. Hence, from (20), $p_2(\Omega_{b\cup C}(v_1, v_2)) = p_2(\Omega^*_b)$. Since $1 - k < 1$, (21) then implies $p_2(\Omega_C(v_1, v_2)) = 0$. Since p_2 is injective we get $\Omega_C(v_1, v_2) = \varnothing$, hence by ex-post efficiency $\Omega_{ab}(v_1, v_2) = \Omega$.

Step 4. We show that $\Omega_{ab}(v) = \Omega$ for all $v \in \mathcal{V}(a) \times \mathcal{V}(b)$.

Simply repeat the argument in Step 3 with the set $\mathcal{V}(a) \times \mathcal{V}(b)$ instead of $\mathcal{V}^*(acb) \times \mathcal{V}(b)$, the set $\mathcal{V}^*(acb) \times \mathcal{V}(b)$ instead of $\mathcal{V}^*(acb) \times \mathcal{V}^*(bca)$, and the word "Step 3" instead of "Step 2".■

5.2 Proof of Proposition 2

Let $\varphi \in \Phi(\{1,2\})$ be a strategy-proof and ex-post efficient rule and let $\sigma : \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}} \to 2^{\Omega}$ be an assignment rule associated with φ. We claim that σ is generically constant in v in the sense that

$$\sigma((v_i, v_{-i}), p) = \sigma((v_i', v_{-i}), p)$$

for all $p \in \mathcal{P}^{\{1,2\}}$, $i \in \{1,2\}$, $v_{-i} \in \mathcal{V}$, and $v_i, v_i' \in \mathcal{V}$ such that $\tau(v_i) \neq \tau(v_{-i})$ and $\tau(v_i') \neq \tau(v_{-i})$.

Fix $p \in \mathcal{P}^{\{1,2\}}$ and $i \in \{1,2\}$, say $i = 1$. Fix $v_2 \in \mathcal{V}$ and let $v_1, v_1' \in \mathcal{V}$ be such that $\tau(v_1) \neq \tau(v_2)$ and $\tau(v_1') \neq \tau(v_2)$. Since p and v_2 are fixed throughout the proof, we drop them from our notation. Thus we must prove that

$$\sigma(v_1) = \sigma(v_1'). \tag{22}$$

Step 1. We prove that (22) holds if $\tau(v_1) = \tau(v_1')$.

Let $\tau(v_1) = \tau(v_1') = a$ and $\tau(v_2) = b$. By our normalization convention, $v_1(a) = v_1'(a) = 1 > v_1(b), v_1'(b)$.

To prevent agent 1 from reporting v_1' when her true valuation is v_1, we must have

$$p_1(\sigma(v_1)) + [1 - p_1(\sigma(v_1))]\, v_1(b) \geq p_1(\sigma(v_1')) + [1 - p_1(\sigma(v_1'))]\, v_1(b)$$

or equivalently

$$[p_1(\sigma(v_1)) - p_1(\sigma(v_1'))] \geq [p_1(\sigma(v_1)) - p_1(\sigma(v_1'))]\, v_1(b),$$

which implies

$$p_1(\sigma(v_1)) \geq p_1(\sigma(v_1')).$$

By a symmetrical argument, preventing agent 1 from reporting v_1 when her true valuation is v_1' requires the opposite weak inequality.

Hence $p_1(\sigma(v_1)) = p_1(\sigma(v_1'))$ and (22) follows because p_1 is injective.

Step 2. We prove that (22) holds if $\tau(v_1) \neq \tau(v_1')$ and $v_1(\tau(v_1'))$ and $v_1'(\tau(v_1))$ are sufficiently close to 1. More precisely, let a, a', b be three distinct outcomes and recall the definition

$$\alpha_1 := \min_{E,E' \in \Omega : E \neq E'} |p_1(E) - p_1(E')| \,.$$

Suppose $\tau(v_1) = a$, $\tau(v_1') = a'$, $\tau(v_2) = b$. Let ε be such that

$$0 < \varepsilon < \alpha_1(1 - \max\{v_1(b), v_1'(b)\})$$

and suppose $v_1(a') = v_1'(a) = 1 - \varepsilon$. We claim that $\sigma(v_1) = \sigma(v_1')$.

