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Représentations et fusion des algèbres de Temperley-Lieb originale et diluée

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Représentations et fusion des algèbres de Temperley-Lieb originale et diluée

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Résumé

Les algèbres de Temperley-Lieb originales, aussi dites *régulières*, apparaissent dans de nombreux modèles statistiques sur réseau en deux dimensions : les modèles d'Ising, de Potts, des dimères, celui de Fortuin-Kasteleyn, etc. L'espace d'Hilbert de l'hamiltonien quantique correspondant à chacun de ces modèles est un module pour cette algèbre et la théorie de ses représentations peut être utilisée afin de faciliter la décomposition de l'espace en blocs ; la diagonalisation de l'hamiltonien s'en trouve alors grandement simplifiée. L'algèbre de Temperley-Lieb diluée joue un rôle similaire pour des modèles statistiques dilués, par exemple un modèle sur réseau où certains sites peuvent être vides ; ses représentations peuvent alors être utilisées pour simplifier l'analyse du modèle comme pour le cas original. Or ceci requiert une connaissance des modules de cette algèbre et de leur structure ; un premier article donne une liste complète des modules projectifs indécomposables de l'algèbre diluée et un second les utilise afin de construire une liste complète de tous les modules indécomposables des algèbres originale et diluée. La structure des modules est décrite en termes de facteurs de composition et par leurs groupes d'homomorphismes.

Le produit de fusion sur l'algèbre de Temperley-Lieb originale permet de « multiplier » ensemble deux modules sur cette algèbre pour en obtenir un autre. Il a été montré que ce produit pouvait servir dans la diagonalisation d'hamiltoniens et, selon certaines conjectures, il pourrait également être utilisé pour étudier le comportement de modèles sur réseaux dans la limite continue. Un troisième article construit une généralisation du produit de fusion pour les algèbres diluées, puis présente une méthode pour le calculer. Le produit de fusion est alors calculé pour les classes de modules indécomposables les plus communes pour les deux familles, originale et diluée, ce qui vient ajouter à la liste incomplète des produits de fusion déjà calculés par d'autres chercheurs pour la famille originale.

Finalement, il s'avère que les algèbres de Temperley-Lieb peuvent être associées à une catégorie monoïdale tressée, dont la structure est compatible avec le produit de fusion décrit ci-dessus. Le quatrième article calcule explicitement ce tressage, d'abord sur la catégorie des algèbres, puis sur la catégorie des modules sur ces algèbres. Il montre également comment ce tressage permet d'obtenir des solutions aux équations de Yang-Baxter, qui peuvent alors être utilisées afin de construire des modèles intégrables sur réseaux.

Mots-Clés : algèbre de Temperley-Lieb, algèbre de Temperley-Lieb diluée, algèbre associative, module indécomposable, produit de fusion, règles de fusion, catégories tressées, théorie des champs conformes, modèles intégrables, théorie de Auslander-Reiten.

Abstract

The original Temperley-Lieb algebra, also called *regular*, appears in numerous integrable statistical models on two dimensional lattices : the Ising model, the Potts model, the dimers model, the Fortuin-Kasteleyn model, etc. The Hilbert space of the corresponding quantum hamiltonian is then a module over this algebra ; its representation theory can be used to split this space in a direct sum of smaller spaces, and thus block diagonalize the corresponding quantum model. The dilute Temperley-Lieb algebra plays a similar role for dilute models, for instance those where lattice sites can be empty ; its representation theory thus plays a similar role for these models. However, doing this requires a detailled knowledge of its modules and their structure ; the first paper presents a complete list of the projective indecomposable modules for the dilute Temperley-Lieb algebra and a second constructs a complete set of indecomposable modules for both the regular and dilute algebras. In both articles the structure of the modules are exposed through their composition factors and homomorphism groups.

The fusion product on the original Temperley-Lieb algebra defines how two modules can be « multiplied » together to obtain a module. It has been shown in some cases that this product can be used to simplify the block diagonalization of quantum hamiltonians, and some speculate that it could be used to determine the continuum limit of the models. A third paper defines a straightforward generalization of this product for the dilute algebra, then introduces an efficient way of computing it. It then calculates this product for the most common classes of indecomposable modules for both the original and dilute algebras ; this fills a hole in the known fusion rules for the original algebra that were left out of previous calculations.

Finally, it happens that the Temperley-Lieb algebras can be grouped together in a braided monoidal category, whose structure is compatible with the fusion product described above. The fourth article builds explicitly this braiding, both for the Temperley-Lieb category, and for its module category. It also shows how this braiding can be used to obtain solutions to the Yang-Baxter equation, which can then be used to build integrable lattice models.

Keywords : Temperley-Lieb algebra, dilute Temperley-Lieb algebra, associative algebra, indecomposable modules, fusion product, fusion rules, braided category, conformal field theory, integrable model, Auslander-Reiten theory.

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Liste des sigles et abbréviations

CFT	Théorie de champs conformes
TL	Temperley-Lieb
YB	Yang-Baxter
BYB	*Boundary* Yang-Baxter
$\mathcal{C}$	Une catégorie
$\mathcal{M}(\mathcal{C})$	La catégorie disjointe de $\mathcal{C}$
$\mathbb{F}$	Un corps algébriquement clos
R	Un anneau commutatif
$\mathcal{A}$	Une algèbre
TL_n	L'algèbre de Temperley-Lieb régulière à n points
dTL_n	L'algèbre de Temperley-Lieb diluée à n-points
ℓ	Le plus petit entier strictement positif tel que $q^{2\ell} = 1$
k_c	Un entier tel que $k_c + 1 \equiv 0 \mod \ell$
k_{+1}	Le plus petit entier strictement plus grand que k tel que $(k + k_{+1})/2$ soit critique
k_{-1}	Le plus grand entier strictement plus petit que k tel que $(k + k_{-1})/2$ soit critique
k_i	$k \equiv k_0$, $k_j \equiv (k_{j\mp1})_{\pm1}$
$D_n(u)$	La matrice double-ligne de TL_n
F_n	Un élément central particulier de TL_n ou de dTL_n
$\mathsf{S}_{n,k}$	Le TL_n ou le dTL_n-module standard à k défauts
$\mathsf{I}_{n,k}$	Le TL_n ou le dTL_n-module irréductible à k défauts
$\mathsf{C}_{n,k}$	Le TL_n ou le dTL_n-module costandard à k défauts
$\mathsf{P}_{n,k}$	La couverture projective de $\mathsf{I}_{n,k}$
$\mathsf{J}_{n,k}$	L'enveloppe injective de $\mathsf{I}_{n,k}$
$\mathsf{B}^0_{n,k}$	Le module irréductible $\mathsf{I}_{n,k}$
$\mathsf{T}^0_{n,k}$	Le module irréductible $\mathsf{I}_{n,k}$
$\mathsf{B}^i_{n,k}$	L'unique extension indécomposable de T^{n,k^1}_{i-1} par $\mathsf{I}_{n,k}$
$\mathsf{T}^i_{n,k}$	L'unique extension indécomposable de $\mathsf{I}_{n,k}$ par B^{n,k^1}_{i-1}

$\mathsf{Hom}_{\mathcal{A}}(U,V)$	Le groupe des homomorphismes de U dans V
$\mathrm{Ext}_{\mathcal{A}}(U,V)$	Le premier groupe d'extension de U par V
$\mathrm{Tor}_{\mathcal{A}}(U,V)$	Le premier groupe de torsion de U et V
$U \times_f V$	Le produit de fusion des deux modules U,V
$U \div_f V$	Le quotient de fusion de U par V
$\mathsf{U}_i(x)$	Le i-ème polynôme de Chebyshev de seconde espèce évalué en x

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Chapitre 1

Introduction

Cette thèse porte sur la théorie des représentations des algèbres de Temperley-Lieb régulière et diluée ainsi que son application à quelques problèmes concrets, comme le calcul de règles de fusion sur réseau et de la structure tressée de la catégorie de Temperley-Lieb. Les algèbres de Temperley-Lieb [25, 39, 45, 66] et leurs diverses généralisations jouent un rôle très important en physique statistique ; de nombreux modèles en deux dimensions, tels ceux d'Ising, de Potts, des six-vertex, des dimères, entre autres, peuvent être vus comme des représentations de ces algèbres. Le calcul de leurs propriétés physiques peut alors être réduit à un problème algébrique, ou dans certains cas être considérablement simplifié par une connaissance fine de leurs représentations. La version diluée de Temperley-Lieb [30, 56] joue alors un rôle similaire pour des versions diluées de ces modèles [15], mais la théorie de ses représentations demeure largement inconnue.

La première partie de cette introduction, la section 1.1 sera consacrée à un rappel rapide d'un problème simple où la théorie des représentations d'une algèbre joue un rôle important : l'oscillateur harmonique simple quantique. Suivra alors un modèle peu connu : la chaîne d'oscillateurs indistinguables, un problème simple qui nous permettra d'introduire un premier exemple de règles de fusion et de bien mesurer leur utilité dans le cas des modèles sur réseaux.

Nous passerons ensuite, à la section 1.2, aux premiers exemples de modèles statistiques, le modèle d'Ising. La version unidimensionnelle du problème sera d'abord présentée afin d'introduire simplement le concept de matrice de transfert et d'hamiltonien quantique. Le modèle d'Ising sur un cylindre suivra ; on y verra l'apparition des algèbres de Temperley-Lieb et de leurs hamiltoniens. On y discutera également du concept de la limite hamiltonienne, qui apparaît souvent dans les conjectures entourant TL.

Suivra un rapide survol des principaux éléments de la théorie des représentations des algèbres de Temperley-Lieb régulières dans la section 1.3. Les définitions classiques en termes de générateurs et en termes de dessins seront données, et la structure des modules standards sera détaillée. Finalement, la filtration de cette famille d'algèbres sera utilisée afin de définir le produit de fusion.

La section 1.4 montrera alors une méthode générale pour construire des modèles statistiques à partir des algèbres régulières et diluées de TL. Nous verrons en particulier l'exemple classique de la chaîne XXZ quantique ; les règles de fusion de Temperley-Lieb pourront alors être utilisées afin de briser l'espace d'Hilbert de la chaîne en somme directe de modules indécomposables.

La section 1.5 donnera une introduction très succincte à la théorie des champs conformes et aux règles de fusion que l'on y retrouve. Nous discuterons alors de la correspondance entre ces fusions et la fusion de TL. Finalement, la section 1.6 présentera un survol rapide de certains concepts en théorie des catégories qui seront utilisés dans les articles.

1.1 L'oscillateur harmonique quantique

1.1.1 Formulation algébrique d'un oscillateur harmonique quantique

L'oscillateur harmonique simple est un modèle omniprésent en mécanique classique. Celui-ci représente une particule qui ressent une force proportionnelle à la distance entre elle-même et un autre point donné, l'origine. Il peut être formulé de façon très simple, mais il revêt tout de même une très grande importance. En effet, dans tout système conservatif, une particule suffisamment proche de l'équilibre se comportera comme dans un oscillateur harmonique simple. De plus, les solutions du modèle, les exponentielles à exposant imaginaire, sont elles aussi très importantes en mathématiques, notamment dans l'analyse de Fourier. Il n'est donc pas surprenant que sa quantification, l'oscillateur harmonique quantique, soit elle aussi très importante en physique. Dans cette version, l'hamiltonien du système devient un opérateur agissant sur un espace d'Hilbert :

$$H = \hbar\omega(\underbrace{a_+a_-}_{a_0} + \frac{1}{2}), \tag{1.1.1}$$

où $\hbar$ est la constante réduite de Planck et ω la fréquence de résonance du système. Les opérateurs $a_\pm$ sont des opérateurs différentiels agissant sur un espace de fonctions suffisamment « lisses » :

$$a_\pm = \sqrt{\frac{m\omega}{2\hbar}}x \mp \sqrt{\frac{\hbar}{2m\omega}}\frac{\partial}{\partial x}. \tag{1.1.2}$$

Ensemble, les opérateurs $\{a_-, a_0 = a_+a_-, a_+, c\}$ engendrent une algèbre de Lie triangulable $\mathfrak{h}$ décrite par les relations suivantes :

$$[a_-, a_+] = c, \qquad [a_0, a_\pm] = (\pm 1)a_\pm, \qquad [a_\pm, c] = [a_0, c] = 0, \tag{1.1.3}$$

$$\mathfrak{h}_\pm = \mathbb{C}a_\pm, \qquad \mathfrak{h}_0 = \mathbb{C}a_0 \oplus \mathbb{C}c. \tag{1.1.4}$$

L'espace de Hilbert du problème, M, est alors un module pour cette algèbre et la solution de l'équation de Schrödinger indépendante du temps consiste alors à trouver les vecteurs propres de H ainsi que leur valeur propre associée. Cependant, quelques observations permettent de simplifier le problème. Supposons que $x \in M$ soit un vecteur propre de a_0, et donc de H, de valeur propre λ. Les relations (1.1.3) permettent alors d'obtenir

$$a_0 a_\pm x = ([a_0, a_\pm] + a_\pm a_0)x = (\lambda \pm 1)a_\pm x. \tag{1.1.5}$$

Le vecteur $a_\pm x$ est donc lui aussi un vecteur propre de a_0 de valeur propre $\lambda \pm 1$, appelé *vecteur descendant* de x. Puisque λ représente l'énergie de l'état x, les opérateurs $a_\pm$ agissent donc en augmentant ou en réduisant cette énergie de 1. Pour cette raison, les opérateurs a_+ et a_- sont appelé *opérateurs de création* et d'*annihilation*, respectivement, ou encore *opérateurs d'échelle*.

Pour que le modèle soit « physique », il faut que l'énergie des états propres soit bornée inférieurement. Il suit donc qu'il existe un vecteur propre $|\lambda\rangle$ de a_0, de valeur propre λ, tel que [1] $a_-|\lambda\rangle = 0$. Puisque dans cette représentation de l'algèbre $a_0 = a_+a_-$, il faut donc que $\lambda = 0$. Il suit alors que tous les états dans M sont descendants d'un vecteur propre de a_0 de valeur propre 0. En utilisant l'expression de a_- dans cette représentation, on obtient alors les fonctions d'ondes propres de l'hamiltonien (1.1.1) :

$$\langle x|k\rangle = \psi_k(x) = \psi_0(0)\frac{1}{\sqrt{k!}}a_+^k e^{-\frac{m\omega}{2\hbar}x^2}, \qquad k = 0, 1, 2, \ldots \tag{1.1.6}$$

où $\psi_k(x)$ possède une énergie de $\hbar\omega\frac{(2k+1)}{2}$, et $\psi_0(0)$ est une constante. Finalement, il faut fixer le produit scalaire sur M afin de fixer la normalisation. Le produit scalaire typique sur l'espace de fonctions considéré est donné par

$$\langle \psi, \phi\rangle = \int_{-\infty}^{\infty} \psi(x)^*\phi(x)dx.$$

En termes d'opérateurs, ceci donne $a_\pm^* = a_\mp$, $a_0^* = a^0$, et $1^* = 1$. On vérifie alors aisément

$$\langle k|q\rangle = \delta_{k,q}\sqrt{\frac{\pi\hbar}{m\omega}}\psi_0(0)^2 = \delta_{k,q}\langle 0|0\rangle. \tag{1.1.7}$$

Dans cette base, l'action de $\mathfrak{h}$ est donnée par :

$$a_+|k\rangle = \sqrt{k+1}|k+1\rangle, \qquad a_-|k\rangle = \sqrt{k}|k-1\rangle, \qquad a_0|k\rangle = k|k\rangle, \qquad c|k\rangle = |k\rangle. \tag{1.1.8}$$

1. C'est un exemple de vecteur *de plus bas poids*.

La solution algébrique précédente repose sur le fait que l'Hamiltonien (1.1.1) soit un élément d'une représentation de l'algèbre de Lie définie par les relations (1.1.3). Alors que le problème fixe la représentation de cette algèbre, il existe de nombreuses autres représentations de celle-ci. À chacune de ces représentations correspond un autre hamiltonien représentant un autre modèle, pas nécessairement physique. Certains de ces autres modèles seront résolubles en utilisant essentiellement les mêmes arguments que pour l'oscillateur harmonique quantique. Classifier l'ensemble des représentations de cette algèbre permettrait donc de trouver l'ensemble des modèles qui se comportent approximativement comme un oscillateur harmonique.

Le produit scalaire choisi coïncide avec la forme bilinéaire de Shapovalov, qui est définie sur les modules de Verma de toute algèbre de Lie triangulable [2]. Or, cette forme ne définit pas toujours un produit scalaire, en particulier celle-ci peut posséder des vecteurs dont la « norme » de Shapovalov est nulle ou même négative. Il sera alors nécessaire d'utiliser une autre forme bilinéaire pour faire du module M un espace d'Hilbert. Ce sera le cas par exemple dans certaines théories de champs conformes.

1.1.2 La chaîne d'oscillateurs harmoniques

Le problème d'une chaîne d'oscillateurs harmoniques quantiques indistinguables peut être traité de manière remarquablement semblable au problème d'un unique oscillateur harmonique quantique. Un problème similaire interviendra dans le traitement des chaînes de spins et montre un exemple simple de ce que nous appellerons des *règles de fusion*. Il est donc important de se familiariser avec cet exemple. On commence par le cas le plus simple : la chaîne de deux oscillateurs.

La situation considérée est la suivante : deux oscillateurs harmoniques quantiques identiques sont placés côte à côte, de manière à ne pas pouvoir interagir entre eux. L'espace d'Hilbert du problème est alors engendré par le produit tensoriel des espaces d'Hilbert respectifs de chaque oscillateur :

$$M_1 \otimes M_2 = \mathrm{span}_{\mathbb{C}}\{|r\rangle \otimes |s\rangle = |r,s\rangle | r,s = 0,1,2,\ldots\}, \tag{1.1.9}$$

où nous avons utilisé la notation $|r\rangle$ introduite à la section précédente. L'espace résultant est un module pour l'algèbre $\mathfrak{h}\otimes\mathfrak{h}$, où $\mathfrak{h} = \{a_-, a_0, a_+, 1\}$ est l'algèbre obtenue à partir de l'oscillateur harmonique quantique seul. Cependant, soit $\mathfrak{g}$ une algèbre de Lie, M_1, M_2 des $\mathfrak{g}$-modules ; le $\mathfrak{g}\otimes\mathfrak{g}$-module $M_1 \otimes M_2$ possède une structure de $\mathfrak{g}$-module donnée par :

$$a(x\otimes y) = (ax)\otimes y + x\otimes(ay). \tag{1.1.10}$$

2. Une algèbre de Lie est triangulable si elle est isomorphe (en tant qu'espace vectoriel) à la somme directe de trois sous-algèbres, dont une est commutative.

Ceci signifie que l'on peut définir une action de l'algèbre de Lie $\mathfrak{h} = \{a_-, a_0, a_+, 1\}$ sur le module $M_1 \otimes M_2$. On a alors, par exemple

$$a_-|r,s\rangle = \sqrt{r}|r-1,s\rangle + \sqrt{s}|r,s-1\rangle, \qquad a_+|r,s\rangle = \sqrt{r+1}|r+1,s\rangle + \sqrt{s+1}|r,s+1\rangle \quad (1.1.11)$$

$$a_0|r,s\rangle = r|r,s\rangle + s|r,s\rangle = (r+s)|r,s\rangle, c|r,s\rangle = 2|r,s\rangle$$

Ceci nous permet d'interpréter physiquement les opérateurs dans $\mathfrak{h}$. Les opérateurs d'échelle créent ou détruisent une particule dans un ou l'autre des oscillateurs. L'opérateur a_0 compte alors le nombre total de particules dans le système ; c'est donc, à une constate près, l'hamiltonien de la paire. En restreignant ainsi les opérateurs agissant sur l'espace d'Hilbert, nous sommes passés d'un modèle représentant deux oscillateurs harmoniques disjoints à une paire d'oscillateurs harmoniques indistinguables.

Afin d'appliquer les mêmes « idées » que dans la section précédente, il faut trouver les vecteurs de plus bas poids, c'est-à-dire les états $v \in M_1 \otimes M_2$ tels que $a_- v = 0$. Une inspection sommaire montre alors qu'il y en a beaucoup :

$$|0,0\rangle, \qquad |1,0\rangle - |0,1\rangle, \qquad (|2,0\rangle + |0,2\rangle) - \sqrt{2}|1,1\rangle,$$

$$(|3,0\rangle - |0,3\rangle) - \sqrt{3}(|2,1\rangle - |1,2\rangle), \qquad |0,4\rangle + |4,0\rangle - 2(|1,3\rangle + |3,1\rangle - \sqrt{\frac{3}{2}}|2,2\rangle), \ldots$$

et ceux-ci ne sont évidemment pas descendants les uns des autres. L'espace d'Hilbert du problème est alors une somme directe de modules V_i, chacun engendré par un vecteur de plus haut poids d'énergie[3] i. On peut alors se convaincre que

$$M \simeq \bigoplus_{i=0}^{\infty} V_i.$$

L'interprétation physique de ce résultat est qu'une paire d'oscillateurs harmoniques quantiques indistinguables est équivalente à une infinité d'oscillateurs harmoniques disjoints, dont les états fondamentaux sont progressivement plus énergétiques. Ce résultat se généralise aisément au cas d'un nombre arbitraire d'oscillateurs. Soit $M(n)$ une série de n oscillateurs harmoniques quantiques indistinguables. Son espace d'Hilbert est

$$M(n) = \underbrace{M(1) \otimes M(1) \otimes \ldots \otimes M(1)}_{n \text{ fois}} \simeq \bigoplus_{i=0}^{\infty} \binom{n+i-2}{i} V_i, \qquad (1.1.12)$$

3. V_i est donc un module de Verma $V(i)$ pour $\mathfrak{h}$.

où V_i est le module de Verma de $\mathfrak{h}$ de poids i.

Notons que l'essence de la solution repose sur le calcul de produits tensoriels de modules de Verma pour $\mathfrak{h}$. Ce type de calcul est courant en physique, apparaissant par exemple dans l'addition de moments cinétiques, et joue un rôle fondamental dans l'étude des représentations d'algèbres de Lie ; le produit tensoriel permet alors de construire des représentations de plus en plus riches à partir de représentations simples. En général, les foncteurs de changement d'anneaux, dont le produit tensoriel est un exemple, jouent un rôle similaire dans l'étude des représentations des algèbres associatives. Ce sera par exemple le cas pour les algèbres de Temperley-Lieb régulière et diluée.

1.2 Des modèles statistiques aux chaînes de spins

1.2.1 Le modèle d'Ising en 1D

On considère une chaîne périodique de N spins classiques σ_i pouvant prendre les valeurs ± 1. Le système est décrit par l'hamiltonien $H = -\sum_i^N (\sigma_i \sigma_{i+1} - 1)$, avec $\sigma_{N+1} \equiv \sigma_1$, et la fonction de partition est

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H} = \sum_{\{\sigma_i\}} \prod_{i=1}^{N} e^{\beta(\sigma_i \sigma_{i+1} - 1)}$$

où la somme est sur toutes les configurations possibles de spins et $\beta = \frac{1}{k_B t}$ avec t la température et k_B la constante de Boltzmann. On introduit l'état $|\sigma_i\rangle$ pour décrire l'état du spin i ; on a alors la relation de complétude $\sum_{\sigma_i = \pm 1} |\sigma_i\rangle\langle\sigma_i| = 1$. La *matrice de transfert* T est la matrice 2×2 telle que

$$\langle\sigma_i|T|\sigma_{i+1}\rangle = e^{\beta(\sigma_i \sigma_{i+1} - 1)}.$$

La fonction de partition devient alors

$$Z = \sum_{\{\sigma_i = \pm 1\}} \langle\sigma_1|T|\sigma_2\rangle\langle\sigma_2|T|\sigma_3\rangle \ldots = \sum_{\sigma_1 = \pm 1} \langle\sigma_1|T^N|\sigma_1\rangle = \text{Tr}\, T^N \tag{1.2.1}$$

Le calcul de la fonction de partition se réduit alors à la détermination des valeurs propres de T. Dans ce cas, on trouve aisément les valeurs propres $\Lambda_\pm = 1 \pm e^{-2\beta}$ ce qui donne

$$Z = 2^N e^{-N\beta} \left(\cosh^N \beta + \sinh^N \beta \right).$$

La matrice de transfert peut également être vue comme un opérateur d'évolution [33] : si on regarde la chaîne d'Ising comme un seul spin représenté à N pas de temps distincts, appliquer la matrice de transfert au système lui fait faire un pas de temps. L'*hamiltonien quantique* $\hat{H}$ décrit le système quantique ainsi obtenue et est défini par

$$T = \exp -\beta \hat{H}.$$

Dans le cas de la chaîne d'Ising, pour un β petit

$$T = \exp -\beta \hat{H} \simeq 1 - \beta \hat{H} \tag{1.2.2}$$

avec

$$\hat{H} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{1.2.3}$$

1.2.2 Le modèle d'Ising en 2D

Le modèle d'Ising en deux dimensions est un des modèles les plus connus en mécanique statistique. Sa définition est relativement simple et sa solution par la méthode de la matrice de transfert est extrêmement élégante. Dans la limite thermodynamique, celle-ci expose la relation entre la théorie des champs conformes, comme le modèle du fermion libre sans masse, et la physique statistique au point critique. Il est défini comme suit. On a un réseau bidimensionnel carré de $N \times M$ sites où les sommets portent chacun une flèche [4] pouvant pointer vers le haut ou vers le bas. L'hamiltonien est alors donné par

$$H_{\text{Ising}} = -\sum_{r,t} (J_s \delta(\theta_{r,t}, \theta_{r+1,t}) + J_\tau \delta(\theta_{r,t}, \theta_{r,t+1})),$$

où (r,t), $r = 1, 2, \ldots N$, $t = 1, 2, \ldots M$, sont les coordonnées sur le réseau, $\theta_{r,t}$ prend la valeur 0 ou 1, selon que la flèche au point (r,t) pointe vers le bas ou vers le haut, et J_s,J_τ mesurent l'énergie d'interaction dans les deux directions du réseau. Enfin, $\delta(x,y)$ est le delta de Kronecker. Avant de procéder avec la solution, il faut fixer les conditions aux bords. Nous choisirons un réseau périodique en une direction et libre dans l'autre : $\theta_{r,M+1} = \theta_{r,1}$ et la somme sur r va jusqu'à $r = N-1$.

Comme pour le modèle en une dimension, on définit alors une matrice de transfert T. Pour $|\theta_t\rangle = |\theta_{1,t}, \ldots, \theta_{N,t}\rangle$,

$$\begin{aligned} \langle \theta_t | T | \theta_{t+1} \rangle &= e^{\beta \sum_r (J_s \delta(\theta_{r,t}, \theta_{r+1,t}) + J_\tau \delta(\theta_{r,t}, \theta_{r,t+1}))} \\ &= \left(\prod_{r=1}^{N-1} e^{\beta J_s \delta(\theta_{r,t}, \theta_{r+1,t})} \right) \left(\prod_{r=1}^{N} e^{\beta J_\tau \delta(\theta_{r,t}, \theta_{r,t+1})} \right). \end{aligned}$$

4. La plupart des auteurs utilisent le mot « spin » plutôt que « flèche », mais il est important de remarquer que le modèle d'Ising en 2D est un modèle *classique*, et non un modèle quantique, comme le suggère le mot « spin ».

Sous forme matricielle, T peut être écrite

$$T = \left(\Pi_{i=1}^{N-1}(1_{2^N} + \Upsilon_s e_{2i})\right)\left(\Pi_{i=1}^{N}\sqrt{2}(\Upsilon_\tau 1_{2^N} + e_{2i-1})\right), \tag{1.2.4}$$

avec $\Upsilon_\tau = \frac{1}{\sqrt{2}}(e^{\beta J_\tau} - 1)$, $\Upsilon_s = \frac{1}{\sqrt{2}}(e^{\beta J_s} - 1)$, et

$$\sqrt{2}e_{2i-1} = 1_{2^N} + \sigma_i^x, \qquad \sqrt{2}e_{2i} = 1_{2^N} + \sigma_i^z\sigma_{i+1}^z,$$

$$\sigma_i^a = \underbrace{1_2 \otimes 1_2 \otimes \ldots \otimes 1_2}_{i-1} \otimes \sigma^a \otimes \underbrace{1_2 \otimes \ldots \otimes 1_2}_{N-i}, \qquad \sigma_{N+1}^a \equiv 0,$$

avec σ^x, σ^z les matrices de Pauli usuelles. Les matrices e_j engendrent alors une représentation de l'algèbre de Temperley-Lieb $\mathsf{TL}_{2N}(\sqrt{2})$:

$$\begin{aligned} e_je_j &= \beta e_j, & \beta &= \sqrt{2} \\ e_ie_j &= e_je_i, & |i-j| &\geq 2 \\ e_je_{j\pm1}e_j &= e_j. \end{aligned}$$

Comme pour le cas unidimensionnel, la fonction de partition du modèle peut alors être écrite

$$Z(J_s, J_\tau) = Tr(T^M),$$

et le problème se réduit à trouver les valeurs propres de T. Or comme T est une matrice $2^N \times 2^N$, trouver ses valeurs propres est une tâche fastidieuse accomplie pour la première fois par Onsager [57]. Heureusement, la théorie des représentations de TL_{2N} peut être utilisée afin de briser T en somme directe de matrices plus petites, et ainsi simplifier partiellement la solution.

1.2.3 La limite hamiltonienne

Puisque le modèle d'Ising est un modèle statistique, nous nous intéressons maintenant à sa limite thermodynamique. Or, puisque le modèle est sur un réseau carré, avec des énergies d'interaction différentes dans les deux directions, il semble évident qu'il existe plusieurs limites possibles : envoyer N à l'infini en gardant M constant, garder J_s constant, mais envoyer J_τ à zéro, envoyer N et M à l'infini en même temps, etc. Il faut donc commencer par définir la façon dont sera prise la limite. La *limite hamiltonienne* est une façon particulière d'envoyer le modèle statistique vers un modèle semi-continu relativement simple ; elle est définie comme suit [33]. On commence par fixer un paramètre scalaire λ, puis on fixe $J_\tau = -\frac{1}{\beta}\log\lambda\beta J_s$. On prend finalement la limite $\beta J_s \to 0$. Un développement en série de (1.2.4) autour de $x = \beta J_s = 0$ donne

$$T \simeq \frac{1}{(x\lambda)^N} 1_{2^N} + \frac{1}{(x\lambda)^{N-1}} \left(\sqrt{2} \sum_{j=1}^{N} e_{2j-1} + \frac{1}{\lambda\sqrt{2}} \sum_{j=1}^{N-1} e_{2j} - (N-1) 1_{2^N} \right) + O(x^{2-N}). \quad (1.2.5)$$

Après le terme proportionnel à l'identité, le second terme dominant est

$$\sum_{j}^{N} \left(\sqrt{2} e_{2j-1} + \frac{1}{\sqrt{2}\lambda} e_{2j} - 1_{2^N} \right) + 1_{2^N} = \frac{N-1}{2\lambda} 1_{2^N} + \frac{1}{2\lambda} \sum_{j=1}^{N} \left(2\lambda \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right).$$

On obtient alors finalement l'hamiltonien quantique du système :

$$\hat{H} = -\sum_{j=1}^{N} \left(2\lambda \sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right). \quad (1.2.6)$$

Il est toutefois important de noter que cet hamiltonien n'est pas unique. Le terme proportionnel à λ peut évidemment être changé simplement en changeant la façon dont la limite est prise, et la forme de l'expression peut être changée en ajoutant une constante à l'hamiltonien initial, ce qui, bien entendu, ne change pas le comportement qualitatif du modèle.

1.2.4 La limite continue au point critique

L'hamiltonien quantique (1.2.6) décrit toujours un modèle discret, il faut donc encore considérer la limite où $N \to \infty$. Afin de simplifier les calculs, nous nous limiterons au cas où le modèle est critique, ce qui se produit lorsque la fonction de partition est « auto-duale » : $Z(\Upsilon_s, \Upsilon_\tau) \equiv Z(\frac{1}{\Upsilon_\tau}, \frac{1}{\Upsilon_s})$. Ceci se produit lorsque $\lambda = 1/2$.

Il est utile de passer au modèle du fermion libre sans masse ; voici un résumé des arguments menant à cette identification. On commence par montrer que la chaîne XZ est équivalente à deux copies du modèle d'Ising [21]. La chaîne quantique XZ est une chaîne de spins $1/2$ décrite par l'hamiltonien

$$H_{XZ} = -\sum_{j=1}^{2N-1} \left(K_x \sigma_j^x \sigma_{j+1}^x + K_z \sigma_j^z \sigma_{j+1}^z \right), \quad (1.2.7)$$

où K_x, K_z sont des constantes et où les bords sont libres : $\sigma_{2N+1}^a \equiv 0$. On définit alors les variables

$$\rho_j^x = \sigma_j^z \sigma_{j+1}^z, \qquad j = 1, 2, \ldots, 2N-1, \qquad \rho_{2N}^x \equiv \sigma_{2N}^z, \quad (1.2.8)$$

$$\rho_j^z = \prod_{i=1}^{j} \sigma_i^x, \qquad j = 1, 2, \ldots, 2N, \quad (1.2.9)$$

qui satisfont les mêmes relations que les matrices σ_j. L'hamiltonien prend alors la forme

$$H_{XZ} = -\sum_{j=1}^{N}(K_z\rho^x_{2j} + K_x\rho^z_{2j}\rho^z_{2j+2}) - \sum_{j=1}^{N}(K_z\rho^x_{2j-1} + K_x\rho^z_{2j-1}\rho^z_{2j+1}) + K_z\rho^x_{2N} - K_x\rho^z_2. \quad (1.2.10)$$

La chaîne XZ se comporte donc comme deux copies indépendantes du modèle d'Ising auxquelles on a rajouté un terme de bord. Dans la limite continue, cette chaîne se comportera donc approximativement de la même façon que le modèle d'Ising. Notons qu'il est possible de traiter directement la chaîne d'Ising sans passer par la chaîne XZ et que le calcul se fait en utilisant des arguments très semblables [33]. Par contre, ceux-ci sont significativement plus simples dans le modèle XZ ; nous emploierons donc ce raccourci afin d'alléger la présentation.

On passe maintenant des opérateurs de Pauli aux opérateurs fermioniques par la transformation de Jordan-Wigner :

$$c_j = e^{i\pi\sum_{k=1}^{j-1}a_k^+a_k}a_j, \qquad c_j^+ = e^{i\pi\sum_{k=1}^{j-1}a_k^+a_k}a_j^+, \quad (1.2.11)$$

où $a_j = 1/2(\sigma^z_j + i\sigma^x_j)$, $a_j^+ = 1/2(\sigma^z_j - i\sigma^x_j)$. En termes des variables fermioniques c, l'hamiltonien XZ au point critique ($K_z = K_x = 1$) s'écrit alors

$$H_{XZ} = -2\sum_{j=1}^{2N-1}(c_j^+c_{j+1} + c_{j+1}^+c_j). \quad (1.2.12)$$

On peut alors diagonaliser directement l'hamiltonien en posant :

$$\Lambda_k = 2\sin\left(\frac{k\pi}{2N+1}\right), \qquad k = \frac{1}{2}, \frac{3}{2}, \dots, \frac{2N-1}{2}, \quad (1.2.13)$$

$$f_k = \frac{1}{Z_k}\sum_{i=1}^{2N}U_{i-1}\left(\frac{\Lambda_k}{2}\right)c_i, \qquad f_{-k} = \frac{1}{Z_k}\sum_{i=1}^{2N}U_{i-1}\left(\frac{\Lambda_k}{2}\right)c_i^+, \quad (1.2.14)$$

$$g_k = \frac{1}{Z_k}\sum_{i=1}^{2N}U_{i-1}\left(\frac{-\Lambda_k}{2}\right)c_i^+, \qquad g_{-k} = \frac{1}{Z_k}\sum_{i=1}^{2N}U_{i-1}\left(\frac{-\Lambda_k}{2}\right)c_i, \quad (1.2.15)$$

où U_i est le i-ième polynôme de Chebyshev de seconde espèce et $(Z_k)^2 = \sum_{i=1}^{2N}\left(U_{i-1}\left(\frac{\Lambda_k}{2}\right)\right)^2$. Ces opérateurs ont été choisis de sorte que pour tous demi-entiers r, s tels que $-\frac{N-1}{2} \le r, s \le \frac{N-1}{2}$,

$$\{f_r, f_s\} = \delta_{r,-s}\,\mathrm{id} = \{g_r, g_s\}, \qquad \{f_r, g_s\} = 0, \quad (1.2.16)$$

$$[H_{XZ}, f_r] = \Lambda_r f_r, \qquad [H_{XZ}, g_r] = \Lambda_r g_r. \quad (1.2.17)$$

On obtient alors la diagonalisation de l'hamiltonien :

$$H_{XZ} = 2\sum_{k=1/2}^{(2N-1)/2} \Lambda_k \,\mathrm{id}_{2^{2N}} - 2\sum_{k=1/2}^{(2N-1)/2} \Lambda_k (f_{-k}f_k + g_{-k}g_k), \tag{1.2.18}$$

où les sommes sont sur les demi-entiers. Pour prendre la limite lorsque $N \to \infty$, il faut également définir

$$\ell_n^{f,N} \equiv \frac{1}{2}\sum_{k\in\mathbb{Z}+1/2} k : f_{n-k}f_k :, \qquad \ell_n^{g,N} \equiv \frac{1}{2}\sum_{k\in\mathbb{Z}+1/2} k : g_{n-k}g_k : \tag{1.2.19}$$

où f_k, g_k sont nuls lorsque $|k| > \frac{2N-1}{2}$ et l'*ordre normal* est donné par la relation

$$f_i f_j =: f_i f_j : +\delta_{i+j,0}\begin{Bmatrix} 1 & \text{si } i > j \\ 0 & \text{sinon} \end{Bmatrix}, \qquad g_i g_j =: g_i g_j : +\delta_{i+j,0}\begin{Bmatrix} 1 & \text{si } i > j \\ 0 & \text{sinon} \end{Bmatrix}.$$

Si l'on suppose que les générateurs f_i, g_i ne sont pas affectés par la limite $N \to \infty$, alors $L_n^f = \lim_{N\to\infty} \ell_n^{f,N}$, $L_n^g = \lim_{N\to\infty} \bar{\ell}_n^{g,N}$ satisfont

$$[L_n^f, L_m^f] = (n-m)L_{n+m}^f + \frac{1}{24}n(n^2-1)\delta_{n+m,0}. \tag{1.2.20}$$

$$[L_n^g, L_m^g] = (n-m)L_{n+m}^g + \frac{1}{24}n(n^2-1)\delta_{n+m,0}. \tag{1.2.21}$$

$$[L_n^f, L_m^g] = 0. \tag{1.2.22}$$

Il suit alors que ces objets engendrent une représentation de l'algèbre de Virasoro avec $c = 1/2$. Puisque dans la limite où N est très grand $\frac{N}{\pi}\Lambda_k \simeq k$, l'hamiltonien renormalisé du modèle devient alors

$$H_{ferm} = L_0^f + L_0^g. \tag{1.2.23}$$

On peut donc conclure que le modèle d'Ising critique en deux dimensions devient invariant conforme dans cette limite continue. Il est généralement conjecturé, à partir d'arguments physiques, que tous les modèles statistiques critiques devraient être invariants conformes dans une certaine limite continue. Le défi consiste alors à construire une approximation des générateurs L_n^f, comme ℓ_n^f, à partir d'opérateurs apparaissant dans le modèle fini. En particulier, il devrait être possible de construire une telle approximation en utilisant les générateurs de Temperley-Lieb.

Plutôt que de construire directement les générateurs, il est souvent plus facile de trouver des « vestiges » de l'invariance conforme, par exemple par la structure des représentations ou par leurs caractères.

1.3 Les algèbres de Temperley-Lieb

Les algèbres de Temperley-Lieb jouent un rôle important dans l'étude des modèles statistiques en deux dimensions. Les modèles d'Ising, Potts, des six vertex, entre autres, peuvent tous être décrits à l'aide de représentations de ces algèbres. La présente section décrit les deux définitions usuelles de ces algèbres de même que les modules les plus communs et leur structure. La section 5.2 présente également un résumé plus détaillé de ces résultats. Finalement, la section 1.3.4 offre une brève introduction à la version diluée de ces algèbres ; un traitement plus détaillé suivra au chapitre 3.

1.3.1 L'algèbre $\mathsf{TL}_n(\beta)$

Pour n un entier strictement positif et $q \in \mathbb{C}^*$, l'algèbre de Temperley-Lieb $\mathsf{TL}_n(\beta)$ est la $\mathbb{C}$-algèbre associative unitaire engendrée par les générateurs e_i, $1 \leq i \leq n-1$ obéissant aux relations

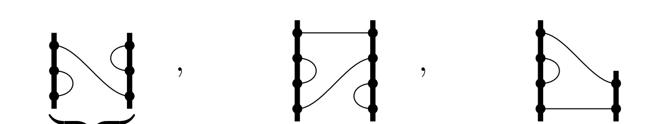
$$\begin{aligned} e_j e_j &= \beta e_j, & j &= 1,2,\dots,N-1, & \beta &= q+q^{-1}, \\ e_i e_j &= e_j e_i, & |i-j| &\geq 2, \\ e_j e_{j\pm 1} e_j &= e_j, & 1 &\leq j, j\pm 1 \leq N-1. \end{aligned}$$

Par exemple,

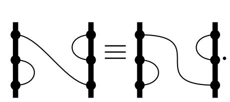
$$\mathsf{TL}_3(q) = \mathrm{span}_{\mathbb{C}}\{1, e_1, e_2, e_1e_2, e_2e_1\}.$$

Il est également utile de définir $\mathsf{TL}_0 \equiv \mathbb{C}$.

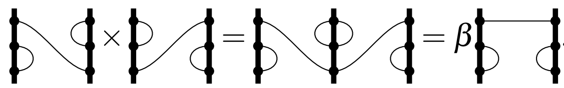
Plutôt que d'être décrite en termes de générateurs et de relations, l'algèbre peut également être définie en termes de (n,n)-*diagrammes*. Un (n,k)-diagramme est composé de deux colonnes verticales de n points à gauche et de k points à droite, les points étant reliés deux à deux par des lignes qui ne se croisent pas, par exemple


$(3,3)$-diagramme, $(4,4)$-diagramme, $(4,2)$-diagramme.

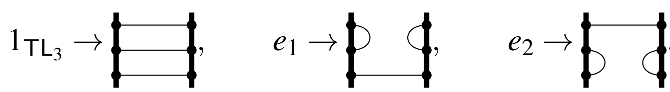
Deux diagrammes sont considérés égaux si et seulement si il est possible de déformer continûment les lignes de l'un vers l'autre. Par exemple,



Le produit d'un (n,k)-diagramme a et d'un (k,m)-diagramme b, noté ab, est défini de la façon suivante : on place d'abord a à gauche de b, puis on identifie les points sur les cotés voisins et on relie les lignes s'y rencontrant. On retire alors ces points et on remplace chaque ligne ne reliant plus aucun point par un facteur β. Par exemple,

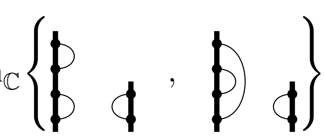


L'algèbre engendrée par les combinaisons linéaires formelles de tout les (n,n)-diagrammes est alors isomorphe à l'algèbre de Temperley-Lieb $\mathsf{TL}_n(\beta)$ avec $\beta = q + q^{-1}$. L'isomorphisme est obtenu en envoyant l'identité vers le diagramme où chaque point est relié à celui en face de lui par une ligne droite, et le générateur e_i est envoyé vers le diagramme où les i-ième et $(i+1)$-ième points à partir du haut sur chaque côté sont reliés entre eux alors que tous les autres points sont reliés à ceux directement en face. Par exemple dans TL_3,

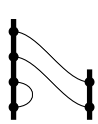
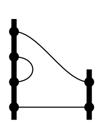
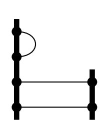


1.3.2 Les modules standards

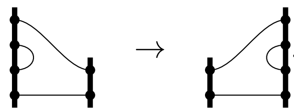
La théorie de la représentation de TL_n se fonde en grande partie sur la structure des *modules standards* $\{\mathsf{S}_{n,k} | k \in [n \mod 2, \dots, n]\}$, ou encore *modules cellulaires* [26] ; le module standard $\mathsf{S}_{n,k}$ est défini à partir des (n,k)-diagrammes. Le bimodule $\mathsf{M}_{n,k}$ est l'espace vectoriel engendré par les combinaisons linéaires formelles des (n,k)-diagrammes avec l'action à gauche de TL_n et à droite de TL_k définie de façon naturelle. Le TL_n-module à gauche $\mathsf{S}_{n,k}$ est alors défini en quotientant $\mathsf{M}_{n,k}$ par l'action à droite de l'idéal $\langle e_1, e_2, \dots, e_{k-1}\rangle \subset \mathsf{TL}_k$, et le TL_k-module à droite $\bar{\mathsf{S}}_{k,n}$ est défini en quotientant par l'action à gauche de TL_n. De façon intuitive le module $\mathsf{S}_{n,k}$ est engendré par les (n,k)-diagrammes où les k points de droite doivent être reliés à des points à gauche, les autres diagrammes étant identifiés à 0. Par exemple pour $n = 4, k = 2$,

$$M_{4,2}\langle e_1 \rangle = \mathrm{Span}_{\mathbb{C}} \Bigg\{$$  $$\Bigg\}, \tag{1.3.1}$$

et $\mathsf{S}_{4,2}$ est engendré par les éléments suivants :

 $+M_{4,2}\langle e_1 \rangle$,  $+M_{4,2}\langle e_1 \rangle$,  $+M_{4,2}\langle e_1 \rangle$.

Les classes d'équivalence dans ces quotients sont appelées des (n,k)-*liens*, et les k lignes rejoignant les deux côtés sont typiquement appelées des *défauts*. Il existe une dualité naturelle entre $\mathsf{S}_{n,k}$ et $\bar{\mathsf{S}}_{n,k}$: chaque (n,k)-lien est envoyé vers le (k,n)-lien obtenus en prenant une réflexion selon l'axe vertical séparant les deux côtés du lien, opération que l'on note $x \to \bar{x}$. Par exemple,



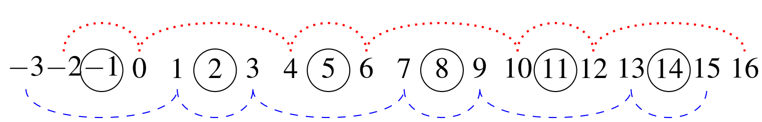
Cette dualité permet de définir une forme bilinéaire sur les modules standards, le *produit de Gram*. En effet, le produit d'un (k,n)-diagramme avec un (n,k)-diagramme s'étend de façon naturelle à un produit d'un (k,n)-lien avec un (n,k)-lien. Il en résulte une transformation $\mathbb{C}$-linéaire $\mathsf{G}_{n,k}$: $\bar{\mathsf{S}}_{n,k} \otimes_{\mathsf{TL}_n} \mathsf{S}_{n,k} \to \mathsf{S}_{k,k}$; puisque le module $\mathsf{S}_{k,k}$ est unidimensionnel, en tant qu'espace vectoriel il est isomorphe à $\mathbb{C}$. On choisit alors l'isomorphisme f qui consiste à envoyer $[1_{\mathsf{TL}_k}] \in \mathsf{S}_{k,k} \to 1$, et il en résulte une forme bilinéaire :

$$\langle x,y\rangle_{n,k} \equiv f(\mathsf{G}_{n,k}(\bar{x} \otimes_{\mathsf{TL}_n} y)). \tag{1.3.2}$$

On vérifie alors aisément que, pour tout $a \in \mathsf{TL}_n$, $\langle x, ay\rangle_{n,k} = \langle \bar{a}x, y\rangle_{n,k}$. Il est important de noter que la forme de Gram n'est pas un produit scalaire ; en particulier il existe parfois des éléments $y \in \mathsf{S}_{n,k}$ tels que $\langle x,y\rangle_{n,k} = 0$ pour tout x. L'ensemble de ces éléments pour lesquels la forme de Gram est triviale est appelé le *radical*[5] de $\mathsf{S}_{n,k}$, et est noté $\mathsf{R}_{n,k}$. De plus, on peut noter que si $x \in \mathsf{S}_{n,k}$ ne fait pas partie du radical, alors $\mathsf{TL}_n x = \mathsf{S}_{n,k}$, et donc $\mathsf{I}_{n,k} \equiv \mathsf{S}_{n,k}/\mathsf{R}_{n,k}$ est irréductible.

La structure des modules standard peut alors être établie à partir de la forme de Gram ; avant de procéder, il faut établir un peu de nomenclature. Si q est une racine de l'unité, on définit ℓ le plus petit entier strictement positif tel que $q^{2\ell} = 1$; si q n'est pas une racine de l'unité on dit que ℓ est infini. Un entier $k \equiv k_c$ est dit *critique* si $k+1$ est un multiple entier de ℓ ; pour un entier non critique k, l'entier k_- (k_+) est défini comme le plus grand (petit) entier strictement plus petit (plus grand) que k tel que $\frac{k+k_-}{2}$ ($\frac{k+k_+}{2}$) soit critique. On définit alors $k_{-2} \equiv (k_-)_-$ et de même pour k_2, k_3, etc. L'ensemble $\{k_i | i \in \mathbb{Z}\}$ est alors appelé *l'orbite de k*. Si k est critique, l'orbite de k ne contient que k lui-même. Par exemple, la figure 1.3.1 montre les orbites entre -3 et 16 lorsque $\ell = 3$. On

FIGURE 1.3.1 – Orbites pour $\ell = 3$: les entiers critiques sont encerclés et les autres éléments sont reliés par les deux types de lignes pointillées qui identifient les deux orbites non critiques.



5. Le mot « radical » réfère ici au radical d'une forme bilinéaire, pas au radical d'un module.

obtient finalement la structure des modules standards.

- L'ensemble $\{\mathsf{I}_{n,k} | 0 \leq k \leq n,\ k \equiv 0 \mod 2\}$ est un ensemble complet de TL_n modules irréductibles non isomorphes, excepté si $\ell = 2$; dans ce cas $\mathsf{I}_{n,0} \equiv 0$.

- Si k est critique ou si $k_+ > n$, alors $\mathsf{I}_{n,k} \simeq \mathsf{S}_{n,k}$; sinon $\mathsf{R}_{n,k} \simeq \mathsf{I}_{n,k_+}$.

- Si k, k' ne sont pas critiques, $\mathsf{Hom}_{\mathsf{TL}_n}(\mathsf{S}_{n,k}, \mathsf{S}_{n,k'}) = (\delta_{k_-,k'} + \delta_{k',k})\mathbb{C}$. En particulier $\mathsf{S}_{n,k}$ est indécomposable.

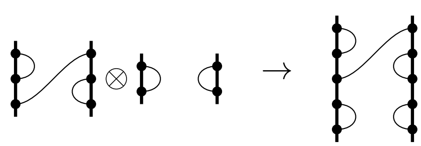
- Si k est critique ou si $k_- < 0$, alors $\mathsf{S}_{n,k}$ est un module projectif ; sinon la couverture projective de $\mathsf{I}_{n,k}$ est le module $\mathsf{P}_{n,k}$ qui est l'unique module indécomposable à satisfaire la suite exacte courte

$$0 \to \mathsf{S}_{n,k_-} \to \mathsf{P}_{n,k} \to \mathsf{S}_{n,k} \to 0. \tag{1.3.3}$$

Plusieurs outils algébriques, comme les modules projectifs, seront introduits lors de la présentation des articles où ils interviennent.

1.3.3 La fusion sur TL

On considère maintenant la famille de Temperley-Lieb $\{\mathsf{TL}_n(q) | n = 0, 1, 2, \ldots\}$. Les divers foncteurs de fusion servent à utiliser l'information sur les représentations de TL_n pour obtenir de l'information sur celles de TL_m, ou vice versa. Pour $m \geq n$, la façon la plus simple de construire de tels foncteurs est de donner à TL_m une structure de $\mathsf{TL}_m - (\mathsf{TL}_n \otimes_{\mathbb{C}} \mathsf{TL}_{m-n})$-bimodule ; cette structure est définie de la façon suivante. On remarque d'abord que l'algèbre $(\mathsf{TL}_n \otimes_{\mathbb{C}} \mathsf{TL}_{m-n})$ est engendrée par les éléments de la forme $a_n \otimes b_{m-n}$ où a_n, b_{m-n} sont des (n,n)-diagrammes et des $(m-n, m-n)$-diagrammes, respectivement. Ces générateurs peuvent alors être transformés en (m,m)-diagrammes en « posant » le diagramme a_n au-dessus du diagramme b_m. Par exemple,



(1.3.4)

Cette transformation permet de définir une action à gauche, ou à droite, de $(\mathsf{TL}_n \otimes_{\mathbb{C}} \mathsf{TL}_{m-n})$ sur TL_m, et ainsi de lui donner une structure de bimodule que nous utiliserons pour définir la fusion. Rappelons que, pour deux algèbres A, B et ${}_AM_B$ un A-module à gauche et un B-module à droite, on fait d'un A-module U un B-module par la transformation :

$$U \to \mathsf{Hom}_A({}_AM_B, U), \qquad bf = (x \to f(xb)), \tag{1.3.5}$$

où $x \in M$, $b \in B$ et $f \in \mathsf{Hom}_A({}_AM_B, U)$. De façon semblable, on fait d'un B-module V un A-module par la transformation

$$V \to_A M_B \otimes_B V. \tag{1.3.6}$$

Soit donc $n, m > 0$ deux entiers, U, V, W des $\mathsf{TL}_n, \mathsf{TL}_m, \mathsf{TL}_{n+m}$-modules, respectivement. On distingue les opérations suivantes :

$$U \times_f V = \mathsf{TL}_{n+m} \otimes_{\mathsf{TL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m} (U \otimes_{\mathbb{C}} V), \qquad \Delta(W) = \mathsf{Hom}_{\mathsf{TL}_{n+m}}(\mathsf{TL}_{n+m}, W), \tag{1.3.7}$$

$$U \bar{\times}_x V = \mathsf{Hom}_{\mathsf{TL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m}(\mathsf{TL}_{n+m}, U \otimes_{\mathbb{C}} V), \qquad \bar{\Delta}(W) = \mathsf{TL}_{n+m} \otimes_{\mathsf{TL}_{n+m}} W, \tag{1.3.8}$$

$$W \div_f V = \mathsf{Hom}_{\mathsf{TL}_{n+m}}(\mathsf{TL}_{n,m} \otimes_{\mathsf{TL}_n \otimes_{\mathbb{C}} \mathsf{TL}_m} (\mathsf{TL}_n \otimes_{\mathbb{C}} V), W). \tag{1.3.9}$$

Le produit $U \times_f V$ définit le *produit de fusion* [24, 61, 62] de U et V, l'opération $\Delta(W)$ définit le *coproduit de fusion* de W et l'opération $W \div_f V$ définit le *quotient de fusion* [9] de W par V. Les produit et quotient de fusion tirent leurs noms de leurs propriétés respectives. Par exemple, on peut vérifier que

$$(U \times_f V) \times_f W \simeq U \times_f (V \times_f W), \qquad (U \div_f V) \div_f W \simeq U \div_f (V \times_f W), \tag{1.3.10}$$

mais qu'en général,

$$(U \times_f V) \div_f V \neq U, \qquad \Delta(U \times_f V) \neq U \otimes_{\mathbb{C}} V. \tag{1.3.11}$$

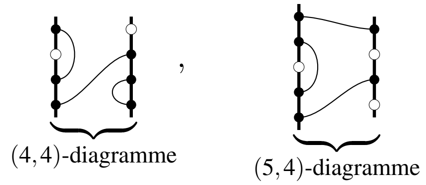
Ces propriétés seront prouvées au chapitre 7. Les produits de fusion des modules projectifs et des modules standards furent calculés dans [24]. Par exemple lorsque q n'est pas une racine de l'unité, le produit de fusion de deux modules standards est donné par

$$\mathsf{S}_{n,r} \times_f \mathsf{S}_{m,s} \simeq \bigoplus_{\substack{t=|r-s| \\ saut=2}}^{r+s} \mathsf{S}_{n+m,t} \quad , \tag{1.3.12}$$

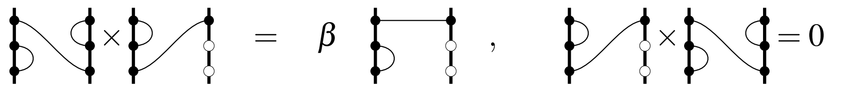
qui rappelle les formules pour l'addition de moments cinétiques en mécanique quantique.

1.3.4 Les algèbres diluées de Temperley-Lieb

Il existe de nombreuses généralisations des algèbres de Temperley-Lieb ; les plus simples sont obtenues en modifiant la définition des algèbres en termes de diagrammes. Par exemple, les algèbres diluées de Temperley-Lieb dTL_n sont définies de la façon suivante. Un (n,k)-diagramme dilué est composé de deux colonnes verticales de n points à gauche et de k points à droite, dont certains sont reliés deux à deux par des lignes qui ne se croisent pas et les autres sont laissés vacants. Par exemple,



Le produit d'un (n,k)-diagramme dilué a et d'un (k,m)-diagramme dilué b est défini de la façon suivante : on place d'abord a à gauche de b, puis on identifie les points sur les cotés voisins et on relit les lignes s'y rencontrant. On retire alors ces points et on remplace chaque ligne ne reliant plus aucun point par un facteur β. Si un point relié par une ligne se trouve identifié à un point vacant, le résultat est l'élément 0. Par exemple,



L'algèbre $\mathsf{dTL}_n(q)$ est alors définie en prenant les combinaisons linéaires formelles des (n,n)-diagrammes dilués avec $\beta = q + q^{-1}$. La structure de l'algèbre et de ses représentations sera établie en détail dans le chapitre 3.

1.4 Les modèles statistiques et les algèbres de Temperley-Lieb

À partir des étapes successives vues à la section 1.2, l'étude du modèle d'Ising en deux dimensions dans la limite hamiltonienne se réduit à l'étude d'une chaîne de spins quantiques. Par le même type d'argument, plusieurs autres modèles statistiques sur réseaux bidimensionnels peuvent se réduire à l'étude de chaînes quantiques en une dimension ; la matrice de transfert du modèle statistique peut alors être interprétée comme un opérateur d'évolution temporelle. Dans ce contexte, les matrices commutant avec la matrice de transfert peuvent être vues comme des quantités conservées. Nous verrons d'abord un exemple de chaîne de spins quantiques décrite par l'algèbre de Temperley-Lieb, puis nous verrons comment dériver ce type de chaîne à partir de la matrice double-ligne. Finalement, nous discuterons d'une généralisation possible : la matrice double-ligne diluée.

1.4.1 La chaîne XXZ quantique.

Une chaîne de spins quantiques ne contenant que des interactions entre voisins immédiats est généralement décrite par un hamiltonien de la forme

$$H = \sum_{i=1} (J_i^x \sigma_i^x \sigma_{i+1}^x + J_i^y \sigma_i^y \sigma_{i+1}^y + J_i^z \sigma_i^z \sigma_{i+1}^z + B_i^x \sigma_i^x + B_i^y \sigma_i^y + B_i^z \sigma^z), \tag{1.4.1}$$

où les matrices σ_j^a sont les matrices de Pauli et les constantes J_i^a et B_i^a mesurent respectivement l'énergie d'interaction et la force du champ magnétique extérieur, dans la direction a à la position i. Les chaînes XXZ sont un cas particulier où $J_i^x = J_i^y$ pour tout i. Nous nous intéressons ici au cas particulier [58]

$$H_{XXZ} = \sum_{j=1}^{N-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \frac{q+q^{-1}}{2} \sigma_j^z \sigma_{j+1}^z + \frac{q-q^{-1}}{2} (\sigma_j^z - \sigma_{j+1}^z) \right),$$

Cet hamiltonien décrit une chaîne de N spins $\frac{1}{2}$ interagissant entre plus proches voisins et dont seuls les bouts ressentent un champ magnétique $\frac{q-q^{-1}}{2}$ orienté selon l'axe des z au bout $j = 1$, et selon $-z$ au bout $j = N$. De plus, les spins aux bords sont supposés libres. On peut alors vérifier que

$$H_{XXZ} = -(N-1)\frac{q+q^{-1}}{4} 1_{2^N} + \sum_{j=1}^{N-1} e_j, \tag{1.4.2}$$

où

$$e_j = \frac{1}{2} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \frac{q+q^{-1}}{2} (\sigma_j^z \sigma_{j+1}^z - 1_{2^N}) + \frac{q-q^{-1}}{2} (\sigma_j^z - \sigma_{j+1}^z) \right), \tag{1.4.3}$$

engendre une représentation [6] de l'algèbre de Temperley-Lieb $\mathsf{TL}_N(\beta)$, avec $\beta = (q+q^{-1})$:

$$\begin{aligned} e_j e_j &= \beta e_j, & j &= 1, 2, \ldots, N-1 \\ e_i e_j &= e_j e_i, & |i-j| &\geq 2 \\ e_j e_{j\pm 1} e_j &= e_j, & 1 &\leq j, j\pm 1 \leq N-1. \end{aligned} \tag{1.4.4}$$

On veut maintenant utiliser la théorie des représentations de $\mathsf{TL}_N(\beta)$ pour décomposer l'espace d'Hilbert de cette chaîne et ainsi simplifier l'étude du problème. Un calcul simple montre alors que

$$[e_k, \sum_{j=1}^{N} \sigma_j^z] = 0, \qquad [H_{XXZ}, \sum_{j=1}^{N} \sigma_j^z] = 0,$$

donc les états propres de l'hamiltonien peuvent être classés selon leur spin total $S_z = \sum_{j=1}^N \sigma_j^z$. De plus, l'espace vectoriel engendré par les états propres ayant un spin total donné est un module de TL_N. Soit alors $M_k(N)$, l'espace vectoriel engendré par les vecteurs $|\theta_1, \theta_2, \ldots, \theta_N\rangle$ tels que k des

6. La littérature consacrée à l' algèbre de Temperley-Lieb utilise deux conventions pour l'expression de β en termes de q : $\beta = q + q^{-1}$ et $\beta \equiv -q - q^{-1}$. Chacun des articles de cette thèse utilise une de ces conventions. Il faut toutefois remarquer que $\mathsf{TL}_N(\beta) \simeq \mathsf{TL}_N(-\beta)$, où l'isomorphisme consiste à envoyer $e_i \to -e_i$; la théorie des représentations de l'algèbre est donc indépendante de la convention choisie.

θ_j valent 1, et $N-k$ valent -1. On vérifie alors directement que

$$(q1_4+e_1)|1,-1\rangle = |-1,1\rangle, \qquad (q^{-1}1_4+e_1)|-1,1\rangle = |1,-1\rangle. \tag{1.4.5}$$

$$e_1|1,1\rangle = e_1|-1,-1\rangle = 0. \tag{1.4.6}$$

Il suit alors que l'algèbre TL_N engendre toutes les permutations des N spins, et donc que

$$M_k(N) = \mathsf{TL}_N|\underbrace{1,\dots,1}_{k},\underbrace{-1,\dots-1}_{N-k}\rangle \simeq M_k(k)\times_f M_0(N-k), \tag{1.4.7}$$

où nous avons utilisé la définition du *produit de fusion* sur les algèbres de Temperley-Lieb. Voici un exemple du calcul de la fusion de $M_3(3)$ avec $M_0(2)$. Par définition,

$$M_3(3) = \mathbb{C}|1,1,1\rangle, \qquad M_0(2) = \mathbb{C}|-1,-1\rangle \tag{1.4.8}$$

$$M_3(3)\otimes_{\mathbb{C}} M_0(2) = \mathbb{C}|1,1,1\rangle \otimes_{\mathbb{C}} |-1,-1\rangle.$$

On fait alors du produit tensoriel un TL_5-module en donnant l'action des générateurs $e_1,\dots,e_4$ sur sa base. Les générateurs e_1,e_2 et e_4 agissent de façon naturelle :

$$e_1(|1,1,1\rangle\otimes|-1,-1\rangle) = (e_1|1,1,1\rangle)\otimes|-1,-1\rangle = 0,$$

$$e_2(|1,1,1\rangle\otimes|-1,-1\rangle) = (e_2|1,1,1\rangle)\otimes|-1,-1\rangle = 0,$$

$$e_4(|1,1,1\rangle\otimes|-1,-1\rangle) = |1,1,1\rangle\otimes(e_1|-1,-1\rangle) = 0.$$

Le générateur e_3 agit seulement de façon formelle : $e_3|1,1,1\rangle\otimes|-1,-1\rangle \neq 0$ est un nouvel élément de $M_3(3)\times_f M_0(2)$. On continue alors à appliquer les générateurs en utilisant les relations (1.4.4) jusqu'à ce que l'on ait une base complète, ce qui donne

$$M_3(3)\times_f M_0(2) = \mathrm{span}_{\mathbb{C}}(\overbrace{|1,1,1\rangle\otimes|-1,-1\rangle}^{x}, t_3x, t_2t_3x, t_1t_2t_3x, t_4t_3x, t_2t_4t_3x, \dots),$$

où $t_i = -q1_{\mathsf{TL}_5}+e_i$. On trouve alors l'isomorphisme en envoyant

$$|1,1,1\rangle\otimes|-1,-1\rangle \to |1,1,1,-1,-1\rangle, \qquad t_3x \to |1,1,-1,1,-1\rangle,$$

$$t_2t_3x \to |1,-1,1,1,-1\rangle, \qquad t_1t_2t_3x \to |-1,1,1,1,-1\rangle, \qquad t_4t_3x \to |1,1,-1,-1,1\rangle,$$

$$t_4t_2t_3x \to |1,-1,1,-1,1\rangle, \qquad t_1t_2t_4t_3x \to |-1,1,1,-1,1\rangle, \qquad t_3t_4t_2t_3x \to |1,-1,-1,1,1\rangle,$$
$$t_3t_4t_1t_2t_3x \to |-1,1,-1,1,1\rangle, \qquad t_2t_3t_4t_1t_2t_3x \to |-1,-1,1,1,1\rangle.$$

La morale de cette discussion est la suivante. Les espaces $M_k(k), M_0(N-k)$ ont une structure extrêmement simple puisqu'ils sont tous deux unidimensionnels ; ils sont respectivement isomorphes à $\mathsf{S}_{k,k}$ et à $\mathsf{S}_{N-k,N-k}$. La structure de $M_k(N)$ peut alors être calculée directement à partir du produit de fusion de ces deux modules. En continuant ainsi, on peut décomposer systématiquement tout l'espace d'Hilbert de la chaîne [24]. Pour diagonaliser H_{XXZ}, il suffira alors de le diagonaliser sur chacun des sommants directs de

$$M(N) = \bigoplus_{j=0}^{N} M_j(N) \simeq \bigoplus_{j=0}^{N} M_j(j) \times_f M_0(N-j). \tag{1.4.9}$$

Par exemple, pour $N = 5$ et q qui n'est pas une racine de l'unité, l'équation (1.3.12) donne :

$$\begin{aligned} M_0(5) &\simeq M_5(5) \simeq \mathsf{S}_{5,5}, \\ M_1(5) &\simeq M_4(5) \simeq \mathsf{S}_{5,5} \oplus \mathsf{S}_{5,3}, \\ M_2(5) &\simeq M_3(5) \simeq \mathsf{S}_{5,5} \oplus \mathsf{S}_{5,3} \oplus \mathsf{S}_{5,1}, \end{aligned}$$

et donc,

$$M(5) \simeq 6\mathsf{S}_{5,5} \oplus 4\mathsf{S}_{5,3} \oplus 2\mathsf{S}_{5,1}. \tag{1.4.10}$$

Les dimensions des modules standards sont

$$\dim(\mathsf{S}_{5,5}) = 1, \qquad \dim(\mathsf{S}_{5,3}) = 4, \qquad \dim(\mathsf{S}_{5,1}) = 5,$$

il suit donc que

$$\dim(M_0(5)) = \dim(M_5(5)) = 1 = \binom{5}{5}, \qquad \dim(M_1(5)) = \dim(M_4(5)) = 1+4 = \binom{5}{1}$$
$$\dim(M_2(5)) = \dim(M_3(5)) = 1+4+5 = \binom{5}{2},$$

comme il se doit. On remarque que la dimension de $M(5)$ est $6+4\times4+2\times5 = 2^5$, alors que le plus grand sommant direct de $M(5)$ est de dimension 5 ; la décomposition (1.4.10) est donc un pas énorme vers la diagonalisation de l'hamiltonien.

Il est toutefois important de remarquer un point subtil de la discussion précédente : qu'est-ce que l'état $|1,1,1,-1,-1\rangle$ a de particulier ? On voit par exemple que l'état $|1,1,-1,1,-1\rangle$ est aussi un générateur de $M_3(5)$, mais que $M_3(5) \neq \mathsf{TL}_5(|1,1,-1\rangle \otimes |-1,1\rangle)$; on vérifie en effet que $\dim(\mathsf{TL}_5(|1,1,-1\rangle \otimes |-1,1\rangle)) = 28 \neq \dim(M_3(5)) = \binom{5}{3} = 10$. En général, on peut vérifier que pour $0 \leq j \leq \min\{k, n-k\}$, il existe un morphisme surjectif

$$f_j : M_{(k-j)}(k) \times_f M_j(n-k) \to M_k(n).$$

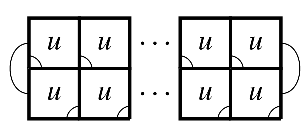
Si on connait déjà la décomposition des $M_{(k-j)}(k)$ et $M_j(n-k)$, il est possible de calculer ces produits de fusion à l'aide des formules développées dans [9,24]. On peut alors vérifier par exemple que pour $j=0$, $\dim(M_{(k-j)}(k) \times_f M_j(n-k)) = \dim(M_k(n))$; puisque f_0 est surjectif, il suit que c'est un isomorphisme. La preuve formelle de ce résultat procède donc par induction sur k pour chaque n. On peut alors remarquer la particularité de l'état $|1,1,1,-1,-1\rangle$: parmi les états $|a_1, a_2, \ldots, a_5\rangle$ c'est celui pour lequel le module $\mathsf{TL}_5(|a_1,a_2,a_3\rangle \otimes |a_4,a_5\rangle)$ est le plus petit.

1.4.2 La matrice double-ligne $D_n(u)$

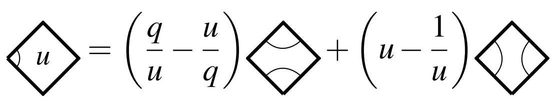
Notre but est maintenant de construire une famille de matrices de transfert commutant entre elles dont les hamiltoniens quantiques sont tous des éléments de l'algèbre de Temperley-Lieb de la forme :

$$H = \sum_{i=1}^{n-1} e_i.$$

Les modèles statistiques décrits par de telles matrices de transfert pourront alors être étudiés en utilisant la théorie de la représentation des algèbres de Temperley-Lieb. On introduit pour ce faire les *matrices double-ligne* $D_n(u)$ [6]. Pour $u \in \mathbb{C}^*$, on a [7]

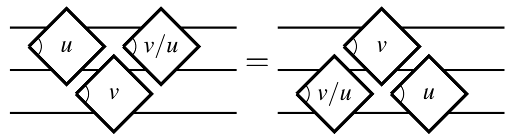
$$D_n(u) = \frac{1}{q^2 - q^{-2}}$$  $\in \mathsf{TL}_n(q)$ (1.4.11)

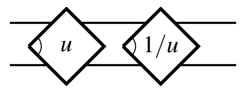
où n est le nombre de colonnes et où on a défini les tuiles

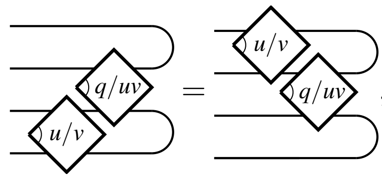
 (1.4.12)

qui représentent des « morceaux » de diagrammes de TL_n et ont été choisies de façon à respecter les trois propriétés suivantes

7. Les diagrammes sont ici tournés de 90° dans le sens anti-horaire, par souci d'espace.

(équation de Yang-Baxter)  (1.4.13)

(relation d'inversion)  $= \rho(q,u)\,\mathrm{id}$ (1.4.14)

(Yang-Baxter aux bords)  (1.4.15)

avec $\rho(q,u) = (qu^{-1} - uq^{-1})(qu - u^{-1}q^{-1})$. Ces propriétés se démontrent toutes de la même façon ; on écrit chacun des côtés de l'égalité en termes de diagrammes et on vérifie que les coefficients de chacun sont les mêmes des deux côtés. On peut utiliser ces équations pour montrer que

$$[D_n(u), D_n(v)] = 0$$

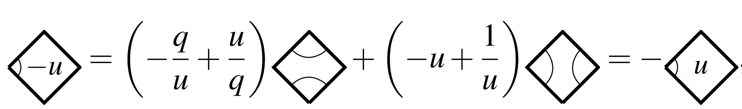
pour tout $u, v \neq 0$. La base de l'argument [6, 60] est d'insérer un produit de tuiles entre deux colonnes en utilisant l'identité d'inversion, puis d'utiliser l'équation de Yang-Baxter pour « pousser » ces tuiles jusqu'aux extrémités. On répète ensuite cette opération pour échanger les lignes dans $D_n(\lambda,u)D_n(\lambda,v)$ jusqu'à obtenir les lignes dans $D_n(v)D_n(u)$.

Autour du point $u = 1$,

$$D_n(u) \simeq (q+q^{-1})\big(q-q^{-1}\big)^{2n}\left(1_{\mathsf{TL}_n} + \frac{4(u-1)}{q-q^{-1}}\big(((q+q^{-1})^{-1} - n(q+q^{-1}))1_{\mathsf{TL}_n} + H_n\big)\right) + O((u-1)^2), \quad (1.4.16)$$

$$H_n = \sum_{i=1}^{n-1} e_i. \quad (1.4.17)$$

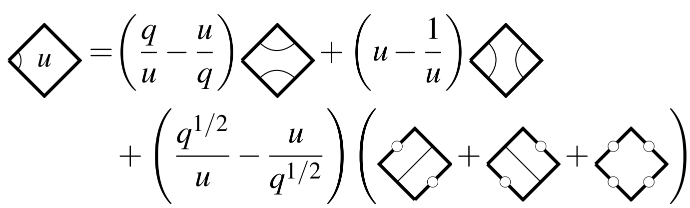
Il suit alors qu'un modèle statistique ayant une matrice de transfert $\rho(D_n(u))$ aura un hamiltonien quantique $\rho(H_n)$. Notons également que $D_n(u)$ peut être vue comme un polynôme de Laurent en u^2 d'ordre n à coefficients dans TL_n. En effet

 (1.4.18)

et puisque $D_n(u)$ contient $2n$ tuiles, $D_n(-u) = D_n(u)$. Il suit donc que bien que u soit un paramètre continu, l'ensemble des $D_n(u)$ engendre un espace vectoriel de dimension au plus $2n+1$.

1.4.3 La matrice double-ligne diluée

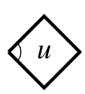
L'algèbre dTL_n étant une généralisation de TL_n, il n'est pas surprenant que celle-ci mène également à des matrices double-ligne ressemblant à celles obtenues dans TL_n. Notons en particulier que l'algèbre dTL_n contient plusieurs sous-algèbres isomorphes à TL_k pour tout $1 \leq k \leq n$; par exemple, pour tout $1 \leq i \leq n$, la sous-algèbre engendrée par tous les diagrammes dilués ayant une position vacante de chaque côté à la position i et aucune autre ailleurs est évidemment isomorphe à TL_{n-1}. Cependant, toutes les matrices résultantes de ces injections décriront des modèles pouvant déjà être décrits par TL_n. Nous tentons donc de construire de nouvelles matrices ne pouvant pas être décrites de façon aussi simple. On procède alors comme pour TL_n en construisant des tuiles respectant les équations spectrales de Yang-Baxter. Une solution simple est de prendre



(1.4.19)

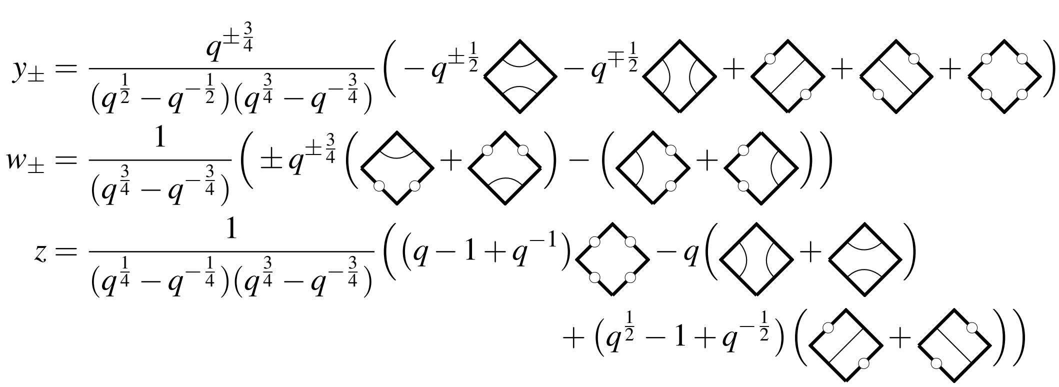
(1.4.20)

Or il appert que ces tuiles produisent des matrices de transfert diagonales par blocs, où chaque bloc est équivalent à une des matrices double-ligne de TL_k. Une solution plus riche est de prendre [37,56]



$$= u^{-2}\cdot y_+ + u^{-1}\cdot w_+ + z + u\cdot w_- + u^2\cdot y_-, \tag{1.4.21}$$

où



Ces tuiles sont utilisées dans certains modèles statistiques dilués tels [17,71].

1.5 Éléments de théorie des champs conformes

La théorie des champs conformes, ci-après CFT, est extrêmement riche ; elle apparaît dans de nombreux domaines, de la théorie des cordes à la physique de la matière condensée. Plutôt que d'en faire une introduction complète et fidèle, nous nous contenterons d'un bref aperçu, basé sur [21], suffisant pour comprendre les enjeux associés aux algèbres de Temperley-Lieb. Après avoir introduit quelques notions de bases, nous discuterons les règles de fusion en CFT et de leur relation conjecturée avec les règles de fusion de la famille de Temperley-Lieb.

1.5.1 Notions de base en CFT

On considère ici un ensemble de champs [8] holomorphes $\Phi = \{\phi^i(z)\}$ définis partout sur le plan complexe, excepté peut-être en un nombre fini de points. Chacun de ces champs peut alors être développé en série de Laurent formelle autour de l'origine

$$\phi(z) = z^{-h} \sum_{j\in\mathbb{Z}} \phi_j z^{-j},$$

où h est la *dimension conforme de* ϕ et les coefficients, appelés *modes*, sont des opérateurs. La dimension conforme d'un champ [9] est telle que pour toute transformation conforme $z \to w$,

$$\phi(w) = \left(\frac{dw}{dz}\right)^{-h} \phi(z). \tag{1.5.1}$$

De façon semblable, on peut développer un produit de champs en série de Laurent formelle dont les coefficients sont eux-même des champs, appelé *développement d'un produit d'opérateurs* (DPO), par exemple

$$\phi(z)\psi(w) = \sum_{j\in\mathbb{Z}} \frac{\{\phi\psi\}_j(w)}{(z-w)^j}. \tag{1.5.2}$$

En général, nous nous intéresserons au comportement d'un développement près du point $z = w$; on introduit alors la relation d'équivalence $\psi(z) \sim \phi(z)$ si et seulement si $\psi_j = \phi_j$ pour tout $j > 0$.

Pour que la théorie soit invariante conforme, il faut quelques hypothèses supplémentaires. On suppose qu'il y a un champ particulier, le *tenseur énergie-impulsion* $\mathsf{T}(z)$ de dimension conforme $h = 2$ et dont les modes L_n forment une représentation de l'algèbre de Virasoro, notée *Vir*, une algèbre de Lie de dimension infinie engendrée par l'ensemble $\{L_n, c | n \in \mathbb{Z}\}$ où le crochet de Lie

8. Pour autant que nous en ayons besoin ici, un champ est une application d'une surface de Riemann vers l'algèbre des endomorphismes d'un espace vectoriel donné, l'espace de Hilbert.

9. Formellement, ceci défini un champ *quasi-primaire* ; nous supposerons que ϕ est un tel champ.

est

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \qquad [c, L_n] = 0. \tag{1.5.3}$$

L'opérateur c est un élément central appelée *charge centrale*, ou *anomalie conforme* ; celui-ci agit toujours comme un multiple de l'identité sur tout Vir-module unitaire, il est donc courant d'identifier c à ce multiple. Nous supposerons également que pour tout champ $\phi(z)$,

$$[L_n, \phi_j] \in \mathrm{Span}_{\mathbb{C}}(\phi_k | k \in \mathbb{Z}). \tag{1.5.4}$$

Étant donné un champ ϕ et un mot $a = L_{i_1}L_{i_2}\dots L_{i_n}$, le champ $a\phi$ est appelé *champ descendant* de ϕ ; la *famille conforme* $[\phi]$ de ϕ est alors donnée par l'ensemble de ses champs descendants.

On peut maintenant considérer l'algèbre W engendrée par l'ensemble des modes de tout les champs $\phi^i(z) \in \Phi$; l'espace de Hilbert de la théorie est alors défini par la donnée d'un W-module M. Typiquement, le module M contient un élément particulier qui engendre tout M : le vide $|0\rangle$; celui-ci est choisi pour que [10] $\lim_{z\to 0}\phi^j(z)|0\rangle$ soit bien défini pour tout les champs dans la théorie. Pour le champ $T(z)$, on demande alors que $L_n|0\rangle = 0$ pour tout $n \geq -1$. Le sous-module engendré par $\lim_{z\to 0}\phi^j(z)|0\rangle$ est parfois appelé le *secteur* $[\phi^j]$; l'équation (1.5.4) revient alors à dire qu'en tant que Vir-module, M est isomorphe une somme directe de secteurs, chacun correspondant à une famille conforme distincte.

Les théories de champs les plus simples sont appelés *modèles minimaux* ; ces théories contiennent un nombre fini de secteurs, et ceux-ci sont irréductibles en tant que Vir-module. Par exemple, la limite continue du modèle d'Ising est un modèle minimal contenant trois secteurs distincts : le secteur identité qui correspond au champ identité (et à $T(z)$), le secteur de spin qui correspond au champ $\sigma(z)$, la limite continue du spin local σ_j, ainsi que le secteur thermique qui correspond à la limite continue de $\sigma_j\sigma_{j+1}$. Dans tous les modèles minimaux l'opérateur L_0, qui correspond à l'hamiltonien de la théorie, est diagonalisable ; ses valeurs propres sont toujours dans $\mathbb{Z}+h$, où h est la dimension conforme du champ qui a engendré le secteur. Par contre, dans une théorie générale, l'opérateur L_0 n'est pas nécessairement diagonalisable. Les *modèles logarithmiques minimaux* sont de ce type ; dans ces théories, les secteurs ne sont pas irréductibles. Ce nom vient du fait que dans une telle théorie, il existe des champs ϕ, ψ dont le DPO est de la forme

$$\phi(z)\psi(w) \sim \log(|z-w|)A(w) + \dots \tag{1.5.5}$$

En particulier, certains calculs numériques suggèrent que la limite continue de la chaîne quantique XXZ (voir section 1.4.1) de même que la limite continue d'autres modèles construit à partir des

10. Cette condition est nécessaire en physique puisque le point $z = 0$ est associé au temps $t = -\infty$; la condition équivaut alors à dire que les champs sont asymptotiquement bien définis.

algèbres de Temperley-Lieb seraient des modèles logarithmiques minimaux.

1.5.2 La fusion en CFT

Les règles de fusion sur une CFT sont définies de la façon suivante. Étant donné deux champs $\phi, \psi \in \Phi$, on considère Ω, l'ensemble de toutes les familles conformes dont au moins un des éléments peut apparaître dans le DPO d'un membre de $[\phi]$ avec un membre de $[\psi]$. La *fusion* de $[\phi]$ et de $[\psi]$ est alors [11]

$$[\phi] \times_f [\psi] = \sum_{[\xi] \in \Omega} [\xi]. \tag{1.5.6}$$

Avec une telle définition, il est évident que le calcul des règles de fusion dans une CFT est une tâche extrêmement non triviale. La fusion de Nahm-Gaberdiel [23,55] est une façon purement algébrique de dériver ces règles ; par contre, elle est définie dans le langage des algèbres d'opérateurs de vertex et son expression exacte dépasse les buts de cette thèse. Nous nous contenterons de présenter un résumé heuristique. Le principe premier est de construire une paire de morphismes $\Delta_1, \Delta_2 : W \to W \otimes_{\mathbb{C}} W$ et de les utiliser pour construire $\bar{W}$ qui est un $(W - W \otimes_{\mathbb{C}} W)$-bimodule. La fusion de deux W-modules U, V alors définie comme étant

$$U \times_f V = \bar{W} \otimes_{W \otimes_{\mathbb{C}} W} U \otimes_{\mathbb{C}} V. \tag{1.5.7}$$

Il est alors possible de montrer, à travers des exemples concrets et des calculs explicites, que le module $\bar{W}$ peut être choisi de façon à ce que cette fusion de W-modules reproduise la fusion (1.5.6). Même sans introduire les morphismes Δ_1, Δ_2, on remarque bien la ressemblance entre les produits de fusion pour les CFT et pour les modules de Temperley-Lieb.

1.5.3 La CFT et les algèbres de Temperley-Lieb

De nombreux exemples portent à croire qu'il y aurait un lien intime entre les théories de champs conformes et les algèbres de Temperley-Lieb [43,58,60]. Le plus simple est certainement le modèle d'Ising : on a vu que le modèle d'Ising sur un réseau cylindrique fini aux bouts libre porte une représentation de TL_n. Dans une certaine limite continue, le modèle devient une théorie de champs conformes : le modèle du fermion libre ; il semble donc qu'il y aurait une façon de construire une CFT à partir de représentations de TL_n par un certain processus limite $n \to \infty$. D'un point de vue algébrique, les espaces de Hilbert du modèle pour une taille de plus en plus grande forment une suite de représentations de $\mathsf{TL}_n, \mathsf{TL}_{n+1}, \mathsf{TL}_{n+2}, \ldots$ Toutefois, puisque TL_n est une sous-algèbre de

11. Formellement, la fusion est définie sur les familles conformes et non sur les champs ; on parle par contre de la fusion de champs par abus de langage.

TL_{n+m} pour tout $m \geq 0$, il est possible d'interpréter ces espaces comme formant une suite de TL_n-modules ; il serait alors raisonnable de penser que la limite continue du modèle soit également un TL_n-module, en plus d'être un Vir-module. Cette limite hypothétique permettrait alors de construire la limite continue de modèles sur réseaux de façon purement algébrique. Il est toutefois important de mentionner que l'existence en soi d'un $\mathsf{TL}_n - Vir$-bimodule n'a rien d'exceptionnel : pour toute paire de R-algèbres A, B, le module $A \otimes_R B$ est un $A - B$-bimodule non nul. Il est donc évident qu'il existe un $\mathsf{TL}_n - Vir$-bimodule qui permette de construire une limite continue de modèles sur réseaux, mais il n'est pas du tout évident que ce sera la limite continue désirée.

Une façon d'essayer de trouver la bonne limite continue d'un TL_n-module repose sur son *caractère*. Pour $\{V_m\}_{m\in\mathbb{N}}$ une suite de $\mathsf{TL}_{n(m)}$-modules, avec $n(m+1) > n(m)$, on construit son caractère de la façon suivante. On commence par diagonaliser l'hamiltonien de Temperley-Lieb $H_n = \sum_{i=1}^{n-1} e_i$ sur V_n, que l'on doit normaliser pour que les plus petites valeurs propres soient indépendantes de n. Il est évident que ceci n'est pas possible sur n'importe quelle suite de modules, mais nous supposerons qu'il est possible de la choisir pour que cette démarche ait du sens. Le caractère de V_n est alors

$$\chi(V_n) = \sum_{\lambda} \dim(V_n(\lambda)) t^{\lambda}, \tag{1.5.8}$$

où la somme porte sur les valeurs propres de H_n et $V_n(\lambda)$ est le sous-espace propre associé à la valeur propre λ. De façon similaire, le caractère d'un Vir-module est défini comme étant

$$\chi(V) = \sum_{\sigma} \dim(V(\sigma)) t^{\sigma}, \tag{1.5.9}$$

où cette fois la somme porte sur les valeurs propres de l'opérateur L_0, qui joue le rôle de l'hamiltonien dans les CFT. On compare alors les deux caractères : si il semble que pour n très grand $\chi(V_n) \simeq \chi(V)$, on conclut que la limite des V_n doit être le module V.

En utilisant cette méthode, Pasquier et Saleur [58] sont parvenus à identifier la limite continue de certains TL_n-modules. On peut alors calculer la fusion de deux TL_n-modules, puis vérifier si la limite de la fusion est bien la fusion des limites. Dans les exemples calculés précédemment [24,61], il semble que ce soit bien le cas, et donc que la fusion de TL_n permettrait peut-être de calculer les règles de fusion dans des CFT. L'article du chapitre 7 montrera explicitement que certains modules de Temperley-Lieb obéissent aux règles de fusion que l'on trouve dans certains modèles minimaux.

1.6 Catégories et foncteurs

Plusieurs de nos résultats seront obtenus, ou décrits, en utilisant des outils tirés de la théorie des catégories ; cette section présente les définitions de base qui seront utilisées dans les articles.

La présentation est inspirée de [1].

Une (petite [12]) catégorie $\mathfrak{C}$ est définie par la donnée d'un ensemble $\mathsf{Ob}(\mathfrak{C})$, les *objets* de la catégorie, et d'un ensemble $\mathsf{Hom}_{\mathfrak{C}}(U,V)$ pour chaque paire d'objets U,V, les *morphismes* de U vers V. De plus, pour tout $f \in \mathsf{Hom}_{\mathfrak{C}}(U,V)$, $g \in \mathsf{Hom}_{\mathfrak{C}}(V,W)$, il existe un morphisme $g \circ f \in \mathsf{Hom}_{\mathfrak{C}}(U,W)$, la *composition* de g et f. Finalement, pour tout objet U il existe un morphisme $\mathrm{id}_U \in \mathsf{Hom}_{\mathfrak{C}}(U,U)$, le *morphisme identité* sur U, et la composition de morphismes est associative. Par exemple, étant donné un groupe fini G, on considère la catégorie $\mathsf{Mod}G$, dont les objets sont les G-modules et les morphismes sont les homomorphismes de groupe. De plus, G lui-même peut être vu comme une catégorie : elle contient un seul objet, l'identité de G, et les endomorphismes de cet objet sont les éléments de G ; la composition de morphismes est alors donnée par le produit d'éléments du groupe.

Pour R un anneau commutatif, la catégorie $\mathfrak{C}$ est dite *R-linéaire* si pour toute paire d'objets U,V, l'ensemble $\mathsf{Hom}_{\mathfrak{C}}(U,V)$ est un R-module, et si la composition de morphismes est aussi R-linéaire. Par exemple, la catégorie $\mathrm{Vect}_{\mathbb{C}}$ dont les objets sont les $\mathbb{C}$-espaces vectoriels et les morphismes sont les transformations $\mathbb{C}$-linéaires est évidemment une catégorie $\mathbb{C}$-linéaire.

Soit alors $\mathfrak{C}$ une catégorie R-linéaire. Pour un morphisme $f \in \mathsf{Hom}_{\mathfrak{C}}(U,V)$, son noyau est une paire $(\mathsf{Ker} f, \mathsf{ker} f)$ où $\mathsf{Ker} f$ est un objet et $\mathsf{ker} f$ est un morphisme de $\mathsf{Ker} f$ dans U tel que pour toute paire (W,g) telle que $f \circ g = 0$, il existe un unique morphisme $g' : W \to \mathsf{Ker} f$ tel que $\mathsf{ker} f \circ g' = g$. Le conoyau de f est alors défini de façon duale : c'est une paire $(\mathsf{Coker} f, \mathsf{coker} f)$ où $\mathsf{Coker} f$ est un objet et $\mathsf{coker} f$ est un morphisme de V dans $\mathsf{Coker} f$ tel que pour toute paire (W,g) telle que $g \circ f = 0$, il existe un unique morphisme $g' : \mathsf{Coker} f \to W$ tel que $g' \circ \mathsf{coker} f = g$. Une catégorie $\mathfrak{C}$ est dite *abélienne* si tous ses morphismes possèdent un noyau et un conoyau, et si pour tout morphisme f, $\mathsf{Ker}(\mathsf{coker} f) \simeq \mathsf{Coker}(\mathsf{ker} f)$, ce qui définit l'*image* de f, par $\mathrm{im}\, f = \mathsf{Ker}(\mathsf{coker} f)$. Par exemple, on vérifie aisément que $\mathrm{Vect}_{\mathbb{C}}$ est abélienne ; pour une transformation linéaire $f : U \to V$, $\mathrm{Ker}\, f$ est le sous-espace de U que f envoie à 0 et le morphisme $\mathrm{ker}\, f$ est simplement l'injection canonique du sous-espace dans U. Le conoyau et l'image de f correspondent de façon similaire aux constructions usuelles.