To see why, suppose $p_1(\sigma(v_1')) > p_1(\sigma(v_1))$. By definition of α_1, there exists a number δ such that

$$p_1(\sigma(v_1')) - p_1(\sigma(v_1)) = \delta > \alpha_1 > 0.$$

By reporting v_1' when her true valuation is v_1, agent 1 gains

$$\begin{aligned}
& [p_1(\sigma(v_1'))v_1(a') + (1 - p_1(\sigma(v_1')))v_1(b)] - [p_1(\sigma(v_1)) + (1 - p_1(\sigma(v_1)))v_1(b)] \\
= \; & [p_1(\sigma(v_1'))v_1(a') - p_1(\sigma(v_1))] + [p_1(\sigma(v_1)) - p_1(\sigma(v_1'))]\, v_1(b) \\
= \; & p_1(\sigma(v_1'))(1 - \varepsilon) - p_1(\sigma(v_1)) - \delta v_1(b) \\
= \; & \delta(1 - v_1(b)) - \varepsilon p_1(\sigma(v_1')) \\
> \; & \alpha_1(1 - v_1(b)) - \varepsilon \\
> \; & 0,
\end{aligned}$$

violating strategy-proofness.

If $p_1(\sigma(v_1')) < p_1(\sigma(v_1))$, a symmetrical argument shows that agent 1 gains by reporting v_1 when her true valuation is v_1'.

We conclude that $p_1(\sigma(v_1)) = p_1(\sigma(v_1'))$ and (22) follows because p_1 is injective.

Step 3. To complete the proof of (22) in full generality, construct $w_1, w_1' \in \mathcal{V}$ such that $\tau(w_1) = \tau(v_1)$, $\tau(w_1') = \tau(v'_1)$, and

$$w_1(\tau(w_1')) = w_1'(\tau(w_1)) = 1 - \varepsilon$$

for some ε such that $0 < \varepsilon < \alpha_1(1 - \max\{w_1(\tau(v_2)), w_1'(\tau(v_2))\})$.

By Step 1, Step 2, and Step 1 again, $\sigma(v_1) = \sigma(w_1) = \sigma(w'_1) = \sigma(v_1')$. This proves that σ is generically constant in v.

The proof is now completed by appealing to Proposition 1. Every two-agent strategy-proof and ex-post efficient rule φ is a top selection and we may assume, without loss of generality, that the assignment rule σ associated with φ is constant in the reported valuations. Indeed, if σ depends upon the valuations when their tops coincide, it can be replaced with a (necessarily unique) function s partitioning Ω on the basis of the reported belief profile only.■

5.3 Proof of Proposition 3

Proof of the "if" statement. The proof of the "if" part of Proposition 3 does not require the assumption that the state space is of size two. Let the Ω-assignment rule $s : \mathcal{P}^{\{1,2\}} \to 2^{\Omega}$ be dictatorial or consensual and define the social choice rule $\varphi \in \Phi(\{1,2\})$ by

$$\varphi(v, p; \omega) = \begin{cases} \tau(v_1) \text{ if } \omega \in s(p), \\ \tau(v_2) \text{ if } \omega \in \complement s(p). \end{cases}$$

Step 1. We check that s is strategy-proof.

This is obvious if s is dictatorial. If s is consensual, there exist non-nested events A, B such that

$$s(p) = \begin{cases} A \text{ if } p_1(A) > p_1(B) \text{ and } p_2(\complement A) > p_2(\complement B), \\ B \text{ otherwise.} \end{cases}$$

Let us check that agent 1 cannot profitably manipulate s at p by reporting p_1'. The proof is similar for agent 2.

Note that the range of s is $\{A, B\}$. If $s(p) = A$ and $s(p_1', p_2) = B$, then $p_1(s(p)) = p_1(A) > p_1(B) = p_1(s(p_1', p_2))$.Suppose next that $s(p) = B$ and $s(p_1', p_2) = A$. The second equality implies $p_2(\complement A) > p_2(\complement B)$, and the first then implies $p_1(A) < p_1(B)$. It follows that $p_1(s(p)) = p_1(B) > p_1(A) = p_1(s(p_1', p_2))$.

Step 2. We check that φ is strategy-proof.