Soit maintenant une paire de catégories $\mathfrak{C}_1$, $\mathfrak{C}_2$, un *foncteur covariant* $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ est une application qui associe à tout $U \in \mathsf{Ob}(\mathfrak{C}_1)$ un objet $F(U) \in \mathsf{Ob}(\mathfrak{C}_2)$, et à tout morphisme $f \in \mathsf{Hom}_{\mathfrak{C}_1}(U,V)$ un morphisme $F(f) \in \mathsf{Hom}_{\mathfrak{C}_2}(F(U),F(V))$ tel que $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$ et $F(f \circ g) = F(f) \circ F(g)$. Par exemple, si on prend deux groupes G_1, G_2 que l'on voit comme des catégories avec un seul élément, un foncteur de $F : G_1 \to G_2$ doit envoyer l'objet de G_1 vers l'objet de G_2 et agir sur les morphismes tel que $F(gh) = F(g)F(h)$. On reconnaît alors ici la définition des morphismes de groupes. De même, étant donné un homomorphisme de groupes $f : G_1 \to G_2$, on fait d'un

12. Une catégorie est *petite* si ses objets forment bien un ensemble ; en général une catégorie peut contenir trop d'objets pour former un ensemble bien défini. Par exemple, la catégorie des catégories n'est pas petite.

G_2-module M un G_1-module en définissant $ax \equiv f(a)x$ pour tout $a \in G_1$, $x \in M$; il en résulte un foncteur de $\mathsf{Mod}(G_2) \to \mathsf{Mod}(G_1)$, le foncteur restriction. Les foncteurs *contravariants* sont définis de manière similaire, excepté que l'on demande que F associe à chaque morphisme $f : U_1 \to U_2$ un morphisme $F(f) : U_2 \to U_1$ et que la règle de composition soit inversée : $F(f \circ g) = F(g) \circ F(f)$.

Soit une paire de catégories $\mathfrak{C}_1$, $\mathfrak{C}_2$ ainsi qu'une paire de foncteurs $F, G : \mathfrak{C}_1 \to \mathfrak{C}_2$. Une *transformation naturelle* $\phi : F \to G$ est définie par la donnée pour chaque objet $U \in \mathfrak{C}_1$ d'un morphisme $\phi_U : F(U) \to G(U)$ tel que pour tout morphisme $f : U \to V$, $\phi_V \circ F(f) = G(f) \circ \phi_U$. Une transformation naturelle pour laquelle chaque morphisme ϕ_U est un isomorphisme est appelé un *isomorphisme naturel*.

Première partie

Les modules indécomposables de l'algèbre de Temperley-Lieb diluée

Chapitre 2

Présentation

2.1 Objectifs et méthodologie

Les algèbres de Temperley-Lieb diluées (dTL_n) furent introduites [30,56] comme des généralisations des algèbres de Temperley-Lieb originales (TL_n) ; l'espoir premier lors de leur introduction fut qu'elles puissent jouer pour des modèles statistiques dilués, le rôle que joue l'algèbre TL_n dans l'étude des modèles denses. Il est ainsi possible de construire des solutions aux équations de Yang-Baxter spectrales à l'aide des éléments de dTL_n, et donc de construire des matrices de transfert diluées. Le comportement des modèles ainsi construits serait similaire à celui des modèles denses, mais pas identique. Bien que ces algèbres étaient définies depuis un certain temps, la théorie de leurs représentations était largement inconnue.

Cet article établit les éléments principaux de l'étude des modules indécomposables de dTL_n : une liste complète des modules irréductibles et des modules projectifs indécomposables, ainsi que des homomorphismes entre eux. La première section donne la définition de l'algèbre $\mathsf{dTL}_n(\beta)$ en termes de diagrammes, comme dans la section 1.3.4. Nous y calculons entre autres la dimension de l'algèbre en fonction de n et la séparation de l'algèbre en somme directe de deux idéaux disjoints : l'idéal pair et l'idéal impair. La section montre également la présence de multiples sous-algèbres isomorphes à des algèbres de Temperley-Lieb régulières ; ces sous-algèbres joueront un rôle très important dans le reste du papier.

Les modules standards dilués sont ensuite introduits de façon analogue au cas régulier ; la décomposition de dTL_n en sous-algèbres isomorphes à $\mathsf{TL}_{n'}$ est alors utilisée afin de calculer leur dimension et pour prouver qu'ils sont indécomposables. Les foncteurs d'induction et de restriction sont ensuite calculés afin de montrer comment passer l'information sur les modules standards entre dTL_n et dTL_{n+1}.

La forme bilinéaire de Gram est ensuite introduite et son déterminant est calculé. Ceci permet de construire les modules irréductibles de dTL_n. Les foncteurs d'induction et de restriction sont alors utilisés pour déterminer la structure de Loewy des modules standards ainsi que les groupes d'homomorphismes entre eux. On poursuit alors le calcul des foncteurs d'induction et de restriction sur les modules irréductibles .

Finalement, nous montrons que les algèbres dTL_n sont des algèbres cellulaires ; les groupes d'homomorphismes entre modules standards sont alors utilisés afin de construire la matrice de Cartan de l'algèbre ; celle-ci est alors utilisée pour construire les modules projectifs indécomposables à l'aide de résultats fondamentaux sur les algèbres cellulaires.

Contributions

J'ai construit les modules standards et les irréductibles de l'algèbre, calculé leur dimension et établi les grandes lignes de leurs structures. J'ai également fait les calculs des foncteurs agissant sur ces modules. Le reste des résultats ont été établis en collaboration avec mon coauteur. Nous avons tous les deux participé à la rédaction.

2.2 Outils algébriques

On fixe ici un corps algébriquement clos $\mathbb{F}$ et une $\mathbb{F}$-algèbre $\mathcal{A}$ de $\mathbb{F}$-dimension finie. Les définitions sont tirées de [1].

Suites exactes et diagrammes commutatifs

Soit $\{U_i\}_{i\in\mathbb{Z}}$ une famille de $\mathcal{A}$-modules à gauche et $\{f_i : U_i \to U_{i+1}\}_{i\in\mathbb{Z}}$ une famille de $\mathcal{A}$-morphismes. On représente cette famille à l'aide d'une suite

$$\dots \xrightarrow{f_{i-2}} U_{i-1} \xrightarrow{f_{i-1}} U_i \xrightarrow{f_i} U_{i+1} \xrightarrow{f_{i+1}} U_{i+2} \xrightarrow{f_{i+2}} \dots$$

où les flèches représentent les morphismes. Cette suite est un *complexe* si $f_i \circ f_{i-1} = 0$ pour tout i ; ce complexe est alors dit *exact* en U_i si $\operatorname{im} f_{i-1} = \operatorname{Ker} f_i$. Une *suite exacte* est un tel complexe exact en tout U_i. Cette suite est dite *courte*, si $U_i = 0$ pour tout $i \neq 0, \pm 1$; elle est dite *longue* si elle n'est pas courte. Par exemple, pour tout morphisme $f : U \to V$, il existe deux suites exactes courtes naturelles :

$$0 \to \operatorname{Ker} f \to U \to \operatorname{im} f \to 0,$$

$$0 \to \operatorname{im} f \to V \to \operatorname{Coker} f \to 0,$$

et une longue

$$0 \to \operatorname{Ker} f \to U \xrightarrow{f} V \to \operatorname{Coker} f \to 0. \tag{2.2.1}$$

Une suite exacte courte de la forme $0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$ est dite *scindée* s'il existe un morphisme $g' : W \to V$ tel que $g \circ g' = \mathrm{id}_W$, ou de façon équivalente, s'il existe un morphisme $f' : V \to U$ tel que $f' \circ f = \mathrm{id}_U$.

Une famille de morphismes entre des $\mathcal{A}$-modules est souvent représentée par un *diagramme* de la forme

$$\begin{array}{ccccccc} & & \mathsf{U}_1 & \xrightarrow{q_1} & \mathsf{V}_1 & \xrightarrow{p_1} & \mathsf{W}_1 & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathsf{U}_2 & \xrightarrow{q_2} & \mathsf{V}_2 & \xrightarrow{p_2} & \mathsf{W}_2 & & \end{array} \tag{2.2.2}$$

Un tel diagramme est alors dit *commutatif* si tous les chemins sur celui-ci sont équivalents. Par exemple, le diagramme (2.2.2) commute si et seulement si $q_2 \circ f = g \circ q_1$ et $p_2 \circ g = h \circ p_1$. Un résultat particulièrement important sur les diagrammes commutatifs de cette forme est le lemme du *serpent* : si les lignes du diagramme (2.2.2) sont exactes, alors il existe une suite exacte longue

$$\operatorname{Ker} f \xrightarrow{\alpha} \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{Coker} f \to \operatorname{Coker} g \xrightarrow{\omega} \operatorname{Coker} h, \tag{2.2.3}$$

et α (resp. ω) est injectif (surjectif) si q_1 (resp. p_2) l'est.

Soit F un foncteur covariant ; il est dit *exact à gauche* si pour tout morphisme $f : U \to V$, $F(\operatorname{Ker} f) \simeq \operatorname{Ker}(F(f))$, et *exact à droite* si $F(\operatorname{Coker} f) \simeq \operatorname{Coker}(F(f))$. Un foncteur exact à gauche et à droite est simplement dit *exact*.

Classes de modules

Un $\mathcal{A}$-module I est dit *irréductible* ou encore *simple*, s'il ne possède pas de sous-module non trivial ; il est plutôt dit *semi-simple* s'il est isomorphe à une somme directe de modules irréductibles. Un $\mathcal{A}$-module U est dit *indécomposable* s'il ne peut pas être exprimé comme une somme directe de deux modules non triviaux. Un module L est dit *libre* s'il est isomorphe à une somme directe de copies de l'algèbre $\mathcal{A}$ vue comme un module sur elle-même. Finalement, un module P est dit *projectif* s'il existe un module libre L et un module quelconque Q tel que $L \simeq P \oplus Q$. De manière équivalente, P est projectif si et seulement si pour tout module U, V et tous morphismes $f : P \to V, g : U \to V$ avec g surjectif, il existe un morphisme $f' : P \to U$ tel que $g \circ f' = f$. On trouve alors la définition catégorique de ces modules : un module P est projectif si le foncteur $\mathsf{Hom}_{\mathcal{A}}(P, -)$ est exact.

Le théorème (généralisé) d'Artin-Wedderburn stipule qu'en tant que module sur elle-même, l'algèbre $\mathcal{A}$ se décompose comme

$$\mathcal{A} \simeq \bigoplus_{\lambda \in \Lambda} (\dim(I_\lambda)) P_\lambda, \tag{2.2.4}$$

où $\{I_\lambda\}_{\lambda \in \Lambda}$ est un ensemble complet [1] de $\mathcal{A}$-modules irréductibles et P_λ est l'unique module projectif indécomposable tel que $\mathsf{Hom}_{\mathcal{A}}(P_\lambda, I_\sigma) \simeq \delta_{\lambda,\sigma}\mathbb{F}$. L'algèbre $\mathcal{A}$ est dite *semi-simple* si $P_\lambda \simeq I_\lambda$ pour tout $\lambda \in \Lambda$.

L'utilité des modules irréductibles est évidente : par le théorème de Jordan-Hölder, les $\mathcal{A}$-modules peuvent être classés à partir de leurs facteurs de composition. L'utilité des modules projectifs est un peu plus subtile : pour tout $\mathcal{A}$-module U, il existe une paire de modules projectifs P, Q et un morphisme [2] $f : P \to Q$ tel que $U \simeq \operatorname{Coker} f$; la suite exacte

$$0 \longrightarrow \operatorname{im} f \longrightarrow Q \overset{\operatorname{coker} f}{\longrightarrow} U \longrightarrow 0 \tag{2.2.5}$$

est appelée *présentation projective* de U. Une fois que les modules projectifs et les morphismes entre eux sont connus, il est possible de construire tous les autres modules de façon algorithmique. D'une certaine façon, les modules projectifs donne l'ensemble des façons dont les modules irréductibles peuvent être « collés » les uns aux autres pour former des modules plus gros ; ces recollements sont étudiés à travers les groupes d'extension (voir sections 4.2 et 5.2.4). Il faut par contre mentionner que bien que la construction soit algorithmique en principe, en pratique rien ne garantit que l'on soit effectivement capable de procéder de façon analytique : les calculs impliqués pourraient bien s'avérer techniquement trop difficile ou encore ne jamais se terminer si l'algèbre possède une infinité de modules indécomposables non isomorphes.

Théorème de Frobenius

Étant donné un bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ et un $\mathcal{A}$-module à gauche ${}_{\mathcal{A}}U$, le groupe d'homomorphismes $\mathsf{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}U)$ est un $\mathcal{B}$-module à gauche où

$$(bf)(x) = f(xb), \qquad \text{et donc } (a(bf))(x) = (bf)(xa) = f(xab) = ((ab)f)(x).$$

De plus, pour tout $\mathcal{B}$-module à gauche ${}_{\mathcal{B}}V$, on a un isomorphisme de $\mathbb{F}$-modules

$$\mathsf{Hom}_{\mathcal{B}}({}_{\mathcal{B}}V, \mathsf{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}U)) \simeq \mathsf{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M_{\mathcal{B}} \otimes_{\mathcal{B}} V, {}_{\mathcal{A}}U). \tag{2.2.6}$$

1. Ceci signifie qu'un module J est irréductible si et seulement s'il est isomorphe à un des I_λ
2. Les modules P, Q et le morphisme f ne sont jamais uniques.

Ce résultat porte le nom de *théorème d'adjonction*. Dans le cas particulier où $\mathcal{B}$ est une sous-algèbre de $\mathcal{A}$ et le bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ est l'algèbre $\mathcal{A}$ vue comme un module à gauche sur elle-même et comme un $\mathcal{B}$-module à droite, le résultat porte plutôt le nom de *théorème de Frobenius*. On note alors les modules

$$V\uparrow_{\mathcal{B}}^{\mathcal{A}} \equiv {}_{\mathcal{A}}\,\mathcal{A}_{\mathcal{B}} \otimes_{\mathcal{B}} V, \qquad U\downarrow_{\mathcal{B}}^{\mathcal{A}} \equiv \mathsf{Hom}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{B}}, {}_{\mathcal{A}}U) \simeq {}_{\mathcal{B}}\,\mathcal{A}_{\mathcal{A}} \otimes_{\mathcal{A}} U, \tag{2.2.7}$$

qui sont l'*induction* de V et la *restriction* de U, respectivement. Avec cette notation le théorème de Frobenius est simplement

$$\mathsf{Hom}_{\mathcal{B}}(V, U\downarrow) \simeq \mathsf{Hom}_{\mathcal{A}}(V\uparrow, U). \tag{2.2.8}$$

Chapitre 3

The principal indecomposable modules of the dilute Temperley-Lieb algebra

The principal indecomposable modules of the dilute Temperley-Lieb algebra

Jonathan Belletête and Yvan Saint-Aubin

ABSTRACT: The Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$ can be defined as the set of rectangular diagrams with n points on each of their vertical sides, with all points joined pairwise by non-intersecting strings. The multiplication is then the concatenation of diagrams. The dilute Temperley-Lieb algebra $\mathsf{dTL}_n(\beta)$ has a similar diagrammatic definition where, now, points on the sides may remain free of strings. Like TL_n, the dilute dTL_n depends on a parameter $\beta \in \mathbb{C}$, often given as $\beta = q + q^{-1}$ for some $q \in \mathbb{C}^\times$. In statistical physics, the algebra plays a central role in the study of dilute loop models. The paper is devoted to the construction of its principal indecomposable modules.

Basic definitions and properties are first given: the dimension of dTL_n, its break up into even and odd subalgebras and its filtration through $n+1$ ideals. The standard modules $\mathsf{S}_{n,k}$ are then introduced and their behaviour under restriction and induction is described. A bilinear form, the Gram product, is used to identify their (unique) maximal submodule $\mathsf{R}_{n,k}$ which is then shown to be irreducible or trivial. It is then noted that dTL_n is a cellular algebra. This fact allows for the identification of complete sets of non-isomorphic irreducible modules and projective indecomposable ones. The structure of dTL_n as a left module over itself is then given for all values of the parameter q, that is, for both q generic and a root of unity.

Keywords dilute Temperley-Lieb algebra · Temperley-Lieb algebra · principal indecomposable modules · dilute loop models · cellular algebras · Nienhuis weights · O(N) models

3.1 Introduction

Since its introduction in the 1970s [66], the Temperley-Lieb algebra has played a central role in several domains of mathematical physics, mainly in the statistical physics description of lattice models and in conformal field theory. But, since its "rediscovery" by mathematicians — Jones' seminal paper [39] comes here to mind, — algebraists have contributed significantly to its understanding. Its representation theory was first described independently by Goodman and Wenzl [25] and by Martin [45] and is now widely used.

Several generalizations have been introduced, many suggested by physical problems: the periodic (affine) Temperley-Lieb algebra [18, 27–29, 47], polychromatic algebras [32], the Birman-Wenzl-Murakami algebra [14, 54] and the dilute Temperley-Lieb algebra [30]. Their role in mathematical physics has developed over the years, particularly since their intimate relationship with infinite-dimensional Lie algebras appearing in the description of continuum limits of lattice models have been recognized. The fact that some hamiltonians or transfer matrices could be seen as representatives, within given modules, of an abstract element of the Temperley-Lieb algebra was already in Temperley and Lieb's work. But the following fact is Pasquier and Saleur's crucial observation [58]: the representation theory of the Temperley-Lieb algebra can be used to understand the Virasoro representations appearing in the limit, when the mesh goes to zero, of the finite-size lattice models.

The origin of the dilute Temperley-Lieb algebra dTL_n can be tied to Nienhuis' work [56]. It was known since early works by Yang and Baxter that some algebraic conditions on Boltzmann weights of statistical lattice models assure some form of integrability. Trying to find integrable $O(N)$ models, Nienhuis introduced a family of such weights satisfying these conditions. He noticed soon after that these weights, labeled by two parameters λ and u, were part of a larger family defined by Izergin and Korepin [37]. With Blöte he explored the large n limit through numerical simulations [15]. (Note that we use small $n \geq 1$ for the size of the lattice. This integer n is independent of the N appearing in the usual name of the $O(N)$ model.) Under the hypothesis that such lattice models would go to conformal field theories in the limit $n \to \infty$, they found a simple relation between the parameter λ and the central charge of these continuum theories. Nienhuis' weights are attached to the tiles forming the lattice. The various states of the tiles of these models are described by non-intersecting links joining their edges pairwise, exactly as in the Temperley-Lieb description of (fully-packed) loop models. But contrarily to the Temperley-Lieb case, some of the edges of the tiles may be left free of links in dilute models. Generalisations of these dilute models [64,67,72] and sets of integrable boundary conditions [5,71] to match the (bulk) Boltzmann weights were found in the years that followed.

Even though the representation theory of the (original) Temperley-Lieb algebra [25,45] and that of the periodic version [27,47] are well-established, that of the dilute Temperley-Lieb lags behind. A few years ago the dichromatic Temperley-Lieb algebra has been studied [31] and one might be able to retrieve, at least partially, some properties of the dilute dTL_n from some quotient of the dichromatic one. But the dilute Temperley-Lieb algebra $\mathsf{dTL}_n(\beta), \beta \in \mathbb{C}$, has now become such an important tool in mathematical physics that a direct and systematic description of its properties is necessary. The structure uncovered and tools developed should be powerful enough to study questions like, for example, the computation of the fusion ring of its standard and projective modules, the possible existence of a Schur-Weyl duality with some other (quantum) algebra, or the identification of modules in which transfer matrices have non-trivial Jordan structure. The present paper is a first step toward this goal. It gives an explicit construction of all its principal indecomposable modules, for both cases when the algebra $\mathsf{dTL}_n(\beta)$ is semisimple and non-semisimple.

Several approaches surrounding the families of Temperley-Lieb algebras are based on diagrammatic techniques. Several rigorous mathematical works resort to them and they are used to define many lattice models. So it is not surprising that the early construction of the principal indecomposable modules of the (original) Temperley-Lieb algebra TL_n by Martin has been reformulated through methods based on link diagrams [63,70]. It is this approach that we choose to follow here. Both the elements and the multiplication of the dilute Temperley-Lieb algebra $\mathsf{dTL}_n(\beta)$ are defined through diagrams in section 3.2. (Another parameter, $q \in \mathbb{C}^\times$, is also used. It is related to the first by $\beta = q + q^{-1}$.) These definitions lead to the identification of a natural subalgebra $S_n \subset \mathsf{dTL}_n$ and several copies of the usual Temperley-Lieb algebras $\mathsf{TL}_{n'}, n' \leq n$, the computation of its dimension and its decomposition into even and odd parts. A natural filtration of dTL_n by ideals is also introduced here. Section 3.3 is devoted to the construction of standard modules. Their basic characteristics are there established: they are cyclic and indecomposable and their dimensions are expressed in terms of those of the standard modules of TL_n. Restriction and induction are used to probe their inner structure. Section 3.4 introduces another classical tool of representation theory. A bilinear form, called the Gram product, is defined on the standard modules. The radical of this form, that is the subspace of vectors with vanishing Gram coupling with all others, is shown to be the unique maximal submodule of the standard module. The determinant of the Gram matrix representing the bilinear form in some basis is easily computed. Its zeroes occur when the parameter q is a root of unity and, consequently, the algebra dTL_n is semisimple when q is generic, that is when it is not a root of unity. Finally the radical, when it is non-trivial, is shown to be irreducible and isomorphic to the irreducible quotient of another standard module.

In section 3.5 the definition of cellular algebras [26, 50] is recalled and results from previous sections show that dTL_n is indeed cellular. The fundamental properties of cellular algebras then provide a complete list of non-isomorphic irreducible modules and of projective indecomposable ones. Some information about the structure of the standard and principal modules can be retrieved as well as the structure of dTL_n as a left module over itself when q is a root of unity. The induction of the principal modules, from dTL_{n-1} to dTL_n, is finally expressed in terms of the principal ones of dTL_n. This gives an explicit way to construct bases for these modules.

The conclusion reviews the main results and discusses possible extensions. Some results of this paper are based on the analogous ones for the algebra TL_n. These are reviewed in appendix 3.A. Appendices 3.B and 3.C contain some technical computations and proofs. Finally appendix 3.D reviews the algebraic tools that are used throughout the paper, but particularly in section 3.5.

3.2 Basic properties of the dilute algebra dTL_n

This section introduces the dilute Temperley-Lieb algebra whose elements and product are defined diagrammatically. It is shown to split naturally into a direct sum of two ideals, its even and odd parts, and to be filtered by ideals labeled by an integer running from 0 to n. Another subalgebra $S_n \subset \mathsf{dTL}_n$ will play a role in the subsequent section and it is also defined. The section ends with the computation of the dimension of dTL_n. Several techniques used here are borrowed from previous studies of the (original) Temperley-Lieb algebra. Appendix 3.A gathers some basic results for this algebra. Reading this appendix in parallel will ease the understanding of this section and of the next one. Some results of this section and the two next ones will be crucial to recognize dTL_n as a cellular algebra. This will be done in subsection 3.5.1.

Techniques and results are borrowed from previous works. First are diagrammatic methods. These were introduced early on in topology (see, e.g. [40, 42] for landmarks of their use). Martin's book [45] contains n-diagrams and n-links (see his chapter 9), but they do not play a major role in the classification of indecomposable projective modules. But both play a crucial one in Martin and Saleur's definition of the Temperley-Lieb algebra of type B, also known in the physics literature as the "blob algebra" [48]. In topology, representation theory and physics, these diagrammatic methods have shown their power. Second the filtration (3.2.2) of dTL_n is a crucial observation. Again it appears in [48] (see their Proposition 1). But it is in Graham and Lehrer's work [26] that the deep consequences of this filtration are recognized. Our lemma

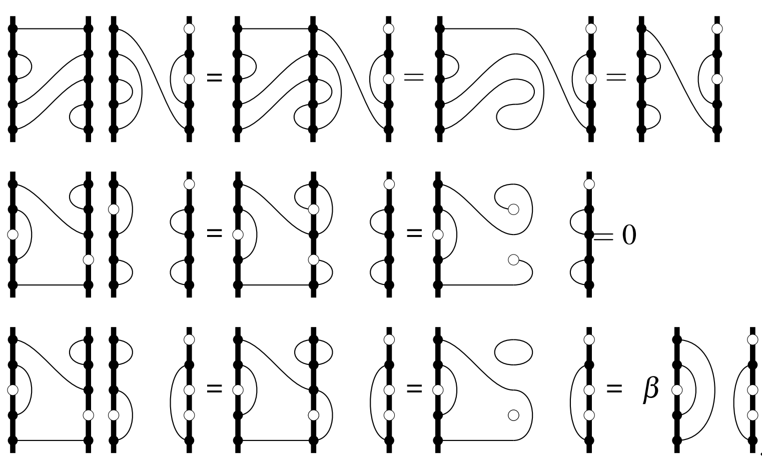
3.4.5 (their proposition 2.6 for cellular algebras) follows from it and is a key step in our study. The filtration's final role will be played in subsection 3.5.1, thanks to Graham and Lehrer's results. A third tool is the Gram bilinear form. It is harder to put a date on its first use in the representation theory of the Temperley-Lieb families, but it was already in use in the early nineties, e.g. [47, 48].

3.2.1 Definition of $\mathsf{dTL}_n(\beta)$

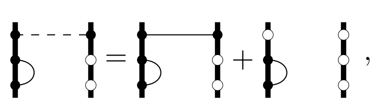
The basic objects, n-diagrams, are first introduced. Draw two vertical lines, each with n points on it, n being a positive integer. Choose first $2m$ points, $0 \le m \le n$ an integer, and put a $\circ$ on each of them. A point with a $\circ$ will be called a *vacancy*. Now connect the remaining points, pairwise, with non-intersecting strings. The resulting object is called a *dilute n-diagram*.

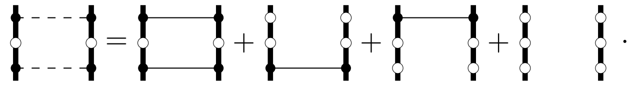
On the set of formal linear combinations of all dilute n-diagrams a product is defined by extending linearly the product of two n-diagrams obtained as follows. The two diagrams are put side by side, the inner borders and the points on them are identified, then removed. A string which no longer ties two points is called a *floating string*. A floating string that closes on itself is called a *closed loop*. If all floating strings are closed loops, the result of the product of the two dilute n-diagrams is then the diagram obtained by reading the vacancies on the left and right vertical lines and the strings between them multiplied by a factor of β for each closed loop. Otherwise, the product is the zero element of the algebra.

The three following products give examples of these definitions. The second contains two floating strings that are not closed and the product is therefore zero, and the third has one closed floating string leading to the factor β:

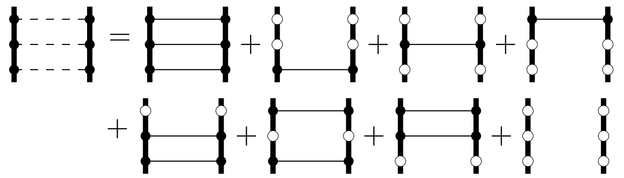


A dashed string represents the formal sum of two diagrams: one where the points are linked by a regular string, and one where the points are both vacancies. For example,



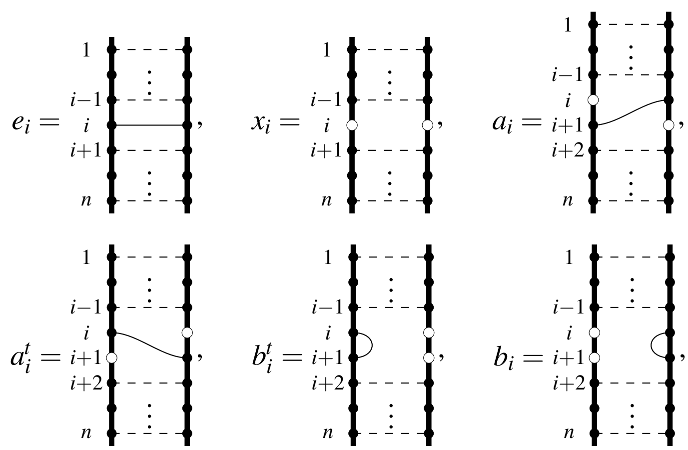


Note that the diagram where each point is linked by a dashed line to the corresponding point on the opposite side acts as the identity on all dilute n-diagrams. It is a sum of 2^n diagrams. For example, when $n = 3$

$\mathsf{id}_3 =$ 

Note finally that the product is clearly associative: the reading of how the left and right sides are connected in a product of three diagrams is blind to the order of glueing, and so is the number of closed loops. The set of n-diagrams with the formal sum and product just introduced is the dilute Temperley-Lieb algebra $\mathsf{dTL}_n = \mathsf{dTL}_n(\beta)$. We also define $\mathsf{dTL}_0 = \mathbb{C}$. When the parameter β is chosen to be a formal one, then the algebra is over $\mathbb{C}[\beta]$. We shall be interested mostly in the case $\beta \in \mathbb{C}$ for which the algebra is over $\mathbb{C}$.

Several generating sets for dTL_n can be found. For instance, the set $\{a_i, a_i^t, b_i, b_i^t, e_j, x_j, i \in [1, n-1], j \in [1, n]\}$ where

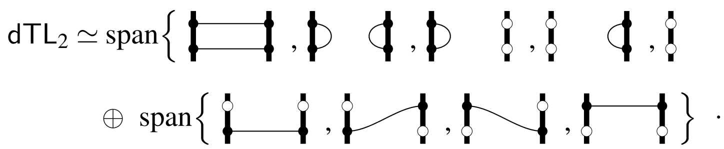


generates the algebra. However, they do not form a minimal set, as for all $1 \leq i \leq n$, $e_i + x_i = \mathsf{id}_n$. Through the identification $u_i = b_i^t b_i$, the connection with the regular n-diagram algebra TL_n should be clear. A set of relations was proposed by Grimm [30] to define dTL_n through generators and relations. The equivalence between the diagrammatic definition, the one used here, and that with relations is stated there without proof.

The numbers of vacancies on either side of a dilute n-diagram always have the same parity. If these numbers are even (odd), the diagram will be called an even (odd) diagram. The subset spanned by only even (odd) diagrams is closed under the product and this subalgebra will be called the even (odd) dilute Temperley-Lieb subalgebra, denoted by edTL_n (odTL_n). Clearly any dilute n-diagram is either even or odd. Since the product of two diagrams of distinct parities is zero, it is clear that the even and odd subalgebras are two-sided ideals of dTL_n and

$$\mathsf{dTL}_n = \mathsf{edTL}_n \oplus \mathsf{odTL}_n.$$

For example



(3.2.1)

The unit $\mathsf{id} \in \mathsf{dTL}_n$ decomposes into $\mathsf{id} = \mathsf{eid} + \mathsf{oid}$ with $\mathsf{eid} \in \mathsf{edTL}_n$ and $\mathsf{oid} \in \mathsf{odTL}_n$. The odd and even units are orthogonal idempotents: $\mathsf{eid}^2 = \mathsf{eid}$, $\mathsf{oid}^2 = \mathsf{oid}$ and $\mathsf{oid} \cdot \mathsf{eid} = \mathsf{eid} \cdot \mathsf{oid} = 0$. (On the previous example of id_3, the four 3-diagrams of the first line of the rhs form eid and the last line is oid.) Let M be a dTL_n-module and decompose it, as vector space, into $M = \mathsf{eid} \cdot M \oplus \mathsf{oid} \cdot M$. Clearly $\mathsf{eid} \cdot (\mathsf{oid} \cdot M) = 0$ and $\mathsf{oid} \cdot (\mathsf{eid} \cdot M) = 0$. But $a = a \cdot \mathsf{eid}$ for any $a \in \mathsf{edTL}_n$ and therefore edTL_n acts trivially on $\mathsf{oid} \cdot M$ and, similarly, so does odTL_n on $\mathsf{eid} \cdot M$. The decomposition into a direct sum of subspaces is thus a direct sum of modules. The two summands $\mathsf{oid} \cdot M$ and $\mathsf{eid} \cdot M$ will be called the odd and even submodules of M. If the odd submodule of M is trivial, M will be said to be *even* and vice versa. An indecomposable module M is either odd or even.

The dilute algebra dTL_n can filtered by a family of ideals defined diagrammatically. Let a *crossing string* be a string in an n-diagram that ties a point on the left vertical line to one on the right and, for each n-diagram $a \in \mathsf{dTL}_n$, define the integer $c = c(a)$ with $0 \leq c \leq n$ to be its number of crossing strings. The diagrammatic definition of the multiplication in dTL_n implies that $c(ab) \leq \min(c(a), c(b))$ for all pairs of n-diagrams a and b for which $ab \neq 0$. Therefore the linear span $I_k \subset \mathsf{dTL}_n$ of all n diagrams a such that $c(a) \leq k$ is an ideal of dTL_n and

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_n = \mathsf{dTL}_n. \tag{3.2.2}$$

Consider now S_n, the subset of dTL_n spanned by dilute n-diagrams having symmetric vacancies, that is, a position on one of their sides is a vacancy if and only if it is also on their other side. Multiplying two symmetric n-diagrams gives either zero if the vacancies do not match perfectly or is a symmetric diagram. The subset S_n is therefore a subalgebra of dTL_n. Now, choose a subset $A \subset \{1, 2, \ldots, n\}$ of ι integers and define $\pi_A = \prod_{i \in A,\, j \notin A} x_j e_i$. Note that $\pi_A^2 = \pi_A$ and thus $\pi_A(\mathsf{dTL}_n)\pi_A$ is a subalgebra of S_n. It is spanned by all n-diagrams with links starting and ending at positions labeled by A and vacancies at all other positions. Therefore $\pi_A(\mathsf{dTL}_n)\pi_A$ is isomorphic to TL_ι and any n-diagram in S_n belongs to precisely one of these subalgebras. For a given ι, there are $\binom{n}{\iota}$ distinct such subalgebras in S_n, all isomorphic to TL_ι. Finally, since the product of two diagrams with different vacancies is always zero, it follows that S_n is isomorphic to the direct sum of all subalgebras $\pi_A(\mathsf{dTL}_n)\pi_A$ obtained from subsets of $A \subset \{1, 2, \ldots, n\}$. We arrive at the following proposition.

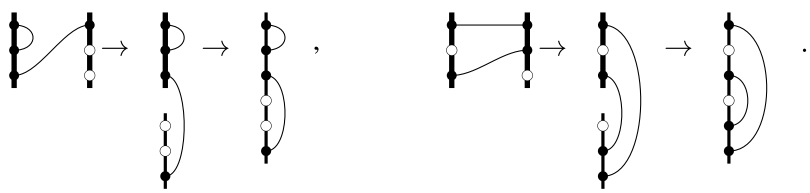
Proposition 3.2.1. *The subalgebra* $S_n \subset \mathsf{dTL}_n$ *is isomorphic to*

$$S_n \simeq \bigoplus_{0\le\iota\le n} \Big(\bigoplus_{1\le p\le\binom{n}{\iota}} \mathsf{TL}_\iota \Big) \tag{3.2.3}$$

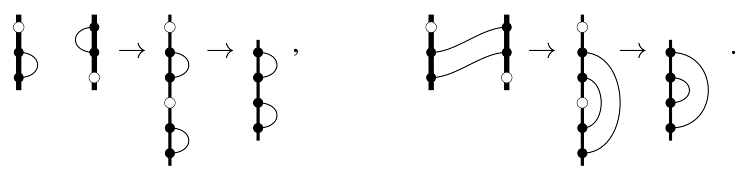
where $\mathsf{TL}_0 = \mathbb{C}$.

3.2.2 The dimension of dTL_n

The ressemblance with the Temperley-Lieb algebra TL_n provides a fairly straightforward method to obtain the dimension of dTL_n. In fact, the same technique of "slicing diagrams" can be used here. The procedure goes as follows: first, take a dilute n-diagram and rotate its right side so that it sits below its left side, stretching the strings so that the points remains connected. Second, connect the two-sides together. For example,



Now, consider the dilute n-diagrams whose vacancies are all at the same places, apply the procedure, then remove the points where the vacancies are. For $n = 3$ and two vacancies located as below, the result look like this:



One should recognize in the results two elements of the link basis of the standard module $V_{4,0}$ of TL_4 or, in general, of the TL_{2m}-module $V_{2m,0}$ with no defects. (See section 3.3 for a formal definition of links and standard modules for dTL_n and also appendix 3.A for their TL_n analogues.) By the reverse procedure just described, it was shown in [63] that $\dim \mathsf{TL}_n = \dim V_{2n,0}$. This leads to the following expression for the dimension of dTL_n.

Proposition 3.2.2. *The dimension of the associative algebra* dTL_n *is*

$$\dim \mathsf{dTL}_n = \sum_{k=0}^{n} \binom{2n}{2k} \dim \mathsf{TL}_k = \sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{2n}{2k}. \tag{3.2.4}$$

where $\mathsf{TL}_0 = \mathbb{C}$.

Proof. Choose $2m \le 2n$ positions and form the subset of dilute n-diagrams that have vacancies at (and only at) these fixed positions. The previous procedure applied to this subset will lead to the link basis of $V_{2(n-m),0}$, irrespective of the chosen positions. Since there are $\binom{2n}{2m}$ different ways of choosing these positions, it follows that the space of dilute n-diagrams with $2m$ vacancies has dimension $\binom{2n}{2m}\dim V_{2(n-m),0}$. The proof is completed by recalling that, for all n, $\dim V_{2n,0} = \dim \mathsf{TL}_n$. ∎

Motzkin numbers $M_n, n \geq 0$, are defined as the number of ways of drawing any number of nonintersecting chords joining n (labeled) points on a circle. The first Motzkin numbers are:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, \ldots$$

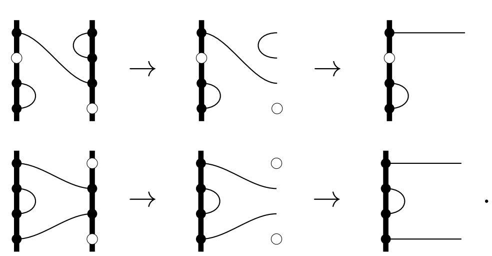
Clearly each n-diagram of dTL_n with its $2n$ points leads to such a drawing of non-intersecting chords on a circle with $2n$ points and vice versa. The dimension of dTL_n is thus the Motzkin number M_{2n} and, for example, $\dim \mathsf{dTL}_8 = M_{16} = 853467$.

3.3 Left (and right) dTL_n-modules

This section introduces some of the basic modules over the dilute Temperley-Lieb algebra dTL_n: the link modules A_n and then the standard modules $\mathsf{S}_{n,k}$. The latter will turn out to form a complete set of non-isomorphic irreducible modules when q is not a root of unity. They will also be seen in subsection 3.5.1 to be the cell modules that are naturally defined for cellular algebras. Some of their properties will be proved here. The modules $\mathsf{S}_{n,k}$ are cyclic and indecomposable, their dimensions can be computed, and both their restriction to dTL_{n-1} and induction to dTL_{n+1} satisfy short exact sequences.

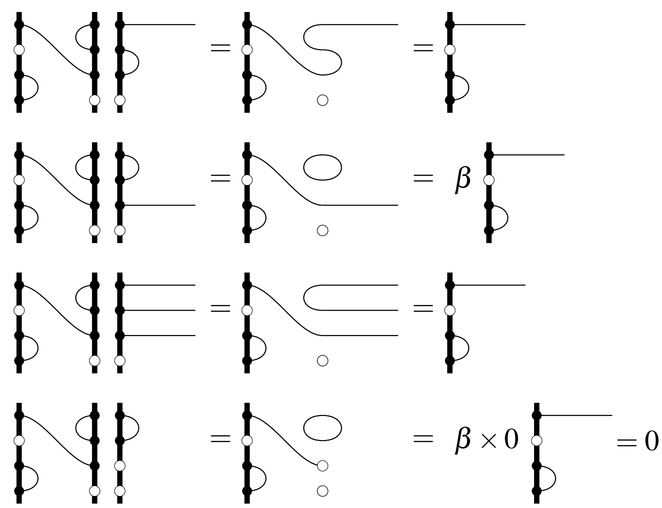
3.3.1 The link modules A_n and $\mathsf{H}_{n,k}$

A left (right) n-link diagram, with $n \geq 1$, is built in the following way. First, take a dilute n-diagram and remove its right (left) side as well as the points that were on it. An object, whether it is a string or a vacancy that no longer touches any point, is simply removed. The other floating strings are straightened out and called *defects*. For example,



The resulting diagram is called a *left n-link* (*right n-link*). It is seen that a dilute n-diagram induces a unique pair of one left and one right n-link diagrams and that, given such a pair, there can be at most one n-diagram, if any, that could have induced them. It will thus be useful to denote an n-diagram by its induced n-links, which we will denote by $b = |lr|$, where l (r) is the left (right) link diagram induced from b. This notation can also be used for linear combinations of n-diagrams as in $b = |(l+j)r| + |uv|$ where l, j, u are left n-links and r, v right ones. If u is a left link, then $\bar{u}$ will denote its (right) mirror image.

A natural action can be defined of dTL_n on left (and right) n-link diagrams. We start with the left action. Draw the n-diagram on the left side of the left n-link, identify the points on its right side with those on the link and remove them. Each floating string that is not connected to the remaining side is removed and yields a factor β if it is closed and zero if it touches a vacancy. If a floating string starting on the remaining side is connected to a defect in the n-link diagram, it becomes a defect. Finally, if a floating string contains two distinct defects of the original diagram, it is simply removed, as any remaining vacancies. The remaining drawing is the resulting n-link diagram, weighted by factors of β, one for each closed floating strings. For example



This action can be extended linearly to any element of dTL_n. Let A_n be the vector space of all formal linear combinations of n-link diagrams. Again the above action can be extended linearly to any element of this space. This action is associative. (The connectivities of each floating string in $(ab)v$ and $a(bv)$, for $a, b \in \mathsf{dTL}_n$ and $v \in \mathsf{A}_n$, are the same.) The vector space A_n is therefore a left dTL_n-module for this action. Right modules can be defined similarly by putting the elements of dTL_n to the right of right n-links. A general element of A_n will be called a *n-link state*. The modules A_n extends the link modules of the Temperley-Lieb algebra. One should note that, unlike for Temperley-Lieb link modules, the number of arcs in dTL_n-link modules can vary freely. However, as in Temperley-Lieb link modules, the action of an element of dTL_n on a link diagram cannot increase its number of defects. The submodule of A_n spanned by n-link diagrams having at most k defects is called $\mathsf{H}_{n,k}$, $0 \le k \le n$, and these submodules $\mathsf{H}_{n,k}$ form a filtration of A_n:

$$\mathsf{H}_{n,0} \subset \mathsf{H}_{n,1} \subset \cdots \subset \mathsf{H}_{n,n} = \mathsf{A}_n. \tag{3.3.1}$$

The submodules $\mathsf{H}_{n,k}$ and the module A_n will be called *link modules*.

3.3.2 The standard modules $\mathsf{S}_{n,k}$

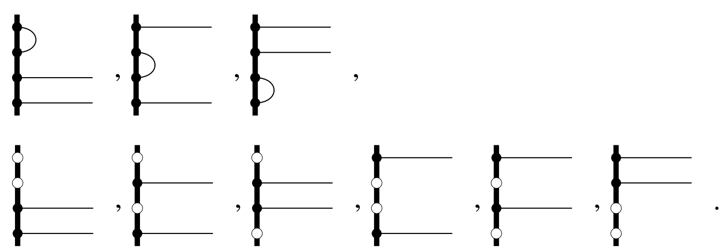
The filtration (3.3.1) leads to the definition of another family of left modules, obtained simply by the quotient of two consecutive link modules $\mathsf{H}_{n,k}$ and $\mathsf{H}_{n,k-1}$, namely:

$$\mathsf{S}_{n,k} \simeq \mathsf{H}_{n,k}/\mathsf{H}_{n,k-1}, \text{ for } 1 \le k \le n, \qquad \text{and} \qquad \mathsf{S}_{n,0} = \mathsf{H}_{n,0}.$$

It will also be useful to set $\mathsf{S}_{n,k} = \{0\}$ for integers $k \in \mathbb{Z}$ not in the set $\{0,1,\dots,n\}$. The left dTL_n-modules $\mathsf{S}_{n,k}$ are called the *standard modules* and extend those of the Temperley-Lieb algebra. (In [63], the standard modules of TL_n were denoted by $\mathcal{V}_{n,p}$ where p stands for the number of arcs. The number of defects is then $n-2p$, as there are no vacancies in the diagrammatic definition of TL_n. As noted before, the number of arcs is not constant in $\mathsf{S}_{n,k}$. This explains the discrepancy in labelling between the present text and [63]. From now on, we shall use defects instead of arcs even for objects related to TL_n and will translate results of [63] accordingly.)

By construction the number of defects is always conserved in $\mathsf{S}_{n,k}$. More precisely, a basis of $\mathsf{H}_{n,k}/\mathsf{H}_{n,k-1}$ can be chosen to be the set of equivalence classes of n-links with precisely k defects. If v is such an n-link diagram, then the class $[v] \in \mathsf{H}_{n,k}/\mathsf{H}_{n,k-1}$ contains a unique n-link with k defects and it is precisely v. For that reason we shall write v for $[v]$.

As an example, the equivalence classes corresponding to the following 4-links form a basis of $\mathsf{S}_{4,2}$:



Note that if $n-k$ is even (odd), then only edTL_n (odTL_n) can act non-trivially on it. That is, only an element of edTL_n (odTL_n) may lead to a non-zero result. For that reason, the standard module $\mathsf{S}_{n,k}$ has a given parity, that of the number $(n-k)$. Also, note that the number of vacancies on a link diagram restricts the elements of dTL_n that can act non-trivially on it. For example, n-diagrams with more than $n-k$ vacancies on either of their sides act as zero on $\mathsf{S}_{n,k}$.

Let $Y_{n,k}$ be the set of left n-links with precisely k defects. By the previous discussion, it is clear that the set $Y_{n,k}$ is a basis of $\mathsf{S}_{n,k}$ (or, more precisely, the set of equivalence classes $[y], y \in Y_{n,k}$, is). Moreover the set $Y_{n,k}, 0 \le k \le n$, can be used to build a basis of dTL_n itself. The glueing of left and right n-links described at the beginning of the section leads to an n-diagram if and only if their number of defects coincide. Note that the ideals appearing in the filtration (3.2.2) are such that the quotient I_k/I_{k-1} has, as a basis, the n-diagrams with precisely k crossing strings. If $C : \sqcup_{0\le k\le n} Y_{n,k} \times Y_{n,k} \to \mathsf{dTL}_n$ denotes the map that sends the pair (x,y) of n-links with k defects onto the n-diagram $|x\bar{y}|$, then the map C is seen to be injective and its image is a basis of dTL_n.

It is also useful to define the subset $X_{n,k} \subset Y_{n,k}$ of n-links having precisely k defects and $n-k$ vacancies. In general the subspace span $X_{n,k}$ *is not* a dTL_n-submodule, but it will be important for the analysis now to be carried.

Let $z \in X_{n,k}$, u and v be any left n-link diagrams in $\mathsf{S}_{n,k}$. For the action in $\mathsf{S}_{n,k}$, the element $|u\bar{v}|$ of dTL_n acts as zero on z unless v and z are equal. Similarly, if $|u\bar{z}|v$ is non-zero in $\mathsf{S}_{n,k}$, then again v and z are equal. (Note that this fails to be true if v is a general link state and not a link diagram. We will see how this property generalizes to link states soon.) Note finally that, for all link states $u \in \mathsf{S}_{n,k}$, $|u\bar{z}|z = u$. This property leads to the following result.

Proposition 3.3.1. *$\mathsf{S}_{n,k}$ is cyclic, with any non-zero element of* span $X_{n,k}$ *being a generator.*

Proof. The property just outlined means that any element z in $X_{n,k}$ is a generator: $(\mathsf{dTL}_n)z = \mathsf{S}_{n,k}$. Let v be a non-zero element in span $X_{n,k}$. Since the elements of $X_{n,k}$ are linearly independent, $v \in$ span $X_{n,k}$ has a non-zero component along some n-link z and $|z\bar{z}|v$ is equal to z up to a non-zero constant. Therefore v is also a generator of $\mathsf{S}_{n,k}$. ■

This property is also used in the following propositions.

Proposition 3.3.2. *$\mathsf{S}_{n,k}$ is indecomposable.*

Proof. Recall that, for any pair of n-link diagrams $u \in \mathsf{S}_{n,k}$ and $z \in X_{n,k}$, $|z\bar{z}|u = z$ if $u = z$ and zero otherwise. So, suppose that $\mathsf{S}_{n,k} \simeq A \oplus B$ for some submodules A and B. Since z generates the whole module, it cannot belong to either A or B, unless one of them is trivial. There must be two non-zero link states $a \in A$ and $b \in B$ such that $z = a + b$, with $z = |z\bar{z}|z = |z\bar{z}|(a+b) = a' + b'$ with $a' = |z\bar{z}|a \in A$ and $b' = |z\bar{z}|b \in B$. If a' is zero, then $b' = z \in B$ and $B = \mathsf{S}_{n,k}$ and $A = \{0\}$. If a' is not zero, it must have a non-zero component along z in the basis of n-links. Therefore $a' = |z\bar{z}|a = \alpha z$ for some $\alpha \in \mathbb{C}^\times$. Again this implies that $A = \mathsf{S}_{n,k}$ and $B = \{0\}$. So $\mathsf{S}_{n,k}$ is indecomposable. ■

Proposition 3.3.3. *$\mathsf{S}_{n,k} \simeq \mathsf{S}_{n,j} \Leftrightarrow k = j$.*

Proof. Only the statement "$\Rightarrow$" is non-trivial. Choose $k \leq j$ and let $\theta : \mathsf{S}_{n,k} \to \mathsf{S}_{n,j}$ be a dTL_n-isomorphism. Choose $x \in X_{n,k}$ and a $\sigma = |u\bar{x}| \in \mathsf{dTL}_n$, with a non-zero $u \in \mathsf{S}_{n,k}$. Then σx is non-zero and, since θ is an isomorphism, so is $\theta(\sigma x) = \sigma\theta(x)$. This means that $\theta(x)$ is a linear combination of states, one of which must have precisely $n-k$ vacancies, all of them coinciding with those of x. Since $j \geq k$, all other positions of this state must be occupied by defects, and j and k must actually be equal. ■

Proposition 3.3.1 has shown that any vector in span $X_{n,k}$ generates the standard module $\mathsf{S}_{n,k}$. But, for the special case $k = n-1$ or n, any n-link diagrams must have precisely 1 and 0 vacancy respectively and $\mathsf{S}_{n,n} =$ span $X_{n,n}$ and $\mathsf{S}_{n,n-1} =$ span $X_{n,n-1}$. Therefore any non-zero vector in these modules generates them and the following result follows.

Corollary 3.3.4. *$\mathsf{S}_{n,n}$ and $\mathsf{S}_{n,n-1}$ are irreducible.*

3.3.3 The dimension of $\mathsf{S}_{n,k}$

The next step is the computation of the dimensions of the standard modules $\mathsf{S}_{n,k}$. This task is made easy by the following ordering of their basis of n-link diagrams. (See below for an example.) First start by ordering the n-link basis by their number of vacancies ι, where $0 \leq \iota \leq n-k$ and $\iota \equiv n-k \bmod 2$. Second, among those with the same number ι of vacancies, gather those whose vacancies are at the same positions. The ordering does not need to be specified any further. Now, for a given number of vacancies and their fixed locations, note that the $(n-\iota)$-link diagrams obtained by omitting the vacant positions are in one-to-one correspondence with elements of the link basis of the Temperley-Lieb standard module $V_{n-\iota,k}$ or, equivalently, $V_{k+2p,k}$ if the number of arcs $p = (n-\iota-k)/2$ is used. The number of arcs must then be in the range $0 \leq p \leq \lfloor (n-k)/2 \rfloor$. For a fixed ι or p, the number of possible positions of the ι vacancies among the n positions is $\binom{n}{\iota} = \binom{n}{k+2p}$. Also, recalling the structure of the subalgebra S_n, the action of this algebra will never change the vacancies of a n-link diagrams. We have therefore proved the following proposition and corollary.

Proposition 3.3.5. *As vector spaces,*

$$\mathsf{S}_{n,k} \simeq \bigoplus_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{k+2p} V_{k+2p,k}. \tag{3.3.2}$$

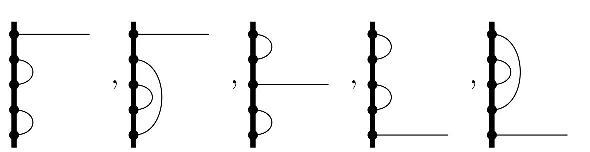
Furthermore, if we consider $\mathsf{S}_{n,k}\!\downarrow_{S_n}^{\mathsf{dTL}_n}$, *the restriction of* $\mathsf{S}_{n,k}$ *to the subalgebra* S_n, *then this isomorphism is also a* S_n*-module isomorphism.*

Corollary 3.3.6. *The dimension of the standard module* $\mathsf{S}_{n,k}$ *is*

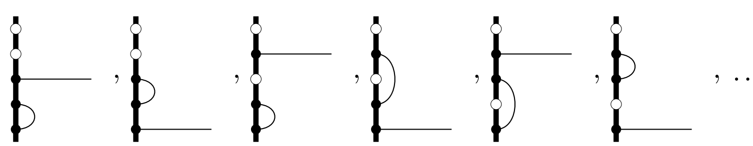
$$\dim \mathsf{S}_{n,k} = \sum_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{k+2p} \dim V_{k+2p,k} \tag{3.3.3}$$

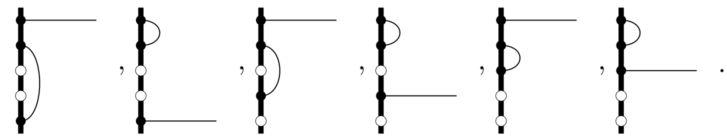
where $\dim V_{n,k} = \binom{n}{(n-k)/2} - \binom{n}{(n-k)/2-1}$.

Here is an example, for the module $\mathsf{S}_{5,1}$, of the ordering used in the proof. The subset of 5-links without any vacancy ($p=2$) form a basis of the TL_5-module $V_{5,1}$:

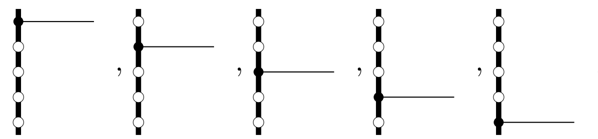


Now the subset with $\iota = 2$ vacancies ($p=1$) contains 20 link diagrams:





Even though some have been omitted, it is clear that, for fixed vacancy positions, the occupied positions are 3-link diagrams and these form a basis of $V_{3,1}$. Finally the subset with $\iota = 4$ vacancies ($p = 0$) is



This subset contains $\binom{5}{4} = 5$ copies of the 1-link state with a single defect, that is a basis of $V_{1,0}$. The dimension of $\mathsf{S}_{5,1}$ is 30.

Another expression [1] is known for the dimensions of the standard modules, namely

$$\dim \mathsf{S}_{n,k} = \binom{n}{k}_2 - \binom{n}{k+2}_2$$

where the trinomial coefficients are defined by the relation $(x+1+x^{-1})^n = \sum_{-n\le k\le n} \binom{n}{k}_2 x^l$. This can be proved by mapping the link basis of $\mathsf{S}_{n,k}$ on leftward walks starting from the origin in $\mathbb{Z}^2$. (See [63].) The ith step of the walk is determined by the ith position of the link: a step $(1,0)$ is taken for a vacancy, a step $(1,1)$ for the opening of the loop or a defect, and $(1,-1)$ for the closing of a loop. All walks corresponding to elements of the basis of $\mathsf{S}_{n,k}$ end at (n,k). All walks visiting only points with non-negative vertical coordinates correspond to n-links with k defects.

The same method of slicing and unfolding n-diagrams used in section 3.2.2 to obtain the dimensions of dTL_n can be used again while keeping track of the number of defects. This leads to another expression for the dimension of the algebra.

Proposition 3.3.7. *The dimension of the dilute Temperley-Lieb algebra* dTL_n *is also given by*

$$\dim \mathsf{dTL}_n = \sum_{k=0}^{n} (\dim \mathsf{S}_{n,k})^2. \tag{3.3.4}$$

3.3.4 The restriction of $\mathsf{S}_{n,k}$

The next two subsections are devoted to the restriction and induction of the standard modules $\mathsf{S}_{n,k}$. The first step, for the study of the restriction, is to decide how the subalgebra dTL_{n-1} is embedded into dTL_n. The embedding that we use is realized by adding a pair of points at the bottom of all $(n-1)$-diagrams (the nth points) and connecting this pair by a dashed line. As the dashed line is seen to act as the identity on the nth points, this is a natural embedding, similar to the one used for the Temperley-Lieb algebra [63, 70]. Any

1. We thank A. Morin-Duchesne for bringing this formula to our attention.

$(n-1)$-diagram of dTL_{n-1} is then embedded as the sum of two n-diagrams of dTL_n. The module $\mathsf{S}_{n,k}$ seen as a dTL_{n-1}-module will be called the *restriction of* $\mathsf{S}_{n,k}$ and denoted by $\mathsf{S}_{n,k}\downarrow$.

Proposition 3.3.8. *With the embedding of* dTL_{n-1} *in* dTL_n *described above, the short sequence*

$$0 \to \mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k-1} \to \mathsf{S}_{n,k}\downarrow \to \mathsf{S}_{n-1,k+1} \to 0 \tag{3.3.5}$$

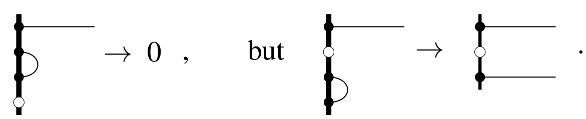
is exact for all $n \geq 2$ *and* $k \in \{0,1,\dots,n\}$ *and therefore*

$$\mathsf{S}_{n,k}\downarrow/(\mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k-1}) \simeq \mathsf{S}_{n-1,k+1}. \tag{3.3.6}$$

Again $\mathsf{S}_{m,j} = \{0\}$ *if* $j \notin \{0,1,\dots,m\}$.

Proof. To show exactness at $\mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k-1}$, an injective map $\phi : \mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k-1} \to \mathsf{S}_{n,k}\downarrow$ needs to be constructed. Consider the operation, defined on $(n-1)$-links with k or $k-1$ defects, that consists in adding a point at the bottom of the diagram and putting a defect there if the diagram had $k-1$ defects, and a vacancy otherwise. The result is an n-link with precisely k defects. Let ϕ be the map that extends linearly this operation to $\mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k-1}$. Since the elements of dTL_{n-1} do not act on the nth point, this is a homomorphism. It should also be clear that it is injective.

To define a homomorphism $\psi : \mathsf{S}_{n,k}\downarrow \to \mathsf{S}_{n-1,k+1}$ such that $\ker\psi = \operatorname{im}\phi$, we again start by defining a diagrammatic operation on n-links. If an n-link diagram has a defect or a vacancy at its nth position, it is sent to zero in $\mathsf{S}_{n-1,k+1}$. Otherwise, its nth point is simply removed and the arch which ended at this point is replaced by a defect at its entry (top) point. For example,



The map ψ is defined as the linear extension to $\mathsf{S}_{n,k}\downarrow$ of this operation defined on links. To see that this is a homomorphism, suppose that an n-diagram in $\mathsf{dTL}_{n-1} \subset \mathsf{dTL}_n$ transforms the bubble ending at position n of an n-link into a defect or a vacancy. This can only be achieved if the opening point of the bubble is linked to a defect by a bubble on the right side of the n-diagram. The same diagram applied to the image of the link would then link two of its defects together and would thus correspond to the zero element in $\mathsf{S}_{n-1,k+1}$. So ψ is indeed a homomorphism. The map has been constructed so that $\operatorname{im}\phi \subset \ker\psi$.

To see that ψ is surjective, we construct a pre-image for a general $(n-1)$-link in $\mathsf{S}_{n-1,k+1}$. Any such a link has at least one defect since $k+1$ is a positive integer. Then add an nth point to the diagram and close the lowest defect in the link onto this new position n. This is then an element of $\mathsf{S}_{n,k}\downarrow$ whose image by ψ is the original $(n-1)$-link. This construction also shows that there is a one-to-one correspondence between n-links in $\mathsf{S}_{n,k}\downarrow$ that have a bubble ending at n and $(n-1)$-links in $\mathsf{S}_{n-1,k+1}$. Therefore $\operatorname{im}\phi$ and $\ker\psi$ must coincide.

Note finally that the previous constructions for ϕ and ψ remain valid when $k = 0, n-1$ or n if the modules

$\mathsf{S}_{n-1,-1}$, $\mathsf{S}_{n-1,n}$ and $\mathsf{S}_{n-1,n+1}$ are taken to be the trivial ones. ■

Note that the exact sequence gives a simple relationship between the dimensions of the $\mathsf{S}_{n,k}$s:

$$\dim \mathsf{S}_{n,k} = \dim \mathsf{S}_{n-1,k} + \dim \mathsf{S}_{n-1,k-1} + \dim \mathsf{S}_{n-1,k+1} \,. \tag{3.3.7}$$

This property could also be proved using the dimension (3.3.3) of $\mathsf{S}_{n,k}$. The module $\mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k-1}$ is a direct sum of two submodules of distinct parities. Since $\mathsf{S}_{n-1,k+1}$ has the parity of $\mathsf{S}_{n-1,k-1}$, the submodule $\mathsf{S}_{n-1,k}$ of $\mathsf{S}_{n,k}\!\downarrow$ is the largest of its parity.

Proposition 3.3.9. *Let $\beta = q + q^{-1}$ with $q \in \mathbb{C}^\times$. If $q^{2(k+1)} \neq 1$, the sequence*

$$0 \to \mathsf{S}_{n-1,k-1} \to \mathsf{S}_{n,k}\!\downarrow / \mathsf{S}_{n-1,k} \to \mathsf{S}_{n-1,k+1} \to 0 \tag{3.3.8}$$

splits and therefore $\mathsf{S}_{n,k}\!\downarrow / \mathsf{S}_{n-1,k} \simeq \mathsf{S}_{n-1,k-1} \oplus \mathsf{S}_{n-1,k+1}$.

Proof. This proof uses the central element F_{n-1} defined in appendix 3.B. Since F_{n-1} is central, its (generalized) eigenspaces in a given dTL_{n-1}-module are submodules. The appendix shows that F_n acts on the standard module $\mathsf{S}_{n,k}$ as $\mathsf{F}_k \times \mathrm{id}$ with $\mathsf{F}_k = q^{k+1} + q^{-(k+1)}$. If $\mathsf{F}_{n-1,k-1}$ and $\mathsf{F}_{n-1,k+1}$ are different, then $\mathsf{S}_{n,k}\!\downarrow / \mathsf{S}_{n-1,k}$ will contain two eigenspaces of F_{n-1} of dimensions $\dim \mathsf{S}_{n-1,k-1}$ and $\dim \mathsf{S}_{n-1,k+1}$ respectively. The exercise consists then in deciding when the two eigenvalues $\mathsf{F}_{n-1,k-1}$ and $\mathsf{F}_{n-1,k+1}$ are distinct. Their difference is:

$$\mathsf{F}_{n-1,k+1} - \mathsf{F}_{n-1,k-1} = q^{k+2} - q^k + q^{-(k+2)} - q^{-k} = (q^2-1)\left(q^{2(k+1)} - 1\right) q^{-k-2}$$

and vanishes if and only if $q^{2(k+1)} = 1$. ■

The condition that $q^{2(k+1)} \neq 1$ will be fundamental for the rest of the text. An integer k will be called *critical* if $q^{2(k+1)} = 1$, and *generic* otherwise. We also say that $\mathsf{S}_{n,k}$ is critical if k is.

3.3.5 The induction of $\mathsf{S}_{n,k}$

After studying the restriction of the dTL_n-module $\mathsf{S}_{n,k}$, it is natural to ask whether its induction is also part of an exact sequence similar to that satisfied by its restriction. This subsection answers this question.

The induction of $\mathsf{S}_{n,k}$, denoted by $\mathsf{S}_{n,k}\!\uparrow$, is defined by the tensor product

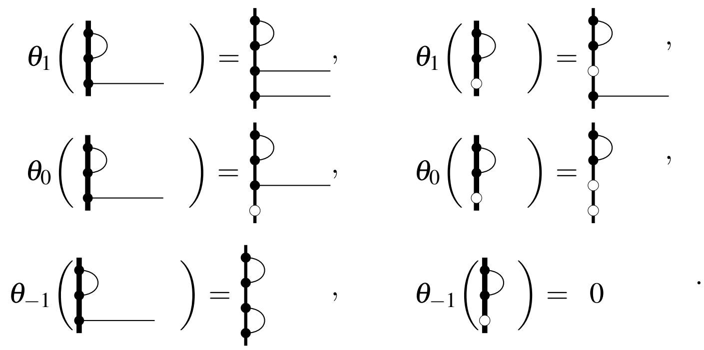
$$\mathsf{S}_{n,k}\!\uparrow = \mathsf{dTL}_{n+1} \otimes_{\mathsf{dTL}_n} \mathsf{S}_{n,k}$$

where the subscript on the tensor product symbol means that the elements of dTL_n (embedded in dTL_{n+1} as in the previous subsection) may pass freely from one of its sides to the other.

The first task is to find a finite generating set for $\mathsf{S}_{n,k}\!\uparrow$ of manageable size. Proposition 3.3.1 provides a first simplification. Let z be an n-link diagram in $X_{n,k}$. Since $\mathsf{S}_{n,k} = \mathsf{dTL}_n z$, then

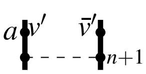
$$\mathsf{S}_{n,k}\!\uparrow = \mathsf{dTL}_{n+1} \otimes_{\mathsf{dTL}_n} (\mathsf{dTL}_n z) = \mathsf{dTL}_{n+1} \otimes_{\mathsf{dTL}_n} z. \tag{3.3.9}$$

A further simplification is possible. We introduce for this purpose three "surgeries" θ_i, $i \in \{-1,0,1\}$, that transforms an n-link diagram $u \in \mathsf{S}_{n,k}$ into an $(n+1)$-link one. The first θ_1 adds to the n-link u a defect at the bottom, at position $n+1$, and the second θ_0 adds there a vacancy. The last one, θ_{-1}, closes the lowest defect of u into an arc ending at $n+1$, if such a defect exists. If there is none, θ_{-1} sends the n-link to zero. The index on the θ_i indicates how the number of defects changes. Here are some examples.



We now argue that any non-zero element of $\mathsf{S}_{n,k}\!\uparrow$ can be written as a sum of terms of the form $|u\overline{\theta_i(z)}| \otimes z$ where $i \in \{-1,0,1\}$ and $u \in \mathsf{S}_{n+1,k+i}$. It is sufficient to study elements of dTL_{n+1} of the form $|u\bar{v}|$ with u and v left $(n+1)$-link diagrams.

The first case to study is when v is in $X_{n+1,j}$ for some j. It is then possible to write $|u\bar{v}| = |u\bar{v}||v\bar{v}|$. Let v' be the n-link diagram obtained from v by deleting its position $n+1$ and the vacancy or the defect at this position. Then

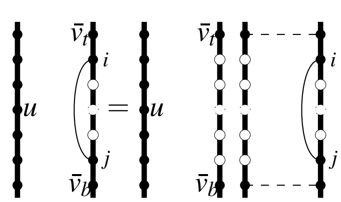
$$|v\bar{v}| = a$$ 

where a stands for the generator x_{n+1} if v has a vacancy at $n+1$ and for the generator e_{n+1} if instead it has a defect there. (The elements x_i and e_i were defined in subsection 3.2.1.) Therefore

$$|u\bar{v}| \otimes z = |u\bar{v}|a \otimes |v'\bar{v}'|z$$

with the appropriate a. This tensor product is zero unless v' is equal to z. That is, when v is an $(n+1)$-link with only defects and vacancies, the vector $|u\bar{v}| \otimes z$ is non-zero only when $v = \theta_0(z)$ if position $n+1$ of v is vacant and when $v = \theta_1(z)$ if it bears a defect.

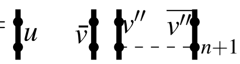
The second case to study is when v contains an arc between two positions above $n+1$. It is then always possible to find an arc between i and j such that $1 \le i < j \le n$ and that all positions k in v with $i < k < j$ are vacant. Then

$$|u\bar{v}| =$$  (3.3.10)

where v_t (v_b) contains the pattern of positions above i in v (below j). There might be arcs going from v_t

to v_b as well as arcs between u and the v_t and v_b. Consider the rightmost factor of (3.3.10). All positions corresponding to those of v_t and v_b are occupied by dashed lines. The n top positions of this factor is an element of dTL_n and annihilates z, because either the arc joins two defects in z or there is a mismatch between the vacancies and defects of z and those of this factor. Element $|u\bar{v}| \otimes z$ with v with such an arc are zero and can therefore be ignored.

The third and last case is when v contains a single arc whose bottom point is at position $n+1$. If this arc joins position i to $n+1$, then all positions in between must be vacancies. A factorization similar to that used in the first case leads to

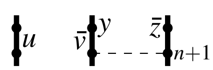
$|u\bar{v}| =$ 

where v'' is obtained from v by deleting its position $n+1$ and putting a defect at position i. Then $|u\bar{v}| \otimes z = |u\bar{v}| \otimes |v''\overline{v''}|z$ and $|u\bar{v}| \otimes z$ is non-zero only if $v'' = z$. Hence, when v has a single arc ending at $n+1$, the element $|u\bar{v}| \otimes z$ is non-zero only if $\theta_{-1}(z) = v$. The analysis of the above three cases is summed up by saying that $\mathsf{S}_{n,k}\!\uparrow = \text{span } B_{n,k}$ where $B_{n,k}$ is the finite set

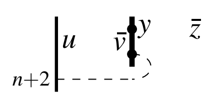
$$B_{n,k} = \Big\{ \ |u\overline{\theta_i(z)}| \otimes z \ \mid \ i \in \{-1,0,1\} \text{ and } u \text{ a link diagram in } \mathsf{S}_{n+1,k+i} \Big\}.$$

The analysis does not prove that $B_{n,k}$ is a basis however. It does not even rule out some of the elements in $B_{n,k}$ being zero. The main remaining result of the present subsection is that $B_{n,k}$ is indeed a basis.

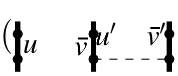
Choose $z \in X_{n,k}$ and let $\phi = \phi_z$ be the linear map $\mathsf{dTL}_{n+1} \otimes_{\mathbb{C}} \mathsf{S}_{n,k} \to \mathsf{S}_{n+2,k}$ defined by the following action on elements of the form $|u\bar{v}| \otimes_{\mathbb{C}} y$ where u and v are $(n+1)$-links with the same numbers of defects and $y \in \mathsf{S}_{n,k}$. (The index on the tensor product sign will be omitted only if it is dTL_n.) To compute $\phi(|u\bar{v}| \otimes_{\mathbb{C}} y)$, first draw



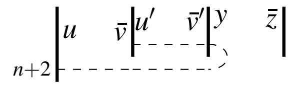
and then detach the dashed line ending at position $(n+1)$ on the far right to attach it at the bottom of u:

 (3.3.11)

The object created has $n+2$ positions on its left edge, n on its right one. There are $n+1$ positions on both sides of the central line and one can use the usual rules to multiply diagrams for this new object. If vacancies do not match, then $\phi(|u\bar{v}| \otimes_{\mathbb{C}} y)$ is set to zero. If they match, then there exists $w \in \mathsf{S}_{n+2,j}$ and $x \in \mathsf{S}_{n,j}$ such that the above diagram is $\beta^{\#}|w\bar{x}|$ where # is the number of closed loops in (3.3.11). The image $\phi(|u\bar{v}| \otimes_{\mathbb{C}} y)$ is non-zero only if $j = k$ and it is then $\beta^{\#}w$. Note that, if u', v' are some n-links with the same numbers of defects, then

$\phi\Big($  $\otimes_{\mathbb{C}} y \Big) = \phi(|u\bar{v}| \otimes_{\mathbb{C}} |u'\bar{v}'|y)$

because the computation of the resulting image is based in both cases on the diagram



Therefore ϕ maps to zero the subspace spanned by $\{ab\otimes_{\mathbb{C}} y - a\otimes_{\mathbb{C}} by, a\in \mathsf{dTL}_{n+1}, b\in\mathsf{dTL}_n, y\in \mathsf{S}_{n,k}\}$. The linear map ϕ thus induces a well-defined linear map

$$\Phi : \mathsf{S}_{n,k}\uparrow \simeq \frac{\mathsf{dTL}_{n+1}\otimes_{\mathbb{C}}\mathsf{S}_{n,k}}{\langle ab\otimes_{\mathbb{C}} y - a\otimes_{\mathbb{C}} by\rangle} \to \mathsf{S}_{n+2,k}\downarrow \tag{3.3.12}$$

Proposition 3.3.10. *Let $n\geq 1$ and $k\in\{0,1,\dots,n\}$. Then*
(i) the set $B_{n,k}$ is a basis of $\mathsf{S}_{n,k}\uparrow$ and
(ii) $\mathsf{S}_{n,k}\uparrow\simeq \mathsf{S}_{n+2,k}\downarrow$ as dTL_{n+1}-modules.

Proof. The linear map Φ defined in (3.3.12) is a dTL_{n+1}-homomorphism. This follows by the observation that, if $\Phi(|u\bar{v}|\otimes y) = \beta^{\#}w$, then $\Phi(a|u\bar{v}|\otimes y)$ and $a\Phi(|u\bar{v}|\otimes y)$ will both give $\beta^{\#}aw$ for all $a\in\mathsf{dTL}_{n+1}$ as can be verified diagrammatically.

No elements of the spanning set $B_{n,k}$ is zero. To see this, it is sufficient to note that their images by Φ are non-zero. Indeed a direct computation shows that, if $u\in\mathsf{S}_{n+1,k+i}$ with $i\in\{-1,0,1\}$, then $\phi(|u\overline{\theta_i(z)}|\otimes z) = \theta_{-i}(u)\in\mathsf{S}_{n+2,k}$ which is non-zero.

To end the proof, it remains to show that the spanning set is linearly independent. Since $|B_{n,k}| = \dim\mathsf{S}_{n+2,k}$, it is sufficient to show that any link diagram in $\mathsf{S}_{n+2,k}$ has a pre-image in $B_{n,k}$. To find the pre-image of u, a $(n+2)$-link in $\mathsf{S}_{n+2,k}$, simply construct $|u\bar{z}|$ and detach the bottom position of u to attach it to z. The result is $|u'\overline{\theta_i(z)}|$ for some $i\in\{-1,0,1\}$. Then $\Phi(|u'\overline{\theta_i(z)}|\otimes z) = u$. The spanning set is therefore linearly independent and Φ is a dTL_{n+1}-isomorphism. ■

The following corollaries are immediate consequences of proposition 3.3.10 and the properties of the restriction of $\mathsf{S}_{n,k}$ obtained in the last subsection.

Corollary 3.3.11. *The short sequence*

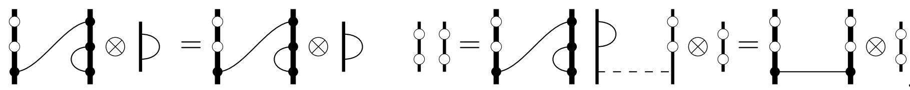
$$0\to\mathsf{S}_{n,k}\oplus\mathsf{S}_{n,k-1}\to\mathsf{S}_{n-1,k}\uparrow\to\mathsf{S}_{n,k+1}\to 0 \tag{3.3.13}$$

is exact for all $n\geq 2$ and $k\in\{0,1,\dots,n-1\}$.

Corollary 3.3.12. *For all $n\geq 2$ and k generic in $\{0,1,\dots,n-1\}$*

$$\mathsf{S}_{n-1,k}\uparrow\simeq\mathsf{S}_{n,k-1}\oplus\mathsf{S}_{n,k}\oplus\mathsf{S}_{n,k+1}\,. \tag{3.3.14}$$

Proposition 3.3.10 and the analogous result for TL_n differs on one small point. For the latter the isomorphism $V_{n,k}\uparrow\simeq V_{n+2,k}\downarrow$ fails in one particular case, namely when $\beta=0$, then $V_{2,0}\uparrow\not\simeq V_{4,0}\downarrow$. Instead $\dim V_{2,0}\uparrow = 3 > \dim V_{4,0}\downarrow = 2$. The difficulty can easily be seen to occur only at $\beta=0$ because, if $\beta\neq 0$, then $u_1u_2 \otimes = \frac{1}{\beta}u_1u_2u_1 \otimes = \frac{1}{\beta}u_1 \otimes = \mathrm{id} \otimes$ where $u_i = b_i^t b_i$. For $\beta = 0$ the vectors $u_1u_2 \otimes$ and $\mathrm{id} \otimes$ are linearly independent. The problem does not occur for the dilute modules. For example, the analogous situation is resolved as follows:

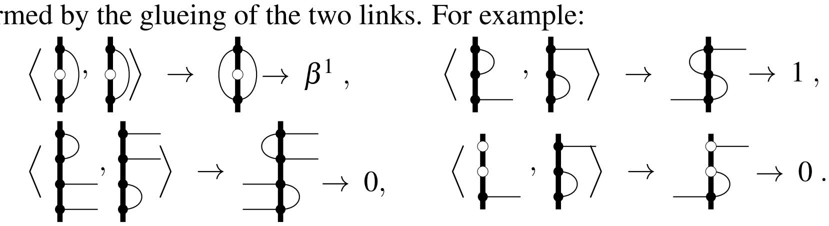


3.4 The Gram product

This section introduces a bilinear form on standard modules that is invariant under the action of the algebra dTL_n (see lemma 3.4.1). It is a familiar tool of representation theory since the radical of this bilinear form is a submodule. For dTL_n, the radical will be the (unique) maximal submodule. Such a submodule can be non-trivial only if the Gram matrix, representing the bilinear form into some basis, is singular. The Gram determinant and its zeroes can be easily computed. These zeroes occur only when q is a root of unity. The structure of dTL_n is then semisimple when q is generic (not a root of unity) and a complete set of non-isomorphic irreducible modules can be identified (theorem 3.4.12). The central result of the section concerns non-trivial radicals at q a root of unity. Proposition 3.4.15 shows that they are then irreducible and isomorphic to the irreducible quotients of another standard modules. The section ends with the description of what the irreducible modules $\mathsf{I}_{n,k}$ become under restriction and induction.

3.4.1 The bilinear form $\langle *, * \rangle_{n,k}$

The *Gram product* $\langle *, * \rangle_{n,k} : \mathsf{S}_{n,k} \times \mathsf{S}_{n,k} \to \mathbb{C}$ is a bilinear form defined on n-link diagrams and extended linearly. To compute the pairing of two (left) link diagrams, first reflect the first link diagram along its vertical axis and then glue it on the left side of the second one, identifying the corresponding points on both diagrams. If a point containing a string in one of the diagrams is identified with a point containing a vacancy in the other, the result is 0. Otherwise, the result is non-zero if and only if every defect of the first diagram is linked to a defect of the second. In such cases, the result is β^m, where m is the number of closed loops formed by the glueing of the two links. For example:



This bilinear form extends that defined on standard modules $V_{n,k}$ of the Temperley-Lieb algebra TL_n (see appendix 3.A). One difference between the two bilinear forms for TL_n and dTL_n is worth mentioning. It concerns the bilinear form on the standard modules $\mathsf{S}_{n,0}$ and $V_{n,0}$ when $\beta = 0$. For $V_{n,0}$ with n even, the bilinear form is strictly zero, as the pairing of link diagrams always closes at least one loop. A special definition has to be introduced to counter this difficulty [63]. The bilinear form on $\mathsf{S}_{n,0}$ as described above is not zero, even when $\beta = 0$, as the pairing of the link diagram with n vacancies with itself gives 1.

The bilinear form is symmetric since exchanging the two arguments amounts to a reflection through a vertical mirror when written in terms of diagrams. We shall say that two elements of $\mathsf{S}_{n,k}$ are *orthogonal* if their Gram product is zero, even though $\langle *,*\rangle_{n,k}$ can be degenerate.

Lemma 3.4.1. *If $x,y \in \mathsf{S}_{n,k}$ and $u \in \mathsf{dTL}_n$ then*

$$\langle x, uy\rangle_{n,k} = \langle u^t x, y\rangle_{n,k} \tag{3.4.1}$$

where u^t is the diagram obtained by reflecting u along its vertical axis. If u is a sum of diagrams, the reflection is done on each diagram of the linear combination separately.

Proof. The proof consists in writing the two sides of the equality in terms of diagrams. ■

Lemma 3.4.2. *If $x,y,z \in \mathsf{S}_{n,k}$, then*

$$|x\bar{y}|z = \langle y,z\rangle_{n,k} x. \tag{3.4.2}$$

Proof. It is sufficient to verify the relation for link diagrams $x,y,z \in \mathsf{S}_{n,k}$, by linearity. Equation (3.4.2) is then non-trivial only if all defects and vacancies of z are respectively linked to defects and vacancies of y. In this case, all defects, arcs and vacancies of x will be preserved and remain at their places, so that $|x\bar{y}|z$ is proportional to x. The proportionality constant is the number of closed loops formed which is precisely $\langle y,z\rangle_{n,k}$. ■

The previous ideas can be extended to the multiplication in dTL_n itself. The ideals I_k that filter dTL_n (see (3.2.2)) are spanned by n-diagrams with at most k crossing strings and the quotient I_k/I_{k-1} by those with precisely k such strings.

Lemma 3.4.3. *Let $x,y \in \mathsf{S}_{n,k}$ be n-links and $u \in \mathsf{dTL}_n$. Then there exist $r_u(z,x) \in \mathbb{C}$ such that*

$$u|x\bar{y}| \equiv \sum_{z\in Y_k} r_u(z,x)|z\bar{y}| \mod I_{k-1}. \tag{3.4.3}$$

Moreover, if $x',y' \in \mathsf{S}_{n,k}$ are two other n-links, then there exists $\phi_u(y,x') \in \mathbb{C}$ such that

$$|x\bar{y}|u|x'\bar{y}'| = \phi_u(y,x')|x\bar{y}'| \mod I_{k-1}. \tag{3.4.4}$$

Clearly $\phi_{\mathrm{id}}(x,y)$ is nothing but $\langle x,y\rangle_{n,k}$.

Proof. Since $|x\bar{y}|$ is an element of the ideal I_k, so is $u|x\bar{y}|$. It can be written as a sum of n-diagrams with k crossing strings or less. Those that have precisely k crossing ones have necessarily $\bar{y}$ as right part by the argument used in the previous proof. Thus $u|x\bar{y}| \equiv \sum_{z\in Y_k} r_u(z,y,x)|z\bar{y}| \mod I_{k-1}$ for some $r_u(z,y,x) \in \mathbb{C}$. But by the diagrammatic definition of the multiplication, the coefficients $r_u(z,y,x)$ may be computed without even drawing $\bar{y}$ and may thus depend only on x and z. The proof of the second statement repeats the argument for the left part. ■

The link diagrams in $X_{n,k}$ enjoy a particular property: the Gram product of any pair is 1 if the two diagrams are the same and 0 otherwise. Proposition 3.3.1 showed that any link diagram in $X_{n,k}$ (or even any non-zero element in its span) is a generator of $\mathsf{S}_{n,k}$. The next lemma explains, in terms of the bilinear form $\langle *,* \rangle_{n,k}$, why these link diagrams are generators and identifies a larger set of generators.

Lemma 3.4.4. *An element x is a generator of $\mathsf{S}_{n,k}$ if there exist $y \in \mathsf{S}_{n,k}$ such that $\langle x,y \rangle_{n,k} \neq 0$.*

Proof. Let $y \in \mathsf{S}_{n,k}$ be such that $\langle y,x \rangle_{n,k} = \alpha \neq 0$. For any $z \in \mathsf{S}_{n,k}$, both z and $\bar{y}$ have the same number of defects and $|z\bar{y}|$ is thus an element of dTL_n. Therefore $\frac{1}{\alpha}|z\bar{y}|x = z$ and $(\mathsf{dTL}_n)x = \mathsf{S}_{n,k}$. ∎

Hence any link state that is not orthogonal to all others is a generator. Those that are orthogonal to all others are known to be as important. Their set

$$\mathsf{R}_{n,k} = \{x \in \mathsf{S}_{n,k} \mid \langle y,x \rangle_{n,k} = 0, \text{ for all } y \in \mathsf{S}_{n,k}\}$$

is called the *(dilute) radical* of $\mathsf{S}_{n,k}$. It is easy to see that it is a submodule. Lemma 3.4.4 actually shows that it is its maximal submodule, that is, every proper submodule of $\mathsf{S}_{n,k}$ is a submodule of its radical. Moreover the module $\mathsf{I}_{n,k} = \mathsf{S}_{n,k}/\mathsf{R}_{n,k}$ is irreducible, since any of its non-zero elements generates it.

The Gram product can also be used to restrict morphisms between quotients of standard modules.

Lemma 3.4.5. *Let N, N' be submodules of $\mathsf{S}_{n,k}$ and $\mathsf{S}_{n,k'}$, respectively, with $k < k'$. Then the only homomorphism $\mathsf{S}_{n,k}/N \to \mathsf{S}_{n,k'}/N'$ is the zero homomorphism.*

Proof. Let γ be the canonical homomorphism from $\mathsf{S}_{n,k}$ to $\mathsf{S}_{n,k}/N$ and θ be a homomorphism from $\mathsf{S}_{n,k}/N$ to $\mathsf{S}_{n,k'}/N'$. Choose $y,z \in \mathsf{S}_{n,k}$ such that $\langle y,z \rangle_{n,k} = 1$. Then for all $x \in \mathsf{S}_{n,k}$,

$$|x\bar{y}|\theta(\gamma(z)) = \theta(\gamma(|x\bar{y}|z)) = \theta(\gamma(x)). \tag{3.4.5}$$

Since $\theta(\gamma(z)) \in \mathsf{S}_{n,k'}/N'$, the usual representative of this conjugacy class has k' defects. But $|x\bar{y}|\theta(\gamma(z))$ can have at most $k < k'$ defects and the left side of (3.4.5) must be zero. Therefore $\theta(\gamma(x))$ is zero for all x and, since γ is surjective, θ is zero. ∎

3.4.2 The structure of the radical

Let $dG_{n,k}$ be the matrix representing the bilinear form $\langle *,* \rangle_{n,k}$ in the basis of link diagrams. Similarly denote by $G_{n,k}$ the matrix for the bilinear form for the corresponding standard TL_n-module, also in its link basis. These matrices will be called *Gram matrices* and, if need be, the adjective *dilute* will be added to the first one. The Gram product of two link diagrams in $\mathsf{S}_{n,k}$ may be non-zero only if their vacancies coincide. In that case, the product does not depend on their positions and it is equal to the Gram product defined for standard modules of $\mathsf{TL}_{n'}$ applied to the two link diagrams obtained from the original ones by deleting their vacancies. (Then n' is $n - \#(\text{vacancies})$.) It is then clear that the matrix $dG_{n,k}$ is block-diagonal if the link basis is ordered, first, by gathering links with the same number of vacancies and, second, those with the same positions for these vacancies. The shape of the Gram matrix $dG_{n,k}$ then appears as a consequence of the decomposition of the dilute standard modules into a direct sum of S_n-modules (see proposition 3.3.5). The next result then follows immediately. (The direct sum symbol is used to indicate the block diagonal decomposition of $dG_{n,k}$ and the binomial factors give the multiplicity of each block or vector space.)

Proposition 3.4.6. *The dilute Gram matrix for the* dTL_n*-modules* $\mathsf{S}_{n,k}$ *is*

$$dG_{n,k} = \bigoplus_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{k+2p} G_{k+2p,k} \tag{3.4.6}$$

where $G_{n,k}$ *is the Gram matrix of the* TL_n*-module* $V_{n,k}$*.*

The following corollaries are immediate consequences.

Corollary 3.4.7. *The determinant of the Gram matrix is*

$$\det dG_{n,k} = \prod_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \left(\det G_{k+2p,k}\right)^{\binom{n}{k+2p}} \tag{3.4.7}$$

Corollary 3.4.8. *The dilute radical* $\mathsf{R}_{n,k}$ *decomposes as*

$$\mathsf{R}_{n,k} \simeq \bigoplus_{p=0}^{\lfloor \frac{n-k}{2} \rfloor}{}' \binom{n}{k+2p} R_{k+2p,k} \qquad \textit{as vector spaces,} \tag{3.4.8}$$

where $R_{n,k}$ *is the radical of the Gram bilinear form on* $V_{n,k}$ *and the "*$\oplus'$*" indicates that the trivial radicals* $(=\{0\})$ *are omitted of the direct sum. Furthermore this decomposition for* $\mathsf{R}_{n,k} \subset \mathsf{S}_{n,k}\downarrow_{S_n}^{\mathsf{dTL}_n}$ *holds as* S_n*-modules.*

Corollary 3.4.9.

$$\dim \mathsf{R}_{n,k} = \sum_{p=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{k+2p} \dim R_{k+2p,k}. \tag{3.4.9}$$

The last corollary leads to various recurrence relations for the dimensions of the dilute radicals and the irreducible modules. They are simple, though neither compact nor particularly enlightening. They will be presented along with their proofs in appendix 3.C.

A distinction between the two algebras TL_n and dTL_n at $\beta = 0$ follows from the above proposition and corollaries. When $\beta = 0$ (and therefore $q = \pm i$), the determinant $G_{n,k}$ vanishes for all even ks and is otherwise non-zero. It follows that $\mathsf{TL}_n(\beta = 0)$ is semisimple if n is odd, because then all its standard modules $V_{n,k}$ have odd ks, and $\mathsf{TL}_n(\beta = 0)$ is non-semisimple if n is even. It will be shown that the dilute $\mathsf{dTL}_n(\beta = 0)$ is non-semisimple for all $n > 1$.

The previous results show that the dilute radical $\mathsf{R}_{n,k}$ is trivial if the radicals $R_{k+2p,k}$, $0 \le p \le \lfloor (n-k)/2 \rfloor$, are all trivial. Since the determinant of $G_{n,k}$ can vanish only at a root of unity distinct from ± 1 (see (3.A.2)), then the following corollaries are straightforward.

Corollary 3.4.10. *The dilute standard module $\mathsf{S}_{n,k}$ is irreducible if q is not a root of unity.*

Corollary 3.4.11. *The dilute standard module $\mathsf{S}_{n,k}$ is irreducible if k is critical.*

Proof. We recall that the radical $R_{n,k}$ of the standard TL_n-module $V_{n,k}$ is trivial whenever k is critical, that is when $q^{2(k+1)} = 1$. (See proposition 3.A.3.) This result is independent of n and all vector spaces $R_{k+2p,k}$ appearing in (3.4.8) are trivial. ∎

Theorem 3.4.12 (Structure of dTL_n for q generic)**.** *If q is not a root of unity, then dTL_n is semisimple, the set $\{\mathsf{S}_{n,k}, 0 \le k \le n\}$ forms a complete set of non-isomorphic irreducible modules and, as a left module, the algebra dTL_n decomposes as*

$$\mathsf{dTL}_n = \bigoplus_{0\le k\le n} (\dim \mathsf{S}_{n,k})\mathsf{S}_{n,k}.$$

Proof. Corollary 3.4.10 states that the $\mathsf{S}_{n,k}$ are irreducible when q is not a root of unity and proposition 3.3.3 that they are non-isomorphic. Weddeburn's theorem 3.D.8 and its generalization 3.D.8 show that, given a subset $\{I_k, k \in K\}$ of its non-isomorphic irreducible modules, the dimension of an algebra is bounded from below by $\sum_{k\in K}(\dim I_k)^2$. In the present case $\sum_{0\le k\le n}(\dim \mathsf{S}_{n,k})^2 = \dim \mathsf{dTL}_n$ by proposition 3.3.7. The three statements then appear as a consequence of Wedderburn's theorem. ∎

3.4.3 Symmetric pairs of standard modules

Let q be a root of unity other than ± 1 and let ℓ be the smallest integer such that $q^{2\ell} = 1$. Then $\ell \ge 2$. Two non-negative integers k and k' form a symmetric pair if they satisfy

$$(k+k')/2+1 \equiv 0 \bmod \ell \qquad \text{and} \qquad 0 < |k-k'|/2 < \ell. \tag{3.4.10}$$

The Bratteli diagram in Figure 3.4.3 explains the meaning of these two conditions. The first equations implies that the average of k and k' falls on a critical line, that is, $k_c = (k+k')/2$ satisfies $q^{2(k_c+1)} = 1$. (On the Bratteli diagram with $\ell = 4$, the critical lines are through $k = 3, 7, \dots$) Consider now the closest critical lines to that going through k_c. (If the latter one is the leftmost, the critical line to its left would be one passing through $k = -1$.) The second condition above means that the integers k and k' are strictly between these two closest critical lines. Hence k and k' fall symmetrically on each side of the line $k_c = (k+k')/2$. A pair of standard modules $\mathsf{S}_{n,k}$ and $\mathsf{S}_{n,k'}$ is also said symmetric if k and k' form a symmetric pair. Note, finally, that there are always a pair of positive integers a, b with $0 < b < \ell$ such that k and k' are $k_\pm = a\ell - 1 \pm b$. When the pair S_{n,k_+} and S_{n,k_-} is symmetric, then the eigenvalues of F_n on these modules coincide (see proposition 3.B.3).

For the (original) Temperley-Lieb algebras TL_n, it is known that $R_{n,k_-} \simeq I_{n,k_+}$ for all symmetric pairs with $k_- < k_+ \le n$ and, if $k_- \le n < k_+$, then $R_{n,k_-} = \{0\}$. The main result of this section is that these isomorphisms still hold for the dilute family. Even without studying the structure of these modules, one can prove readily the coincidence of their dimensions.

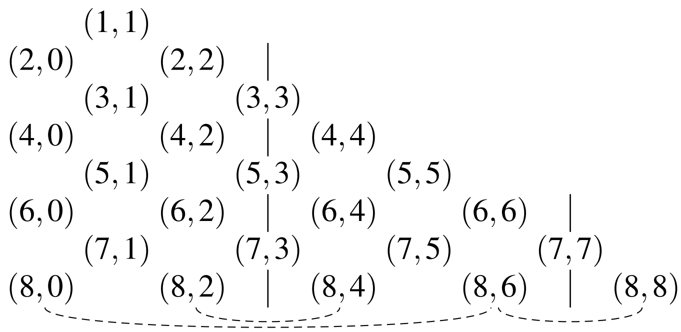


Figure 3.4.1 – The indices (n,k) of even standard modules are presented on a Bratteli diagram. Each line corresponds to a given n and therefore a given dTL_n. The vertical lines are the critical lines when $\ell = 4$. Symmetric pairs for $n = 8$ are joined by dashed lines.

Lemma 3.4.13. *If $k_- < k_+$ is a symmetric pair, then* $\dim \mathsf{R}_{n,k_-} = \dim \mathsf{I}_{n,k_+}$.

Proof. As above set $b = (k_+ - k_-)/2$. Then $\lfloor (n-k_-)/2 \rfloor = \lfloor (n-k_+)/2 \rfloor + b$. Since $\dim \mathsf{I}_{n,k_+} = \dim \mathsf{S}_{n,k_+} - \dim \mathsf{R}_{n,k_+}$, corollaries 3.3.6 and 3.4.9 provide the first equality below. The third one uses the equality of dimensions of radical and irreducible for TL_n-modules.

$$\begin{aligned}
\dim \mathsf{I}_{n,k_+} &= \sum_{p=0}^{\lfloor \frac{n-k_+}{2} \rfloor} \binom{n}{k_+ + 2p} (\dim V_{k_+ + 2p, k_+} - \dim R_{k_+ + 2p, k_+}) \\
&= \sum_{p=0}^{\lfloor \frac{n-k_+}{2} \rfloor} \binom{n}{k_+ + 2p} \dim I_{k_+ + 2p, k_+} = \sum_{p=0}^{\lfloor \frac{n-k_+}{2} \rfloor} \binom{n}{k_+ + 2p} \dim R_{k_+ + 2p, k_-} \\
&= \sum_{q=b}^{\lfloor \frac{n-k_-}{2} \rfloor} \binom{n}{k_- + 2q} \dim R_{k_- + 2q, k_-} \\
&= \sum_{q=0}^{\lfloor \frac{n-k_-}{2} \rfloor} \binom{n}{k_- + 2q} \dim R_{k_- + 2q, k_-} = \dim \mathsf{R}_{n,k_-}.
\end{aligned}$$

In the last line b terms were added. But they are all zero as they are the dimensions of radicals $R_{n',k'}$ indexed by $k' \leq n'$ whose symmetric partners fall beyond n'. ■

To probe the structure of the radicals, a few tools will be useful.

Let $z \in X_{n,k}$ be an n-link diagram and set $\pi_z = |z\bar{z}| \in \mathsf{dTL}_n$. (Note that π_z coincides with the projectors π_A introduced in subsection 3.2.1 if A is taken to be the set of positions of the defects of z.) Here are some simple observations about π_z. The set $T_z = \pi_z \mathsf{dTL}_n \pi_z$ is spanned by n-diagrams that have precisely $(n-k)$ vacancies on their sides, located where the vacancies of z are. The vector space T_z is a subalgebra of dTL_n isomorphic to TL_k. This leads to a reformulation of proposition 3.2.1, namely:

$$S_n \simeq \bigoplus_{0 \leq k \leq n} \bigoplus_{z \in X_{n,k}} \pi_z \mathsf{dTL}_n \pi_z.$$

Similarly the identity $\mathsf{id} \in \mathsf{dTL}_n$ can be written as $\mathsf{id} = \sum_{0\le k\le n}\sum_{z\in X_{n,k}} \pi_z$.

Note that $\pi_z\pi_z = \pi_z$ so that π_z, $z \in X_{n,k}$, acts as a projector. Moreover, for two distinct link diagrams $z \in X_{n,i}$ and $z' \in X_{n,j}$, $0 \le i < j \le n$, $\pi_z\pi_{z'} = \pi_{z'}\pi_z = 0$ and $\pi_{z'}M \cap \pi_z M = \{0\}$ for any module M.

Lemma 3.4.14. *Let q be a root of unity other than ± 1 and k be critical for this q. Let ψ be the endomorphism of $\mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k}$ defined by left multiplication by the central element $\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id}$. Then ψ is non-zero.*

Proof. The proof builds on that for the Temperley-Lieb algebra. To make contact with this previous result, we need to choose a link diagram $z \in X_{n-1,k}$. The actual one is irrelevant, but the explanations are simpler when z has all its vacancies at the top and its defects at the bottom positions. The vector $v = \pi_{\theta_1(z)} \otimes_{\mathsf{dTL}_{n-1}} z$ is then an element of the basis constructed in subsection 3.3.5 for the induced module $\mathsf{S}_{n-1,k}\!\uparrow$. We claim that $(\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})v$ is non-zero. Note first that

$$(\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})v = (\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})\pi_{\theta_1(z)}v = \big(\pi_{\theta_1(z)}(\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})\pi_{\theta_1(z)}\big)v$$

since $\pi_{\theta_1(z)}\pi_{\theta_1(z)} = \pi_{\theta_1(z)}$. Due to proposition 3.B.2, $\pi_{\theta_1(z)}\mathsf{F}_n\pi_{\theta_1(z)}$ corresponds to the action of F_{k+1} on the bottom $k+1$ positions, the top ones being forced to be vacancies. The fact that these vacancies do not play any role is useful. Recall that $T_{\pi_{\theta_1(z)}} = \pi_{\theta_1(z)}\mathsf{dTL}_n\pi_{\theta_1(z)}$ is a subalgebra isomorphic to TL_{k+1}. Similarly $T_z = \pi_z\mathsf{dTL}_{n-1}\pi_z \simeq \mathsf{TL}_k$ and $\pi_z\mathsf{S}_{n-1,k}$ is a TL_k-module (with the restricted action) isomorphic to $V_{k,k}$. With these isomorphisms, the computation of $(\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})v$ amounts to computing the action of $(F_k - \mathsf{F}_{k-1}\cdot\mathsf{id}_{\mathsf{TL}_{k+1}})$ on $\mathsf{id}_{\mathsf{TL}_{k+1}} \otimes_{\mathsf{TL}_k} z_k \in V_{k,k}\!\uparrow$ where z_k is the k-link state with k defects. Note that the criterion for criticality does not depend on n and the TL_k-module $V_{k,k}$ also sits on the critical line. Proposition 3.A.4 then states readily that $(F_k - \mathsf{F}_{k-1}\cdot\mathsf{id}_{\mathsf{TL}_{k+1}})\mathsf{id}_{\mathsf{TL}_{k+1}} \otimes_{\mathsf{TL}_k} z_k$ is non-zero. One can then conclude that $(\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})\pi_{\theta_1(z)} \otimes_{\mathsf{dTL}_{n-1}} z$ is non-zero since $T_{\pi_{\theta_1(z)}}\cdot v$ and $V_{k,k}\!\uparrow$ are isomorphic as modules over the subalgebra $T_{\pi_{\theta_1(z)}} \subset \mathsf{dTL}_n$. Clearly the vector $v \in \mathsf{S}_{n,k}\!\uparrow$ lies in the submodule of $\mathsf{S}_{n-1,k}\!\uparrow$ that has the parity of $\mathsf{S}_{n,k+1}$ and thus projects onto a non-zero vector in $\mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k}$. ■

Proposition 3.4.15. *Let q be a root of unity other than ± 1 and let S_{n,k_-} and S_{n,k_+} be two standard dTL_n-modules where k_- and k_+ form a symmetric pair ($k_- < k_+$). Then*

$$\mathsf{R}_{n,k_-} \simeq \mathsf{I}_{n,k_+}. \tag{3.4.11}$$

Proof. Let $k = (k_- + k_+)/2$ be the critical k between k_- and k_+ and let b such that $k_\pm = k \pm b$. If $b = 1$, the short sequence

$$0 \to \mathsf{S}_{n,k-1} \xrightarrow{\alpha} \mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k} \xrightarrow{\gamma} \mathsf{S}_{n,k+1} \to 0 \tag{3.4.12}$$

is exact by corollary 3.3.11. Let ψ be the endomorphism obtained by left multiplying a dTL_n-module by $(\mathsf{F}_n - \mathsf{F}_{k-1}\cdot\mathsf{id})$. By the previous lemma, this is a non-zero endomorphism on $\mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k}$. But it does act as zero on $\mathsf{S}_{n,k-1}$ and therefore $\operatorname{im}\alpha \subset \ker\psi$. It also acts as zero on $\mathsf{S}_{n,k+1}$ and $\operatorname{im}\psi \subset \ker\gamma = \operatorname{im}\alpha$. Since γ is surjective, for any $w \in \mathsf{S}_{n,k+1}$, there is a $v \in \mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k}$ such that $\gamma(v) = w$. If $v' \in \mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k}$ is another vector satisfying $\gamma(v') = w$, then $v - v' \in \ker\gamma \subset \ker\psi$. It thus follows that the map $w \mapsto \psi(v)$ is well-defined. It can be seen to be a module homomorphism $\Psi : \mathsf{S}_{n,k+1} \to \operatorname{im}\alpha \subset \mathsf{S}_{n-1,k}\!\uparrow/\mathsf{S}_{n,k}$. Since α is injective, it has an inverse on $\operatorname{im}\psi \subset \operatorname{im}\alpha$. Therefore $\alpha^{-1}\circ\Psi : \mathsf{S}_{n,k+1} \to \mathsf{S}_{n,k-1}$ is a non-zero homomorphism and $\mathsf{Hom}_{\mathsf{dTL}_n}(\mathsf{S}_{n,k+1}, \mathsf{S}_{n,k-1}) \neq 0$.

Let b be an integer such that $1 < b < \ell$ where ℓ is the smallest integer such that $q^{2\ell} = 1$. Then

$$\begin{aligned}
&\mathsf{Hom}_{\mathsf{dTL}_{n+b}}(\mathsf{S}_{n+b,k+b}, \mathsf{S}_{n+b,k-b}) \\
&\quad = \mathsf{Hom}_{\mathsf{dTL}_{n+b}}(\mathsf{S}_{n+b,k+b} \oplus \mathsf{S}_{n+b,k+b-1} \oplus \mathsf{S}_{n+b,k+b-2}, \mathsf{S}_{n+b,k-b}) \\
&\quad = \mathsf{Hom}_{\mathsf{dTL}_{n+b}}(\mathsf{S}_{n+b-1,k+b-1}\!\uparrow, \mathsf{S}_{n+b,k-b}) \\
&\quad = \mathsf{Hom}_{\mathsf{dTL}_{n+b-1}}(\mathsf{S}_{n+b-1,k+b-1}, \mathsf{S}_{n+b,k-b}\!\downarrow) \\
&\quad = \mathsf{Hom}_{\mathsf{dTL}_{n+b-1}}(\mathsf{S}_{n+b-1,k+b-1}, \mathsf{S}_{n+b-1,k-b} \oplus \mathsf{S}_{n+b-1,k-b+1} \oplus \mathsf{S}_{n+b-1,k-b-1}) \\
&\quad = \mathsf{Hom}_{\mathsf{dTL}_{n+b-1}}\big(\mathsf{S}_{n+b-1,k+(b-1)}, \mathsf{S}_{n+b-1,k-(b-1)}\big).
\end{aligned}$$

The third equality is due to Frobenius reciprocity theorem and the second and the fourth follow from corollary 3.3.12 and the fact that neither $(k+b-1)$ nor $(k-b)$ are critical. The first equality rests upon two slightly different observations. Lemma 3.B.4 shows that F_{n+b} act upon the two modules $\mathsf{S}_{n+b,k+b-2}$ and $\mathsf{S}_{n+b,k-b}$ with distinct eigenvalues and therefore any homomorphism between them is zero. Similarly, there cannot be a homomorphism between two standard modules of distinct parities and $\mathsf{Hom}_{\mathsf{dTL}_{n+b}}(\mathsf{S}_{n+b,k+b-1}, \mathsf{S}_{n+b,k-b}) = 0$. The last equality follows from the same two observations. Therefore

$$\mathsf{Hom}_{\mathsf{dTL}_{n+b}}(\mathsf{S}_{n+b,k+b}, \mathsf{S}_{n+b,k-b}) = \mathsf{Hom}_{\mathsf{dTL}_n}(\mathsf{S}_{n,k+1}, \mathsf{S}_{n,k-1}) \neq \{0\}. \tag{3.4.13}$$

Let k_- and k_+ be a symmetric pair and $f : \mathsf{S}_{n,k_+} \to \mathsf{S}_{n,k_-}$ a non-zero homomorphism. Its kernel is a proper submodule of S_{n,k_+} and, since the radical of a standard module is a maximal submodule, $\ker f \subset \mathsf{R}_{n,k_+}$. Now, if f is surjective, $\mathsf{S}_{n,k_-} \simeq \mathsf{S}_{n,k_+}/\ker f$, which contradicts proposition 3.4.5. We thus conclude that $\operatorname{im} f$ is a proper sub-module of S_{n,k_-} and is thus a sub-module of R_{n,k_-} by maximality of the radical. But, we have

$$\dim \mathsf{R}_{n,k_-} \geq \dim \operatorname{im} f = \dim \mathsf{S}_{n,k_+} - \dim\ker f \geq \dim \mathsf{S}_{n,k_+} - \dim \mathsf{R}_{n,k_+} = \dim \mathsf{I}_{n,k_+}.$$

But $\dim \mathsf{I}_{n,k_+} = \dim \mathsf{R}_{n,k_-}$ by lemma 3.4.13. It then follows that $\dim\operatorname{im} f = \dim \mathsf{R}_{n,k_-}$ and $\dim\ker f = \dim \mathsf{S}_{n,k_+} - \dim\operatorname{im} f = \dim \mathsf{S}_{n,k_+} - \dim \mathsf{I}_{n,k_+} = \dim \mathsf{R}_{n,k_+}$. Thus $\ker f \simeq \mathsf{R}_{n,k_+}$ and the first isomorphism theorem then concludes the proof. ■

Suppose that (k_-, k_+) is a symmetric pair with $k_- \leq n < k_+$. Then the radical R_{n,k_-} is trivial and S_{n,k_-} irreducible. This can be proved either by extending the previous proof (allowing $\mathsf{S}_{n,j} = \{0\}$ whenever $j > n$), or by a careful analysis of the zeroes of $\det dG_{n,k}$ (corollary 3.4.7), or by checking with (3.A.3) which radicals of TL_n occuring in corollary 3.4.9 are non-trivial. The last results of this section follow easily from the previous result.

Corollary 3.4.16. *The radical $\mathsf{R}_{n,k}$ is either irreducible or trivial.*

Corollary 3.4.17. *If k_- and k_+ form a symmetric pair ($k_- < k_+$), then the following short sequence is exact:*

$$0 \longrightarrow \mathsf{I}_{n,k_+} \longrightarrow \mathsf{S}_{n,k_-} \longrightarrow \mathsf{I}_{n,k_-} \longrightarrow 0. \tag{3.4.14}$$

Proof. The radical is the unique maximal submodule and, by definition, $0 \to \mathsf{R}_{n,k_-} \to \mathsf{S}_{n,k_-} \to \mathsf{I}_{n,k_-} \to 0$. The statement then follows from proposition 3.4.15. ■

Corollary 3.4.18. *If $f \in \mathsf{Hom}(\mathsf{S}_{n,k}, \mathsf{S}_{n,k})$, then f is an isomorphism or zero.*

Proof. If $\mathsf{S}_{n,k}$ is irreducible, the result is trivial. If $\mathsf{S}_{n,k}$ is reducible, then k forms a symmetric pair with some $k_+ > k = k_-$. Choose a non-zero element $f \in \mathsf{Hom}(\mathsf{S}_{n,k}, \mathsf{S}_{n,k})$. If $\ker f$ is non-zero, then $\ker f = \operatorname{im} f = \mathsf{R}_{n,k}$, since $\mathsf{R}_{n,k}$ is the only non-trivial proper submodule. Then the first isomorphism theorem says $\mathsf{S}_{n,k_-}/\mathsf{R}_{n,k_-} \simeq \mathsf{R}_{n,k_-} \simeq \mathsf{I}_{n,k_+} = \mathsf{S}_{n,k_+}/\mathsf{R}_{n,k_+}$, contradicting lemma 3.4.5. So f must be an isomorphism. ■

A similar argument gives the following corollary.

Corollary 3.4.19. *If $\mathsf{S}_{n,k}$ is reducible, then*

$$\mathsf{Hom}(\mathsf{I}_{n,k}, \mathsf{S}_{n,k}) \simeq \mathsf{Hom}(\mathsf{S}_{n,k}, \mathsf{R}_{n,k}) \simeq 0. \tag{3.4.15}$$

3.4.4 Restriction and induction of irreducible modules

We complete the analysis of the restriction and induction of the fundamental modules by giving those of the radicals and the irreducible quotients. The results are simple and elegant. Their proofs are straightforward but somewhat long and repetitive.

Proposition 3.4.20. *If $\mathsf{R}_{n+1,k} \neq 0$, then*

$$\mathsf{R}_{n+1,k}\!\downarrow\, \simeq \mathsf{R}_{n,k-1} \oplus \mathsf{R}_{n,k} \oplus \left\{ \begin{array}{ll} \mathsf{S}_{n,k+1} & \text{if } k+1 \text{ is critical} \\ \mathsf{R}_{n,k+1} & \text{otherwise} \end{array} \right\}. \tag{3.4.16}$$

Some of the direct summands may be trivial.

Proof. If $\mathsf{R}_{n+1,k} \neq 0$, proposition 3.4.15 gives the exactness of the following short sequence of dTL_{n+1}-modules:

$$0 \to \mathsf{R}_{n+1,k} \longrightarrow \mathsf{S}_{n+1,k} \longrightarrow \mathsf{I}_{n+1,k} \to 0 \tag{3.4.17}$$

and therefore of its restriction to dTL_n:

$$0 \to \mathsf{R}_{n+1,k}\!\downarrow \longrightarrow \mathsf{S}_{n+1,k}\!\downarrow \longrightarrow \mathsf{I}_{n+1,k}\!\downarrow \to 0. \tag{3.4.18}$$

It follows that $\mathsf{R}_{n+1,k}\!\downarrow$ is isomorphic to a submodule of $\mathsf{S}_{n+1,k}\!\downarrow$ which splits in a direct sum of three modules which are distinct eigenspaces of F_n of different parity: $\mathsf{R}_{n+1,k}\!\downarrow\, \simeq R_0 \oplus R_- \oplus R_+$ where R_0 and the $R_\pm$ are submodules of $\mathsf{S}_{n,k}$ and $\mathsf{S}_{n,k\pm1}$ respectively. One or more of the Rs may vanish. (See propositions 3.3.9 and 3.B.3.)

We first study R_0. Consider the (restriction of) the injective homomorphism $\phi : \mathsf{S}_{n,k} \to \mathsf{S}_{n+1,k}\!\downarrow$ introduced in the proof of proposition 3.3.8 that simply adds a vacancy at the bottom of every link diagram. Let $u \in \mathsf{S}_{n+1,k}\!\downarrow$ and write it as $u' + v'$ where all terms in u' have a vacancy at the position $n+1$ while those in v' do not. Then, if r is in the radical $\mathsf{R}_{n,k} \subset \mathsf{S}_{n,k}$

$$\langle \phi(r), u\rangle_{n+1,k} = \langle \phi(r), u'\rangle_{n+1,k} = \langle r, \phi^{-1}(u')\rangle_{n,k} = 0. \tag{3.4.19}$$

The image $\phi(\mathsf{R}_{n,k})$ is thus in $\mathsf{R}_{n+1,k}\!\downarrow$. Since R_0 is the only summand of $\mathsf{R}_{n+1,k}\!\downarrow$ having the parity of $\mathsf{R}_{n,k}$, it must contain a submodule isomorphic to $\mathsf{R}_{n,k}$.

We turn to the other two submodules R_- and R_+. Corollary 3.3.9 has established the exactness of the short sequence

$$0 \to \mathsf{S}_{n+1,k-1} \oplus \mathsf{S}_{n+1,k} \longrightarrow \mathsf{S}_{n,k}\!\uparrow \longrightarrow \mathsf{S}_{n+1,k+1} \to 0 \tag{3.4.20}$$

which implies the exactness of (see proposition 3.D.4)

$$0 \to \mathsf{Hom}(\mathsf{S}_{n+1,k+1}, \mathsf{R}_{n+1,k}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{n,k}\!\uparrow, \mathsf{R}_{n+1,k}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{n+1,k-1} \oplus \mathsf{S}_{n+1,k}, \mathsf{R}_{n+1,k}). \tag{3.4.21}$$

Corollary 3.4.19, the linearity of Hom and the parity of the modules involved lead to

$$\mathsf{Hom}(\mathsf{S}_{n+1,k-1} \oplus \mathsf{S}_{n+1,k}, \mathsf{R}_{n+1,k}) = 0 \qquad \text{and} \qquad \mathsf{Hom}(\mathsf{S}_{n+1,k+1}, \mathsf{R}_{n+1,k}) = 0. \tag{3.4.22}$$

Frobenius theorem then gives

$$\mathsf{Hom}(\mathsf{S}_{n,k}\!\uparrow, \mathsf{R}_{n+1,k}) \simeq \mathsf{Hom}(\mathsf{S}_{n,k}, \mathsf{R}_{n+1,k}\!\downarrow) \simeq 0. \tag{3.4.23}$$

Therefore $\mathsf{R}_{n+1,k}\!\downarrow$ has no (non-trivial) submodule isomorphic to a quotient of $\mathsf{S}_{n,k}$. This proves that R_0 is isomorphic to $\mathsf{R}_{n,k}$.

Similarly the short exact sequences

$$0 \to \mathsf{S}_{n+1,k\pm1-1} \oplus \mathsf{S}_{n+1,k\pm1} \longrightarrow \mathsf{S}_{n,k\pm1}\!\uparrow \longrightarrow \mathsf{S}_{n+1,k\pm1+1} \to 0 \tag{3.4.24}$$

give rise to the exact sequences

$$0 \to \mathsf{Hom}(\mathsf{S}_{n+1,k\pm1+1}, \mathsf{R}_{n+1,k}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{n,k\pm1}\!\uparrow, \mathsf{R}_{n+1,k}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{n+1,k\pm1-1} \oplus \mathsf{S}_{n+1,k\pm1}, \mathsf{R}_{n+1,k}). \tag{3.4.25}$$

Note that $\mathsf{S}_{n+1,k\pm1}$ and $\mathsf{R}_{n+1,k}$ always have different parities and

$$\mathsf{Hom}(\mathsf{S}_{n+1,k\pm1-1}\oplus\mathsf{S}_{n+1,k\pm1},\mathsf{R}_{n+1,k})\simeq\mathsf{Hom}(\mathsf{S}_{n+1,k\pm1-1},\mathsf{R}_{n+1,k})\simeq 0, \tag{3.4.26}$$

where the second equality follows from either proposition 3.4.5 or corollary 3.4.19. The argument now splits according to whether $k+1$ is critical or not.

If $k+1$ is not critical, the central element F_{n+1} takes distinct eigenvalues on $\mathsf{S}_{n+1,k+2}$ and $\mathsf{S}_{n+1,k}$ which forces $\mathsf{Hom}(\mathsf{S}_{n+1,k+2},\mathsf{R}_{n+1,k})=0$. Corollary 3.4.19 also gives $\mathsf{Hom}(\mathsf{S}_{n+1,k},\mathsf{R}_{n+1,k})=0$, so Frobenius theorem leads to

$$\mathsf{Hom}(\mathsf{S}_{n,k\pm1}\uparrow,\mathsf{R}_{n+1,k})\simeq\mathsf{Hom}(\mathsf{S}_{n,k\pm1},\mathsf{R}_{n+1,k}\downarrow)\simeq 0. \tag{3.4.27}$$

Therefore, the $\mathsf{S}_{n,k\pm1}$ are not isomorphic to submodules of $\mathsf{R}_{n+1,k}\downarrow$ and in particular $R_\pm\neq\mathsf{S}_{n,k\pm1}$.

If $k+1$ is critical, proposition 3.4.15 gives $\mathsf{R}_{n,k}\simeq\mathsf{I}_{n,k+2}$ so that

$$\mathsf{Hom}(\mathsf{S}_{n+1,k+2},\mathsf{R}_{n+1,k})\simeq\mathsf{Hom}(\mathsf{S}_{n,k+1}\uparrow,\mathsf{R}_{n+1,k})\simeq\mathsf{Hom}(\mathsf{S}_{n,k+1},\mathsf{R}_{n+1,k}\downarrow)\neq 0 \tag{3.4.28}$$

by the exactness of (3.4.25). Since $\mathsf{S}_{n,k+1}$ is irreducible when $k+1$ is critical, the restriction $\mathsf{R}_{n+1,k}\downarrow$ has a submodule isomorphic to $\mathsf{S}_{n,k+1}$. But since the parity of $\mathsf{S}_{n,k}$ and $\mathsf{S}_{n,k+1}$ are different, this submodule cannot be in R_0. Again F_n takes distinct eigenvalues on $\mathsf{S}_{n,k+1}$ and $\mathsf{S}_{n,k-1}$ so that $\mathsf{S}_{n,k+1}$ cannot be a submodule of R_-. (This statement remains true in the special case when $k-1$ is also critical. Then $l=2$, $q=\pm i$ and $\mathsf{F}_{k+1}=-\mathsf{F}_{k-1}$.) This proves that $\mathsf{S}_{n,k+1}$ must be a submodule of R_+, which is itself a submodule of $\mathsf{S}_{n,k+1}$ and thus $\mathsf{S}_{n,k+1}\simeq R_+$.

So far, we have narrowed down the possible submodules of $\mathsf{R}_{n+1,k}\downarrow$ to

$$\mathsf{R}_{n+1,k}\downarrow\simeq\mathsf{R}_{n,k}\oplus\{0\text{ or }\mathsf{R}_{n,k-1}\}\oplus\left\{\begin{array}{cc}\mathsf{S}_{n,k+1} & \text{if } k+1 \text{ is critical}\\ 0\text{ or }\mathsf{R}_{n,k+1} & \text{otherwise}\end{array}\right\}. \tag{3.4.29}$$

Equation (3.C.1) and proposition 3.C.1 give a formula for the dimension of $\mathsf{R}_{n+1,k}$. The proof ends with a comparison of this dimension with the above possibilities. ■

Note that equation (3.4.18) gives $\mathsf{I}_{n+1,k}\downarrow\simeq\mathsf{S}_{n+1,k}\downarrow/\mathsf{R}_{n+1,k}\downarrow$. Combining this observation with the preceding proposition then gives the following corollary.

Corollary 3.4.21. *If* $\mathsf{R}_{n+1,k}\neq 0$ *then*

$$\mathsf{I}_{n+1,k}\downarrow\simeq\mathsf{I}_{n,k-1}\oplus\mathsf{I}_{n,k}\oplus\left\{\begin{array}{cc}0 & \textit{if } k+1 \textit{ is critical}\\ \mathsf{I}_{n,k+1} & \textit{otherwise}\end{array}\right\}. \tag{3.4.30}$$

Now that we have formulas for the restriction of the irreducible modules, we can use them to prove formulas for their induction.

Proposition 3.4.22. *If* $\mathsf{R}_{n-1,k} \neq 0$ *then*

$$\mathsf{I}_{n-1,k}\uparrow \simeq \mathsf{I}_{n,k-1} \oplus \mathsf{I}_{n,k} \oplus \left\{ \begin{array}{cl} 0 & \textit{if } k+1 \textit{ is critical} \\ \mathsf{I}_{n,k+1} & \textit{otherwise} \end{array} \right\}. \tag{3.4.31}$$

Proof. The argument is similar to that of proposition 3.4.20 and uses systematically Frobenius theorem, the parity of the modules and the eigenspaces of the central element F_n (or F_{n-1}). If $\mathsf{R}_{n-1,k} \neq 0$, the exactness of

$$0 \to \mathsf{R}_{n-1,k} \longrightarrow \mathsf{S}_{n-1,k} \longrightarrow \mathsf{I}_{n-1,k} \to 0. \tag{3.4.32}$$

implies the exactness of the sequence of dTL_{n+1}-modules:

$$\mathsf{R}_{n-1,k}\uparrow \longrightarrow \mathsf{S}_{n-1,k}\uparrow \longrightarrow \mathsf{I}_{n-1,k}\uparrow \to 0. \tag{3.4.33}$$

Since $\mathsf{S}_{n-1,k}\uparrow$ splits in a direct sum of three modules of distinct parities or on which F_n has different eigenvalues, the module $\mathsf{I}_{n-1,k}\uparrow$ splits accordingly into $L_- \oplus L_0 \oplus L_+$ where L_0 and the $L_\pm$ are quotients of $\mathsf{S}_{n,k}$ and $\mathsf{S}_{n,k\pm 1}$ respectively.

We first study L_0. Corollary 3.4.21 gives

$$\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow, \mathsf{I}_{n,k}) \simeq \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{I}_{n,k}\downarrow) \simeq \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{I}_{n-1,k}) \neq 0. \tag{3.4.34}$$

Therefore L_0, the only submodule of $\mathsf{I}_{n-1,k}\uparrow$ of the parity of $\mathsf{I}_{n,k}$, is non-trivial. Moreover proposition 3.4.19 gives

$$\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow, \mathsf{S}_{n,k}) \simeq \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k-1} \oplus \mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k+1}) \simeq 0. \tag{3.4.35}$$

Hence L_0 is non-trivial, distinct from $\mathsf{S}_{n,k}$ and must be isomorphic to $\mathsf{I}_{n,k}$.

We now turn to L_-. If $k-1$ is not critical, corollary 3.4.21 shows again that $\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow, \mathsf{I}_{n,k-1})$ is non-trivial and $\mathsf{I}_{n,k-1}$ must be isomorphic to a quotient of L_-. The short exact sequence

$$0 \to \mathsf{S}_{n-1,k-2} \oplus \mathsf{S}_{n-1,k-1} \longrightarrow \mathsf{S}_{n,k-1}\downarrow \longrightarrow \mathsf{S}_{n-1,k} \to 0 \tag{3.4.36}$$

gives rise to the exactness of

$$0 \to \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k-2} \oplus \mathsf{S}_{n-1,k-1}) \longrightarrow \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n,k-1}\downarrow) \longrightarrow \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k}). \tag{3.4.37}$$

Corollary 3.4.19 gives $\mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k}) \simeq 0$ and therefore

$$\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow, \mathsf{S}_{n,k-1}) \simeq \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k-2} \oplus \mathsf{S}_{n-1,k-1}) \simeq 0 \tag{3.4.38}$$

since the three eigenvalues F_{k-2}, F_{k-1} and F_k of F_{n-1} are distinct if both $k-1$ and k are non-critical. The module $\mathsf{S}_{n,k-1}$ is not a quotient of $\mathsf{I}_{n-1,k}\uparrow$ and L_- must therefore be distinct of $\mathsf{S}_{n-1,k}$. Hence L_- must be isomorphic to $\mathsf{I}_{n,k-1}$. Finally, if $k-1$ is critical, then $\mathsf{I}_{n-1,k} \simeq \mathsf{R}_{n-1,k-2}$ by proposition 3.4.15 and $\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow$

$, \mathsf{S}_{n,k-1}) \neq 0$. Since $\mathsf{S}_{n,k-1}$ is then irreducible, $L_- \simeq \mathsf{I}_{n,k-1}$.

It remains to study L_+. The exact sequence

$$0 \to \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k} \oplus \mathsf{S}_{n-1,k+1}) \longrightarrow \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n,k+1}\downarrow) \longrightarrow \mathsf{Hom}(\mathsf{I}_{n-1,k}, \mathsf{S}_{n-1,k+2}) \tag{3.4.39}$$

follows from the exact sequence for $\mathsf{S}_{n,k+1}\downarrow$. The two outer Hom spaces are trivial because of corollary 3.4.19 and lemma 3.4.5. This proves that $\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow, \mathsf{S}_{n,k+1}) \simeq 0$ and that no submodules of $\mathsf{S}_{n,k+1}$ are isomorphic to a quotient of $\mathsf{I}_{n-1,k}\uparrow$ and in particular that $L_+ \neq \mathsf{S}_{n,k+1}$. Now, if $k+1$ is not critical, then $\mathsf{Hom}(\mathsf{I}_{n-1,k}\uparrow, \mathsf{I}_{n,k+1})$ is non-trivial by corollary 3.4.21 and L_+ must therefore be $\mathsf{I}_{n,k+1}$. If $k+1$ is critical, then $\mathsf{S}_{n,k+1} \simeq \mathsf{I}_{n,k+1}$ is irreducible and, since $L_+ \neq \mathsf{S}_{n,k+1}$, the submodule L_+ must be trivial. ■

The use of symmetric pairs and proposition 3.4.15 give the last result of this section.

Corollary 3.4.23. *If k_c is critical, $0 < i < \ell$ and $\mathsf{R}_{n-1,k_c+i} \neq 0$, then*

$$\mathsf{R}_{n-1,k_c-i}\uparrow \simeq \left\{ \begin{array}{ll} 0 & \text{if } i = \ell - 1 \\ \mathsf{R}_{n,k_c-i-1} & \text{otherwise} \end{array} \right\} \oplus \mathsf{R}_{n,k_c-i} \oplus \left\{ \begin{array}{ll} \mathsf{S}_{n,k_c} & \text{if } i = 1 \\ \mathsf{R}_{n,k_c-i+1} & \text{otherwise} \end{array} \right\}. \tag{3.4.40}$$

Some of these radicals may vanish.

Note that if $\mathsf{R}_{n-1,k_c+i} = 0$ we have $\mathsf{R}_{n-1,k_c-i} \simeq \mathsf{I}_{n-1,k_c+i} \simeq \mathsf{S}_{n-1,k_c+i}$. We therefore find simply $\mathsf{R}_{n-1,k_c-i}\uparrow \simeq \mathsf{S}_{n-1,k_c+i}\uparrow$.

3.5 The structure of dTL_n at a root of unity

In this section, q is a root of unity and ℓ the smallest positive integer such that $q^{2\ell} = 1$.

When q is not a root of unity, every standard module of the dilute Temperley-Lieb algebra is irreducible. The algebra is then semisimple and the standard modules form a complete set of irreducible modules (theorem 3.4.12). However, when q is a root of unity, some of them will be reducible, yet indecomposable. That is, if q is a root of unity, the algebra dTL_n is not always semisimple. To probe its structure the first subsection first shows that dTL_n is a *cellular algebra*, that is, an example of the associative algebras introduced in 1996. (See also chapter 2 of [50] for a complete overview.) As examples of their cellular algebras Graham and Lehrer [26] displayed the Hecke algebra, the Brauer's centralizer algebra, the Temperley-Lieb algebra and the Jones algebra. Their results on cellular algebras give a straightforward description of the structure of the principal indecomposable modules; applying these results to dTL_n will be the content of the second subsection. The third and last will show how to give a fairly explicit construction of principal modules using induction from dTL_{n-1} to dTL_n.

3.5.1 The dilute Temperley-Lieb algebra as a cellular algebra

This section recalls the definition of *cellular algebras* over $\mathbb{C}$ and the results crucial for our task, and shows how the dilute Temperley-Lieb algebra dTL_n satisfies the defining axioms. We refer to [26, 50] for details and proofs.

Définition 3.5.1. *A* cellular algebra *over $\mathbb{C}$ is an associative unital algebra A, together with* cell datum $(\Lambda, Y, C, {}^t)$ *where*

(i) *Λ is a partially ordered set and for each $\lambda \in \Lambda$, Y_λ is a finite set such that $C : \sqcup_{\lambda\in\Lambda} Y_\lambda \times Y_\lambda \to A$ is an injective map whose image is a basis of A;*

(ii) *if $\lambda \in \Lambda$ and $x, y \in Y_\lambda$, write $C(x,y) = C^\lambda_{x,y} \in A$. The map "^{t}" is a linear anti-involution of A such that $(C^\lambda_{x,y})^t = C^\lambda_{y,x}$;*

(iii) *if $\lambda \in \Lambda$ and $x, y \in Y_\lambda$, then for any element $u \in A$ we have*

$$uC^\lambda_{x,y} \equiv \sum_{z\in Y_\lambda} r_u(z,x) C^\lambda_{z,y} \qquad \mathrm{mod}\, A(<\lambda),$$

where $r_u(z,x) \in \mathbb{C}$ is independent of y and where $A(<\lambda)$ is the submodule of A generated by $\{C^\mu_{x',y'} | \mu < \lambda; x', y' \in Y_\mu\}$.

The verification that dTL_n is a cellular algebra has essentially been done. The set Λ is simply $\{0, 1, 2, \ldots, n\}$ with the usual (total) order $<$ on integers and the sets $Y_k, k \in \Lambda$, are to be identified with the bases $Y_{n,k}$ of the standard modules, that is, the n-links with k defects (see subsection 3.3.2). The injective map C has also been defined in that subsection: The pair $x, y \in Y_k$ is mapped by C to the n-diagram $|x\bar{y}|$. Since every n-diagram can be cut into its left part x and right one $\bar{y}$, the image of C contains all n-diagrams and is therefore a basis of dTL_n. The anti-involution "^{t}" is defined on n-diagrams by $|x\bar{y}| \mapsto |y\bar{x}|$ and was introduced in lemma 3.4.1. It corresponds to a reflection of any diagram through a vertical mirror. Finally the last axiom is lemma 3.4.3.

The following lemma [26], a direct consequence of axiom (3), allows for the definition of a bilinear form on some natural modules to be introduced next.

Lemma 3.5.2. *Let A be a cellular algebra. For $\lambda \in \Lambda$, $x, y, x', y' \in Y_\lambda$ and $u \in A$, there exists $\phi_u(y,x') \in \mathbb{C}$ such that*

$$C^\lambda_{x,y} u C^\lambda_{x',y'} \equiv \phi_u(y,x') C^\lambda_{x,y'} \qquad \mathrm{mod}\, A(<\lambda). \tag{3.5.1}$$

The coefficient $\phi_u(y,x')$ is independent of x and y'.

Définition 3.5.3. *For each $\lambda \in \Lambda$, the* cell module *of A corresponding to λ is*

$$\mathsf{S}_\lambda = span_{\mathbb{C}}\{m_x | x \in Y_\lambda\},$$

where the action of A is defined by

$$um_x = \sum_{x'\in Y_\lambda} r_u(x,x') m_{x'}.$$

For $\lambda \in \Lambda$, define the bilinear form $\langle \cdot, \cdot \rangle_\lambda : \mathsf{S}_\lambda \times \mathsf{S}_\lambda \to \mathbb{C}$ by $\langle m_x, m_y \rangle_\lambda = \phi_{\mathrm{id}}(x,y)$ for all $x, y \in Y_\lambda$, and extend it linearly.

Since the elements of the basis of S_λ are in one-to-one correspondence with elements of Y_λ, we shall identify $m_x \leftrightarrow x$, $x \in Y_\lambda$. Then, for $A = \mathsf{dTL}_n$, the n-links with k defects form a basis of its cell module $Y_k, k \in \Lambda$. That the actions on S_k and on $\mathsf{S}_{n,k}$ coincide follows from lemma 3.4.3. Finally the bilinear form $\langle \cdot, \cdot \rangle_k$ on S_k and the Gram product $\langle \cdot, \cdot \rangle_{n,k}$ coincide because of lemmas 3.4.1 and 3.4.3. To sum up:

Proposition 3.5.4. *The dilute Temperley-Lieb algebra $\mathsf{dTL}_n(\beta)$ is cellular. Its cell modules $\mathsf{S}_k, k \in \{0, 1, \ldots, n\}$, are isomorphic to the standard modules $\mathsf{S}_{n,k}$ and the bilinear form $\langle \cdot, \cdot \rangle_k$ coincides with the Gram product $\langle \cdot, \cdot \rangle_{n,k}$ on $\mathsf{S}_{n,k}$.*

These observations simplify enormously the task of identifying the principal indecomposable modules because of the next theorem. The following notation is needed. The radical rad_λ of the form $\langle \cdot, \cdot \rangle_\lambda$ is defined exactly as the radical of the Gram product. The quotient $\mathsf{S}_\lambda / \mathrm{rad}_\lambda = \mathsf{I}_\lambda$ can be shown to be (absolutely) irreducible. (Lemma 3.4.4 did it in a direct way for dTL_n.) The modules are now assumed to be finite-dimensional.

Theorem 3.5.5 (Graham and Lehrer [26] (see also Mathas [50]))**.** *Let A be cellular algebra and $\Lambda_0 = \{\lambda \in \Lambda \mid \langle \cdot, \cdot \rangle_\lambda \neq 0\} \subset \Lambda$. Then*

(i) *The set $\{\mathsf{I}_\lambda | \lambda \in \Lambda_0\}$ is a complete set of non-isomorphic irreducible A-modules.*

(ii) *The set $\left\{\mathsf{P}_\lambda | \lambda \in \Lambda_0\right\}$ is a complete set of non-isomorphic projective indecomposable modules, where P_λ is the projective cover of S_λ.*

(iii) *Let $d_{\lambda,\mu}$, $\lambda \in \Lambda, \mu \in \Lambda_0$, be the multiplicity of I_μ in any composition series of S_λ and $c_{\lambda,\mu}$ that of I_μ in P_λ. Then*

$$c_{\lambda,\mu} = \sum_{\nu \leq \mu, \lambda} d_{\nu,\mu} d_{\nu,\lambda}.$$

(iv) *For any projective module P, there is a filtration*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{d-1} \subset M_d = \mathsf{P},$$

such that $d \leq |\Lambda|$ and M_i / M_{i-1} is isomorphic to a direct sum of cell modules.

The next section will show how this theorem reveals the structure of the dilute algebra dTL_n.

3.5.2 The indecomposable modules $\mathsf{P}_{n,k}$ and the structure of dTL_n for q a root of unity

Hereafter the pairs of integers (k_-, k) and (k, k_+) are symmetric pairs with $k_- < k < k_+$.

The first step is to determine the subset Λ_0 for dTL_n. But this was done at the beginning of section 3.4.1 where it was noted that $\langle\cdot,\cdot\rangle_{n,k}$ is never identically zero (contrarily to the Gram product of TL_n). So $\Lambda_0 = \Lambda = \{0,1,\dots,n\}$. The algebra $\mathsf{dTL}_n(\beta = q+q^{-1})$ has thus $n+1$ non-isomorphic irreducible modules and as many principal indecomposable ones. Since this is the number of standard modules $\mathsf{S}_{n,k}$, each having a distinct irreducible quotient $\mathsf{I}_{n,k}$ by lemma 3.4.5, all irreducibles are known. A complete set of projective indecomposable is given by the projective covers of the standard modules $\mathsf{S}_{n,k}$. Their structure can be partially revealed by statements (3) and (4) of theorem 3.5.5.

Let $d_{k,j}$ be the number of (irreducible) composition factors $\mathsf{I}_{n,j}$ in $\mathsf{S}_{n,k}$ and $c_{k,j}$ their number in $\mathsf{P}_{n,k}$, the projective cover of $\mathsf{S}_{n,k}$. By (3)

$$c_{k,j} = \sum_{i\le k,j} d_{i,j}d_{i,k}. \tag{3.5.2}$$

Corollary 3.4.17 has identified the one or two composition factors of the standard module $\mathsf{S}_{n,k}$ when k in non-critical. The matrix d is thus known:

$$d_{k,j} = \delta_{k,j} + \begin{cases} \delta_{j,k_+}, & \text{if } k \text{ is not critical and } k_+ \le n \\ 0, & \text{otherwise} \end{cases} \tag{3.5.3}$$

and the composition factors of $\mathsf{P}_{n,k}$ are obtained by matrix multiplication $c = d^t d$:

$$c_{k,j} = \delta_{k,j} + \begin{cases} \delta_{j,k_+} + \delta_{j,k} + \delta_{j,k_-} & \text{if } k \text{ is not critical, } k_- \ge 0 \\ \delta_{j,k_+} & \text{if } k \text{ is not critical, } k_- < 0\,, \\ 0 & \text{if } k \text{ is critical} \end{cases} \tag{3.5.4}$$

where it is understood that, if $k_+ > n$, δ_{j,k_+} is always equal to zero. (Note that the restriction $i \le k, j$ on the sum index is superfluous here as d is an upper triangular matrix.)

We now turn to the structure of the principal indecomposable modules $\mathsf{P}_{n,k}, 0 \le k \le n$. If k is critical, $\mathsf{P}_{n,k}$ has a single composition factor and $\mathsf{P}_{n,k} \simeq \mathsf{I}_{n,k} \simeq \mathsf{S}_{n,k}$. Thus $\mathsf{S}_{n,k}$ is projective when k is critical. When $k < \ell - 1$, $\mathsf{P}_{n,k}$ has two composition factors, $\mathsf{I}_{n,k}$ and I_{n,k_+}. Since these are precisely the composition factors of $\mathsf{S}_{n,k}$ and $\mathsf{P}_{n,k}$ is its projective cover, $\mathsf{S}_{n,k}$ and $\mathsf{P}_{n,k}$ are thus isomorphic for $k < \ell - 1$.

When $k > \ell - 1$ is not critical, $\mathsf{P}_{n,k}$ has either three or four composition factors among I_{n,i_+},I_{n,i_-} and $\mathsf{I}_{n,i}$ which always appears twice. (If $k_+ > n$, then the module I_{n,k_+} is trivial.) The filtration $0 = M_0 \subset \dots \subset M_{d-1} \subset M_d = \mathsf{P}_{n,k}$ of statement (4) shows that $\mathsf{P}_{n,k}$ has a submodule M_{d-1} such that $\mathsf{P}_{n,k}/M_{d-1}$ is isomorphic to a direct sum of standard modules. Since $\mathsf{P}_{n,k}$ is the projective cover of $\mathsf{S}_{n,k}$, this direct sum must contain at least a copy of $\mathsf{S}_{n,k}$. The composition factors of $\mathsf{S}_{n,k}$ are $\mathsf{I}_{n,k}$ and I_{n,k_+} and the remaining ones are $\mathsf{I}_{n,k}$ and I_{n,k_-} which are precisely those of S_{n,k_-}. No standard module contains only the irreducible I_{n,k_-} and S_{n,k_-} is the unique one with the remaining factors. Therefore S_{n,k_-} must be a submodule of $\mathsf{P}_{n,k}$, the quotient $\mathsf{P}_{n,k}/M_{d-1}$ must be isomorphic to $\mathsf{S}_{n,k}$ and all composition factors are thus accounted for. The filtration is then simply $0 \subset M_1 \simeq \mathsf{S}_{n,k_-} \subset M_2 = \mathsf{P}_{n,k}$. Note finally that F_n takes the same eigenvalue $\mathsf{F}_{n,k}$ on both S_{n,k_-} and $\mathsf{S}_{n,k}$. The only eigenvalue of F_n on $\mathsf{P}_{n,k}$ is thus $\mathsf{F}_{n,k}$ as well. This discussion ends the proof of the following statement.

Proposition 3.5.6. *The set $\{\mathsf{P}_{n,k}\,|\,0\leq k\leq n\}$ forms a complete set of non-isomorphic projective indecomposable modules. If k is critical or $k<\ell-1$, $\mathsf{P}_{n,k}\simeq\mathsf{S}_{n,k}$. Otherwise, it is the (indecomposable) projective module, unique up to isomorphism, satisfying the short exact sequence*

$$0\longrightarrow\mathsf{S}_{n,k_-}\longrightarrow\mathsf{P}_{n,k}\longrightarrow\mathsf{S}_{n,k}\longrightarrow 0. \tag{3.5.5}$$

The central element F_n has a single eigenvalue acting on $\mathsf{P}_{n,k}$, namely $\mathsf{F}_{n,k}=q^{k+1}-q^{-(k+1)}$.

The regular module of dTL_n is the algebra itself seen as module for the action given by left multiplication. Wedderburn's theorem states that this module is a direct sum of irreducible modules if the algebra is semisimple, and of principal indecomposable ones if it is not. (See theorems 3.D.8 and 3.D.9 in appendix 3.D.) Theorem 3.4.12 gave the decomposition of dTL_n in terms of non-isomorphic irreducible modules for q such that dTL_n is semisimple. The following theorem completes the description of the dilute Temperley-Lieb algebras for the case when q is a root of unity. It is a corollary of the previous classification of principal modules and Wedderburn's theorem 3.D.9.

Theorem 3.5.7 (Structure of dTL_n for q a root of unity)**.** *Let q be a root of unity other than ± 1 and ℓ the smallest positive integer such that $q^{2\ell}=1$. Let K be the set of critical integers smaller or equal to n. Then*

$$\mathsf{dTL}_n\simeq\Big(\bigoplus_{0\leq k<\ell-1} i_{n,k}\,\mathsf{S}_{n,k}\Big)\oplus\Big(\bigoplus_{k_c\in K}\big(i_{n,k_c}\,\mathsf{S}_{n,k_c}\oplus\big(\bigoplus_{1\leq i<\ell} i_{n,k_c+i}\,\mathsf{P}_{n,k_c+i}\big)\big)\Big) \tag{3.5.6}$$

where $i_{n,k}=\dim\mathsf{I}_{n,k}$ if $k\in\{0,1,\dots,n\}$ and 0 otherwise.

3.5.3 Induction of $\mathsf{P}_{n-1,k}$

We now describe how the projective modules of dTL_{n-1} are related to those of dTL_n through induction. When $k<\ell-1$, then $\mathsf{P}_{n,k}\simeq\mathsf{S}_{n,k}$ and corollary 3.3.12 gives the result. We concentrate on the other cases. The crucial property here will be that, if P is a projective dTL_{n-1}-module, then $P\!\uparrow$ is a projective dTL_n-module (see Appendix 3.D).

If $k=k_c$ is critical, proposition 3.3.11 gives the short exact sequence

$$0\longrightarrow\mathsf{S}_{n,k_c}\oplus\mathsf{S}_{n,k_c-1}\longrightarrow\mathsf{S}_{n-1,k_c}\!\uparrow\longrightarrow\mathsf{S}_{n,k_c+1}\longrightarrow 0.$$

Now, since the parity of $\mathsf{S}_{n,k_c\pm 1}$ is different from that of S_{n,k_c}, the module $\mathsf{S}_{n-1,k_c}\!\uparrow$ must be isomorphic to $\mathsf{S}_{n,k_c}\oplus P$, where P is some projective module satisfying the sequence

$$0\longrightarrow\mathsf{S}_{n,k_c-1}\longrightarrow P\longrightarrow\mathsf{S}_{n,k_c+1}\longrightarrow 0.$$

Since P_{n,k_c+1} is the projective cover of S_{n,k_c+1} and satisfies the same sequence, the induced module $\mathsf{S}_{n-1,k_c}\!\uparrow$ $\simeq\mathsf{P}_{n-1,k_c}\!\uparrow$ is

$$\mathsf{S}_{n-1,k_c}\!\uparrow\simeq\mathsf{S}_{n,k_c}\oplus\mathsf{P}_{n,k_c+1}.$$

Suppose then that k is non-critical and larger than $\ell-1$. Because induction is right-exact, the sequence (3.5.5) yields the exact sequence

$$\mathsf{S}_{n,k_--1}\oplus\mathsf{S}_{n,k_-}\oplus\mathsf{S}_{n,k_-+1}\longrightarrow\mathsf{P}_{n-1,k}\!\uparrow\longrightarrow\mathsf{S}_{n,k-1}\oplus\mathsf{S}_{n,k}\oplus\mathsf{S}_{n,k+1}\longrightarrow 0,$$

where proposition 3.3.12 was used. This already gives an upper bound on the dimension of $\mathsf{P}_{n-1,k}\!\uparrow$:

$$\dim\mathsf{P}_{n-1,k}\!\uparrow\ \le \dim(\mathsf{S}_{n,k_--1}\oplus\mathsf{S}_{n,k_-}\oplus\mathsf{S}_{n,k_-+1})+\dim(\mathsf{S}_{n,k-1}\oplus\mathsf{S}_{n,k}\oplus\mathsf{S}_{n,k+1}), \tag{3.5.7}$$

where the upper bound is reached if the leftmost morphism is injective. Since the projective cover of the standard module $\mathsf{S}_{n,k'}$ is $\mathsf{P}_{n,k'}$ for all $k'\in\Lambda$, the induced $\mathsf{P}_{n-1,k}\!\uparrow$ must be isomorphic to $P\oplus\mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k}\oplus\mathsf{P}_{n,k+1}$ for some projective module P. If neither $k+1$ nor $k-1$ is critical, then $\dim\mathsf{P}_{n,i}=\dim\mathsf{S}_{n,i}+\dim\mathsf{S}_{n,i_-}$ for $i\in\{k-1,k,k+1\}$ and $\dim\mathsf{P}_{n-1,k}\!\uparrow\ =\dim(P\oplus\mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k}\oplus\mathsf{P}_{n,k+1})\ge\dim(\mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k}\oplus\mathsf{P}_{n,k+1})$ and the upper bound above is also a lower bound. This yields $P\simeq 0$.

A special treatment is required when either $k-1$ or $k+1$ is critical. Suppose $k+1$ is (and then so is k_--1). Then $\mathsf{P}_{n,k+1}\simeq\mathsf{S}_{n,k+1}$ and the bound only gives $\dim P\le\dim\mathsf{S}_{n,k_--1}$. Since $\mathsf{P}_{n-1,k}$ is projective, the functor $\mathsf{Hom}(\mathsf{P}_{n-1,k},-)$ is exact and, applied to the sequence (3.3.5), gives

$$\begin{aligned}0\longrightarrow&\mathsf{Hom}(\mathsf{P}_{n-1,k},\mathsf{S}_{n-1,k_--1}\oplus\mathsf{S}_{n-1,k_--2})\\&\longrightarrow\mathsf{Hom}(\mathsf{P}_{n-1,k},\mathsf{S}_{n,k_--1}\!\downarrow)\longrightarrow\mathsf{Hom}(\mathsf{P}_{n-1,k},\mathsf{S}_{n-1,k_-})\longrightarrow 0.\end{aligned}\tag{3.5.8}$$

Because of the sequences (3.4.14) and (3.5.5), the third term in the sequence is non-zero (k and k_- form a symmetric pair) and thus the middle term cannot vanish. Therefore, by Frobenius reciprocity, $\mathsf{Hom}(\mathsf{P}_{n-1,k}\!\uparrow,\mathsf{S}_{n,k_--1})\simeq\mathsf{Hom}(\mathsf{P}_{n-1,k},\mathsf{S}_{n,k_--1}\!\downarrow)$ is also non-trivial. Since S_{n,k_--1} is irreducible and projective (again, k_--1 is critical), there must be a surjective morphism from $\mathsf{P}_{n-1,k}\!\uparrow$ onto S_{n,k_--1}. But the addition of the composition factors of S_{n,k_--1} saturates the upper bound for the dimension of $\mathsf{P}_{n-1,k}\!\uparrow$ and thus $\mathsf{P}_{n-1,k}\!\uparrow\ \simeq\mathsf{S}_{n,k_--1}\oplus\mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k}\oplus\mathsf{P}_{n,k+1}$. The case when $k-1$ is critical (or when both $k-1$ and $k+1$ are) is treated similarly. We summarize our results in the following proposition.

Proposition 3.5.8. *For $0\le k\le n-1$,*

$$\mathsf{P}_{n-1,k}\!\uparrow\ \simeq\mathsf{P}_{n,k}\oplus\mathsf{P}_{n,k+1}\oplus\begin{cases}0, & \text{if } k \text{ is critical}\\ \mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k_--1}, & \text{if } k\pm1 \text{ are both critical}\\ \mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k_--1}, & \text{if only } k+1 \text{ is critical}\\ \mathsf{P}_{n,k-1}\oplus\mathsf{P}_{n,k-1}, & \text{if only } k-1 \text{ is critical}\\ \mathsf{P}_{n,k-1}, & \text{otherwise},\end{cases}\tag{3.5.9}$$

where it is understood that $\mathsf{P}_{n,k'}\simeq 0$ if $k'<0$.

This result together with proposition 3.3.10 give a simple diagrammatic basis for P_{n,k_c+1}. Similar arguments can then be used to build basis for the other projective indecomposable modules by inducing repeatedly from S_{n,k_c}.

3.6 Conclusion

The main results of this paper are now reviewed. The dilute Temperley-Lieb algebras $\mathsf{dTL}_n(\beta), n \geq 0$, form a family of algebras parametrized by a complex (or formal) parameter β, often written as $\beta = q + q^{-1}$ with $q \in \mathbb{C}^\times$. The dimension of $\mathsf{dTL}_n(\beta)$ is the Motzkin number M_{2n}. These algebras decompose into a sum of even and odd parts, and so do their modules. They are examples of cellular algebras [26].

Their representation theory is largely based on the study of the standard modules $\mathsf{S}_{n,k}$, $0 \leq k \leq n$. These are shown to be indecomposable (proposition 3.3.2) and cyclic (proposition 3.3.1). Their diagrammatic definition is a technical advantage: it allows for quick computations and observing several of their properties. For example, in the link basis, the matrices representing the generators have at most one non-zero element per column and this element is then a power of β. It is also easy to observe that any link state with only defects and vacancies is actually a generator. Finally the standard modules are all distinct ($\mathsf{S}_{n,k} \simeq \mathsf{S}_{n,j} \Leftrightarrow k = j$) and, for neighbouring ns, they are related by restriction and induction and $\mathsf{S}_{n,k}\!\uparrow \simeq \mathsf{S}_{n+2,k}\!\downarrow$ for all n and $0 \leq k \leq n$ (proposition 3.3.10).

The latter property, together with the natural bilinear form $\langle *, * \rangle_{n,k}$ and a particular central element F_n, is sufficient to unravel the structure of the algebra dTL_n when the complex number q is generic, that is not a root of unity. Then $\mathsf{dTL}_n(\beta = q + q^{-1})$ is semisimple and the standard modules form a complete set of non-isomorphic irreducible modules (theorem 3.4.12).

If however q is a root of unity, distinct from ± 1, a finer analysis is required. Let ℓ be the smallest positive integer such that $q^{2\ell} = 1$. An integer k_c is said critical if $k_c + 1 \equiv 0 \bmod \ell$ and a pair (k_-, k_+) of distinct integers form a symmetric pair if their average is critical and $0 < (k_+ - k_-)/2 < \ell$. With this notation the standard module $\mathsf{S}_{n,k}$ is reducible, but indecomposable, if k is the smallest element k_- of a symmetric pair with $0 \leq k_- < k_+ \leq n$. In that case, its maximal proper submodule $\mathsf{R}_{n,k} \subset \mathsf{S}_{n,k}$ is the radical of the Gram pairing $\langle *, * \rangle_{n,k}$ and is irreducible. In fact, if k is the k_- of a symmetric pair (k_-, k_+), then $\mathsf{R}_{n,k=k_-} \simeq \mathsf{I}_{n,k_+}$ where $\mathsf{I}_{n,k}$ is the irreducible quotient $\mathsf{S}_{n,k}/\mathsf{R}_{n,k}$.

The indecomposable projective modules, that is the principal indecomposable ones, can be identified and linked to standard modules thanks to general results that hold for all cellular algebras. Moreover the principal indecomposable modules of dTL_n are related to those of dTL_{n-1} through the induction functor described in section 3.3.5. Alternatively, one could have followed another path, first determining the action of the induction functor on the projective modules of dTL_{n-1} and then using it to build the projective indecomposable modules of dTL_n. A given principal indecomposable module of dTL_n is then characterized as a direct summand of $(\mathsf{S}_{n',k_c})\underbrace{\uparrow\uparrow\ldots\uparrow}_{n-n'}$ (with $n - n' < \ell$) completely determined by its F_n-eigenvalue and parity. Starting with $\mathsf{dTL}_1 \simeq \mathsf{S}_{1,1} \oplus \mathsf{S}_{1,0}$, this process can be used recursively to construct the principal modules. For example, this approach was used to study the regular Temperley-Lieb algebra TL_n in [63].

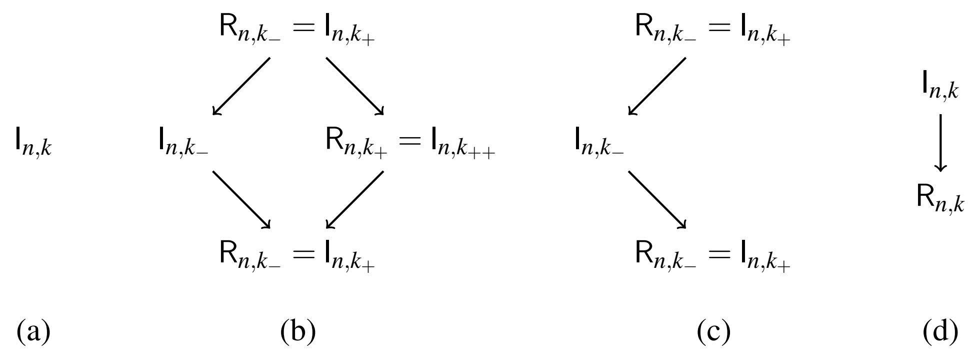


Figure 3.6.2 – The Loewy diagrams of the principal indecomposable modules

Loewy diagrams are often used in the mathematical physics literature and it is useful to draw them for the principal indecomposable modules. (The construction of the Loewy diagrams for dTL_n is identical to that for TL_n which is described in [63].) If k is critical, the projective is simply the (irreducible) standard module $\mathsf{S}_{n,k}$ and its Loewy diagram contains a single node (figure 3.6 (a)). For k non-critical, let k_-, k_+ and k_{++} be such that $k_- < k_+ = k < k_{++}$ and both (k_-, k_+) and (k_+, k_{++}) are symmetric pairs. Then, if $k_-, k_+, k_{++} \in \{0, 1, \dots, n\}$, the Loewy diagrams of the principal modules with irreducible quotient $\mathsf{I}_{n,k}$ has the form (b) in the figure. If $k_{++} > n$, then the right node is deleted and the resulting Loewy diagram is of type (c) on the figure. Finally, if k is at the left of the first critical line, then its Loewy diagram is that of the standard $\mathsf{S}_{n,k}$ and appears as (d) on the figure.

The similarity of these Loewy diagrams with those of the Temperley-Lieb algebra leads to a natural correspondence between their respective irreducible modules:

$$\mathsf{I}_{n,i} \to \begin{cases} I_{n,i}, & \text{if } n \equiv i \mod 2 \\ I_{n-1,i}, & \text{otherwise} \end{cases}.$$

Under this transformation it is clear that the indecomposable projective modules of dTL_n are sent to those of TL_n and TL_{n-1}, except when $\ell = 2$, because then $I_{2n,0} = \{0\}$. This suggests that dTL_n is Morita equivalent to the direct sum $\mathsf{TL}_n \oplus \mathsf{TL}_{n-1}$ when $\beta \neq 0$. In fact it can be shown that dTL_n and the direct sum $\mathsf{TL}_n \oplus \mathsf{TL}_{n-1}$ are Morita-equivalent when $\beta \neq 0$. [2]

What can one learn from these results about limiting structures appearing in physical models like conformal field theories (CFT)? The original Temperley-Lieb algebras, whose representation theory the dilute

2. Though it goes beyond the goals of the paper, here are two paths toward a proof of their Morita-equivalence. In the first, one can put the projective modules of $\mathsf{TL}_n \oplus \mathsf{TL}_{n-1}$ in correspondence with those of the even and odd parts of dTL_n, respectively, and show that this one-to-one correspondence preserves all morphisms between them, if $\beta \neq 0$. In the second, suggested by the referee, one may use the (finite) Temperley-Lieb category $\mathbf{T} = \mathbf{T}_{\mathbb{C},q}$ introduced by Graham and Lehrer (see Definition (2.1) in [27]). A (full) subcategory is obtained by restraining to diagrams $i \to j$ for $i, j \leq n$. One the one hand this truncated category can be shown to be Morita-equivalent to $\mathsf{TL}_n \oplus \mathsf{TL}_{n-1}$ when $\beta \neq 0$. And on the other hand, one can establish its Morita-equivalence to dTL_n by noticing that diagrams $i \to j$ with i or j smaller than n can be understood as diagrams with vacancies.

ones mimic so closely, has been used to understand the representation theory of the Virasoro algebra appearing in the continuum limit of lattice models whose transfer matrix is an element of TL_n. The fusion ring, defined formally in [24], is a natural outcome of the representation theory of these finite-dimensional associative algebras. The result announced there for the TL_n fusion ring is paralleled to the CFT fusion for Virasoro modules, with staggered ones sharing the Loewy structure of the principal indecomposable modules of type (b) in figure 3.6. We hope that the results reported here may help reveal the fusion ring of the dilute Temperley-Lieb algebras.

Acknowledgements

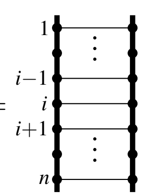
The authors would like to thank David Ridout for useful discussions and the referee whose constructive suggestions brought us to use cellular algebras as the main tool in section 3.5 and make it easier to understand. JB holds a scholarship from Fonds de recherche Nature et Technologies (Québec). YSA holds a grant from the Canadian Natural Sciences and Engineering Research Council. This support is gratefully acknowledged.

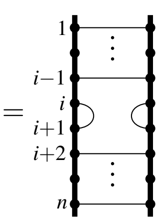
Appendix

3.A The Temperley-Lieb algebra

The original Temperley-Lieb algebras [66] were introduced much before the dilute ones. Since the present text studies the latter, the former will be presented starting from the definitions for the dilute objects. Only the results needed here are recalled. They are taken from an article by Ridout and one of the authors [63]. It must be underlined that their paper uses the number p of arcs instead of the number k of defects to characterize link states and modules. The results stated below have been adapted to the labelling in terms of defects.

The one-parameter family of Temperley-Lieb algebras $\mathsf{TL}_n(\beta)$ is spanned by all n-diagrams, defined in subsection 3.2.1, that contain no vacancies. The product is given by the same rules as for dTL_n, the factor β also weighting each closed loop generated through concatenation of diagrams. The algebra TL_n has a compact definition in terms of generators $u_i, 1 \le i \le n-1$, and the unit id. These correspond to the following diagrams:

$\mathsf{id} =$ 

, $\quad u_i =$ 

(3.A.1)

and satisfy the relations:

$$\begin{aligned} u_i^2 &= \beta u_i, \\ u_i &= u_i u_{i\pm 1} u_i, && \text{if } 1 \le i, i\pm 1 \le n-1, \\ u_i u_j &= u_j u_i, && |j-i| > 1. \end{aligned}$$

The dimension of TL_n is the Catalan number $C_{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

The standard module $V_{n,k}$ is spanned by the basis of $\mathsf{S}_{n,k}$ from which all n-link states that bear vacancies are discarded. They are defined only when n and k have the same parity. The action of TL_n on $V_{n,k}$ is defined as that of dTL_n on $\mathsf{S}_{n,k}$. Their dimension is given by $\dim V_{n,k} = \binom{n}{(n-k)/2} - \binom{n}{(n-k)/2-1}$.

The Gram bilinear form $\langle *, * \rangle_{n,k} : V_{n,k} \times V_{n,k} \to \mathbb{C}$ is introduced exactly as the Gram product on $\mathsf{S}_{n,k}$. Now the rule stating that $\langle u, v \rangle_{n,k}$ is zero whenever unmatched vacancies arise upon glueing of $\bar{u}$ and v

can be ignored safely as no link states with vacancies occur in $V_{n,k}$. (The same observation holds for the multiplication in TL_n and the action of TL_n on the standard modules $V_{n,k}$, discussed above.) It is possible to compute the determinant of the matrix $G_{n,k}$ representing the bilinear form $\langle *,* \rangle_{n,k}$ in the basis of n-link states with k defects. (See for example [63, 70].)

Proposition 3.A.1. *The Gram determinant for the bilinear form on $V_{n,k}$ when $\beta = q + q^{-1}$ is given, up to a sign, by*

$$\det G_{n,k} = \prod_{j=1}^{(n-k)/2} \left(\frac{[k+j+1]_q}{[j]_q} \right)^{\dim V_{n,k+2j}} \tag{3.A.2}$$

where q-numbers are used: $[m]_q = (q^m - q^{-m})/(q - q^{-1})$.

Note that $\det G_{n,k}$, $n \geq 1, 0 \leq k \leq n$, does not vanish at $\beta = \pm 2$, that is, at $q = \pm 1$.

The radical $R_{n,k} = \{ v \in V_{n,k} \mid \langle v, w \rangle_{n,k} = 0 \text{ for all } w \in V_{n,k} \}$ is a submodule of the standard module $V_{n,k}$. It has the following properties.

Proposition 3.A.2. *The radical $R_{n,k}$ is the maximal proper submodule of $V_{n,k}$. It is either trivial ($\simeq \{0\}$) or irreducible.*

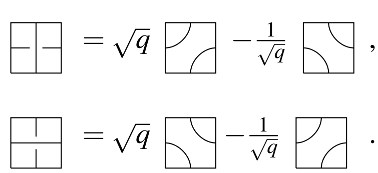
Like for the dilute ones, the radicals of the Temperley-Lieb standard modules are nontrivial only when q is a root of unity distinct than ± 1. Let ℓ be the smallest positive integer such that $q^{2\ell} = 1$. An integer k is called critical if $k + 1 \equiv 0 \bmod \ell$ and non-critical otherwise. Let $I_{n,k}$ stand for the irreducible quotient $V_{n,k}/R_{n,k}$ of the standard module $V_{n,k}$.

Proposition 3.A.3. *With the notation just introduced, the dimensions of the irreducible quotients can be obtained from the following recurrence equations:*

$$\dim I_{n,k} = \begin{cases} \dim V_{n,k}, & \textit{if } k \textit{ is critical}, \\ \dim I_{n-1,k-1}, & \textit{if } k+1 \textit{ is critical}, \\ \dim I_{n-1,k-1} + \dim I_{n-1,k+1}, & \textit{otherwise}, \end{cases} \tag{3.A.3}$$

with initial conditions $\dim I_{n,n} = 1$ *for all n and* $\dim I_{n,0} = 0$ *when n is odd.*

The algebra $\mathsf{TL}_n(\beta)$ has a central element F_n whose eigenvalues can distinguish any pair of standard modules whose labels k and k' fall between two consecutive critical lines. It is defined diagrammatically as the analogous element in dTL_n (see appendix 3.B) by equation (3.B.1). Here, however, the building tiles are defined by



Here are the basic properties of F_n.

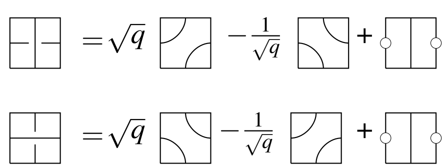
Proposition 3.A.4. *(i) The element $F_n \in \mathsf{TL}_n$ is central, satisfies $F^t = F$ and acts on $V_{n,k}$ as the identity times $\mathsf{F}_k = q^{k+1} + q^{-(k+1)}$.*
(ii) Let q be a root of unity distinct from ± 1 and $k \in \{0,1,\dots,n\}$ be critical for this q. Let $z_k \in V_{n,k}$ be the link state with k defects at the lowest positions and arcs between positions $2i-1$ and $2i$ for $1 \le i \le (n-k)/2$. Then the action of F_{n+1} on $\mathsf{id} \otimes z_k \in V_{n,k}\uparrow$ has a non-zero component along the vector $y_k = u_1 u_2 \dots u_n \otimes z_k$ in a basis containing both linearly independent elements y_k and $z_k \in V_{n,k}\uparrow$.

A basis $\mathcal{S}_{n,k}$ of the induced module $V_{n,k}\uparrow$ is constructed explicitly in [63]. It is with this basis that the above result *(ii)* is stated in that paper. The simpler statement above establishes that the action of F_{n+1} is not a multiple of the identity on $V_{n,k}\uparrow$. This is what will be used in the proof of lemma 3.4.14.

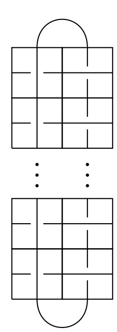
3.B The central element F_n

One central element of dTL_n plays an important role in the text, starting with the proof of proposition 3.3.9. If q is generic, it has distinct eigenvalues on non-isomorphic standard modules. If q is a root of unity, for certain indecomposable modules, it is not a multiple of the identity, a property that allows one to probe their structure. A similar element for TL_n appeared in [42]. A rather different formulation, in terms of diagrams, was used in [53] to probe the Jordan structure of the transfer matrix of loop models. It was also used in [63] to discuss the representation theory of TL_n.

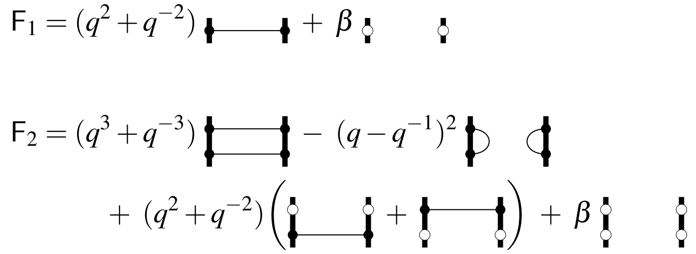
The central element F_n is defined graphically through the following tiles



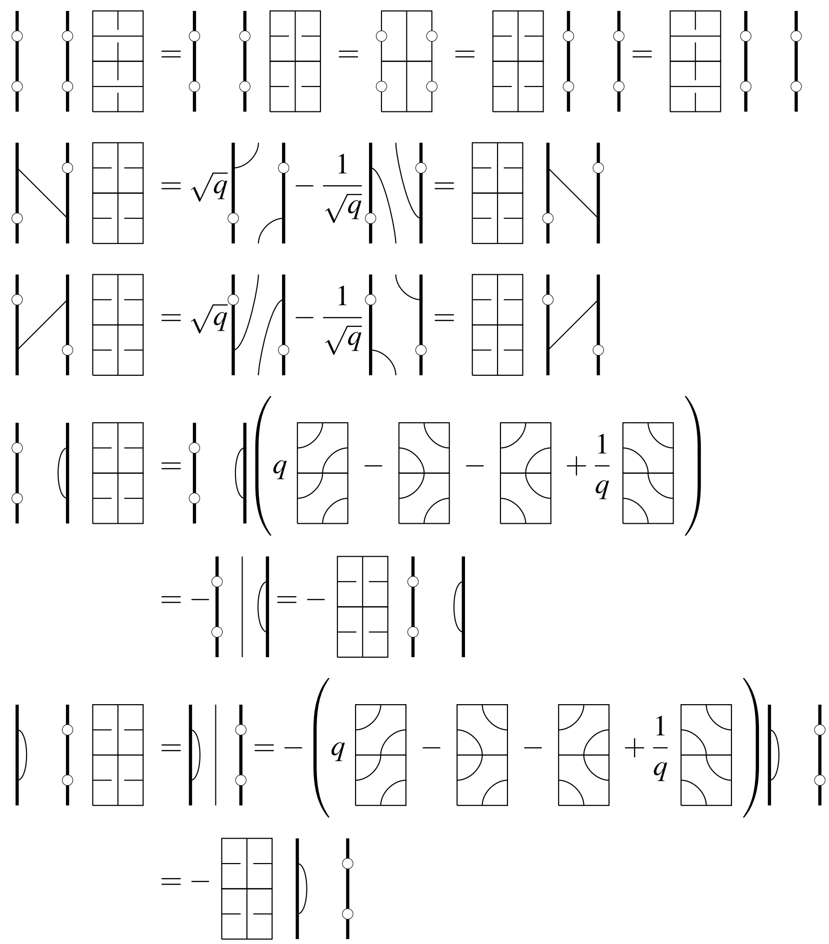
which are multiplied according to the rules used for diagrams. It is important to recall that, if a vacancy and a string meet at an edge, the n-diagram to which they belong is zero. Note that the dashed horizontal segments on these defining tiles underline the fact that, for the three "states" of the right-hand sides, the vertical edges are the same: either both receive a link or both are vacancies. A stronger proprety, equation (3.B.2), will be satisfied by the element F_n whose definition is

$\mathsf{F}_n =$  . (3.B.1)

The expansion of the $2n$ tiles leads to 3^{2n} different diagrams, most of them being zero. The two first F_n are



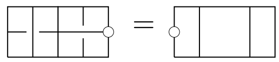
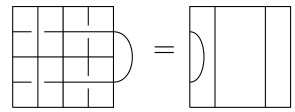
To verify that it is indeed a central element, we compute its products with the generators of dTL_n. We start by expanding the tiles of the left column:



Note that a sign appears during the commutation of the two last generators. The computation for the right column is obtained from that for the left by the exchange $\sqrt{q} \leftrightarrow -1/\sqrt{q}$. The following result is thus proved.

Proposition 3.B.1. *F_n is a central element of dTL_n.*

The eigenvalues of F_n on standard modules $\mathsf{S}_{n,k}$ are easily computed. Before doing so, it is useful to note that rows of the defining tiles, acting on vacancies or arcs of a link state, have the following properties

 and  (3.B.2)

as a direct expansion of the tiles shows. The first property above indicates that F_n is actually an element of the subalgebra S_n. To see this, let $u \in \mathsf{dTL}_n$ be an n-diagram. If $I = \{i_1, i_2, \dots, i_{n-k}\}$ is the set of positions of its vacancies on its left side, then $u = \pi_z u$ where $z \in X_{n,k}$ is the link diagram with vacancies at these same positions. (The element $\pi_z = |z\bar{z}| \in \mathsf{dTL}_n$ is introduced and discussed in Section 3.4.) The product $\mathsf{F}_n u = \mathsf{F}_n \pi_z u$ is simplified by the observation that all vacancies of π_z go through F_n due to (3.B.2) and $\mathsf{F}_n u = (\pi_z \mathsf{F}_n \pi_z)u$. The sums in the remaining tiles may omit the tile, as a link in π_z is connected on either side of each tile to be summed. These sums are then precisely those intervening in the definition of F_k of TL_k.

Proposition 3.B.2. *The central element $\mathsf{F}_n \in S_n \subset \mathsf{dTL}_n$ can be written as*

$$\mathsf{F}_n = \sum_{0 \le k \le n} \sum_{z \in X_{n,k}} \pi_z \mathsf{F}_n \pi_z \tag{3.B.3}$$

where each summand $\pi_z \mathsf{F}_n \pi_z$ is constructed by insertion in F_k of $(n-k)$ lines of vacancies to match those in z.

Proposition 3.B.3. *On $\mathsf{S}_{n,k}$, the element F_n acts as $\mathsf{F}_k \cdot \mathrm{id}$ where $\mathsf{F}_k = q^{k+1} + q^{-(k+1)}$.*

Proof. Since F_n is central and the modules $\mathsf{S}_{n,k}$ are indecomposable (proposition 3.3.2), the endomorphism defined by left multiplication by F_n can only have one eigenvalue. By the previous proposition and the properties (3.B.2), the tiles on lines of F_n acting on vacancies or on arcs of a link state are therefore completely determined, they contribute an overall factor of 1 and can be left out of the computation. For z a link diagram in $\mathsf{S}_{n,k}$, the computation of $\mathsf{F}_n z$ thus reduces to that of $\mathsf{F}_k z_0$ where z_0 is the unique k-link diagram with k defects. Moreover the computation of $\mathsf{F}_k z_0$ does not involve anymore the tile and it becomes identical to that for the action of the central element F_k on $V_{k,k}$. This computation was done in [63]: $\mathsf{F}_k z_0 = F_k z_0 = (q^{k+1} + q^{-(k+1)}) z_0$. (See proposition 3.A.4.) Since the link diagrams form a basis of $\mathsf{S}_{n,k}$, the element F_n acts as a multiple of the identity and the result follows. ∎

These eigenvalues of the central element F_n provide a good way to distinguish between standard modules. More precisely:

Lemma 3.B.4. *Let n be a positive integer.*
(i) If q is not a root of unity, then $\mathsf{F}_j \neq \mathsf{F}_k$ if $j \neq k$.
Let q be a root of unity other than ± 1 and ℓ be the smallest positive integer such that $q^{2\ell} = 1$. Let k_c be critical or -1 and K_{e} (resp. K_{o}) denote the set of ks such that $k_c < k \le k_c + \ell$ and k has the parity of $n - k_c$ (resp. of $n - k_c - 1$).

(ii) If j and k are distinct and both in K_{e} (or both in K_{o}), then $\mathsf{F}_j \neq \mathsf{F}_k$.

(iii) The intersection $K_{\mathsf{e}} \cap K_{\mathsf{o}}$ is non-empty if and only if q is of the form $e^{2i\pi m/\ell}$ with $\gcd(m,l)=1$ and l odd.

(iv) The function F_k is even with respect to a mirror reflection through a critical line.

Proof. If $\theta \in \mathbb{C}$ is chosen such that $q = e^{i\theta}$, then $\mathsf{F}_j = \mathsf{F}_k$ is equivalent to $\cos((j+1)\theta) = \cos((k+1)\theta)$ which in turn amounts to either *(a)* $(k+1)\theta = (j+1)\theta + 2\pi p$ or *(b)* $(k+1)\theta = -(j+1)\theta + 2\pi p$ for some integer p. If $j \neq k$, then θ must be a (real) rational multiple of π and *(i)* follows. If $q^{2\ell} = 1$ with ℓ the smallest possible, then either *(c)* $q = e^{2\pi i m/\ell}$ with $\gcd(m,\ell)=1$ and ℓ odd or *(d)* $q = e^{\pi i(2m+1)/\ell}$ with $\gcd(2m+1,\ell)=1$. The equation *(a)* requires that $(k-j)\theta$ be an integer multiple of 2π. But $(k-j)\theta$ is either $2\pi(k-j)m/\ell$ or $\pi(k-j)(2m+1)/\ell$. Both forms require that the difference $k-j$ be a multiple of ℓ which is impossible since $k_c < k, j \leq k_c + \ell$.

To study the case *(b)*, write $k = k_c + \bar{k}$ and $j = k_c + \bar{j}$ with $0 < \bar{k}, \bar{j} \leq \ell$. Since $k_c + 1 \equiv 0 \mod \ell$, the equation *(b)* forces $(k+j+2)\theta$ (or equivalently $(\bar{k}+\bar{j})\theta$) to be an integer multiple of 2π. For the case (d), this is impossible since $(\bar{k}+\bar{j})(2m+1)/\ell = 2p$ would mean that $(\bar{k}+\bar{j})$ is an even multiple of ℓ. However, in the case *(c)*, ℓ is always odd (and ≥ 3) and the equation $(\bar{k}+\bar{j})m/\ell = p$ has always the solution $\bar{k}=1$ and $\bar{j} = \ell - 1 (\neq \bar{k})$. Note that this solution and all others ($\bar{k} = i$ and $\bar{j} = \ell - i$) have k and ℓ of distinct parity. This proves both *(ii)* and *(iii)*.

If $k_\pm = k_c \pm m$, then $q^{k_+ + 1} = q^{k_c+m+1} = q^{-k_c+m-1} = q^{-(k_-+1)}$ where the criticality of k_c was used. The last statement follows. ■

3.C The dimensions of the irreducible modules $\mathsf{I}_{n,k}$

The dimensions of the irreducible quotients $\mathsf{I}_{n,k}$ satisfy recurrence relations that can be used efficiently to compute them. We gather here these relations and their proofs. Tables containing the dimensions of standard modules and of irreducible quotients for $\ell = 3$ and $\ell = 4$ are also given.

As usual q is a root of unity other than ± 1 and ℓ is the smallest positive integer such that $q^{2\ell} = 1$ (and $\ell \geq 2$). The notation

$$i_{n,k} = \dim \mathsf{I}_{n,k}, \quad r_{n,k} = \dim \mathsf{R}_{n,k}, \quad s_{n,k} = \dim \mathsf{S}_{n,k} \text{ and } \tilde{i}_{n,k} = \dim I_{n,k}.$$

is used throughout. We recall that the symbol $\mathsf{I}_{n,k}$ is used for the irreducible module over dTL_n and $I_{n,k}$ for that over TL_n.

The module $\mathsf{I}_{n,k}$ is defined to be the irreducible quotient $\mathsf{S}_{n,k}/\mathsf{R}_{n,k}$. Corollaries 3.3.6 and 3.4.9 then give a simple formula for its dimension in terms of those for the irreducibles $I_{n,k}$ of the (original) Temperley-Lieb algebra:

$$i_{n,k} = \dim \mathsf{S}_{n,k} - \dim \mathsf{R}_{n,k} = \sum_{p=0}^{\lfloor (n-k)/2 \rfloor} \binom{n}{k+2p} \tilde{i}_{n,k}. \tag{3.C.1}$$

Proposition 3.C.1. *Let $n \geq 1$ and k_c be an integer critical for q. Then the three following recurrence relations hold:*

$$i_{n+1,k_c-1} = i_{n,k_c-2} + i_{n,k_c-1}, \tag{3.C.2}$$

$$i_{n+1,k_c+i} = i_{n,k_c+i+1} + i_{n,k_c+i} + i_{n,k_c+i-1}, \qquad 1 \leq i \leq \ell - 2, \tag{3.C.3}$$

$$i_{n+1,k_c} = i_{n,k_c-1} + i_{n,k_c} + 2i_{n,k_c+1} + i_{n,k_c+2\ell-1}, \tag{3.C.4}$$

where any $i_{n,j}$ with a j outside the set $\{0,1,\dots,n\}$ is zero. With this convention, the second equation also holds for $k_c = -1$. The boundary conditions are $i_{n,-1} = 0$ and $i_{n,n} = 1$.

Proof. For the first of these recurrences, use (3.C.1) to write i_{n+1,k_c-1} in terms of the $\tilde{i}$s and split the sum into two using the binomial identity $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$. The summation index of the second sum, that containing the binomial $\binom{n}{j-1}$, is then shifted using (3.A.3) of proposition 3.A.2. Terms that are missing at either end of the sums can be added as they are weighted by a binomial that vanishes. The proof of the second recurrence follows the same lines.

The last recurrence is proved as follows:

$$\begin{aligned} i_{n+1,k_c} &= s_{n+1,k_c} \\ &= s_{n,k_c-1} + i_{n,k_c} + s_{n,k_c+1} \\ &= i_{n,k_c-1} + r_{n,k_c-1} + i_{n,k_c} + i_{n,k_c+1} + r_{n,k_c+1} \\ &= i_{n,k_c-1} + i_{n,k_c+1} + i_{n,k_c} + i_{n,k_c+1} + i_{n,k_c+2\ell-1}. \end{aligned}$$

The first equation is simply the irreducibility of S_{n+1,k_c}, the second line follows from the restriction of S_{n+1,k_c} (see (3.3.7)), the last line is a consequence of proposition (3.4.15). ■

The dimensions of $\mathsf{S}_{n,k}$ are showed in table 3.C.1 for $n \leq 10$, and the dimension of $\mathsf{I}_{n,k}$ are showed in tables 3.C.2 and 3.C.3 for $\ell = 3$ and $\ell = 4$, respectively.

3.D Tools from algebra

We review here concepts and results in algebra that are used in the article and might not be familiar to some readers. We start by presenting short exact sequences and proceed to projective modules. The interplay between induction and the tensor product is then recalled. We finally recall Wedderburn's theorem, and its generalization, and Frobenius reciprocity theorem.

Throughout the appendix, A is a unital associative algebra over $\mathbb{C}$, B a subalgebra of A. Unless otherwise stated, L, M, N and P are A-modules.

n / k	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	2	2	1								
3	4	5	3	1							
4	9	12	9	4	1						
5	21	30	25	14	5	1					
6	51	76	69	44	20	6	1				
7	127	196	189	133	70	27	7	1			
8	323	512	518	392	230	104	35	8	1		
9	835	1353	1422	1140	726	369	147	44	9	1	
10	2188	3610	3915	3288	2235	1242	560	200	54	10	1

Table 3.C.1 – Dimensions $s_{n,k} = \dim \mathsf{S}_{n,k}$

n / k	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	2	2	1								
3	4	4	3	1							
4	8	8	9	4	1						
5	16	16	25	14	5	1					
6	32	32	69	44	19	6	1				
7	64	64	189	132	63	27	7	1			
8	128	128	518	384	195	104	35	8	1		
9	256	256	1422	1097	579	369	147	43	9	1	
10	512	512	3915	3098	1676	1242	559	190	54	10	1

Table 3.C.2 – Dimensions $i_{n,k} = \dim \mathsf{I}_{n,k}$ for $\ell = 3$

n / k	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	2	2	1								
3	4	5	3	1							
4	9	12	8	4	1						
5	21	29	20	14	5	1					
6	50	70	49	44	20	6	1				
7	120	169	119	133	70	27	7	1			
8	289	408	288	392	230	104	34	8	1		
9	697	985	696	1140	726	368	138	44	9	1	
10	1682	2378	1681	3288	2234	1232	506	200	54	10	1

Table 3.C.3 – Dimensions $i_{n,k} = \dim \mathsf{I}_{n,k}$ for $\ell = 4$.

3.D.1 Short exact sequences

Let $L \xrightarrow{f} M$ and $M \xrightarrow{g} N$ be two module homomorphisms. The sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is said to be *exact* (or *exact at* M) if the kernel of g is equal to the image of f. A *short exact sequence* is a sequence of homomorphisms

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

that is exact at L, M and N. This is equivalent to saying that the sequence is exact at M with f and g being injective and surjective, respectively.

Proposition 3.D.1. *A sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is exact if and only if it verifies the three following conditions:

(i) $gf = 0$

(ii) if there is a module U *and a homomorphism* $u : U \to M$ *such that* $gu = 0$, *then there is a unique homomorphism* $\bar{u} : U \to L$ *such that* $f\bar{u} = u$;

(iii) if there is a module V *and a homomorphism* $v : M \to V$ *such that* $vf = 0$, *then there is a unique homomorphism* $\bar{v} : N \to V$ *such that* $\bar{v}g = v$.

The short exact sequence of proposition 3.D.1 is called *split* if $M \simeq L \oplus N$.

Proposition 3.D.2. *If the short sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is exact, the three following statements are equivalent:

(i) the sequence splits;

(ii) there is a homomorphism $\bar{f} : M \to L$ *such that* $\bar{f}f = \mathrm{id}_L$;

(iii) there is a homomorphism $\bar{g} : N \to M$ *such that* $g\bar{g} = \mathrm{id}_N$.

3.D.2 Projective modules

A module P is said to be *projective* if for all modules M and N and all homomorphisms $f : M \to N$ and $g : P \to N$ with f surjective, there is a homomorphism $h : P \to M$ such that $f \circ h = g$. In other words, given homomorphisms f and g as in the diagram below with an exact horizontal row, then there exist h that makes the diagram commute.

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \downarrow g & & \\ M & \xrightarrow[f]{} & N & \longrightarrow & 0. \end{array}$$

Because of proposition (3.D.2), the above definition (with $P = N$ and $g = \mathrm{id}_P$) gives:

Proposition 3.D.3. *A module P is projective if and only if all short exact sequences*

$$0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$$

split.

Direct sums and direct summands of projective modules are also projective. Note also that an algebra seen as a module over itself is always projective.