Let $(v, p) \in \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}}$. We check that agent 1 cannot profitably manipulate φ at (v, p) by reporting (v_1', p_1'). The proof is similar for agent 2. Agent 1's expected utility from $\varphi(v, p)$ is

$$p_1(s(p))v_1(\tau(v_1)) + [1 - p_1(s(p))]\, v_1(\tau(v_2)).$$

Her expected utility from $\varphi((v_1', p_1'), (v_2, p_2))$ is

$$\begin{aligned} & p_1(s(p_1', p_2))v_1(\tau(v_1')) + [1 - p_1(s(p_1', p_2))]\, v_1(\tau(v_2)) \\ \leq\ & p_1(s(p_1', p_2))v_1(\tau(v_1)) + [1 - p_1(s(p_1', p_2))]\, v_1(\tau(v_2)) \\ \leq\ & p_1(s(p))v_1(\tau(v_1)) + [1 - p_1(s(p))]\, v_1(\tau(v_2)), \end{aligned}$$

where the second inequality holds because $p_1(s(p_1', p_2)) \leq p_1(s(p))$ by strategy-proofness of s and $v_1(\tau(v_1)) = 1 \geq v_1(\tau(v_2))$.

Proof of the "only if" statement.

Let $\Omega = \{\omega_1, \omega_2\}$ and let $\varphi \in \Phi(\{1,2\})$ be a strategy-proof and ex-post efficient social choice rule. By Proposition 2, there exists an Ω-assignment rule s such that, for all $(v, p) \in \mathcal{V}^{\{1,2\}} \times \mathcal{P}^{\{1,2\}}$

and all $\omega \in \Omega$,

$$\varphi(v,p;\omega) = \begin{cases} \tau(v_1) \text{ if } \omega \in s(p), \\ \tau(v_2) \text{ if } \omega \in \complement s(p). \end{cases}$$

Because φ is strategy-proof, s must be strategy-proof.

Since s is strategy-proof, the events belonging to its range must be non-nested. Since $\Omega = \{\omega_1, \omega_2\}$, it follows that $|\mathcal{R}_s| = 1$ or $|\mathcal{R}_s| = 2$. If $|\mathcal{R}_s| = 1$, then s is trivially dictatorial. If $|\mathcal{R}_s| = 2$, then $\mathcal{R}_s = \{\{\omega_1\}, \{\omega_2\}\}$ and we distinguish two cases.

Case 1: s is constant in p_1 or p_2.

Assume without loss of generality that s is constant in p_2. We claim that agent 1 is a dictator. Suppose not: there exists $q \in \mathcal{P}^{\{1,2\}}$ such that, say, $s(q) = \{\omega_1\}$ and $q_1(\omega_1) < q_1(\omega_2)$. Since s is strategy-proof, $s(p_1, q_2) = \{\omega_1\}$ for all $p_1 \in \mathcal{P}$, that is, $s(., q_2)$ is a constant function of p_1 at q_2. Since by assumption $s(p_1, .)$ is a constant function of p_2 at every p_1, it follows that $s(.,.)$ is constant, contradicting the fact that $\mathcal{R}_s = \{\{\omega_1\}, \{\omega_2\}\}$.

Case 1: s is varies with p_1 and p_2.

Then we may assume without loss of generality that there exist $p_1, p_1', p_2, p_2' \in \mathcal{P}$ such that

$$s(p) = \{\omega_1\} \text{ and } s(p_1', p_2) = s(p_1, p_2') = \{\omega_2\}. \tag{23}$$

Since s is strategy-proof, (23) implies that for all $q_1, q_2 \in \mathcal{P}$,

$$s(q_1, p_2) = \begin{cases} \{\omega_1\} \text{ if } q_1(\omega_1) > q_1(\omega_2), \\ \{\omega_2\} \text{ if } q_1(\omega_1) < q_1(\omega_2), \end{cases} \tag{24}$$

$$s(p_1, q_2) = \begin{cases} \{\omega_1\} \text{ if } q_2(\omega_1) < q_2(\omega_2), \\ \{\omega_2\} \text{ if } q_2(\omega_1) > q_2(\omega_2), \end{cases} \tag{25}$$

where we recall that the event assigned to agent 2 at (p_1, q_2) is $\Omega \setminus s(p_1, q_2)$. To complete the proof, we now check that

$$s(q_1, q_2) = \begin{cases} \{\omega_1\} \text{ if } q_1(\omega_1) > q_1(\omega_2) \textit{ and } q_2(\omega_1) < q_2(\omega_2), \\ \{\omega_2\} \text{ if } q_1(\omega_1) < q_1(\omega_2) \textit{ or } q_2(\omega_1) > q_2(\omega_2). \end{cases}$$

If $q_1(\omega_1) > q_1(\omega_2)$ and $q_2(\omega_1) < q_2(\omega_2)$, then (24) and (25) imply $s(q_1, p_2) = s(p_1, q_2) = \{\omega_1\}$. Strategy-proofness then implies $s(q_1, q_2) = \{\omega_1\}$. Indeed, if $s(q_1, q_2) = \{\omega_2\}$, then $q_1(s(p_1, q_2)) = q_1(\omega_1) > q_1(\omega_2) = q_1(s(q_1, q_2))$ and agent 1 has an incentive to manipulate s at (q_1, q_2) (and agent 2 has a similar incentive).