A *projective cover* of a module M is a pair (P, f) with P a projective module and $P \xrightarrow{f} M$ a surjective morphism having the following property. If (P', g) is another pair where P' is projective and g is a surjective morphism from P' to M, then $P' \simeq P \oplus Q$ for some module Q. Therefore projective covers are unique up to isomorphism.

Write $\mathsf{Hom}(M,N)$ for the vector space of A-homomorphisms of M into N.

Proposition 3.D.4. *If $0 \to M \to N \to P \to 0$ is exact, then, for any other module L, so are*

$$0 \longrightarrow \mathsf{Hom}(L,M) \longrightarrow \mathsf{Hom}(L,N) \longrightarrow \mathsf{Hom}(L,P),$$

and

$$0 \longrightarrow \mathsf{Hom}(P,L) \longrightarrow \mathsf{Hom}(N,L) \longrightarrow \mathsf{Hom}(M,L).$$

Moreover, if L is projective, then the last homomorphism in the first sequence above is surjective.

3.D.3 Restriction and induction

If an algebra A has a subalgebra B, it is natural to ask how a given A-module M would behave as a B-module. Since B is a subalgebra of A, the space M can be seen as a B-module for the same action and the B-module thus obtained is called the *restriction* of M to B, and is noted $M{\downarrow}$ (or $M{\downarrow}_B^A$). It can be shown that restriction preserves short exact sequences, that is:

Proposition 3.D.5. *Let A be an associative algebra and B a sub-algebra of A, the sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$$

of A-modules is exact if and only if the sequence

$$0 \longrightarrow L{\downarrow}_B^A \longrightarrow M{\downarrow}_B^A \longrightarrow P{\downarrow}_B^A \longrightarrow 0$$

of B-modules is exact.

It is also natural to do the "reverse process", that is, to transform a B-module into an A-module. This process is slightly more complex, and the resulting module is called the *induction* of M to A, noted $M{\uparrow}$ (or $M{\uparrow}_B^A$). It is defined as the tensor product of $A \otimes_B M$. The regular module structure then carries over to the tensor product: $a'(a \otimes m) = (a'a) \otimes m$ for all $a, a' \in A$ and $m \in M$. It can be shown that the induction preserves parts of exact short sequences. More precisely, one has:

Proposition 3.D.6. *Let A be an associative algebra and B a subalgebra of A, the sequence*

$$A \otimes L \longrightarrow A \otimes M \longrightarrow A \otimes P \longrightarrow 0$$

of A-modules is exact if the sequence

$$L \longrightarrow M \longrightarrow P \longrightarrow 0$$

of B-modules is exact.

There are cases when the homomorphism $A \otimes L \to A \otimes M$ fails to be injective even when the B-module homomorphism $L \to M$ is.

3.D.4 Frobenius reciprocity, Wedderburn and Jordan-Hölder theorem

The operations of restriction and induction were presented as “reverse processes". This is particularly meaningful in view of the next result.

Proposition 3.D.7 (Frobenius reciprocity theorem)**.** *Let A be a finite-dimensional associative algebra over $\mathbb{C}$ and B a subalgebra of A. Let M be an A-module and N be a B-module. Then, as vector spaces*

$$\mathsf{Hom}_B(N, M\!\downarrow) \simeq \mathsf{Hom}_A(N\!\uparrow, M). \tag{3.D.1}$$

The algebra A can be seen as a left A-module where the action is simply left multiplication. This module is called the *regular* module and one may write ${}_AA$ to emphasize the left module structure. The algebra is called *semisimple* if its regular module is completely reducible, that is, it is isomorphic to a direct sum of irreducible modules. A key property of semisimple algebras is the following.

Theorem 3.D.8 (Wedderburn's theorem)**.** *Let A be a finite-dimensional associative algebra over $\mathbb{C}$. A is semisimple if and only if the regular module decomposes as*

$${}_AA \simeq \bigoplus_i (\dim L_i) L_i$$

where the set $\{L_i\}$ forms a complete set of non-isomorphic irreducible A-modules.

It can also be shown that A is semisimple if and only if every A-module is projective. If an algebra is not semisimple, there will be indecomposable yet reducible modules. When A is not semisimple, Wedderburn's theorem no longer holds, and it is replaced by the following generalisation.

Theorem 3.D.9. *Let A be a finite-dimensional associative algebra over $\mathbb{C}$. The regular module decomposes as*

$$_AA \simeq \bigoplus_i (\dim L_i) P_i$$

where the set $\{P_i\}$ forms a complete set of non-isomorphic projective indecomposable A-modules, and L_i is the unique irreducible quotient of P_i.

The projective indecomposables in this last proposition are called *principal indecomposable modules*. It can be shown that any projective module is a direct sum of principal indecomposable ones.

Note that induction of the regular module $_BB$ is simply $B{\uparrow}_B^A = {}_AA \otimes_B {}_BB \simeq {}_AA$. If $_BB = \oplus_i B_i$ is the decomposition of B into its principal indecomposable modules, then $_AA \simeq \oplus_i B_i{\uparrow}$ and, since they appear as direct summands of the free module $_AA$, the A-modules $B_i{\uparrow}$ are projective. (They might not be indecomposable.) Therefore the induction of a projective module is projective.

A *composition series* of the module M is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = M$ such that all quotients M_i/M_{i-1}, $0 < i \leq k$, are irreducible. The quotients M_i/M_{i-1} are called the *composition factors* of M.

Theorem 3.D.10 (Jordan-Hölder's theorem)**.** *Any finite-dimensional module M has a composition series. Moreover, if $0 = M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = M$ and $0 = N_0 \subset N_1 \subset \cdots \subset N_{l-1} \subset N_l = M$ are two composition series of M, then $k = l$ and the two sets of composition factors coincide up to permutation.*

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Deuxième partie

Restriction et induction des modules indécomposables des algèbres de Temperley-Lieb régulière et diluée

Chapitre 4

Présentation

4.1 Objectifs et méthodologie

Le but principal de cet article est de terminer le calcul des foncteurs d'induction et de restriction des modules indécomposables pour les algèbres de Temperley-Lieb régulière et diluée. Comme le produit de fusion défini sur les algèbres TL_n requiert le calcul de tels foncteurs, il s'agit d'une étape essentielle. De plus, bien que la théorie des représentations de TL_n était déjà très avancée, les preuves de certains de ses résultats [70] n'étaient pas suffisamment robustes pour permettre le calcul des foncteurs d'induction. Nous avons donc entrepris de construire une liste complète des modules indécomposables de TL_n et dTL_n avant de poursuivre les calculs.

L'article commence par un rappel des principaux résultats obtenus pour dTL_n dans [11] et leurs analogues pour le cas régulier [63]. Ensuite les foncteurs de dualité sont utilisés afin de construire les modules indécomposables injectifs de ces algèbres. Ceci permet le calcul des foncteurs d'extension entre les classes de modules les plus rudimentaires : les irréductibles, les projectifs, les injectifs et les standards.

Nous pouvons ensuite construire les modules indécomposables restants à partir de suites exactes judicieusement choisies et trouver leur couverture projective et leur enveloppe injective. Ces couvertures peuvent alors être utilisées pour calculer les inductions et les restrictions de tous les modules indécomposables restants.

La dernière section de l'article présente une introduction à la théorie d'Auslander-Reiten [2–4]. Il s'agit d'une méthode algorithmique pour construire les modules indécomposables d'une algèbre donnée ; le *carquois d'Auslander-Reiten* permet alors de présenter cette information de façon graphique. Nous montrons alors comment calculer le carquois d'Auslander-Reiten de TL_n et de dTL_n.

Contributions

J'ai établi les premiers résultats : structure des modules B et T ainsi que l'action des foncteurs d'induction et de restriction sur ceux-ci. J'ai également obtenu les arguments originaux pour les prouver. La rédaction se fait à parts égales entre les coauteurs.

4.2 Outils algébriques

On fixe ici un corps algébriquement clos $\mathbb{F}$ et une $\mathbb{F}$-algèbre $\mathcal{A}$ de $\mathbb{F}$-dimension finie. Les définitions présentées ici sont basées sur celles trouvées dans [1, 2].

Coiffe, socle et diagrammes de Loewy

Soit un $\mathcal{A}$-module U, et soit l'ensemble I_U de tous les morphismes de U vers un module irréductible quelconque ; le *radical* de U, noté $\operatorname{rad} U$, est l'intersection de tous les noyaux des éléments dans I_U. Autrement dit, c'est le plus petit [1] sous-module de U tel que $U/\operatorname{rad} U$ soit semi-simple ; ce quotient est la *coiffe* de U. De façon analogue, le socle de U, noté $\operatorname{soc} U$, est le plus grand [2] sous-module semi-simple de U.

Soit un $\mathcal{A}$-module U de dimension finie. La *suite de radicaux* de U est la suite d'inclusions

$$0 \equiv \operatorname{rad}^n U \subset \operatorname{rad}^{n-1} U \subset \ldots \subset \operatorname{rad} U \subset U, \tag{4.2.1}$$

où n est le plus petit entier tel que $\operatorname{rad}^{n-1} U = \operatorname{soc} U$ et est la *longueur de Loewy* de U. On construit alors le *diagramme de Loewy* de U comme suit. On trace un carquois [3] dont les points sont les facteurs de compositions de U ; ceux faisant partie du socle sont placés sur une ligne en bas complètement du carquois, puis ceux faisant partie de $(\operatorname{rad}^{n-2} U)/(\operatorname{rad}^{n-1} U)$ sont placés sur une ligne au-dessus et ainsi de suite jusqu'à ce que tous les facteurs de composition soient placés. Étant donné deux facteurs de composition I_i, I_j, on place une flèche dans le carquois pointant de I_i vers I_j et portant une étiquette M si U possède un sous-quotient [4] indécomposable M n'ayant que ces deux facteurs de composition particulier. On continue alors ainsi jusqu'à ce que tous les sous-quotients de U soient identifiés ; le carquois résultant est appelé *diagramme de Loewy* de U.

Foncteurs de dualité et module injectifs

Les foncteurs de dualité servent à construire des $\mathcal{A}$-modules à gauche à partir de $\mathcal{A}$-modules à droite, et vice versa. Au passage, certaines propriétés des modules seront changées, mais d'autres non. La dualité la plus simple est la dualité de $\mathbb{F}$-modules : en effet, puisque tout $\mathcal{A}$-module à droite est aussi un $\mathbb{F}$-module à gauche, on peut construire un foncteur de changement d'anneau et passer d'un $\mathbb{F}$-module à un $\mathcal{A}$-module. Soit donc U un $\mathcal{A}$-module à droite que l'on voit comme un $\mathbb{F}-\mathcal{A}$-bimodule ; le *dual vectoriel* de U est le $\mathcal{A}$-module à gauche

$$U^* = \mathsf{Hom}_{\mathbb{F}}(U, \mathbb{F}). \tag{4.2.2}$$

1. Si $V \subset U$ est tel que U/V est semisimple, alors $\operatorname{rad} U \subset V$.
2. Si $V \subset U$ est un sous-module semisimple, alors $V \subset \operatorname{soc} U$.
3. Un réseau où les côtés ont une orientation.
4. Un *sous-quotient* d'un module U est un module M qui est quotient d'un sous-module de U.

La représentation associée à U^* est parfois appelée *duale* ou *contragrédiente*. Puisque $\mathbb{F}$ est algébriquement clos, cette dualité est un foncteur exact et contravariant qui est son propre inverse : $(U^*)^* \simeq U$. On rappelle qu'un foncteur contravariant inverse la direction des morphismes ; un morphisme de $\mathcal{A}$-modules à droite $f : U \to V$ deviendra donc un morphisme de $\mathcal{A}$-modules à gauche $f^* : V^* \to U^*$. De plus, puisque cette dualité inverse le sens des morphismes, le dual vectoriel d'un module projectif est un module *injectif*.

Un module J est injectif si, pour toute paire de modules U,V et toute paire de morphismes $f : U \to V$, $g : U \to J$ avec f injectif, il existe un morphisme $f' : V \to J$ tel que $f' \circ f = g$. Dans notre cas, un module à gauche J est injectif si et seulement s'il existe un module à droite P qui est projectif tel que $P^* \simeq J$. Ces modules jouent un rôle similaire aux modules projectifs : pour tout module U, il existe une paire de modules injectifs J_1,J_2 et un morphisme $f : J_1 \to J_2$ tel que $\operatorname{Ker} f \simeq U$; la suite exacte courte

$$0 \longrightarrow U \xrightarrow{\ker f} J_1 \longrightarrow \operatorname{im} f \longrightarrow 0, \tag{4.2.3}$$

est appelée une *présentation injective* de U. Il est également possible de montrer que J est injectif si et seulement si le foncteur $\mathsf{Hom}_{\mathcal{A}}(-,J)$ est exact.

Le dual algébrique est défini de façon similaire : puisque l'algèbre $\mathcal{A}$ peut être vue comme un bimodule sur elle même, on peut s'en servir pour construire un foncteur de changement d'anneau qui prend un $\mathcal{A}$-module à droite U, et en fait un $\mathcal{A}$-module à gauche :

$$U^t \equiv \mathsf{Hom}_{\mathcal{A}}(U,\mathcal{A}). \tag{4.2.4}$$

Le dual algébrique est aussi un foncteur contravariant, mais il n'est pas exact en général, puisque $\mathcal{A}$ n'est en général pas injective vue comme module sur elle-même. De plus, le dual algébrique d'un module projectif est aussi projectif, et le foncteur dual algébrique n'est en général pas fidèle [5].

Groupe d'extension et modules indécomposables

Les *groupes d'extension* servent à classifier les modules indécomposables d'une algèbre donnée ; de façon intuitive, ceux-ci donnent une « recette » permettant de construire de nouveaux modules indécomposables en collant deux autres modules plus petits ensemble. Étant donné deux $\mathcal{A}$-modules à gauche U,V, une *extension* de U par V est une suite exacte courte de la forme

$$0 \to V \to M \to U \to 0, \tag{4.2.5}$$

où M est aussi un $\mathcal{A}$-module. Deux extensions

$$e_1 : 0 \longrightarrow V \xrightarrow{f} M \xrightarrow{g} U \longrightarrow 0, \qquad e_2 : 0 \longrightarrow V \xrightarrow{f'} M' \xrightarrow{g} U \longrightarrow 0, \tag{4.2.6}$$

5. Un foncteur F est fidèle si $F(U) \simeq F(V)$ implique que $U \simeq V$.

sont dites *équivalentes* s'il existe un isomorphisme $h : M \to M'$ tel que $h \circ f = f'$ et $g' \circ h = g$. En particulier, l'*extension triviale* est donnée par la suite scindée $0 \to V \to V \oplus U \to U \to 0$. Il est toutefois important de noter qu'il existe des extensions non équivalentes avec $M \simeq M'$.

Afin de comprendre la classification de ces extensions, il est utile de procéder comme suit. On commence par prendre une présentation projective[6] de U, $0 \longrightarrow U' \xrightarrow{q} P \xrightarrow{p} U \longrightarrow 0$ et on construit le diagramme commutatif suivant à l'aide de l'extension e_1 dans (4.2.6) :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U' & \xrightarrow{q} & P & \xrightarrow{p} & U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & V & \xrightarrow[f]{} & M & \xrightarrow[g]{} & U & \longrightarrow & 0 \end{array} . \tag{4.2.7}$$

Dans ce diagramme, le morphisme g^* faisant commuter le carré de droite existe car P est un module projectif, et le morphisme f^* existe par l'universalité du noyau[7] de g. On peut alors montrer que M est isomorphe à la *somme amalgamée* de q et de f^*, c'est-à-dire que $M \simeq \mathrm{Coker}(\kappa_1 \circ q - \kappa_2 \circ f^*)$, où κ_1, κ_2 sont les injections canoniques de P et de V dans $P \oplus V$, respectivement. La morale de cette discussion est que les extensions d'un module donné U par un module V sont déterminées par les morphismes de U' dans V ; classifier ces extensions peut donc se réduire à l'étude de ces morphismes. Il ne reste plus alors qu'à caractériser la relation d'équivalence entre extensions en termes de morphismes : on peut montrer que $e_1 \sim e_2$ si et seulement si le morphisme $f^* - (f')^*$ se *relève par* P, c'est-à-dire qu'il existe un morphisme $h : P \to V$ tel que $h \circ q = f^* - (f')^*$. Le nom de *groupe d'extension* vient alors du fait que l'on peut donner à l'ensemble des extensions de U par V une structure de $\mathbb{F}$-module (et donc de groupe abélien) à travers celle des homomorphismes de U' dans V.

Les couvertures projectives et les enveloppes injectives

Soit U un $\mathcal{A}$-module à gauche, et

$$0 \longrightarrow U' \xrightarrow{q} P \xrightarrow{p} U \longrightarrow 0$$

une de ses présentations projectives. On dira de cette présentation qu'elle est *minimale*, ou encore que (P, p) est une *couverture projective* de U, si $q(U') \subset \mathrm{rad}\, P$. Si (P', p') est une couverture projective de U', on trouve alors une *résolution projective* de U :

$$P' \xrightarrow{q \circ p'} P \xrightarrow{p} U \to 0.$$

De façon similaire, étant donnée une présentation injective de U, $0 \longrightarrow U \xrightarrow{j} J \xrightarrow{q} U' \longrightarrow 0$, on dira

6. Les présentations projectives d'un module ne sont jamais uniques ; on peut par contre montrer que la construction qui suit donne le même résultat, peu importe la présentation choisie.

7. Le noyau de g est le plus grand sous-module de M tel que $g(\mathrm{Ker}\, f) = 0$; puisque $g \circ g^* \circ q = 0$, il suit que $\mathrm{im}(g^* \circ q) \subset \mathrm{Ker}\, g \simeq V$.

qu'elle est minimale, ou encore que (J, j) est une enveloppe injective de U, si $\operatorname{soc} J \subset \operatorname{im}(j)$.

Supposons maintenant que nous désirions calculer $F(U)$ pour un certain foncteur $\mathbb{F}$-linéaire et exact à droite F. On commence par calculer $F(L)$ pour les modules libres L. Puisque F est linéaire, ceci revient à calculer F sur $\mathcal{A}$ vue comme un module sur elle-même. Ensuite, on calcule $F(Q)$ pour tous les modules projectifs indécomposables ; on peut alors utiliser le fait que l'on connaisse déjà $F(\mathcal{A})$ pour simplifier les calculs. Finalement, on applique F sur la résolution projective de U pour obtenir

$$F(P') \xrightarrow{F(q)\circ F(p')} F(P) \xrightarrow{F(p)} F(U) \to 0. \tag{4.2.8}$$

Puisque l'on connaît $F(P')$ et $F(P)$, on peut utiliser cette suite exacte pour trouver $F(p)$. Les résolutions injectives s'utilisent de façon similaire pour calculer des foncteurs $\mathbb{F}$-linéaires et exacts à gauche. Nous utiliserons ces deux méthodes pour calculer les foncteurs d'induction et de restriction.

Chapitre 5

Restriction and induction of indecomposable modules over the Temperley-Lieb algebras

Restriction and induction of indecomposable modules over the Temperley-Lieb algebras

Jonathan Belletête, David Ridout and Yvan Saint-Aubin

ABSTRACT:Both the original Temperley-Lieb algebras TL_n and their dilute counterparts dTL_n form families of filtered algebras: $\mathsf{TL}_n \subset \mathsf{TL}_{n+1}$ and $\mathsf{dTL}_n \subset \mathsf{dTL}_{n+1}$, for all $n \geq 0$. For each such inclusion, the restriction and induction of every finite-dimensional indecomposable module over TL_n (or dTL_n) is computed. To accomplish this, a thorough description of each indecomposable is given, including its projective cover and injective hull, some short exact sequences in which it appears, its socle and head, and its extension groups with irreducible modules. These data are also used to prove the completeness of the list of indecomposable modules, up to isomorphism. In fact, two completeness proofs are given, the first based on elementary homological methods, the second uses Auslander-Reiten theory. The latter proof offers a detailed example of this algebraic tool that may be of independent interest.

Keywords Temperley-Lieb algebra, dilute Temperley-Lieb algebra, indecomposable modules, standard modules, projective modules, injective modules, extension groups, Auslander-Reiten theory, non-semisimple associative algebras.

5.1 Introduction

One of the two main goals of this paper is to construct a complete list of non-isomorphic indecomposable modules of the Temperley-Lieb algebras $\mathsf{TL}_n(\beta)$ and their dilute counterparts $\mathsf{dTL}_n(\beta)$. Both families of algebras are parametrised by a positive integer n and a complex number β. These indecomposable modules are characterised using Loewy diagrams, from which socles and heads may be deduced, and their projective covers and injective hulls are determined. We also compute some of their extension groups and detail some of the short exact sequences in which they appear. Many of these results are new, but a subset of these extensive data has appeared in the literature, most of the time without proofs. Due to the increasing role played by the representation theory of these two families of algebras, both in mathematics and physics, we believe that a self-contained treatment is called for.

The second main goal concerns the fact that both the Temperley-Lieb and dilute Temperley-Lieb families of algebras have a natural filtration: $\mathsf{TL}_n(\beta) \subset \mathsf{TL}_{n+1}(\beta)$ and $\mathsf{dTL}_n(\beta) \subset \mathsf{dTL}_{n+1}(\beta)$, for $n \geq 1$. This first inclusion, for example, is realised by using all the generators e_i, $1 \leq i \leq n$, of $\mathsf{TL}_{n+1}(\beta)$, except e_n, to generate a subalgebra isomorphic to $\mathsf{TL}_n(\beta)$. The restriction and induction of $\mathsf{TL}_n(\beta)$- and $\mathsf{dTL}_n(\beta)$-modules along these filtrations has grown to be a useful tool and the paper identifies the results, up to isomorphism, for all indecomposable modules. To the best of our knowledge, these results are also new.

The study of the representation theory of the Temperley-Lieb algebra was launched by Jones [39], Martin [45] and Goodman and Wenzl [25]. They provided descriptions of the projective and irreducible $\mathsf{TL}_n(\beta)$-

modules. Shortly thereafter, Graham and Lehrer [26] introduced a large class of algebras, the "cellular algebras", that included both $\mathsf{TL}_n(\beta)$ and $\mathsf{dTL}_n(\beta)$. They proved many general properties and obtained a beautiful description of the role that their "cell representations" play as intermediates between the irreducible and projective modules. Many cellular algebras have diagrammatic descriptions that are closely related to their applications in statistical mechanics. In this setting, the cell representations are often referred to as the "standard modules". More recent works on the representation theory of $\mathsf{TL}_n(\beta)$ and $\mathsf{dTL}_n(\beta)$ have profited by combining Graham and Lehrer's insights into these standard modules with diagrammatic presentations [11, 41, 63, 70].

The recent literature attests to the necessity of having a thorough understanding of the indecomposable modules of the Temperley-Lieb algebras. We recall a few examples arising in the mathematical physics literature. First, due to the close relationship between statistical lattice models and conformal field theories, it is believed that an operation analogous to the fusion product of conformal field theory should exist at the level of the lattice. One proposal for such a "lattice fusion product" appeared in the work of Read and Saleur [61] and explicit computations were subsequently carried out over the Temperley-Lieb algebra by Gainutdinov and Vasseur [24]. Their method exploited the well known relationship between the representation theories of TL_n and the quantum group $U_q(sl_2)$, leading them to the following simple but crucial observation: The lattice fusion product of a TL_n-module with a TL_m-module does not depend on n or m, if both are sufficiently large. [1] This observation is welcome since these integers have no natural interpretation in the continuous limit where conformal field theory is believed to take over. Further examples of lattice fusion products were computed in [9], for both TL_n- and dTL_n-modules, using powerful arguments based on restriction and induction. The classification of indecomposables is important here because the lattice fusion of an irreducible and a standard module may result in indecomposables that are not irreducible, standard or projective. These computations assume some of the results that will be proved here (and thereby provide one of the primary motivations for the work reported in this paper).

As a second example illustrating the importance of having a complete list of indecomposable modules, we mention the work [52] of Morin-Duchesne, Rasmussen and Ruelle, who used the map between dimer and XXZ spin configurations to introduce a new representation of $\mathsf{TL}_n(\beta)$, with $\beta = 0$. In most models described by the XXZ spin-chain, the Temperley-Lieb generators act on two neighbouring sites. In their new representation, the generators act on three consecutive sites and the decomposition of this representation into a direct sum of indecomposable modules involves modules that are, generically, neither irreducible, standard nor projective.

Our last example points even further down the path that the present paper starts to explore. The Temperley-Lieb algebras and their dilute counterparts are only two of the families of diagrammatic algebras encountered in mathematics and physics. Other important variants of the Temperley-Lieb algebras include the affine (or periodic) families and the one- and two-boundary families. The one-boundary Temperley-Lieb algebra (or blob algebra) was introduced by Martin and Saleur [48] to describe statistical models wherein the physical

1. The fact that lattice fusion operates between modules over different algebras indicates that it should be thought of as an operation on modules over the Temperley-Lieb categories.

field takes on various states along the domain boundary. A discussion of the one-boundary representation theory may be found in [49]. Recent work [51] has shown that in order to properly account for certain examples of integrable boundary conditions, one needs to consider a quotient of the one-boundary Temperley-Lieb algebra instead. By examining the bilinear forms on their standard modules and studying numerical data of the statistical models, the authors inferred the structures of the Virasoro modules appearing in their conformal field-theoretic limits, finding that the Virasoro modules were identified as certain finitely generated submodules of Feigin-Fuchs modules. Even if these observations remain at the level of conjecture, the complexity of the modules over the one-boundary Temperley-Lieb algebras and their quotients seems to go well beyond that of the indecomposable modules that will be classified in the present paper. Hopefully, the methods developed here will also be fruitful for these other families.

The paper is organised as follows. Section 5.2 recalls the basic representation theory of the original and dilute Temperley-Lieb algebras. It also introduces (twisted) dual modules, gives the restriction and induction of the standard, irreducible and projective modules, and computes their Hom- and Ext-groups. Section 5.3 constructs recursively two families of new indecomposable modules and computes their extension groups with all the irreducibles. These calculations show that, together with the projectives and irreducibles, these new families form a complete list of non-isomorphic indecomposable modules (Theorem 5.3.10). Section 5.4 gives the restriction and induction of these new families of modules, relative to the inclusions described above. Finally, Section 5.5 repeats the construction of all indecomposable modules and the proof of their completeness using a more advanced tool, namely Auslander-Reiten theory. We hope that reading both Sections 5.3 and 5.5 in parallel will provide a comparison ground of standard methods for classifying indecomposable modules over finite-dimensional associative algebras and show their relative advantages.

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5.2 Standard, irreducible and projective modules

The definition of both the original and dilute Temperley-Lieb algebras, $\mathsf{TL}_n(\beta)$ and $\mathsf{dTL}_n(\beta)$, depends on a parameter β taking values in a commutative ring. In what follows, this ring will always be the complex numbers $\mathbb{C}$. Another parameter $q \in \mathbb{C}^\times$, related to the first by $\beta = q + q^{-1}$, is also used. The standard modules of these algebras may be introduced in several ways. In [45], bases for the standard modules are formed from walk diagrams, similar to Dyck paths. In [25, 69], the ties between the Temperley-Lieb, Hecke and symmetric group algebras lead naturally to standard tableaux. Following early work [41] on tangles and knots, both the algebra and the standard modules were given diagrammatic forms for which the action is simply concatenation of diagrams and the parameter β appears when a closed loop is formed. This resulted in simplified proofs of many structural results [70] and led to the creation of the theory of cellular algebras [26,27]. This diagrammatic definition of standard modules was used in [11,63] to construct the indecomposable projective modules — these are also called the principal indecomposables — and their irreducible quotients. The present section summarises the properties upon which the construction of the remaining indecomposable modules and the computation of their restrictions and inductions will be based. (Many results are quoted in this section without proof; our presentation follows [11, 63] and we direct the reader to these works for the missing proofs.)

5.2.1 Standards, irreducibles and projectives

We introduce a set of integers Λ_n for each algebra TL_n or dTL_n. This set naturally parametrises the standard modules $\mathsf{S}_{n,k}$, with $k \in \Lambda_n$. For dTL_n, this set is simply $\{0,1,\dots,n\}$; for TL_n, it is the subset of $\{0,1,\dots,n\}$ whose elements have the same parity as that of n. When q is not a root of unity or when it is ± 1, both algebras $\mathsf{TL}_n(\beta)$ and $\mathsf{dTL}_n(\beta)$ are semisimple. Then, the standard modules $\mathsf{S}_{n,k}$, for $k \in \Lambda_n$, form a complete set of non-isomorphic irreducible modules of TL_n (dTL_n). Since the algebras are semisimple, these irreducible modules exhaust the list of indecomposable modules. We will thus assume throughout that q is a root of unity other than ± 1.

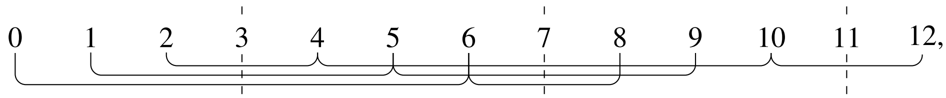
Let $\ell \geq 2$ be the smallest positive integer such that $q^{2\ell} = 1$.

Unless otherwise specified, all modules will be complex finite-dimensional left modules.

The set Λ_n is partitioned as follows. If an element k satisfies $k \equiv \ell - 1 \bmod \ell$, then k is said to be *critical* and it forms its own class in the partition. If the element k is not critical, then its class $[k]$ consists of the images (in Λ_n) of k generated by reflections with respect to the critical integers. Here, if k_c is a critical integer, then $2k_c - k$ is the reflection of k through k_c. The class of a non-critical k thus contains precisely one integer between each pair of consecutive critical ones. We shall often need to refer to neighbouring elements in a non-critical class $[k]$. They will be ordered as $k_L < \cdots < k^{--} < k^- < k < k^+ < k^{++} < \cdots < k_R$, so that $k_L \geq 0$ and $k_R \leq n$ are the smallest and largest elements in $[k] \subset \Lambda_n$. The notation k^j (k^{-j}) is also used to refer to the j-th element to the right of k (to its left) so that, for example, $k^{--} = k^{-2}$ and $k^{+++} = k^3$. We shall often refer to non-critical classes as *orbits*.

As an example of a partition, we take $\ell = 4$ and $n = 12$ so that the critical classes for dTL_{12} are $[3] = \{3\}$, $[7] = \{7\}$ and $[11] = \{11\}$, whereas the (non-critical) orbits are $\{0,6,8\}$, $\{1,5,9\}$ and $\{2,4,10,12\}$. Note that the partition for TL_{12}, with $\ell = 4$, consists of just two non-critical classes, namely $\{0,6,8\}$ and $\{2,4,10,12\}$.

These are easily obtained from the diagram



where the dashed lines indicate the critical k. In what follows, we shall also find it convenient to reflect about a critical k when the result does not belong to Λ_n, extending the notation k^-, k^+, and so on, in the obvious fashion. For instance, if $k = 9$ in the above example, then $k^+ = 13 \notin \Lambda_n$.

The partition of the set Λ_n into classes under reflection is intimately related to the existence of distinguished central elements. Both TL_n and dTL_n have a central element F_n whose exact form will not be needed (see [63] for its definition for TL_n and [11] for dTL_n). Its crucial property here is the following ([63, Prop. A.2] and [11, Prop. B.3]):

Proposition 5.2.1. *The element F_n acts as scalar multiplication by $\delta_k = q^{k+1} + q^{-k-1}$ on the standard module $\mathsf{S}_{n,k}$.*

It is easily shown that k and k' belong to the same orbit in Λ_n if and only if they have the same parity and $\delta_k = \delta_{k'}$. Thus, all irreducible and standard modules labelled by an element of $[k]$ have the same F_n-eigenvalue. Note that F_n has only one eigenvalue when acting on an arbitrary indecomposable module, although it need not then act as a multiple of the identity.

The dilute algebra dTL_n has other important central elements. In particular, there are central idempotents eid and oid for which $\mathsf{eid} + \mathsf{oid}$ is the unit of dTL_n. (The subalgebra $\mathsf{eid} \cdot \mathsf{dTL}_n$ ($\mathsf{oid} \cdot \mathsf{dTL}_n$) contains elements whose diagrammatic representations have an even (odd) number of vacancies on each of their sides; see [11].) Any indecomposable module M can be given a parity. This is defined to be even if $\mathsf{eid} \cdot \mathsf{M} = \mathsf{M}$ and odd otherwise. The standard module $\mathsf{S}_{n,k}$ over dTL_n has the parity of $n - k$.

Proposition 5.2.2. *The composition factors of an indecomposable TL_n- or dTL_n-module are irreducible modules I_{n,k_i} whose labels k_i belong to a single orbit $[k]$ and, for dTL_n, have the same parity.*

We will construct these irreducible modules shortly.

It follows that if M and N are indecomposable, then $\operatorname{Hom}(\mathsf{M}, \mathsf{N}) = 0$ whenever M and N have distinct parities (for dTL_n) or their F_n-eigenvalues are distinct. [2] Similarly, $\operatorname{Ext}(\mathsf{M}, \mathsf{N}) = 0$ under the same conditions, as otherwise there would exist indecomposable modules possessing more than one F_n-eigenvalue or parity (see Section 5.2.4 for a primer on these extension groups and Theorem 5.2.20 for additional conditions that imply $\operatorname{Ext}(\mathsf{M}, \mathsf{N}) = 0$). This is how these central elements will be used hereafter.

We now recall the theorems describing the basic structure of the standard modules. When not stated explicitly, these results hold for the standards of both TL_n and dTL_n, as long as the index k on $\mathsf{S}_{n,k}$ belongs to the set Λ_n of the algebra.

2. We remark that the elements of Hom-groups will be understood, unless otherwise specified, to be TL_n- or dTL_n-module homomorphisms, as appropriate.

We will use the symbol $\mathcal{A}_n$ to stand for either TL_n or dTL_n. Moreover, we will generally omit the label n on the algebra $\mathcal{A}_n$, its modules, and the set Λ_n, except when two different values are needed in the same statement.

The structure of the standard modules $\mathsf{S}_k \equiv \mathsf{S}_{n,k}$ is conveniently investigated by employing a symmetric bilinear form $\langle \cdot, \cdot \rangle$ naturally defined on each. This form is invariant with respect to the algebra action in the sense that there exists an involutive antiautomorphism † of $\mathcal{A} \equiv \mathcal{A}_n$ such that

$$\langle x, u \cdot y \rangle = \langle u^\dagger \cdot x, y \rangle, \qquad \text{for all } u \in \mathcal{A} \text{ and } x, y \in \mathsf{S}_k. \tag{5.2.1}$$

We refer to [63, Sec. 3], for $\mathcal{A} = \mathsf{TL}$, and [11, Sec. 4.1], for $\mathcal{A} = \mathsf{dTL}$, for the definitions of $\langle \cdot, \cdot \rangle$ and †, mentioning only that † corresponds diagrammatically to a reflection. The point is that the invariance of this bilinear form means that its radical R_k is a submodule of S_k. We denote the quotient by $\mathsf{I}_k = \mathsf{S}_k / \mathsf{R}_k$.

Proposition 5.2.3.

(i) *The standard modules S_k are indecomposable with $\mathsf{Hom}(\mathsf{S}_k, \mathsf{S}_k) \simeq \mathbb{C}$.*

(ii) *R_k is the maximal proper submodule of S_k, for all $k \in \Lambda$, unless $\mathcal{A} = \mathsf{TL}$, $\beta = 0$ and $k = 0$. In this latter case, the form $\langle \cdot, \cdot \rangle$ is identically zero, so $\mathsf{R}_0 = \mathsf{S}_0$ and $\mathsf{I}_0 = 0$.*

(iii) *If k is critical, then the form $\langle \cdot, \cdot \rangle$ is non-degenerate, so $\mathsf{R}_k = 0$ and $\mathsf{S}_k = \mathsf{I}_k$ is irreducible.*

(iv) *If k is non-critical, then $\mathsf{R}_k \simeq \mathsf{I}_{k^+}$, for all $k < k_R$, and $\mathsf{R}_{k_R} = 0$. The standard module S_{k_R} is therefore irreducible: $\mathsf{S}_{k_R} \simeq \mathsf{I}_{k_R}$.*

We remark that unless $\mathcal{A} = \mathsf{TL}$, $\beta = 0$ and $k = 0$, the module I_k is irreducible (and non-zero). In fact, all the irreducible modules, up to isomorphism, may be obtained in this fashion.

Proposition 5.2.4. *For all $n \geq 1$ and $\beta \in \mathbb{C}$, the irreducible modules I_k, with $k \in \Lambda$, form a complete set of non-isomorphic irreducible modules, except in the case of TL_n with $\beta = 0$ and n even. In this latter case, $\mathsf{I}_0 = 0$ and the set of irreducibles I_k with $k \in \Lambda \setminus \{0\}$ is complete.*

We deal with the annoying possibility that I_k may be zero by introducing [26] a set $\Lambda_{n,0} \equiv \Lambda_0$ which is defined to be $\Lambda \setminus \{0\}$, if $\mathcal{A} = \mathsf{TL}$, $\beta = 0$ and n is even, and Λ otherwise. The set $\{\mathsf{I}_k \ : \ k \in \Lambda_0\}$ is then a complete set of non-isomorphic irreducible modules.

We conclude our discussion of standard modules by noting that Theorem 5.2.3 implies that the following sequence is exact and non-split for k non-critical and $k < k_R$:

$$0 \longrightarrow \mathsf{I}_{k^+} \longrightarrow \mathsf{S}_k \longrightarrow \mathsf{I}_k \longrightarrow 0. \tag{5.2.2}$$

This sequence is also exact for $k = k_R$ if we assume that any module with a label $k \notin \Lambda$ is zero. For $k = k_R$, so $k^+ \notin \Lambda$, it reads $0 \to 0 \to \mathsf{S}_{k_R} \to \mathsf{I}_{k_R} \to 0$, which simply states that S_{k_R} is its own irreducible quotient. With this understanding, the composition factors of the non-critical standard module S_k are therefore I_k and I_{k^+}. This leads to a quick proof of the following result, needed in Section 5.2.2.

Proposition 5.2.5. *For each non-critical $k \in \Lambda$ with $k \neq k_L$,* $\mathsf{Hom}(\mathsf{S}_k, \mathsf{S}_{k^-}) \simeq \mathbb{C}$.

Proof. By (5.2.2), S_k and S_{k^-} only have one composition factor in common: I_k. It follows that if $f \in \mathsf{Hom}(\mathsf{S}_k, \mathsf{S}_{k^-})$, then $\mathsf{S}_k / \ker f \simeq \operatorname{im} f$ is either 0 or isomorphic to I_k. Any non-zero f is therefore unique up to rescaling. But, I_k is a quotient of S_k and a submodule of S_{k^-}, so there exists a non-zero f given by the composition $\mathsf{S}_k \twoheadrightarrow \mathsf{I}_k \simeq \mathsf{R}_{k^-} \hookrightarrow \mathsf{S}_{k^-}$. ■

We turn now to the structure of the projective modules. For each $k \in \Lambda_0$, let $\mathsf{P}_k \equiv \mathsf{P}_{n,k}$ be the projective cover of the irreducible module I_k. (The module P_k is thus indecomposable.) When $\mathsf{I}_k = 0$, thus $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = k = 0$, we could define $\mathsf{P}_k = 0$ as well, but this is more trouble than it is worth.

Proposition 5.2.6.

(i) *If $k \in \Lambda_0$ is critical, then the irreducible I_k is projective: $\mathsf{I}_k \simeq \mathsf{S}_k \simeq \mathsf{P}_k$.*

(ii) *If $k \in \Lambda_0$ is non-critical, then the following sequence is exact:*

$$0 \longrightarrow \mathsf{S}_{k^-} \longrightarrow \mathsf{P}_k \longrightarrow \mathsf{S}_k \longrightarrow 0. \tag{5.2.3}$$

If $k = k_L$, then $k^- \notin \Lambda$, so S_{k^-} is understood to be zero. The sequence, in this case, reads $0 \to 0 \to \mathsf{P}_{k_L} \to \mathsf{S}_{k_L} \to 0$, indicating that S_{k_L} is projective and, due to (5.2.2), has (at most) two composition factors, I_k and I_{k^+}. If $k = k_R \neq k_L$, then $\mathsf{S}_{k_R} \simeq \mathsf{I}_{k_R}$, by Theorem 5.2.3(iv), and P_{k_R} has exactly three composition factors, I_{k^-} and two copies of I_k. For all other non-critical $k \in \Lambda_0$, the projective P_k has precisely four composition factors, namely I_{k^-}, I_k (twice) and I_{k^+}.[3] We will discuss this further in Section 5.2.6.

Finally, we record the following result for future use.

Proposition 5.2.7. *For all $k \in \Lambda_0$,* $\mathsf{Hom}(\mathsf{S}_k, \mathsf{P}_k) \simeq \mathbb{C}$.

Proof. If k is critical or $k = k_L$, then $\mathsf{P}_k \simeq \mathsf{S}_k$, so $\mathsf{Hom}(\mathsf{S}_k, \mathsf{P}_k) \simeq \mathbb{C}$ by Theorem 5.2.3(i). So, we may assume that k is non-critical with $k > k_L$. Then, P_k has precisely two composition factors isomorphic to I_k.

Let $f \in \mathsf{Hom}(\mathsf{S}_k, \mathsf{P}_k)$ and consider the composition $\mathsf{S}_k \xrightarrow{f} \mathsf{P}_k \overset{\pi}{\twoheadrightarrow} \mathsf{I}_k$, where π is the canonical quotient map. If $\pi f \neq 0$, then it is a surjection, hence there exists a homomorphism g making

$$\begin{array}{ccccc} \mathsf{P}_k & & & & \\ {\scriptstyle g}\downarrow & \searrow^{\pi} & & & \\ \mathsf{S}_k & \xrightarrow[\pi f]{} & \mathsf{I}_k & \longrightarrow & 0 \end{array} \tag{5.2.4}$$

commute, by the projectivity of P_k. Now, $\pi f g = \pi$ requires that f be surjective, as $\ker \pi$ is the maximal proper submodule of P_k. However, this is impossible as P_k has more composition factors than S_k.

3. When $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = 0$, these two statements must be modified for P_2 because $\mathsf{I}_0 = 0$: $\mathsf{P}_{2,2}$ has only two composition factors and, for $n > 2$, $\mathsf{P}_{n,2}$ has three. The sequence (5.2.3) remains exact and non-split in both these cases.

We must therefore have $\pi f = 0$, hence that $\operatorname{im} f \subseteq \ker \pi$. Now, $\ker \pi$ has only one composition factor isomorphic to I_k and it is a submodule: $\mathsf{I}_k \simeq \mathsf{R}_{k^-} \subseteq \mathsf{S}_{k^-} \subseteq \ker \pi$. Thus, $\operatorname{im} f$ is either 0 or isomorphic to I_k, by (5.2.2). f is therefore unique up to rescaling and the (non-zero) composition $\mathsf{S}_k \twoheadrightarrow \mathsf{I}_k \hookrightarrow \mathsf{S}_{k^-} \hookrightarrow \mathsf{P}_k$ gives the result. ■

5.2.2 Their restriction and induction

This subsection describes how the families of modules introduced so far behave under restriction and induction. We first fix an inclusion of algebras $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ whose image is the subalgebra of diagrams whose $(n+1)$-th nodes are connected by an identity strand (see [63, Sec. 4] and [11, Sec. 3.4] for details). If M is an $\mathcal{A}_n$-module, its restriction to an $\mathcal{A}_{n-1}$-module will be denoted by $\mathsf{M}{\downarrow}$ and its induction to an $\mathcal{A}_{n+1}$-module by $\mathsf{M}{\uparrow}$.

Proposition 5.2.8.

(i) *The restriction of the standard module $\mathsf{S}_{n+1,k}$ satisfies the exact sequence*

$$0 \longrightarrow \mathsf{S}_{n,k-1} \longrightarrow \mathsf{S}_{n+1,k}{\downarrow} \longrightarrow \mathsf{S}_{n,k+1} \longrightarrow 0 \qquad (\mathcal{A}_n = \mathsf{TL}_n), \tag{5.2.5a}$$

$$0 \longrightarrow \mathsf{S}_{n,k-1} \oplus \mathsf{S}_{n,k} \longrightarrow \mathsf{S}_{n+1,k}{\downarrow} \longrightarrow \mathsf{S}_{n,k+1} \longrightarrow 0 \qquad (\mathcal{A}_n = \mathsf{dTL}_n). \tag{5.2.5b}$$

(ii) *For all non-critical k, these sequences split:*

$$\mathsf{S}_{n+1,k}{\downarrow} \simeq \begin{cases} \mathsf{S}_{n,k-1} \oplus \mathsf{S}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{TL}_n, \\ \mathsf{S}_{n,k-1} \oplus \mathsf{S}_{n,k} \oplus \mathsf{S}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{dTL}_n. \end{cases} \tag{5.2.6}$$

(iii) *For critical k, the result is almost always projective:*

$$\mathsf{S}_{n+1,k}{\downarrow} \simeq \begin{cases} \mathsf{P}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{TL}_n \text{ and } k \neq n+1, \\ \mathsf{P}_{n,k} \oplus \mathsf{P}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{dTL}_n \text{ and } k \neq n, n+1. \end{cases} \tag{5.2.7}$$

The exceptions are $\mathsf{S}_{n+1,n+1}{\downarrow} \simeq \mathsf{S}_{n,n}$ for TL_n and dTL_n, as well as $\mathsf{S}_{n+1,n}{\downarrow} \simeq \mathsf{S}_{n,n-1} \oplus \mathsf{S}_{n,n}$ for dTL_n.

(iv) *The induction and restriction of standard modules are related by*

$$\mathsf{S}_{n-1,k}{\uparrow} \simeq \mathsf{S}_{n+1,k}{\downarrow}, \qquad \text{for all } n \geq 2, \tag{5.2.8}$$

except when $\beta = 0$ for the module $\mathsf{S}_{2,0}{\uparrow}$ over TL_3. In that case,

$$\mathsf{S}_{2,0}{\uparrow} \simeq \mathsf{S}_{3,1} \oplus \mathsf{S}_{3,3}, \qquad \textit{but} \qquad \mathsf{S}_{4,0}{\downarrow} \simeq \mathsf{S}_{3,1}. \tag{5.2.9}$$

Again, if one of the indices of the direct summands does not belong to Λ_n, then this summand is understood to be 0. We remark that the submodule $\mathsf{S}_{n,k}$ of $\mathsf{S}_{n+1,k}{\downarrow}$, for $\mathcal{A}_n = \mathsf{dTL}_n$, is always a direct summand, because its parity differs from that of $\mathsf{S}_{n,k-1}$ and $\mathsf{S}_{n,k+1}$. This proposition appeared in [63] for TL_n and in [11] for dTL_n (see also [70] where a part of the proposition was first stated). Analogous results were proved in [11] for the restriction and induction of the irreducibles of dTL_n. The proofs can be extended straightforwardly to TL_n.

Proposition 5.2.9. *Suppose that $k \in \Lambda_{n+1,0}$.*

(i) *If $\mathcal{A}_n = \mathsf{dTL}_n$ and $\mathsf{R}_{n+1,k} \neq 0$, then*

$$\mathsf{I}_{n+1,k}\!\downarrow \simeq \begin{cases} \mathsf{I}_{n,k-1} \oplus \mathsf{I}_{n,k}, & \text{if } k+1 \text{ is critical,} \\ \mathsf{I}_{n,k-1} \oplus \mathsf{I}_{n,k} \oplus \mathsf{I}_{n,k+1}, & \text{otherwise.} \end{cases} \tag{5.2.10}$$

If, moreover, $\mathsf{R}_{n-1,k} \neq 0$, then $\mathsf{I}_{n-1,k}\!\uparrow \simeq \mathsf{I}_{n+1,k}\!\downarrow$.

(ii) *If $\mathcal{A}_n = \mathsf{TL}_n$, then the same statements hold if one deletes the $\mathsf{I}_{n,k}$ appearing on the right-hand side of (5.2.10).*

Of course, $\mathsf{R}_{n\pm1,k} = 0$ implies that $\mathsf{I}_{n\pm1,k} = \mathsf{S}_{n\pm1,k}$, so these results were already given in Theorem 5.2.8. This proposition also gives the behaviour of the critical principal indecomposables under restriction and induction, because then $\mathsf{P}_{n,k} \simeq \mathsf{S}_{n,k}$. The non-critical cases are rather more delicate and are covered by the following result.

Proposition 5.2.10.

(i) *For non-critical $k \in \Lambda_{n+1,0}$, the principal indecomposables satisfy*

$$\mathsf{P}_{n+1,k}\!\downarrow \simeq \begin{cases} \mathsf{P}_{n,k-1} \oplus \mathsf{P}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{TL}_n, \\ \mathsf{P}_{n,k-1} \oplus \mathsf{P}_{n,k} \oplus \mathsf{P}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{dTL}_n. \end{cases} \tag{5.2.11}$$

Here, any modules with indices not in $\Lambda_{n,0}$ are not set immediately to zero. We first make the following corrections to the right-hand side of the above formula:

- *If $k+1$ is critical, then $\mathsf{P}_{n,k+1}$ is replaced by $\mathsf{P}_{n,k+1} \oplus \mathsf{P}_{n,k^- -1} \simeq \mathsf{S}_{n,k+1} \oplus \mathsf{S}_{n,k^- -1}$.*
- *If $k-1$ is critical, then $\mathsf{P}_{n,k-1}$ is replaced by $2\,\mathsf{P}_{n,k-1} \simeq 2\,\mathsf{S}_{n,k-1}$.*
- *Any remaining $\mathsf{P}_{n,k'}$ with $k' > n$ non-critical is replaced by S_{n,k'^-}.*

Now, any modules with indices not in $\Lambda_{n,0}$ are set to zero.

(ii) *In all cases (with $n \geq 2$), $\mathsf{P}_{n-1,k}\!\uparrow \simeq \mathsf{P}_{n+1,k}\!\downarrow$.*

Proof. We work with dTL_n for definiteness, the argument for TL_n being identical after removing modules whose indices are not in Λ_n. We will moreover omit the routine checks that nothing untoward happens for $k = k_L, k_R$, as long as $k+1 \leq n$.

Since restriction is an exact functor, (5.2.3) gives the exactness of

$$0 \longrightarrow \mathsf{S}_{n+1,k^-}\!\downarrow \longrightarrow \mathsf{P}_{n+1,k}\!\downarrow \longrightarrow \mathsf{S}_{n+1,k}\!\downarrow \longrightarrow 0, \tag{5.2.12}$$

hence, by (5.2.2), that of

$$0 \longrightarrow \mathsf{S}_{n,k^--1} \oplus \mathsf{S}_{n,k^-} \oplus \mathsf{S}_{n,k^-+1} \longrightarrow \mathsf{P}_{n+1,k}\!\downarrow \longrightarrow \mathsf{S}_{n,k-1} \oplus \mathsf{S}_{n,k} \oplus \mathsf{S}_{n,k+1} \longrightarrow 0. \tag{5.2.13}$$

Considerations of parity and F_n-eigenvalues now force $\mathsf{P}_{n+1,k}\!\downarrow$ to decompose as $\mathsf{M}_{-1} \oplus \mathsf{M}_0 \oplus \mathsf{M}_{+1}$, where F_n has eigenvalue δ_{k+i} on M_i, $i \in \{-1,0,+1\}$. The exact sequence (5.2.13) therefore decomposes into three exact sequences:

$$0 \longrightarrow \mathsf{S}_{n,k^--i} \longrightarrow \mathsf{M}_i \longrightarrow \mathsf{S}_{n,k+i} \longrightarrow 0, \qquad i \in \{-1,0,+1\}. \tag{5.2.14}$$

Note that $(k+i)^- = k^- - i$ since neighbouring elements of an orbit are obtained from one another by reflection (see the beginning of Section 5.2.1). The goal is to prove that $\mathsf{M}_i \simeq \mathsf{P}_{n,k+i}$, for each $i \in \{-1,0,+1\}$, taking into account the replacements noted in the statement of the proposition. These replacements are easily dealt with: Let $i = \pm 1$ and assume that $k+i$ is critical with $\ell \neq 2$. Then, the exact sequence (5.2.14) shows that M_i is $\mathsf{S}_{n,k+1} \oplus \mathsf{S}_{n,k^--1}$, if $i=+1$, and a direct sum of two copies of the projective $\mathsf{S}_{n,k-1} = \mathsf{S}_{n,k^-+1}$, if $i=-1$. Furthermore, if $k+i>n$, then $\mathsf{S}_{n,k+i} \simeq 0$ and (5.2.13) tells us that $\mathsf{P}_{n+1,k}\!\downarrow$ contains S_{n-1,k^--i}. In the case $\ell = 2$, the four summands S_{n,k^--1}, $\mathsf{S}_{n,k^-+1} \simeq \mathsf{S}_{n,k-1}$ and $\mathsf{S}_{n,k+1}$ of (5.2.13) are all projective and they will thus be direct summands of $\mathsf{P}_{n+1,k}\!\downarrow$. This takes care of the replacements. From now on, we will therefore fix $i \in \{-1,0,+1\}$ and assume that $k+i \le n$ is not critical.

Consider the following diagram in which the rows are exact by (5.2.3) and (5.2.14):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{S}_{n,k^--i} & \xrightarrow{\alpha} & \mathsf{P}_{n,k+i} & \xrightarrow{\beta} & \mathsf{S}_{n,k+i} & \longrightarrow & 0 \\ & & \Big\downarrow g & & \Big\downarrow f & \swarrow h & \Big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \mathsf{S}_{n,k^--i} & \xrightarrow{\gamma} & \mathsf{M}_i & \xrightarrow{\delta} & \mathsf{S}_{n,k+i} & \longrightarrow & 0. \end{array} \tag{5.2.15}$$

A map f must exist making the right square commute, $\delta f = \beta$, as $\mathsf{P}_{n,k+i}$ is projective. Moreover, f then satisfies $\delta f \alpha = \beta\alpha = 0$. As $\operatorname{im} f\alpha \subseteq \ker\delta = \operatorname{im}\gamma$ and γ is injective, one may construct a unique homomorphism g that also makes the left square commute. Because of Theorem 5.2.3(i), the map g is either 0 or an isomorphism. If it is the latter, then the short five lemma implies that f is also an isomorphism, hence that $\mathsf{M}_i \simeq \mathsf{P}_{n,k+i}$ as desired. The proof of (i) will thus be complete if the case $g=0$ can be ruled out.

Suppose then that $g=0$. Because $0 = \gamma g = f\alpha$, we have $\ker f \supseteq \operatorname{im}\alpha = \ker\beta$, so there exists a homomorphism h such that $h\beta = f$. The commutativity of the right square then gives $\delta h \beta = \delta f = \beta$, so that δh acts as the identity on $\operatorname{im}\beta = \mathsf{S}_{n,k+i}$. Therefore, h splits the bottom row so that $\mathsf{M}_i \simeq \mathsf{S}_{n,k^--i} \oplus \mathsf{S}_{n,k+i}$. It follows that

$$\operatorname{Hom}(\mathsf{S}_{n,k+i}, \mathsf{M}_i) \simeq \operatorname{Hom}(\mathsf{S}_{n,k+i}, \mathsf{S}_{n,k^--i}) \oplus \operatorname{Hom}(\mathsf{S}_{n,k+i}, \mathsf{S}_{n,k+i}) = \mathbb{C} \oplus \mathbb{C}, \tag{5.2.16}$$

the first Hom-group following from Theorem 5.2.5 and the second following from Theorem 5.2.3(i). However,

$$\begin{aligned}\mathsf{Hom}(\mathsf{S}_{n,k+i},\mathsf{M}_i) &\overset{(1)}{\simeq} \mathsf{Hom}(\mathsf{S}_{n,k+i},\mathsf{M}_{-1}\oplus\mathsf{M}_0\oplus\mathsf{M}_1) \simeq \mathsf{Hom}(\mathsf{S}_{n,k+i},\mathsf{P}_{n+1,k}\!\downarrow) \\ &\overset{(2)}{\simeq} \mathsf{Hom}(\mathsf{S}_{n,k+i}\!\uparrow,\mathsf{P}_{n+1,k}) \simeq \mathsf{Hom}(\mathsf{S}_{n+1,k+i-1}\oplus\mathsf{S}_{n+1,k+i}\oplus\mathsf{S}_{n+1,k+i+1},\mathsf{P}_{n+1,k}) \\ &\overset{(3)}{\simeq} \mathsf{Hom}(\mathsf{S}_{n+1,k},\mathsf{P}_{n+1,k}), \end{aligned} \tag{5.2.17}$$

where, for (1), adding the two other modules does not change the Hom-group as their parities or F_{n+1}-eigenvalues are different, (2) is Frobenius reciprocity, and (3) again follows from parity and F_n-eigenvalue considerations. But, Theorem 5.2.7 gives $\mathsf{Hom}(\mathsf{S}_{n,k+i},\mathsf{M}_i) \simeq \mathsf{Hom}(\mathsf{S}_{n+1,k},\mathsf{P}_{n+1,k}) \simeq \mathbb{C}$, contradicting (5.2.16). This rules out $g = 0$.

For (ii), the isomorphism of $\mathsf{P}_{n-1,k}\!\uparrow$ and $\mathsf{P}_{n+1,k}\!\downarrow$ follows by comparing with the induction results of [11, 63]. ∎

Recall that induction functors are always right-exact. Here, we show that the functor $\uparrow$ is not left-exact in general. Choose $k \in \Lambda_n$ such that $k^{++} = n+1$ or $n+2$, as parity dictates, so that $\mathsf{S}_{n,k^+} \simeq \mathsf{I}_{n,k^+}$ and $0 \to \mathsf{S}_{n,k^+} \to \mathsf{S}_{n,k} \to \mathsf{I}_{n,k} \to 0$ is exact. Inducing results in the following exact sequence:

$$\mathsf{S}_{n,k^+}\!\uparrow \longrightarrow \mathsf{S}_{n,k}\!\uparrow \longrightarrow \mathsf{I}_{n,k}\!\uparrow \longrightarrow 0. \tag{5.2.18}$$

By Theorem 5.2.8, $\mathsf{S}_{n,k^+}\!\uparrow$ has a direct summand isomorphic to S_{n+1,k^++1} which has a submodule isomorphic to $\mathsf{I}_{n+1,k^{++}-1}$. Because $k^{++} = n+1$ or $n+2$, this submodule is $\mathsf{I}_{n+1,n}$ or $\mathsf{I}_{n+1,n+1}$, hence is non-zero. However, the same proposition shows that $\mathsf{S}_{n,k}\!\uparrow$ does not have any composition factor isomorphic to $\mathsf{I}_{n+1,n}$ or $\mathsf{I}_{n+1,n+1}$, so the left-most map of (5.2.18) cannot be an inclusion and we conclude that $\uparrow$ is not left-exact.

5.2.3 Their duals

Given a left module M over $\mathcal{A} \equiv \mathcal{A}_n$, the dual vector space $\mathsf{Hom}_{\mathbb{C}}(\mathsf{M},\mathbb{C})$ is naturally equipped with the structure of a right $\mathcal{A}$-module which we shall denote by M^*. However, the involutive antiautomorphism † may be used to twist this action to obtain a left module structure on $\mathsf{Hom}(\mathsf{M},\mathbb{C})$; the corresponding twisted dual will be denoted by $\mathsf{M}^\vee$. Explicitly, this twisted action is

$$(uf)(m) = f\big(u^\dagger m\big); \qquad u \in \mathcal{A}, \quad f \in \mathsf{M}^\vee, \quad m \in \mathsf{M}. \tag{5.2.19}$$

We will have occasion to consider (untwisted) dual modules M^* in Sections 5.4 and 5.5. Until then, it is understood that when we speak of a dual module, it is the twisted dual that we are referring to. As above, all modules are therefore assumed to be complex finite-dimensional left modules, unless otherwise specified.

Proposition 5.2.11. *Every irreducible $\mathcal{A}$-module is self-dual: $\mathsf{I}_k^\vee \simeq \mathsf{I}_k$, for all $k \in \Lambda_0$.*

Proof. By Theorems 5.2.3 and 5.2.4, every irreducible module may be constructed as the quotient of a standard module by the radical of its invariant bilinear form. Therefore, every irreducible module carries an invariant *non-degenerate* bilinear form $\langle \cdot, \cdot \rangle$.

For each $x \in \mathsf{I}$, define $f_x \in \mathsf{I}^\vee$ by $f_x(y) = \langle x, y \rangle$. Then, $x \mapsto f_x$ is a module homomorphism:

$$(uf_x)(y) = f_x\big(u^\dagger y\big) = \big\langle x, u^\dagger y \big\rangle = \big\langle ux, y \big\rangle = f_{ux}(y). \tag{5.2.20}$$

Moreover, it is injective as $f_x = 0$ implies that $0 = f_x(y) = \langle x, y \rangle$ for all $y \in \mathsf{I}$, hence $x = 0$ by non-degeneracy. Since $\dim \mathsf{I}^\vee = \dim \mathsf{I}$, this map is an isomorphism. ∎

Proposition 5.2.12. *Duality is reflexive:* $\mathsf{M} \simeq (\mathsf{M}^\vee)^\vee$.

Proof. The proof is quite similar. Define the map $x \in \mathsf{M} \mapsto \phi_x \in (\mathsf{M}^\vee)^\vee$ by $\phi_x(f) = f(x)$, for all $f \in \mathsf{M}^\vee$. This is a module homomorphism,

$$(u\phi_x)(f) = \phi_x\big(u^\dagger f\big) = \big(u^\dagger f\big)(x) = f\big(u^{\dagger\dagger} x\big) = f(ux) = \phi_{ux}(f), \tag{5.2.21}$$

and it is injective,

$$\phi_x = 0 \quad \Rightarrow \quad \phi_x(f) = 0 \quad \text{for all } f \in \mathsf{M}^\vee \quad \Rightarrow \quad f(x) = 0 \quad \text{for all } f \in \mathsf{M}^\vee \quad \Rightarrow \quad x = 0, \tag{5.2.22}$$

hence it is an isomorphism for dimensional reasons. ∎

Proposition 5.2.13. *Duality is an exact contravariant functor: The sequence*

$$0 \longrightarrow \mathsf{L} \xrightarrow{\iota} \mathsf{M} \xrightarrow{\pi} \mathsf{N} \longrightarrow 0 \tag{5.2.23}$$

is exact if and only if the sequence

$$0 \longrightarrow \mathsf{N}^\vee \xrightarrow{\pi^\vee} \mathsf{M}^\vee \xrightarrow{\iota^\vee} \mathsf{L}^\vee \longrightarrow 0 \tag{5.2.24}$$

is exact, where $\iota^\vee$ *and* $\pi^\vee$ *are defined by*

$$\iota^\vee(g)(\ell) = g(\iota(\ell)), \quad \textit{for all } \ell \in \mathsf{L} \textit{ and } g \in \mathsf{M}^\vee; \qquad \pi^\vee(h)(m) = h(\pi(m)), \quad \textit{for all } m \in \mathsf{M} \textit{ and } h \in \mathsf{N}^\vee. \tag{5.2.25}$$

Proof. Assume that (5.2.23) is exact. We first check that $\pi^\vee$ is a module homomorphism. This follows because

$$\big(u\pi^\vee(h)\big)(m) = \pi^\vee(h)\big(u^\dagger m\big) = h\big(\pi\big(u^\dagger m\big)\big) = h\big(u^\dagger \pi(m)\big) = (uh)(\pi(m)) = \pi^\vee(uh)(m), \tag{5.2.26}$$

for all $m \in \mathsf{M}$ and $h \in \mathsf{N}^\vee$. The check for $\iota^\vee$ is similar.

Second, we prove that $\pi^\vee$ is injective. Take $h \in \ker \pi^\vee$, so that $0 = \pi^\vee(h)(m) = h(\pi(m))$ for all $m \in \mathsf{M}$. This implies that $h = 0$ because π is a surjection.

Third, we show that $\ker \iota^\vee = \operatorname{im} \pi^\vee$. Exactness gives $\iota^\vee(\pi^\vee(h))(\ell) = \pi^\vee(h)(\iota(\ell)) = h(\pi(\iota(\ell))) = 0$, for all $\ell \in \mathsf{L}$ and $h \in \mathsf{N}^\vee$. Thus, $\operatorname{im} \pi^\vee \subseteq \ker \iota^\vee$. To prove the reverse inclusion, we take $g \in \ker \iota^\vee$, so that $0 = \iota^\vee(g)(\ell) = g(\iota(\ell))$ for all $\ell \in \mathsf{L}$. Thus, g annihilates $\operatorname{im} \iota = \ker \pi$. Now, define a functional $h \in \mathsf{N}^\vee$ by $h(n) = g(m)$, where $\pi(m) = n$. This is well defined, because $\pi(m) = \pi(m')$ implies that $m - m' \in \ker \pi$, hence that $g(m - m') = 0$, thus $g(m) = g(m')$. But, $\pi^\vee(h)(m) = h(\pi(m)) = g(m)$ for all $m \in \mathsf{M}$, so we see that $g \in \operatorname{im} \pi^\vee$. This proves that $\ker \iota^\vee \subseteq \operatorname{im} \pi^\vee$.

Last, $\iota^\vee$ is surjective by the rank-nullity theorem of linear algebra:

$$\begin{aligned} \dim \operatorname{im} \iota^\vee &= \dim \mathsf{M}^\vee - \dim \ker \iota^\vee = \dim \mathsf{M} - \dim \operatorname{im} \pi^\vee = \dim \mathsf{M} - \dim \mathsf{N}^\vee = \dim \mathsf{M} - \dim \mathsf{N} = \dim \mathsf{L} \\ &= \dim \mathsf{L}^\vee. \end{aligned} \tag{5.2.27}$$

This completes the proof, Theorem 5.2.12 providing the converse. ■

Indeed, taking (twisted) duals defines an autoequivalence of the category of finite-dimensional $\mathcal{A}$-modules. Such equivalences are always exact.

We will refer to the duals of the standard modules as *costandard modules*, denoting them by $\mathsf{C}_k = \mathsf{S}_k^\vee$. From Theorems 5.2.11 and 5.2.13, we learn that the dual of the exact sequence (5.2.2) is

$$0 \longrightarrow \mathsf{I}_k \longrightarrow \mathsf{C}_k \longrightarrow \mathsf{I}_{k^+} \longrightarrow 0 \tag{5.2.28}$$

and that this sequence is also exact and non-split, by Theorem 5.2.12. Note that this failure to split uses the basic fact that $(\mathsf{M} \oplus \mathsf{N})^\vee \simeq \mathsf{M}^\vee \oplus \mathsf{N}^\vee$ as modules. It follows that a standard module S_k is self-dual if and only if it is irreducible. The corresponding result for the principal indecomposables is as follows:

Proposition 5.2.14. *The critical principal indecomposables are self-dual, as are the non-critical P_k with $k > k_L$. The non-critical P_{k_L} are self-dual if and only if they are irreducible.*

Proof. The case where k is critical or where k is non-critical with $k = k_L$, hence $\mathsf{P}_k \simeq \mathsf{S}_k$, has already been dealt with. We therefore assume that k is critical and prove that P_{k+i} is self-dual, for all $n \geq k+i$, where $i = 1, \dots, \ell - 1$. The proof is by induction and we shall detail it for $\mathcal{A} = \mathsf{dTL}$, omitting the simple modifications required for TL.

The base cases of this induction are $i = 0$ and 1. The former is the critical case already dealt with, so we turn to $i = 1$. The key tool for this case, and indeed for the extension to higher i, is the realisation that duality commutes with restriction: $(\mathsf{M}^\vee)\!\downarrow$ and $(\mathsf{M}\!\downarrow)^\vee$ are the same vector space with the same algebra action. For $i = 1$, we use the criticality of P_k and Theorem 5.2.8(iii) twice to arrive at

$$\mathsf{P}_{n,k+1}^\vee \oplus \mathsf{P}_{n,k} \simeq \mathsf{P}_{n,k+1}^\vee \oplus \mathsf{P}_{n,k}^\vee \simeq (\mathsf{S}_{n+1,k}\!\downarrow)^\vee = (\mathsf{S}_{n+1,k}^\vee)\!\downarrow = \mathsf{S}_{n+1,k}\!\downarrow = \mathsf{P}_{n,k+1} \oplus \mathsf{P}_{n,k}, \tag{5.2.29}$$

hence the self-duality of P_{k+1}, for all $n \geq k+1$ (noting that $k \neq n, n+1$). Assuming that self-duality holds for P_{k+i} and P_{k+i-1}, where $i \leq \ell - 2$, a similar calculation using Theorem 5.2.10 now proves that P_{k+i+1} is self-dual. ∎

For k non-critical with $k > k_L$, the dual of the non-split exact sequence (5.2.3) is therefore

$$0 \longrightarrow \mathsf{C}_k \longrightarrow \mathsf{P}_k \longrightarrow \mathsf{C}_{k^-} \longrightarrow 0, \tag{5.2.30}$$

which is likewise exact and non-split. For $k = k_L$, $\mathsf{C}_{k^-} = 0$ but C_{k_L} is not projective, in general, because P_{k_L} need not be self-dual. Rather, $\mathsf{C}_{k_L} \simeq \mathsf{P}_{k_L}^{\vee}$ is *injective*.

Corollary 5.2.15.

(i) *When $k \in \Lambda$ is critical, the standard module S_k is projective and injective.*

(ii) *When $k \in \Lambda_0$ is non-critical and $k > k_L$, the projective module P_k is injective.*

(iii) *When $k \in \Lambda_0$ is non-critical and $k = k_L$, the costandard module C_k is injective.*

Proof. These follow from the exactness of duality, the modules being self-dual for (i) and (ii), and the reflexivity of duality for (iii) (Theorems 5.2.12 to 5.2.14). For P being projective is equivalent to $\mathsf{M} \to \mathsf{P} \to 0$ splitting for all M, which is equivalent to $0 \to \mathsf{P}^{\vee} \to \mathsf{N}$ splitting for all N, which means that $\mathsf{P}^{\vee}$ is injective. ∎

We remark that for $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = 0$, the costandard module C_0 coincides with I_2, by (5.2.2), which is not injective.

For each $k \in \Lambda_0$, we let J_k denote the *injective hull* of I_k. The critical standard modules are irreducible, projective and injective: $\mathsf{I}_k = \mathsf{S}_k = \mathsf{P}_k = \mathsf{J}_k$. Each critical S_k is therefore the unique indecomposable object in its block, this block being semisimple. These statements continue to hold for non-critical k if the class $[k] \subset \Lambda$ contains a single element: $[k] = \{k\}$. Otherwise, the non-critical blocks are non-semisimple and we identify the injective hulls and projective covers of the modules introduced so far in the following proposition.

Proposition 5.2.16. *If $k \in \Lambda_0$ is critical, then $\mathsf{I}_k = \mathsf{S}_k = \mathsf{P}_k = \mathsf{J}_k$. If $k \in \Lambda_0$ is non-critical, then:*

(i) *The module J_k is P_k, if $k > k_L$, and C_{k_L}, if $k = k_L$.*

(ii) *The projective cover of I_k, S_k and P_k is P_k. That of C_k is P_k, if $k = k_R$, and otherwise it is P_{k^+}. That of J_k is P_{k^+}, if $k = k_L$, and otherwise it is P_k.*

(iii) *The injective hull of I_k, C_k and J_k is J_k. That of S_k is J_k, if $k = k_R$, and otherwise it is J_{k^+}. That of P_k is J_{k^+}, if $k = k_L$, and otherwise it is J_k.*

Proof. All injective and projective modules appearing in Theorem 5.2.15 are indecomposable. Therefore, these injectives (projectives) will automatically be the injective hull (projective cover) of their submodules (quotients). Moreover, the proof of the previous corollary shows that the injective hull of a module M is the projective cover of its dual $\mathsf{M}^{\vee}$; the statement (iii) is thus a reformulation of (ii). ∎

We remark that the definition of the projective cover (injective hull) of M should include the surjective (injective) homomorphism: $\mathsf{P} \twoheadrightarrow \mathsf{M}$ ($\mathsf{M} \hookrightarrow \mathsf{J}$). However, we shall see in Theorem 5.2.17 that these homomorphisms are unique, up to rescaling, in each case addressed by Theorem 5.2.16, hence we shall usually leave them implicit.

5.2.4 Interlude: Ext-groups

In order to classify indecomposable modules, we need to know when we can "stitch" existing modules together to build bigger ones. This knowledge is encoded in the extension group $\text{Ext}^1(\mathsf{M}, \mathsf{N})$, where M and N are the modules being stitched. We begin with a quick definition of these groups, usually referred to as Ext-groups for short, before showing how they may be computed and what their relation is to indecomposable modules. In this section, we assume that $\mathcal{A}$ is an arbitrary finite-dimensional associative algebra over some field $\mathbb{K}$.

We have noted in the previous sections that the operations of restricting and taking the dual of a module are exact. This means that, if $0 \to \mathsf{K} \to \mathsf{L} \to \mathsf{M} \to 0$ is a short exact sequence, applying these operations (functors) to each of the constituent modules gives another short exact sequence (see Theorem 5.2.13 for an example). Not all functors are exact however. At the end of Section 5.2.2, we observed that induction is right-exact, but not left-exact, by giving an explicit example. The functors under study in the present section are the Hom-functors $\mathsf{Hom}_{\mathcal{A}}(\mathsf{N}, -)$ and $\mathsf{Hom}_{\mathcal{A}}(-, \mathsf{N})$. The first is covariant and the second contravariant. Both are left-exact, but neither need be right-exact in general. In a sense, extension groups measure the failure of these functors to be right-exact. (A complete discussion of Ext-groups, covering what is needed here, can be found, for example, in chapter III of [34] and chapter IX of [1].)

Let N be an $\mathcal{A}$-module. The n-th extension functors $\text{Ext}^n_{\mathcal{A}}(\mathsf{N}, -)$ and $\text{Ext}^n_{\mathcal{A}}(-, \mathsf{N},)$ are the n-th right derived functors of $\mathsf{Hom}_{\mathcal{A}}(\mathsf{N}, -)$ and $\mathsf{Hom}_{\mathcal{A}}(-, \mathsf{N})$, respectively. This means that for any short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \mathsf{K} \longrightarrow \mathsf{L} \longrightarrow \mathsf{M} \longrightarrow 0, \tag{5.2.31}$$

there exist two long exact sequences of $\mathbb{K}$-vector spaces called the *Hom-Ext long exact sequences*:[4]

$$\begin{aligned} 0 \longrightarrow \mathsf{Hom}_{\mathcal{A}}(\mathsf{N}, \mathsf{K}) \to \mathsf{Hom}_{\mathcal{A}}(\mathsf{N}, \mathsf{L}) \to \mathsf{Hom}_{\mathcal{A}}(\mathsf{N}, \mathsf{M}) \\ \hookrightarrow \text{Ext}^1_{\mathcal{A}}(\mathsf{N}, \mathsf{K}) \longrightarrow \text{Ext}^1_{\mathcal{A}}(\mathsf{N}, \mathsf{L}) \longrightarrow \text{Ext}^1_{\mathcal{A}}(\mathsf{N}, \mathsf{M}) \longrightarrow \cdots \\ \hookrightarrow \cdots \to \text{Ext}^n_{\mathcal{A}}(\mathsf{N}, \mathsf{K}) \longrightarrow \text{Ext}^n_{\mathcal{A}}(\mathsf{N}, \mathsf{L}) \longrightarrow \text{Ext}^n_{\mathcal{A}}(\mathsf{N}, \mathsf{M}) \longrightarrow \cdots \end{aligned} \tag{5.2.32a}$$

4. As we shall see, these Hom- and Ext-groups are actually $\mathbb{K}$-vector spaces because we work over a field. If we were instead working over a general commutative ring, then the Hom- and Ext-groups would only be abelian groups, whence their names.

and

$$\begin{aligned} 0 \longrightarrow \mathsf{Hom}_{\mathcal{A}}(\mathsf{M},\mathsf{N}) \to \mathsf{Hom}_{\mathcal{A}}(\mathsf{L},\mathsf{N}) \to \mathsf{Hom}_{\mathcal{A}}(\mathsf{K},\mathsf{N}) \\ \to \operatorname{Ext}^1_{\mathcal{A}}(\mathsf{M},\mathsf{N}) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(\mathsf{L},\mathsf{N}) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(\mathsf{K},\mathsf{N}) \longrightarrow \cdots \\ \to \cdots \to \operatorname{Ext}^n_{\mathcal{A}}(\mathsf{M},\mathsf{N}) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(\mathsf{L},\mathsf{N}) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(\mathsf{K},\mathsf{N}) \longrightarrow \cdots . \end{aligned} \tag{5.2.32b}$$

In what follows, we shall only be concerned with the first extension groups $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$; we therefore simply write $\operatorname{Ext}_{\mathcal{A}}$ for $\operatorname{Ext}^1_{\mathcal{A}}$. We shall also omit the subscript $\mathcal{A}$, as we do for Hom-groups, when it is clear from the context.

The following results are crucial tools for computing extension groups: $\operatorname{Ext}(\mathsf{P},-) = 0$ if and only if P is projective, while $\operatorname{Ext}(-,\mathsf{J}) = 0$ if and only if J is injective. To see how these are used, let K and M be $\mathcal{A}$-modules and let

$$0 \longrightarrow \mathsf{K} \longrightarrow \mathsf{J} \longrightarrow \mathsf{L} \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \mathsf{R} \longrightarrow \mathsf{P} \longrightarrow \mathsf{M} \longrightarrow 0 \tag{5.2.33}$$

be an injective presentation of K and a projective presentation of M, respectively.[5] Once specialised to these cases, the exact sequences (5.2.32a) and (5.2.32b) truncate, becoming

$$0 \longrightarrow \mathsf{Hom}(\mathsf{N},\mathsf{K}) \longrightarrow \mathsf{Hom}(\mathsf{N},\mathsf{J}) \longrightarrow \mathsf{Hom}(\mathsf{N},\mathsf{L}) \longrightarrow \operatorname{Ext}(\mathsf{N},\mathsf{K}) \longrightarrow 0, \tag{5.2.34a}$$

$$0 \longrightarrow \mathsf{Hom}(\mathsf{M},\mathsf{N}) \longrightarrow \mathsf{Hom}(\mathsf{P},\mathsf{N}) \longrightarrow \mathsf{Hom}(\mathsf{R},\mathsf{N}) \longrightarrow \operatorname{Ext}(\mathsf{M},\mathsf{N}) \longrightarrow 0. \tag{5.2.34b}$$

If the relevant Hom groups are known, then $\operatorname{Ext}(\mathsf{N},\mathsf{K})$ and $\operatorname{Ext}(\mathsf{M},\mathsf{N})$ can be identified.

The reason to introduce these extension groups here is the following: The (first) extension group $\operatorname{Ext}(\mathsf{M},\mathsf{N})$ describes, roughly speaking, the inequivalent ways to "stitch" the modules M and N together to obtain a new module E with N isomorphic to a submodule of E and M isomorphic to the quotient E/N. In other words, $\operatorname{Ext}(\mathsf{M},\mathsf{N})$ characterises the (inequivalent) short exact sequences $0 \to \mathsf{N} \to \mathsf{E} \to \mathsf{M} \to 0$. We then say that E is an extension of M by N. To state what it means for two extensions to be equivalent, let E and E' be two extensions of M by N:

$$e\colon 0 \longrightarrow \mathsf{N} \xrightarrow{f} \mathsf{E} \xrightarrow{g} \mathsf{M} \longrightarrow 0 \qquad \text{and} \qquad e'\colon 0 \longrightarrow \mathsf{N} \xrightarrow{f'} \mathsf{E}' \xrightarrow{g'} \mathsf{M} \longrightarrow 0. \tag{5.2.35}$$

Then, e and e' are said to be equivalent if there exists $h\colon \mathsf{E} \to \mathsf{E}'$ such that $f' = hf$ and $g = g'h$. It can be shown that this is an equivalence relation and that the set of inequivalent extensions of M by N is in one-to-one correspondence with $\operatorname{Ext}(\mathsf{M},\mathsf{N})$. In this correspondence, the origin of the vector space $\operatorname{Ext}(\mathsf{M},\mathsf{N})$ corresponds to the split extension $\mathsf{E} = \mathsf{N} \oplus \mathsf{M}$. Moreover, multiplying e by $\alpha \in \mathbb{K}^\times$ yields the extensions

$$\alpha e\colon 0 \longrightarrow \mathsf{N} \xrightarrow{\alpha f} \mathsf{E} \xrightarrow{g} \mathsf{M} \longrightarrow 0 \qquad \text{or} \qquad 0 \longrightarrow \mathsf{N} \xrightarrow{f} \mathsf{E} \xrightarrow{\alpha g} \mathsf{M} \longrightarrow 0, \tag{5.2.36}$$

5. A injective (projective) presentation is a short exact sequence of the form (5.2.33), where the middle term is injective (projective).

which are easily seen to be equivalent. Finally, the sum $e+e'$ is defined through an operation sometimes known as the Baer sum, completing the $\mathbb{K}$-vector space structure on the set of inequivalent extensions. As we will have no need of this sum, we omit its definition and refer the reader to any standard text on homological algebra, for example [68, Sec. 3.4].

We remark that if the two extensions e and e' of (5.2.35) are equivalent, then the middle modules E and E' are isomorphic, by the short five lemma. However, the converse is not true: e and αe are not equivalent, for $\alpha \neq 1$, despite both their middle modules being E. In particular, if $\operatorname{Ext}(\mathsf{M},\mathsf{N}) \simeq \mathbb{K}$, then there are precisely two extensions, up to isomorphism, one split and one non-split. We shall use this conclusion many times in what follows.

5.2.5 Their Hom- and Ext-groups

We now have enough information to compute the homomorphism groups between the modules I_k, S_k, C_k, P_k and J_k. This is quite straightforward, but there are various cases that have to be considered for each pair of module types. For example, $\mathsf{Hom}(\mathsf{I}_k,\mathsf{S}_{k'}) = \delta_{k',k^-}\mathbb{C}$ because the image can only be the submodule $\mathsf{R}_{k'} \simeq \mathsf{I}_{k'^+}$ (or zero), unless $k' = k'_R$, in which case $\mathsf{S}_{k'} = \mathsf{I}_{k'}$ and $\mathsf{Hom}(\mathsf{I}_k,\mathsf{S}_{k'_R}) = \delta_{k,k'_R}\mathbb{C}$. We can avoid much of this case analysis by agreeing to use the following conventions for the rest of this subsection:

- We will not consider J_k explicitly because, for every k, J_k is either P_k or C_k.
- The index k in S_k, C_k and P_k will be assumed to be non-critical because $\mathsf{P}_k \simeq \mathsf{C}_k \simeq \mathsf{S}_k \simeq \mathsf{I}_k$ for critical k.
- The index k in S_k and C_k will exclude $k = k_R$ because $\mathsf{C}_{k_R} \simeq \mathsf{S}_{k_R} \simeq \mathsf{I}_{k_R}$.
- The index k in P_k will exclude $k = k_L$ because $\mathsf{P}_{k_L} \simeq \mathsf{S}_{k_L}$.
- When $\mathcal{A} = \mathsf{TL}$, $\beta = 0$ and n is even, the index k in S_k and C_k will exclude $k = 0$ because $\mathsf{C}_0 = \mathsf{S}_0 = \mathsf{I}_2$.

In the case $\mathcal{A} = \mathsf{TL}$, $\beta = 0$ and n even, we will also obviously exclude $\mathsf{I}_0 = 0$. The computation of Hom-groups is then straightforward and uses the exact sequences (5.2.2), (5.2.3) and their duals.

Proposition 5.2.17. *With these conventions, the groups* $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ *between the irreducible, standard, co-standard and projective modules are summarised in the following table:*

$\mathsf{Hom}(\mathsf{M},\mathsf{N})$		N: $\mathsf{I}_{k'}$	$\mathsf{S}_{k'}$	$\mathsf{C}_{k'}$	$\mathsf{P}_{k'}$
M	I_k	$\delta_{k',k}\mathbb{C}$	$\delta_{k',k^-}\mathbb{C}$	$\delta_{k',k}\mathbb{C}$	$\delta_{k',k}\mathbb{C}$
	S_k	$\delta_{k',k}\mathbb{C}$	$(\delta_{k',k}+\delta_{k',k^-})\mathbb{C}$	$\delta_{k',k}\mathbb{C}$	$(\delta_{k',k}+\delta_{k',k^+})\mathbb{C}$
	C_k	$\delta_{k',k^+}\mathbb{C}$	$\delta_{k',k}\mathbb{C}$	$(\delta_{k',k}+\delta_{k',k^+})\mathbb{C}$	$(\delta_{k',k}+\delta_{k',k^+})\mathbb{C}$
	P_k	$\delta_{k',k}\mathbb{C}$	$(\delta_{k',k}+\delta_{k',k^-})\mathbb{C}$	$(\delta_{k',k}+\delta_{k',k^-})\mathbb{C}$	$(2\,\delta_{k',k}+\delta_{k',k^-}+\delta_{k',k^+})\mathbb{C}$.

We remark that the entries of this table are related by the vector space isomorphisms $\mathsf{Hom}(\mathsf{M}^\vee,\mathsf{N}^\vee) \simeq \mathsf{Hom}(\mathsf{N},\mathsf{M})$, implemented by sending $\phi\colon \mathsf{N} \to \mathsf{M}$ to $\phi^\vee\colon \mathsf{M}^\vee \to \mathsf{N}^\vee$, where $\phi^\vee(f)(n) = f(\phi(n))$ (see Theorem 5.2.13).

The extension groups involving these modules are now straightforward to compute. We assume the same conventions on the index k as for the Hom-groups and note that this means that each P_k is projective and injective, hence any Ext-group involving a P_k is zero.

Proposition 5.2.18. *With these conventions, the groups* $\operatorname{Ext}(\mathsf{M},\mathsf{N})$ *between the irreducible, standard and costandard modules may be summarised, with three exceptions, in the following table:*

$\operatorname{Ext}(\mathsf{M},\mathsf{N})$		N: $\mathsf{I}_{k'}$	N: $\mathsf{S}_{k'}$	N: $\mathsf{C}_{k'}$
M	I_k	$(\delta_{k',k^-}+\delta_{k',k^+})\mathbb{C}$	$(\delta_{k',k^-}\delta_{k,k_R}+\delta_{k',k^{--}})\mathbb{C}$	$\delta_{k',k^+}\mathbb{C}$
	S_k	$\delta_{k',k^-}\mathbb{C}$	$(\delta_{k',k^-}+\delta_{k',k^{--}})\mathbb{C}$	0
	C_k	$(\delta_{k',k^+}\delta_{k',k'_R}+\delta_{k',k^{++}})\mathbb{C}$	0	$(\delta_{k',k^+}+\delta_{k',k^{++}})\mathbb{C}$.

The exceptions occur for $\mathcal{A}=\mathsf{TL}_2$ *and* $\beta=0$*, for which* $\operatorname{Ext}(\mathsf{I}_{2,2},\mathsf{I}_{2,2})\simeq\mathbb{C}$ *instead of* 0*, and for* $\mathcal{A}=\mathsf{TL}$ *and* $\beta=0$*, for which* $\operatorname{Ext}(\mathsf{I}_2,\mathsf{C}_2)\simeq\operatorname{Ext}(\mathsf{S}_2,\mathsf{I}_2)\simeq\mathbb{C}$ *instead of* 0*.*

Proof. We compute these extension groups using the Hom-Ext long exact sequences (5.2.32). To see how this works, consider $\operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k'})$, for k non-critical, $k\neq k_R$ and $k'\neq k'_R$. We start from the short exact sequence (5.2.3), which is a projective presentation of S_k, and examine the contravariant Hom-Ext long exact sequence (5.2.34b):

$$0\longrightarrow \mathsf{Hom}(\mathsf{S}_k,\mathsf{S}_{k'})\longrightarrow \mathsf{Hom}(\mathsf{P}_k,\mathsf{S}_{k'})\longrightarrow \mathsf{Hom}(\mathsf{S}_{k^-},\mathsf{S}_{k'})\longrightarrow \operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k'})\longrightarrow 0. \tag{5.2.37}$$

Here, we have noted that the rightmost term is $\operatorname{Ext}(\mathsf{P}_k,\mathsf{S}_{k'})=0$ because of projectivity. Substituting in the homomorphism groups from Theorem 5.2.17 gives

$$0\longrightarrow (\delta_{k',k}+\delta_{k',k^-})\mathbb{C}\longrightarrow (\delta_{k',k}+\delta_{k',k^-})\mathbb{C}\longrightarrow (\delta_{k',k^-}+\delta_{k',k^{--}})\mathbb{C}\longrightarrow \operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k'})\longrightarrow 0. \tag{5.2.38}$$

The Ext-group is therefore zero unless $k'=k^-$ or k^{--}. In these cases, the exact sequence becomes

$$0\longrightarrow\mathbb{C}\longrightarrow\mathbb{C}\longrightarrow\mathbb{C}\longrightarrow\operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k^-})\longrightarrow 0 \quad\text{and}\quad 0\longrightarrow 0\longrightarrow 0\longrightarrow\mathbb{C}\longrightarrow\operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k^{--}})\longrightarrow 0, \tag{5.2.39}$$

respectively, and the Euler-Poincaré principle gives $\operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k^-})\simeq\operatorname{Ext}(\mathsf{S}_k,\mathsf{S}_{k^{--}})\simeq\mathbb{C}$.

The computations for $\operatorname{Ext}(\mathsf{I}_k,\mathsf{S}_{k'})$, $\operatorname{Ext}(\mathsf{I}_k,\mathsf{C}_{k'})$, $\operatorname{Ext}(\mathsf{S}_k,\mathsf{I}_{k'})$, $\operatorname{Ext}(\mathsf{S}_k,\mathsf{C}_{k'})$, $\operatorname{Ext}(\mathsf{C}_k,\mathsf{I}_{k'})$, $\operatorname{Ext}(\mathsf{C}_k,\mathsf{S}_{k'})$ and $\operatorname{Ext}(\mathsf{C}_k,\mathsf{C}_{k'})$ are almost identical, utilising (5.2.2) or its dual (5.2.28) (which is an injective presentation of C_k). We only remark that, for $\operatorname{Ext}(\mathsf{I}_k,\mathsf{S}_{k'})$, one has to consider the case $k'=k_R^-$ separately because then $\mathsf{Hom}(\mathsf{I}_k,\mathsf{S}_{k'^+})=\mathsf{Hom}(\mathsf{I}_k,\mathsf{I}_{k'^+})$. There is a similar case to consider for $\operatorname{Ext}(\mathsf{C}_k,\mathsf{I}_{k'})$.

The computation for $\operatorname{Ext}(\mathsf{I}_k,\mathsf{I}_{k'})$ is slightly different in that we instead start from the short exact sequence (5.2.2), which is not a projective or injective presentation, to derive the Hom-Ext long exact sequence

$$\begin{aligned}0\to \mathsf{Hom}(\mathsf{I}_k,\mathsf{I}_{k'})\to \mathsf{Hom}(\mathsf{S}_k,\mathsf{I}_{k'})\to \mathsf{Hom}(\mathsf{I}_{k^+},\mathsf{I}_{k'})\\ \to \operatorname{Ext}(\mathsf{I}_k,\mathsf{I}_{k'})\longrightarrow \operatorname{Ext}(\mathsf{S}_k,\mathsf{I}_{k'})\longrightarrow \operatorname{Ext}(\mathsf{I}_{k^+},\mathsf{I}_{k'})\longrightarrow\cdots\,.\end{aligned} \tag{5.2.40}$$

Substituting for $\mathsf{Hom}(\mathsf{I}_{k^+},\mathsf{I}_{k'})$ and $\mathrm{Ext}(\mathsf{S}_k,\mathsf{I}_{k'})$, we learn that $\mathrm{Ext}(\mathsf{I}_k,\mathsf{I}_{k'})=0$, unless $k'=k^{\pm}$, and that the result is $\mathbb{C}$ in either of the interesting cases.

Finally, the exceptions noted in the table are all related to the degenerate structure of the projective P_2 when $\mathcal{A}=\mathsf{TL}$, $\beta=0$ and n is even. Specifically, the exact sequence (5.2.3) becomes

$$0 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{P}_2 \longrightarrow \mathsf{S}_2 \longrightarrow 0, \tag{5.2.41}$$

explaining the non-triviality of $\mathrm{Ext}(\mathsf{S}_2,\mathsf{I}_2)$ and $\mathrm{Ext}(\mathsf{I}_2,\mathsf{C}_2)$. When $n=2$, $\mathsf{S}_2\simeq\mathsf{I}_2$, explaining the non-triviality of $\mathrm{Ext}(\mathsf{I}_2,\mathsf{I}_2)$. The dimensionality of these Ext-groups is easily verified using projective presentations and Hom-Ext long exact sequences, as above. ∎

It is easy to show that the entries of the table in Theorem 5.2.18 are related by the vector space isomorphisms $\mathrm{Ext}(\mathsf{M}^\vee,\mathsf{N}^\vee)\simeq\mathrm{Ext}(\mathsf{N},\mathsf{M})$. This follows by noting that a projective presentation $0\to\mathsf{R}\to\mathsf{P}\to\mathsf{M}\to 0$ of M gives an injective presentation $0\to\mathsf{M}^\vee\to\mathsf{P}^\vee\to\mathsf{R}^\vee\to 0$ of $\mathsf{M}^\vee$, hence the exact sequence (5.2.34a) becomes

$$0 \longrightarrow \mathsf{Hom}(\mathsf{N}^\vee,\mathsf{M}^\vee) \longrightarrow \mathsf{Hom}(\mathsf{N}^\vee,\mathsf{P}^\vee) \longrightarrow \mathsf{Hom}(\mathsf{N}^\vee,\mathsf{R}^\vee) \longrightarrow \mathrm{Ext}(\mathsf{N}^\vee,\mathsf{M}^\vee) \longrightarrow 0, \tag{5.2.42}$$

upon replacing N by $\mathsf{N}^\vee$. As $\mathsf{Hom}(\mathsf{N}^\vee,\mathsf{K}^\vee)\simeq\mathsf{Hom}(\mathsf{K},\mathsf{N})$, comparing with (5.2.34b) completes the proof.

An example of an application of these extension groups is the following result.

Corollary 5.2.19. *If a module* M *is indecomposable with exact sequence* $0\to\mathsf{I}_{k^+}\to\mathsf{M}\to\mathsf{I}_k\to 0$*, then* $\mathsf{M}\simeq\mathsf{S}_k$*. Similarly, if* N *is indecomposable with exact sequence* $0\to\mathsf{I}_k\to\mathsf{N}\to\mathsf{I}_{k^+}\to 0$*, then* $\mathsf{N}\simeq\mathsf{C}_k$*.*

Proof. Since $\mathrm{Ext}(\mathsf{I}_k,\mathsf{I}_{k^+})\simeq\mathbb{C}$, there is a single isomorphism class of non-trivial extensions of I_k by I_{k^+}. Since S_k is indecomposable and is therefore one such extension, M must be isomorphic to S_k. The second statement now follows by duality. ∎

It follows from Theorem 5.2.18 and Theorem 5.2.19 that we have classified all $\mathcal{A}$-modules with two composition factors, up to isomorphism. The complete list consists of the direct sums of two irreducibles, the (reducible) standard and costandard modules, and the projective TL_2-module $\mathsf{P}_{2,2}$ at $\beta=0$.

We end this subsection by proving two useful lemmas. The first limits the number of non-trivial extension groups.

Lemma 5.2.20.

(i) *Let* M *and* N *be* $\mathcal{A}$*-modules whose composition factors have indices* k^i *and* k^j*, for some reference index* k*, where* i *and* j *run over (multi)sets* I *and* J*, respectively, of integers. If* $|i-j|>1$ *for all* $i\in I$ *and* $j\in J$*, then* $\mathrm{Ext}(\mathsf{M},\mathsf{N})\simeq\mathrm{Ext}(\mathsf{N},\mathsf{M})\simeq 0$.

(ii) *Let* M *be an* $\mathcal{A}$*-module such that* $\mathrm{Ext}(\mathsf{M},\mathsf{I})\simeq 0$ *(*$\mathrm{Ext}(\mathsf{I},\mathsf{M})\simeq 0$*) for all semisimple modules* I*. Then* $\mathrm{Ext}(\mathsf{M},\mathsf{N})\simeq 0$ *(*$\mathrm{Ext}(\mathsf{N},\mathsf{M})\simeq 0$*) for all modules* N.

Proof. The proof of (i) is by double induction, first on the length of M, that is the number of its composition factors (including multiplicities), then on the length of N. First, suppose that $\mathsf{N} \simeq \mathsf{I}_{kj}$ is irreducible. If the length of M is 1, then M is also irreducible and Theorem 5.2.18 gives the result. If its length is greater than 1, let I be an irreducible submodule of M. Then, $0 \to \mathsf{I} \to \mathsf{M} \to \mathsf{M}/\mathsf{I} \to 0$ is exact and the covariant Hom-Ext long exact sequence (5.2.32a) says that so is $\operatorname{Ext}(\mathsf{I}_{kj}, \mathsf{I}) \to \operatorname{Ext}(\mathsf{I}_{kj}, \mathsf{M}) \to \operatorname{Ext}(\mathsf{I}_{kj}, \mathsf{M}/\mathsf{I})$. The two extreme extension groups are 0 by the induction hypothesis, hence $\operatorname{Ext}(\mathsf{I}_{kj}, \mathsf{M}) = 0$ as well. The contravariant Hom-Ext long exact sequence (5.2.32b) similarly yields $\operatorname{Ext}(\mathsf{M}, \mathsf{I}_{ki}) = 0$. This now provides the base case for a similar induction on the length of N which completes the proof of (i). A similar induction argument proves (ii). ■

A non-trivial extension group $\operatorname{Ext}(\mathsf{N}, \mathsf{L})$ implies the existence of a module M such that the exact sequence

$$0 \longrightarrow \mathsf{L} \longrightarrow \mathsf{M} \longrightarrow \mathsf{N} \longrightarrow 0 \tag{5.2.43}$$

does not split. This, however, does not prove that M is indecomposable. Indeed, Theorem 5.3.9 will give examples of non-trivial extensions that are decomposable. The second lemma gives an easy criterion to prove the indecomposability of a given extension.

Lemma 5.2.21. *Let* L*,* M *and* N *be the* $\mathcal{A}$*-modules appearing in the short exact sequence* (5.2.43)*. Suppose furthermore that* L *and* N *are indecomposable and that* $\mathsf{Hom}(\mathsf{L}, \mathsf{N}) = 0$*. Then,* M *is decomposable if and only if* (5.2.43) *splits.*

Proof. Since the definition of the short exact sequence splitting is that $\mathsf{M} \simeq \mathsf{L} \oplus \mathsf{N}$, we only need show that decomposability implies splitting under these hypotheses. Suppose then that M is decomposable. This implies that there exists a non-trivial projection $q\colon \mathsf{M} \to \mathsf{M}$, meaning that the morphism q satisfies $q^2 = q$ but is neither zero nor the identity.

Consider now the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{L} & \xrightarrow{f} & \mathsf{M} & \xrightarrow{g} & \mathsf{N} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \\ 0 & \longrightarrow & \mathsf{L} & \xrightarrow{f} & \mathsf{M} & \xrightarrow{g} & \mathsf{N} & \longrightarrow & 0, \end{array} \tag{5.2.44}$$

in which both rows are exact. Because $\mathsf{Hom}(\mathsf{L}, \mathsf{N}) \simeq 0$, the composition gqf must vanish, implying that there exist unique morphisms p and r that make (5.2.44) commute. Now, f is injective and $fpp = qfp = qqf = qf = fp$, so it follows that $p^2 = p$. Similarly, g being surjective implies that $r^2 = r$. The indecomposability of L and N therefore requires that p and r must be either the zero morphism or the identity morphism. There are thus four subcases to study.

If both p and r are the identity, then so is q by the short five lemma. If, contrarily, both p and r are zero, then we have $qf = 0$, hence $\operatorname{im} f \subseteq \ker q$, and $gq = 0$, hence $\operatorname{im} q \subseteq \ker g$. Combining these with exactness now gives $\operatorname{im} q \subseteq \ker q$, which yields $q = q^2 = 0$, a contradiction.

The remaining two cases are more interesting. If p is the identity and r is zero, then the snake lemma gives $\ker q \simeq \mathsf{N}$. On the other hand, the commuting left square of (5.2.44) implies that q acts as the identity on $\operatorname{im} f \simeq \mathsf{L}$. The two eigenspaces of q are therefore isomorphic to L and N, hence $\mathsf{M} \simeq \mathsf{L} \oplus \mathsf{N}$ and the sequence (5.2.43) splits. A similar argument shows that (5.2.43) also splits when p is zero and r is the identity. ■

We remark that the conclusion of Theorem 5.2.21 is also true under the hypotheses that L and N are indecomposable and that M has no subquotient [6] isomorphic to a direct sum of two isomorphic irreducibles. For then the submodules of M obey the distributive laws [13]

$$(\mathsf{A}+\mathsf{B})\cap\mathsf{C} \simeq (\mathsf{A}\cap\mathsf{C})+(\mathsf{B}\cap\mathsf{C}), \qquad (\mathsf{A}\cap\mathsf{B})+\mathsf{C} \simeq (\mathsf{A}+\mathsf{C})\cap(\mathsf{B}+\mathsf{C}), \tag{5.2.45}$$

from which the lemma follows rather trivially. However, the hypotheses of Theorem 5.2.21 given above have the advantage that they only require knowledge of L and N, and not of the subquotient structure of the extension M itself.

We will generally use Theorem 5.2.21 when the two indecomposable modules N and L have no composition factors in common, so the Hom-groups between them necessarily vanish. Then the extension (5.2.43) will be non-trivial if and only if M is indecomposable.

To illustrate a typical usage of this result, consider the non-trivial extension group $\operatorname{Ext}(\mathsf{S}_{k^+},\mathsf{S}_{k^-}) \simeq \mathbb{C}$ of Theorem 5.2.18 (for k non-critical). As S_{k^+} and S_{k^-} are indecomposable modules with no composition factors in common, $\mathsf{Hom}(\mathsf{S}_{k^-},\mathsf{S}_{k^+}) = 0$ and thus any non-trivial extension will be indecomposable by Theorem 5.2.21. This example demonstrates the existence of indecomposable $\mathcal{A}$-modules with (three or) four composition factors I_{k^-}, I_k, I_{k^+} and $\mathsf{I}_{k^{++}}$. We remark that the only indecomposable modules with four composition factors that have been encountered thus far are projective and that these projectives always have a composition factor of multiplicity two. The indecomposable extension described above cannot be one of these projectives, hence must be a new indecomposable $\mathcal{A}$-module. This observation will be the starting point of Section 5.3.

5.2.6 Their Loewy diagrams

It is sometimes convenient to visualise the structure of non-semisimple modules diagrammatically, particularly when computing Hom-groups. A convention popular in the mathematical physics community is to represent the structure as a graph in which the vertices are the composition factors of the module M and the arrows represent the "action of the algebra". More precisely, an arrow is drawn from the factor I to the factor I' if M has a subquotient isomorphic to a non-trivial extension of I by I'. In principle, one can also decorate the arrow with an extra label if $\dim \operatorname{Ext}(\mathsf{I},\mathsf{I}') > 1$, but Theorem 5.2.18 ensures that this never happens for $\mathcal{A} = \mathsf{TL}$ or dTL.

The utility of this arrow notation is evidently limited. For example, suppose that M has a composition factor I appearing as a submodule with multiplicity 2: $\mathsf{I} \overset{\iota_1}{\hookrightarrow} \mathsf{M}$ and $\mathsf{I} \overset{\iota_2}{\hookrightarrow} \mathsf{M}$. Then, it has an infinite number

6. We recall that a subquotient of a module M is a submodule of a quotient of M or, equivalently, a quotient of a submodule of M.

of submodules isomorphic to I, corresponding to linear combinations of ι_1 and ι_2 (modulo rescalings). An arrow indicating a submodule $\mathsf{L} \subset \mathsf{M}$ that is isomorphic to a non-trivial extension in $\mathrm{Ext}(\mathsf{I}', \mathsf{I})$ will then also require a label to identify which linear combination of ι_1 and ι_2 describes the submodule $\mathsf{I} \subset \mathsf{L}$. The labelling of the arrows can therefore be unpleasantly complicated in general. However, this issue also turns out to not be a problem for the indecomposable modules of TL or dTL, so we will always decorate our diagrams with arrows in order to maximise the information conveyed.

We will refer to these structure graphs as *Loewy diagrams*. For the reasons already mentioned, the Loewy diagrams defined by mathematicians tend not to have arrows; instead, the composition factors are arranged in horizontal layers that have structural meaning. To make this precise, one introduces the *radical* $\mathrm{rad}\,\mathsf{M}$ and the *socle* $\mathrm{soc}\,\mathsf{M}$ of a module M as the intersection of its maximal proper submodules and the sum of its simple submodules, respectively.[7] The *head* $\mathrm{hd}\,\mathsf{M}$ of M is then the quotient by the radical: $\mathrm{hd}\,\mathsf{M} = \mathsf{M}/\mathrm{rad}\,\mathsf{M}$. Just as $\mathrm{soc}\,\mathsf{M}$ is the (unique) maximal semisimple submodule of M, $\mathrm{hd}\,\mathsf{M}$ is the (unique) maximal semisimple quotient of M.

Radicals and socles lead to important examples of filtrations. Given a module M, define its *radical series* and *socle series* to be the following strictly descending and strictly ascending chains of submodules:

$$\begin{gathered} \mathsf{M} = \mathrm{rad}^0\,\mathsf{M} \supset \mathrm{rad}^1\,\mathsf{M} \supset \mathrm{rad}^2\,\mathsf{M} \supset \cdots \supset \mathrm{rad}^{n-1}\,\mathsf{M} \supset \mathrm{rad}^n\,\mathsf{M} = 0, \\ 0 = \mathrm{soc}^0\,\mathsf{M} \subset \mathrm{soc}^1\,\mathsf{M} \subset \mathrm{soc}^2\,\mathsf{M} \subset \cdots \subset \mathrm{soc}^{n-1}\,\mathsf{M} \subset \mathrm{soc}^n\,\mathsf{M} = \mathsf{M}. \end{gathered} \tag{5.2.46}$$

Here, $\mathrm{rad}^j\,\mathsf{M}$ and $\mathrm{soc}^j\,\mathsf{M}$ are defined recursively, for $j \geq 1$, to be $\mathrm{rad}(\mathrm{rad}^{j-1}\,\mathsf{M})$ and the unique submodule satisfying $\mathrm{soc}^j\,\mathsf{M}/\mathrm{soc}^{j-1}\,\mathsf{M} = \mathrm{soc}(\mathsf{M}/\mathrm{soc}^{j-1}\,\mathsf{M})$, respectively. Note that both chains contain the same number n of non-zero submodules (this number is called the *Loewy length* of the module M) and that the successive quotients $\mathrm{rad}^j\,\mathsf{M}/\mathrm{rad}^{j+1}\,\mathsf{M}$ and $\mathrm{soc}^j\,\mathsf{M}/\mathrm{soc}^{j-1}\,\mathsf{M}$ are maximal semisimple.

For $\mathcal{A} = \mathsf{TL}$ or dTL, the radical and socle series of any given $\mathcal{A}$-module coincide: $\mathrm{rad}^j\,\mathsf{M} = \mathrm{soc}^{n-j}\,\mathsf{M}$. We may therefore draw *the* Loewy diagram so that its composition factors are partitioned (uniquely) into horizontal layers according to the following convention: the j-th layer (counting from bottom to top) indicates the composition factors that appear in the maximal semisimple quotient $\mathrm{soc}^j\,\mathsf{M}/\mathrm{soc}^{j-1}\,\mathsf{M} = \mathrm{rad}^{n-j}\,\mathsf{M}/\mathrm{rad}^{n-j+1}\,\mathsf{M}$. In addition to arranging our composition factors thusly, we shall also use arrows as a means to indicate further refinements to the substructure. It moreover proves convenient to arrange the composition factors so that their indices increase from left to right, as in the non-critical orbits of Section 5.2.1.

As an example of the "annotated" Loewy diagrams that we shall use, we present the diagrams for the standard modules:

$$\underset{(k\text{ critical})}{\mathsf{S}_k} : \quad \mathsf{I}_k \qquad\qquad \underset{(k\text{ non-critical})}{\mathsf{S}_k} : \quad \begin{array}{ccc} \mathsf{I}_k & & \\ & \searrow & \\ & & \mathsf{I}_{k^+} \end{array} . \tag{5.2.47}$$

The left diagram obviously reflects the fact that critical standard modules are irreducible; the arrow in the right diagram means that the action of $\mathcal{A}$ can map any element of S_k associated with the composition factor

7. For the algebras TL and dTL, this notion of radical generalises that which was introduced in Section 5.2.1 for the standard modules S_k.

I_k to an element of the factor I_{k^+}, but not vice versa. The factor I_{k^+} is then a submodule of S_k and the factor I_k represents the quotient $\mathsf{S}_k/\mathsf{I}_{k^+}$, as in the exact sequence (5.2.2). Note that we are employing the convention that modules with $k \notin \Lambda$ are zero: when $k = k_R$, the diagram on the right degenerates to that of the left because $\mathsf{I}_{k_R^+} = 0$. In the language introduced above, the standard modules S_k, with k non-critical and $k \neq k_R$, have $\operatorname{soc}\mathsf{S}_k = \operatorname{rad}\mathsf{S}_k = \mathsf{R}_k \simeq \mathsf{I}_{k^+}$ and $\operatorname{hd}\mathsf{S}_k \simeq \mathsf{I}_k$.

The short exact sequence (5.2.28) then gives the Loewy diagrams of the costandard modules:

$$\underset{(k \text{ critical})}{\mathsf{C}_k} : \quad \mathsf{I}_k \qquad\qquad \underset{(k \text{ non-critical})}{\mathsf{C}_k} : \quad \begin{array}{ccc} & & \mathsf{I}_{k^+} \\ & \swarrow & \\ \mathsf{I}_k & & \end{array} . \tag{5.2.48}$$

This illustrates the general rule that the Loewy diagram for $\mathsf{M}^\vee$ is obtained from that of M by reversing all arrows and flipping the diagram upside-down. In principle, one should also replace each composition factor by its dual as well, but for $\mathcal{A} = \mathsf{TL}$ and dTL, every irreducible is self-dual (Theorem 5.2.11). Moreover, because duality is an exact contravariant functor (Theorem 5.2.13), it exchanges a module's radical and socle series. In particular, it swaps the socle with the head: $\operatorname{soc}(\mathsf{M}^\vee) \simeq \operatorname{hd}\mathsf{M}$.

The Loewy diagrams for the projective modules are also easily constructed. The following cases are easy:

$$\underset{(k \text{ critical})}{\mathsf{P}_k} : \quad \mathsf{I}_k \qquad\qquad \underset{(k \text{ non-critical with } k = k_L)}{\mathsf{P}_k} : \quad \begin{array}{ccc} \mathsf{I}_k & & \\ & \searrow & \\ & & \mathsf{I}_{k^+} \end{array} . \tag{5.2.49a}$$

The diagram for k non-critical and larger than k_L follows from the exact sequence (5.2.3) and its dual (5.2.30), recalling that these projectives are self-dual (Theorem 5.2.14). The submodules and quotients from these exact sequences lead to the following Loewy diagram:

$$\underset{(k \text{ non-critical with } k > k_L)}{\mathsf{P}_k} : \quad \begin{array}{ccccc} & & \mathsf{I}_k & & \\ & \swarrow & & \searrow & \\ \mathsf{I}_{k^-} & & & & \mathsf{I}_{k^+} \\ & \searrow & & \swarrow & \\ & & \mathsf{I}_k & & \end{array} . \tag{5.2.49b}$$

There is no arrow between the two I_k factors because such a self-extension would be a direct sum: $\operatorname{Ext}(\mathsf{I}_k, \mathsf{I}_k) = 0$ (the single exception for TL_2, $\beta = 0$ and $k = 2$ is not relevant here as $\mathsf{P}_{2,2}$ has only two composition factors) and any arrows between the $\mathsf{I}_{k^\pm}$ are likewise ruled out by extension groups. More fundamentally, such an arrow would contradict the fact that P_k has submodules isomorphic to S_{k^-} and C_k. For example, an arrow from I_{k^-} to I_{k^+} in (5.2.49b) would mean that the submodule generated by the I_{k^-} factor is not isomorphic to S_{k^-}.

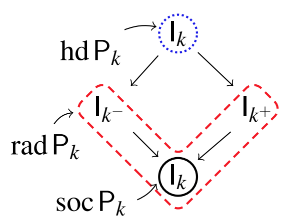
We mention the degenerate case $k = k_R$, for which $\mathsf{I}_{k_R^+} = 0$. This factor and its incident arrows are therefore removed from the Loewy diagram (5.2.49b) to obtain that of P_{k_R}. A different degeneration occurs when $\mathcal{A} = \mathsf{TL}$, with n even and $\beta = 0$, as then $\mathsf{I}_{k^-} = 0$ for $k = 2$. Moreover, both degenerations occur simultaneously if, in addition, $n = 2$. For completeness, we draw the Loewy diagrams for each of these cases:

$$\begin{array}{c}\mathsf{P}_k\\ (k\text{ non-critical with}\\ k=k_R\text{ and }n\neq 2)\end{array} : \begin{array}{ccc} & & \mathsf{I}_k \\ & \swarrow & \\ \mathsf{I}_{k^-} & & \\ & \searrow & \\ & & \mathsf{I}_k\end{array} \qquad \begin{array}{c}\mathsf{P}_2\\ (n\neq 2\text{ even,}\\ \mathcal{A}=\mathsf{TL},\ \beta=0)\end{array} : \begin{array}{ccc} \mathsf{I}_2 & & \\ & \searrow & \\ & & \mathsf{I}_4 \\ & \swarrow & \\ \mathsf{I}_2 & & \end{array} \qquad \begin{array}{c}\mathsf{P}_{2,2}\\ (\mathcal{A}=\mathsf{TL}_2,\ \beta=0)\end{array} : \begin{array}{c}\mathsf{I}_{2,2}\\ \downarrow\\ \mathsf{I}_{2,2}\end{array} . \tag{5.2.49c}$$

It may be useful to translate the Loewy diagrams (5.2.49b) and (5.2.49c) of the P_k, with k non-critical and $k \neq k_L$, into the language of radicals and socles. The Loewy length of these modules is 3 (except for $\mathsf{P}_{2,2}$, for $\mathcal{A} = \mathsf{TL}$ and $\beta = 0$). The (unique) Loewy series for P_k takes the form

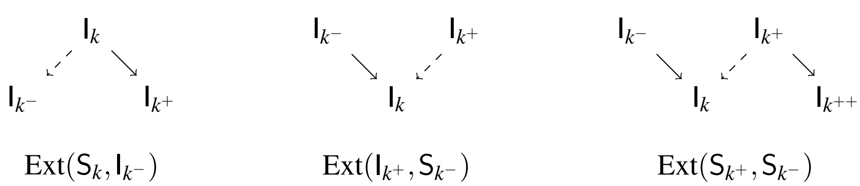
$$0 \subset \mathsf{I}_k \subset \mathsf{V}_{k^-} \subset \mathsf{P}_k, \tag{5.2.50}$$

where I_k is (isomorphic to) the socle of P_k and the, as yet undescribed, module V_{k^-} is its radical.[8] The semisimple quotient $\mathsf{V}_k/\mathsf{I}_k = \operatorname{rad}\mathsf{P}_k/\operatorname{rad}^2\mathsf{P}_k = \operatorname{soc}^2\mathsf{P}_k/\operatorname{soc}\mathsf{P}_k$ is (isomorphic to) $\mathsf{I}_{k^-} \oplus \mathsf{I}_{k^+}$. Finally, the head is also (isomorphic to) I_k, consistent with the self-duality of P_k. We summarise this as follows:



. (5.2.51)

We close this section by describing a suggestive use for Loewy diagrams: they help in determining the structure of non-trivial extensions. In Section 5.2.5, we observed that Theorem 5.2.18 predicted the existence of indecomposable modules with three and four composition factors. For example, the extension groups $\operatorname{Ext}(\mathsf{S}_k, \mathsf{I}_{k^-})$, $\operatorname{Ext}(\mathsf{I}_{k^+}, \mathsf{S}_{k^-})$ and $\operatorname{Ext}(\mathsf{S}_{k^+}, \mathsf{S}_{k^-})$ are each isomorphic to $\mathbb{C}$. We use dashed arrows in these drawings to indicate the extension itself; solid arrows describing the submodule and quotient:



. (5.2.52)

We shall prove in Section 5.3.4 that the Loewy diagrams of these extensions are obtained from these diagrams by replacing the dashed arrows with solid ones. Moreover, representatives for the non-trivial isomorphism classes of the first two Ext-groups will be constructed, in Section 5.3.3, as subquotients of the principal indecomposables, as their Loewy diagrams suggest. In contrast, the third Loewy diagram is new, indicating that there are more indecomposables to be discovered beyond the subquotients of the projectives and injectives.

8. The module V_{k^-} will be studied in Section 5.3.3 where the reason for the chosen notation will become apparent.

5.3 A complete set of indecomposable modules

This section constructs a complete set of classes of indecomposable modules for TL_n and dTL_n, up to isomorphism, using relatively elementary theory. Section 5.5 obtains the same set using a more advanced tool, namely Auslander-Reiten theory. There, the main results of this theory will be reviewed (without proof) and then applied to TL_n and dTL_n. It will turn out that both of these algebras are representation-finite, meaning that their inequivalent indecomposable (finite-dimensional) modules are finite in number.

Here, we pick up from the observation that closed Section 5.2.5. The table of extension groups of Theorem 5.2.18 proves the existence of non-trivial extensions with three and four composition factors that, by Theorem 5.2.21, are indecomposable. Our first step will exploit this observation further. It will define, recursively, a family of indecomposable modules, the existence of the next member being granted by a non-trivial extension group involving the present one. The second step will reveal the structure of these modules which we will summarise by computing their socles, radicals and heads. This information suffices to draw their Loewy diagrams. The third step uses this information to construct projective and injective presentations of the new indecomposables. These presentations allow us to compute their extension groups with irreducible modules in the fourth step. Theorem 5.2.20 tells us that these groups will detect if *any* further extensions are possible. The fifth and last step will prove that the modules introduced in the first step form, together with the projective and critical standard modules, a complete set of inequivalent indecomposable modules.

5.3.1 The modules $\mathsf{B}^l_{n,k}$ and $\mathsf{T}^l_{n,k}$

Let $k \in \Lambda_0$ be non-critical and define $\mathsf{B}^0_k \equiv \mathsf{I}_k$. We state the obvious fact that this module is irreducible, hence indecomposable, with a single composition factor I_k. From B^0_k, we construct recursively a family of indecomposable modules B^{2j}_k, $j = 0, 1, \ldots$, as follows (the range of j will be clarified below). The j-th step constructs the module B^{2j}_k which has $2j+1$ composition factors I_k, I_{k^+}, $\mathsf{I}_{k^{++}}$, $\ldots$, $\mathsf{I}_{k^{2j}}$. The following step of the recursion then shows that the extension group $\mathrm{Ext}(\mathsf{S}_{k^{2j+1}}, \mathsf{B}^{2j}_k)$ is isomorphic to $\mathbb{C}$ and we name the non-trivial extension $\mathsf{B}^{2(j+1)}_k$.

Note that the first step of this recursive definition follows easily from Theorem 5.2.18. Indeed, it shows that the extension group $\mathrm{Ext}(\mathsf{S}_{k^+}, \mathsf{B}^0_k) = \mathrm{Ext}(\mathsf{S}_{k^+}, \mathsf{I}_k)$ is $\mathbb{C}$, hence that the short exact sequence $0 \to \mathsf{I}_k \to \mathsf{B}^2_k \to \mathsf{S}_{k^+} \to 0$ completely characterises the non-trivial extension B^2_k, up to isomorphism. Moreover, since the composition factors of I_k and S_{k^+} are distinct, Theorem 5.2.21 proves that B^2_k is indecomposable. Finally, its composition factors are clearly I_k, I_{k^+} and $\mathsf{I}_{k^{++}}$, as required.

Suppose then that the j-th recursive step has been completed, so that there exists an indecomposable module B^{2j}_k with composition factors I_k, I_{k^+}, $\ldots$, $\mathsf{I}_{k^{2j}}$ and non-split short exact sequence

$$0 \longrightarrow \mathsf{B}^{2(j-1)}_k \longrightarrow \mathsf{B}^{2j}_k \longrightarrow \mathsf{S}_{k^{2j-1}} \longrightarrow 0. \tag{5.3.1}$$

Our goal is to compute $\mathrm{Ext}(\mathsf{S}_{k^{2j+1}}, \mathsf{B}^{2j}_k)$. To do this, we first show that $\mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{B}^{2j}_k) \simeq \mathbb{C}$. This follows from the Hom-Ext long exact sequence (5.2.32a), derived from (5.3.1):

$$0 \longrightarrow \mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{B}_k^{2(j-1)}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{B}_k^{2j}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{S}_{k^{2j-1}}) \longrightarrow \mathrm{Ext}(\mathsf{S}_{k^{2j}}, \mathsf{B}_k^{2(j-1)}) \longrightarrow \cdots. \quad (5.3.2)$$

Indeed, $\mathsf{S}_{k^{2j}}$ and $\mathsf{B}_k^{2(j-1)}$ have no common composition factor, hence their Hom-group is zero, and their Ext-group is also zero because their composition factors are sufficiently separated (see Theorem 5.2.20(i)). As $\mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{S}_{k^{2j-1}}) \simeq \mathbb{C}$, by Theorem 5.2.17, we obtain $\mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{B}_k^{2j}) \simeq \mathbb{C}$, as claimed.

The desired Ext-group is now computed from the exact sequence

$$0 = \mathsf{Hom}(\mathsf{P}_{k^{2j+1}}, \mathsf{B}_k^{2j}) \longrightarrow \mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{B}_k^{2j}) \longrightarrow \mathrm{Ext}(\mathsf{S}_{k^{2j+1}}, \mathsf{B}_k^{2j}) \longrightarrow \mathrm{Ext}(\mathsf{P}_{k^{2j+1}}, \mathsf{B}_k^{2j}) = 0. \quad (5.3.3)$$

This follows from the Hom-Ext sequence (5.2.32b), based on (5.2.3), by noting that the first Hom-group is zero, because $\mathsf{I}_{k^{2j+1}}$ is not a composition factor of B_k^{2j}, and that the last Ext-group is zero, because $\mathsf{P}_{k^{2j+1}}$ is projective. It follows that $\mathrm{Ext}(\mathsf{S}_{k^{2j+1}}, \mathsf{B}_k^{2j}) \simeq \mathsf{Hom}(\mathsf{S}_{k^{2j}}, \mathsf{B}_k^{2j}) \simeq \mathbb{C}$. This result leads to the definition of $\mathsf{B}_k^{2(j+1)}$ as any representative of the corresponding non-trivial extension. Its composition factors are clearly I_k, I_{k^+}, $\ldots$, $\mathsf{I}_{k^{2j}}$, $\mathsf{I}_{k^{2j+1}}$, $\mathsf{I}_{k^{2(j+1)}}$. Indecomposability follows from Theorem 5.2.21 as usual, hence the $(j+1)$-th step of the recursion is complete.

We remark that this recursion may be continued as long as there are integers larger than k^{2j} in the (non-critical) orbit of k, that is, as long as $k^{2j+1} \in \Lambda$. In the case where $k^{2j+1} = k_R$, so that $k^{2(j+1)} \notin \Lambda$, the above computation remains valid, but we shall denote the resulting module by B_k^{2j+1} to underline the fact that it only contains $2j+2$ composition factors, instead of the $2j+3$ factors possessed by the other $\mathsf{B}_k^{2(j+1)}$.

A similar recursive construction can be used to construct a second family of modules B_k^{2j+1}, $j = 0, 1, \ldots$, starting now from $\mathsf{B}_k^1 \equiv \mathsf{C}_k$, for any non-critical k in Λ_0 smaller than k_R^-. The module B_k^{2j+1} is then defined, if $k^{2j+1} < k_R$, to be a non-trivial extension described by $\mathrm{Ext}(\mathsf{C}_k, \mathsf{B}_{k^{++}}^{2j-1}) \simeq \mathbb{C}$ (when $k \in \Lambda_0$ of course). The computation of this Ext-group is again done recursively, uses now the sequence (5.2.30), and copies otherwise the previous argument. Note that, in the first family, composition factors are added to the right of existing ones, but in this new family they are added to the left. The composition factors of B_k^{2j+1} are thus $\mathsf{I}_k, \mathsf{I}_{k^+}, \ldots, \mathsf{I}_{k^{2j}}, \mathsf{I}_{k^{2j+1}}$ and are even in number. For this second family, the process stops whenever the index of the costandard module C_k that would be used to extend $\mathsf{B}_{k^{++}}^{2j-1}$ falls outside Λ_0. The constraints $k < k_R^-$ for B_k^1 and $k^{2j+1} < k_R$ for B_k^{2j+1} ensure that I_{k_R} is never a composition factor of an indecomposable module of this second family. Note that the first family contains modules B_k^{2j+1} with an even number of composition factors, but that they all have I_{k_R} as composition factor. In this way, these constructions never produce modules with the same labels using different means: the notation B_k^l is well defined.

Finally, the duals of the modules that we have constructed above will be denoted by $\mathsf{T}_k^j \equiv (\mathsf{B}_k^j)^\vee$. As duality is exact contravariant (Theorem 5.2.13), this is equivalent to dualising the above inductive definitions: T_k^{2j} is realised through the non-trivial extensions of $\mathrm{Ext}(\mathsf{T}_k^{2(j-1)}, \mathsf{C}_{k^{2j-1}})$ and T_k^{2j+1} through those of $\mathrm{Ext}(\mathsf{T}_{k^{++}}^{2j-1}, \mathsf{S}_k)$.

Proposition 5.3.1. *Given the recursive constructions above, the B_k^l and T_k^l are indecomposable and appear* *in the following non-split exact sequences:*

$$0 \longrightarrow \mathsf{B}_k^{2(j-1)} \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{S}_{k^{2j-1}} \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{C}_{k^{2j-1}} \longrightarrow \mathsf{T}_k^{2j} \longrightarrow \mathsf{T}_k^{2(j-1)} \longrightarrow 0, \tag{5.3.4a}$$

$$0 \longrightarrow \mathsf{B}_{k^{++}}^{2j-1} \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow \mathsf{C}_k \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{S}_k \longrightarrow \mathsf{T}_k^{2j+1} \longrightarrow \mathsf{T}_{k^{++}}^{2j-1} \longrightarrow 0 \qquad (k^{2j+1} \neq k_R). \tag{5.3.4b}$$

Moreover, the non-split exact sequences for B_k^{2j+1} *and* T_k^{2j+1}*, when* $k^{2j+1} = k_R$*, are instead*

$$0 \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow \mathsf{I}_{k^{2j+1}} \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{I}_{k^{2j+1}} \longrightarrow \mathsf{T}_k^{2j+1} \longrightarrow \mathsf{T}_k^{2j} \longrightarrow 0 \qquad (k^{2j+1} = k_R). \tag{5.3.4c}$$

The B_k^l *and* T_k^l *have* $l+1$ *composition factors, namely* I_k*,* I_{k^+}*,* $\dots$*,* I_{k^l}*, and they represent mutually non-isomorphic classes of indecomposable modules, except for* $\mathsf{B}_k^0 = \mathsf{T}_k^0 = \mathsf{I}_k$*.*

Proof. Only the last statement remains to be proved. The indices of the composition factors of the B_k^l and the T_k^l are consecutive integers, from k to k^l, in the non-critical orbit $[k]$. If two of these modules have the same indices, then they must have the same values of k and l. Thus, only B_k^l and T_k^l could be isomorphic. For $l > 0$, B_k^l and T_k^l are both reducible, but indecomposable, so they do not coincide with their socles. Let us first study the case with l even. Because $\mathsf{B}_k^0 \simeq \mathsf{I}_k$ and $\mathsf{B}_k^{2(j-1)} \subset \mathsf{B}_k^{2j}$, it follows that I_k is in the socle of B_k^{2j}. Since $\mathsf{T}_k^{2j} = (\mathsf{B}_k^{2j})^\vee$, I_k is in its head (Section 5.2.6). Thus, B_k^{2j} and T_k^{2j} cannot be isomorphic for $j > 0$. The same argument also takes care of the pair B_k^{2j+1} and T_k^{2j+1} when $k^{2j+1} = k_R$. Finally, the argument for l odd copies the previous one, but uses the irreducible $\mathsf{I}_{k^{2j+1}}$ that is in the head of B_k^{2j+1} but in the socle of T_k^{2j+1}. ∎

5.3.2 Their Loewy diagrams

This section clarifies the structure of the new modules B_k^l and T_k^l by identifying their socles, radicals and heads (see Section 5.2.6). The Loewy diagrams for the B_k^l and T_k^l are easily drawn from these data. Moreover, their injective hulls and projective covers will be obtained as immediate consequences of Theorem 5.2.16.

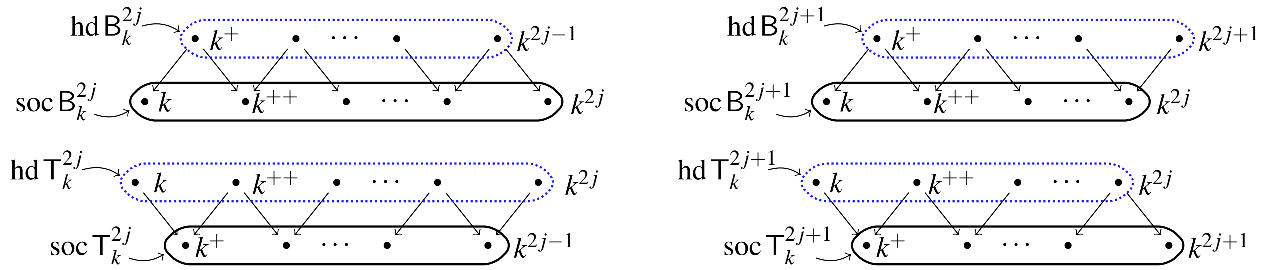
Proposition 5.3.2. *The modules* B_k^l *and* T_k^l*, for* $l \geq 1$*, all have Loewy length* 2*. The head, socle and radical of each are given in the following table:*

M	B_k^{2j}	B_k^{2j+1}	T_k^{2j}	T_k^{2j+1}
$\operatorname{hd}\mathsf{M}$	$\bigoplus_{i=0}^{j-1} \mathsf{I}_{k^{2i+1}}$	$\bigoplus_{i=0}^{j} \mathsf{I}_{k^{2i+1}}$	$\bigoplus_{i=0}^{j} \mathsf{I}_{k^{2i}}$	$\bigoplus_{i=0}^{j} \mathsf{I}_{k^{2i}}$
$\operatorname{soc}\mathsf{M} = \operatorname{rad}\mathsf{M}$	$\bigoplus_{i=0}^{j} \mathsf{I}_{k^{2i}}$	$\bigoplus_{i=0}^{j} \mathsf{I}_{k^{2i}}$	$\bigoplus_{i=0}^{j-1} \mathsf{I}_{k^{2i+1}}$	$\bigoplus_{i=0}^{j} \mathsf{I}_{k^{2i+1}}$.

Of course, $\mathsf{B}_k^0 \simeq \mathsf{T}_k^0 \simeq \mathsf{I}_k$*, hence in this case, the radical is* 0 *and the socle and the head are* I_k*.*

Note that this table is consistent with the general result $\operatorname{soc}\mathsf{M} \simeq \operatorname{hd}\mathsf{M}^\vee$.

Before giving a proof, we use the proposition to draw the Loewy diagrams of the B_k^l and T_k^l, with $l > 0$, thus revealing their "zigzag" structure:



(5.3.5)

To lighten the notation, we have replaced the composition factors $\mathsf{I}_{k'}$ by dots, labelling (some of) them by the corresponding index k' (the missing labels should be clear). As before, the socle forms the bottom row and it is circled by a solid line, while the head forms the top row and is circled by a dotted line. There can only be arrows from the head to the socle between neighbouring composition factors as the extension groups $\operatorname{Ext}(\mathsf{I}_k, \mathsf{I}_{k'})$ (Theorem 5.2.18) forbid other possibilities. Finally, none of the arrows indicated in these diagrams may be omitted as the result would indicate a decomposable module. We remark that the notation B_k^l (T_k^l) was chosen as a reminder that the composition factor I_k appears in the bottom (top) layer of the Loewy diagram. The composition factor with the highest index is I_{k^l}.

Proof of Theorem 5.3.2. We provide details for the first family of modules B_k^{2j}, those for the second family B_k^{2j+1} being similar and the results for the T_k^l then being obtained by duality.

The socle of B_k^{2j} is obtained by induction on j. For $j = 0$, we have $\mathsf{B}_k^0 \equiv \mathsf{I}_k$, hence $\operatorname{soc} \mathsf{B}_k^0 = \mathsf{I}_k$ as required. For general j, we begin with the short exact sequence

$$0 \longrightarrow \mathsf{B}_k^{2(j-1)} \xrightarrow{\iota} \mathsf{B}_k^{2j} \xrightarrow{\pi} \mathsf{S}_{k^{2j-1}} \longrightarrow 0. \tag{5.3.6}$$

Since the image of a semisimple module is semisimple, $\iota(\operatorname{soc} \mathsf{B}_k^{2(j-1)}) \subseteq \operatorname{soc} \mathsf{B}_k^{2j}$ and so the latter must contain the composition factors I_k, $\mathsf{I}_{k^{++}}$, ..., $\mathsf{I}_{k^{2(j-1)}}$. Moreover, $\mathsf{L} \subseteq \mathsf{M}$ implies that $\operatorname{soc} \mathsf{L} = \mathsf{L} \cap \operatorname{soc} \mathsf{M}$, so $\operatorname{soc} \mathsf{B}_k^{2j}$ cannot contain any of the composition factors I_{k^+}, $\mathsf{I}_{k^{+++}}$, ..., $\mathsf{I}_{k^{2j-3}}$. If $\mathsf{I}_{k^{2j-1}}$ were in $\operatorname{soc} \mathsf{B}_k^{2j}$, then π would map it into $\operatorname{soc} \mathsf{S}_{k^{2j-1}} \simeq \mathsf{I}_{k^{2j}}$, hence $\mathsf{I}_{k^{2j-1}}$ would be in $\ker \pi = \operatorname{im} \iota$. This contradicts the fact that $\mathsf{B}_k^{2(j-1)}$ does not have $\mathsf{I}_{k^{2j-1}}$ as a composition factor, so it follows that $\mathsf{I}_{k^{2j-1}}$ is not in $\operatorname{soc} \mathsf{B}_k^{2j}$. Suppose finally that $\mathsf{I}_{k^{2j}}$ is not in $\operatorname{soc} \mathsf{B}_k^{2j}$. Then, we would have

$$\operatorname{soc} \mathsf{B}_k^{2j} \simeq \operatorname{soc} \mathsf{B}_k^{2(j-1)} \simeq \mathsf{I}_k \oplus \mathsf{I}_{k^{++}} \oplus \cdots \oplus \mathsf{I}_{k^{2(j-1)}}. \tag{5.3.7}$$

As the injective hull of a module coincides with that of its socle, that of B_k^{2j} would now be $\mathsf{J}_k \oplus \mathsf{J}_{k^{++}} \oplus \cdots \oplus \mathsf{J}_{k^{2(j-1)}}$. But, this hull does not have $\mathsf{I}_{k^{2j}}$ as a composition factor, whereas B_k^{2j} does, another contradiction. We therefore conclude that the socle of B_k^{2j} is $\mathsf{I}_k \oplus \mathsf{I}_{k^{++}} \oplus \cdots \oplus \mathsf{I}_{k^{2j}}$, as required.

We now prove that the radical and socle of B_k^{2j} coincide, for $j > 0$, a consequence of this being that the head is the direct sum of the composition factors that are not in the socle: $\operatorname{hd} \mathsf{B}_k^{2j} \simeq \mathsf{I}_{k^+} \oplus \cdots \oplus \mathsf{I}_{k^{2j-1}}$. The proof will follow immediately upon constructing an injection

$$\frac{\mathsf{B}_k^{2j}}{\operatorname{soc}\mathsf{B}_k^{2j}} \hookrightarrow \frac{\operatorname{rad}\mathsf{J}[\mathsf{B}_k^{2j}]}{\operatorname{soc}\mathsf{J}[\mathsf{B}_k^{2j}]}, \tag{5.3.8}$$

where $\mathsf{J}[\mathsf{B}_k^{2j}] \simeq \mathsf{J}_k \oplus \mathsf{J}_{k^{++}} \oplus \cdots \oplus \mathsf{J}_{k^{2j}}$ denotes the injective hull of B_k^{2j}. To see why, recall from Theorem 5.2.16 that the injective modules all have Loewy lengths at most 3. The quotient $\operatorname{rad}\mathsf{J}[\mathsf{B}_k^{2j}]/\operatorname{soc}\mathsf{J}[\mathsf{B}_k^{2j}]$ therefore has Loewy length at most 1, meaning that it is semisimple. The injection (5.3.8) will therefore establish that $\mathsf{B}_k^{2j}/\operatorname{soc}\mathsf{B}_k^{2j}$ is semisimple, hence that B_k^{2j} has Loewy length 2 (it does not coincide with its socle, for $j > 0$), so its radical equals its socle.

It remains to construct the injection (5.3.8). Suppose first that $k \neq k_L$. Then, $\mathsf{J}_{k^{2i}} \simeq \mathsf{P}_{k^{2i}}$ has two composition factors isomorphic to $\mathsf{I}_{k^{2i}}$, one contributing to the socle and the other to the head. As B_k^{2j} has a single composition factor isomorphic to $\mathsf{I}_{k^{2i}}$, for each $0 \le i \le j$, any morphism $f\colon \mathsf{B}_k^{2j} \to \mathsf{J}[\mathsf{B}_k^{2j}] \simeq \mathsf{J}_k \oplus \mathsf{J}_{k^{++}} \oplus \cdots \oplus \mathsf{J}_{k^{2j}}$ will send this composition factor to that of the *socle* of $\mathsf{J}_{k^{2i}}$ (or to zero). Indeed, this composition factor belongs to the socle of B_k^{2j}, so it follows that $\operatorname{soc}\mathsf{B}_k^{2j}$ is mapped into $\operatorname{soc}\mathsf{J}[\mathsf{B}_k^{2j}]$. Moreover, the image of f thus never includes the composition factors corresponding to the heads of the $\mathsf{J}_{k^{2i}}$, hence it will lie in $\operatorname{rad}\mathsf{J}[\mathsf{B}_k^{2j}]$. This shows that any $f\colon \mathsf{B}_k^{2j} \to \mathsf{J}[\mathsf{B}_k^{2j}]$ will induce a map as in (5.3.8). To find an injective map and complete the proof, it suffices to take f injective (which is always possible by the definition of injective hulls) because then f maps $\operatorname{soc}\mathsf{B}_k^{2j}$ onto $\operatorname{soc}\mathsf{J}[\mathsf{B}_k^{2j}]$.

If $k = k_L$, the injective J_{k_L} is isomorphic to C_{k_L} whose Loewy length is 2. In this case, the argument goes through if $\operatorname{rad}\mathsf{J}[\mathsf{B}_k^{2j}] \simeq \operatorname{rad}(\mathsf{J}_k \oplus \mathsf{J}_{k^{++}} \oplus \cdots \oplus \mathsf{J}_{k^{2j}})$ is replaced throughout by $\mathsf{J}_k \oplus \operatorname{rad}(\mathsf{J}_{k^{++}} \oplus \cdots \oplus \mathsf{J}_{k^{2j}})$. ∎

As the projective cover $\mathsf{P}[\mathsf{M}]$ (injective hull $\mathsf{J}[\mathsf{M}]$) of a module M is isomorphic to that of its head (socle), Theorem 5.3.2 immediately identifies these data for the B_k^l and T_k^l.

Corollary 5.3.3. *The projective covers and injective hulls of the B_k^l and T_k^l, with $l > 0$, are as follows:*

M	B_k^{2j}	B_k^{2j+1}	T_k^{2j}	T_k^{2j+1}
$\mathsf{P}[\mathsf{M}]$	$\bigoplus_{i=0}^{j-1}\mathsf{P}_{k^{2i+1}}$	$\bigoplus_{i=0}^{j}\mathsf{P}_{k^{2i+1}}$	$\bigoplus_{i=0}^{j}\mathsf{P}_{k^{2i}}$	$\bigoplus_{i=0}^{j}\mathsf{P}_{k^{2i}}$
$\mathsf{J}[\mathsf{M}]$	$\bigoplus_{i=0}^{j}\mathsf{J}_{k^{2i}}$	$\bigoplus_{i=0}^{j}\mathsf{J}_{k^{2i}}$	$\bigoplus_{i=0}^{j-1}\mathsf{J}_{k^{2i+1}}$	$\bigoplus_{i=0}^{j}\mathsf{J}_{k^{2i+1}}$.

As $\mathsf{B}_k^0 \simeq \mathsf{T}_k^0 \simeq \mathsf{I}_k$, the projective covers and injective hulls, when $l = 0$, are P_k and J_k, respectively.

We remark that, as in Section 5.2.3, we are again neglecting to specify the surjective (injective) morphisms that complete the description of these projective covers (injective hulls). However, it is easy to check that these morphisms are unique, up to a scaling factor for each projective (injective) indecomposable appearing in the cover (hull).

5.3.3 Their projective and injective presentations

This section establishes projective and injective presentations of the B_k^l and T_k^l. More precisely, we determine the kernel (cokernel) of the projection (inclusion) from each of these modules to its projective cover (injective hull). This will require precise relations between these families and the indecomposable projectives and injectives. Define then, for each non-critical $k \neq k_L, k_R$, the $\mathcal{A}$-modules A_{k^-} and V_{k^-} to be $\mathsf{P}_k / \operatorname{soc} \mathsf{P}_k$ and $\operatorname{rad} \mathsf{P}_k$, respectively. These definitions immediately yield the following short exact sequences

$$0 \longrightarrow \mathsf{I}_k \longrightarrow \mathsf{P}_k \longrightarrow \mathsf{A}_{k^-} \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{V}_{k^-} \longrightarrow \mathsf{P}_k \longrightarrow \mathsf{I}_k \longrightarrow 0, \tag{5.3.9}$$

because $\operatorname{soc} \mathsf{P}_k \simeq \mathsf{P}_k / \operatorname{rad} \mathsf{P}_k \simeq \mathsf{I}_k$ for all non-critical k with $k \neq k_L$.

Lemma 5.3.4. *For $k \neq k_L$, k_R, we have the following non-split short exact sequences:*

$$0 \longrightarrow \mathsf{I}_{k^-} \longrightarrow \mathsf{A}_{k^-} \longrightarrow \mathsf{S}_k \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{S}_{k^-} \longrightarrow \mathsf{V}_{k^-} \longrightarrow \mathsf{I}_{k^+} \longrightarrow 0, \tag{5.3.10a}$$

$$0 \longrightarrow \mathsf{I}_{k^+} \longrightarrow \mathsf{A}_{k^-} \longrightarrow \mathsf{C}_{k^-} \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{C}_k \longrightarrow \mathsf{V}_{k^-} \longrightarrow \mathsf{I}_{k^-} \longrightarrow 0. \tag{5.3.10b}$$

In particular, we have the identifications $\mathsf{A}_{k^-} \simeq \mathsf{B}^2_{k^-}$ and $\mathsf{V}_{k^-} \simeq \mathsf{T}^2_{k^-}$.

Proof. We first prove that the sequence

$$0 \longrightarrow \mathsf{I}_{k^-} \longrightarrow \mathsf{A}_{k^-} \longrightarrow \mathsf{S}_k \longrightarrow 0 \tag{5.3.11}$$

is exact. The top row of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{I}_k & \xrightarrow{\ \iota\ } & \mathsf{S}_{k^-} & \xrightarrow{\ \pi\ } & \mathsf{I}_{k^-} & \longrightarrow & 0 \\ & & \Big\downarrow \mathrm{id} & & \Big\downarrow i & & \phi \Big\downarrow \Big\uparrow \bar{\phi} & & \\ 0 & \longrightarrow & \mathsf{I}_k & \xrightarrow{\ \bar{\iota}\ } & \mathsf{P}_k & \xrightarrow{\ \bar{\pi}\ } & \mathsf{A}_{k^-} & \longrightarrow & 0 \end{array} \tag{5.3.12}$$

is the exact sequence (5.2.2). Let i denote the inclusion of (5.2.3), so that $\bar{\iota} \equiv i\iota$ is injective. The left square thus commutes and we may choose $\bar{\pi}$ so that the bottom row is exact, because of $\operatorname{Hom}(\mathsf{I}_k, \mathsf{P}_k) \simeq \mathbb{C}$ (Theorem 5.2.17) and Equation (5.3.9). Because $\bar{\pi} i \iota = \bar{\pi} \bar{\iota} = 0$, one may now define ϕ so that the right square commutes. The snake lemma then gives

$$\ker i = 0 \longrightarrow \ker \phi \longrightarrow \operatorname{Coker} \mathrm{id} = 0 \longrightarrow \operatorname{Coker} i = \mathsf{S}_k \longrightarrow \operatorname{Coker} \phi \longrightarrow 0, \tag{5.3.13}$$

hence $\ker \phi \simeq 0$ and $\operatorname{Coker} \phi \simeq \mathsf{S}_k$, which settles the exactness of (5.3.11).

We next show that (5.3.11) does not split. If it did, then there would exist $\bar{\phi} : \mathsf{A}_{k^-} \to \mathsf{I}_{k^-}$ such that $\bar{\phi}\phi = \mathrm{id}$ on I_{k^-}. But then $\bar{\phi}\bar{\pi}$ cannot be zero (both are surjective), contradicting $\operatorname{Hom}(\mathsf{P}_k, \mathsf{I}_{k^-}) = 0$ (Theorem 5.2.17). As non-split extensions of S_k by I_{k^-} are unique up to isomorphism (Theorem 5.2.18), it follows from the definitions in Section 5.3.1 that $\mathsf{A}_{k^-} \simeq \mathsf{B}^2_{k^-}$.

The method is easily adapted to prove the remaining short exact sequences. The only conceptual difference for the two in (5.3.10b) is that we use the duals of (5.2.2) and (5.2.3), remembering that duality is exact contravariant (Theorem 5.2.13), and that the dual of the first sequence becomes the bottom row of the diagram rather than the top. ■

Before stating the main result of this section, Theorem 5.3.6, we need another lemma. It indicates that the two recursively defined families, the B_k^{2j} and B_k^{2j+1}, are intimately related by proving the exactness of two sequences. The first sequence shows that the first family may be constructed recursively by extending a costandard module (as in the definition given for the second family in Section 5.3.1). The second sequence then shows that the first family may be constructed by extending members of the second family by an irreducible.

Lemma 5.3.5. *For $j \geq 1$ such that $k^{2j} \in \Lambda$, the following short sequences are exact:*

$$0 \longrightarrow \mathsf{B}_{k^{++}}^{2(j-1)} \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{C}_k \longrightarrow 0 \qquad \textit{and} \qquad 0 \longrightarrow \mathsf{I}_{k^{2j}} \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{B}_k^{2j-1} \longrightarrow 0. \tag{5.3.14}$$

Proof. We consider the exactness of the first of the two sequences, proceeding by induction on j. The first sequence of (5.3.10b) establishes the case $j = 1$. Assume then the exactness for j and suppose that $k^{2(j+1)} \in \Lambda$. In the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathsf{B}_{k^{++}}^{2(j-1)} & \xrightarrow{\gamma} & \mathsf{B}_k^{2j} & \xrightarrow{\overline{\gamma}} & \mathsf{C}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & {\scriptstyle \delta}\downarrow\uparrow{\scriptstyle \delta^*} & & \\
0 & \longrightarrow & \mathsf{B}_{k^{++}}^{2(j-1)} & \xrightarrow{\beta} & \mathsf{B}_k^{2(j+1)} & \xrightarrow{\overline{\beta}} & \operatorname{coker}\beta & \longrightarrow & 0\,, \\
 & & & & \downarrow{\scriptstyle \overline{\alpha}} & & & & \\
 & & & & \mathsf{S}_{k^{2j+1}} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{5.3.15}$$

the first row is thus assumed exact. We moreover define $\beta = \alpha\gamma$, so that the left square commutes, and then $\overline{\beta}$, so that the second row is exact. Finally, the second column is exact, by the definition (5.3.4) of $\mathsf{B}_k^{2(j+1)}$. This setup guarantees that there exists a morphism δ making the right square commute. The snake lemma then gives $\ker\delta = 0$ and $\operatorname{coker}\delta \simeq \mathsf{S}_{k^{2j+1}}$, proving that $0 \to \mathsf{C}_k \to \operatorname{coker}\beta \to \mathsf{S}_{k^{2j+1}} \to 0$ is exact. Since $\operatorname{Ext}(\mathsf{S}_{k^{2j+1}}, \mathsf{C}_k) = 0$ (Theorem 5.2.18), this sequence splits and there exists a surjection $\delta^* \colon \operatorname{coker}\beta \to \mathsf{C}_k$ such that $\delta^*\delta$ is the identity on C_k.

Consider now the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{B}_{k^{++}}^{2(j-1)} & \xrightarrow{\gamma} & \mathsf{B}_{k}^{2j} & \xrightarrow{\overline{\gamma}} & \mathsf{C}_k & \longrightarrow & 0 \\ & & \phi \,\vdots & & \alpha \big\downarrow & & \big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \ker f & \longrightarrow & \mathsf{B}_{k}^{2(j+1)} & \xrightarrow{f} & \mathsf{C}_k & \longrightarrow & 0 \end{array} , \tag{5.3.16}$$

in which $f = \delta^* \overline{\beta}$. Both rows are thus exact and the right square commutes: $f\alpha = \delta^*\overline{\beta}\alpha = \delta^*\delta\overline{\gamma} = \overline{\gamma}$. It now follows from $f\alpha\gamma = \gamma\overline{\gamma} = 0$ that there is a morphism ϕ making the left square commute. The snake lemma then gives the exactness of $0 \to \mathsf{B}_{k^{++}}^{2(j-1)} \to \ker f \to \mathsf{S}_{k^{2j+1}} \to 0$, as before. Theorem 5.2.21 and $\mathrm{Ext}(\mathsf{S}_{k^{2j+1}}, \mathsf{B}_{k^{++}}^{2(j-1)}) = \mathbb{C}$ (Section 5.3.1) now imply that there are only two possibilities: either this sequence is non-split, in which case it gives $\ker f \simeq \mathsf{B}_{k^{++}}^{2j}$, by the definition of the latter, or it splits and $\ker f \simeq \mathsf{B}_{k^{++}}^{2(j-1)} \oplus \mathsf{S}_{k^{2j+1}}$. However, if $\ker f \simeq \mathsf{B}_{k^{++}}^{2(j-1)} \oplus \mathsf{S}_{k^{2j+1}}$, then $\mathsf{S}_{k^{2j+1}}$ would be a submodule of $\ker f$ and thus also of $\mathsf{B}_{k}^{2(j+1)}$. It would then follow that $\mathsf{B}_{k}^{2(j+1)}$ is the sum of two submodules, B_{k}^{2j} and $\mathsf{S}_{k^{2j+1}}$, whose intersection is zero, contradicting its indecomposability. We conclude that $\ker f \simeq \mathsf{B}_{k^{++}}^{2j}$, hence that the second row of (5.3.16) is the desired exact sequence.

The proof of the exactness of the second sequence proceeds in a similar inductive fashion. As the two sequences of (5.3.14) coincide for $j = 1$, the base case has already been established. Assuming the exactness for $j-1$, we consider the following diagram, similar to (5.3.15):

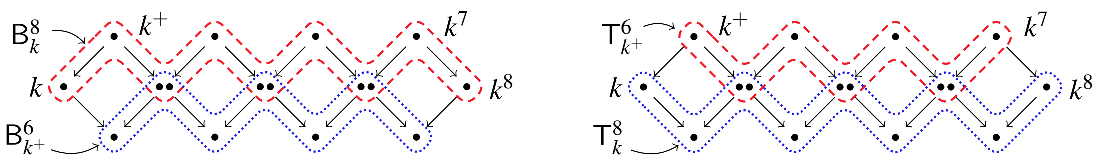
$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \big\downarrow & & & & \\ 0 & \longrightarrow & \mathsf{I}_{k^{2j}} & \xrightarrow{\gamma} & \mathsf{B}_{k^{++}}^{2(j-1)} & \xrightarrow{\overline{\gamma}} & \mathsf{B}_{k^{++}}^{2j-3} & \longrightarrow & 0 \\ & & \mathrm{id} \big\downarrow & & \alpha \big\downarrow & & \vdots\, \delta & & \\ 0 & \longrightarrow & \mathsf{I}_{k^{2j}} & \xrightarrow{\beta} & \mathsf{B}_{k}^{2j} & \longrightarrow & \operatorname{coker} \beta & \longrightarrow & 0\,. \\ & & & & \overline{\alpha} \big\downarrow & & & & \\ & & & & \mathsf{C}_k & & & & \\ & & & & \big\downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{5.3.17}$$

The first row is exact by assumption and the second column is the first sequence of (5.3.14) (whose exactness has just been proved). Since the left square commutes, there exists a morphism δ that makes the right square commute. The snake lemma computes its kernel and cokernel which give the exact sequence $0 \to \mathsf{B}_{k^{++}}^{2j-3} \to \operatorname{coker} \beta \to \mathsf{C}_k \to 0$. This time, Theorem 5.2.21 and $\mathrm{Ext}(\mathsf{C}_k, \mathsf{B}_{k^{++}}^{2j-3}) = \mathbb{C}$ (Section 5.3.1) show that either $\operatorname{coker} \beta \simeq \mathsf{B}_{k}^{2j-1}$, by the definition of the latter, or $\operatorname{coker} \beta$ splits as $\mathsf{C}_k \oplus \mathsf{B}_{k^{++}}^{2j-3}$. Now, $\operatorname{coker} \beta$ splitting would entail the existence of a non-zero morphism from C_k to $\operatorname{coker} \beta$. However, Theorems 5.2.17 and 5.2.18 imply that the second row of (5.3.17) yields the following exact sequence:

$$0 = \mathsf{Hom}(\mathsf{C}_k, \mathsf{I}_{k^{2j}}) \longrightarrow \mathsf{Hom}(\mathsf{C}_k, \mathsf{B}_k^{2j}) \longrightarrow \mathsf{Hom}(\mathsf{C}_k, \operatorname{coker}\beta) \longrightarrow \mathsf{Ext}(\mathsf{C}_k, \mathsf{I}_{k^{2j}}) = 0. \tag{5.3.18}$$

Thus, $\mathsf{Hom}(\mathsf{C}_k, \mathsf{B}_k^{2j}) \simeq \mathsf{Hom}(\mathsf{C}_k, \operatorname{coker}\beta)$ could not be 0. But, any non-zero map of $\mathsf{Hom}(\mathsf{C}_k, \mathsf{B}_k^{2j})$ must be injective since $\operatorname{hd}\mathsf{C}_k \simeq \mathsf{I}_{k^+}$ is not a submodule of B_k^{2j} (Theorem 5.3.2). Thus, $\operatorname{coker}\beta$ splitting would imply that the indecomposable B_k^{2j} is the sum of two submodules, C_k and $\mathsf{B}_{k^{++}}^{2(j-1)}$, whose intersection is zero. This contradiction means that $\operatorname{coker}\beta \simeq \mathsf{B}_k^{2j-1}$, hence that the second row of (5.3.17) is the desired exact sequence. ■

With these lemmas in hand, we now turn to projective (injective) presentations. More specifically, we compute the kernels (cokernels) of the projections (inclusions) that define the projective covers (injective hulls) of the modules B_k^l and T_k^l. As usual, Loewy diagrams provide an intuitive description of the result to come.



(5.3.19)

The diagram on the left depicts the inclusion of $\mathsf{B}_{k^+}^6$ in its injective hull $\mathsf{J}[\mathsf{B}_{k^+}^6]$, the one on the right that of T_k^8 in $\mathsf{J}[\mathsf{T}_k^8]$. These injective hulls are direct sums of the indecomposable injectives $\mathsf{J}_{k'}$, with indices increasing in steps of 2, each bringing (generically) four composition factors to the hull. Some are repeated, for example the composition factor $\mathsf{I}_{k^{++}}$ has multiplicity 2 in both $\mathsf{J}[\mathsf{B}_{k^+}^6]$ and $\mathsf{J}[\mathsf{T}_k^8]$, and we indicate this above by drawing two dots close together. The images of the inclusion maps are depicted by dotted lines. Where it passes through a pair of "double dots", this image will contain a proper subspace, equivalent to one dot, of the subspace represented by these dots. The cokernels of these inclusions are depicted by dashed lines with the same proviso regarding their passing through double dots. We note that the cokernel $\mathsf{J}[\mathsf{B}_{k^+}^6]/\mathsf{B}_{k^+}^6$ has the same composition factors as B_k^8 and that $\mathsf{J}[\mathsf{T}_k^8]/\mathsf{T}_k^8$ has the same composition factors as $\mathsf{T}_{k^+}^6$.

Proposition 5.3.6. *For $k \in \Lambda_0$ and $k^l \in \Lambda$, let f_k^l and g_k^l denote the natural inclusions $\mathsf{B}_k^l \hookrightarrow \mathsf{J}[\mathsf{B}_k^l]$ and $\mathsf{T}_k^l \hookrightarrow \mathsf{J}[\mathsf{T}_k^l]$, respectively. With a few exceptions, the cokernels of these inclusions are*

$$\begin{aligned} \operatorname{Coker} f_k^{2j} &\simeq \mathsf{B}_{k^{2\delta_L-1}}^{2(j+1-\delta_L)-\delta_R^{2j}}, & \operatorname{Coker} g_k^{2j} &\simeq \mathsf{T}_{k^+}^{2(j-1)}, \\ \operatorname{Coker} f_k^{2j+1} &\simeq \mathsf{B}_{k^{2\delta_L-1}}^{2(j-\delta_L)+1}, & \operatorname{Coker} g_k^{2j+1} &\simeq \mathsf{T}_{k^+}^{2j+1-\delta_R^{2j+1}}, \end{aligned} \tag{5.3.20a}$$

where $\delta_L \equiv \delta_{k,k_L}$ and $\delta_R^l \equiv \delta_{k^l,k_R}$. The exceptions occur for $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = 0$, specifically

$$\operatorname{Coker} f_2^{2j} \simeq \mathsf{T}_2^{2j+1-\delta_R^{2j}}, \qquad \operatorname{Coker} f_2^{2j+1} \simeq \mathsf{T}_2^{2j}. \tag{5.3.20b}$$

Proof. We prove, by induction on j, the result for $\operatorname{Coker} f_k^{2j} \simeq \mathsf{J}[\mathsf{B}_k^{2j}]/\mathsf{B}_k^{2j}$, ignoring the exceptional cases at first. When $j=0$, the goal is to identify the cokernel in

$$0 \longrightarrow \mathsf{I}_k \xrightarrow{f_k^0} \mathsf{J}_k \longrightarrow \operatorname{Coker} f_k^0 \longrightarrow 0. \tag{5.3.21}$$

If $k=k_L$, then $\mathsf{J}_k \simeq \mathsf{C}_k$ and $\operatorname{Coker} f_k^0 \simeq \mathsf{I}_{k^+} = \mathsf{B}^0_{k^+}$, by (5.2.28). If $k_L < k < k_R$, then $\mathsf{J}_k \simeq \mathsf{P}_k$ and we obtain $\operatorname{Coker} f_k^0 \simeq \mathsf{B}^2_{k^-}$ from (5.3.9) and Theorem 5.3.4. If $k = k_R$, then $\mathsf{J}_k \simeq \mathsf{P}_k$ again, but $\mathsf{I}_k \simeq \mathsf{C}_k$, thus (5.2.30) gives $\operatorname{Coker} f_k^0 \simeq \mathsf{C}_{k^-} = \mathsf{B}^1_{k^-}$. Note that if $k_L = k = k_R$, then $\mathsf{B}^0_{k^+} = \mathsf{B}^1_{k^-} = 0$ which is the correct cokernel (f_k^0 is an isomorphism in this case). It is easy to check that the result in (5.3.20a) for $\operatorname{Coker} f_k^0$ unifies all of these cases. The exceptional case $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = 0$ uses instead the exact sequence (5.2.41) to conclude that $\operatorname{Coker} f_2^0 \simeq \mathsf{S}_2 \simeq \mathsf{T}_2^{1-\delta_R^0}$, if $n>2$, and $\mathsf{I}_2 \simeq \mathsf{T}^0_2$, if $n=2$ (see also the Loewy diagrams (5.2.49c)).

Suppose now that $j \geq 1$. The top row of the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{B}_k^{2(j-1)} & \xrightarrow{\ a\ } & \mathsf{B}_k^{2j} & \xrightarrow{\ \alpha\ } & \mathsf{S}_{k^{2j-1}} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f_k^{2(j-1)}} & & \big\downarrow{\scriptstyle f_k^{2j}} & & \big\downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \mathsf{J}[\mathsf{B}_k^{2(j-1)}] & \overset{b}{\dashrightarrow} & \mathsf{J}[\mathsf{B}_k^{2j}] & \overset{\beta}{\dashrightarrow} & \mathsf{J}_{k^{2j}} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle h_k^{2(j-1)}} & & \big\downarrow{\scriptstyle h_k^{2j}} & & \big\downarrow{\scriptstyle \pi} & & \\
0 & \rightarrow & \operatorname{Coker} f_k^{2(j-1)} & \overset{c}{\dashrightarrow} & \operatorname{Coker} f_k^{2j} & \overset{\gamma}{\dashrightarrow} & \mathsf{S}_{k^{2j}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.3.22}$$

is the exact sequence (5.3.4) defining B_k^{2j}, the two leftmost columns describe the inclusions of the appropriate B_k^l into their injective hulls (the projections h_k^l being chosen to make these columns exact), and the rightmost column is the exact sequence (5.2.3) (note that $k^{2j} \neq k_L$). Given $\operatorname{coker} f_k^{2(j-1)}$, the goal is to identify $\operatorname{coker} f_k^{2j}$.

Because $f_k^{2(j-1)}$ is injective, the injectivity of $\mathsf{J}[\mathsf{B}_k^{2j}]$ ensures that there exists b making the top-left square of (5.3.22) commute. If b were not injective, then it would annihilate some composition factor in $\operatorname{soc} \mathsf{J}[\mathsf{B}_k^{2(j-1)}] \simeq \operatorname{soc} \mathsf{B}_k^{2(j-1)}$. But then, $f_k^{2j} a$ would annihilate this factor in $\mathsf{B}_k^{2(j-1)}$, contradicting the injectivity of the latter morphism. Thus, b is injective. Similarly, the injectivity of f_k^{2j} and $\mathsf{J}_{k^{2j}}$ ensures that there exists β making the top-right square commute. If β were not surjective, then it must annihilate the composition factor of $\operatorname{soc} \mathsf{J}[\mathsf{B}_k^{2j}]$ that is isomorphic to $\operatorname{hd} \mathsf{J}_{k^{2j}} \simeq \mathsf{I}_{k^{2j}}$. As $\iota\alpha$ does not annihilate this factor in B_k^{2j}, this is a contradiction, hence β is surjective. Finally, we conclude that the middle row of the diagram (5.3.22) is exact by comparing composition factors.

The snake lemma now gives the exactness of the bottom row by defining morphisms c and γ that make the bottom two squares commute. If the bottom row of the commutative diagram (5.3.22) does not split, then $\operatorname{Coker} f_k^{2j}$ will be a non-trivial extension of $\mathsf{S}_{k^{2j}}$ by $\operatorname{Coker} f_k^{2(j-1)}$. By the induction hypothesis, this latter cokernel will be isomorphic to $\mathsf{B}^{2(j-\delta_L)}_{k^{2\delta_L-1}}$ (as $k^{2(j-1)} \neq k_R$), so the bottom row will give $\operatorname{coker} f_k^{2j} = \mathsf{B}^{2(j+1-\delta_L)-\delta_R^{2j}}_{k^{2\delta_L-1}}$, by definition, the correction $-\delta_R^{2j}$ being necessary when $k^{2j} = k_R$, hence $\mathsf{S}_{k^{2j}} \simeq \mathsf{I}_{k^{2j}}$. It thus remains to prove that the bottom row of (5.3.22) does not split.

Suppose then that the bottom row does split, so that there exists an injection $\gamma^* \colon \mathsf{S}_{k^{2j}} \to \operatorname{Coker} f_k^{2j}$ such that $\gamma\gamma^*$ is the identity on $\mathsf{S}_{k^{2j}}$. Moreover, this splitting means that

$$\mathsf{Hom}(\mathsf{J}_{k^{2j}}, \operatorname{Coker} f_k^{2j}) \simeq \mathsf{Hom}(\mathsf{J}_{k^{2j}}, \operatorname{Coker} f_k^{2(j-1)}) \oplus \mathsf{Hom}(\mathsf{J}_{k^{2j}}, \mathsf{S}_{k^{2j}}) \simeq \mathbb{C}, \tag{5.3.23}$$

by Theorem 5.2.17 and the induction hypothesis ($\operatorname{Coker} f_k^{2(j-1)} \simeq \mathsf{B}_{k^{2\delta_L-1}}^{2(j-\delta_L)}$). Since the modules in the middle row are all injective, this row splits and there exists an injection $\beta^* \colon \mathsf{J}_{k^{2j}} \to \mathsf{J}[\mathsf{B}_k^{2j}]$ such that $\beta\beta^*$ is the identity on $\mathsf{J}_{k^{2j}}$. If $h_k^{2j}\beta^*$ were identically zero, then we would have $\mathsf{J}_{k^{2j}} \simeq \operatorname{im}\beta^* \subseteq \ker h_k^{2j} = \operatorname{im} f_k^{2j} \simeq \mathsf{B}_k^{2j}$. However, $\mathsf{J}_{k^{2j}}$ has two composition factors isomorphic to $\mathsf{I}_{k^{2j}}$ whereas B_k^{2j} has but one, a contradiction. It follows that $h_k^{2j}\beta^* \neq 0$.

As $\gamma^*\pi \neq 0$, by the surjectivity of π, (5.3.23) shows that $\gamma^*\pi$ and $h_k^{2j}\beta^*$ are equal, up to some non-zero multiplicative constant. In particular, $h_k^{2j}\beta^*\iota = 0$, so $\operatorname{im}(\beta^*\iota) \subseteq \operatorname{im} f_k^{2j}$ and there exists a morphism $\alpha^* \colon \mathsf{S}_{k^{2j-1}} \to \mathsf{B}_k^{2j}$ such that $f_k^{2j}\alpha^* = \beta^*\iota$. But now, $\iota = \beta\beta^*\iota = \beta f_k^{2j}\alpha^* = \iota\alpha\alpha^*$ and, since ι is injective, $\alpha\alpha^*$ is the identity on $\mathsf{S}_{k^{2j-1}}$. The top row therefore splits, contradicting the indecomposability of B_k^{2j}. This contradiction shows that the bottom row is not split, completing the identification of $\operatorname{coker} f_k^{2j}$.

This identification of $\operatorname{coker} f_k^{2j}$ proceeds inductively until k^{2j} approaches k_R. It is easy to check that the above argument requires no significant changes if $k^{2j} = k_R$; however, changes are required if $k^{2j-1} = k_R$. In the latter case, the recursive construction of Section 5.3.1 produces B_k^{2j-1} from $\mathsf{B}_k^{2(j-1)}$ and so the first row of (5.3.22) has to be replaced by the defining exact sequence $0 \to \mathsf{B}_k^{2(j-1)} \to \mathsf{B}_k^{2j-1} \to \mathsf{I}_{k^{2j-1}} \to 0$. Unfortunately, the first two injective hulls of the second row then become $\mathsf{J}[\mathsf{B}_k^{2(j-1)}]$ and $\mathsf{J}[\mathsf{B}_k^{2j-1}]$, which are isomorphic by Theorem 5.3.3. We can still deduce that there exists an injection, hence an isomorphism, b making the top left square commute. However, to make the second row of (5.3.22) exact, we must replace $\mathsf{J}_{k^{2j}}$ by 0. Applying the snake lemma to these replaced rows yields the morphism c that makes the bottom left square commute and the short exact sequence $0 \to \mathsf{I}_{k^{2j-1}} \to \operatorname{coker} f_k^{2(j-1)} \to \operatorname{coker} f_k^{2j-1} \to 0$. But, induction identifies $\operatorname{coker} f_k^{2(j-1)}$ as $\mathsf{B}_{k^{2\delta_L-1}}^{2(j-\delta_L)}$. As this module has a unique submodule isomorphic to $\mathsf{I}_{k^{2j-1}}$, comparing this exact sequence with the second of Theorem 5.3.5 yields the desired conclusion: $\operatorname{coker} f_k^{2j-1} \simeq \mathsf{B}_{k^{2\delta_L-1}}^{2(j-\delta_L)-1}$.

Similar arguments and duality identify the other cokernels in (5.3.20). We remark that the proof for $\operatorname{coker} f_k^{2j+1}$ is somewhat easier because the restriction put on k in the definition of the remaining B_k^{2j+1} avoids the technicalities that would arise should k^{2j+1} approach k_R. ■

These cokernels give injective presentations for the B_k^l, for example $0 \to \mathsf{B}_k^{2j} \to \mathsf{J}[\mathsf{B}_k^{2j}] \to \mathsf{B}_{k^{2\delta_L-1}}^{2(j+1-\delta_L)-\delta_R^{2j}} \to 0$. Analogous projective presentations now follow by taking duals.

Proposition 5.3.7. *For $k \in \Lambda_0$ and $k^l \in \Lambda$, let p_k^l and q_k^l denote the natural projections $\mathsf{P}[\mathsf{B}_k^l] \twoheadrightarrow \mathsf{B}_k^l$ and $\mathsf{P}[\mathsf{T}_k^l] \twoheadrightarrow \mathsf{T}_k^l$, respectively. With a few exceptions, the kernels of these projections are*

$$\begin{aligned} \ker p_k^{2j} &\simeq \mathsf{B}_{k^+}^{2(j-1)}, & \ker q_k^{2j} &\simeq \mathsf{T}_{k^{2\delta_L-1}}^{2(j+1-\delta_L)-\delta_R^{2j}}, \\ \ker p_k^{2j+1} &\simeq \mathsf{B}_{k^+}^{2j+1-\delta_R^{2j+1}}, & \ker q_k^{2j+1} &\simeq \mathsf{T}_{k^{2\delta_L-1}}^{2(j-\delta_L)+1}, \end{aligned} \tag{5.3.24a}$$

in the notation of Theorem 5.3.6. The exceptions occur for $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = 0$, specifically

$$\ker q_2^{2j} \simeq \mathsf{B}_2^{2j+1-\delta_R^{2j}}, \qquad \ker q_2^{2j+1} \simeq \mathsf{B}_2^{2j}. \tag{5.3.24b}$$

5.3.4 Their extension groups with irreducible modules

Theorem 5.2.20(ii) has shown that computing $\operatorname{Ext}(\mathsf{M}, \mathsf{I}_k)$ and $\operatorname{Ext}(\mathsf{I}_k, \mathsf{M})$ is sufficient to see when a module M can appear in any non-trivial extension. We shall therefore limit ourselves to these extension groups. With the presentations derived in the previous section, it is easy to compute them.

Proposition 5.3.8. *Let $k, k' \in \Lambda_0$. The extension groups of the B_k^l, for $l \geq 2$, with an irreducible are given by*

$$\begin{aligned} &\text{(a)} \quad \operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{B}_k^{2j}) \simeq \bigoplus_{i=0}^{j+1} \delta_{k',k^{2i-1}} \mathbb{C}, \qquad &&\text{(c)} \quad \operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{B}_k^{2j+1}) \simeq \bigoplus_{i=0}^{j} \delta_{k',k^{2i-1}} \mathbb{C}, \\ &\text{(b)} \quad \operatorname{Ext}(\mathsf{B}_k^{2j}, \mathsf{I}_{k'}) \simeq \bigoplus_{i=1}^{j-1} \delta_{k',k^{2i}} \mathbb{C}, \qquad &&\text{(d)} \quad \operatorname{Ext}(\mathsf{B}_k^{2j+1}, \mathsf{I}_{k'}) \simeq \bigoplus_{i=1}^{j+1} \delta_{k',k^{2i}} \mathbb{C}. \end{aligned} \tag{5.3.25}$$

Those of the T_k^l with an irreducible are then given by $\operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{T}_k^l) \simeq \operatorname{Ext}(\mathsf{B}_k^l, \mathsf{I}_{k'})$ and $\operatorname{Ext}(\mathsf{T}_k^l, \mathsf{I}_{k'}) \simeq \operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{B}_k^l)$. The extension groups for $l = 0$ and 1 were given in Theorem 5.2.18.

Proof. The relationship between the extension groups involving the B_k^l and the T_k^l is just duality. The computations for each of the four cases above are very similar, so we shall only present the details for (a), ignoring the exceptional case where $\mathcal{A} = \mathsf{TL}$ with n even, $\beta = 0$ and $k = 2$ (the results for these exceptional cases are also as given in (5.3.25) above).

Recall that Theorem 5.3.6 gives an injective presentation of B_k^{2j}:

$$0 \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{J}[\mathsf{B}_k^{2j}] \longrightarrow \mathsf{B}_{k^{2\delta_L - 1}}^{2(j+1-\delta_L)-\delta_R^{2j}} \longrightarrow 0. \tag{5.3.26}$$

This gives the long exact sequence

$$0 \longrightarrow \mathsf{Hom}(\mathsf{I}_{k'}, \mathsf{B}_k^{2j}) \longrightarrow \mathsf{Hom}(\mathsf{I}_{k'}, \mathsf{J}[\mathsf{B}_k^{2j}]) \longrightarrow \mathsf{Hom}(\mathsf{I}_{k'}, \mathsf{B}_{k^{2\delta_L-1}}^{2(j+1-\delta_L)-\delta_R^{2j}}) \longrightarrow \operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{B}_k^{2j}) \longrightarrow 0, \tag{5.3.27}$$

where we recall that $\operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{J}[\mathsf{B}_k^{2j}]) = 0$, by injectivity. Now, $\mathsf{Hom}(\mathsf{I}_{k'}, \mathsf{M}) \simeq \mathsf{Hom}(\mathsf{I}_{k'}, \operatorname{soc} \mathsf{M})$, for any module M, and $\operatorname{soc} \mathsf{B}_k^{2j} \simeq \operatorname{soc} \mathsf{J}[\mathsf{B}_k^{2j}]$. The two leftmost Hom-groups of (5.3.27) are therefore isomorphic, hence we have

$$\operatorname{Ext}(\mathsf{I}_{k'}, \mathsf{B}_k^{2j}) \simeq \mathsf{Hom}(\mathsf{I}_{k'}, \mathsf{B}_{k^{2\delta_L-1}}^{2(j+1-\delta_L)-\delta_R^{2j}}) \simeq \mathsf{Hom}(\mathsf{I}_{k'}, \operatorname{soc} \mathsf{B}_{k^{2\delta_L-1}}^{2(j+1-\delta_L)-\delta_R^{2j}}) \simeq \bigoplus_{i=0}^{j+1-\delta_L-\delta_R^{2j}} \delta_{k',k^{2i-1+2\delta_L}} \mathbb{C}, \tag{5.3.28}$$

by Theorems 5.2.17 and 5.3.2. It is easy to check that the condition $k' \in \Lambda_0$ allows this to be simplified to the statement of (a). The proof of (c) follows the same argument whilst (b) and (d) instead use projective presentations. ∎

5.3.5 Identification of non-trivial extensions and completeness

Theorem 5.3.8 shows that the modules B_k^l and T_k^l have non-trivial extensions with irreducibles. The next proposition shows that, in fact, every one of these non-trivial extensions is isomorphic to one (or a direct sum) of the modules that have already been introduced. This will then imply that we have identified a complete list of indecomposable $\mathcal{A}$-modules.

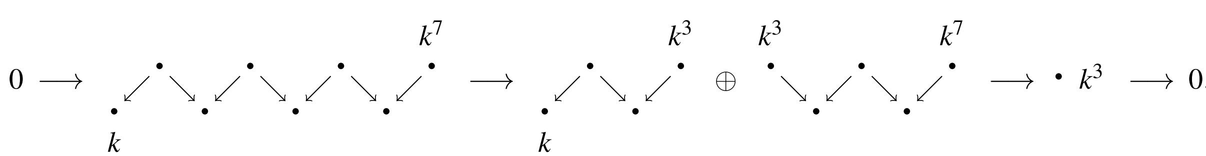
Proposition 5.3.9. *Let $k, k' \in \Lambda_0$ and $k^{2j} \in \Lambda$. The following short exact sequences identify a representative, unique up to isomorphism, of the non-trivial extensions of the B_k^l and $\mathsf{I}_{k'}$ defined by* (a), (b), (c) *and* (d) *of Theorem 5.3.8:*

$$\begin{array}{llr}
\text{(a1)} & 0 \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{B}_k^{2i-1} \oplus \mathsf{T}_{k'}^{2(j-i)+1} \longrightarrow \mathsf{I}_{k'} \longrightarrow 0, & \text{for } k' = k^{2i-1},\ 1 \le i \le j, \\
\text{(a2)} & 0 \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{T}_{k'}^{2j+1} \longrightarrow \mathsf{I}_{k'} \longrightarrow 0, & \text{for } k' = k^-, \\
\text{(a3)} & 0 \longrightarrow \mathsf{B}_k^{2j} \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow \mathsf{I}_{k'} \longrightarrow 0, & \text{for } k' = k^{2j+1}, \\
\text{(b)} & 0 \longrightarrow \mathsf{I}_{k'} \longrightarrow \mathsf{B}_k^{2i} \oplus \mathsf{B}_{k'}^{2(j-i)} \longrightarrow \mathsf{B}_k^{2j} \longrightarrow 0, & \text{for } k' = k^{2i},\ 1 \le i \le j-1, \\
\text{(c1)} & 0 \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow \mathsf{B}_k^{2i-1} \oplus \mathsf{T}_{k'}^{2(j-i+1)} \longrightarrow \mathsf{I}_{k'} \longrightarrow 0, & \text{for } k' = k^{2i-1},\ 1 \le i \le j, \\
\text{(c2)} & 0 \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow \mathsf{T}_{k'}^{2(j+1)} \longrightarrow \mathsf{I}_{k'} \longrightarrow 0, & \text{for } k' = k^-, \\
\text{(d1)} & 0 \longrightarrow \mathsf{I}_{k'} \longrightarrow \mathsf{B}_k^{2i} \oplus \mathsf{B}_{k'}^{2(j-i)+1} \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow 0, & \text{for } k' = k^{2i},\ 1 \le i \le j, \\
\text{(d2)} & 0 \longrightarrow \mathsf{I}_{k'} \longrightarrow \mathsf{B}_k^{2(j+1)} \longrightarrow \mathsf{B}_k^{2j+1} \longrightarrow 0, & \text{for } k' = k^{2(j+1)}.
\end{array} \tag{5.3.29}$$

Dualising these sequences gives representatives for the non-trivial extensions of the T_k^l and $\mathsf{I}_{k'}$.

We remark that we have already established (d2) as the second exact sequence of (5.3.14).

Before turning to the proof, we exemplify one of these exact sequences with Loewy diagrams. Consider (c1), with $i = 2$, $j = 3$ and $k' = k^3$. It takes the form $0 \to \mathsf{B}_k^7 \to \mathsf{B}_k^3 \oplus \mathsf{T}_{k^3}^4 \to \mathsf{I}_{k^3} \to 0$ and may be depicted thus:



(5.3.30)

The composition factors are depicted by dots, as usual, and only the extreme ones are labelled. It is easy to see that morphisms from B_k^7 to B_k^3 and $\mathsf{T}_{k^3}^4$ exist (the latter are quotients of the former). If we can show that the direct sum $\mathsf{B}_k^7 \to \mathsf{B}_k^3 \oplus \mathsf{T}_{k^3}^4$ of these morphisms is injective, then its cokernel must be isomorphic to the irreducible I_{k^3}, by counting composition factors. There cannot be a non-zero morphism $\mathsf{I}_{k^3} \to \mathsf{B}_k^3 \oplus \mathsf{T}_{k^3}^4$ since $\operatorname{soc}(\mathsf{B}_k^3 \oplus \mathsf{T}_{k^3}^4)$ has no submodule isomorphic to I_{k^3}. The sequence (5.3.30) is therefore non-split and $\mathsf{B}_k^3 \oplus \mathsf{T}_{k^3}^4$ represents the (isomorphism class of the) non-trivial extensions of $\operatorname{Ext}(\mathsf{I}_{k^3}, \mathsf{B}_k^7) \simeq \mathbb{C}$, even though it is reducible.

Proof of Theorem 5.3.9. Each of these sequences are established using similar arguments, though the proofs differ slightly depending on whether the irreducible $\mathsf{I}_{k'}$ is a submodule or a quotient. We will illustrate the method of proof by detailing the arguments for (b), where the irreducible is a submodule, and only comment on the changes required when the irreducible is a quotient.

To prove (b), take $1 \leq i \leq j-1$ and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{B}_{k^+}^{2(j-1)} & \xrightarrow{r} & \mathsf{P}[\mathsf{B}_k^{2j}] & \xrightarrow{p} & \mathsf{B}_k^{2j} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathsf{I}_{k^{2i}} & \xrightarrow{\iota} & \mathsf{B}_k^{2i} \oplus \mathsf{B}_{k^{2i}}^{2(j-i)} & \xrightarrow{\pi} & \operatorname{Coker}\iota & \longrightarrow & 0, \end{array} \tag{5.3.31}$$

in which the top row is the projective presentation of B_k^{2j}, given in Theorem 5.3.7, and is therefore exact. As $\mathsf{I}_{k^{2i}}$ is a composition factor of $\operatorname{hd}\mathsf{B}_{k^+}^{2(j-1)}$, we may choose α to be non-zero, hence surjective. Since $\mathsf{B}_k^{2i} \oplus \mathsf{B}_{k^{2i}}^{2(j-i)}$ and B_k^{2j} have isomorphic heads, their projective covers are isomorphic. We may therefore choose β to also be surjective in (5.3.31). Indeed, we may choose the kernel so that

$$\mathsf{B}_{k^+}^{2(i-1)} \oplus \mathsf{B}_{k^{2i+1}}^{k^{2(j-i-1)}} \simeq \ker\beta \subset \ker p \simeq \mathsf{B}_{k^+}^{2(j-1)} \tag{5.3.32}$$

and the kernels only differ in that $\ker p$ has one composition factor isomorphic to $\mathsf{I}_{k^{2i}}$ while $\ker\beta$ has none.

As $\operatorname{soc}\mathsf{B}_k^{2i} \oplus \mathsf{B}_{k^{2i}}^{2(j-i)}$ has two composition factors isomorphic to $\mathsf{I}_{k^{2i}}$, the inclusion ι belongs to a two-dimensional Hom-group. We will choose ι so that the left square of (5.3.31) commutes. To see that this is possible, note that $\ker\beta \subset \ker p = \operatorname{im} r$, so that $\ker\beta r = \operatorname{im} r \cap \ker\beta = \ker\beta \simeq \mathsf{B}_{k^+}^{2(i-1)} \oplus \mathsf{B}_{k^{2i+1}}^{k^{2(j-i-1)}}$, hence

$$\operatorname{im}\beta r \simeq \frac{\mathsf{B}_{k^+}^{2(j-1)}}{\mathsf{B}_{k^+}^{2(i-1)} \oplus \mathsf{B}_{k^{2i+1}}^{k^{2(j-i-1)}}} \simeq \mathsf{I}_{k^{2i}}. \tag{5.3.33}$$

It follows that there exists ι making the left square commute. π can now be chosen to make the bottom row of (5.3.31) exact. Since $\pi\beta r = \pi\iota\alpha = 0$, there is a map γ that makes the right square of (5.3.31) commute. It is surjective, as both β and π are, hence it is an isomorphism because the composition factors of B_k^{2j} and $\operatorname{Coker}\iota$ coincide. The bottom row is thus the required short exact sequence (b).

When the irreducible $\mathsf{I}_{k'}$ appearing in a sequence in (5.3.29) is a quotient, instead of a submodule, there is a minor change to the method of proof. The diagram (5.3.31) is replaced by one in which the bottom row is an injective presentation of the module that is extending $\mathsf{I}_{k'}$ and the top row is the sequence whose exactness is to be established. For example the proof of the dual of (d2) would use the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \xrightarrow{\iota} & \mathsf{T}_k^{2(j+1)} & \xrightarrow{\pi} & \mathsf{I}_{k^{2(j+1)}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \mathsf{T}_k^{2j+1} & \xrightarrow{f} & \mathsf{J}[\mathsf{T}_k^{2j+1}] & \xrightarrow{h} & \mathsf{T}_{k^+}^{2j+1} & \longrightarrow & 0. \end{array} \tag{5.3.34}$$

Otherwise, one proves the commutativity of the diagram as before. ■

Theorem 5.3.10. *Let $\mathcal{A}$ be TL_n or dTL_n. Then, any finite-dimensional indecomposable module over $\mathcal{A}$ is isomorphic to one of the following:*

(i) S_k*, for k a critical integer in Λ;*

(ii) P_k*, for $k \in \Lambda$ non-critical and larger than k_L in the orbit $[k]$;*

(iii) B_k^l *or* $\mathsf{T}_k^{l'}$*, for k non-critical in Λ_0 and $l \geq 0$ and $l' > 0$ such that k^l, $k^{l'} \in \Lambda$.*

These indecomposables are distinct in that there are no isomorphisms among different elements of the above list.