Next, suppose that $q_1(\omega_1) < q_1(\omega_2)$ or $q_2(\omega_1) > q_2(\omega_2)$. Without loss of generality, assume the first inequality. From (24),

$$s(q_1, p_2) = \{\omega_2\}. \tag{26}$$

Since from (23) $s(p_1, p_2) = \{\omega_1\}$ and $s(p_1, p_2') = \{\omega_2\}$, Strategy-proofness implies

$$p_2(\omega_1) < p_2(\omega_2).$$

It follows from this inequality and (26) that $s(q_1, q_2) = \{\omega_2\}$: if $s(q_1, q_2) = \{\omega_1\}$, then $p_2(s(q_1, q_2)) = p_2(\omega_1) < p_2(\omega_2) = p_2(s(q_1, p_2))$ and agent 2 has an incentive to manipulate s at (q_1, p_2) by reporting q_2■.

6 References

Barberà, S., Gul, F., and Stacchetti, E. (1993), "Generalized median voter schemes and committees," *Journal of Economic Theory,* **61**, 262–289.

Barberà, S., Sonnenschein, H., and Zhou, L. (1991), "Voting by committees," *Econometrica,* **59**, 595–609.

Border, K. and Jordan, J. (1983), "Straightforward elections, unanimity and phantom agents," *Review of Economic Studies,* **50**, 153–170.

de Finetti, B. (1974), *Theory of Probability*, Vol. 1. New York: Wiley.

Dutta, B., Peters, H., and Sen, A. (2007), "Strategy-proof cardinal schemes," *Social Choice and Welfare,* **28**, 163–179.

Dutta, B., Peters, H., and Sen, A. (2008), "Erratum to: Strategy-proof cardinal schemes," *Social Choice and Welfare,* **30**, 701–702.

Gibbard, A. (1973). "Manipulation of voting schemes: a general result," *Econometrica,* **41**, 587–601.

Gibbard, A. (1977). "Manipulation of schemes that mix voting with chance," *Econometrica,* **45**, 665–681.

Gilboa, I., Samet, D., and Schmeidler, D. (2004), "Utilitarian aggregation of beliefs and tastes," *Journal of Political Economy,* **112,** 932–938.

Gilboa, I., Samuelson, L., and Schmeidler, D. (2014), "No-betting Pareto dominance," *Econometrica,* **82**, 1405–1442.

Gneiting, T. and Raftery, A. E. (2007), "Strictly proper scoring rules, prediction, and estimation," *Journal of the American Statistical Association,* **102: 477**, 359–378.

Hylland, A. (1980), "Strategy proofness of voting procedures with lotteries as outcomes and infinite sets of strategies," *Working Paper,* University of Oslo, Institute of Economics.

Hylland, A. and Zeckhauser, R. (1979), "The impossibility of Baeysian group decision making with separate aggregation of beliefs and values," *Econometrica*, **47**, 1321–1336.

Karni, E. (2009), "A mechanism for eliciting probabilities," *Econometrica* **77**, 603–606.

Le Breton, M. and Sen, A. (1999), "Separable preferences, strategyproofness and decomposability," *Econometrica,* **67**, 605–628.

Mongin, P. (1995), "Consistent Bayesian aggregation," *Journal of Economic Theory,* **66**, 313–351.

Nandeibam, S. (2013), "The structure of decision schemes with cardinal preferences," *Review of Economic Design,* **17**, 205–238.

Pápai, S. (2007), "Exchange in a general market with indivisible goods," *Journal of Economic Theory,* **132**, 208–235.

Satterthwaite, M. (1975). "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory,* **10**, 198–217.

Savage, L. J. (1954), *The foundations of statistics.* New York: Wiley.

Savage, L. J. (1971), "Elicitation of personal probabilities and expectations," *Journal of the American Statistical Association,* **66**, 783–801.

Sperner, E. (1928), "Ein Satz über Untermengen einer endlichen Menge," *Mathematische Zeitschrift,* **27**, 544–548.