Proof. The modules appearing in the above list have already been shown to be indecomposable and pairwise non-isomorphic. To show that the list given is complete, note that any finite-dimensional indecomposable module M may be constructed iteratively by adding one composition factor at a time. Indeed, one could start from the head of M, which is semisimple, and then add the composition factors of the head of its radical, and then add those of the head of the radical of the radical, and so on. Therefore, every finite-dimensional indecomposable $\mathcal{A}$-module may be constructed by adding, one at time, irreducible modules to a direct sum of modules in the above list. However, Theorems 5.2.18 and 5.3.9 show that any such extension of these modules is already a direct sum of modules in the list. This list therefore constitutes a complete set of finite-dimensional indecomposable $\mathcal{A}$-modules, up to isomorphism.

The above list is also complete for the exceptional case $\mathcal{A} = \mathsf{TL}$ with n even and $\beta = 0$. Two remarks are useful to reach this conclusion in this case. First, the sublist (i) is empty. Second, k_L is omitted from (ii) not to avoid coincidence with $\mathsf{B}_{k_L}^1$, as in the generic case, but because k_L is then 0 and does not belong to Λ_0 (P_0 is not defined). The leftmost irreducible module of the orbit is then I_2 and its projective cover P_2 is distinct from B_2^1. ■

5.4 The restriction and induction of the modules $\mathsf{C}_{n,k}$, $\mathsf{B}_{n,k}^l$ and $\mathsf{T}_{n,k}^l$

The action of the induction and restriction functors on the standard, irreducible and projective modules was obtained (or recalled) in Section 5.2.2. This section extends those calculations to the remaining classes of indecomposable modules, namely the costandards C_k and the B_k^l and T_k^l, with $l > 1$, thus completing the description of these functors on all indecomposable $\mathcal{A}$-modules. In this section, the algebra label n will be made explicit, so we shall write, for example, $\mathsf{B}_{n,k}^j$ instead of B_k^j. We also write $\Lambda_{n,0}$ for the set Λ_0 corresponding to $\mathcal{A}_n$.

5.4.1 Restriction

We begin with the restriction of the $\mathsf{C}_{n+1,k}$. This follows immediately from Theorem 5.2.8 and the fact that restriction commutes with duality.

Proposition 5.4.1. *For all non-critical $k \in \Lambda_{n+1,0}$, the restriction of the costandard modules is given by*

$$\mathsf{C}_{n+1,k}\!\downarrow \simeq \begin{cases} \mathsf{C}_{n,k-1} \oplus \mathsf{C}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{TL}_n, \\ \mathsf{C}_{n,k-1} \oplus \mathsf{C}_{n,k} \oplus \mathsf{C}_{n,k+1}, & \text{if } \mathcal{A}_n = \mathsf{dTL}_n. \end{cases} \tag{5.4.1}$$

For critical k, $\mathsf{C}_{n+1,k} \simeq \mathsf{S}_{n+1,k}$ and the result was given in Theorem 5.2.8(iii).

Identifying the restrictions of the $\mathsf{B}^l_{n+1,k}$ requires considerably more work.

Proposition 5.4.2. *For k non-critical and $k, k^l \in \Lambda_{n+1,0}$, the restriction of $\mathsf{B}^l_{n+1,k}$ is given by*

$$\begin{aligned} \mathsf{B}^l_{n+1,k}\!\downarrow \simeq & \left\{ \begin{matrix} \displaystyle\bigoplus_{j=0}^{\lfloor l/2 \rfloor} \mathsf{P}_{n,k^{2j}-1}, & \text{if } k-1 \text{ is critical,} \\ \mathsf{B}^l_{n,k-1}, & \text{otherwise} \end{matrix} \right\} \oplus \left\{ \begin{matrix} \mathsf{B}^l_{n,k}, & \text{if } \mathcal{A} = \mathsf{dTL}, \\ 0, & \text{otherwise} \end{matrix} \right\} \\ & \oplus \left\{ \begin{matrix} \displaystyle\bigoplus_{j=0}^{\lfloor (l-1)/2 \rfloor} \mathsf{P}_{n,k^{2j}+1}, & \text{if } k+1 \text{ is critical,} \\ \mathsf{B}^l_{n,k+1}, & \text{otherwise} \end{matrix} \right\}, \end{aligned} \tag{5.4.2}$$

where it is understood that each summand of the form $\mathsf{B}^l_{n,\kappa}$ should be replaced by $\mathsf{B}^{l-1}_{n,\kappa}$ whenever $\kappa^l \notin \Lambda_{n,0}$.

We remark that the projectives appearing in (5.4.2) are all critical, hence irreducible.

Proof. The proofs for l even and l odd are (slightly) different; we first detail that for l even and then explain how the arguments may be changed for l odd.

The proof for l even proceeds by induction on l. The case $l = 0$ is covered by Theorem 5.2.9, so assume that $l \geq 2$. Since restriction is an exact covariant functor, (5.3.1) gives the exact sequence

$$0 \longrightarrow \mathsf{B}^{l-2}_{n+1,k}\!\downarrow \longrightarrow \mathsf{B}^l_{n+1,k}\!\downarrow \longrightarrow \mathsf{S}_{n+1,k^{l-1}}\!\downarrow \longrightarrow 0. \tag{5.4.3}$$

As in the proof of Theorem 5.2.10, this sequence can be decomposed into two or three exact sequences, according as to whether $\mathcal{A} = \mathsf{TL}$ or dTL, respectively, by selecting a parity for the direct summands of the modules (for dTL) and distinguishing their F_n-eigenvalues. These sequences are analysed using similar arguments, so we shall only focus on one of them, namely

$$0 \longrightarrow (\mathsf{B}^{l-2}_{n+1,k}\!\downarrow)_{+1} \longrightarrow (\mathsf{B}^l_{n+1,k}\!\downarrow)_{+1} \longrightarrow \mathsf{S}_{n,k^{l-1}-1} \longrightarrow 0, \tag{5.4.4}$$

where $(\mathsf{M})_i$, for $i \in \{-1, 0, +1\}$, is the direct summand of M on which F_n has the same eigenvalue as on $\mathsf{S}_{n,k+i}$ (see the proof of Theorem 5.2.10 where this notation was first introduced). This choice of F_n-eigenvalue will lead to the third direct summand (enclosed in braces) of (5.4.2).

The diagram below illustrates the argument. It assumes that $\ell = 4$ and describes the restriction of the TL_{n+1}-module $\mathsf{B}^4_{n+1,2}$, hence $k = 2$ and $l = 4$. The indices of the composition factors of this module are typeset in bold and appear in the bottom line; those of $\mathsf{B}^4_{n+1,2}\!\downarrow$ appear in the top line.

	1		3		5		7		9		11		13		15		17		19		$\cdots$	
0		**2**		**4**		6		8		**10**		**12**		14		16		**18**		20		$\cdots$
		k		k^+						k^{l-2}		k^{l-1}						k^l				

We first discuss the subcase in which $k+1$ is critical (as in the diagram). Then so is $k^{l-1}-1 = k^{l-2}+1$, hence the standard module $\mathsf{S}_{n,k^{l-1}-1}$ is projective and (5.4.4) must split. The induction hypothesis therefore gives

$$(\mathsf{B}^l_{n+1,k}\!\downarrow)_{+1} \simeq \mathsf{S}_{n,k^{l-1}-1} \oplus (\mathsf{B}^{l-2}_{n+1,k}\!\downarrow)_{+1} \simeq \mathsf{P}_{n,k^{l-2}+1} \oplus \bigoplus_{j=0}^{l/2-2} \mathsf{P}_{n,k^{2j}+1} = \bigoplus_{j=0}^{l/2-1} \mathsf{P}_{n,k^{2j}+1}, \tag{5.4.5}$$

as in (5.4.2). If $k+1$ is not critical, then the sequence (5.4.4) cannot split because

$$\mathsf{Hom}(\mathsf{S}_{n,k^{l-1}-1}, \mathsf{B}^l_{n+1,k}\!\downarrow) \overset{(1)}{\simeq} \mathsf{Hom}(\mathsf{S}_{n,k^{l-1}-1}\!\uparrow, \mathsf{B}^l_{n+1,k}) \overset{(2)}{\simeq} \mathsf{Hom}(\mathsf{S}_{n+1,k^{l-1}}, \mathsf{B}^l_{n+1,k}) \overset{(3)}{=} 0. \tag{5.4.6}$$

Here, the isomorphism (1) is Frobenius reciprocity, (2) follows from Theorem 5.2.8 and the eigenvalues of F_n, and (3) amounts to noting that any such non-zero morphism would have to be injective, contradicting the fact that $\mathsf{B}^l_{n+1,k}$ has no submodule isomorphic to $\mathsf{S}_{n+1,k^{l-1}}$. (If it did, then the inclusion would split the defining exact sequence (5.3.1) and $\mathsf{B}^l_{n,k}$ would be decomposable.) The induction hypothesis gives $(\mathsf{B}^{l-2}_{n+1,k}\!\downarrow)_{+1} \simeq \mathsf{B}^{l-2}_{n,k+1}$, so the non-split exact sequence (5.4.4) is then that defining $\mathsf{B}^l_{n,k+1}$, whence we conclude that $(\mathsf{B}^l_{n+1,k}\!\downarrow)_{+1} \simeq \mathsf{B}^l_{n,k+1}$. Note that this assumes that $(k+1)^l \leq n$, for otherwise (5.4.4) gives instead$(\mathsf{B}^l_{n+1,k}\!\downarrow)_{+1} \simeq \mathsf{B}^{l-1}_{n,k+1}$, by Theorem 5.3.9. This completes the identification of $(\mathsf{B}^l_{n+1,k}\!\downarrow)_{+1}$ and similar arguments identify $(\mathsf{B}^l_{n+1,k}\!\downarrow)_0$ and $(\mathsf{B}^l_{n+1,k}\!\downarrow)_{-1}$, completing the proof for l even.

When l is odd, the induction instead starts at $l=1$, for which the statement is obtained from Theorem 5.2.8 by noting that

$$\mathsf{C}_{n+1,k}\!\downarrow \simeq \big(\mathsf{S}^\vee_{n+1,k}\big)\!\downarrow \simeq \big(\mathsf{S}_{n+1,k}\!\downarrow\big)^\vee, \tag{5.4.7}$$

since restriction and duality commute. In the inductive step, the sequence $0 \to \mathsf{B}^{l-2}_{n+1,k^2} \to \mathsf{B}^l_{n+1,k} \to \mathsf{C}_{n+1,k} \to 0$ is used instead of (5.3.1) (see Theorem 5.3.1) and the rest of the arguments proceed as before. ∎

The restrictions of the $\mathsf{T}^l_{n+1,k}$ are now obtained by duality.

Proposition 5.4.3. *For k non-critical and $k, k^l \in \Lambda_{n+1,0}$, the restriction of $\mathsf{T}^l_{n+1,k}$ is given by*

$$\mathsf{T}^l_{n+1,k}\!\downarrow \simeq \begin{Bmatrix} \bigoplus_{j=0}^{\lfloor l/2 \rfloor} \mathsf{P}_{n,k^{2j}-1}, & \text{if } k-1 \text{ is critical,} \\ \mathsf{T}^l_{n,k-1}, & \text{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \mathsf{T}^l_{n,k}, & \text{if } \mathcal{A} = \mathsf{dTL}, \\ 0, & \text{otherwise} \end{Bmatrix}$$

$$\oplus \left\{ \begin{matrix} \bigoplus_{j=0}^{\lfloor (l-1)/2 \rfloor} \mathsf{P}_{n,k^{2j}+1}, & \text{if } k+1 \text{ is critical,} \\ \mathsf{T}^l_{n,k+1}, & \text{otherwise} \end{matrix} \right\}, \tag{5.4.8}$$

where it is understood that each summand of the form $\mathsf{T}^l_{n,\kappa}$ *should be replaced by* $\mathsf{T}^{l-1}_{n,\kappa}$ *whenever* $\kappa^l \notin \Lambda_{n,0}$.

5.4.2 Interlude: Tor-groups

The method used to compute these restrictions, as well as those of Section 5.2.2, relies on the fact that the restriction functor is exact, so restricting each module in a short exact sequence results in another short exact sequence. But, as we saw at the end of Section 5.2.2, the induction functor is only right-exact, meaning that identifying induced modules will require more sophisticated arguments. We also note that induction does not commute with duality, in general, hence Theorem 5.2.8 does not immediately identify, for instance, the induced costandard modules.

Inducing a $\mathcal{A}_n$-module M to an $\mathcal{A}_{n+1}$-module amounts to taking the tensor product $\mathcal{A}_{n+1} \otimes_{\mathcal{A}_n} \mathsf{M}$, where $\mathcal{A}_{n+1}$ is regarded as a left $\mathcal{A}_{n+1}$-module and a right $\mathcal{A}_n$-module. Just as the failure of Hom-functors to be exact is measured by extension groups, the failure of tensor products to be exact is measured by *torsion groups*. In particular, a short exact sequence

$$0 \longrightarrow \mathsf{N}'' \longrightarrow \mathsf{N} \longrightarrow \mathsf{N}' \longrightarrow 0 \tag{5.4.9}$$

of left $\mathcal{A}_n$-modules gives rise, upon induction, to the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}(\mathcal{A}_{n+1}\!\downarrow, \mathsf{N}'') \longrightarrow \operatorname{Tor}(\mathcal{A}_{n+1}\!\downarrow, \mathsf{N}) \longrightarrow \operatorname{Tor}(\mathcal{A}_{n+1}\!\downarrow, \mathsf{N}') \longrightarrow \mathsf{N}''\!\uparrow \longrightarrow \mathsf{N}\!\uparrow \longrightarrow \mathsf{N}'\!\uparrow \longrightarrow 0, \tag{5.4.10}$$

in which $\mathcal{A}_{n+1}$ is viewed as a right $\mathcal{A}_n$-module. As with the Hom-Ext long exact sequences (5.2.32a) and (5.2.32b), this sequence continues with higher torsion groups $\operatorname{Tor}_m(\mathcal{A}_{n+1}\!\downarrow, -)$. Because we have no need for these higher groups, we omit the index m and write $\operatorname{Tor}_1 \equiv \operatorname{Tor}$, for brevity. We remark that as $\mathcal{A}_{n+1}$ is also a left $\mathcal{A}_{n+1}$-module, each of the torsion groups in (5.4.10) is also a left $\mathcal{A}_{n+1}$-module (as are the induced modules).

The following two facts about Tor-groups will be essential. First, Tor-groups are trivial when either of their arguments is flat. A projective module P is always flat, so $\operatorname{Tor}(\mathsf{P}, -) \simeq \operatorname{Tor}(-, \mathsf{P}) \simeq 0$. Second, if M is a finite-dimensional right module and N a left one over the same finite-dimensional algebra $\mathcal{A}$ (over a field), then the Tor-groups and Ext-groups are related by [1, Cor. IX.4.12]

$$\operatorname{Tor}(\mathsf{M}, \mathsf{N}) \simeq \operatorname{Ext}(\mathsf{M}, \mathsf{N}^*)^* \simeq \operatorname{Ext}(\mathsf{N}, \mathsf{M}^*), \tag{5.4.11}$$

where M^* denotes the vector space dual module of M (see Section 5.2.3). [9]

9. With no other hypotheses, this is an isomorphism of vector spaces (over the field), hence the additional dual on the right-hand side is superfluous. However, in the case (5.4.10) of interest, M is also a left module over a different

Now, $\mathcal{A}_{n+1}$ is free, hence projective, as a left $\mathcal{A}_{n+1}$-module. However, it might not be projective as a right $\mathcal{A}_n$-module. If it is, then the torsion groups in (5.4.10) vanish and the induction functor is exact. To analyse this, note that the left and right representation theories of TL_n and dTL_n are identical, [10] so we may combine Theorems 5.2.8 and 5.2.10 with the (left $\mathcal{A}_n$-module) decomposition

$$\mathcal{A}_{n+1}\!\downarrow = \bigoplus_{k\in\Lambda_{n+1,0}} (\dim \mathsf{I}_{n+1,k})\,\mathsf{P}_{n+1,k}\!\downarrow \tag{5.4.12}$$

to explore the projectivity of the left $\mathcal{A}_n$-module $\mathcal{A}_{n+1}$ (the answer will be the same as a right $\mathcal{A}_n$-module). Introducing the symbol $\overset{\mathsf{P}}{\simeq}$ to indicate an isomorphism up to projective direct summands, the results of this exploration may be summarised by noting that $\mathsf{P}_{n+1,k}\!\downarrow$ is always projective for $k<n$ and is otherwise given by

$$\mathsf{P}_{n+1,n}\!\downarrow \overset{\mathsf{P}}{\simeq} \begin{cases} 0, & \text{if } n+1 \text{ is critical,} \\ \mathsf{I}_{n,(n+1)^-}, & \text{otherwise,} \end{cases} \qquad \mathsf{P}_{n+1,n+1}\!\downarrow \overset{\mathsf{P}}{\simeq} \begin{cases} \mathsf{I}_{n,n}, & \text{if } n+1 \text{ is critical,} \\ \mathsf{I}_{n,(n+1)^-}, & \text{if } n+2 \text{ is critical,} \\ \mathsf{I}_{n,(n+1)^-} \oplus \mathsf{I}_{n,(n+2)^-}, & \text{otherwise.} \end{cases} \tag{5.4.13}$$

Here, we of course ignore modules whose indices have different parities if $\mathcal{A} = \mathsf{TL}$. Since $\dim \mathsf{I}_{n+1,n} = n+1$ and $\dim \mathsf{I}_{n+1,n+1} = 1$, (5.4.12) becomes

$$\mathcal{A}_{n+1}\!\downarrow \overset{\mathsf{P}}{\simeq} \begin{cases} \mathsf{I}_{n,n}, & \text{if } n+1 \text{ is critical,} \\ (n+2)\,\mathsf{I}_{n,(n+1)^-}, & \text{if } n+2 \text{ is critical,} \\ (n+2)\,\mathsf{I}_{n,(n+1)^-} \oplus \mathsf{I}_{n,(n+2)^-}, & \text{otherwise.} \end{cases} \tag{5.4.14}$$

Thus, $\mathsf{dTL}_{n+1}\!\downarrow$ is projective as a dTL_n-module, hence the induction functor for dTL_n-modules is exact, if and only if dTL_n is semisimple. On the other hand, this holds for $\mathsf{TL}_{n+1}\!\downarrow$ if and only if TL_{n+1} is semisimple *or* $n+2$ is critical.

In any case, the key observation to take away from these computations is that whenever $\mathcal{A}_{n+1}\!\downarrow$ has non-projective summands, they are irreducibles $\mathsf{I}_{n,k}$ with $k = k_R$. This observation will be crucial when we identify the inductions of the $\mathsf{C}_{n,k}$, B^l_k and T^l_k, a task to which we now turn.

5.4.3 Induction

We begin with the inductions of the costandard modules, recalling our convention that any module with an index k not in $\Lambda_{n+1,0}$ is set to zero, as are those, for $\mathcal{A}_{n+1} = \mathsf{TL}_{n+1}$, whose indices $n+1$ and k have different parities.

algebra $\mathcal{B}$. This means that $\mathrm{Tor}(\mathsf{M},\mathsf{N})$ is naturally a left $\mathcal{B}$-module, whilst $\mathrm{Ext}(\mathsf{M},\mathsf{N}^*)$ is a right $\mathcal{B}$-module, whence the required extra dual.

10. This is clear from the diagrammatic definitions of these algebras.

Proposition 5.4.4. *If k is non-critical and $k,k^+ \in \Lambda_{n,0}$, then the induction of $\mathsf{C}_{n,k}$ is given by*

$$\mathsf{C}_{n,k}\uparrow \simeq \begin{Bmatrix} \mathsf{P}_{n+1,k-1}, & \text{if } k-1 \text{ is critical,} \\ \mathsf{B}^2_{n+1,k-1}, & \text{if } k-1 \text{ is non-critical and } k^{++}=n+1 \text{ or } n+2, \\ \mathsf{C}_{n+1,k-1}, & \text{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \mathsf{B}^2_{n+1,k}, & \text{if } k^{++}=n+1, \\ \mathsf{C}_{n+1,k}, & \text{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \mathsf{P}_{n+1,k+1}, & \text{if } k+1 \text{ is critical,} \\ \mathsf{C}_{n+1,k+1}, & \text{otherwise} \end{Bmatrix}. \tag{5.4.15}$$

Proof. The long exact sequence (5.4.10) derived from the exact sequence (5.2.28) is

$$\cdots \longrightarrow \operatorname{Tor}(\mathcal{A}_{n+1}\downarrow, \mathsf{I}_{n,k}) \longrightarrow \operatorname{Tor}(\mathcal{A}_{n+1}\downarrow, \mathsf{C}_{n,k}) \longrightarrow \operatorname{Tor}(\mathcal{A}_{n+1}\downarrow, \mathsf{I}_{n,k^+}) \longrightarrow \mathsf{I}_{n,k}\uparrow \longrightarrow \mathsf{C}_{n,k}\uparrow \longrightarrow \mathsf{I}_{n,k^+}\uparrow \longrightarrow 0. \tag{5.4.16}$$

Since Tor-groups involving projective modules vanish, we may replace $\mathcal{A}_{n+1}\downarrow$ by the right-hand side of (5.4.14), or rather the right module version of it, when calculating the Tor-groups in this sequence. Let I^* denote the right module version, that is the vector space dual, of the right-hand side of (5.4.14). Then, (5.4.11) gives $\operatorname{Tor}(\mathcal{A}_{n+1}\downarrow, \mathsf{M}) \simeq \operatorname{Ext}(\mathsf{M}, \mathsf{I})$. We therefore have to compute $\operatorname{Ext}(\mathsf{I}_{n,k}, \mathsf{I})$, $\operatorname{Ext}(\mathsf{C}_{n,k}, \mathsf{I})$ and $\operatorname{Ext}(\mathsf{I}_{n,k^+}, \mathsf{I})$.

Recall from Section 5.4.2 that I is a direct sum of irreducibles whose indices are always the rightmost in their orbit. Consulting Theorem 5.2.18, we see that $\operatorname{Ext}(\mathsf{I}_{n,k^+}, \mathsf{I})$ may only be non-zero if $k = k_R^{--}$. But then, $\operatorname{Ext}(\mathsf{I}_{n,k}, \mathsf{I}) = 0$ and $\operatorname{Ext}(\mathsf{C}_{n,k}, \mathsf{I}) \simeq \operatorname{Ext}(\mathsf{I}_{n,k^+}, \mathsf{I}) \simeq \mathsf{E}$ (say). Thus, if $k \neq k_R^{--}$, then (5.4.16) reduces to the short exact sequence

$$0 \longrightarrow \mathsf{I}_{n,k}\uparrow \longrightarrow \mathsf{C}_{n,k}\uparrow \longrightarrow \mathsf{I}_{n,k^+}\uparrow \longrightarrow 0. \tag{5.4.17}$$

However, if $k = k_R^{--}$, then (5.4.16) reduces to

$$0 \longrightarrow \mathsf{E} \longrightarrow \mathsf{E} \longrightarrow \mathsf{I}_{n,k}\uparrow \longrightarrow \mathsf{C}_{n,k}\uparrow \longrightarrow \mathsf{I}_{n,k^+}\uparrow \longrightarrow 0, \tag{5.4.18}$$

which also implies the exactness of (5.4.17).

From this point on, the proof follows familiar arguments. The exact sequence (5.4.17) is decomposed into (two or) three exact sequences corresponding to the eigenvalues of the central element F_n. Here is an example. Suppose first that neither $k-1$ nor $k+1$ are critical and that $k^+ \neq k_R$, so that $\mathsf{S}_{n,k^+} \not\simeq \mathsf{I}_{n,k^+}$. Then, all three short exact sequences will have the same form, namely

$$0 \longrightarrow \mathsf{I}_{n+1,k+i} \longrightarrow (\mathsf{C}_{n,k}\uparrow)_i \longrightarrow \mathsf{I}_{n+1,k^+-i} \longrightarrow 0 \qquad (i = 0, \pm 1), \tag{5.4.19}$$

by Theorem 5.2.9. None of these sequences can split because

$$\operatorname{Hom}(\mathsf{C}_{n,k}\uparrow, \mathsf{I}_{n+1,k+i}) \simeq \operatorname{Hom}(\mathsf{C}_{n,k}, \mathsf{I}_{n+1,k+i}\downarrow) \simeq \operatorname{Hom}(\mathsf{C}_{n,k}, \mathsf{I}_{n,k+i-1} \oplus \mathsf{I}_{n,k+i} \oplus \mathsf{I}_{n,k+i+1}) \simeq 0, \tag{5.4.20}$$

by Frobenius reciprocity and Theorems 5.2.9 and 5.2.17. Theorems 5.2.18 and 5.2.19 now give the conclusion: $(\mathsf{C}_{n,k}\uparrow)_i \simeq \mathsf{C}_{n,k+i}$, hence $\mathsf{C}_{n,k}\uparrow \simeq \mathsf{C}_{n,k-1} \oplus \mathsf{C}_{n,k} \oplus \mathsf{C}_{n,k+1}$.

When either $k-1$ or $k+1$ is critical, but still $k^+ \neq k_R$, some of the irreducibles in the three exact sequences (5.4.19) are critical, hence projective, and some are replaced by 0, according to Theorem 5.2.9. The results are as before, except that $(\mathsf{C}_{n,k}\uparrow)_{-1} \simeq \mathsf{P}_{n+1,k-1}$ or $(\mathsf{C}_{n,k}\uparrow)_{+1} \simeq \mathsf{P}_{n+1,k^+-1} = \mathsf{P}_{n+1,k+1}$, respectively.

Finally, the case $k^+ = k_R$ leads to the novel summands $\mathsf{B}^2_{n+1,k'}$ in (5.4.15). Theorem 5.2.9 may be used only when $\mathsf{I}_{n,k^+} \not\simeq \mathsf{S}_{n,k^+}$; otherwise, Theorem 5.2.8 applies instead and $\mathsf{I}_{n,k^+}\uparrow \simeq \mathsf{S}_{n+1,k^+-1} \oplus \mathsf{S}_{n+1,k^+} \oplus \mathsf{S}_{n+1,k^++1}$. The question is whether these standard modules are irreducible or not, for if so, then the analysis proceeds as above. Now, S_{n+1,k^++1} is only reducible when $k^+ + 1$ is non-critical and $(k^+ + 1)^+ = k^{++} - 1 \in \Lambda_{n+1,0}$, that is, when $k^{++} = n+1$ or $n+2$. Similarly, S_{n+1,k^+} is only reducible when $k^{++} = n+1$ and S_{n+1,k^+-1} is never reducible. In these few cases, some of the three short exact sequences, obtained by decomposing (5.4.17), describe non-split extensions of an irreducible by a reducible standard. Theorems 5.2.18, 5.2.21 and 5.3.1 identify the extensions as being isomorphic to $\mathsf{B}^2_{n+1,k-1}$ or $\mathsf{B}^2_{n+1,k}$, completing the proof. ∎

Proposition 5.4.5. *Let k be non-critical with $k, k^l \in \Lambda_{n,0}$. If l is even, then the induction of $\mathsf{B}^l_{n,k}$ is given by*

$$\mathsf{B}^l_{n,k}\uparrow \simeq \mathsf{B}^l_{n+2,k}\downarrow. \tag{5.4.21a}$$

If $l = 2i+1$ is odd, then the induction is instead given by

$$\begin{aligned}\mathsf{B}^l_{n,k}\uparrow \simeq &\begin{Bmatrix} \bigoplus_{j=0}^{i} \mathsf{P}_{n+1,k^{2j}-1}, & \text{if } k-1 \text{ is critical,} \\ \mathsf{B}^{l+1}_{n+1,k-1}, & \text{if } k-1 \text{ is non-critical and } k^{l+1} = n+1 \text{ or } n+2, \\ \mathsf{B}^l_{n+1,k-1}, & \text{otherwise} \end{Bmatrix} \\ &\oplus \begin{Bmatrix} \mathsf{B}^{l+1}_{n+1,k}, & \text{if } k^{l+1} = n+1, \\ \mathsf{B}^l_{n+1,k}, & \text{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \bigoplus_{j=0}^{i-1} \mathsf{P}_{n+1,k^{2j}+1}, & \text{if } k+1 \text{ is critical,} \\ \mathsf{B}^l_{n+1,k+1}, & \text{otherwise} \end{Bmatrix}. \end{aligned} \tag{5.4.21b}$$

Proof. The proof is by induction on l, distinguishing the two parities. If $l = 0$, then $\mathsf{B}^0_{n,k} = \mathsf{I}_{n,k}$ and the result was given in Theorems 5.2.8 and 5.2.9. If $l = 1$, then $\mathsf{B}^1_{n,k} = \mathsf{C}_{n,k}$ and the result was given in Theorem 5.4.4.

So, let $l \geq 2$ be an even integer. The long exact sequence obtained by inducing the defining sequence

$$0 \longrightarrow \mathsf{B}^{l-2}_{n,k} \longrightarrow \mathsf{B}^l_{n,k} \longrightarrow \mathsf{S}_{n,k^{l-1}} \longrightarrow 0 \tag{5.4.22}$$

of Theorem 5.3.1 has $\mathrm{Tor}(\mathcal{A}_{n+1}\downarrow, \mathsf{S}_{n,k^{l-1}}) = 0$. Indeed, this torsion group is isomorphic to $\mathrm{Ext}(\mathsf{S}_{n,k^{l-1}}, \mathsf{I})$, where I is a direct sum of non-critical irreducibles whose indices have the form k'_R, and $\mathrm{Ext}(\mathsf{S}_{n,k^{l-1}}, \mathsf{I}_{k'_R})$ is always zero, by Theorem 5.2.18. We therefore arrive at the short exact sequence

$$0 \longrightarrow \mathsf{B}^{l-2}_{n,k}\uparrow \longrightarrow \mathsf{B}^l_{n,k}\uparrow \longrightarrow \mathsf{S}_{n,k^{l-1}}\uparrow \longrightarrow 0. \tag{5.4.23}$$

The submodule here is known, by the induction hypothesis, and the quotient is known by Theorem 5.2.8.

The identification of $\mathsf{B}^l_{n,k}\!\uparrow$ now follows the same arguments as in the proof of Theorem 5.4.4. In particular, (5.4.23) is decomposed into (at most) three exact sequences whose nature, split or non-split, is established through the presence of projectives or by calculating Hom-groups. For example, if $k-1$ and k are not critical, then

$$\mathsf{Hom}(\mathsf{B}^l_{n,k}\!\uparrow, \mathsf{B}^{l-2}_{n+1,k-1}) \simeq \mathsf{Hom}(\mathsf{B}^l_{n,k}, \mathsf{B}^{l-2}_{n+1,k-1}\!\downarrow) \simeq \mathsf{Hom}(\mathsf{B}^l_{n,k}, \mathsf{B}^{l-2}_{n,k}) = 0, \tag{5.4.24}$$

as otherwise $\mathsf{B}^l_{n,k}$ would be decomposable. Theorems 5.2.21 and 5.3.1 then complete the identification.

One case with l odd also fits into this induction argument, that with $k^l = k_R$. Then, (5.4.22) is replaced by the defining exact sequence

$$0 \longrightarrow \mathsf{B}^{l-1}_{n,k} \longrightarrow \mathsf{B}^l_{n,k} \longrightarrow \mathsf{S}_{n,k^l} \longrightarrow 0 \tag{5.4.25}$$

and the subsequent analysis follows similar lines to that performed for $\mathsf{C}_{n,k}\!\uparrow$ (for these values of k). This case leads to the result for $\mathsf{B}^l_{n,k}\!\uparrow$, when $k^{l+1} = n+1$ or $n+2$, in (5.4.21b).

The induction process for $l = 2i+1 \geq 3$ odd starts with the computation of the Tor-groups related to the defining sequence of $\mathsf{B}^{2i+1}_{n,k}$:

$$0 \longrightarrow \mathsf{B}^{2i-1}_{n,k^{++}} \longrightarrow \mathsf{B}^{2i+1}_{n,k} \longrightarrow \mathsf{C}_{n,k} \longrightarrow 0. \tag{5.4.26}$$

Theorem 5.2.20 indicates that each $\mathrm{Ext}(\mathsf{C}_{n,k}, \mathsf{I}_{k'_R})$ is always zero because the composition factors of the two modules are sufficiently separated in their orbit (for example, k^{++} stands between them). The long exact sequence thus reduces to

$$0 \longrightarrow \mathsf{B}^{2i-1}_{n,k}\!\uparrow \longrightarrow \mathsf{B}^{2i+1}_{n,k}\!\uparrow \longrightarrow \mathsf{C}_{n,k}\!\uparrow \longrightarrow 0 \tag{5.4.27}$$

and, from this point on, the proof closely follows that for l even. ∎

Our final induction result does not require any new techniques, so we omit the proof.

Proposition 5.4.6. *Let k be non-critical with $k, k^l \in \Lambda_{n,0}$. If l is odd, then the induction of $\mathsf{T}^l_{n,k}$ is given by*

$$\mathsf{T}^l_{n,k}\!\uparrow \simeq \mathsf{T}^l_{n+2,k}\!\downarrow. \tag{5.4.28a}$$

If $l = 2i$ is even, then the induction is instead given by

$$\mathsf{T}^l_{n,k}\!\uparrow \simeq \left\{ \begin{array}{ll} \bigoplus_{j=0}^{i} \mathsf{P}_{n+1,k^{2j}-1}, & \text{if } k-1 \text{ is critical,} \\ \mathsf{T}^l_{n+1,k-1}, & \text{otherwise} \end{array} \right\} \oplus \left\{ \begin{array}{ll} \mathsf{T}^{l+1}_{n+1,k}, & \text{if } k^{l+1} = n+1, \\ \mathsf{T}^l_{n+1,k}, & \text{otherwise} \end{array} \right\}$$
$$\oplus \left\{ \begin{array}{ll} \bigoplus_{j=0}^{i-1} \mathsf{P}_{n+1,k^{2j}+1}, & \text{if } k+1 \text{ is critical,} \\ \mathsf{T}^{l+1}_{n+1,k+1}, & \text{if } k+1 \text{ is non-critical and } k^{l+1} = n+1 \text{ or } n+2, \\ \mathsf{T}^l_{n+1,k+1}, & \text{otherwise} \end{array} \right\}. \tag{5.4.28b}$$

5.5 The Auslander-Reiten quiver for TL_n and dTL_n

Section 5.3.5 completed the classification of all indecomposable modules, up to isomorphism, over $\mathcal{A} = \mathsf{TL}_n$ and dTL_n. The tools required included basic homological algebra and the representation theory of associative algebras: properties of injective and projective modules, extension groups and diagram chasing. The input, summed up in Section 5.2, was the list of irreducible, projective and injective modules, their structures, and those of the standard and costandard modules, as captured either by short exact sequences or Loewy diagrams.

There are more advanced techniques to perform the same task. One of them is Auslander-Reiten theory. The method it underlies is purely algorithmic, at least in the case of TL_n and dTL_n, and its input is again the data recalled in Section 5.2. The current section presents the application of this method to TL_n and dTL_n.

First, we review the theoretical results of Auslander-Reiten theory. Then, we show how these abstract results may be applied algorithmically and carry this out on the simple case of an orbit $[k]$ of non-critical integers that contains only 3 elements. We conclude by giving the result of applying this algorithm in the general case, providing only sketches of proofs, and thereby recover the classification of indecomposable $\mathcal{A}$-modules.

5.5.1 The main results of Auslander-Reiten theory

This subsection reviews the main ideas and results of Auslander-Reiten theory that we shall need to build a complete list of indecomposable modules of the algebras TL_n and dTL_n. Our summary closely follows Chapter IV of [2], though the results are not necessarily presented in the same order that they are proved there.

In this subsection, $\mathcal{A}$ stands for any finite-dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is an algebraically closed field.

Homological algebra studies modules through their Hom-groups. We first recall that a monomorphism (injective homomorphism) $f\colon \mathsf{U} \to \mathsf{V}$ is split if there exists $g\colon \mathsf{V} \to \mathsf{U}$ such that gf is the identity on U. Similarly, an epimorphism (surjective homomorphism) $f\colon \mathsf{U} \to \mathsf{V}$ is split if there exists $g\colon \mathsf{V} \to \mathsf{U}$ such that fg is the identity on V. A morphism is said to be split if it is either a split monomorphism or a split epimorphism. Auslander-Reiten theory studies refinements of these concepts.

Définition 5.5.1. *Let $f\colon \mathsf{U} \to \mathsf{V}$ be a morphism between two $\mathcal{A}$-modules.*

- *The morphism f is* left minimal almost split *if*
  - (i) *f is not a split monomorphism;*
  - (ii) *for any morphism $g\colon \mathsf{U} \to \mathsf{W}$ that is not a split monomorphism, there exists a morphism $\bar{f}\colon \mathsf{V} \to \mathsf{W}$ such that $\bar{f}f = g$;*
  - (iii) *for any $h\colon \mathsf{V} \to \mathsf{V}$, $hf = f$ implies that h is an isomorphism.*

- *The morphism f is* right minimal almost split *if*

  (i) *f is not a split epimorphism;*

  (ii) *for any morphism $g\colon \mathsf{W} \to \mathsf{V}$ that is not a split epimorphism, there exists a morphism $\bar{f}\colon \mathsf{W} \to \mathsf{U}$ such that $f\bar{f} = g$;*

  (iii) *for any $h\colon \mathsf{U} \to \mathsf{U}$, $fh = f$ implies that h is an isomorphism.*

- *The morphism f is* irreducible *if*

  (i) *f is not a split morphism;*

  (ii) *given any morphisms $f_1\colon \mathsf{U} \to \mathsf{Z}$ and $f_2\colon \mathsf{Z} \to \mathsf{V}$ satisfying $f = f_2 f_1$, then either f_2 is a split epimorphism or f_1 is a split monomorphism.*

Définition 5.5.2. *A short exact sequence*

$$0 \longrightarrow \mathsf{U} \xrightarrow{\iota} \mathsf{V} \xrightarrow{\pi} \mathsf{V}/\mathsf{U} \longrightarrow 0 \tag{5.5.1}$$

is said to be an almost split exact sequence *if*

- *ι is left minimal almost split;*

- *π is right minimal almost split.*

In fact, it can be shown that either of these two conditions implies the other.

It is useful to draw a parallel between irreducible morphisms and semisimple modules. Intuitively, a morphism is irreducible if it cannot be expressed as a composition of non-split morphisms, just as a module is semisimple if it cannot be expressed as a non-trivial extension of one module by another. Similarly, finite-dimensional modules can be studied through their Loewy diagrams in terms of extensions of simple modules, while morphisms between modules in a representation-finite algebra can be expressed as sums of compositions of irreducible morphisms. As for the left (right) minimal almost split morphisms, they turn out to be in one-to-one correspondence with the non-injective (non-projective) finite-dimensional indecomposable modules. In particular, the following proposition notes that the domain of every left minimal almost split morphism is indecomposable, while Theorem 5.5.10 below states that every non-injective finite-dimensional indecomposable module is the domain of a left minimal almost split morphism.

Proposition 5.5.3. *Let U and V be $\mathcal{A}$-modules. If $f\colon \mathsf{U} \to \mathsf{V}$ is a left minimal almost split morphism, then*

(i) *U is indecomposable;*

(ii) *f is irreducible;*

(iii) *if $f'\colon \mathsf{U} \to \mathsf{V}'$ is also left minimal almost split, for some $\mathcal{A}$-module V', then there exists an isomorphism $g\colon \mathsf{V} \to \mathsf{V}'$ such that $gf = f'$;*

(iv) *a morphism* $f'\colon \mathsf{U} \to \mathsf{V}'$ *of* $\mathcal{A}$*-modules is irreducible if and only if* $\mathsf{V}' \neq 0$*,* $\mathsf{V} \simeq \mathsf{V}' \oplus \mathsf{V}''$ *for some module* V''*, and there exists* $f''\colon \mathsf{U} \to \mathsf{V}''$ *such that* $f' \oplus f''\colon \mathsf{U} \to \mathsf{V}' \oplus \mathsf{V}''$ *is left minimal almost split. In particular, composing* f *with the projection onto a (non-zero) direct summand always gives an irreducible morphism.*

A similar result holds for right minimal almost split morphisms.

Note that while irreducible and almost split morphisms are very useful, identifying and constructing them is far from trivial. Fortunately Auslander-Reiten theory provides a few irreducible morphisms to get one started (the next result), a way of building new ones out of old ones, and tools for classifying them. First, however, it is convenient to introduce two more definitions.

Définition 5.5.4.

- *For two* $\mathcal{A}$*-modules* U *and* V*, define the* radical *of* $\mathsf{Hom}_{\mathcal{A}}(\mathsf{U},\mathsf{V})$ *to be the vector space* $r(\mathsf{U},\mathsf{V})$ *spanned by the morphisms* $f\colon \mathsf{U} \to \mathsf{V}$ *such that for all indecomposable* $\mathcal{A}$*-modules* Z *and all morphisms* $g\colon \mathsf{Z} \to \mathsf{U}$ *and* $h\colon \mathsf{V} \to \mathsf{Z}$*, the composition* hfg *is not an isomorphism.*

- *Define the* second radical *of* $\mathsf{Hom}_{\mathcal{A}}(\mathsf{U},\mathsf{V})$ *to be the vector subspace* $r^2(\mathsf{U},\mathsf{V}) \subseteq r(\mathsf{U},\mathsf{V})$ *consisting of the morphisms* $f\colon \mathsf{U} \to \mathsf{V}$ *which may be factored as* $f = gh$*, where* $g \in r(\mathsf{Z},\mathsf{V})$ *and* $h \in r(\mathsf{U},\mathsf{Z})$*, for some* $\mathcal{A}$*-module* Z*.*

We remark that if U and V are indecomposable, then $r(\mathsf{U},\mathsf{V})$ is the subspace of non-isomorphisms in $\mathsf{Hom}(\mathsf{U},\mathsf{V})$.

Proposition 5.5.5.

(i) *If* P *is a non-simple projective indecomposable and* J *a non-simple injective indecomposable, then the canonical inclusion and projection,*

$$\operatorname{rad}\mathsf{P} \hookrightarrow \mathsf{P} \qquad \textit{and} \qquad \mathsf{J} \twoheadrightarrow \frac{\mathsf{J}}{\operatorname{soc}\mathsf{J}}, \tag{5.5.2}$$

are irreducible. Furthermore, these are the only irreducible morphisms with target P and source J, respectively, up to rescaling.

(ii) *If* Q *is a non-simple projective and injective indecomposable with* $\operatorname{rad}\mathsf{Q} \neq \operatorname{soc}\mathsf{Q}$*, then the canonical projection and inclusion,*

$$\operatorname{rad}\mathsf{Q} \twoheadrightarrow \frac{\operatorname{rad}\mathsf{Q}}{\operatorname{soc}\mathsf{Q}} \qquad \textit{and} \qquad \frac{\operatorname{rad}\mathsf{Q}}{\operatorname{soc}\mathsf{Q}} \hookrightarrow \frac{\mathsf{Q}}{\operatorname{soc}\mathsf{Q}}, \tag{5.5.3}$$

are irreducible.

(iii) *When U and V are indecomposable,* $f\colon \mathsf{U} \to \mathsf{V}$ *is irreducible if and only if* $f \in r(\mathsf{U},\mathsf{V})$ *but* $f \notin r^2(\mathsf{U},\mathsf{V})$*.*

Recall that the radicals of the projectives and the socles of the injectives are known for $\mathcal{A} = \mathsf{TL}_n$ and dTL_n. Moreover, if $[k]$ is a non-critical orbit, then each projective $\mathsf{P}_{k'}$, with $k' \in [k] \setminus \{k_L\}$, is also injective. The previous proposition therefore provides several irreducible morphisms for these algebras.

Définition 5.5.6. *Let* U *be a left (or right)* $\mathcal{A}$*-module.*

- *The vector space dual* $\mathsf{U}^* \equiv \mathsf{Hom}_{\mathbb{K}}(\mathsf{U},\mathbb{K})$ *(introduced in Section 5.2.3 for* $\mathbb{K} = \mathbb{C}$*) is a right (or left)* $\mathcal{A}$*-module with action* $(fa)(x) = f(ax)$ *(or* $(af)(x) = f(xa)$*), for all* $f \in \mathsf{U}^*$ *and* $a \in \mathcal{A}$*.*
- *The algebra dual* $\mathsf{U}^t \equiv \mathsf{Hom}_{\mathcal{A}}(\mathsf{U},\mathcal{A})$ *is a right (or left)* $\mathcal{A}$*-module with action* $(fa)(x) = f(x)a$ *(or* $(af)(x) = af(x)$*), for all* $f \in \mathsf{U}^t$ *and* $a \in \mathcal{A}$*.*

As with the twisted dual of Section 5.2.3, the vector space dual defines a contravariant exact functor. The functor for the algebra dual, on the other hand, is contravariant but only left-exact.

Recall that a projective presentation of a module U is a short exact sequence $0 \to \ker p \to \mathsf{P} \xrightarrow{p} \mathsf{U} \to 0$ in which P is projective. Replacing $\ker p$ by another projective Q, its projective cover for example, gives another exact sequence: $\mathsf{Q} \xrightarrow{q} \mathsf{P} \xrightarrow{p} \mathsf{U} \to 0$. This sequence is said to be a *minimal* projective presentation of U if $p\colon \mathsf{P} \to \mathsf{U}$ is a projective cover of U and $q\colon \mathsf{Q} \to \ker p$ is a projective cover of $\ker p$. Apply the functor $(-)^t$ to obtain one last exact sequence:

$$0 \longrightarrow \mathsf{U}^t \xrightarrow{p^t} \mathsf{P}^t \xrightarrow{q^t} \mathsf{Q}^t \longrightarrow \operatorname{Coker} q^t \longrightarrow 0. \tag{5.5.4}$$

Définition 5.5.7. *The* Auslander-Reiten transpose *of the left (right)* $\mathcal{A}$*-module* U *is the right (left)* $\mathcal{A}$*-module*

$$\operatorname{Tr}\mathsf{U} = \operatorname{Coker} q^t. \tag{5.5.5}$$

We remark that the isomorphism class of the Auslander-Reiten transpose does not depend upon the choice of minimal projective presentation.

The algebra dual U^t may be quite different to the vector space dual U^*, even as vector spaces. Here is a simple example for $\mathcal{A} = \mathsf{TL}_n$. Consider the irreducible left TL_n-module I_{k_L} corresponding to the smallest integer in a non-critical orbit $[k]$ that contains at least two integers. Of course, $\mathsf{I}^*_{k_L}$ is the corresponding irreducible right TL_n-module, a fact that we exploited in Section 5.4.3. On the other hand, $\mathsf{I}^t_{k_L}$ is the space of all homomorphisms from I_{k_L} to the left TL_n-module TL_n. However TL_n is a direct sum of indecomposable projectives, none of which contain I_{k_L} in their socles. Therefore, the only such homomorphism is zero and $\mathsf{I}^t_{k_L} = 0$.

Computing U^t can thus be tricky. Fortunately, we shall only need to compute this dual when U is projective and, in this case, the next result gives the answer.

Proposition 5.5.8.

(i) *If* P *is a finitely generated projective* $\mathcal{A}$*-module, then so is* P^t*. In particular, if* $\mathsf{P} = \mathcal{A}e$ *for some idempotent* $e \in \mathcal{A}$*, then* $\mathsf{P}^t \simeq e\mathcal{A}$*.*

(ii) *An indecomposable* U *is projective if and only if* $\operatorname{Tr}\mathsf{U} = 0$*;*

(iii) *If* U *is indecomposable and not projective, with minimal projective presentation* $\mathsf{Q} \xrightarrow{q} \mathsf{P} \xrightarrow{p} \mathsf{U} \longrightarrow 0$, *then*

- *the exact sequence*
$$\mathsf{P}^t \xrightarrow{q^t} \mathsf{Q}^t \longrightarrow \operatorname{Tr}\mathsf{U} \longrightarrow 0 \tag{5.5.6}$$
is a minimal projective presentation of $\operatorname{Tr}\mathsf{U}$*;*
- *the module* $\operatorname{Tr}\mathsf{U}$ *is indecomposable and not projective;*
- $\operatorname{Tr}(\operatorname{Tr}\mathsf{U}) \simeq \mathsf{U}$.

Of course, the Auslander-Reiten transpose of a left module is a right module, when we are really only interested in classifying left modules. We therefore modify this construction one last time.

Définition 5.5.9. *Let* U *be an indecomposable left (right) module. The* Auslander-Reiten translation *of* U *is the left (right) module*
$$\tau\mathsf{U} = (\operatorname{Tr}\mathsf{U})^* \tag{5.5.7a}$$
and the inverse Auslander-Reiten translation *of* U *is the left (right) module*
$$\tau^{-1}\mathsf{U} = \operatorname{Tr}(\mathsf{U}^*). \tag{5.5.7b}$$

Since $(-)^*$ preserves indecomposability, Theorem 5.5.8 shows that a non-projective (non-injective) indecomposable U has an indecomposable translation $\tau\mathsf{U}$ (inverse translation $\tau^{-1}\mathsf{U}$). This is a key feature of Auslander-Reiten translation because it allows one to construct new indecomposable modules from known ones. Another key feature is that it can also reveal new irreducible morphisms. The next proposition sums up these features.

Proposition 5.5.10. *Let* U *and* V *be indecomposable* $\mathcal{A}$*-modules and denote the set of isomorphism classes of finite-dimensional indecomposable* $\mathcal{A}$*-modules by* Ω*.*

(i) *If* V *is not projective, then*

- $\tau\mathsf{V}$ *is indecomposable and not injective, with* $\tau^{-1}(\tau\mathsf{V}) \simeq \mathsf{V}$*;*
- *As vector spaces,* $r(\mathsf{U},\mathsf{V})/r^2(\mathsf{U},\mathsf{V}) \simeq r(\tau\mathsf{V},\mathsf{U})/r^2(\tau\mathsf{V},\mathsf{U})$*;*
- *there exists a unique, up to isomorphism, almost split short exact sequence*
$$0 \longrightarrow \tau\mathsf{V} \longrightarrow \bigoplus_{\mathsf{M}\in\Omega} s(\mathsf{M})\cdot\mathsf{M} \longrightarrow \mathsf{V} \longrightarrow 0, \tag{5.5.8a}$$
where $s(\mathsf{M}) = \dim(r(\mathsf{M},\mathsf{V})/r^2(\mathsf{M},\mathsf{V}))$.

(ii) *If* U *is not injective, then*

- $\tau^{-1}\mathsf{U}$ *is indecomposable and not projective, with* $\tau(\tau^{-1}\mathsf{U}) \simeq \mathsf{U}$*;*
- *As vector spaces,* $r(\mathsf{U},\mathsf{V})/r^2(\mathsf{U},\mathsf{V}) \simeq r(\mathsf{V},\tau^{-1}\mathsf{U})/r^2(\mathsf{V},\tau^{-1}\mathsf{U})$*;*
- *there exists a unique, up to isomorphism, almost split short exact sequence*

$$0 \longrightarrow \mathsf{U} \longrightarrow \bigoplus_{\mathsf{M}\in\Omega} t(\mathsf{M})\cdot\mathsf{M} \longrightarrow \tau^{-1}\mathsf{U} \longrightarrow 0, \tag{5.5.8b}$$

where $t(\mathsf{M}) = \dim(r(\mathsf{U},\mathsf{M})/r^2(\mathsf{U},\mathsf{M}))$.

As we shall see, the non-negative integers $s(\mathsf{M})$ and $t(\mathsf{M})$ usefully measure the "sizes" of the sets $r(\mathsf{M},\mathsf{V}) \setminus r^2(\mathsf{M},\mathsf{V})$ and $r(\mathsf{U},\mathsf{M}) \setminus r^2(\mathsf{U},\mathsf{M})$ of irreducible morphisms from M to V and from U to M, respectively. [11] One of the key assertions of the previous proposition is then that the set of irreducible morphisms from U to V has the same size as that from $\tau\mathsf{V}$ to U, if V is not projective, and as that from V to $\tau^{-1}\mathsf{U}$, if U is not injective.

Starting with a single irreducible morphism $\mathsf{U} \to \mathsf{V}$, Auslander-Reiten translation therefore infers an iterative sequence of irreducible morphisms between indecomposable modules that may be composed to form a chain:

$$\cdots \longrightarrow \tau(\tau\mathsf{V}) \longrightarrow \tau\mathsf{U} \longrightarrow \tau\mathsf{V} \longrightarrow \mathsf{U} \longrightarrow \mathsf{V} \longrightarrow \tau^{-1}\mathsf{U} \longrightarrow \tau^{-1}\mathsf{V} \longrightarrow \tau^{-1}(\tau^{-1}\mathsf{U}) \longrightarrow \cdots. \tag{5.5.9}$$

This chain terminates if either $\tau^m\mathsf{V}$ or $\tau^m\mathsf{U}$ is projective or if $\tau^{-m}\mathsf{V}$ or $\tau^{-m}\mathsf{U}$ is injective. Note that it may also happen that $\tau^m\mathsf{U} \simeq \mathsf{U}$ and $\tau^m\mathsf{V} \simeq \mathsf{V}$, for some $m \in \mathbb{Z}$, in which case the chain (5.5.9) becomes a cycle of irreducible morphisms. The algebras TL_n and dTL_n will provide examples of both possibilities.

Theorem 5.5.10 gives more than just a way to construct new indecomposable modules and irreducible morphisms. It can also be used to verify the completeness of a set of indecomposable modules. To see this, suppose that a set Ω' of inequivalent finite-dimensional indecomposable modules has been identified, along with the irreducible morphisms between them. We can test for its completeness as follows. For each non-projective module $\mathsf{V} \in \Omega'$, one can list the known (linearly independent) irreducible morphisms whose targets are V. If $s'(\mathsf{M})$ is the number of these morphisms with source M, then one can count composition factors to check whether the sequence

$$0 \longrightarrow \tau\mathsf{V} \longrightarrow \bigoplus_{\mathsf{M}\in\Omega'} s'(\mathsf{M})\cdot\mathsf{M} \longrightarrow \mathsf{V} \longrightarrow 0 \tag{5.5.10}$$

could be exact. If $\tau\mathsf{V}$ and V have too many composition factors, (5.5.8a) tells us that the set Ω' is not complete or that we have not found all the irreducible morphisms.

A second test, based on (5.5.8b), checks if, for every non-injective U, the composition factors of U and $\tau^{-1}\mathsf{U}$ match those of $\bigoplus_{\mathsf{M}\in\Omega'} t'(\mathsf{M})\cdot\mathsf{M}$, where $t'(\mathsf{M})$ is the number of known (linearly independent)

11. Recall that the set of irreducible morphisms does not form a vector space, see Theorem 5.5.5(iii), explaining the slightly awkward language employed here.

irreducible morphisms with source U and target M. Again, if this test fails for a single U, then either Ω' is not complete or some irreducible morphisms are missing.

Suppose however, that the algebra $\mathcal{A}$ is *connected*, meaning that it cannot be written as a direct sum of more than one *block* (non-trivial indecomposable two-sided ideal). If both tests pass, for all non-injectives U and non-projectives V, then Ω' indeed yields a complete set of isomorphism classes of indecomposable modules. If $\mathcal{A}$ is not connected, then one simply restricts these tests to each block.

For clarity, the indecomposable modules and irreducible morphisms of $\mathcal{A}$ are often represented graphically as a quiver (graph).

Définition 5.5.11. *Let $\mathcal{A}$ be a finite-dimensional associative $\mathbb{K}$-algebra. The* Auslander-Reiten quiver $\Gamma(\mathcal{A}\text{-mod})$ *of the category* $\mathcal{A}$-mod *of finite-dimensional left $\mathcal{A}$-modules is defined as follows:*

- *The vertices of $\Gamma(\mathcal{A}\text{-mod})$ are the isomorphism classes $[\mathsf{U}]$ of indecomposable modules U in $\mathcal{A}$-mod.*
- *The arrows $[\mathsf{U}] \to [\mathsf{V}]$, for U, V in $\mathcal{A}$-mod, are in one-to-one correspondence with basis vectors of the $\mathbb{K}$-vector space $r(\mathsf{U},\mathsf{V})/r^2(\mathsf{U},\mathsf{V})$.*

Proposition 5.5.12.

(i) *If $\mathcal{A}$ is a connected finite-dimensional $\mathbb{K}$-algebra, then $\Gamma(\mathcal{A}\text{-mod})$ is connected and the number of arrows between any two vertices is finite.*

(ii) *$\mathcal{A}$ is of finite-representation type if and only if $\Gamma(\mathcal{A}\text{-mod})$ is a finite quiver.*

(iii) *If the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ have equivalent module categories, then their Auslander-Reiten quivers are identical.*

(iv) *If $\mathcal{A}$ is not connected, then $\Gamma(\mathcal{A}\text{-mod})$ is the disjoint union of the Auslander-Reiten quivers of its blocks.*

Of course, one can similarly study the Auslander-Reiten quiver $\Gamma(\text{mod-}\mathcal{A})$ of the category mod-$\mathcal{A}$ of finite-dimensional right $\mathcal{A}$-modules. For TL_n and dTL_n, the left and right quivers are isomorphic.

5.5.2 A detailed example

As an example of the algorithmic construction afforded by the abstract results of the previous subsection, we compute the Auslander-Reiten quiver for a block of $\mathcal{A}$ corresponding to a non-critical orbit containing three elements $\{k_1 = k_L, k_2, k_3 = k_R\}$. (We omit the case where n is even and $\beta = 0$, for $\mathcal{A} = \mathsf{TL}_n$, deferring its study to the end of the next subsection.) In what follows, we shall use the notation B_k^l and T_k^l for every module that is not projective, injective or irreducible, slightly modified for brevity so that the index $k = k_i$ is replaced by the label i. The same modification will be applied to the projectives, injectives and irreducibles. Thus, I_2, P_3, B_2^1 and T_1^2 now stand for I_{k_2}, P_{k_3}, $\mathsf{B}_{k_2}^1$ and $\mathsf{T}_{k_1}^2$, respectively, whilst we shall prefer P_1 and J_1

over $\mathsf{S}_1 \simeq \mathsf{T}_1^1$ and $\mathsf{C}_1 \simeq \mathsf{B}_1^1$, respectively. This notation extends in an obvious fashion to orbits of arbitrary length.

We break down the construction of the Auslander-Reiten quiver into three steps: The computation of the translation τ on indecomposable modules, the identification of irreducible morphisms, and the drawing and check of completeness of the Auslander-Reiten quiver (and therefore of the list of isomorphism classes of indecomposable $\mathcal{A}$-modules).

The action of τ on indecomposable modules — We assume the results of Section 5.2, that is the existence of the irreducible, standard, costandard, injective and projective modules, as well as their Loewy diagrams. The translation τ will be applied to all non-projectives of this list, and then on the new ones thus obtained, until the process does not introduce any new non-projective indecomposable modules.

Let us start with the (left) injective module $\mathsf{J}_1 = \mathsf{C}_1 = \mathsf{B}_1^1$. The first step is to construct a minimal projective presentation $\mathsf{Q} \to \mathsf{P} \to \mathsf{J}_1 \to 0$. It is

$$\begin{array}{ccc} & & 3 \\ & \swarrow & \\ 2 & & \\ & \searrow & \\ & & 3 \end{array} \xrightarrow{q} \begin{array}{ccccc} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{array} \xrightarrow{p} \begin{array}{ccc} & & 2 \\ & \swarrow & \\ 1 & & \end{array} \longrightarrow 0. \tag{5.5.11}$$

(We have replaced the names of the modules by their Loewy diagrams and each composition factor by its index i. We have also omitted the arrowheads on the Loewy diagrams for clarity — they all point down.) To see that this is a minimal projective presentation, note first that the projective cover of J_1 is $\mathsf{P}_2 \xrightarrow{p} \mathsf{J}_1$, by Theorem 5.2.16(ii), and that $\ker p \simeq \mathsf{B}_2^1$, by (5.2.30). Then, $\mathsf{Q} = \mathsf{P}[\mathsf{B}_2^1] \simeq \mathsf{P}_3$, again by Theorem 5.2.16(ii). The next step is to identify the cokernel of $\mathsf{P}^t \xrightarrow{q^t} \mathsf{Q}^t$. By Theorem 5.5.8(i), the modules P_2^t and P_3^t are the (right) modules P_2^* and P_3^*, respectively. Moreover, the morphism q^t is non-zero because $q^t = 0$ would imply that $\mathrm{Tr}\,\mathsf{J}_1$ was projective, by part (iii) of the same proposition, contradicting part (ii). The cokernel $\mathrm{Coker}\,q^t$ is thus easily identified:

$$\mathrm{Tr}\,\mathsf{J}_1 = \mathrm{coker}\left[\begin{array}{ccccc} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{array} \xrightarrow{q^t} \begin{array}{ccc} & & 3 \\ & \swarrow & \\ 2 & & \\ & \searrow & \\ & & 3 \end{array} \right] \simeq \mathsf{I}_3^*. \tag{5.5.12}$$

The Auslander-Reiten translation of J_1 is therefore $\tau\mathsf{J}_1 = (\mathrm{Tr}\,\mathsf{J}_1)^* \simeq \mathsf{I}_3$. Replacing J_1 by I_3, a similar computation shows that $\tau\mathsf{I}_3 = (\mathrm{Tr}\,\mathsf{I}_3)^* \simeq \mathsf{P}_1$. Since P_1 is projective, the process started with J_1 stops here. Moreover, since J_1 is injective, we cannot use τ^{-1} to construct any other new indecomposables. If we indicate $\mathsf{M}_1 = \tau\mathsf{M}_2$ by $\mathsf{M}_1 \rightsquigarrow \mathsf{M}_2$, then these computations may be summarised as the following chain:

$$\mathsf{P}_1 \rightsquigarrow \mathsf{I}_3 \rightsquigarrow \mathsf{J}_1. \tag{t_0}$$

(The label (t_0) will be explained in the next subsection.)

The choice of arrow direction, following the action of τ^{-1} rather than τ, was made so as to agree with the direction of the irreducible morphisms in the chain (5.5.9). Indeed, we may redraw this chain in a zigzag pattern, adding squiggly arrows representing τ^{-1} as follows:

$$\begin{array}{ccccccccccccc} & \tau^2\mathsf{U} & \rightsquigarrow & \tau\mathsf{U} & \rightsquigarrow & \mathsf{U} & \rightsquigarrow & \tau^{-1}\mathsf{U} & \rightsquigarrow & \tau^{-2}\mathsf{U} & & & \\ \cdots & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \cdots . \\ & & \tau^2\mathsf{V} & \rightsquigarrow & \tau\mathsf{V} & \rightsquigarrow & \mathsf{V} & \rightsquigarrow & \tau^{-1}\mathsf{V} & \rightsquigarrow & \tau^{-2}\mathsf{V} & \end{array} \tag{5.5.13}$$

In this way, we see how to combine the τ-orbits of U and V, when we have prior knowledge of an irreducible morphism from U to V, into a chain of irreducible morphisms. We will refer to this combination process as *weaving*.

We continue with the identification of the τ-orbits of indecomposables, illustrating the method one more time by computing the repeated action of τ on I_2. The projective cover of I_2 is P_2 and the kernel of the covering map p has head $\mathsf{I}_1 \oplus \mathsf{I}_3$ (we know that $\ker p \simeq \mathsf{T}_1^2$, by (5.3.9) and Theorem 5.3.4, but we are only using the results of Section 5.2 here). The projective cover of $\ker p$ is therefore $\mathsf{P}_1 \oplus \mathsf{P}_3$ (in agreement with Theorem 5.3.3), so the minimal projective presentation of I_2 is

$$\mathsf{Q} = \begin{array}{cc} 1 & \\ & 2 \end{array} \oplus \begin{array}{cc} & 3 \\ 2 & \\ & 3 \end{array} \xrightarrow{\ q\ } \mathsf{P} = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \xrightarrow{\ p\ } \mathsf{I}_2 \longrightarrow 0. \tag{5.5.14}$$

We know that the transpose $\operatorname{Tr}\mathsf{I}_2$ is indecomposable, by Theorem 5.5.8(iii), and is characterised by the following exact sequence (of right $\mathcal{A}$-modules):

$$\mathsf{P}^t = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \xrightarrow{\ q^t\ } \mathsf{Q}^t = \begin{array}{cc} 1 & \\ & 2 \end{array} \oplus \begin{array}{cc} & 3 \\ 2 & \\ & 3 \end{array} \longrightarrow \operatorname{Tr}\mathsf{I}_2 \longrightarrow 0. \tag{5.5.15}$$

As before, $q^t \neq 0$, so $\operatorname{Tr}\mathsf{I}_2$ must have precisely three composition factors: I_1^* and I_3^* in its head and I_2^* in its socle. Indeed, the only other possibility is that q^t maps $\operatorname{hd}\mathsf{P}^t$ onto $\operatorname{soc}\mathsf{P}_1^*$, contradicting the indecomposability of $\operatorname{Tr}\mathsf{I}_2$. $\operatorname{Tr}\mathsf{I}_2$ is therefore not one of the indecomposables considered in Section 5.2. It must be, of course, the (right version of the) module T_1^2 introduced in Section 5.3. Because taking the (vector space) dual of a module exchanges its socle and head, the dual of the right module version of T_1^2 is the left module B_1^2 and, thus, $\tau\mathsf{I}_2 = (\operatorname{Tr}\mathsf{I}_2)^* \simeq \mathsf{B}_1^2$. Iterating the action of τ, we obtain

$$\cdots \rightsquigarrow \mathsf{I}_2 \rightsquigarrow \mathsf{T}_1^2 \rightsquigarrow \mathsf{B}_1^2 \rightsquigarrow \mathsf{I}_2 \rightsquigarrow \cdots \tag{i_2}$$

Note that the Auslander-Reiten translation of T_1^2 is the irreducible I_2 that we started with: this sequence of translated modules forms a cycle. The computations for the translations of I_1 are similar to those detailed above and result in

$$\cdots \rightsquigarrow \mathsf{I}_1 \rightsquigarrow \mathsf{T}_2^1 \rightsquigarrow \mathsf{B}_2^1 \rightsquigarrow \mathsf{I}_1 \rightsquigarrow \cdots \tag{t_2}$$

At this point, all of the indecomposable modules known from Section 5.2 (the “input data") have appeared in either (t_0), (i_2) or (t_2), except for those that are both projective and injective: P_2 and P_3. By

Theorem 5.5.8(ii), $\tau\mathsf{M}$ and $\tau^{-1}\mathsf{M}$ are both the zero module if M is projective and injective. We have thus exhausted the possibility of constructing new indecomposables from the ones we know. It is not clear at this point whether the list of indecomposables that we have constructed,

$$\{\mathsf{I}_1, \mathsf{I}_2, \mathsf{I}_3, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3, \mathsf{J}_1, \mathsf{B}_1^2, \mathsf{B}_2^1, \mathsf{T}_1^2, \mathsf{T}_2^1\}, \tag{5.5.16}$$

is complete. Proving completeness is the goal of the third step. But first, the irreducible morphisms between the known indecomposables must be counted.

Irreducible morphisms and weaving — Theorem 5.5.5(i) gives some irreducible morphisms: including the radical in a projective indecomposable and projecting an injective indecomposable onto the quotient by its socle. In the present case, we obtain six irreducibles morphisms this way, conveniently summarised thus:

$$\mathsf{J}_1 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{P}_1, \qquad \mathsf{T}_1^2 \longrightarrow \mathsf{P}_2 \longrightarrow \mathsf{B}_1^2, \qquad \mathsf{T}_2^1 \longrightarrow \mathsf{P}_3 \longrightarrow \mathsf{B}_2^1. \tag{5.5.17a}$$

When an indecomposable module is projective and injective, with its socle strictly contained in its radical, Theorem 5.5.5(ii) gives further irreducible morphisms. Only the P_i, with $i > 1$, have these properties. We compute the quotients $\operatorname{rad}\mathsf{P}_2/\operatorname{soc}\mathsf{P}_2 \simeq \mathsf{I}_1 \oplus \mathsf{I}_3$ and $\operatorname{rad}\mathsf{P}_3/\operatorname{soc}\mathsf{P}_3 \simeq \mathsf{I}_2$, thereby adding six irreducible morphisms to those of (5.5.17a):

$$\mathsf{T}_1^2 \longrightarrow \mathsf{I}_1 \longrightarrow \mathsf{B}_1^2, \qquad \mathsf{T}_2^1 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{B}_2^1, \qquad \mathsf{T}_1^2 \longrightarrow \mathsf{I}_3 \longrightarrow \mathsf{B}_1^2. \tag{5.5.17b}$$

We remark that the decomposability of $\operatorname{rad}\mathsf{P}_2/\operatorname{soc}\mathsf{P}_2$ allowed us to construct four irreducible morphisms for P_2, instead of two, using Theorem 5.5.3(iv).

The non-projective modules J_1 and I_2 from the first irreducible morphism of (5.5.17a) belong to the distinct translation sequences (t_0) and (i_2), respectively. Weaving these sequences, as in (5.5.13), thus gives many other irreducible morphisms:

$$\begin{array}{ccccccccccccccc} & & \mathsf{P}_1 & \rightsquigarrow & \mathsf{I}_3 & \rightsquigarrow & \mathsf{J}_1 & & & & \mathsf{P}_1 & \rightsquigarrow & \mathsf{I}_3 & \rightsquigarrow & \mathsf{J}_1 & & \\ \cdots & \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow & & \nearrow & & \searrow \; \nearrow & & \searrow \; \nearrow & & \searrow & \cdots\,. \\ \mathsf{I}_2 & \rightsquigarrow & & \mathsf{T}_1^2 & \rightsquigarrow & \mathsf{B}_1^2 & \rightsquigarrow & & \mathsf{I}_2 & \rightsquigarrow & \mathsf{T}_1^2 & \rightsquigarrow & \mathsf{B}_1^2 & \rightsquigarrow & \mathsf{I}_2 \end{array} \tag{5.5.18}$$

Here, the top row is the translation chain (t_0) (repeated) and the bottom row is the translation cycle (i_2). The irreducible morphisms form the following cycle:

$$\cdots \longrightarrow \mathsf{I}_3 \longrightarrow \mathsf{B}_1^2 \longrightarrow \mathsf{J}_1 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{P}_1 \longrightarrow \mathsf{T}_1^2 \longrightarrow \mathsf{I}_3 \longrightarrow \cdots\,. \tag{5.5.19}$$

Of these irreducible morphisms, $\mathsf{J}_1 \to \mathsf{I}_2 \to \mathsf{P}_1$ and $\mathsf{T}_1^2 \to \mathsf{I}_3 \to \mathsf{B}_1^2$ have appeared already in (5.5.17), but $\mathsf{B}_1^2 \to \mathsf{J}_1$ and $\mathsf{P}_1 \to \mathsf{T}_1^2$ are new.

We continue weaving to find further irreducible morphisms. Of those in (5.5.17a), the first two were analysed above and the remaining four involve an indecomposable that is both projective and injective, so give nothing new. We therefore turn to the first four morphisms in (5.5.17b), the last two having also appeared in the previous analysis. From $\mathsf{T}_1^2 \to \mathsf{I}_1$, we weave the translation cycles (i_2) and (t_2) together and arrive at the following cycle of irreducible morphisms:

$$\cdots \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{B}_2^1 \longrightarrow \mathsf{T}_1^2 \longrightarrow \mathsf{I}_1 \longrightarrow \mathsf{B}_1^2 \longrightarrow \mathsf{T}_2^1 \longrightarrow \mathsf{I}_2 \longrightarrow \cdots . \tag{5.5.20}$$

These include the remaining morphisms of (5.5.17b) as well as $\mathsf{B}_2^1 \to \mathsf{T}_1^2$ and $\mathsf{B}_1^2 \to \mathsf{T}_2^1$, which are new. Having exhausted our stock of irreducible morphisms, it is reasonable to conjecture that we have found a complete set. Testing this is the content of the last step.

The Auslander-Reiten quiver and completeness — All indecomposable modules constructed using Auslander-Reiten translation and all irreducible morphisms obtained in the previous step are drawn in the quiver of Figure 5.5.1, along with the quivers for orbits of lengths 2, 4 and 5. This quiver combines the morphisms of the cycles (5.5.19) and (5.5.20) with those of the chains in (5.5.17a) involving the projective injectives P_2 and P_3.

Theorem 5.5.10 gives two sets of checks to perform: whether, for all non-projective indecomposables V, every irreducible morphism with target V has been obtained and whether, for all non-injective indecomposables U, every irreducible morphism with source U has been obtained. Here is one example of the many verifications required by these checks. The module B_2^1 is not projective, its translation is $\tau\mathsf{B}_2^1 = \mathsf{T}_2^1$, and it is the target of at least two irreducible morphisms: $\mathsf{P}_3 \to \mathsf{B}_2^1$ and $\mathsf{I}_2 \to \mathsf{B}_2^1$. Theorem 5.5.10 now states that there is an (almost split) exact sequence of the form

$$0 \longrightarrow \mathsf{T}_2^1 \longrightarrow \mathsf{I}_2 \oplus \mathsf{P}_3 \oplus ? \longrightarrow \mathsf{B}_2^1 \longrightarrow 0, \tag{5.5.21}$$

where “?” will be non-zero if and only there are additional independent irreducible morphisms with target B_2^1. However, the composition factors of T_2^1 and B_2^1 together are I_2 and I_3, both appearing with multiplicity two, which precisely matches the factors of $\mathsf{I}_2 \oplus \mathsf{P}_3$. Thus, “?" is the zero module and a complete set of irreducible morphisms with target B_1^2 has been obtained. The remaining verifications are numerous, but the quiver makes them expeditious.

We remark that as P_2 and P_3 are both projective and injective, they escape the checks of Theorem 5.5.10. However, Theorem 5.5.5(i) assures us that all irreducible morphisms with these modules as source or target have already appeared in (5.5.17a). Thus, the top right quiver drawn in Figure 5.5.1 depicts all the irreducible morphisms between those indecomposables that have been identified, thus far.

Our last task is to ascertain whether the list of (isomorphism classes of) indecomposable modules that has been obtained is complete. Since the list of projectives is complete, any missing indecomposable V would have to be non-projective. Theorem 5.5.10 would then state the existence of an almost split exact sequence

$$0 \longrightarrow \tau\mathsf{V} \longrightarrow \oplus_i \mathsf{M}_i \longrightarrow \mathsf{V} \longrightarrow 0, \tag{5.5.22}$$

for some modules M_i, and more irreducible morphisms. None of these new morphisms could have any of

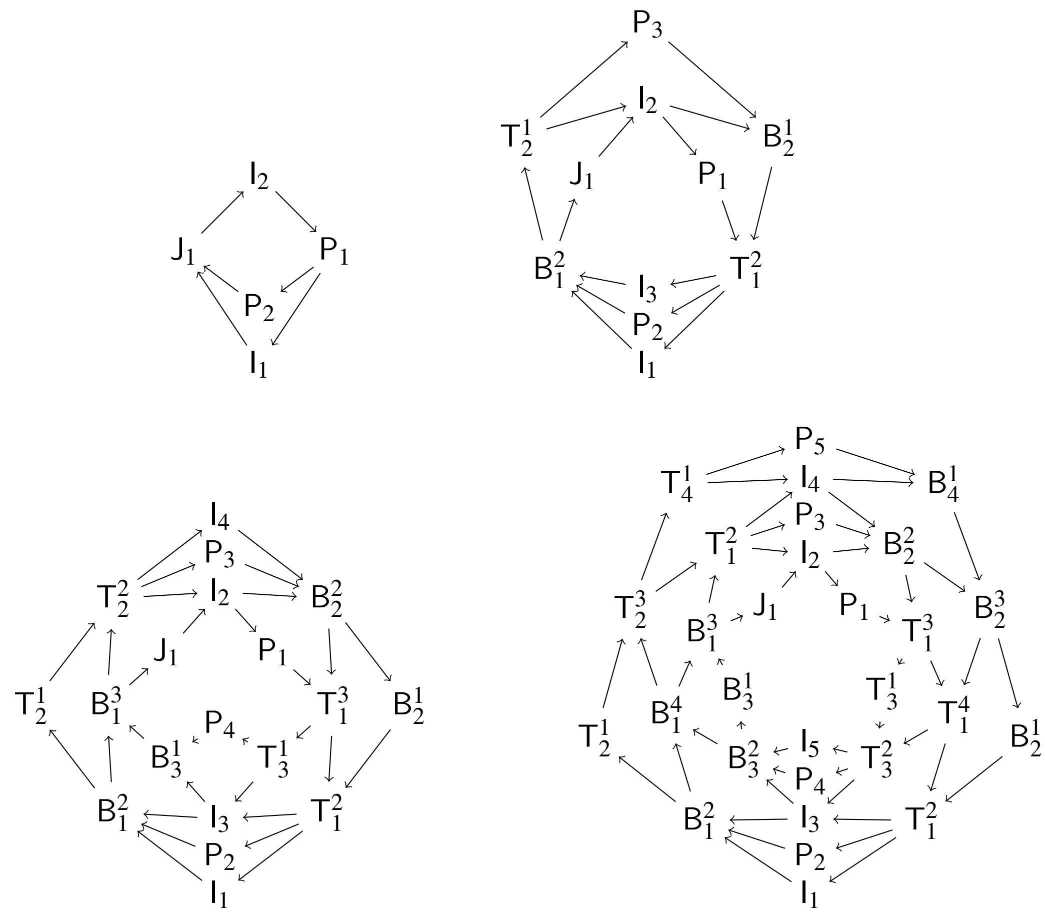


Figure 5.5.1 – The Auslander-Reiten quivers of TL_n and dTL_n for orbits $[k]$ of lengths 2, 3, 4 and 5. The quivers for TL_n, with n even and $\beta = 0$, are different and are instead illustrated in Figure 5.5.2. Reflecting about vertical bisectors amounts to taking the twisted dual of Section 5.2.3.

the indecomposables already found as source or target, because we have already determined that the second quiver in Figure 5.5.1 already contains all such irreducible morphisms. A new irreducible morphism would therefore imply that the quiver for this block of $\mathcal{A}$ is disconnected, contradicting Theorem 5.5.12(i). Thus, we have obtained a complete list of isomorphism classes of indecomposables.

5.5.3 Algorithmic construction of Auslander-Reiten quivers for TL_n and dTL_n

The goal of this last subsection is to present an algorithmic construction of the Auslander-Reiten quiver for the algebras TL_n and dTL_n. A detailed proof of this result is long, but does not involve any argument not yet covered in the example of the previous subsection. We shall only indicate how to construct the quiver and sketch the proofs. As usual, we shall assume that the case of TL_n, with n even and $\beta = 0$, is not under consideration, its study being deferred to the end of the section.

Let s denote the length of the non-critical orbit of $k \in \Lambda_n$. If $s = 1$, then $\mathsf{P}_k = \mathsf{J}_k = \mathsf{I}_k$ and this is the only indecomposable module for the corresponding block. We shall therefore assume that $s \geq 2$. Order the integers in $[k]$ from the smallest to the largest and label them by $k_1 = k_L$, k_2, $\dots$, $k_s = k_R$. Our goal is to construct the Auslander-Reiten quiver of the corresponding block. Its vertices are precisely the indecomposable $\mathcal{A}$-modules which have the property that each of their composition factors is isomorphic to one of the irreducibles I_1, I_2, $\dots$, or I_s (as in Section 5.5.2, we shall replace k_j by j for clarity). We will also find it convenient to write $s = 2i$, if s is even, and $s = 2i+1$, if it is odd.

The orbits of the translation τ — As in the previous example, the first step is to characterise the τ-orbits. Recall that the Auslander-Reiten translation $\tau\mathsf{M}$ of an indecomposable module M is zero if (and only if) M is projective. The use of the word "orbit" for the action of the translation τ is thus somewhat abusive. We shall use the word regardless, qualifying it as a *τ-orbit* to avoid confusion with the orbit of a non-critical k. We will also describe τ-orbits as chains and cycles, as in Section 5.5.2. We note that τ-orbits are disjoint: no indecomposable appears in more than one.

It is easy to show that there is always a τ-orbit, denoted by (t_0), that takes the form of the following chain:

$$\mathsf{P}_1 = \mathsf{T}^1_1 \rightsquigarrow \mathsf{T}^1_3 \rightsquigarrow \mathsf{T}^1_5 \rightsquigarrow \cdots \begin{Bmatrix} \mathsf{T}^1_{2i-1} \rightsquigarrow \mathsf{B}^1_{2i-1} \\ \mathsf{T}^1_{2i-1} \rightsquigarrow \mathsf{I}_{2i+1} \rightsquigarrow \mathsf{B}^1_{2i-1} \end{Bmatrix} \rightsquigarrow \cdots \rightsquigarrow \mathsf{B}^1_5 \rightsquigarrow \mathsf{B}^1_3 \rightsquigarrow \mathsf{B}^1_1 = \mathsf{J}_1, \qquad \begin{cases} s \text{ even}, \\ s \text{ odd}. \end{cases} \tag{t_0}$$

We recall that $\rightsquigarrow$ denotes the action of τ^{-1}. For $s = 3$ $(i = 1)$, this chain indeed reduces to the chain (t_0) obtained in the previous subsection. Explicit examples of τ-orbits are given in Section 5.A.

The description of the other τ-orbits requires a systematic use of the following "trick". Let B^j_k, with $k \in \mathbb{Z}$ and j a positive odd integer, denote the zigzag represented graphically as

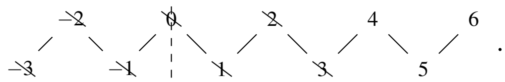
$$\begin{array}{ccccccccc} & k+1 & & & & & & j+k & \\ \diagup & & \diagdown & \diagup & \cdots & \diagdown & \diagup & & \\ k & & & k+2 & & & j+k-1 & & \end{array} . \tag{5.5.23}$$

Of course, this need not be the Loewy diagram of an indecomposable module as, in general, the integers appearing in this zigzag might be smaller than 1 or larger than s. In order to simplify the following discussion, we shall suppose that j and k satisfy

$$0 \leq j \leq 2s+1, \qquad -s \leq k \leq s, \qquad 1 \leq j+k \leq 2s+1. \tag{5.5.24}$$

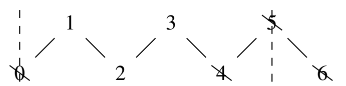
These conditions will be satisfied in the applications to come. For all indices $m \leq 0$ appearing in this zigzag, construct the pair $(m, r(m))$, where $r(m) = -m$ is the reflection of m through a mirror at 0. Similarly, for all $m \geq s+1$ in the zigzag, construct the pair $(m, r'(m))$, where $r'(m) = 2(s+1) - m$ is now the reflection of m through a mirror at $s+1$. Finally, delete from the zigzag every integer that appears in one of the pairs $(m, r(m))$ or $(m, r'(m))$ and denote the resulting zigzag by $\mathrm{rd}\big[\mathsf{B}^j_k\big]$. The conditions (5.5.24) ensure that the result is either empty or that it is the Loewy diagram of one of the indecomposable modules found in Section 5.3.[12] In particular, the composition factors of this module have labels in the set $\{1,2,\dots,s\}$. Using the nomenclature of Section 5.3, this module can be of type B, T or I. We shall call the process of constructing $\mathrm{rd}\big[\mathsf{B}_k^j\big]$ from B_k^j the *reflection-deletion trick*.

Here are two examples of this process. If $s=8$, then $\mathrm{rd}\big[\mathsf{B}_{-3}^9\big]$ is T_4^2 because the pairs $(0,0)$, $(-1,1)$, $(-2,2)$ and $(-3,3)$ are to be deleted, leaving only the composition factors labelled by $4,5$ and 6:



(5.5.25)

(The dashed lines indicate the mirror through which integers have been reflected.) Similarly, $\mathrm{rd}\big[\mathsf{B}_0^6\big]=\mathsf{T}_1^2$, if $s=4$, since the pairs to be deleted are now $(0,0)$, $(5,5)$ and $(6,4)$:

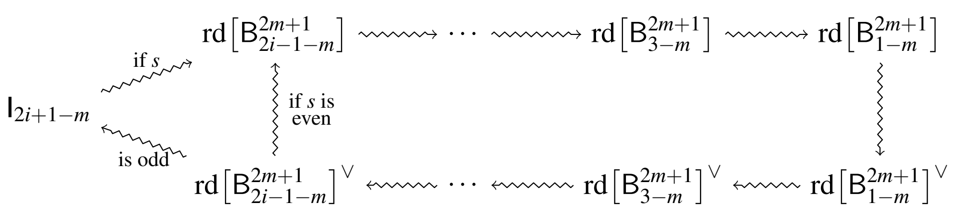


(5.5.26)

We now construct the remaining τ-orbits, beginning with those that we denote by (t_{2l}), where $1\le l\le \lfloor\frac{1}{2}(s-1)\rfloor$. First, for each l in this range, construct the following sequence of i or $i+1$ zigzags, according as to whether s is even or odd, respectively:

$$\underbrace{\mathsf{B}_{2i+1-m}^{2m+1}\rightsquigarrow}_{\text{if } s=2i+1 \text{ is odd}} \mathsf{B}_{2i-1-m}^{2m+1}\rightsquigarrow\cdots\rightsquigarrow \mathsf{B}_{3-m}^{2m+1}\rightsquigarrow \mathsf{B}_{1-m}^{2m+1}. \tag{5.5.27}$$

Here, we write $m=2l$ for later convenience. Second, apply the reflection-deletion trick to each zigzag in (5.5.27). Note that $\mathrm{rd}\big[\mathsf{B}_{1-m}^{2m+1}\big]=\mathsf{T}_m^2=\mathsf{T}_{2l}^2$ (except when s is odd and $m=s-1$, in which case $\mathrm{rd}\big[\mathsf{B}_{1-m}^{2m+1}\big]=\mathsf{T}_m^1=\mathsf{T}_{2l}^1$), explaining our choice of label (t_{2l}). Note also that when s is odd, $\mathrm{rd}\big[\mathsf{B}_{2i+1-m}^{2m+1}\big]$ is I_{2i+1-m}. Third, extend the reflected-deleted sequence to the left by adding the (twisted) duals of the $\mathrm{rd}\big[\mathsf{B}_j^{2m+1}\big]$ in reverse order (if s is odd, the self-dual I_{2i+1-m} is not repeated). The complete τ-orbit is then obtained from this extended sequence by closing it to form a cycle:



(t_{2l})

It contains s distinct indecomposables. Taking $s=3$ and $l=1$, hence $i=1$ and $m=2$, we note that $\mathrm{rd}\big[\mathsf{B}_{-1}^5\big]=\mathsf{T}_1^2$, thereby recovering the cycle (t_2) obtained in the previous subsection.

---

12. It is clear that this reflection-deletion trick has its origins in a signed action of the affine Weyl group $\mathsf{A}_1^{(1)}=\mathbb{Z}_2\ltimes\mathbb{Z}$ on the integers $\mathbb{Z}$ with "fundamental alcove" $\{1,2,\dots,s\}$ (see [25]). This action reflects and translates each zigzag vertex into the fundamental alcove, picking up a formal sign with every reflection. However, the resulting zigzag may have vertices appearing with negative coefficients in general, ruining its interpretation as the Loewy diagram of a module. We have imposed the conditions (5.5.24) in order to ensure that this does not happen.

The remaining τ-orbits will be denoted by (i_{2l}), where $1 \leq l \leq \lfloor \frac{1}{2}s \rfloor$, because they contain the irreducible module I_{2l}. Their construction is similar to that of the (t_{2l}). First, we have the following sequences of $i+1$ zigzags:

$$\mathsf{B}^{2m-1}_{2i+1-m} \rightsquigarrow \cdots \rightsquigarrow \mathsf{B}^{2m-1}_{3-m} \rightsquigarrow \mathsf{B}^{2m-1}_{1-m}. \tag{5.5.28}$$

If s is odd, then m takes the values $m = 2l$, for $1 \leq l \leq i$. If it is even, then $m = 2i+1-2l$ with, again, $1 \leq l \leq i$. Second, apply the reflection-deletion trick to each zigzag of each sequence. Note that $\mathrm{rd}\big[\mathsf{B}^{2m-1}_{1-m}\big] = \mathsf{I}_m$, which is I_{2l}, if s is odd, and I_{s+1-2l}, if it is even. Moreover, $\mathrm{rd}\big[\mathsf{B}^{2m-1}_{2i+1-m}\big] = \mathsf{I}_{2l}$, if s is even. Third, we again extend the reflected-deleted sequence to the left by adding the (twisted) duals in reverse order. When an endpoint of the sequence is irreducible, hence self-dual, it does not get repeated. The result is the cycle

$$\begin{array}{ccccccccc}
 & & \mathrm{rd}\big[\mathsf{B}^{2m-1}_{2i-1-m}\big] & \rightsquigarrow & \cdots & \rightsquigarrow \ \mathrm{rd}\big[\mathsf{B}^{2m-1}_{5-m}\big] & \rightsquigarrow & \mathrm{rd}\big[\mathsf{B}^{2m-1}_{3-m}\big] & \\
\mathrm{rd}\big[\mathsf{B}^{2m-1}_{2i+1-m}\big] & \overset{\text{if}}{\rightsquigarrow} & \Big\updownarrow{\scriptstyle \text{if } s \text{ is}} & & & & & & \searrow \\
{\scriptstyle s \text{ is}} \ \Big\updownarrow & & \mathsf{I}_{2l} & & & & & & \mathsf{I}_m \ . \\
\mathrm{rd}\big[\mathsf{B}^{2m-1}_{2i+1-m}\big]^\vee & \underset{\text{odd}}{\leftsquigarrow} & \Big\updownarrow{\scriptstyle \text{even}} & & & & & & \swarrow \\
 & & \mathrm{rd}\big[\mathsf{B}^{2m-1}_{2i-1-m}\big]^\vee & \leftsquigarrow & \cdots & \leftsquigarrow \ \mathrm{rd}\big[\mathsf{B}^{2m-1}_{5-m}\big]^\vee & \leftsquigarrow & \mathrm{rd}\big[\mathsf{B}^{2m-1}_{3-m}\big]^\vee &
\end{array} \tag{i_{2l}}$$

It likewise contains s distinct indecomposables.

Lemma 5.5.13. *Let $[k]$ be a non-critical orbit of length $s \geq 2$ (omitting the case of $\mathcal{A} = \mathsf{TL}_n$, with n even and $\beta = 0$) and write $s = 2i$ or $2i+1$, according as to whether s is even or odd, respectively. Then, the s τ-orbits*

$$(t_0), (t_2), \ldots, (t_{2i-2}), \underbrace{(t_{2i}),}_{\textit{if } s \textit{ is odd}} (i_2), (i_4), \ldots, (i_{2i}) \tag{5.5.29}$$

are all cycles, except for (t_0) which is a chain. They each contain s distinct indecomposable modules and are disjoint in the sense that no indecomposable appears in more than one such τ-orbit.

Sketch of proof. First, the Auslander-Reiten translation of the modules that are “far" from the boundary of the class $[k]$ are computed. For a module to be “far enough", it is sufficient that neither the composition factor $k_L = k_1$ nor $k_R = k_s$ appear in the minimal projective presentation used to compute its Auslander-Reiten translation. This computation is then straightforward and does not depend on the parity of s. Second, the translation of the modules whose projective presentation involves k_1 or k_s is studied. In most cases, the result depends on the parity of s. Consequently, the analysis involves many subcases (and is rather tedious). Third, when sufficiently many subcases have been computed, it is straightforward to check that the “boundary cases” in the τ-orbits (t_{2l}) and (i_{2l}), with $l > 0$, are correctly predicted by the reflection-deletion trick. Finally, the disjointness, number and lengths of the τ-orbits are obtained by inspection. Again, the parity of s plays a role. ∎

It will turn out that every indecomposable $\mathcal{A}$-module, except the projective injectives P_j, $j > 1$, will appear in one of the τ-orbits (t_{2l}) or (i_{2l}). As was noted above, P_1 and J_1 both appear in the chain (t_0).

Irreducible morphisms and the weaving of τ-orbits — The second goal is to construct the irreducible morphisms between the indecomposable modules that have been constructed.

As $[k]$ contains s elements, Theorem 5.5.5(i) gives $2s$ irreducible morphisms, namely $\mathrm{rad}(\mathsf{P}_j) \hookrightarrow \mathsf{P}_j$ and $\mathsf{J}_j \twoheadrightarrow \mathsf{J}_j/\operatorname{soc}(\mathsf{J}_j)$, for $1 \le j \le s$. Explicitly, these are

$$\mathsf{I}_2 \longrightarrow \mathsf{P}_1, \quad \mathsf{J}_1 \longrightarrow \mathsf{I}_2; \qquad \mathsf{T}^2_{j-1} \longrightarrow \mathsf{P}_j \longrightarrow \mathsf{B}^2_{j-1} \quad (1<j<s); \qquad \mathsf{T}^1_{s-1} \longrightarrow \mathsf{P}_s \longrightarrow \mathsf{B}^1_{s-1}. \tag{5.5.30}$$

Moreover, since the P_j, with $j>1$, are non-simple, projective and injective, part (ii) of the same proposition, combined with Theorem 5.5.3(iv), gives another $4s-6$ irreducible morphisms:

$$\mathsf{T}^2_{j-1} \longrightarrow \mathsf{I}_{j-1} \longrightarrow \mathsf{B}^2_{j-1}, \quad \mathsf{T}^2_{j-1} \longrightarrow \mathsf{I}_{j+1} \longrightarrow \mathsf{B}^2_{j-1} \quad (1<j<s); \qquad \mathsf{T}^1_{s-1} \longrightarrow \mathsf{I}_{s-1} \longrightarrow \mathsf{B}^1_{s-1}. \tag{5.5.31}$$

The next step is to identify pairs of τ-orbits that may be weaved together. This happens when there exists an irreducible morphism $U \to V$ from an indecomposable U of one τ-orbit to an indecomposable V of another. Such pairs are then weaved together as in (5.5.13).

Suppose first that s is odd. Then, the sequence (t_0) contains the irreducible $\mathsf{I}_{2i+1} = \mathsf{I}_s$ and the sequence (i_2) contains the indecomposable $\mathrm{rd}\big[\mathsf{B}^3_{s-2}\big] = \mathsf{B}^2_{s-2}$ and its dual T^2_{s-2}. The irreducible morphisms $\mathsf{T}^2_{s-2} \to \mathsf{I}_s \to \mathsf{B}^2_{s-2}$ of (5.5.31) therefore allow us to weave (t_0) and (i_2) together. Moreover, (t_2) contains I_{s-2} and (5.5.31) includes $\mathsf{T}^2_{s-2} \to \mathsf{I}_{s-2} \to \mathsf{B}^2_{s-2}$, hence we may also weave (i_2) and (t_2) together. Continuing, we find that we can recursively weave contiguous pairs in the following sequence, read from left to right:

$$(t_0) \longleftrightarrow (i_2) \longleftrightarrow (t_2) \longleftrightarrow (i_4) \longleftrightarrow (t_4) \longleftrightarrow \cdots \longleftrightarrow (i_{2i}) \longleftrightarrow (t_{2i}). \tag{5.5.32}$$

For s even, we can also recursively weave contiguous pairs from (5.5.32), though the final τ-orbit (t_{2i}) is omitted. The justification for this weaving is slightly different to that for s odd because the (t_{2l}) no longer contain any irreducibles. To start, note that the irreducible morphisms $\mathsf{T}^1_{s-1} \to \mathsf{I}_{s-1} \to \mathsf{B}^1_{s-1}$ of (5.5.31) allow us to weave (t_0) with (i_2), as before. To obtain the weave with (t_2), we note that Theorem 5.5.10 implies the exactness of

$$0 \longrightarrow \begin{array}{ccccc} s-3 & & s-1 & & \\ & \searrow \; \swarrow & & \searrow & \\ & s-2 & & & s \end{array} \longrightarrow ? \oplus \begin{array}{ccc} s-1 & & \\ & \searrow & \\ & & s \end{array} \longrightarrow \mathsf{I}_{s-1} \longrightarrow 0, \tag{5.5.33}$$

for some unknown module ?, because I_{s-1} is not projective, $\tau\mathsf{I}_{s-1} \simeq \mathsf{T}^3_{s-3}$, and there is an irreducible morphism from T^1_{s-1} to I_{s-1}. The identity of the unknown module is now determined by the fact that its composition factors are I_{s-3}, I_{s-2}, I_{s-1} and the fact that $? \oplus \mathsf{T}^1_{s-1}$ has a submodule isomorphic to T^4_{s-3}: the only possible module is $? \simeq \mathsf{T}^2_{s-3}$. But now, Theorem 5.5.10 gives the existence of another irreducible morphism, this time from T^2_{s-3} to I_{s-1}. This morphism allows us to weave (i_2) and (t_2).

The iteration for s even now proceeds in the following fashion: When weaving (t_{2l}) with $(i_{2(l+1)})$, the procedure is the same as that described in the s odd case. However, when weaving (i_{2l}) with (t_{2l}), the procedure follows that described in the previous paragraph. We remark that this latter procedure would obviously fail when $s=2$; however, the only τ-orbits in this case are (t_0) and (i_2), so no such weaving is required.

Lemma 5.5.14. *The only weaves between τ-orbits are those between contiguous pairs of the sequence*

$$(t_0) \longleftrightarrow (i_2) \longleftrightarrow (t_2) \longleftrightarrow (i_4) \longleftrightarrow (t_4) \longleftrightarrow \cdots \longleftrightarrow (i_{2i}) \underbrace{\longleftrightarrow (t_{2i})}_{\text{if } s \text{ is odd}}. \tag{5.5.34}$$

Together with the morphisms (5.5.30), the morphisms obtained from these weaves form a complete list of all irreducible morphisms between indecomposable modules. The list of indecomposable modules is likewise complete.

Sketch of proof. There are two steps to this proof. First, one has to check that the irreducible morphisms described in the above procedures do actually appear in the previously woven pair. Second, one has to check that the tests described after Theorem 5.5.10 are satisfied, for all non-projective and non-injective indecomposable modules. While both steps are required to conclude that all the irreducible morphisms have been constructed, they each require nothing but patience. As in the example of Section 5.5.2, completeness follows from Theorem 5.5.12 and the fact that blocks are connected, by definition. The proof of completeness of the set of indecomposables also follows as in Section 5.5.2. ■

The complete Auslander-Reiten quiver of TL_n *and* dTL_n — The last step of the algorithm is to put together the quivers for each block of $\mathcal{A}$. Recall that a block corresponds to the partition into classes of critical and non-critical integers (see Section 5.2.1). Recall that if k is critical or if its (non-critical) orbit has length 1 ($[k] = \{k\}$), then all its extension groups are trivial and the only indecomposable with I_k as a composition factor is I_k itself. [13]

Theorem 5.5.15. *The Auslander-Reiten quiver of $\mathcal{A} = \mathsf{TL}_n$ or dTL_n (omitting TL_n, when n is even and $\beta = 0$) is the disjoint union of c connected graphs where c is the number of distinct classes $[k]$, for $k \in \Lambda_n$.*

(i) *If a class $[k]$ contains the single integer k, then the corresponding connected subgraph consists of a single vertex, labelled by the irreducible I_k, and has no arrows.*

(ii) *If a class $[k]$ contains $s \geq 2$ integers, then the connected subgraph associated with this class is constructed by computing the τ-orbits, weaving them, and then adding the injective projectives and their irreducible morphisms, as described in Theorems 5.5.13 and 5.5.14.*

We have put aside the case of $\mathcal{A} = \mathsf{TL}_n$, with n even and $\beta = 0$. However, the result for this case is now easily stated. For $\mathcal{A} = \mathsf{TL}_n$, with n even and $\beta = 0$, every $k \in \Lambda_{n,0}$ is even and belongs to the same class $\{2, 4, \ldots, n\}$. In the notation of this section, these integers are replaced by the labels $1, 2, \ldots, n/2$. The quiver for this class is then identical to the quiver described above for a class of length $s = n/2$, except for the following changes: P_1 and J_1 are replaced by T_1^1 and B_1^1, respectively (in this case, neither are projective nor injective), a new P_1 is introduced in the center of the quiver (under I_2), and the irreducible morphisms

13. In general, the case TL_2, with $\beta = 0$, provides the only counterexample to this statement.

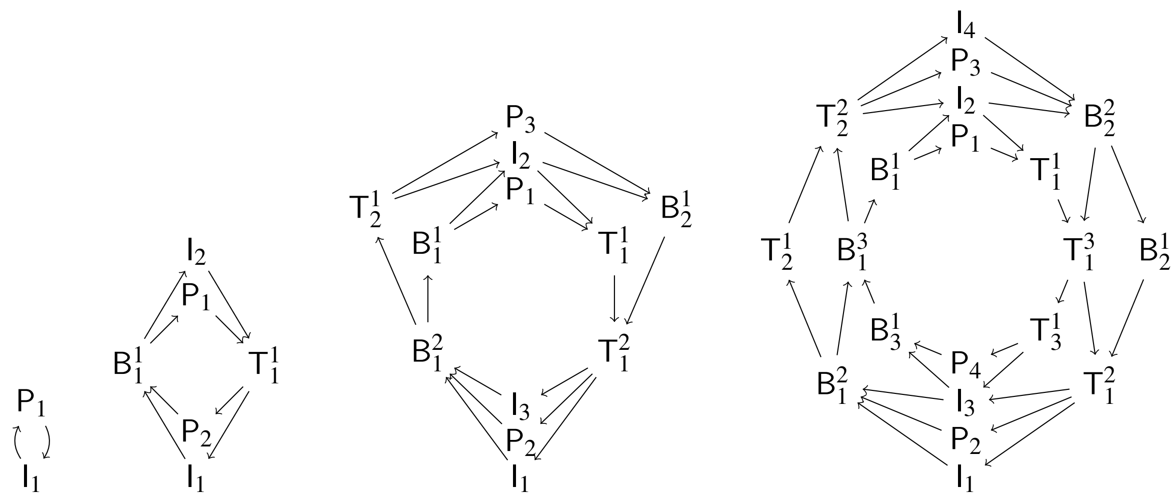


Figure 5.5.2 – The Auslander-Reiten quivers of TL_n, with $\beta = 0$, for $n = 2$, 4, 6 and 8 (left to right). Reflecting about vertical bisectors again amounts to taking the twisted dual of Section 5.2.3.

$\mathsf{B}_1^1 \to \mathsf{P}_1 \to \mathsf{T}_1^1$ are added. For completeness, we draw the quivers for $n = 2$, 4, 6 and 8 in Figure 5.5.2. Interestingly, they coincide with those of a zigzag algebra discussed in [36].

Appendix

5.A τ-orbits for $s = 6$ and 7

The τ-orbits for $s = 6$ are

$$\begin{aligned}
&\mathsf{P}_1 \rightsquigarrow \mathsf{T}_3^1 \rightsquigarrow \mathsf{T}_5^1 \rightsquigarrow \mathsf{B}_5^1 \rightsquigarrow \mathsf{B}_3^1 \rightsquigarrow \mathsf{J}_1, \quad (t_0) \qquad && \mathsf{T}_1^3 \rightsquigarrow \mathsf{T}_3^3 \rightsquigarrow \mathsf{I}_5 \rightsquigarrow \mathsf{B}_3^3 \rightsquigarrow \mathsf{B}_1^3 \rightsquigarrow \mathsf{I}_2, \quad (i_2)\\
&\mathsf{B}_2^2 \rightsquigarrow \mathsf{T}_1^5 \rightsquigarrow \mathsf{T}_3^2 \rightsquigarrow \mathsf{B}_3^2 \rightsquigarrow \mathsf{B}_1^5 \rightsquigarrow \mathsf{T}_2^2, \quad (t_2) && \mathsf{B}_1^4 \rightsquigarrow \mathsf{T}_2^4 \rightsquigarrow \mathsf{I}_4 \rightsquigarrow \mathsf{B}_2^4 \rightsquigarrow \mathsf{T}_1^4 \rightsquigarrow \mathsf{I}_3, \quad (i_4)\\
&\mathsf{B}_1^2 \rightsquigarrow \mathsf{T}_2^3 \rightsquigarrow \mathsf{T}_4^2 \rightsquigarrow \mathsf{B}_4^2 \rightsquigarrow \mathsf{B}_2^3 \rightsquigarrow \mathsf{T}_1^2, \quad (t_4) && \mathsf{T}_2^1 \rightsquigarrow \mathsf{T}_4^1 \rightsquigarrow \mathsf{I}_6 \rightsquigarrow \mathsf{B}_4^1 \rightsquigarrow \mathsf{B}_2^1 \rightsquigarrow \mathsf{I}_1. \quad (i_6)
\end{aligned} \tag{5.A.1}$$

All τ-orbits are cycles except (t_0), that is, the τ-translation of the leftmost module in each orbit is the rightmost one. Their weaves are as follows:

$$\mathsf{P}_1 \longrightarrow \mathsf{T}_1^3 \longrightarrow \mathsf{T}_3^1 \longrightarrow \mathsf{T}_3^3 \longrightarrow \mathsf{T}_5^1 \longrightarrow \mathsf{I}_5 \longrightarrow \mathsf{B}_5^1 \longrightarrow \mathsf{B}_3^3 \longrightarrow \mathsf{B}_3^1 \longrightarrow \mathsf{B}_1^3 \longrightarrow \mathsf{J}_1 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{P}_1, \quad ((t_0) \leftrightarrow (i_2))$$

$$\mathsf{B}_2^2 \longrightarrow \mathsf{T}_1^3 \longrightarrow \mathsf{T}_1^5 \longrightarrow \mathsf{T}_3^3 \longrightarrow \mathsf{T}_3^2 \longrightarrow \mathsf{I}_5 \longrightarrow \mathsf{B}_3^2 \longrightarrow \mathsf{B}_3^3 \longrightarrow \mathsf{B}_1^5 \longrightarrow \mathsf{B}_1^3 \longrightarrow \mathsf{T}_2^2 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{B}_2^2, \quad ((i_2) \leftrightarrow (t_2))$$

$$\mathsf{B}_2^2 \longrightarrow \mathsf{B}_2^4 \longrightarrow \mathsf{T}_1^5 \longrightarrow \mathsf{T}_1^4 \longrightarrow \mathsf{T}_3^2 \longrightarrow \mathsf{I}_3 \longrightarrow \mathsf{B}_3^2 \longrightarrow \mathsf{B}_1^4 \longrightarrow \mathsf{B}_1^5 \longrightarrow \mathsf{T}_2^4 \longrightarrow \mathsf{T}_2^2 \longrightarrow \mathsf{I}_4 \longrightarrow \mathsf{B}_2^2, \quad ((t_2) \leftrightarrow (i_4))$$

$$\mathsf{B}_4^2 \longrightarrow \mathsf{B}_2^4 \longrightarrow \mathsf{B}_2^3 \longrightarrow \mathsf{T}_1^4 \longrightarrow \mathsf{T}_1^2 \longrightarrow \mathsf{I}_3 \longrightarrow \mathsf{B}_1^2 \longrightarrow \mathsf{B}_1^4 \longrightarrow \mathsf{T}_2^3 \longrightarrow \mathsf{T}_2^4 \longrightarrow \mathsf{T}_4^2 \longrightarrow \mathsf{I}_4 \longrightarrow \mathsf{B}_4^2, \quad ((i_4) \leftrightarrow (t_4))$$

$$\mathsf{B}_4^2 \longrightarrow \mathsf{B}_4^1 \longrightarrow \mathsf{B}_2^3 \longrightarrow \mathsf{B}_2^1 \longrightarrow \mathsf{T}_1^2 \longrightarrow \mathsf{I}_1 \longrightarrow \mathsf{B}_1^2 \longrightarrow \mathsf{T}_2^1 \longrightarrow \mathsf{T}_2^3 \longrightarrow \mathsf{T}_4^1 \longrightarrow \mathsf{T}_4^2 \longrightarrow \mathsf{I}_6 \longrightarrow \mathsf{B}_4^2. \quad ((t_4) \leftrightarrow (i_6))$$

The τ-orbits for $s = 7$ are

$$\begin{aligned}
&\mathsf{P}_1 \rightsquigarrow \mathsf{T}_3^1 \rightsquigarrow \mathsf{T}_5^1 \rightsquigarrow \mathsf{I}_7 \rightsquigarrow \mathsf{B}_5^1 \rightsquigarrow \mathsf{B}_3^1 \rightsquigarrow \mathsf{J}_1, \quad (t_0) \qquad && \mathsf{T}_1^3 \rightsquigarrow \mathsf{T}_3^3 \rightsquigarrow \mathsf{T}_5^2 \rightsquigarrow \mathsf{B}_5^2 \rightsquigarrow \mathsf{B}_3^3 \rightsquigarrow \mathsf{B}_1^3 \rightsquigarrow \mathsf{I}_2, \quad (i_2)\\
&\mathsf{B}_2^2 \rightsquigarrow \mathsf{T}_1^5 \rightsquigarrow \mathsf{T}_3^4 \rightsquigarrow \mathsf{I}_5 \rightsquigarrow \mathsf{B}_3^4 \rightsquigarrow \mathsf{B}_1^5 \rightsquigarrow \mathsf{T}_2^2, \quad (t_2) && \mathsf{B}_2^4 \rightsquigarrow \mathsf{T}_1^6 \rightsquigarrow \mathsf{T}_3^2 \rightsquigarrow \mathsf{B}_3^2 \rightsquigarrow \mathsf{B}_1^6 \rightsquigarrow \mathsf{T}_2^4 \rightsquigarrow \mathsf{I}_4, \quad (i_4)\\
&\mathsf{B}_4^2 \rightsquigarrow \mathsf{B}_2^5 \rightsquigarrow \mathsf{T}_1^4 \rightsquigarrow \mathsf{I}_3 \rightsquigarrow \mathsf{B}_1^4 \rightsquigarrow \mathsf{T}_2^5 \rightsquigarrow \mathsf{T}_4^2, \quad (t_4) && \mathsf{B}_4^3 \rightsquigarrow \mathsf{B}_2^3 \rightsquigarrow \mathsf{T}_1^2 \rightsquigarrow \mathsf{B}_1^2 \rightsquigarrow \mathsf{T}_2^3 \rightsquigarrow \mathsf{T}_4^3 \rightsquigarrow \mathsf{I}_6. \quad (i_6)\\
&\mathsf{B}_6^1 \rightsquigarrow \mathsf{B}_4^1 \rightsquigarrow \mathsf{B}_2^1 \rightsquigarrow \mathsf{I}_1 \rightsquigarrow \mathsf{T}_2^1 \rightsquigarrow \mathsf{T}_4^1 \rightsquigarrow \mathsf{T}_6^1, \quad (t_6) &&
\end{aligned} \tag{5.A.2}$$

Again, all τ-orbits are cyclic except (t_0). Their weaves are:

$$\mathsf{P}_1 \to \mathsf{T}_1^3 \to \mathsf{T}_3^1 \to \mathsf{T}_3^3 \to \mathsf{T}_5^1 \to \mathsf{T}_5^2 \to \mathsf{I}_7 \to \mathsf{B}_5^2 \to \mathsf{B}_5^1 \to \mathsf{B}_3^3 \to \mathsf{B}_3^1 \to \mathsf{B}_1^3 \to \mathsf{B}_1^1 \to \mathsf{I}_2 \to \mathsf{P}_1, \quad ((t_0) \leftrightarrow (i_2))$$
$$\mathsf{B}_2^2 \to \mathsf{T}_1^3 \to \mathsf{T}_1^5 \to \mathsf{T}_3^3 \to \mathsf{T}_3^4 \to \mathsf{T}_5^2 \to \mathsf{I}_5 \to \mathsf{B}_5^2 \to \mathsf{B}_3^4 \to \mathsf{B}_3^3 \to \mathsf{B}_1^5 \to \mathsf{B}_1^3 \to \mathsf{T}_2^2 \to \mathsf{I}_2 \to \mathsf{B}_2^2, \quad ((i_2) \leftrightarrow (t_2))$$
$$\mathsf{B}_2^2 \to \mathsf{B}_2^4 \to \mathsf{T}_1^5 \to \mathsf{T}_1^6 \to \mathsf{T}_3^4 \to \mathsf{T}_3^2 \to \mathsf{I}_5 \to \mathsf{B}_3^2 \to \mathsf{B}_3^4 \to \mathsf{B}_1^6 \to \mathsf{B}_1^5 \to \mathsf{T}_2^4 \to \mathsf{T}_2^2 \to \mathsf{I}_4 \to \mathsf{B}_2^2, \quad ((t_2) \leftrightarrow (i_4))$$
$$\mathsf{B}_4^2 \to \mathsf{B}_2^4 \to \mathsf{B}_2^5 \to \mathsf{T}_1^6 \to \mathsf{T}_1^4 \to \mathsf{T}_3^2 \to \mathsf{I}_3 \to \mathsf{B}_3^2 \to \mathsf{B}_1^4 \to \mathsf{B}_1^6 \to \mathsf{T}_2^5 \to \mathsf{T}_2^4 \to \mathsf{T}_4^2 \to \mathsf{I}_4 \to \mathsf{B}_4^2, \quad ((i_4) \leftrightarrow (t_4))$$
$$\mathsf{B}_4^2 \to \mathsf{B}_4^3 \to \mathsf{B}_2^5 \to \mathsf{B}_2^3 \to \mathsf{T}_1^4 \to \mathsf{T}_1^2 \to \mathsf{I}_3 \to \mathsf{B}_1^2 \to \mathsf{B}_1^4 \to \mathsf{T}_2^3 \to \mathsf{T}_2^5 \to \mathsf{T}_4^3 \to \mathsf{T}_4^2 \to \mathsf{I}_6 \to \mathsf{B}_4^2, \quad ((t_4) \leftrightarrow (i_6))$$
$$\mathsf{B}_6^1 \to \mathsf{B}_4^3 \to \mathsf{B}_4^1 \to \mathsf{B}_2^3 \to \mathsf{B}_2^1 \to \mathsf{T}_1^2 \to \mathsf{I}_1 \to \mathsf{B}_1^2 \to \mathsf{T}_2^1 \to \mathsf{T}_2^3 \to \mathsf{T}_4^1 \to \mathsf{T}_4^3 \to \mathsf{T}_6^1 \to \mathsf{I}_6 \to \mathsf{B}_6^1. \quad ((i_6) \leftrightarrow (t_6))$$

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Troisième partie

Règles de fusion dans les familles de Temperley-Lieb

Chapitre 6

Présentation

6.1 Objectifs et méthodologie

Les règles de fusion sur la famille régulière de Temperley-Lieb, présentée dans la section 1.3, furent introduites dans [61, 62], puis raffinées dans [24]. Appuyés par de nombreuses analyses numériques ainsi que des arguments physiques, ces auteurs ont conjecturé que, dans la limite continue, ces règles de fusions devraient correspondre à la fusion physique d'une théorie de champs conformes (voir section 1.5). Leur calcul se base essentiellement sur l'argument suivant. L'espace d'Hilbert de la chaîne quantique XXZ (voir section 1.4.1) est à la fois un module pour l'algèbre de Temperley-Lieb $\mathsf{TL}_n(\beta)$, et pour le groupe quantique $U_q(\mathfrak{sl}_2)$; de plus l'action des deux algèbres commute. Il suit alors qu'il est possible de faire de cet espace d'Hilbert un $\mathsf{TL}_n - U_q(\mathfrak{sl}_2)$-bimodule. Il en découle une correspondance fonctorielle entre la catégorie des TL_n-modules et la catégorie des $U_q(\mathfrak{sl}_2)$-modules, la correspondance de Schur-Weyl quantique [38, 44, 46]. Or, il est possible de montrer que la catégorie des modules sur n'importe quel groupe quantique peut toujours être munie d'un produit de fusion. Leur stratégie consiste donc à faire d'un module de Temperley-Lieb un module pour $U_q(\mathfrak{sl}_2)$, de calculer la fusion à l'aide de la fusion naturelle sur ce groupe quantique, puis de réutiliser la correspondance de Schur-Weyl quantique pour retrouver un module de Temperley-Lieb. Ils parviennent alors à argumenter que, dans certains cas précis, cette procédure donne bien le même résultat que la fusion définie à l'aide des diagrammes (section 1.3).

Cette méthode a cependant deux faiblesses. La première est que la correspondance de Schur-Weyl quantique n'est pas un foncteur fidèle. Autrement dit, il existe des TL_n-modules non isomorphes qui correspondent au même $U_q(\mathfrak{sl}_2)$-module. Il existe donc des cas où leur stratégie ne fonctionnera pas, ce qu'ils ont bien remarqué. Ces cas devront donc être calculés différemment. La seconde faiblesse est la suivante : alors que la définition du produit de fusion se généralise aisément à d'autres algèbres de diagrammes, comme les algèbres diluées de Temperley-Lieb, la correspondance entre ces algèbres et les groupes quantiques est beaucoup plus obscure.

Notre premier objectif est donc de trouver une façon de généraliser ces règles de fusion pour d'autres algèbres, dont la version diluée de l'algèbre de Temperley-Lieb. Ensuite, il nous faut trouver une façon de calculer le produit de fusion directement à partir de la définition, sans recourir à des correspondances avec des groupes quantiques. Nous pouvons alors montrer la force de la méthode en calculant les produits de fusion pour les modules de Temperley-Lieb qui n'ont pas été couverts par les résultats de [24, 61], ainsi que la fusion des modules dilués.

L'article est structuré comme suit. La première section contient un rappel des résultats obtenus dans [10, 11, 63] concernant la théorie des représentations des algèbres de Temperley-Lieb régulière et diluée. Ensuite, nous décrivons la fusion généralisée et prouvons les propriétés élémentaires dont nous aurons besoin pour calculer les produits de fusion. On peut alors calculer les fusions entre modules projectifs et modules standards, reproduisant ainsi les résultats de [24, 61]. Les fusions entre modules irréductibles sont alors calculées ; celles-ci requièrent le calcul des produits de fusion de modules beaucoup plus exotiques (les $\mathsf{B}_j^{n,k}$ et les $\mathsf{T}_j^{n,k}$), qui est relégué dans un appendice, comme l'est aussi le calcul de certains quotients de fusion.

Un des résultats important de l'article prouve les règles de fusion pour les modules irréductibles $\mathsf{I}_{n,k}$ avec $k < \ell - 1$ (voir la section 1.3), qui échappaient à la procédure de [24, 61]. Nous montrons ainsi que ceux-ci obéissent à des règles de fusion identiques à celles satisfaites par un sous-ensemble des champs primaires d'une théorie de champs conformes minimale.

Les outils algébriques utilisés dans le papier sont essentiellement les mêmes que ceux utilisés dans les deux premières parties.

Chapitre 7

The fusion rules for the Temperley-Lieb algebra and its dilute generalisation

The fusion rules for the Temperley-Lieb algebra and its dilute generalisation

Jonathan Belletête

ABSTRACT:The Temperley-Lieb (TL) family of algebras is well known for its role in building integrable lattice models. Even though a proof is still missing, it is agreed that these models should go to conformal field theories in the thermodynamic limit and that the limiting vector space should carry a representation of the Virasoro algebra. The fusion rules are a notable feature of the Virasoro algebra. One would hope that there is an analogous construction for the TL family. Such a construction was proposed by Read and Saleur [Nucl. Phys. B 777, 316 (2007)] and partially computed by Gainutdinov and Vasseur [Nucl. Phys. B 868, 223-270 (2013)] using the bimodule structure over the Temperley-Lieb algebras and the quantum group Uq(sl2).

We use their definition for the dilute Temperley-Lieb (dTL) family, a generalisation of the original TL family. We develop a new way of computing fusion by using induction and show its power by obtaining fusion rules for both dTL and TL. We recover those computed by Gainutdivov and Vasseur and new ones that were beyond their scope. In particular, we identify a set of irreducible TL- or dTL-representations whose behavior under fusion is that of some irreducibles of the CFT minimal models.

Keywords dilute Temperley-Lieb algebra · Temperley-Lieb algebra · fusion rules · dilute loop models · Virasoro algebra

7.1 Introduction

The Temperley-Lieb family of algebras $\{\mathsf{TL}_n(q)\}_{n\in\mathbb{Z}_{>0}}$, introduced in [66], is well-known for its use in building integrable lattice models that correspond to a large variety of different physical systems [45, 56], particularly to quantum spin chains. Many properties of these physical models can be interpreted in terms of the algebraic properties of the family, which can be obtained by studying the representation theory of these algebras. As such, it has received a lot of attention over the years. Since its introduction, many generalizations have been proposed: the periodic Temperley-Lieb algebras [18, 27–29], the boundary or blob algebras [31], the multi-colored Temperley-Lieb algebras [32], etc... One such generalization which is of particular interest is the dilute Temperley-Lieb family $\{\mathsf{dTL}_n(q)\}_{n\in\mathbb{Z}_{>0}}$ [11, 30], which has been introduced to build dilute lattice models, i.e., ones where lattice sites can be empty.

It has been conjectured that the lattice models built from TLn(q), for example loop models with particular boundary conditions, should correspond, in the continuum limit, to logarithmic conformal field theories [43, 58, 60] when the parameter q is a root of unity. A consequence of these conjectures is that modules over the Temperley-Lieb family should bear a structure of Virasoro-module when n goes to infinity. If these conjectures were true, it would give a relatively simple method for building and studying LCFTs. In order to study these conjectures, or at least to support them, there has been a lot of interest towards identifying similar algebraic structures between TL_n and the Virasoro algebra, like module structure [10, 25, 63, 70] and fusion rules [24, 59, 61, 62].

The word “fusion” was first used for the operator product expansion of two conformal fields [8]. Fusion, from that physical point of view, describes how fields interact at short distance. Because of the state-operator correspondence, it quickly evolved into a more mathematical construction defining a product between modules over the algebras underlying the theory. There is now a definition, within the context of vertex operator algebra, of fusion over some simple chiral algebras. But for more general conformal field theories such a definition remains a field of active research [35]. Even for those algebras where the definition is agreed upon, computing fusion explicitly has proven to be very challenging. The recursive algorithm described by Nahm [55] and developed by Gaberdiel and Kausch [23] remains the leading tool.

Because of the conjecture relating the continuum limit of lattice models built out of the Temperley-Lieb algebra on the one hand, and conformal field theories on the other hand, there have been efforts to construct a “fusion” between Temperley-Lieb modules that would mimic the fusion product between Virasoro modules. One such suggestion, proposed by Read and Saleur [61, 62] and later studied by Gainutnidov and Vasseur [24], is built around the following description. To compute the fusion product between two spin chains, one joins them together at one of their extremities and then one lets them evolve. While heuristic, they used this idea to build a purely categorical description of the fusion rules which, while motivated from spin chain analysis, rely entirely on algebraic properties of the algebras. It is with this meaning that the word “fusion” is used in the present paper.

Instead of computing these fusion rules directly, Gainutnidov and Vasseur opted to follow a route closer to how these rules are defined in the Virasoro algebra. There, fusion is defined by first pushing modules to modules over a quantum group, using the co-multiplication on Virasoro, and are then pulled back to modules over the Lie algebra. However, there is no co-multiplication on TL_n, so they instead used the quantum Schur-Weyl duality between TL_n and the quantum group $U_q(sl_2)$ [38, 44, 46]. Modules over TL_n are first pushed to modules over this quantum group, where the co-multiplication naturally defines a fusion product, and the result is then pulled back to TL_n. They then argued that the resulting construction was equivalent to Read and Saleur’s original one. Using this argument, they were able to compute fusion rules for most of the main classes of Temperley-Lieb modules [24].

We are interested in generalizing this construction for the other, more exotic Temperley-Lieb algebras, in particular, the dilute dTL_n. While generalizing Read and Saleur’s construction is simple enough, generalizing Gainutnidov and Vasseur’s argument is not, mainly because the duality between dTL_n and $U_q(sl_2)$ is not so clear. Our goal is thus to compute directly this fusion product, without using this duality. We instead rely purely on category theory and the representation theory of TL_n and dTL_n.

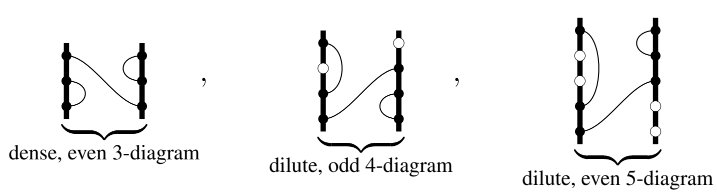
The outline of the paper is as follows. In section 7.2, we present a quick overview of the the representation theory of the regular TL_n and the dilute dTL_n families. None of these results are proved here; the reader can consult [10, 11, 63] for their proofs. In section 7.3, we present the generalization of Read and Saleur’s construction for general family of algebras and then for dilute and regular cases. A natural consequence of this construction is the existence of a dual product, the *fusion quotient*. Studying this new operation is beyond the scope of this paper but some results are nevertheless presented in appendix 7.C. The fusion of projective modules is studied in section 7.4. These turn out to admit a representation in terms of Chebyshev polynomials of the second kind. In section 7.5, we study the fusion of standard modules, first with projective modules and then with other standard modules. Fusion rules for irreducible modules are first studied in sections 7.6.1 and 7.6.2. These show the appearance of two other classes of modules, the Bs and the Ts. The fusion rules for those are studied in appendix 7.B. Fusion rules for pairs of irreducible modules are finally computed in section 7.6.3. In particular, a subset of irreducible modules is shown to behave under fusion like primary fields in a minimal model of the Virasoro algebra.

7.2 Temperley-Lieb algebras

The results of this section first appeared in [25, 45, 70]. The definitions and results presented here are based on [10, 11, 63].

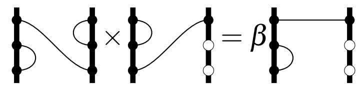
The Temperley-Lieb algebras can be defined in terms of generators or in terms of diagrams. The later is presented here and will be used throughout the paper as it gives a more intuitive description of the fusion product. After introducing this definition, the classes of indecomposable modules are introduced in terms of extensions. Loewy diagrams are given and can be used as a quick way of assessing the various properties of these modules. Finally, the algebra's families are described in terms of the induction and restriction functors.

The basic objects, n-diagrams, are first introduced. Draw two vertical lines, each with n points on it, n being a positive integer. Choose first $2m$ points, $0 \leq m \leq n$ an integer, and put a $\circ$ on each of them. A point with a $\circ$ will be called a *vacancy*. Now connect the remaining points, pairwise, with non-intersecting strings. The resulting object is called a *n-diagram*. If the diagram contains no vacancy, it is said to be *dense*, and is called *dilute* otherwise. If the number of vacancies on the left side of a n-diagram is odd (even), it is called *odd*, (*even*). For example,



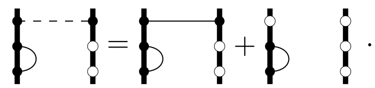
On the set of formal linear combinations of all n-diagrams a product is defined by extending linearly the product of two n-diagrams obtained as follows. The two diagrams are put side by side, the inner borders and the points on them are identified, then removed. A string which no longer ties two points is called a *floating string*. A floating string that closes on itself is called a *closed loop*. If all floating strings are closed loops, the result of the product of the two n-diagrams is then the diagram obtained by reading the vacancies on the left and right vertical lines and the strings between them multiplied by a factor of $\beta = q + q^{-1}$, q a non-zero complex number, for each closed loop. Otherwise, the product is the zero element of the algebra.

For example,



If q is a root of unity, the integer ℓ is defined as the smallest strictly positive integer such that $q^{2\ell} = 1$. If q is not a root of unity, ℓ is said to be infinite.

A dashed string represents the formal sum of two diagrams: one where the points are linked by a regular string, and one where the points are both vacancies. For example,



Note that the diagram where each point is linked by a dashed line to the corresponding point on the opposite side acts as the identity on all n-diagrams and is a sum of 2^n n-diagrams.

Note finally that the product is clearly associative: the reading of how the left and right sides are connected in a product of three diagrams is blind to the order of glueing, and so is the number of closed loops. The set of n-diagrams with the formal sum with complex number coefficients and the product just introduced is the dilute Temperley-Lieb algebra $\mathsf{dTL}_n = \mathsf{dTL}_n(\beta)$. The subset spanned by only even (odd) diagrams is closed under the product and this subalgebra will be called the even (odd) dilute Temperley-Lieb subalgebra, denoted by edTL_n (odTL_n). Clearly any dilute n-diagram is either even or odd. Since the product of two diagrams of distinct parities is zero, it is clear that the even and odd subalgebras are two-sided ideals of dTL_n and

$$\mathsf{dTL}_n = \mathsf{edTL}_n \oplus \mathsf{odTL}_n.$$

A module on which every odd (even) diagram acts as zero is called even (odd). It follows that every module can be split into a direct sum of an even, and an odd modules.

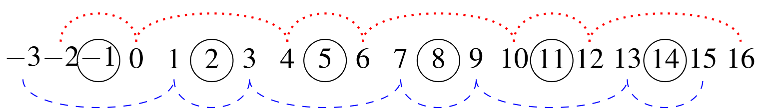
The regular Temperley-Lieb algebra $\mathsf{TL}_n = \mathsf{TL}_n(\beta)$ is obtained by considering only dense diagrams, that is, those containing no vacancies. As such, every non-zero TL_n-module is even. In the case $\beta = 0$ ($\ell = 2$), the structure of TL_n will be slightly more complicated than for the other cases. It will thus be treated separately in many calculations and definitions.

7.2.1 The indecomposable modules

Since the Temperley-Lieb algebras are finite dimensional associative algebras over the complex numbers, they have finitely many non-isomorphic, irreducible modules. In both algebras, these can be indexed by a single integer $0 \leq k \leq n$, which must be of the same parity as n in TL_n, and are written $\mathsf{I}_{n,k}$. The only exception is when $\ell = 2$ in TL_n, where $\mathsf{I}_{n,0} \equiv 0$.

These integers k are first classified in orbits. If ℓ is a finite number, an integer $k \geq 0$ is said to be *critical*, and is written k_c if $k + 1 \equiv 0 \mod \ell$. If ℓ is not a finite number, every integer is said to be critical; this is also the case if $\ell = 1$. For a non-critical integer k, define k_1 to be the smallest non-critical integer strictly bigger than k such that $(k_1 + k)/2 + 1 \equiv 0 \mod \ell$. Similarly, define k_{-1} to be the biggest non-critical integer strictly smaller than k such that $(k_{-1}+k)/2+1 \equiv 0 \mod \ell$. Define inductively $(k_i)_j = k_{i+j}$, so that for instance $(k_1)_{-1} = k_0 = k$. Two integers r,k are then said to be in the same orbit if there exist $i \in \mathbb{Z}$ such that $r = k_i$; the modules $\mathsf{I}_{n,k}, \mathsf{I}_{n,r}$ are also said to be on the same orbit. The irreducible modules I_{n,k_c} are each alone on their orbit. For instance, when $\ell = 3$, figure 7.2.1 shows the orbits between -3 and 16.

Figure 7.2.1 – Orbits when $\ell = 3$: the critical numbers are circled, and the two other orbits are linked with dashed, and dotted lines respectively.



Proposition 7.2.1. *In* TL_n*, for* $0 \leq r,k \leq n$*,*

$$\mathrm{Ext}(\mathsf{I}_{n,r},\mathsf{I}_{n,k}) \simeq \begin{cases} 0 & \text{if } \ell = 2 \text{ and } r = 0 \text{ or } k = 0, \\ \mathbb{C} & \text{if } \ell = n = 2 \text{ and } r = k = 2, \\ \mathbb{C}\delta_{r,k_{\pm 1}}. & \text{otherwise} \end{cases} \tag{7.2.1}$$

In dTL_n*, for* $0 \leq r,k \leq n$*,*

$$\mathrm{Ext}(\mathsf{I}_{n,r},\mathsf{I}_{n,k}) \simeq \mathbb{C}\delta_{r,k_{\pm 1}}. \tag{7.2.2}$$

There is then a unique indecomposable module $\mathsf{S}_{n,k}$, up to isomorphism [1], satisfying the short exact sequence

$$0 \longrightarrow \mathsf{I}_{n,k_1} \longrightarrow \mathsf{S}_{n,k} \longrightarrow \mathsf{I}_{n,k} \longrightarrow 0. \tag{7.2.3}$$

This defines the *standard module* $\mathsf{S}_{n,k}$. In TL_n, when $\ell = 2$, $\mathsf{I}_{n,0} = 0$, so that $\mathsf{S}_{n,0} \equiv \mathsf{I}_{n,2}$. Note also that if $k_1 > n$, the module I_{n,k_1} is simply not defined, in which case $\mathsf{S}_{n,k} \simeq \mathsf{I}_{n,k}$. It is generally consistent to set undefined irreducible modules to the zero module; we shall use this convention unless otherwise noted.

There is also a unique indecomposable module $\mathsf{C}_{n,k}$, satisfying the short exact sequence

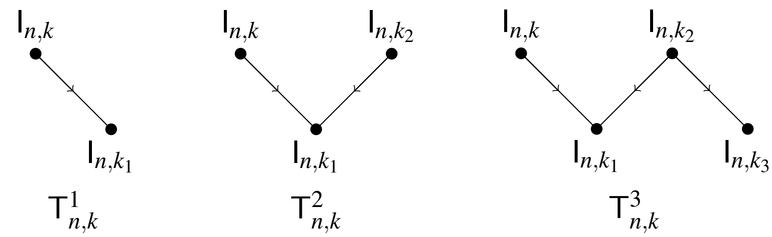
$$0 \longrightarrow \mathsf{I}_{n,k} \longrightarrow \mathsf{C}_{n,k} \longrightarrow \mathsf{I}_{n,k_1} \longrightarrow 0. \tag{7.2.4}$$

This defines the *dual standard module* $\mathsf{C}_{n,k}$.

Let $\mathsf{T}^1_{n,k} = \mathsf{S}_{n,k}$, then $\mathsf{T}^{2i}_{n,k}$ is defined as the unique indecomposable extension of $\mathsf{I}_{n,k_{2i}}$ by $\mathsf{T}^{2i-1}_{n,k}$ and $\mathsf{T}^{2i+1}_{n,k}$ as the unique indecomposable extension of $\mathsf{T}^{2i}_{n,k}$ by $\mathsf{I}_{n,k_{2i+1}}$. Figure 7.2.2 shows the Loewy diagrams of the smallest T modules.

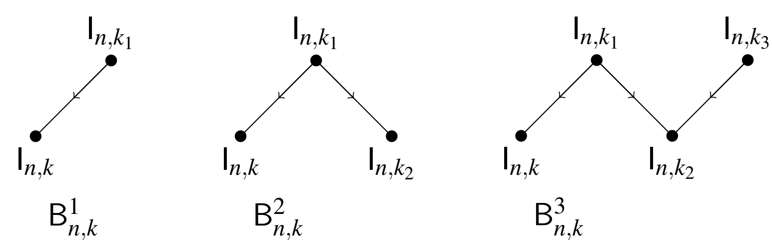
1. Whenever we say that a module is unique, we will always mean "up to isomorphism", but it will not always be mentioned.

Figure 7.2.2 – Loewy diagrams of some T modules.



Similarly, let $\mathsf{B}^1_{n,k} = \mathsf{C}_{n,k}$ and define $\mathsf{B}^{2i}_{n,k}$ as the unique indecomposable extension of $\mathsf{B}^{2i-1}_{n,k}$ by $\mathsf{I}_{n,k_{2i}}$, and $\mathsf{B}^{2i+1}_{n,k}$ as the unique indecomposable extension of $\mathsf{I}_{n,k_{2i+1}}$ by $\mathsf{B}^{2i}_{n,k}$. Figure 7.2.3 shows the Loewy diagrams of a few B modules.

Figure 7.2.3 – Loewy diagrams of some B modules.



The P modules are defined a bit differently. In the case $\ell = 2$ of TL_n, P_0 is the zero module and if $n = 2$, P_2 is the unique indecomposable self-extension of $\mathsf{I}_{2,2}$. For all other cases, when k is critical or smaller than $\ell - 1$, $\mathsf{P}_{n,k} = \mathsf{S}_{n,k}$; otherwise, $\mathsf{P}_{n,k}$ is the unique indecomposable extension of $\mathsf{S}_{n,k}$ by $\mathsf{S}_{n,k_{-1}}$. Figure 7.2.4 shows the Loewy diagrams of the P modules except for the case $\ell = 2$ in TL_n which are shown in figure 7.2.5.

Figure 7.2.4 – The Loewy diagrams of the P modules.

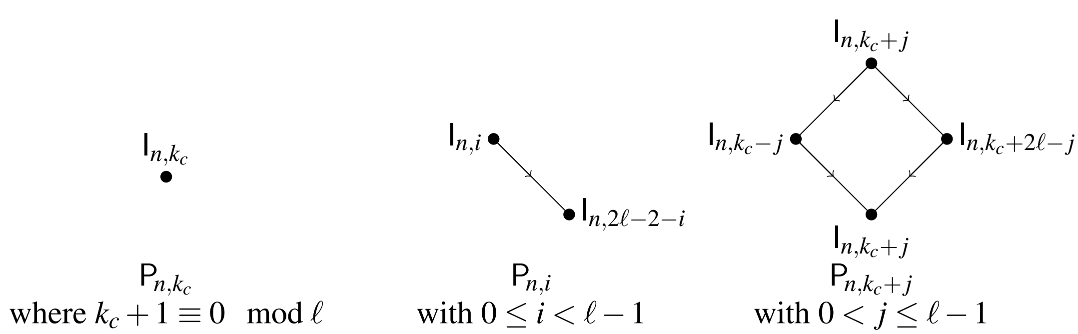
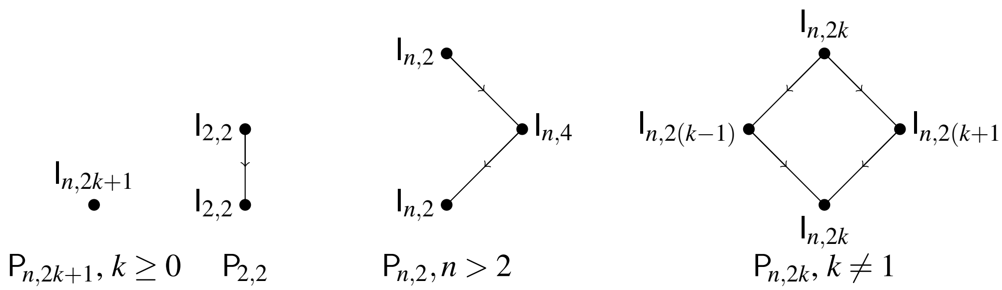


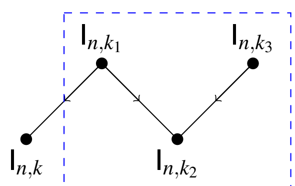
Figure 7.2.5 – The Loewy diagrams of the P modules in TL_n with $\ell = 2$.



These modules satisfy several exact sequences which can all be read from their Loewy diagrams. For example, the short exact sequence

$$0 \longrightarrow \mathsf{I}_{n,k} \longrightarrow \mathsf{B}^3_{n,k} \longrightarrow \mathsf{T}^2_{n,k_1} \longrightarrow 0,$$

can be seen by noticing that in the Loewy diagram of $\mathsf{B}^3_{n,k}$, the part circled in a dashed line is precisely the Loewy diagram of T^2_{n,k_1}:



The Hom spaces can similarly be read off their diagrams.

Proposition 7.2.2. • *The P modules are all projective; they form a complete set of non-isomorphic indecomposable projective modules.*

- $\mathsf{P}_{n,k}$ *is the projective cover of* $\mathsf{I}_{n,k}$.

- *If* $f : \mathsf{B}^{2i}_{n,k} \to \bigoplus_{j=0}^{i} \mathsf{P}_{n,k_{2j}}$ *is injective, then* $\mathrm{Coker}\, f \simeq \mathsf{B}^{2(i+1)}_{n,k_{-1}}$. *If* $k_{-1} < 0$, *there are no such morphism.*

- *If* $g : \mathsf{T}^{2i+1}_{n,k_{-1}} \to \bigoplus_{j=0}^{i} \mathsf{P}_{n,k_{2j}}$ *is injective, then* $\mathrm{Coker}\, g \simeq \mathsf{T}^{2i+1}_{n,k}$.

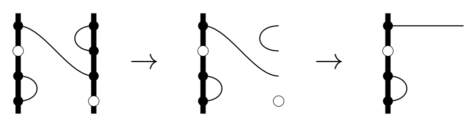
- *The modules* $\mathsf{P}_{n,k}$ *are injective for all* $k \geq \ell - 1$, *and the modules* $\mathsf{B}^1_{n,k}$ *are also for all* $k < \ell - 1$, *except if* $\ell = 2$ *in* TL_n *in which case* $\mathsf{B}^1_{n,0}$ *is not injective. They form a complete set of non-isomorphic indecomposable injective modules.*

- *The injective hull of* $\mathsf{I}_{n,k}$ *is* $\mathsf{B}^1_{n,k}$ *if* $k < \ell - 1$ *and* $\mathsf{P}_{n,k}$ *otherwise.*

7.2.2 A basis of $\mathsf{S}_{n,k}$

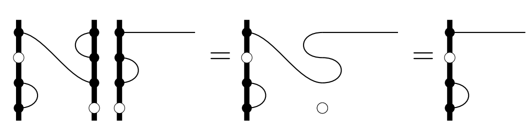
Our computations will almost all be based on the short-exact sequences satisfied by the various modules and on their homological properties, they will therefore be completely independent of a choice of basis. However, a basis of the standard module $\mathsf{S}_{n,k}$ will be needed. The bases we present here are the usual ones used in the Temperley-Lieb algebras so the reader should feel free to skip this section if they are already familiar with them.

Start by defining the basic objects, the n-link diagrams, which are built in the following way. First, take a dilute n-diagram and remove its right (left) side as well as the points that were on it. An object, whether it is a string or a vacancy that no longer touches any point, is simply removed. The other floating strings are straightened out and called *defects*. For example,



The resulting diagram is called a *left n-link* (*right n-link*). It is seen that a dilute n-diagram induces a unique pair of one left and one right n-link diagrams and that, given such a pair, there can be at most one n-diagram, if any, that could have induced them. It will thus be useful to denote an n-diagram by its induced n-links, $b = |lr|$, where l (r) is the left (right) link diagram induced from b. This notation can also be used for linear combinations of n-diagrams as in $b = |(l+j)r| + |uv|$ where l, j, u are left n-links and r, v right ones. If u is a left link, then $\bar{u}$ will denote its (right) mirror image.

A natural action can be defined of n-diagrams on left (and right) n-link diagrams. We start with the left action. Draw the n-diagram on the left side of the left n-link, identify the points on its right side with those on the link and remove them. Each floating string that is not connected to the remaining side is removed and yields a factor β if it is closed and zero if it opened, or touches a vacancy. If a floating string starting on the remaining side is connected to a defect in the n-link diagram, it becomes a defect. Finally, remove any remaining vacancies on the right side. The remaining drawing is the resulting n-link diagram, weighted by factors of β, one for each closed floating strings. For example



Proposition 7.2.3. *Over* dTL_n*, the formal sums of all n-link diagrams having exactly k-defects, with the action defined above, defines a basis of* $\mathsf{S}_{n,k}$.

Over TL_n*, the formal sums of all n-link diagrams having exactly k-defects and no vacancies, with the action defined above, defines a basis of* $\mathsf{S}_{n,k}$.

7.2.3 The Temperley-Lieb families

There is a natural inclusion of the symmetric group S_n into S_{n+1}. There are similar inclusion for the Temperley-Lieb algebras. Consider the following transformation: take a n-diagram and add a dashed line at its bottom. The result is an element of dTL_{n+1}. Similarly, taking a dense n-diagram and adding a straight line at its bottom yields a dense $(n+1)$-diagram which is an element of TL_{n+1}. Extending the first transformation linearly gives a subalgebra of dTL_{n+1} isomorphic to dTL_n, while doing the same thing to the second yields a subalgebra of TL_{n+1} isomorphic to TL_n. There are thus two ascending families of algebras

$$\mathsf{dTL}_1 \subset \mathsf{dTL}_2 \subset \mathsf{dTL}_3 \subset \dots, \qquad \text{and } \mathsf{TL}_1 \subset \mathsf{TL}_2 \subset \mathsf{TL}_3 \subset \dots$$

The functor $-\uparrow_n^{n+1}$ is the induction functor from dTL_n to dTL_{n+1}, or from TL_n to TL_{n+1}. While this really defines multiple functors, they will have similar properties so we write them all $-\uparrow$, unless it is not clear which one we are talking about from the context. The induction functor from a subalgebra B to an algebra A is always a right-exact linear functor defined on all B-module U by

$$U\uparrow = A \otimes_B U,$$

where A is seen as a left A-module and a right B-module, and the index B next to the tensor product sign means that elements of B can pass freely through it.

As the induction functors "moves up" along the families, its adjoint, the restriction functor $-\downarrow$ "moves down", taking dTL_{n+1}-modules to dTL_n-modules or TL_{n+1}-modules to TL_n-modules. The restriction functor from an algebra A to a subalgebra B is always an exact, linear functor defined on an A-module V by

$$V\downarrow = \mathrm{Hom}_A(A, V),$$

where A is seen as a left A-module and a right B-module.

These functors have been computed before for all indecomposable modules over either family of Temperley-Lieb algebras in [10, 11, 63]. These results will be very important for computing the fusion rules and they will be stated where they are needed.

7.3 The fusion ring

Fusion is first defined for left modules over a general family of algebras. This definition is a straightforward generalization of the definition in [24, 61, 62], which works for the regular Temperley-Lieb family. Some general results are then proven before studying fusion in the Temperley-Lieb families.

7.3.1 The fusion product

Consider $(A_i)_{i\in\mathbb{N}}$ a family of associative algebras over $\mathbb{C}$ such that for all positive integers i, j the tensor algebra $A_i \otimes_{\mathbb{C}} A_j$ is isomorphic to a subalgebra of A_{i+j}. The tensor algebra $A_i \otimes_{\mathbb{C}} A_j$ is defined such that $(a\otimes b)(c\otimes d) = ac\otimes bd$ for all $a,c\in A_i$ and all $b,d\in A_j$. Given U a A_i-module and V a A_j-module, the *fusion* of U and V is defined as

$$U\times_f V = A_{i+j}\otimes_{A_i\otimes_{\mathbb{C}}A_j}(U\otimes_{\mathbb{C}} V). \tag{7.3.1}$$

Note that $U\otimes_{\mathbb{C}} V$ is naturally a $A_i\otimes_{\mathbb{C}} A_j$-module. The fusion can thus be seen as a simple induction from $A_i\otimes_{\mathbb{C}} A_j$ to A_{i+j} and, hence, $U\times_f V$ is an A_{i+j}-module. Note that to each induction functor corresponds an adjoint restriction functor. As such, there exists a construction adjoint to the fusion product which is called the *fusion quotient*. This construction will only be used while computing the fusion product of irreducible modules, and the argument to obtain the needed fusion quotients are slightly different from those used to compute fusion products. These results will therefore be presented in appendix 7.C.

The following propositions follow readily from the properties of tensor products.

Proposition 7.3.1. *For U,V two A_i-modules and W,Z two A_j-modules,*

$$(U\oplus V)\times_f(W\oplus Z)\simeq (U\times_f W)\oplus(U\times_f Z)\oplus(V\times_f W)\oplus(V\times_f Z).$$

Furthermore, if U and W are projective then so is $U\times_f W$.

Proof. The first result follows readily from the linearity of tensor products.

Suppose now that U and W are two projective A_i- and A_j-modules respectively. By definition, this means that there are two sets Λ and Σ and two projective modules P and Q such that $A_i^\Lambda \simeq U\oplus P$ and $A_j^\Sigma\simeq W\oplus Q$. Here A_i^Λ is a direct sum of copies of A_i indexed by the elements of Λ and similarly for A_j^Σ. Using the first result,

$$A_i^\Lambda\times_f A_j^\Sigma \simeq (U\times_f W)\oplus(U\times_f Q)\oplus(P\times_f W)\oplus(P\times_f Q)\simeq A_{i+j}^\Gamma, \tag{7.3.2}$$

were Γ is a set whose elements are the pairs (λ,σ) with $\lambda\in\Lambda,\sigma\in\Sigma$. The second equality is obtained by noting that the induction to an algebra A of a subalgebra B is always isomorphic to A. Since A_{i+j}^Γ is a free module by definition, $U\times_f W$ is projective. ∎

Proposition 7.3.2. *If the sequence*

$$0\longrightarrow U\xrightarrow{f} V\xrightarrow{g} W\longrightarrow 0$$

of A_i-modules is exact, the sequence of A_{i+j}-modules

$$U\times_f S\xrightarrow{\bar f} V\times_f S\xrightarrow{\bar g} W\times_f S\longrightarrow 0$$

is also exact for all A_j-modules S.

Proof. Note that $\mathbb{C}$ is semi-simple, so that all $\mathbb{C}$-modules are flat. The sequence of $A_i \otimes_{\mathbb{C}} A_j$-modules

$$0 \longrightarrow U \otimes_{\mathbb{C}} S \stackrel{f\otimes_{\mathbb{C}}\mathrm{id}_{A_j}}{\longrightarrow} V \otimes_{\mathbb{C}} S \stackrel{g\otimes_{\mathbb{C}}\mathrm{id}_{A_j}}{\longrightarrow} W \otimes_{\mathbb{C}} S \longrightarrow 0,$$

is therefore exact. The conclusion is obtained by using the fact that induction is right-exact. ■

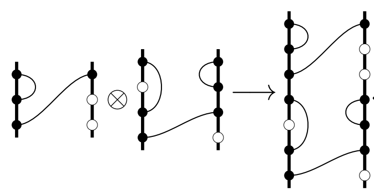
It should also be noted that for any A_i-module U,

$$U \times_f A_j = A_{i+j} \otimes_{A_i \otimes_{\mathbb{C}} A_j} U \otimes_{\mathbb{C}} A_j \simeq A_{i+j} \otimes_{A_i} U, \tag{7.3.3}$$

which is simply the induction functor from A_i to A_{i+j}. Note also that just like the induction functor, it will depend on the actual embedding $A_i \to A_{i+j}$.

7.3.2 Fusion on the dilute Temperley-Lieb family

Of the many ways of including dTL_n as a subalgebra of dTL_{n+p}, we focus on two. The first is to insert p dashed lines at the bottom of every diagram in dTL_n and the other is to add them at the top. The simplest way to define the inclusion of $\mathsf{dTL}_n \otimes \mathsf{dTL}_p$ in dTL_{n+p} is thus to draw the diagram $a \in \mathsf{dTL}_n$ on top of $b \in \mathsf{dTL}_p$. For example,



(7.3.4)

Notice that we could have defined it the other way around, drawing b on top of a. It can be shown that the two inclusions yield isomorphic bi-module structures on dTL_{n+m}. It follows that fusion is commutative on the dilute Temperley-Lieb family.

Proposition 7.3.3. *For U,V, modules over dTL_n and dTL_p, respectively,*

$$U \times_f V \simeq V \times_f U.$$

Note that the inclusion used is compatible with the parity of diagrams. Take a,b two diagrams with well-defined parity in dTL_n and dTL_m, respectively. If a is odd but b is even, $a \otimes b$ is odd while if they are both odd or even, $a \otimes b$ is even. It follows that fusing two modules with the same parity yields an even module while if their parities are different it yields an odd one. Note also that fusing a module with dTL_1 gives the induction of this module as defined in 7.2.3. Since $\mathsf{dTL}_1 \simeq \mathsf{P}_{1,1} \oplus \mathsf{P}_{1,0}$ the following proposition is obtained.

Proposition 7.3.4. *For a dTL_n-module V with a well-defined parity,*

$$V \times_f \mathsf{dTL}_1 \simeq V\!\uparrow \simeq V \times_f \mathsf{P}_{1,0} \oplus V \times_f \mathsf{P}_{1,1},$$

$V \times_f \mathsf{P}_{1,1}$ has the same parity as V, while $V \times_f \mathsf{P}_{1,0}$ has a different parity.

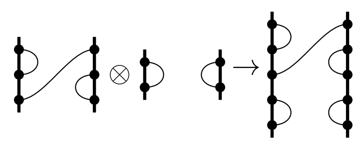
Furthermore, tensor products are associative and it is easy to verify that the chosen inclusion process is also. It thus follows that the fusion algebra of the dilute Temperley-Lieb family is associative.

Proposition 7.3.5. *For U a dTL_n-module, V a dTL_m-module and W a dTL_p-module,*

$$(U \times_f V) \times_f W \simeq U \times_f (V \times_f W). \tag{7.3.5}$$

7.3.3 Fusion on the regular Temperley-Lieb algebra

Fusion for the regular Temperley-Lieb family is very similar to that on the dilute family. Again inclusion of TL_n in TL_{n+p} can be obtained by adding straight lines below or above an n-diagram and inclusion of $\mathsf{TL}_n \otimes \mathsf{TL}_p$ in TL_{n+p} by drawing n-diagrams atop p-diagrams. For example,



(7.3.6)

The definition mimics very closely that on the dilute family and the proofs of the various results will be nearly identical. In particular, the same arguments yields the following proposition.

Proposition 7.3.6. *For U a TL_n-module, V a TL_m-module and W a TL_k-module,*

$$U \times_f \mathsf{TL}_1 \simeq U \times_f \mathsf{S}_{1,1} \simeq U\uparrow, \tag{7.3.7}$$

$$U \times_f V \simeq V \times_f U, \tag{7.3.8}$$

$$(U \times_f V) \times_f W \simeq U \times_f (V \times_f W). \tag{7.3.9}$$

7.4 Fusion of projective modules

It was proved in proposition 7.3.1 that the fusion of two projective modules always yields a projective module. Since the projective modules of the Temperley-Lieb algebras are all known, it is natural to start by computing their fusion rules. The projective indecomposables of dTL_n and TL_n falls in three different classes (see section 7.2.1), the standard modules $\mathsf{S}_{n,k} = \mathsf{P}_{n,k}$ with $k < \ell - 1$, which we will often call the *small projectives*, the standard modules $\mathsf{S}_{n,k_c} = \mathsf{P}_{n,k_c}$ where k_c is critical and the projective indecomposable P_{n,k_c+i} for $0 < i < \ell$. We use the same notation for the two families, but recall that in dTL_n, modules such as $\mathsf{P}_{n,k}$ are defined for all integer $k \in [0,n]$, while in TL_n, they are only defined when $k \equiv n \mod 2$. Propositions 7.3.4 and 7.3.6 show that fusion is closely related to the process of induction; the following proposition gives the induction of projective modules [10, 11].

Proposition 7.4.1. *For all critical k_c, $0 < i < \ell$, and $n-1 \geq k_c + i$,*

$$\mathsf{P}_{n-1,i-1}\!\uparrow \simeq \begin{cases} \mathsf{P}_{n,i-2} \oplus \mathsf{P}_{n,i-1} \oplus \mathsf{P}_{n,i}, & \text{on } \mathsf{dTL}_n \\ \mathsf{P}_{n,i-2} \oplus \mathsf{P}_{n,i}, & \text{on } \mathsf{TL}_n \end{cases}, \tag{7.4.1}$$

$$\mathsf{P}_{n-1,k_c}\!\uparrow \simeq \begin{cases} \mathsf{P}_{n,k_c} \oplus \mathsf{P}_{n,k_c+1}, & \text{on } \mathsf{dTL}_n \\ \mathsf{P}_{n,k_c+1}, & \text{on } \mathsf{TL}_n \end{cases}, \tag{7.4.2}$$

$$\begin{aligned} \mathsf{P}_{n-1,k_c+i}\!\uparrow \simeq & \begin{Bmatrix} \mathsf{P}_{n,k_c+i}, & \text{on } \mathsf{dTL}_n \\ 0, & \text{on } \mathsf{TL}_n \end{Bmatrix} \oplus \begin{Bmatrix} \mathsf{P}_{n,k_c} \oplus \mathsf{P}_{n,k_c}, & \text{if } i=1 \\ \mathsf{P}_{n,k_c+i-1}, & \text{otherwise} \end{Bmatrix} \\ & \oplus \begin{Bmatrix} \mathsf{P}_{n,k_c-\ell} \oplus \mathsf{P}_{n,k_c+\ell}, & \text{if } i=\ell-1 \\ \mathsf{P}_{n,k_c+i+1}, & \text{otherwise} \end{Bmatrix}, \end{aligned} \tag{7.4.3}$$

where it is understood that $\mathsf{P}_{n,j} \simeq 0$ if $j < 0$.

Proposition 7.3.4 described how fusion behaves regarding parity of modules: the fusion of two odd or even modules yields an even module while the fusion of an odd and an even module yields an odd one. A projective module $\mathsf{P}_{n,k}$ is odd (even) if $n-k$ is odd (even); the following proposition is thus easily proven.

Proposition 7.4.2. *For all critical k_c, $0 < i < \ell$, and $n-1 \geq k_c + i$, on the dilute family*

$$\mathsf{P}_{n-1,i-1} \times_f \mathsf{P}_{1,0} \simeq \mathsf{P}_{n,i-1}, \qquad \mathsf{P}_{n-1,k_c} \times_f \mathsf{P}_{1,0} \simeq \mathsf{P}_{n,k_c}, \tag{7.4.4}$$

$$\mathsf{P}_{n-1,k_c+i} \times_f \mathsf{P}_{1,0} \simeq \mathsf{P}_{n,k_c+i}, \tag{7.4.5}$$

and in both families

$$\mathsf{P}_{n-1,i-1} \times_f \mathsf{P}_{1,1} \simeq \mathsf{P}_{n,i-2} \oplus \mathsf{P}_{n,i}, \qquad \mathsf{P}_{n-1,k_c} \times_f \mathsf{P}_{1,1} \simeq \mathsf{P}_{n,k_c+1}, \tag{7.4.6}$$

$$\mathsf{P}_{n-1,k_c+i} \times_f \mathsf{P}_{1,1} \simeq \begin{cases} \mathsf{P}_{n,k_c} \oplus \mathsf{P}_{n,k_c} & \text{if } i=1 \\ \mathsf{P}_{n,k_c+i-1} & \text{otherwise} \end{cases} \oplus \begin{cases} \mathsf{P}_{n,k_c-\ell} \oplus \mathsf{P}_{n,k_c+\ell} & \text{if } i=\ell-1 \\ \mathsf{P}_{n,k_c+i+1} & \text{otherwise} \end{cases}. \tag{7.4.7}$$

Proof. It follows from the previous proposition together with the linearity of fusion, the breakdown according to parity and the fact that $\mathsf{dTL}_1 \simeq \mathsf{P}_{1,0} \oplus \mathsf{P}_{1,1}$ and $\mathsf{TL}_1 \simeq \mathsf{P}_{1,1}$. ■

For all projective modules in the dilute family, fusion of projectives with $\mathsf{P}_{1,0}$ simply increases the parameter n by one. Since fusion is associative, fusions can be computed using the smallest n for which the modules make sense, and fuse the result with the appropriate number of $\mathsf{P}_{1,0}$ needed to reach the required n. For instance

$$\mathsf{P}_{10,3} \times_f \mathsf{P}_{6,4} \simeq \mathsf{P}_{1,0} \times_f (\mathsf{P}_{9,3} \times_f \mathsf{P}_{6,4}) \simeq \mathsf{P}_{2,0} \times_f (\mathsf{P}_{8,3} \times_f \mathsf{P}_{6,4}) \simeq \ldots \simeq \mathsf{P}_{9,0} \times_f (\mathsf{P}_{3,3} \times_f \mathsf{P}_{4,4}).$$

In the regular family, this role is played by $\mathsf{P}_{2,0}$, when $\ell \neq 2$. Then

$$\mathsf{P}_{2,0} \times_f \mathsf{P}_{n,p} \simeq \mathsf{P}_{n+2,p},$$

for all p. The proof is much more involved and based on diagrammatic arguments; it is presented in appendix 7.A. When $\ell = 2$, it will be proved as a corollary of proposition 7.6.4 that this role is played by $\mathsf{I}_{4,2} \simeq \mathsf{S}_{4,0}$. The results could therefore depend on the parity of $n/2$. Nevertheless, most of our proofs will be independent of n, so we will simply write $\mathsf{P}_p = \mathsf{P}_{n,p}$ and assume that n is big enough for the module to exist. Proofs where n is important will be dealt with separately.

7.4.1 The fusion matrix

For a projective module P, define the *fusion matrix* $\mathsf{F}(\mathsf{P})$ by

$$\mathsf{P} \times_f \mathsf{P}_j \simeq \bigoplus_k (\mathsf{F}(\mathsf{P}))_j^k \mathsf{P}_k$$

where it is understood that a non-negative integer multiple of a module stands for that many copies of this module. To simplify the notation, k is allowed to run over all non-negative integers, but it is assumed that $\mathsf{P}_{n,k} \simeq 0$ when $k > n$, or when $k \not\equiv n \mod 2$ in the regular family. Define also $X = \mathsf{F}(\mathsf{P}_{1,1})$, $F_i = \mathsf{F}(\mathsf{P}_i)$ and write $(F_i)_j^k = F_{i,j}^k$. This definition will reduce the computation of fusion rules to simple products of matrices. Note that since fusion is commutative, $F_{i,j}^k = F_{j,i}^k$.

Proposition 7.4.2 already gives the fusion matrices of P_0 and P_1:

$$F_{0,i}^j = \delta_{i,j}, \tag{7.4.8}$$

$$F_{1,i}^j = \begin{cases} \delta_{j,i+1}, & \text{if } i = 0 \text{ or } i+1 \equiv 0 \mod \ell \\ 2\delta_{j,i-1} + \delta_{j,i+1}, & \text{if } i \equiv 0 \mod \ell \text{ and } \ell \neq 2 \\ \delta_{j,i-1} + \delta_{j,i+1} + \delta_{j,i-2\ell+1}, & \text{if } i > \ell - 1 \text{ and } i+2 \equiv 0 \mod \ell \text{ and } \ell \neq 2 \\ 2\delta_{j,i-1} + \delta_{j,i+1} + \delta_{j,i-2\ell+1}, & \text{if } i > \ell - 1 \text{ and } i+2 \equiv 0 \mod \ell \text{ and } \ell = 2 \\ \delta_{j,i-1} + \delta_{j,i+1}, & \text{otherwise} \end{cases} \tag{7.4.9}$$

where $\delta_{i,j}$ is the Kronecker delta.

The following proposition shows that a finite projective module is uniquely determined by its fusion matrix.

Proposition 7.4.3. *For P,Q two finite projective* dTL_n*- or* TL_n*-modules,*

$$F(P) = F(Q) \iff P \simeq Q.$$

Proof. Every finite projective module is isomorphic to a direct sum of principal indecomposable modules.

For a projective module T, define the set $\alpha(T)$ as the set of integers such that

$$T \simeq \oplus_{i\in\alpha(T)} \mathsf{P}_i,$$

where each integer can occur more than once. Define $i(T)$ as the maximum of $\alpha(T)$, and $\#i(T)$ as the number of times this maximum appears. From proposition 7.4.2, it is clear that $i(T \times_f \mathsf{P}_{1,1}) = i(T)+1$ and $\#i(T \times_f \mathsf{P}_{1,1}) = \#i(T)$.

Now, for P,Q two projective dTL_n- or TL_n-modules, if $\mathrm{F}(P) = \mathrm{F}(Q)$, then in particular $P \times_f \mathsf{P}_{1,1} \simeq Q \times_f \mathsf{P}_{1,1}$. Thus $i(P\times_f \mathsf{P}_{1,1}) = i(Q\times_f \mathsf{P}_{1,1})$ and $\#i(P\times_f \mathsf{P}_{1,1}) = \#i(Q\times_f \mathsf{P}_{1,1})$. Therefore $i(P) = i(Q)$, $\#i(P) = \#i(Q)$, and

$$P \simeq P' \oplus \#i(P)\mathsf{P}_{i(P)}, \qquad Q \simeq Q' \oplus \#i(P)\mathsf{P}_{i(P)}, \tag{7.4.10}$$

where $i(P') < i(P)$ and $i(Q') < i(Q)$. Since fusion is linear, $P' \times_f \mathsf{P}_{1,1} \simeq Q' \times_f \mathsf{P}_{1,1}$. Proceeding by recursion on the cardinality of $i(P)$, the result is obtained. ■

7.4.2 Fusion matrices of small projectives

By using the formulas in proposition 7.4.2, for $0 \leq i < \ell - 1$ and all j,

$$\mathsf{P}_1 \times_f (\mathsf{P}_i \times_f \mathsf{P}_j) \simeq (\mathsf{P}_{i-1} \oplus \mathsf{P}_{i+1}) \times_f \mathsf{P}_j. \tag{7.4.11}$$

In terms of the fusion matrices, this is simply

$$\sum_m F^p_{1,m} F^m_{i,j} = F^p_{i-1,j} + F^p_{i+1,j} \tag{7.4.12}$$

and this gives the recurrence relation

$$XF_i = F_{i-1} + F_{i+1}, \qquad F_0 = \mathrm{id},\, F_1 = X, \tag{7.4.13}$$

where

$$(XF_i)^p_j = \sum_{m=0}^{n} X^p_m (F_i)^m_j = \sum_{m=0}^{n} F^p_{1,m} F^m_{i,j}.$$

One should recognize the recurrence relation [2] of Chebyshev polynomials of the second kind $U_i\left(\frac{X}{2}\right)$ and thus find

$$F_i = U_i\left(\frac{X}{2}\right), \qquad 0 \leq i \leq l-1.$$

Since the matrix X is known, this can be used in principle to compute the fusion matrix of all small projectives. Note that this proof fails when $\ell = 2$ on the regular family because in this case, there are no small projectives.

2. Note that the Chebyshev solution to this recurrence relation is valid on $\mathbb{C}[X]$ even when X is a matrix.

7.4.3 Fusion matrices for the indecomposable projective P_{k_c+i}

Using again proposition 7.4.2, for $0 \le i \le \ell - 1$

$$\mathsf{P}_1 \times_f \mathsf{P}_{k_c+i} \simeq \left. \begin{cases} 0, & \text{if } i = 0 \\ \mathsf{P}_{k_c} \oplus \mathsf{P}_{k_c} & \text{if } i = 1 \\ \mathsf{P}_{k_c+i-1} & \text{otherwise} \end{cases} \right\} \oplus \left. \begin{cases} \mathsf{P}_{k_c-\ell} \oplus \mathsf{P}_{k_c+\ell} & \text{if } i = \ell - 1 \\ \mathsf{P}_{k_c+i+1} & \text{otherwise} \end{cases} \right\}. \tag{7.4.14}$$

Expressing this in terms of fusion matrices gives the following recurrence relation

$$XF_{k_c} = F_{k_c+1}, \tag{7.4.15}$$

$$XF_{k_c+1} = 2F_{k_c} + F_{k_c+2}, \tag{7.4.16}$$

$$XF_{k_c+i} = F_{k_c+i-1} + F_{k_c+i+1}, \qquad \text{if } k_c + i \pm 1 \text{ are not critical} \tag{7.4.17}$$

$$XF_{k_c+\ell-1} = F_{k_c+\ell-2} + F_{k_c+\ell} + F_{k_c-\ell}, \tag{7.4.18}$$

where it was implicitly assumed that $\ell \neq 2$. When $\ell = 2$, equations (7.4.16) and (7.4.18) becomes

$$XF_{k_c+\ell-1} = XF_{k_c+1} = 2F_{k_c} + F_{k_c+2} + F_{k_c-2}. \tag{7.4.19}$$

Using the fact that $F_{\ell-1} = U_{\ell-1}\left(\frac{X}{2}\right)$, it can be checked directly that the solution to this system is

Proposition 7.4.4. *For $0 \le i \le \ell - 1$,*

$$F_{k_c+i} = \begin{cases} U_{k_c}\left(\frac{X}{2}\right), & \text{if } i = 0 \\ U_{k_c-i}\left(\frac{X}{2}\right) + U_{k_c+i}\left(\frac{X}{2}\right), & \text{otherwise} \end{cases}. \tag{7.4.20}$$

7.4.4 A closed expression for $\mathsf{P}_i \times_f \mathsf{P}_j$

Using fusion matrices, computing fusion rules is reduced to evaluating a Chebyshev polynomial at a matrix X, but since this matrix is not diagonal, computing this polynomial may be far from trivial. However, since the projective indecomposable modules are all finite dimensional, proposition 7.4.3 implies that if

$$F_i F_j = F(G), \tag{7.4.21}$$

where $F(G)$ is the fusion matrix of some finite-dimensional projective module G, then

$$\mathsf{P}_i \times_f \mathsf{P}_j \simeq G.$$

Computing fusion rules thus reduces to expressing a product of Chebyshev polynomials as a linear combination of other Chebyshev polynomials. Using this fact will greatly simplify the proof of the following explicit formulas. These are written in a particular way to express the fact that they are identical to those obtained by Gainutnidov and Vasseur [24].

Proposition 7.4.5. *If* $k, r \geq 1$, $0 < i, j < \ell$,

$$\mathsf{P}_i \times_f \mathsf{P}_j \simeq \bigoplus_{\sigma=(i+j+\ell+1) \bmod 2}^{i+j-\ell+1} \mathsf{P}_{\ell-1+\sigma} \oplus \bigoplus_{\sigma=|i-j|}^{\mathrm{Min}(i+j,2\ell-(i+j)-4)} \mathsf{P}_{\sigma}, \tag{7.4.22}$$

$$\mathsf{P}_i \times_f \mathsf{P}_{k\ell-1} \simeq \bigoplus_{\sigma=i \bmod 2}^{i} \mathsf{P}_{k\ell-1+\sigma}, \tag{7.4.23}$$

$$\mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{r\ell-1} \simeq \bigoplus_{\rho=|k-r|+1}^{k+r-1} \bigoplus_{\sigma=(\ell+1) \bmod 2}^{\ell-1} \mathsf{P}_{\rho\ell-1+\sigma} \tag{7.4.24}$$

$$\simeq \bigoplus_{\rho=|k-r|+1}^{k+r-1} \left(\mathsf{P}_{\rho\ell-1} \times_f \mathsf{P}_{\ell-1}\right), \tag{7.4.25}$$

$$\begin{aligned} \mathsf{P}_j \times_f \mathsf{P}_{k\ell-1+i} \simeq & \bigoplus_{\sigma=(i+j+\ell) \bmod 2}^{i+j-\ell} \left(\mathsf{P}_{(k-1)\ell-1+\sigma} + \mathsf{P}_{(k+1)\ell-1+\sigma}\right) \oplus 2 \bigoplus_{\sigma=|i-j| \bmod 2}^{j-i} \mathsf{P}_{k\ell-1+\sigma} \\ & \oplus \bigoplus_{\sigma=\mathrm{Max}(i-j,j-i+2)}^{\mathrm{Min}(i+j,2\ell-(i+j)-2)} \mathsf{P}_{k\ell-1+\sigma}, \end{aligned} \tag{7.4.26}$$

$$\begin{aligned} \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{k\ell-1+i} \simeq & \bigoplus_{\sigma=(i+1) \bmod 2}^{i-1} \left(\mathsf{P}_{|k-r|\ell-1+\sigma} + \mathsf{P}_{(k+r)\ell-1+\sigma}\right) \\ & \oplus 2 \bigoplus_{\rho=|k-r|+1}^{k+r-1} \bigoplus_{\sigma=(i+\ell+1) \bmod 2}^{\ell-i-1} \mathsf{P}_{\rho\ell-1+\sigma} \oplus 2 \bigoplus_{\rho=|k-r|+2}^{k+r-2} \bigoplus_{\sigma=(i-1) \bmod 2}^{i-1} \mathsf{P}_{\rho\ell-1+\sigma} \end{aligned} \tag{7.4.27}$$

$$\begin{aligned} & \simeq \left(\mathsf{P}_{|k-r|\ell-1} + \mathsf{P}_{(k+r)\ell-1}\right) \times_f \mathsf{P}_{i-1} \\ & \oplus 2 \bigoplus_{\rho=|k-r|+1}^{k+r-1} \left(\mathsf{P}_{\rho\ell-1} \times_f \mathsf{P}_{\ell-i-1}\right) \oplus 2 \bigoplus_{\rho=|k-r|+2}^{k+r-2} \left(\mathsf{P}_{\rho\ell-1} \times_f \mathsf{P}_{i-1}\right) \end{aligned} \tag{7.4.28}$$

$$\begin{aligned} \mathsf{P}_{k\ell-1+i} \times_f \mathsf{P}_{r\ell-1+j} \simeq & \bigoplus_{\rho=|k-r|-1}^{k+r+1} \left(4\phi_\rho \bigoplus_{\sigma=2\ell-(i+j)+1}^{\ell-(l+i+j+1) \bmod 2} \mathsf{P}_{(\rho+1)\ell-1-\sigma}\right) \\ & \oplus 2 \bigoplus_{\rho=|k-r|+1}^{k+r-1} \left(\bigoplus_{\sigma=|i-j|+1}^{\mathrm{Min}(i+j-1,2\ell-i-j-1)} \mathsf{P}_{(\rho+1)\ell-1-\sigma} \oplus 2 \bigoplus_{\sigma=i+j+1}^{\ell-(l+i+j+1) \bmod 2} \mathsf{P}_{(\rho+1)\ell-1-\sigma}\right) \end{aligned}$$

$$\oplus 2 \bigoplus_{\rho=|k-r|}^{k+r} \psi_\rho \left(\bigoplus_{\sigma=|\ell-i-j|+1}^{\mathrm{Min}(\ell-i+j-1,\ell+i-j-1)} \mathsf{P}_{(\rho+1)\ell-1-\sigma} \oplus 2 \bigoplus_{\sigma=\mathrm{Min}(\ell-i+j+1,\ell+i-j+1)}^{\ell-\gamma_1} \mathsf{P}_{(\rho+1)\ell-1-\sigma} \right), \quad (7.4.29)$$

where $\gamma_1 = (i+j+1) \bmod 2, \gamma_2 = (i+j+q+\ell) \bmod 2, \phi_\rho = 1 - \frac{3}{4}\delta_{\rho,|k-r|-1} - \frac{1}{4}\delta_{\rho,|k-r|+1} - \frac{1}{4}\delta_{\rho,k+r-1} - \frac{3}{4}\delta_{\rho,k+r+1}$, $\psi_\rho = 1 - \frac{1}{2}\delta_{\rho,|k-r|} - \frac{1}{2}\delta_{\rho,k+r}$ *and all sums have "step=2".*

The proof of all these are done using the same argument. Start by using the identity

$$U_i(y)U_j(y) = \sum_{\substack{k=|i-j| \\ step=2}}^{i+j} U_k(y),$$

to write the product of the fusion matrices as a sum of Chebyshev polynomials then gather them in appropriate combinations to obtain a linear combination of fusion matrices. Using the fact that a fusion matrix uniquely determines a projective module and that fusion of two projective modules always yields a projective modules, the conclusion is obtained. Here are a few examples on how this is done. Since the argument of the polynomials involved will always be $\frac{X}{2}$, we will simply omit them and write U_i instead of $U_i\left(\frac{X}{2}\right)$.

For $\ell = 5$, here are some fusion of small projectives.

$$F_3F_2 = U_3U_2 = U_1 + U_3 + U_5 = F_1 + F_5, \quad (7.4.30)$$

$$F_3F_4 = U_1 + U_3 + U_5 + U_7 = (U_1 + U_7) + (U_3 + U_5) = F_5 + F_7, \quad (7.4.31)$$

$$F_2F_4 = U_2 + U_4 + U_6 = U_4 + (U_2 + U_6) = F_4 + F_6. \quad (7.4.32)$$

For the fusion of a small projective and a projective indecomposable,

$$\begin{aligned} F_4F_8 \quad &= U_4(U_0 + U_8) = 2U_4 + U_6 + U_8 + U_{10} + U_{12} \\ &= 2U_4 + (U_8 + U_{10}) + (U_6 + U_{12}) = 2F_4 + F_{10} + F_{12} \end{aligned} \quad (7.4.33)$$

giving the fusion rule

$$\mathsf{P}_4 \times_f \mathsf{P}_8 \simeq 2\mathsf{P}_4 \oplus \mathsf{P}_{10} \oplus \mathsf{P}_{12}. \quad (7.4.34)$$

The fusion matrix of $\mathsf{P}_{11} \times_f \mathsf{P}_{28}$ is

$$\begin{aligned} F_{11}F_{28} = (U_7 + U_{11})(U_{20} + U_{28}) &= \sum_{\substack{i=13 \\ step=2}}^{27} (U_i) + \sum_{\substack{i=9 \\ step=2}}^{31} (U_i) + \sum_{\substack{i=21 \\ step=2}}^{35} (U_i) + \sum_{\substack{i=17 \\ step=2}}^{39} (U_i) \\ &= U_9 + (U_{11} + U_{17}) + 2(U_{13} + U_{15}) + 3U_{19} + 2(U_{17} + U_{21}) + 4(U_{23} + U_{25}) \\ &+ 2(U_{21} + U_{27}) + 3U_{29} + 2(U_{27} + U_{31}) + 2(U_{33} + U_{35}) + (U_{31} + U_{37}) + U_{39} \\ &= F_9 + F_{17} + 2F_{15} + 3F_{19} + 2F_{21} + 4F_{25} + 2F_{27} + 3F_{29} \\ &+ 2F_{31} + 2F_{35} + F_{37} + F_{39} \end{aligned} \quad (7.4.35)$$

giving the fusion rule

$$\mathsf{P}_{11} \times_f \mathsf{P}_{28} \simeq \mathsf{P}_9 \oplus 2\mathsf{P}_{15} \oplus \mathsf{P}_{17} \oplus 3\mathsf{P}_{19} \oplus 2\mathsf{P}_{21} \oplus 4\mathsf{P}_{25} \oplus 2\mathsf{P}_{27} \oplus 3\mathsf{P}_{29} \oplus 2\mathsf{P}_{31} \oplus 2\mathsf{P}_{35} \oplus \mathsf{P}_{37} \oplus \mathsf{P}_{39}. \tag{7.4.36}$$

Note that we used the same notation in this proposition than in [24], where they compute the fusion rules in TL_n. This makes it obvious that the two fusion rules are identical.

7.4.5 The semi-simple case

When q is not a root of unity different from ± 1, the algebras TL_n and dTL_n are semi-simple and the standard modules $\mathsf{S}_{n,i}$ are all irreducible and projective. They satisfy the induction rules

$$\mathsf{S}_{n,i}\!\uparrow\, \simeq \mathsf{S}_{n+1,i-1} \oplus \mathsf{S}_{n+1,i} \oplus \mathsf{S}_{n+1,i+1},$$

where it is understood that $\mathsf{S}_{n,i+1} = 0$ if $n \neq i+1 \mod 2$ in the regular family. Using arguments identical to those in section 7.4.2 yields

$$\mathsf{S}_{n,i} \times_f \mathsf{S}_{1,1} \simeq \mathsf{S}_{n+1,i-1} \oplus \mathsf{S}_{n+1,i+1}, \qquad \mathsf{S}_{n,i} \times_f \mathsf{S}_{1,0} \simeq \mathsf{S}_{n+1,i},$$

where the second rule is replaced by

$$\mathsf{S}_{n,i} \times_f \mathsf{S}_{2,0} \simeq \mathsf{S}_{n+2,i},$$

in the regular family. This gives the following recurrence relation for the fusion matrices

$$XF_i = F_{i+1} + F_{i-1}, \qquad F_0 = \mathrm{id}, \qquad F_1 = X, \tag{7.4.37}$$

where X is simply $(X)_i^j = \delta_i^{j+1} + \delta_i^{j-1}$. Using the same argument as in section 7.4.2 then gives the following fusion rules.

Theorem 7.4.6. *If q is not a root of unity different from ± 1, then for $0 \leq i \leq n$, $0 \leq j \leq m$,*

$$\mathsf{S}_{n,i} \times_f \mathsf{S}_{m,j} \simeq \bigoplus_{\substack{k=|i-j| \\ step=2}}^{i+j} \mathsf{S}_{n+m,k}.$$

7.5 Fusion of standard modules

It was noted in section 7.3 that fusion is closely related with induction, we thus start by giving the behaviour of the non-projective standard modules under the induction functor [10, 11, 63].

Proposition 7.5.1. *If i with $0 \le i \le n-1$ is not critical,*

$$\mathsf{S}_{n-1,i}\uparrow \simeq \begin{cases} \mathsf{S}_{n,i-1} \oplus \mathsf{S}_{n,i} \oplus \mathsf{S}_{n,i+1}, & \text{in the dilute family},\\ \mathsf{S}_{n,i-1} \oplus \mathsf{S}_{n,i+1}, & \text{in the regular family}, \end{cases} \tag{7.5.1}$$

where it is understood that $\mathsf{S}_{n,-1} = 0$.

Using the same arguments as in proposition 7.4.2, this gives the following fusion rules.

Proposition 7.5.2. *If i with $0 \le i \le n-1$ is not critical, in the dilute family*

$$\mathsf{S}_{n-1,i} \times_f \mathsf{P}_{1,0} \simeq \mathsf{S}_{n,i}, \tag{7.5.2}$$

while in both families

$$\mathsf{S}_{n-1,i} \times_f \mathsf{P}_{1,1} \simeq \mathsf{S}_{n,i-1} \oplus \mathsf{S}_{n,i+1}, \tag{7.5.3}$$

where $\mathsf{S}_{n,j} \simeq \mathsf{P}_{n,j}$ *if j is critical.*

Using the same argument as in the projective case with the first fusion rule, and proposition 7.A.1 in the regular case,

$$\mathsf{S}_{n,i} \times_f \mathsf{S}_{m,j} \simeq \mathsf{P}_{n-i+m-j,0} \times_f (\mathsf{S}_{i,i} \times_f \mathsf{S}_{j,j}).$$

We will therefore always omit the parameter n, writing $\mathsf{S}_{n,i} = \mathsf{S}_i$, and assume that n is big enough and of the right parity, in the regular case, for the module to exists. Note that in the regular case when $\ell = 2$, the module $\mathsf{S}_{n,0}$ is very particular because $\mathsf{S}_{n,0} \simeq \mathsf{I}_{n,2}$. This module will therefore be treated in section 7.5.3.

Once a formula for the fusion of $\mathsf{S}_{k\ell}$, $k \in \mathbb{N}$, with some module M is obtained, the second fusion rules (7.5.3) will be used to obtain a formula for the fusion of M with the other standard modules by simple induction. We start by studying the fusion of a standard module with a projective module then consider the fusion of two standard modules. Finally, we give a simple rule that can be used to quickly compute the fusion of standard modules.

7.5.1 Fusion of a standard and a projective module

The general formula that will be obtained is quite complex and the inductive proof is very technical. The argument is thus split in four propositions that will be simpler to prove. Each one will be preceded by an example with $\ell = 5$ before moving to the general case. The proof for general ℓ is very straightforward once these examples are understood so we highly suggest that the reader works them out carefully.

Consider the case $\ell = 5$ and the standard module $\mathsf{S}_{n,25} = \mathsf{S}_{25}$ which is not projective. Proposition 7.5.2 then gives

$$\mathsf{S}_{25} \times_f \mathsf{P}_1 \simeq \mathsf{P}_{24} \oplus \mathsf{S}_{26}. \tag{7.5.4}$$

Note that $\mathsf{S}_{24} \simeq \mathsf{P}_{24}$ is projective. Fusing the left side of this isomorphism with P_1 and using the associativity of fusion with proposition 7.5.2 and 7.4.2 gives

$$\mathsf{S}_{25} \times_f (\mathsf{P}_1 \times_f \mathsf{P}_1) \simeq \mathsf{S}_{25} \times_f \mathsf{P}_0 \oplus \mathsf{S}_{25} \times_f \mathsf{P}_2 \simeq \mathsf{S}_{25} \oplus \mathsf{S}_{25} \times_f \mathsf{P}_2, \tag{7.5.5}$$

while fusing its right side with P_1 and using the same propositions gives

$$(\mathsf{P}_{24} \oplus \mathsf{S}_{26}) \times_f \mathsf{P}_1 \simeq \mathsf{P}_{24} \times_f \mathsf{P}_1 \oplus \mathsf{P}_{26} \times_f \mathsf{P}_1 \simeq \mathsf{P}_{25} \oplus \mathsf{S}_{25} \oplus \mathsf{S}_{27}. \tag{7.5.6}$$

Comparing the two results yields

$$\mathsf{S}_{25} \times_f \mathsf{P}_2 \simeq \mathsf{P}_{25} \oplus \mathsf{S}_{27}.$$

Repeating the same arguments gives the fusion rules

$$\mathsf{S}_{25} \times_f \mathsf{P}_3 \simeq \mathsf{P}_{24} \oplus \mathsf{P}_{26} \oplus \mathsf{S}_{28}, \tag{7.5.7}$$

$$\mathsf{S}_{25} \times_f \mathsf{P}_4 \simeq \mathsf{P}_{25} \oplus \mathsf{P}_{27} \oplus \mathsf{S}_{29} \tag{7.5.8}$$

where $\mathsf{S}_{29} = \mathsf{P}_{29}$ is projective. A pattern can be identified here: for all $i < 5$,

$$\mathsf{S}_{25} \times_f \mathsf{P}_i \simeq \mathsf{P}_{24} \times_f \mathsf{P}_{i-1} \oplus \mathsf{S}_{25+i}.$$

Proposition 7.5.3. *For $i < \ell$, $\ell > 2$ in TL_n, and $k \in \mathbb{N}$,*

$$\mathsf{S}_{k\ell} \times_f \mathsf{P}_i \simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{i-1} \oplus \mathsf{S}_{k\ell+i}. \tag{7.5.9}$$

Proof. The proof proceeds by induction on i. Proposition 7.5.2 already gives the case $i = 0$ and $i = 1$. Suppose therefore the result for $i < \ell - 1$ and $i - 1$. Applying propositions 7.5.2 and 7.4.2 on the left side of equation (7.5.9) gives

$$\mathsf{S}_{k\ell} \times_f \mathsf{P}_i \times_f \mathsf{P}_1 \simeq \mathsf{S}_{k\ell} \times_f (\mathsf{P}_i \times_f \mathsf{P}_1) \simeq \mathsf{S}_{k\ell} \times_f \mathsf{P}_{i-1} \oplus \mathsf{S}_{k\ell} \times_f \mathsf{P}_{i+1}. \tag{7.5.10}$$

Using the same proposition on the right side of (7.5.9) yields

$$\mathsf{P}_{i-1} \times_f \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_1 \oplus \mathsf{S}_{k\ell+i} \times_f \mathsf{P}_1 \simeq \mathsf{P}_{k\ell-1} \times_f (\mathsf{P}_{i-2} \oplus \mathsf{P}_i) \oplus \mathsf{S}_{k\ell+i-1} \oplus \mathsf{S}_{k\ell+i+1}. \tag{7.5.11}$$

Comparing the two results and using the induction hypothesis for $i - 1$ gives the conclusion. Note that we implicitly assumed that $\ell \neq 2$. In this case, there is only $i = 0$ and $i = 1$, which are both covered by proposition 7.5.2. ■

Let us return to the preceding $\ell = 5$ example. Using again the associativity and commutativity of fusion with proposition 7.4.2 if equation (7.5.8) is fused with P_1, the left side gives

$$\mathsf{S}_{25} \times_f (\mathsf{P}_4 \times_f \mathsf{P}_1) \simeq \mathsf{S}_{25} \times_f \mathsf{P}_5, \tag{7.5.12}$$

while the right one becomes

$$(\mathsf{P}_{24}\times_f\mathsf{P}_1)\times_f\mathsf{P}_3\oplus\underbrace{\mathsf{S}_{29}}_{\simeq\mathsf{P}_{29}}\times_f\mathsf{P}_1\simeq\mathsf{P}_{25}\times_f\mathsf{P}_3\oplus\mathsf{P}_{30}. \tag{7.5.13}$$

Comparing the two gives

$$\mathsf{S}_{25}\times_f\mathsf{P}_5\simeq\mathsf{P}_{25}\times_f\mathsf{P}_3\oplus\mathsf{P}_{30}. \tag{7.5.14}$$

Repeating this operation yields

$$\mathsf{S}_{25}\times_f\mathsf{P}_6\simeq\mathsf{P}_{26}\times_f\mathsf{P}_3\oplus\mathsf{P}_{31}, \tag{7.5.15}$$

$$\mathsf{S}_{25}\times_f\mathsf{P}_7\simeq\mathsf{P}_{27}\times_f\mathsf{P}_3\oplus\mathsf{P}_{32}, \tag{7.5.16}$$

$$\mathsf{S}_{25}\times_f\mathsf{P}_8\simeq\mathsf{P}_{28}\times_f\mathsf{P}_3\oplus\mathsf{P}_{33}. \tag{7.5.17}$$

Fusing P_1 again on the last rule [3], the left side becomes

$$\mathsf{S}_{25}\times_f(\mathsf{P}_8\times_f\mathsf{P}_1)\simeq\mathsf{S}_{25}\times_f(\mathsf{P}_7\oplus\mathsf{P}_9), \tag{7.5.18}$$

while the right one becomes

$$(\mathsf{P}_{28}\times_f\mathsf{P}_1)\times_f\mathsf{P}_3\oplus\mathsf{P}_{33}\times_f\mathsf{P}_1\simeq(\mathsf{P}_{19}\oplus\mathsf{P}_{27}\oplus\mathsf{P}_{29})\times_f\mathsf{P}_3\oplus\mathsf{P}_{24}\oplus\mathsf{P}_{32}\oplus\mathsf{P}_{34} \tag{7.5.19}$$

$$\simeq\underbrace{(\mathsf{P}_{27}\times_f\mathsf{P}_3\oplus\mathsf{P}_{32})}_{\simeq\mathsf{S}_{25}\times_f\mathsf{P}_7}\oplus(\mathsf{P}_{19}\oplus\mathsf{P}_{29})\times_f\mathsf{P}_3\oplus\mathsf{P}_{24}\oplus\mathsf{P}_{34}. \tag{7.5.20}$$

This is simply

$$\mathsf{S}_{25}\times_f\mathsf{P}_9\simeq(\mathsf{P}_{19}\oplus\mathsf{P}_{29})\times_f\mathsf{P}_3\oplus\mathsf{P}_{24}\oplus\mathsf{P}_{34}. \tag{7.5.21}$$

We can then proceed with the general case.

Proposition 7.5.4. *For $0\le i<\ell$, $k,s\in\mathbb{Z}_{>0}$,*

$$\mathsf{S}_{k\ell}\times_f\mathsf{P}_{s\ell-1+i}\simeq\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}(\mathsf{P}_{r\ell-1+i}\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s}\mathsf{P}_{r\ell-1+i}. \tag{7.5.22}$$

In the case $\ell=2$ in TL_n, the fusion with $\mathsf{P}_{\ell-2}$ must be removed.

Proof. The proof proceeds by induction on s and i. Let us start by proving that for a given k, if the result stands for $i=0$ then it will also stands for all $i\le\ell$. Note that the case $i=1$ follows directly from the case $i=0$ since for all $p\ge1$, $\mathsf{P}_{p\ell}\simeq\mathsf{P}_1\times_f\mathsf{P}_{p\ell-1}$. Suppose therefore that the result stands for $i-1,i<\ell$. Fusing (7.5.22) with P_1 and using proposition 7.4.2 with the associativity and commutativity of fusion then gives,

3. Note that $9,14,19,24,29$ and 34 are critical.

on the left side

$$\mathsf{S}_{k\ell}\times_f(\mathsf{P}_{s\ell-1+i}\times_f\mathsf{P}_1)\simeq(1+\delta_{i,1})\mathsf{S}_{k\ell}\times_f\mathsf{P}_{s\ell-2+i}\oplus\mathsf{S}_{k\ell}\times_f\left(\mathsf{P}_{s\ell+i}\oplus\delta_{i,\ell-1}\mathsf{P}_{(s-1)\ell-1}\right),\tag{7.5.23}$$

and on the right side

$$\begin{aligned}
&\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}((\mathsf{P}_{r\ell-1+i}\times_f\mathsf{P}_1)\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s}(\mathsf{P}_{r\ell-1+i}\times_f\mathsf{P}_1)\\
&\quad\simeq\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}\left(\left((1+\delta_{i,1})\mathsf{P}_{r\ell-2+i}\oplus\mathsf{P}_{r\ell+i}\oplus\delta_{i,\ell-1}\mathsf{P}_{(r-1)\ell-1}\right)\times_f\mathsf{P}_{\ell-2}\right)\\
&\quad\oplus\bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s}\left((1+\delta_{i,1})\mathsf{P}_{r\ell-2+i}\oplus\mathsf{P}_{r\ell+i}\oplus\delta_{i,\ell-1}\mathsf{P}_{(r-1)\ell-1}\right)\\
&\quad\simeq(1+\delta_{i,1})\left(\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}(\mathsf{P}_{r\ell-2+i}\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s}\mathsf{P}_{r\ell-2+i}\right)\\
&\quad\oplus\delta_{i,\ell-1}\left(\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}\left(\mathsf{P}_{(r-1)\ell-1}\times_f\mathsf{P}_{\ell-2}\right)\oplus\bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s}\mathsf{P}_{(r-1)\ell-1}\right)\\
&\quad\oplus\left(\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}(\mathsf{P}_{r\ell-1+i+1}\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s}\mathsf{P}_{r\ell-1+i+1}\right).
\end{aligned}$$

If $i\neq\ell-1$, collecting the relevant terms, comparing the two sides and applying the induction hypothesis then gives the result for $i+1$. If $i=\ell-1$, there is a slight subtlety involved. In the preceding expression, collect the terms being fused with $\mathsf{P}_{\ell-2}$ and note that

$$\begin{aligned}
\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1}\left(\mathsf{P}_{(r-1)\ell-1}\oplus\mathsf{P}_{r\ell-1+\ell}\right)&\simeq\bigoplus_{\substack{r'=|k-s|\\ step=2}}^{k+s-2}\mathsf{P}_{r'\ell-1}\oplus\bigoplus_{\substack{r'=|k-s|+2\\ step=2}}^{k+s}\mathsf{P}_{r'\ell-1}\\
&\simeq\bigoplus_{\substack{r=|k-(s-1)|+1\\ step=2}}^{k+(s-1)-1}\mathsf{P}_{r\ell-1}\oplus\bigoplus_{\substack{r=|k-(s+1)|+1\\ step=2}}^{k+s+1-1}\mathsf{P}_{r\ell-1},
\end{aligned}$$

where we rearranged the terms between the two sums and used the fact that $\mathsf{P}_{-1}\equiv 0$. The exact rearranging required depends on the value of $k-s$. Doing the same rearranging on the other terms gives

$$\mathsf{S}_{k\ell}\times_f\left(\mathsf{P}_{(s+1)\ell-1}\oplus\mathsf{P}_{(s-1)\ell-1}\right)\simeq\bigoplus_{\substack{r=|k-(s-1)|+1\\ step=2}}^{k+(s-1)-1}(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|s-1-(k+1)|+1\\ step=2}}^{k+s-1}\mathsf{P}_{r\ell-1}$$
$$\oplus\bigoplus_{\substack{r=|k-(s+1)|+1\\ step=2}}^{k+s+1-1}(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|s+1-(k+1)|+1\\ step=2}}^{k+s+1}\mathsf{P}_{r\ell-1}.$$

It follows that if the statement holds for $(s-1,i=0)$, $(s,i=0)$, it will also stand for (s,i) for all $i\leq\ell-1$ and $(s+1,i=0)$.

The only remaining step is to prove that the result stands for $k=1$, $i=0$. This is precisely proposition 7.5.3. In the case $\ell=2$ of TL_n, the result and its proof are slightly different, because then $1=\ell-1$, so that

$$\mathsf{P}_{s\ell}\times_f\mathsf{P}_1\simeq 2\mathsf{P}_{s\ell-1}\oplus\mathsf{P}_{(s+1)\ell-1}\oplus\mathsf{P}_{(s-1)\ell-1}.$$

However, the same arguments can be used to induce on s and on i. ■

Now that the expression for the fusion of $\mathsf{S}_{k\ell}$ is known, proposition 7.5.2 can be used to compute the fusion of the other standard modules with the projective. We return to the $\ell=5$ example. It was previously found that

$$\mathsf{S}_{25}\times_f\mathsf{P}_8\simeq\mathsf{P}_{28}\times_f\mathsf{P}_3\oplus\mathsf{P}_{33}.$$

Fusing the left side with P_1 gives,

$$\mathsf{S}_{25}\times_f\mathsf{P}_8\times_f\mathsf{P}_1\simeq(\mathsf{S}_{25}\times_f\mathsf{P}_1)\times_f\mathsf{P}_8\simeq(\mathsf{P}_{24}\oplus\mathsf{S}_{26})\times_f\mathsf{P}_8,\tag{7.5.24}$$

while fusing the right side with P_1 yields

$$\begin{aligned}\mathsf{P}_{28}\times_f(\mathsf{P}_3\times_f\mathsf{P}_1)\oplus\mathsf{P}_{33}\times_f\mathsf{P}_1&\simeq\mathsf{P}_{28}\times_f(\mathsf{P}_2\oplus\mathsf{P}_4)\oplus\mathsf{P}_{33}\times_f\mathsf{P}_1\\&\simeq(\mathsf{P}_{28}\times_f\mathsf{P}_2\oplus\mathsf{P}_{33}\times_f\mathsf{P}_1)\oplus\mathsf{P}_{28}\times_f\mathsf{P}_4.\end{aligned}\tag{7.5.25}$$

Using proposition 7.4.5, notice that

$$\mathsf{P}_{24}\times_f\mathsf{P}_8\simeq(\mathsf{P}_{19}\oplus\mathsf{P}_{29})\times_f\mathsf{P}_3\oplus 2\mathsf{P}_{24}\simeq\mathsf{P}_4\times_f\mathsf{P}_{28}.$$

Comparing (7.5.24) with (7.5.25) then gives the fusion rule

$$\mathsf{S}_{26}\times_f\mathsf{P}_8\simeq\mathsf{P}_{28}\times_f\mathsf{P}_2\oplus\mathsf{P}_{33}\times_f\mathsf{P}_1.\tag{7.5.26}$$

Repeating the same steps gives

$$\mathsf{S}_{27}\times_f\mathsf{P}_8\simeq\mathsf{P}_{28}\times_f\mathsf{P}_1\oplus\mathsf{P}_{33}\times_f\mathsf{P}_2,\tag{7.5.27}$$

$$\mathsf{S}_{28} \times_f \mathsf{P}_8 \simeq \mathsf{P}_{28} \times_f \mathsf{P}_0 \oplus \mathsf{P}_{33} \times_f \mathsf{P}_3. \tag{7.5.28}$$

Theorem 7.5.5. *For $0 < i < \ell$, $0 \leq j < \ell$, $\ell \neq 2$ in TL_n, and $k, s \in \mathbb{Z}_{>0}$,*

$$\mathsf{S}_{k\ell-1+i} \times_f \mathsf{P}_{s\ell-1+j} \simeq \bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_{\ell-1-i}\big) \oplus \bigoplus_{\substack{r=|s-(k+1)|+1 \\ step=2}}^{k+s} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_{i-1}\big). \tag{7.5.29}$$

In the case $\ell = 2$ in TL_n, the fusions with $\mathsf{P}_{\ell-1-i}$ and P_{i-1} must be removed.

Proof. In this case the proof is a simple induction on i. The case $i = 1$ is covered by proposition 7.5.4. Fusing the left side of (7.5.29) ($i = 1$) with P_1 and using proposition 7.5.2 gives

$$(\mathsf{S}_{k\ell} \times_f \mathsf{P}_1) \times_f \mathsf{P}_{s\ell-1+j} \simeq (\mathsf{P}_{k\ell-1} \oplus \mathsf{S}_{k\ell-1+2}) \times_f \mathsf{P}_{s\ell-1+j}, \tag{7.5.30}$$

while fusing the right side of the same equation with P_1 and using proposition 7.4.2 yields

$$\begin{aligned}
&\bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f (\mathsf{P}_{\ell-2} \times_f \mathsf{P}_1)\big) \oplus \bigoplus_{\substack{r=|s-(k+1)|+1 \\ step=2}}^{k+s} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_1\big) \\
&\quad \simeq \bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f (\mathsf{P}_{\ell-1} \oplus \mathsf{P}_{\ell-3})\big) \oplus \bigoplus_{\substack{r=|s-(k+1)|+1 \\ step=2}}^{k+s} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_1\big) \\
&\quad \simeq \Big(\bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_{\ell-3}\big) \oplus \bigoplus_{\substack{r=|s-(k+1)|+1 \\ step=2}}^{k+s} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_1\big)\Big) \\
&\quad \oplus \bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_{\ell-1}\big).
\end{aligned} \tag{7.5.31}$$

However, proposition 7.4.5 gives

$$\begin{aligned}
\mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{s\ell-1+j} \simeq{}& \big(\mathsf{P}_{|s-k|\ell-1} + \mathsf{P}_{(s+k)\ell-1}\big) \times_f \mathsf{P}_{j-1} \\
&\oplus 2 \bigoplus_{\rho=|k-s|+1}^{k+s-1} \big(\mathsf{P}_{\rho\ell-1} \times_f \mathsf{P}_{\ell-j-1}\big) \oplus 2 \bigoplus_{\rho=|k-s|+2}^{k+s-2} \big(\mathsf{P}_{\rho\ell-1} \times_f \mathsf{P}_{j-1}\big),
\end{aligned} \tag{7.5.32}$$

and

$$\begin{aligned}
\bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_{\ell-1}\big) &\simeq \bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \Big(\big(\mathsf{P}_{(r-1)\ell-1} \oplus \mathsf{P}_{(r+1)\ell-1}\big) \times_f \mathsf{P}_{j-1} \oplus 2\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-(j+1)}\Big) \\
&\simeq 2 \bigoplus_{\substack{r=|k-s|+1 \\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-(j+1)}\big) \oplus \bigoplus_{\substack{r'=|k-s| \\ step=2}}^{k+s-2} \big(\mathsf{P}_{r'\ell-1} \times_f \mathsf{P}_{j-1}\big)
\end{aligned}$$

$$\oplus \bigoplus_{\substack{r'=|k-s|+2\\ step=2}}^{k+s} \big(\mathsf{P}_{r'\ell-1} \times_f \mathsf{P}_{j-1}\big). \tag{7.5.33}$$

Collecting identical terms in the last two sums then gives the identity

$$\mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{s\ell-1+j} \simeq \bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_{\ell-1}\big). \tag{7.5.34}$$

Comparing equations (7.5.30), and (7.5.31) and using this identity then gives the result for $i=2$.

Suppose now that the result stands for $i-1, i$ with $1<i<\ell$. Fusing the left side of equation (7.5.29) with P_1 and using proposition 7.5.2 gives

$$\big(\mathsf{S}_{k\ell-1+i} \times_f \mathsf{P}_1\big) \times_f \mathsf{P}_{s\ell-1+j} \simeq \big(\mathsf{S}_{k\ell-1+(i-1)} \oplus \mathsf{S}_{k\ell-1+(i+1)}\big) \times_f \mathsf{P}_{s\ell-1+j}, \tag{7.5.35}$$

while fusing the right side of equation (7.5.29) with P_1 and using proposition 7.4.2 yields

$$\begin{aligned}
&\bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f (\mathsf{P}_{\ell-1-i} \times_f \mathsf{P}_1)\big) \oplus \bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s} \big(\mathsf{P}_{r\ell-1+j} \times_f (\mathsf{P}_{i-1} \times_f \mathsf{P}_1)\big)\\
&\quad\simeq \bigoplus_{\substack{r=|k-s|+1\\ step=2}}^{k+s-1} \big(\mathsf{P}_{r\ell-1+j} \times_f \big(\mathsf{P}_{\ell-1-(i-1)} \oplus \mathsf{P}_{\ell-1-(i+1)}\big)\big)\\
&\quad\oplus \bigoplus_{\substack{r=|s-(k+1)|+1\\ step=2}}^{k+s} \big(\mathsf{P}_{r\ell-1+j} \times_f (\mathsf{P}_{i-2} \oplus \mathsf{P}_i)\big).
\end{aligned} \tag{7.5.36}$$

Comparing these two results and using the induction hypothesis then gives the result for $i+1$.

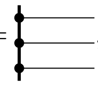
Note that it was implicitly assumed that $\ell \neq 2$, because this case is covered by proposition 7.5.4. ■

7.5.2 Fusion of two standard modules

The action of P_1 has played a central role so far in the proofs. Projective modules can all be expressed as “polynomials" in P_1 and even the standard modules $\mathsf{S}_{k\ell+i}$ could be obtained by fusing S_{kl} with it. However, fusing $\mathsf{S}_{k\ell}$ repeatedly with P_1 produced a sum of projective module, so that $\mathsf{S}_{(k+1)\ell}$ cannot be obtained from $\mathsf{S}_{k\ell}$. Another argument will thus be needed to “cross" the critical lines without obtaining projective modules. It will eventually be proved that this is done by fusing with S_{ℓ}. The proofs are identical for the dilute and the regular family, except when $\ell=2$. The proof of proposition 7.5.7 below is then very different. The result still stands in this case, but the proof will be presented in section 7.5.3.

The first step is to compute the dimension of $\mathsf{S}_{k,k} \times_f \mathsf{S}_{r,r}$ as it will make the proof of proposition 7.5.7 much easier. Note that the parameter n in $\mathsf{S}_{n,k}$ is now important as the dimension of the modules depends on it. The general case is very simple but somewhat long. We compute the dimension of $\mathsf{S}_{3,3} \times_f \mathsf{S}_{3,3}$. Define

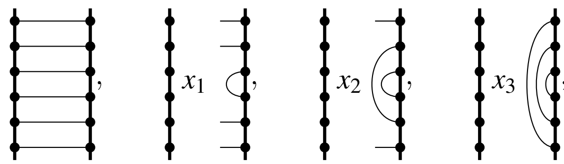
(see section 7.2.2)

$z =$  ,

which is such that $\mathsf{S}_{3,3} = A_3 z$, where $A_n = \mathsf{TL}_n$ or dTL_n. Then

$$\mathsf{S}_{3,3} \times_f \mathsf{S}_{3,3} \simeq A_6(\mathrm{id}_{A_6} \otimes_{A_3 \otimes A_3} (z \otimes_{\mathbb{C}} z)).$$

Furthermore, notice that the only diagram in A_3 which does not act as zero on z is the identity. It follows that the only diagrams of A_6 which do not act as zero on $\mathrm{id}_{A_6} \otimes_{A_3 \otimes A_3} (z \otimes_{\mathbb{C}} z)$ are those of the following form



(7.5.37)

where x_i is a link diagram in $\mathsf{S}_{6,6-2i}$. It also follows that for $b \in A_3 \otimes A_3$, $b(z \otimes_{\mathbb{C}} z) = 0$, unless b can be expressed as $b = (\mathrm{id} \otimes \mathrm{id}) + c$, for some $c \in \mathsf{dTL}_3 \otimes_{\mathbb{C}} \mathsf{dTL}_3$. We thus conclude that these diagrams form a basis of $\mathsf{S}_{3,3} \times_f \mathsf{S}_{3,3}$ and thus that

$$\dim \mathsf{S}_{3,3} \times_f \mathsf{S}_{3,3} = \dim \mathsf{S}_{6,6} + \dim \mathsf{S}_{6,4} + \dim \mathsf{S}_{6,2} + \dim \mathsf{S}_{6,0}.$$

The general case is obtained by a straightforward generalisation of this argument.

Lemma 7.5.6. *For all $k, r \in \mathbb{N}$,*

$$\dim(\mathsf{S}_{k,k} \times_f \mathsf{S}_{r,r}) = \sum_{i=0}^{\min(k,r)} \dim \mathsf{S}_{k+r,k+r-2i}. \tag{7.5.38}$$

The proof of the general case $\mathsf{S}_{k\ell+i} \times_f \mathsf{S}_{r\ell+j}$ will be done by induction on k,r,i and j. Fusion with P_1 will be used to induce from i to $i+1$, and from j to $j+1$, while fusion with S_ℓ will be used to induce from k to $k+1$,and r to $r+1$. The inductive proof is split into numerous lemmas so that the various steps are clearer. Each lemma will be accompanied by an example to illustrate the result.

Use again the particular case $\ell = 5$, and recall (see section 7.2.1) that the projective module $\mathsf{P}_{5,5}$ satisfies the short exact sequence

$$0 \longrightarrow \mathsf{P}_{5,3} \longrightarrow \mathsf{P}_{5,5} \longrightarrow \mathsf{S}_{5,5} \longrightarrow 0,$$

and using the right-exactness of fusion, proposition 7.3.2, this implies the exact sequence

$$\mathsf{P}_{5,3} \times_f \mathsf{S}_{5,5} \xrightarrow{f} \mathsf{P}_{5,5} \times_f \mathsf{S}_{5,5} \longrightarrow \mathsf{S}_{5,5} \times_f \mathsf{S}_{5,5} \longrightarrow 0. \tag{7.5.39}$$

Using the previously obtained fusion rules, note that

$$(\mathsf{P}_{5,5} \times_f \mathsf{S}_{5,5})/(\mathsf{P}_{5,3} \times_f \mathsf{S}_{5,5}) \simeq \frac{2\mathsf{P}_{10,4} \oplus 2\mathsf{P}_{10,6} \oplus \mathsf{P}_{10,8} \oplus \mathsf{P}_{10,10}}{\mathsf{P}_{10,4} \oplus \mathsf{P}_{10,6} \oplus \mathsf{S}_{10,8}} \simeq \mathsf{P}_{10,4} \oplus \mathsf{P}_{10,6} \oplus \mathsf{P}_{10,8} \oplus \mathsf{S}_{10,10} \tag{7.5.40}$$

by using the fact that $\mathsf{P}_{10,10}/\mathsf{S}_{10,8} \simeq \mathsf{S}_{10,10}$. However, lemma 7.5.6 gives

$$\dim \mathsf{S}_{5,5} \times_f \mathsf{S}_{5,5} = \dim \mathsf{P}_{10,4} \oplus \mathsf{P}_{10,6} \oplus \mathsf{P}_{10,8} \oplus \mathsf{S}_{10,10},$$

so it follows that f must be injective and thus

$$\mathsf{S}_{5,5} \times_f \mathsf{S}_{5,5} \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{4,4} \oplus \mathsf{S}_{10,10}.$$

Fusing the left side of this result with $\mathsf{P}_{1,1}$ and using proposition 7.5.2 gives

$$\mathsf{S}_{5,5} \times_f (\mathsf{S}_{5,5} \times_f \mathsf{P}_{1,1}) \simeq \mathsf{S}_{5,5} \times_f (\mathsf{P}_{6,4} \oplus \mathsf{S}_{6,6}), \tag{7.5.41}$$

while fusing its right side with $\mathsf{P}_{1,1}$ and using propositions 7.5.2, and 7.4.2 yields

$$\mathsf{P}_{6,4} \times_f (\mathsf{P}_{4,4} \times_f \mathsf{P}_{1,1}) \oplus (\mathsf{S}_{10,10} \times_f \mathsf{P}_{1,1}) \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{5,5} \oplus \mathsf{P}_{11,9} \oplus \mathsf{S}_{11,11}. \tag{7.5.42}$$

Using proposition 7.4.5, and 7.5.4, note that

$$\mathsf{P}_{6,4} \times_f \mathsf{P}_{5,5} \simeq 2\mathsf{P}_{6,4} \times_f \mathsf{P}_{5,3} \oplus \mathsf{P}_{11,9} \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{5,3} \oplus \mathsf{S}_{5,5} \times_f \mathsf{P}_{6,4}.$$

Comparing equations (7.5.41), and (7.5.42) and using this observation give the fusion rule

$$\mathsf{S}_{6,6} \times_f \mathsf{S}_{5,5} \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{5,3} \oplus \mathsf{P}_{11,9} \oplus \mathsf{S}_{11,11} \tag{7.5.43}$$

Repeating these arguments yields

$$\mathsf{S}_{7,7} \times_f \mathsf{S}_{5,5} \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{6,2} \oplus \mathsf{P}_{11,9} \times_f \mathsf{P}_{1,1} \oplus \mathsf{S}_{12,12}, \tag{7.5.44}$$

$$\mathsf{S}_{8,8} \times_f \mathsf{S}_{5,5} \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{7,1} \oplus \mathsf{P}_{11,9} \times_f \mathsf{P}_{2,2} \oplus \mathsf{S}_{13,13}, \tag{7.5.45}$$

$$\mathsf{S}_{9,9} \times_f \mathsf{S}_{5,5} \simeq \mathsf{P}_{6,4} \times_f \mathsf{P}_{8,0} \oplus \mathsf{P}_{11,9} \times_f \mathsf{P}_{3,3} \oplus \mathsf{S}_{14,14}. \tag{7.5.46}$$

Note that since $\mathsf{S}_{9,9} \simeq \mathsf{P}_{9,9}$ is projective, the last one could be obtained from proposition 7.5.4.

Proposition 7.5.7. *For $0 < i < \ell$, $k \in \mathbb{Z}_{>0}$ and in the regular family $\ell > 2$,*

$$\mathsf{S}_{k\ell-1+i} \times_f \mathsf{S}_\ell \simeq (\mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{\ell-i}) \oplus \left(\mathsf{P}_{(k+1)\ell-1} \times_f \mathsf{P}_{i-2}\right) \oplus \mathsf{S}_{(k+1)\ell-1+i}. \tag{7.5.47}$$

Proof. The proof proceeds by induction on k and i. Let us start by proving that for a given k, if the result stands for $i = 1$, it will also stand for all $i \leq \ell - 1$.

Suppose that the result stands for $i = 1$. Fusing the left side of equation (7.5.47) with P_1 and using proposition 7.5.2 gives

$$(\mathsf{S}_{k\ell} \times_f \mathsf{P}_1) \times_f \mathsf{S}_\ell \simeq (\mathsf{P}_{k\ell-1} \oplus \mathsf{S}_{k\ell+1}) \times_f \mathsf{S}_\ell \tag{7.5.48}$$

while fusing its right side and using propositions 7.5.2, and 7.4.2 yields

$$(\mathsf{P}_{k\ell-1}\times_f(\mathsf{P}_{\ell-1}\times_f\mathsf{P}_1))\oplus\mathsf{S}_{(k+1)\ell}\times_f\mathsf{P}_1\simeq(\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_\ell)\oplus\mathsf{P}_{(k+1)\ell-1}\oplus\mathsf{S}_{(k+1)\ell-1+2}. \tag{7.5.49}$$

However, proposition 7.4.5 gives

$$\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_\ell\simeq 2\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_{\ell-2}\oplus\mathsf{P}_{(k-1)\ell-1}\oplus\mathsf{P}_{(k+1)\ell-1},$$

and proposition 7.5.5,

$$\mathsf{P}_{k\ell-1}\times_f\mathsf{S}_\ell\simeq\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_{\ell-2}\oplus\mathsf{P}_{(k-1)\ell-1}\oplus\mathsf{P}_{(k+1)\ell-1}.$$

Comparing equations (7.5.48), and (7.5.49), and using these two results gives the result for $i=2$. Suppose then that the result stands for $i-1,i$, with $2\leq i<\ell-1$. Fusing the left side of (7.5.47) with P_1 and using proposition 7.5.2 gives

$$(\mathsf{S}_{k\ell-1+i}\times_f\mathsf{P}_1)\times_f\mathsf{S}_\ell\simeq(\mathsf{S}_{k\ell-1+i-1}\oplus\mathsf{S}_{k\ell-1+i+1})\times_f\mathsf{S}_\ell,$$

while fusing its right side with P_1 and using propositions 7.5.2, and 7.4.2 yields

$$\begin{aligned}(\mathsf{P}_{k\ell-1}\times_f&(\mathsf{P}_{\ell-i}\times_f\mathsf{P}_1))\oplus\big(\mathsf{P}_{(k+1)\ell-1}\times_f(\mathsf{P}_{i-2}\times_f\mathsf{P}_1)\big)\oplus\mathsf{S}_{(k+1)\ell-1+i}\times_f\mathsf{P}_1\\
&\simeq\big(\mathsf{P}_{k\ell-1}\times_f\big(\mathsf{P}_{\ell-(i-1)}\oplus\mathsf{P}_{\ell-(i+1)}\big)\big)\oplus\big(\mathsf{P}_{(k+1)\ell-1}\times_f(\mathsf{P}_{i-3}\oplus\mathsf{P}_{i-1})\big)\\
&\qquad\oplus\mathsf{S}_{(k+1)\ell-1+i-1}\oplus\mathsf{S}_{(k+1)\ell-1+i+1}\\
&\simeq\big(\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_{\ell-(i-1)}\big)\oplus\big(\mathsf{P}_{(k+1)\ell-1}\times_f\mathsf{P}_{(i-1)-2}\big)\oplus\mathsf{S}_{(k+1)\ell-1+(i-1)}\\
&\qquad\oplus\big(\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_{\ell-(i+1)}\big)\oplus\big(\mathsf{P}_{(k+1)\ell-1}\times_f\mathsf{P}_{(i+1)-2}\big)\oplus\mathsf{S}_{(k+1)\ell-1+(i+1)}\end{aligned} \tag{7.5.50}$$

Comparing the two and using the induction hypothesis yields the result for $i+1$.

We must now do the induction on k. Note that when $k=0$, $\mathsf{S}_{0\ell-1+i}\simeq\mathsf{P}_{i-1}$ and is thus projective. Proposition 7.5.3 then gives the result when $k=0$. Suppose now that the result holds for k and $i=\ell-1$. There is a short-exact sequence

$$0\longrightarrow\mathsf{S}_{(k+1)\ell,k\ell-1+(\ell-1)}\longrightarrow\mathsf{P}_{(k+1)\ell,(k+1)\ell}\longrightarrow\mathsf{S}_{(k+1)\ell,(k+1)\ell}\longrightarrow 0. \tag{7.5.51}$$

Note that the $n=(k+1)\ell$ is important in this case so it is written explicitly. Fusing this sequence with $\mathsf{S}_{\ell,\ell}$ gives the exact sequence

$$\mathsf{S}_{(k+1)\ell,k\ell-1+(\ell-1)}\times_f\mathsf{S}_{\ell,\ell}\xrightarrow{f}\mathsf{P}_{(k+1)\ell,(k+1)\ell}\times_f\mathsf{S}_{\ell,\ell}\longrightarrow\mathsf{S}_{(k+1)\ell,(k+1)\ell}\times_f\mathsf{S}_{\ell,\ell}\longrightarrow 0. \tag{7.5.52}$$

We thus have the following inequality

$$\begin{aligned}\dim\mathsf{S}_{(k+1)\ell,(k+1)\ell}\times_f\mathsf{S}_{\ell,\ell}&\leq\dim\mathsf{P}_{(k+1)\ell,(k+1)\ell}\times_f\mathsf{S}_{\ell,\ell}-\dim\mathsf{S}_{(k+1)\ell,k\ell-1+(\ell-1)}\times_f\mathsf{S}_{\ell,\ell}\\
&=\dim\mathsf{P}_{(k+1)\ell-1,(k+1)\ell-1}\times_f\mathsf{P}_{\ell+1,\ell-1}\oplus\mathsf{S}_{(k+2)\ell,(k+2)\ell},\end{aligned} \tag{7.5.53}$$

where equality stands if and only if $\ker f = 0$, and the second line is obtained by using proposition 7.5.4 and the induction hypothesis with the structure of the projective modules (see section 7.2.1). However, lemma 7.5.6 gives

$$\dim \mathsf{S}_{(k+1)\ell,(k+1)\ell} \times_f \mathsf{S}_{\ell,\ell} = \dim \mathsf{P}_{(k+1)\ell+1,(k+1)\ell-1} \times_f \mathsf{P}_{\ell-1,\ell-1} \oplus \mathsf{S}_{(k+2)\ell,(k+2)\ell}.$$

It follows that $\ker f = 0$, and thus that

$$\begin{aligned} \mathsf{S}_{(k+1)\ell,(k+1)\ell} \times_f \mathsf{S}_{\ell,\ell} &\simeq \big(\mathsf{P}_{(k+1)\ell,(k+1)\ell} \times_f \mathsf{S}_{\ell,\ell}\big) / \big(\mathsf{S}_{(k+1)\ell,k\ell-1+(\ell-1)} \times_f \mathsf{S}_{\ell,\ell}\big) \\ &\simeq \mathsf{P}_{(k+1)\ell-1,(k+1)\ell-1} \times_f \mathsf{P}_{\ell+1,\ell-1} \oplus \mathsf{S}_{(k+2)\ell,(k+2)\ell}, \end{aligned} \tag{7.5.54}$$

where the second equality is obtained by using proposition 7.5.4 and the induction hypothesis with the structure of the projective modules. Note that once the result stands for $n = (k+1)\ell$, fusing it repeatedly with $\mathsf{P}_{2,0}$ will give the result for all $n \geq (k+1)\ell$. It follows that if the result stands for k and $i = \ell - 1$, it stands for $k+1$ and $i = 1$. Using the first part of the proof, the conclusion is obtained. ■

Fusion with S_ℓ can thus be used to “cross" the critical lines. The following continuation of the $\ell = 5$ example illustrate how the argument works. Proposition 7.5.7 gives

$$\mathsf{S}_{10} \times_f \mathsf{S}_5 \simeq \mathsf{P}_9 \times_f \mathsf{P}_4 \oplus \mathsf{S}_{15}. \tag{7.5.55}$$

Fusing the left side of this equation with S_5 and using propositions 7.5.7, and 7.5.5 produces

$$\begin{aligned} \mathsf{S}_{10} \times_f (\mathsf{S}_5 \times_f \mathsf{S}_5) &\simeq \mathsf{S}_{10} \times_f ((\mathsf{P}_4 \times_f \mathsf{P}_4) \oplus \mathsf{S}_{10}) \\ &\simeq (\mathsf{P}_9 \times_f \mathsf{P}_3 \oplus \mathsf{P}_{14}) \times_f \mathsf{P}_4 \oplus \mathsf{S}_{10} \times_f \mathsf{S}_{10}, \end{aligned} \tag{7.5.56}$$

while fusing its right side with S_5 and using the same propositions gives

$$\begin{aligned} (\mathsf{P}_9 \times_f \mathsf{S}_5) \times_f \mathsf{P}_4 \oplus \mathsf{S}_{15} \times_f \mathsf{S}_5 &\simeq (\mathsf{P}_9 \times_f \mathsf{P}_3 \oplus \mathsf{P}_4 \oplus \mathsf{P}_{14}) \times_f \mathsf{P}_4 \oplus (\mathsf{P}_{14} \times_f \mathsf{P}_4 \oplus \mathsf{S}_{20}) \\ &\simeq (\mathsf{P}_9 \times_f \mathsf{P}_3 \oplus \mathsf{P}_{14}) \times_f \mathsf{P}_4 \oplus (\mathsf{P}_4 \oplus \mathsf{P}_{14}) \times_f \mathsf{P}_4 \oplus \mathsf{S}_{20}. \end{aligned} \tag{7.5.57}$$

Comparing the two gives the fusion rule

$$\mathsf{S}_{10} \times_f \mathsf{S}_{10} \simeq (\mathsf{P}_4 \oplus \mathsf{P}_{14}) \times_f \mathsf{P}_4 \oplus \mathsf{S}_{20}.$$

Proposition 7.5.8. *For $q,k \in \mathbb{Z}_{>0}$,*

$$\mathsf{S}_{q\ell} \times_f \mathsf{S}_{k\ell} \simeq \bigoplus_{\substack{r=|q-k|+1 \\ step=2}}^{q+k-1} (\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1}) \oplus \mathsf{S}_{(q+k)\ell}. \tag{7.5.58}$$

Proof. Since fusion is commutative, suppose without loss of generality that $k \leq q$. The proof then proceeds by induction on k. For $k=1$, propositions 7.5.7,7.5.11 give the result for all q. Suppose then that the result holds for some $k<q$. Fusing the left side of equation (7.5.58) with S_ℓ and using propositions 7.5.7, 7.5.11 and 7.5.4 gives

$$\begin{aligned}\mathsf{S}_{q\ell}\times_f(\mathsf{S}_{k\ell}\times_f\mathsf{S}_\ell)&\simeq\mathsf{S}_{q\ell}\times_f\left(\mathsf{P}_{k\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{(k+1)\ell}\right)\\&\simeq\Bigg(\bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1}(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-2})\oplus\bigoplus_{\substack{r=|k-(q+1)|+1\\ step=2}}^{k+q}\mathsf{P}_{r\ell-1}\Bigg)\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{q\ell}\times_f\mathsf{S}_{(k+1)\ell}.\end{aligned}\tag{7.5.59}$$

Fusing its right side with S_ℓ and using the same propositions yields

$$\begin{aligned}&\bigoplus_{\substack{r=q-k+1\\ step=2}}^{q+k-1}(\mathsf{P}_{r\ell-1}\times_f(\mathsf{P}_{\ell-1}\times_f\mathsf{S}_\ell))\oplus\mathsf{S}_{(q+k)\ell}\times_f\mathsf{S}_\ell\\&\quad\simeq\bigoplus_{\substack{r=q-k+1\\ step=2}}^{q+k-1}(\mathsf{P}_{r\ell-1}\times_f(\mathsf{P}_{\ell-1}\times_f\mathsf{P}_{\ell-2}\oplus\mathsf{P}_{2\ell-1}))\oplus\mathsf{P}_{(q+k)\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{(q+k+1)\ell-1}\\&\quad\simeq\bigoplus_{\substack{r=q-k+1\\ step=2}}^{q+k-1}\left(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-1}\times_f\mathsf{P}_{\ell-2}\oplus\left(\mathsf{P}_{(r-1)\ell-1}\oplus\mathsf{P}_{(r+1)\ell-1}\right)\times_f\mathsf{P}_{\ell-1}\right)\\&\qquad\oplus\mathsf{P}_{(q+k)\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{(q+k+1)\ell-1}.\end{aligned}\tag{7.5.60}$$

Comparing these two equations gives

$$\begin{aligned}&\bigoplus_{\substack{r=q+2-k\\ step=2}}^{k+q}\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{q\ell}\times_f\mathsf{S}_{(k+1)\ell}\\&\quad\simeq\bigoplus_{\substack{r=q-k+1\\ step=2}}^{q+k-1}\left(\mathsf{P}_{(r-1)\ell-1}\oplus\mathsf{P}_{(r+1)\ell-1}\right)\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{P}_{(q+k)\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{(q+k+1)\ell-1}\\&\quad\simeq\bigoplus_{\substack{r=q-k+2\\ step=2}}^{q+k}\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\bigoplus_{\substack{r=q-(k+1)+1\\ step=2}}^{q+k}\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{(q+k+1)\ell-1},\end{aligned}$$

where the second equality is obtained by rearranging the terms in the sum. Comparing the two sides of this equation gives the conclusion. ∎

Knowing the fusion $\mathsf{S}_{q\ell}\times_f\mathsf{S}_{k\ell}$, the fusion $\mathsf{S}_{q\ell+i}\times_f\mathsf{S}_{k\ell+j}$ can be computed by using the fusion of $\mathsf{S}_{k\ell+j'}$ with P_1.

Proposition 7.5.9. *For $q,k\in\mathbb{Z}_{>0}$, $0\le i,j<\ell$,*

$$\mathsf{S}_{q\ell-1+i}\times_f\mathsf{S}_{k\ell-1+j}\simeq\bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1}\left(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-|i-j|-1}\right)\oplus\bigoplus_{\substack{r=|k-q-sign(i-j)|+1\\ step=2}}^{q+k}\left(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{|i-j|-1}\right)$$
$$\oplus\bigoplus_{\substack{s=|i-j|+1\\ step=2}}^{\ell-|\ell-(i+j)|-1}\left(\mathsf{S}_{(q+k)\ell-1+s}\right)\oplus\mathsf{P}_{(k+q+1)\ell-1}\times_f\mathsf{P}_{i+j-\ell-1}.\quad(7.5.61)$$

Proof. The proof proceeds by induction on i,j and involves many different particular cases: $i<j$, $i=j$ or $i>j$ with $i+j<\ell$ or $i+j>\ell$. Note that when $\ell=2$, this proposition is equivalent to proposition 7.5.8, so we assume that $\ell\neq 2$.

Without loss of generality, suppose $q\geq k$. Proposition 7.5.8 gives the case $i=j=1$, proposition 7.5.5 gives the case $j=0$ for all $i>0$, and $i=0$ for all $j>0$, while proposition 7.4.5 gives the case $i=j=0$. Suppose now $j\geq 1$, fusing the left side of equation (7.5.61) with P_1 and using propositions 7.5.2 and 7.5.5 gives

$$\mathsf{S}_{q\ell-1+i}\times_f\left(\mathsf{S}_{k\ell-1+j}\times_f\mathsf{P}_1\right)\simeq\mathsf{S}_{q\ell-1+i}\times_f\mathsf{S}_{k\ell-1+j-1}\oplus\mathsf{S}_{q\ell-1+i}\times_f\mathsf{S}_{k\ell-1+(j+1)},\quad(7.5.62)$$

while fusing the right side of this equation with P_1 yields

$$\overbrace{\bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1}\left(\mathsf{P}_{r\ell-1}\times_f\left(\mathsf{P}_{\ell-|i-j|-1}\times_f\mathsf{P}_1\right)\right)}^{a}\oplus\overbrace{\bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-j)|+1\\ step=2}}^{q+k}\left(\mathsf{P}_{r\ell-1}\times_f\left(\mathsf{P}_{|i-j|-1}\times_f\mathsf{P}_1\right)\right)}^{b}$$
$$\oplus\underbrace{\bigoplus_{\substack{s=|i-j|+1\\ step=2}}^{\ell-|\ell-(i+j)|-1}\left(\mathsf{S}_{(q+k)\ell-1+s}\times_f\mathsf{P}_1\right)}_{c}\oplus\underbrace{\mathsf{P}_{(k+q+1)\ell-1}\times_f\left(\mathsf{P}_{i+j-\ell-1}\times_f\mathsf{P}_1\right)}_{d}.\quad(7.5.63)$$

The terms in a can be written

$$\bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1}\left(\mathsf{P}_{r\ell-1}\times_f\left(\mathsf{P}_{\ell-|i-j|-1}\times_f\mathsf{P}_1\right)\right)$$
$$\overset{1}{\simeq}\bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1}\mathsf{P}_{r\ell-1}\times_f\left\{\begin{matrix}\mathsf{P}_\ell, & \text{if } i=j\\ \mathsf{P}_{\ell-|i-j|-1-1}\oplus\mathsf{P}_{\ell-|i-j|-1+1}, & \text{otherwise}\end{matrix}\right\}$$
$$\overset{2}{\simeq}\bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1}\left\{\begin{matrix}2\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-2}\oplus\mathsf{P}_{(r-1)\ell-1}\oplus\mathsf{P}_{(r+1)\ell-1}, & \text{if } i=j,\\ \mathsf{P}_{r\ell-1}\times_f\left(\mathsf{P}_{\ell-|i-j-1|-1}\oplus\mathsf{P}_{\ell-|i-j+1|-1}\right), & \text{otherwise}\end{matrix}\right\}$$

$$
\begin{aligned}
&\overset{3}{\simeq} \bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1} \left(\mathsf{P}_{r\ell-1} \times_f \left(\mathsf{P}_{\ell-|i-j-1|-1} \oplus \mathsf{P}_{\ell-|i-j+1|-1}\right)\right)\\
&\quad \oplus \bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1} \left\{\begin{array}{ll} \mathsf{P}_{(r-1)\ell-1} \oplus \mathsf{P}_{(r+1)\ell-1}, & \text{if } i=j,\\ 0, & \text{otherwise} \end{array}\right\}\\
&\overset{4}{\simeq} \bigoplus_{\substack{r=|q-k|+1\\ step=2}}^{q+k-1} \left(\mathsf{P}_{r\ell-1} \times_f \left(\mathsf{P}_{\ell-|i-j-1|-1} \oplus \mathsf{P}_{\ell-|i-j+1|-1}\right)\right)\\
&\quad \oplus \left\{\begin{array}{ll} \displaystyle\bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-(j-1))|+1\\ step=2}}^{q+k} \left(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\underbrace{|i-(j-1)|-1}_{=0}}\right) & \\ \oplus \displaystyle\bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-(j+1))|+1\\ step=2}}^{q+k-2} \left(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\underbrace{|i-(j+1)|-1}_{=0}}\right) & \text{if } i=j,\\ 0 & \text{otherwise.} \end{array}\right\}
\end{aligned} \tag{7.5.64}
$$

The first equality is obtained by using proposition 7.4.2, the second by using proposition 7.4.5 and the third and fourth are obtained by noting that $|i-j+1| = |i-j-1| = 1$ when $i = j$ and rearranging the terms in the sums, respectively.

The terms in b can be written

$$
\begin{aligned}
&\bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-j)|+1\\ step=2}}^{q+k} \left(\mathsf{P}_{r\ell-1} \times_f \left(\mathsf{P}_{|i-j|-1} \times_f \mathsf{P}_1\right)\right)\\
&\simeq \bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-j)|+1\\ step=2}}^{q+k} \left\{\begin{array}{ll} 0, & \text{if } i=j,\\ \mathsf{P}_{r\ell-1} \times_f \left(\mathsf{P}_{|i-j|-2} \oplus \mathsf{P}_{|i-j|}\right), & \text{otherwise} \end{array}\right\}\\
&\simeq \bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-j)|+1\\ step=2}}^{q+k} \left\{\begin{array}{ll} 0, & \text{if } i=j,\\ \mathsf{P}_{r\ell-1} \times_f \left(\mathsf{P}_{|i-j-1|-1} \oplus \mathsf{P}_{|i-j+1|-1}\right), & \text{otherwise} \end{array}\right\}\\
&\simeq \left\{\begin{array}{ll} 0, & \text{if } i=j,\\ \displaystyle\bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-j-1)|+1\\ step=2}}^{q+k} \left(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{|i-j-1|-1}\right) & \\ \oplus \displaystyle\bigoplus_{\substack{r=|k-q-\mathrm{sign}(i-j+1)|+1\\ step=2}}^{q+k} \left(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{|i-j+1|-1}\right), & \text{otherwise} \end{array}\right\}.
\end{aligned} \tag{7.5.65}
$$

The first equality is obtained by using proposition 7.4.2 and the fact that $\mathsf{P}_{-1} \equiv 0$, while the second one is obtained by noting that if $i > j$, $|i-j|-2 = |i-(j+1)|-1$, $|i-j| = |i-(j-1)|-1$ while if $i < j$, $|i-j|-2 = |i-(j-1)|-1$, $|i-j| = |i-(j+1)|-1$. The third one is obtained by noting that if $i \neq j$ and $\mathrm{sign}(i-j) \neq \mathrm{sign}(i-j\pm 1)$, then $|i-j\pm 1|-1 < 0$, and thus $\mathsf{P}_{|i-j\pm 1|-1} \equiv 0$.

The terms in c can be written

$$\begin{aligned}
&\bigoplus_{\substack{s=|i-j|+1\\ step=2}}^{\ell-|\ell-(i+j)|-1} \left(\mathsf{S}_{(q+k)\ell-1+s}\times_f \mathsf{P}_1\right)\\
&\simeq \bigoplus_{\substack{s=|i-j|+1\\ step=2}}^{\ell-|\ell-(i+j)|-1} \left(\mathsf{S}_{(q+k)\ell-1+s-1}\oplus \mathsf{S}_{(q+k)\ell-1+s+1}\right)\\
&\simeq \bigoplus_{\substack{s=|i-j|\\ step=2}}^{\ell-|\ell-(i+j)|-2} \mathsf{S}_{(q+k)\ell-1+s}\oplus \bigoplus_{\substack{s=|i-j|+2\\ step=2}}^{\ell-|\ell-(i+j)|} \mathsf{S}_{(q+k)\ell-1+s}\\
&\simeq \bigoplus_{\substack{s=|i-j-1|+1\\ step=2}}^{\ell-|\ell-(i+j+1)|-1} \mathsf{S}_{(q+k)\ell-1+s}\oplus \bigoplus_{\substack{s=|i-j+1|+1\\ step=2}}^{\ell-|\ell-(i+j-1)|-1} \mathsf{S}_{(q+k)\ell-1+s}\\
&\oplus \delta_{0,|i-j|}\mathsf{P}_{(k+q)\ell-1}\oplus \delta_{i+j,\ell}\mathsf{P}_{(k+q+1)\ell-1}.
\end{aligned} \tag{7.5.66}$$

The first equality is obtained by applying proposition 7.5.2, the second by splitting the sum in two and renaming the indices while the third is obtained by considering the different possibilities for the absolute values and rearranging the two sums accordingly.

The terms in d can be written

$$\begin{aligned}
&\mathsf{P}_{(k+q+1)\ell-1}\times_f\left(\mathsf{P}_{i+j-\ell-1}\times_f \mathsf{P}_1\right)\\
&\qquad\simeq \mathsf{P}_{(k+q+1)\ell-1}\times_f \left\{\begin{matrix} 0, & \text{if } i+j<\ell+1,\\ (\mathsf{P}_{i+j-1-\ell-1}\oplus\mathsf{P}_{i+j+1-\ell-1}), & \text{otherwise}\end{matrix}\right\}
\end{aligned} \tag{7.5.67}$$

by simply using proposition 7.4.2 and the fact that $\mathsf{P}_t\equiv 0$ when $t<0$.

Putting all of these together, grouping the terms in the appropriate manner and comparing the result with equation (7.5.62) yields the conclusion for $i, j+1$, provided that it stands for i, j, $i, j-1$. The induction to $i+1, j$ from $i-1, j$ is done using the same arguments, except that in equation (7.5.62), P_1 is fused with $\mathsf{S}_{q\ell-1+i}$ instead of $\mathsf{S}_{k\ell-1+j}$, and the rearranging used to reorder the sums in the different terms is slightly different. ■

7.5.3 The case $\ell=2$ in TL_n

We treat here the regular Temperley-Lieb family when $\ell=2$. Recall that in this case the module P_0 which was used to remove the dependence on n is trivial, so the proof of proposition 7.5.7 does not work. The method used here is more tedious than that of the previous section but it will ultimately give the same results.

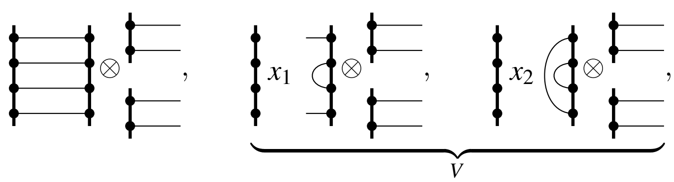
Proposition 7.5.10. *When $\ell = 2$, in the regular family,*

$$\mathsf{S}_{n,2} \times_f \mathsf{S}_{m,2} \simeq \mathsf{P}_{n+m,2} \oplus \mathsf{S}_{n+m,4}. \tag{7.5.68}$$

If $n \geq 4$,

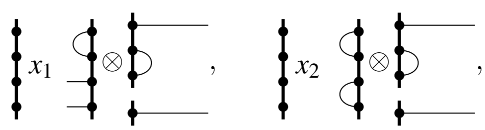
$$\mathsf{I}_{n,2} \times_f \mathsf{S}_{m,2} \simeq \mathsf{S}_{n+m,2}. \tag{7.5.69}$$

Proof. The case $n = m = 2$ is particular and it must be computed by hand. Using the same arguments as in lemma 7.5.6 the following set is a basis of $\mathsf{S}_{2,2} \times_f \mathsf{S}_{2,2}$:



(7.5.70)

where x_i are the link diagrams in $\mathsf{S}_{4,4-2i}$ and it can be seen directly that the elements of V spans a submodule of $\mathsf{S}_{2,2} \times_f \mathsf{S}_{2,2}$. However when $\ell = 2$, $\mathsf{P}_{4,2} \simeq \mathsf{P}_{3,1} \times_f \mathsf{P}_{1,1}$ is spanned by



(7.5.71)

where x_i are the link diagrams in $\mathsf{S}_{4,4-2i}$. A simple verification shows that $\text{span}\{V\} \simeq \mathsf{P}_{4,2}$, and that $(\mathsf{S}_{2,2} \times_f \mathsf{S}_{2,2})/\text{Span}\{V\} \simeq \mathsf{S}_{4,4}$. Using the fact that $\mathsf{P}_{4,2}$ is injective (see section 7.2.1) yields the conclusion.

Suppose that $n \geq m$, $n \geq 4$ and start with the exact sequence

$$0 \longrightarrow \mathsf{I}_{n,2} \longrightarrow \mathsf{P}_{n,2} \longrightarrow \mathsf{S}_{n,2} \longrightarrow 0,$$

which becomes

$$\mathsf{I}_{n,2} \times_f \mathsf{S}_{m,2} \xrightarrow{f} \mathsf{P}_{n+m,2} \oplus \mathsf{P}_{n+m,4} \longrightarrow \mathsf{S}_{n,2} \times_f \mathsf{S}_{m,2} \longrightarrow 0, \tag{7.5.72}$$

by using the right-exactness of fusion with the fusion rules 7.5.4. To find $\mathsf{I}_{n,2} \times_f \mathsf{S}_{m,2}$, fuse the sequence

$$\mathsf{P}_{m,2} \longrightarrow \mathsf{P}_{m,2} \longrightarrow \mathsf{S}_{m,2} \longrightarrow 0,$$

with $\mathsf{I}_{n,2}$ to obtain,,

$$\mathsf{P}_{n+m,2} \longrightarrow \mathsf{P}_{n+m,2} \longrightarrow \mathsf{S}_{m,2} \times_f \mathsf{I}_{n,2} \longrightarrow 0, \tag{7.5.73}$$

where proposition 7.6.4 was used. Note that the proof of this proposition is independent of this one so it can safely be used. It follows that there are three possibilities

$$\mathsf{S}_{m,2} \times_f \mathsf{I}_{n,2} \simeq \begin{cases} \mathsf{P}_{n+m,2}, \\ \mathsf{S}_{n+m,2}, \\ 0 \end{cases}.$$

But, proposition 7.6.4 gives $\mathsf{S}_{m,2} \times_f \mathsf{I}_{n,2} \times_f \mathsf{P}_{1,1} \simeq \mathsf{S}_{m,2} \times_f \mathsf{P}_{n+1,1} \simeq \mathsf{P}_{n+m+1,1} \oplus \mathsf{P}_{n+m+1,3}$. Since $\mathsf{P}_{n+m,2} \times_f \mathsf{P}_{1,1} \simeq 2\mathsf{P}_{n+m+1,1} \oplus \mathsf{P}_{n+m+1,3}$, it follows that

$$\mathsf{S}_{m,2} \times_f \mathsf{I}_{n,2} \simeq \mathsf{S}_{n+m,2}.$$

Now, the morphisms from $\mathsf{S}_{n+m,2}$ to $\mathsf{P}_{n+m,2} \oplus \mathsf{P}_{n+m,4}$ are known (see their Loewy diagrams) and the cokernel of f must be one of the following modules

$$\mathsf{P}_{n+m,2} \oplus \mathsf{P}_{n+m,4}, \qquad \mathsf{S}_{n+m,2} \oplus \mathsf{P}_{n+m,4}, \qquad \mathsf{P}_{n+m,2} \oplus \mathsf{S}_{n+m,4}.$$

Using propositions 7.4.2 and 7.5.2,

$$(\mathsf{P}_{n+m,2} \oplus \mathsf{P}_{n+m,4}) \times_f \mathsf{P}_{1,1} \simeq 3\mathsf{P}_{n+m+1,1} \oplus 3\mathsf{P}_{n+m+1,3} \oplus \mathsf{P}_{n+m+1,5},$$

$$(\mathsf{S}_{n+m,2} \oplus \mathsf{P}_{n+m,4}) \times_f \mathsf{P}_{1,1} \simeq 2\mathsf{P}_{n+m+1,1} \oplus 3\mathsf{P}_{n+m+1,3} \oplus \mathsf{P}_{n+m+1,5},$$

$$(\mathsf{P}_{n+m,2} \oplus \mathsf{S}_{n+m,4}) \times_f \mathsf{P}_{1,1} \simeq 2\mathsf{P}_{n+m+1,1} \oplus 2\mathsf{P}_{n+m+1,3} \oplus \mathsf{P}_{n+m+1,5},$$

while

$$\mathsf{S}_{n,2} \times_f \mathsf{S}_{m,2} \times_f \mathsf{P}_{1,1} \simeq \mathsf{S}_{n,2} \times_f (\mathsf{P}_{m+1,1} \oplus \mathsf{P}_{m+1,3}) \simeq 2\mathsf{P}_{n+m+1,1} \oplus 2\mathsf{P}_{n+m+1,3} \oplus \mathsf{P}_{n+m+1,5}.$$

It thus follows that $\mathsf{S}_{n,2} \times_f \mathsf{S}_{m,2} \simeq \mathsf{P}_{n+m,2} \oplus \mathsf{S}_{n+m,4}$, as long as one of n or m is bigger or equal to 4. ■

Now that the fusion of $\mathsf{S}_{n,2}$ with itself is known, it can be used to compute the fusion of the other standard modules. Note that the fusion of $\mathsf{S}_{2,2}$ with standard modules other than $\mathsf{S}_{2,2}$ can be obtained by the same arguments as in proposition 7.5.7, so we will only give the proof for $\mathsf{S}_{n,2} = \mathsf{S}_2$ with $n \geq 4$. We present a few examples before proving the general case. There is an exact sequence

$$0 \longrightarrow \mathsf{S}_2 \longrightarrow \mathsf{P}_4 \longrightarrow \mathsf{S}_4 \longrightarrow 0,$$

which becomes

$$\mathsf{S}_2 \xrightarrow{f} \mathsf{P}_4 \longrightarrow \mathsf{I}_2 \times_f \mathsf{S}_4 \longrightarrow 0, \tag{7.5.74}$$

by fusing it with I_2 and using the preceding proposition with proposition 7.6.4. Note that $\mathsf{I}_2 \times_f \mathsf{S}_4 \times_f \mathsf{P}_1 \simeq \mathsf{S}_4 \times_f \mathsf{P}_1 \simeq \mathsf{P}_3 \oplus \mathsf{P}_5$. Since the cokernel of f is either P_4 or S_4, it follows that $\mathsf{I}_2 \times_f \mathsf{S}_4 \simeq \mathsf{S}_4$. Now, the exact sequence

$$0 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{P}_2 \longrightarrow \mathsf{S}_2 \longrightarrow 0,$$

when fused with S_4, yield the exact sequence

$$\mathsf{S}_4 \longrightarrow \mathsf{P}_4 \oplus \mathsf{P}_6 \longrightarrow \mathsf{S}_2 \times_f \mathsf{S}_4 \longrightarrow 0,$$

by using proposition 7.5.5. There are thus three possibilities

$$\mathsf{S}_2 \times_f \mathsf{S}_4 \simeq \mathsf{P}_4 \oplus \mathsf{P}_6, \qquad \mathsf{B}_2^2 \oplus \mathsf{P}_6 \qquad \text{or } \mathsf{P}_4 \oplus \mathsf{S}_6.$$

But, using proposition 7.5.5, we get

$$\mathsf{S}_2 \times_f \mathsf{S}_4 \times_f \mathsf{P}_1 \simeq (\mathsf{P}_1 \oplus \mathsf{P}_3) \times_f \mathsf{S}_4 \simeq \mathsf{P}_1 \oplus 2\mathsf{P}_3 \oplus 2\mathsf{P}_5 \oplus \mathsf{P}_7.$$

Then, we verify which of the three possibilities satisfies this rule:

$$(\mathsf{P}_4 \oplus \mathsf{P}_6) \times_f \mathsf{P}_1 \simeq \mathsf{P}_1 \oplus 3\mathsf{P}_3 \oplus 3\mathsf{P}_5 \oplus \mathsf{P}_7,$$

$$(\mathsf{B}_2^2 \oplus \mathsf{P}_6) \times_f \mathsf{P}_1 \simeq \mathsf{P}_1 \oplus 2\mathsf{P}_3 \oplus 3\mathsf{P}_5 \oplus \mathsf{P}_7,$$

$$(\mathsf{P}_4 \oplus \mathsf{S}_6) \times_f \mathsf{P}_1 \simeq \mathsf{P}_1 \oplus 2\mathsf{P}_3 \oplus 2\mathsf{P}_5 \oplus \mathsf{P}_7,$$

where propositions 7.4.2, 7.5.2, and 7.B.3 were used. We are allowed to do so because the proofs of these propositions are independent of the fusion rules for standard modules. Comparing these fusion with $\mathsf{S}_2 \times_f \mathsf{S}_4 \times_f \mathsf{P}_1$, it follows that

$$\mathsf{S}_2 \times_f \mathsf{S}_4 \simeq \mathsf{P}_4 \oplus \mathsf{S}_6.$$

The proof of the general result that follows is obtained by induction and repeats the preceding arguments.

Proposition 7.5.11. *For $n \geq 4$, $m/2 \geq k \geq 1$,*

$$\mathsf{S}_{n,2} \times_f \mathsf{S}_{m,2k} \simeq \mathsf{P}_{n+m,2k} \oplus \mathsf{S}_{n+m,2(k+1)},$$

and

$$\mathsf{I}_{n,2} \times_f \mathsf{S}_{m,2k} \simeq \mathsf{S}_{n+m,2k}.$$

Note that a simple corollary of this proposition is that $\mathsf{I}_{n,2} \simeq \mathsf{S}_{n,0}$ when $\ell = 2$, plays the role of $\mathsf{P}_{n,0}$ when $\ell \neq 2$, except that in this case $n \geq 4$ instead of $n \geq 2$.

7.5.4 A simple rule for fusion

The fusion rules for standard modules and projective modules can be hard to apply in practice because of the numerous direct sum and fusions involved; we thus present a simple “rule of thumb" to quickly compute fusion of standard modules.

Proposition 7.5.12. *To a standard module* S_i *(i can be critical), associate the Chebyshev polynomial of the second kind* $U_i(\frac{x}{2})$ *where x is a formal parameter. To a projective module* P_{k_c+j}*,* $\ell > j > 0$*, associate the sum of Chebyshev polynomials* $U_{k_c-j}(\frac{x}{2}) + U_{k_c+j}(\frac{x}{2})$*. Call this association the* polynomial representation *of the modules. Furthermore, since the polynomials all have the same argument, it will simply be omitted. To take the fusion of two modules, multiply their polynomial representations and split the result by using the product rule*

$$U_iU_j = \sum_{\substack{k=|i-j| \\ step=2}}^{i+j} U_k.$$

Collect the terms in this sum to form the polynomial representation of projective modules, starting with the smallest k. Remaining terms are then identified with the corresponding standard modules.

It is straightforward but tedious to prove that all fusion rules obtained so far respect this simple rule.

7.6 Fusion of irreducible modules

We are now trying to compute the fusion of two irreducible modules. We begin by explaining the general idea which we will use to compute them. Suppose there are two modules U, V and two resolutions

$$U_2 \longrightarrow U_1 \longrightarrow U \longrightarrow 0, \qquad V_2 \longrightarrow V_1 \longrightarrow V \longrightarrow 0,$$

by modules U_i, V_i. It is a standard exercise in diagram chasing to obtain the exact sequence

$$U_2 \times_f V_1 \oplus U_1 \times_f V_2 \longrightarrow U_1 \times_f V_1 \xrightarrow{\phi} U \times_f V \longrightarrow 0. \tag{7.6.1}$$

If ϕ can be computed somehow, the knowledge of the fusion rules for $U_1 \times_f V_1$, $U_2 \times_f V_1$ and $U_1 \times_f V_2$ will give $U \times_f V$. If U_1, V_1 are “close" to U and V, the kernel of ϕ will be small, and its image will be much easier to compute. The idea is therefore to find the U_1, V_1 that are the “closest" to U and V but such that their fusion can be computed. Of course the “closest" module to an irreducible $\mathsf{I}_{n,i}$ is $\mathsf{I}_{n,i}$ itself, the second closest would be the standard module $\mathsf{S}_{n,i}$ and the third would be the projective module $\mathsf{P}_{n,i}$. The goal is thus to find the fusion of irreducible modules with projective ones, which will then be used to compute the fusion of irreducible modules with standard modules. This is where the modules B_k^{2i}s appear. The fusion rules for these modules introduce yet another class of modules, the T_k^{2i+1}s. Fusion rules for these two classes of modules are presented in appendix 7.B. Note that the same arguments will be used over and over again so we will not detail the proofs as much as in the preceding sections.

7.6.1 Fusion of irreducible and projective modules

We start by giving the rules for the induction of $\mathsf{I}_{n,k}$ [11].

Proposition 7.6.1. *If $n \geq k\ell - 1 + i$, $0 < i < \ell$,*

$$\mathsf{I}_{n,k\ell-1-i}\!\uparrow \simeq \left\{\begin{matrix} \mathsf{I}_{n+1,k\ell-1-i} & \text{in } \mathsf{dTL}_n \\ 0 & \text{in } \mathsf{TL}_n \end{matrix}\right\} \oplus \begin{cases} \mathsf{P}_{n+1,(k-1)\ell-1} & \text{if } i = \ell-1 \\ \mathsf{I}_{n+1,k\ell-1-i-1} & \text{otherwise} \end{cases} \oplus \begin{cases} 0 & \text{if } i = 1 \\ \mathsf{I}_{n+1,k\ell-1-i+1} & \text{otherwise} \end{cases}. \tag{7.6.2}$$

The condition on n ensures that the module under study is not a standard module. Using proposition 7.3.4 with the parity of the irreducibles gives the following fusion rules.

Proposition 7.6.2. *If $n \geq k\ell - 1 + i$, $0 < i < \ell$,then in the dilute Temperley-Lieb family*

$$\mathsf{I}_{n,k\ell-1-i} \times_f \mathsf{P}_{1,0} \simeq \mathsf{I}_{n+1,k\ell-1-i}, \tag{7.6.3}$$

while in both families

$$\mathsf{I}_{n,k\ell-1-i} \times_f \mathsf{P}_{1,1} \simeq \begin{cases} \mathsf{P}_{n+1,(k-1)\ell-1} & \text{if } i = \ell-1 \\ \mathsf{I}_{n+1,k\ell-1-i-1} & \text{otherwise} \end{cases} \oplus \begin{cases} 0 & \text{if } i = 1 \\ \mathsf{I}_{n+1,k\ell-1-i+1} & \text{otherwise} \end{cases}. \tag{7.6.4}$$

In the standard family, when $\ell \neq 2$,

$$\mathsf{I}_{n,i} \times_f \mathsf{P}_{2,0} \simeq \mathsf{I}_{n+2,i},$$

which is proven in proposition 7.A.3. The proofs in this section will be independent of n as long as it is big enough for the irreducible modules to be distinct from the standard modules; we will therefore simply omit the n. Note now that

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_1 \simeq \mathsf{I}_{k\ell-3}.$$

Fusing the left side of this equation with P_1 and using proposition 7.4.2 gives

$$\mathsf{I}_{k\ell-2} \times_f (\mathsf{P}_1 \times_f \mathsf{P}_1) \simeq \mathsf{I}_{k\ell-2} \times_f (\mathsf{P}_0 \oplus \mathsf{P}_2), \tag{7.6.5}$$

while fusing its right side with P_1 and using proposition 7.6.2 gives

$$\mathsf{I}_{k\ell-3} \times_f \mathsf{P}_1 \simeq \mathsf{I}_{k\ell-2} \oplus \mathsf{I}_{k\ell-4}. \tag{7.6.6}$$

Comparing the two results then yields the fusion rule

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_2 \simeq \mathsf{I}_{k\ell-4}.$$

The following proposition is then obtained by simply repeating these arguments.

Proposition 7.6.3. *For all* $0 \leq i < \ell - 1$,

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_i \simeq \mathsf{I}_{k\ell-2-i}. \tag{7.6.7}$$

Once the fusion rules for $\mathsf{I}_{k\ell-2}$ are known, this proposition will be used to quickly compute the fusion of the other irreducible modules, since for all $0 < i < \ell$ and any module M,

$$\mathsf{I}_{k\ell-1-i} \times_f M \simeq (\mathsf{I}_{k\ell-2} \times_f M) \times_f \mathsf{P}_{i-1}.$$

For $k > 1$, $i = \ell - 1$, the same arguments give

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_{\ell-1} \simeq \mathsf{P}_{(k-1)\ell-1}.$$

Fusing this repeatedly with P_1 then yields

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_{\ell-1} \times_f \mathsf{P}_1 \simeq \mathsf{I}_{k\ell-2} \times_f \mathsf{P}_\ell \simeq \mathsf{P}_{(k-1)\ell},$$

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_{\ell+1} \simeq \mathsf{P}_{(k-1)\ell+1},$$

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_{\ell+2} \simeq \mathsf{P}_{(k-1)\ell+2}.$$

Continuing in this manner eventually yields

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_{\ell+\ell-1} \simeq \mathsf{P}_{(k-2)\ell-1} \oplus \mathsf{P}_{k\ell-1}.$$

Note that if $k = 2$, $\mathsf{P}_{(k-2)\ell-1} \simeq \mathsf{P}_{-1} \simeq 0$. The following proposition gives the general formula.

Proposition 7.6.4. *For all* $k > 1, r \geq 1$, $0 \leq i < \ell - 1$, $0 \leq j < \ell$,

$$\mathsf{I}_{k\ell-2-i} \times_f \mathsf{P}_{r\ell-1+j} \simeq \bigoplus_{\substack{p=\max(k-r,r-k+2) \\ step=2}}^{k+r-2} \mathsf{P}_{p\ell-1+j} \times_f \mathsf{P}_i. \tag{7.6.8}$$

Proof. The proof proceeds by induction on r and j. The cases $r = 1$ (for all j) and $r = 2$, $j = 0$ were proved in the preceding discussion, so suppose that the result stands for some r and $j = 0$. Fusing the left side of equation (7.6.8) with P_1 and using proposition 7.4.2 then gives

$$\mathsf{I}_{k\ell-2-i} \times_f (\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_1) \simeq \mathsf{I}_{k\ell-2-i} \times_f \mathsf{P}_{r\ell},$$

while fusing its right side with P_1 and using the same proposition yields

$$\bigoplus_{\substack{p=\max(k-r,r-k+2) \\ step=2}}^{k+r-2} \big(\mathsf{P}_{p\ell-1} \times_f \mathsf{P}_1\big) \times_f \mathsf{P}_i \simeq \bigoplus_{\substack{p=\max(k-r,r-k+2) \\ step=2}}^{k+r-2} \mathsf{P}_{p\ell} \times_f \mathsf{P}_i.$$

The case $j=1$ is then obtained by simply comparing the two results. Now, assume that the result stands for this q and $j-1, j$, $1 \leq j < \ell-1$. Fusing the left side of equation (7.6.8) and using proposition 7.4.2 gives

$$\mathsf{I}_{k\ell-2-i} \times_f \left(\mathsf{P}_{r\ell-1+j} \times_f \mathsf{P}_1\right) \simeq \mathsf{I}_{k\ell-2-i} \times_f \left((1+\delta_{j,1})\mathsf{P}_{r\ell-1+(j-1)} \oplus \mathsf{P}_{r\ell-1+j+1}\right),$$

while fusing its right side and using the same proposition yields

$$\begin{aligned}
&\bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1+j} \times_f \mathsf{P}_1\right) \times_f \mathsf{P}_i \\
&\simeq \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left((1+\delta_{j,1})\mathsf{P}_{p\ell-1+j-1} \oplus \mathsf{P}_{p\ell-1+j+1}\right) \times_f \mathsf{P}_i \\
&\simeq (1+\delta_{j,1}) \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1+j-1} \times_f \mathsf{P}_i\right) \oplus \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1+j+1} \times_f \mathsf{P}_i\right).
\end{aligned}$$

Comparing these two results and using the induction hypothesis then yields the conclusion for $j+1$. Note that doing the same thing for the case $j=\ell-1$ gives, for the left side

$$\mathsf{I}_{k\ell-2-i} \times_f \left(\mathsf{P}_{(r+1)\ell-2} \times_f \mathsf{P}_1\right) \simeq \mathsf{I}_{k\ell-2-i} \times_f \left((1+\delta_{\ell,2})\mathsf{P}_{r\ell-1+(\ell-2)} \oplus \mathsf{P}_{(r+1)\ell-1} \oplus \mathsf{P}_{(r-1)\ell-1}\right),$$

and for the right side

$$\begin{aligned}
&\bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1+\ell-1} \times_f \mathsf{P}_1\right) \times_f \mathsf{P}_i \\
&\simeq (1+\delta_{\ell,2}) \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1+\ell-2} \times_f \mathsf{P}_i\right) \oplus \bigoplus_{\substack{p=\max\{k-r,r-k+2\}\\ step=2}}^{k+r-2} \left(\mathsf{P}_{(p+1)\ell-1} \times_f \mathsf{P}_i\right) \\
&\quad \oplus \bigoplus_{\substack{p=\max\{k-r,r-k+2\}\\ step=2}}^{k+r-2} \left(\mathsf{P}_{(p-1)\ell-1} \times_f \mathsf{P}_i\right) \\
&\simeq (1+\delta_{\ell,2}) \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1+\ell-2} \times_f \mathsf{P}_i\right) \oplus \bigoplus_{\substack{p=\max(k-(r+1),r+1-k+2)\\ step=2}}^{k+r+1-2} \left(\mathsf{P}_{p\ell-1} \times_f \mathsf{P}_i\right) \\
&\quad \oplus \bigoplus_{\substack{p=\max\{k-(r-1),r-1-k+2\}\\ step=2}}^{k+r-1-2} \left(\mathsf{P}_{p\ell-1} \times_f \mathsf{P}_i\right),
\end{aligned}$$

where the last equality is obtained by rearranging the terms between the sums and considering the different values of $r-k$. Comparing the two sides, it follows that if the conclusion holds for $r-1, j=0$, $r, j=\ell-1, \ell-2$, it will also hold for $r+1, j=0$. ■

Note that if $k = 1$, repeating the arguments leading to proposition 7.6.3 gives

$$\mathsf{I}_{\ell-2} \times_f \mathsf{P}_{\ell-3} \simeq \mathsf{I}_1,$$

$$\mathsf{I}_{\ell-2} \times_f \mathsf{P}_{\ell-2} \simeq \mathsf{I}_0,$$

$$\mathsf{I}_{\ell-2} \times_f \mathsf{P}_{\ell-1} \simeq 0.$$

This implies of course that $\mathsf{I}_{\ell-2} \times_f \mathsf{P}_i \simeq 0$ for all $i \geq \ell - 1$.

Proposition 7.6.5. *For all* $i \geq \ell - 1,\ j < \ell - 1$,

$$\mathsf{I}_j \times_f \mathsf{P}_i \simeq 0. \tag{7.6.9}$$

Note also that since fusion is right-exact, fusing $\mathsf{I}_{\ell-2}$ with any quotient of P_k will always yield 0. This include the standard non-projective modules as well as all irreducibles I_k with $k > \ell - 1$.

7.6.2 Fusion of irreducible and standard modules

Proposition 7.5.7 can be used to obtain the non-projective standard modules by repeatedly fusing $\mathsf{S}_{n,\ell}$ with itself and small projectives. The first step to obtain the fusion of irreducible modules with standard modules is thus to compute $\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_\ell$, for $k > 1$. There is a short exact sequence

$$0 \longrightarrow \mathsf{P}_{\ell-2} \longrightarrow \mathsf{P}_\ell \longrightarrow \mathsf{S}_\ell \longrightarrow 0.$$

Using the right-exactness of fusion together with known fusion rules, this yields the exact sequence

$$\mathsf{I}_{(k-1)\ell} \xrightarrow{f} \mathsf{P}_{(k-1)\ell} \longrightarrow \mathsf{I}_{k\ell-2} \times_f \mathsf{S}_\ell \longrightarrow 0.$$

Since $\mathsf{I}_{(k-1)\ell}$ is irreducible, f is either zero or injective. If it is injective, then $\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_\ell \simeq \mathsf{B}^2_{(k-1)\ell-2}$ by proposition 7.2.2 while if $f = 0$, $\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_\ell \simeq \mathsf{P}_{(k-1)\ell}$. However, note that by propositions 7.6.4 and 7.5.3

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_\ell \times_f \mathsf{P}_{\ell-1} \simeq \mathsf{P}_{(k-1)\ell-1} \times_f \mathsf{S}_\ell,$$

while by proposition 7.4.5

$$\mathsf{P}_{(k-1)\ell} \times_f \mathsf{P}_{\ell-1} \simeq \mathsf{P}_{(k-1)\ell-1} \times_f \mathsf{S}_\ell \oplus \mathsf{P}_{(k-1)\ell-1} \times_f \mathsf{P}_{\ell-2}.$$

It follows that f cannot be zero, and thus that $\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_l \simeq \mathsf{B}^2_{(k-1)\ell-2}$.

Note that the case $\ell = 2$ in the regular family cannot be obtained from this discussion, since in this case the exact sequence satisfied by P_2 is instead

$$0 \longrightarrow \mathsf{I}_2 \longrightarrow \mathsf{P}_2 \longrightarrow \mathsf{S}_2 \longrightarrow 0. \tag{7.6.10}$$

In this case, proposition 7.5.10 gives

$$\mathsf{I}_2 \times_f \mathsf{S}_{2k} \simeq \mathsf{S}_{2k}.$$

Instead, use the exact sequence

$$\mathsf{S}_{2(k+1)} \longrightarrow \mathsf{S}_{2k} \longrightarrow \mathsf{I}_{2k} \longrightarrow 0,$$

which becomes

$$\mathsf{S}_{2(k+1)} \longrightarrow \mathsf{S}_{2k} \longrightarrow \mathsf{I}_{2k} \times_f \mathsf{I}_2 \longrightarrow 0, \tag{7.6.11}$$

by fusing it with I_2. Since $\mathsf{I}_{2k} \times_f \mathsf{I}_2 \times_f \mathsf{P}_1 \simeq \mathsf{I}_{2k} \times_f \mathsf{P}_1 \simeq \mathsf{P}_{2k-1}$, it follows that

$$\mathsf{I}_{2k} \times_f \mathsf{I}_2 \simeq \mathsf{I}_{2k}.$$

Using this fact with the exact sequence (7.6.10) gives

$$\mathsf{I}_{2k} \longrightarrow \mathsf{P}_{2k} \longrightarrow \mathsf{I}_{2k} \times_f \mathsf{S}_2 \longrightarrow 0. \tag{7.6.12}$$

Now, since $\mathsf{I}_{2k} \times_f \mathsf{S}_2 \times_f \mathsf{P}_1 \simeq \mathsf{P}_{2k-1} \times_f \mathsf{S}_2 \simeq \mathsf{P}_{2(k-1)-1} \oplus \mathsf{P}_{2(k+1)-1}$, while $\mathsf{P}_{2k} \times_f \mathsf{P}_1 \simeq \mathsf{P}_{2k-1} \oplus \mathsf{P}_{2(k+1)-1} \oplus 2\mathsf{P}_{2(k-1)-1}$, it follows that

$$\mathsf{I}_{2k} \times_f \mathsf{S}_2 \simeq \mathsf{B}^2_{2(k-1)}.$$

Proposition 7.6.6. *For all $k > 1$, and $\ell \geq 2$*

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_\ell \simeq \mathsf{B}^2_{(k-1)\ell-2}. \tag{7.6.13}$$

To proceed and compute the fusion of the irreducibles with the other standard modules, we therefore need the fusion of $\mathsf{B}^2_{(k-1)\ell-2}$ with S_ℓ, obtained in section 7.B.2.

Proposition 7.6.7. *For $k > 1, r > 0$, and in the regular family if $\ell \neq 2$,*

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{S}_{r\ell} \simeq \begin{cases} \mathsf{B}^{2r}_{(k-r)\ell-2} & \text{if } k > q \\ \mathsf{T}^{2k-3}_{(2+r-k)\ell} & \text{if } k \leq q \end{cases}. \tag{7.6.14}$$

In the regular family, if $\ell = 2$,

$$\mathsf{I}_{k\ell} \times_f \mathsf{S}_{r\ell} \simeq \begin{cases} \mathsf{B}^{2r}_{(k-r)\ell} & \text{if } k > r \\ \mathsf{T}^{2k+1}_{(1+r-k)\ell} & \text{if } k \leq r \end{cases}. \tag{7.6.15}$$

Proof. We proceed by induction on r. Proposition 7.6.6 already gives the case $r = 1$, so suppose that the result holds for some $1 \leq r < k-1$. Fuse the left side of equation (7.6.14) with S_ℓ and use propositions 7.5.7, and 7.6.4 to obtain

$$\mathsf{I}_{k\ell-2} \times_f (\mathsf{S}_{r\ell} \times_f \mathsf{S}_\ell) \simeq \mathsf{I}_{k\ell-2} \times_f \left(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{S}_{(r+1)\ell}\right)$$

$$\simeq \bigoplus_{p=0}^{r-1} \mathsf{P}_{(k+2p-r)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{I}_{k\ell-2} \times_f \mathsf{S}_{(r+1)\ell}. \tag{7.6.16}$$

Then, fuse the right side of equation (7.6.14) with S_ℓ and use propositions 7.B.5, and 7.B.11 to obtain

$$\mathsf{B}^{2q}_{(k-r)\ell-2} \times_f \mathsf{S}_\ell \simeq \bigoplus_{p=0}^{r-1} \mathsf{P}_{(k+2p-r)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \begin{Bmatrix} \mathsf{B}^{2(r+1)}_{(k-1-r)\ell-2}, & \text{if } r < k-1 \\ \mathsf{T}^{2(k-2)+1}_{2\ell}, & \text{if } r = k-1 \end{Bmatrix}. \tag{7.6.17}$$

Comparing the two results gives the conclusion for $r+1$. In particular, this gives the conclusion for all $r \leq k$.

Suppose now that the result holds for some $r \geq k$. Fuse the left side of equation (7.6.14) with S_ℓ and use propositions 7.5.7, and 7.6.4 to obtain

$$\begin{aligned} \mathsf{I}_{k\ell-2} \times_f (\mathsf{S}_{r\ell} \times_f \mathsf{S}_\ell) &\simeq \mathsf{I}_{k\ell-2} \times_f \left(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{S}_{(r+1)\ell}\right) \\ &\simeq \bigoplus_{p=0}^{k-2} \left(\mathsf{P}_{(2+r-k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\right) \oplus \mathsf{I}_{k\ell-2} \times_f \mathsf{S}_{(r+1)\ell}. \end{aligned} \tag{7.6.18}$$

Fusing the right side of equation (7.6.14) with S_ℓ and using propositions 7.B.11 instead gives

$$\mathsf{T}^{2k-3}_{(2+r-k)\ell} \times_f \mathsf{S}_\ell \simeq \bigoplus_{p=0}^{k-2} \left(\mathsf{P}_{(2+r-k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\right) \oplus \mathsf{T}^{2k-3}_{(2-k+r+1)\ell}. \tag{7.6.19}$$

Comparing the two results then give the conclusion for $r+1$.

In the regular family, the case where $\ell = 2$ is slightly different because then $\mathsf{B}^{2i}_0 \simeq \mathsf{T}^{2i-1}_2$. Nevertheless, the arguments are nearly identical. ■

Proposition 7.6.8. *For $k > 1$, $r \geq 1$, $0 < i, j < \ell$,*

$$\mathsf{I}_{k\ell-1-i} \times_f \mathsf{S}_{r\ell-1+j} \simeq \begin{cases} \mathsf{B}^{2r}_{(k-r)\ell-1-j} \times_f \mathsf{P}_{i-1} & \text{if } k > r \\ \mathsf{T}^{2k-3}_{(2+r-k)\ell+(j-1)} \times_f \mathsf{P}_{i-1} & \text{if } k \leq r \end{cases}. \tag{7.6.20}$$

Proof. The proof mimics those of previous sections so we will only give a rough outline. Proceed by induction on i, j, using proposition 7.6.7 for the case $i = j = 1$. To induce on i, fuse both sides of equation (7.6.20) with P_1, use propositions 7.6.3, 7.B.2, and 7.B.8 and compare the two results. Then, induce on j by doing the same thing but with propositions 7.5.3, 7.B.2, and 7.B.8, instead. ■

7.6.3 Fusion of two irreducible modules

Now that the fusion of standard modules with irreducible ones are known, the fusion of two irreducible modules can be directly computed.

Proposition 7.6.9. *For $k \geq r > 1$, and in the regular family, $\ell \neq 2$,*

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{I}_{r\ell-2} \simeq \begin{cases} \displaystyle\bigoplus_{\substack{p=k-r+2 \\ step=2}}^{k+r-2} \mathsf{I}_{p\ell-2} & \text{if } r < k \\ \mathsf{B}_0^1 \oplus \displaystyle\bigoplus_{\substack{p=4 \\ step=2}}^{2k-2} \mathsf{I}_{p\ell-2} & \text{if } r = k \end{cases}. \tag{7.6.21}$$

Proof. Start with the exact sequence

$$\mathsf{S}_{r\ell} \longrightarrow \mathsf{S}_{r\ell-2} \longrightarrow \mathsf{I}_{r\ell-2} \longrightarrow 0, \tag{7.6.22}$$

which becomes

$$\begin{Bmatrix} \mathsf{B}^{2r}_{(k-r)\ell-2} & \text{if } r < k \\ \mathsf{T}^{2k-3}_{2\ell} & \text{if } r = k \end{Bmatrix} \xrightarrow{g} \mathsf{B}^{2(r-1)}_{(k-r)\ell} \longrightarrow \mathsf{I}_{r\ell-2} \times_f \mathsf{I}_{k\ell-2} \longrightarrow 0 \tag{7.6.23}$$

by using the right-exactness of fusion together with proposition 7.6.8. Then, build the following exact commuting diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & \ker f & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \begin{Bmatrix} \mathsf{B}^{2r}_{(k-r)\ell-2} & \text{if } r < k \\ \mathsf{T}^{2k-3}_{2\ell} & \text{if } r = k \end{Bmatrix} & \xrightarrow{g} & \mathsf{B}^{2(r-1)}_{(k-r)\ell} & \xrightarrow{\bar{g}} & \mathsf{I}_{r\ell-2} \times_f \mathsf{I}_{k\ell-2} & \longrightarrow & 0 \\
 & & \gamma\downarrow & & \downarrow \mathrm{id} & & \downarrow f & & \\
0 & \longrightarrow & \displaystyle\bigoplus_{\substack{p=k-r \\ step=2}}^{k+r-2} \mathsf{I}_{p\ell} & \xrightarrow{\alpha} & \mathsf{B}^{2(r-1)}_{(k-r)\ell} & \xrightarrow{\bar{\alpha}} & \displaystyle\bigoplus_{\substack{p=k-r+2 \\ step=2}}^{k+r-2} \mathsf{I}_{p\ell-2} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} . \tag{7.6.24}$$

Here, f exists by universality of the Cokernel of g because

$$\mathrm{Hom}\big(\mathsf{B}^{2r}_{(k-r)\ell-2}, \bigoplus_{\substack{p=k-r+2 \\ step=2}}^{k+r-2} \mathsf{I}_{p\ell-2}\big) \simeq \mathrm{Hom}\big(\mathsf{T}^{2k-3}_{2\ell}, \bigoplus_{\substack{p=2 \\ step=2}}^{2k-2} \mathsf{I}_{p\ell-2}\big) \simeq 0, \tag{7.6.25}$$

and thus $\bar{\alpha}g = 0$, which also give the existence of γ by universality of $\ker \bar{\alpha}$. The snake lemma then gives $\mathrm{Coker}\, f \simeq 0$ and $\ker f \simeq \mathrm{Coker}\,\gamma$. Our goal is now to prove that

$$\mathrm{Hom}\big(\bigoplus_{\substack{p=k-r \\ step=2}}^{k+r-2} \mathsf{I}_{p\ell}, \mathsf{I}_{r\ell-2} \times_f \mathsf{I}_{k\ell-2}\big) \simeq 0, \tag{7.6.26}$$

because that would imply that $\ker f = 0$, and thus that f is an isomorphism. But, if there is a non-zero morphism from some $\mathsf{I}_{p\ell}$ to $\mathsf{I}_{r\ell-2} \times_f \mathsf{I}_{k\ell-2}$, it has to be injective since $\mathsf{I}_{p\ell}$ is irreducible, and there must thus be a morphism from $\mathsf{I}_{r\ell-2}\times_f\mathsf{I}_{k\ell-2}$ to $\mathsf{P}_{p\ell}$, the injective hull of $\mathsf{I}_{p\ell}$ (when $p\neq 0$). We are therefore trying to compute

$$\mathrm{Hom}\big(\mathsf{I}_{r\ell-2}\times_f\mathsf{I}_{k\ell-2},\mathsf{P}_{p\ell}\big),$$

for $k-r\leq p\leq k+r-2$.

Now, recall that $\mathsf{I}_{r\ell-2}\times_f\mathsf{P}_{\ell-1}\simeq\mathsf{P}_{(r-1)\ell-1}$ which implies that

$$\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2}\times_f\mathsf{P}_{\ell-1}\simeq\mathsf{I}_{k\ell-2}\times_f\mathsf{P}_{(r-1)\ell-1}\simeq\bigoplus_{\substack{s=k-r+2\\ step=2}}^{k+r-2}\mathsf{I}_{s\ell-2}\times_f\mathsf{P}_{\ell-1}. \tag{7.6.27}$$

Using this observation with the definition of the fusion quotient (see section 7.C) and proposition 7.C.12 give

$$\begin{aligned}\mathrm{Hom}\big(\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2}\times_f\mathsf{P}_{\ell-1},\mathsf{P}_{p\ell-(\ell-1)}\big)&\simeq\mathrm{Hom}\Big(\bigoplus_{\substack{s=k-r+2\\ step=2}}^{k+r-2}\mathsf{I}_{s\ell-2}\times_f\mathsf{P}_{\ell-1},\mathsf{P}_{p\ell-(\ell-1)}\Big)\\ &\simeq\mathrm{Hom}\Big(\bigoplus_{\substack{s=k-r+2\\ step=2}}^{k+r-2}\mathsf{I}_{s\ell-2},\mathsf{P}_{p\ell-(\ell-1)}\times_f\mathsf{P}_{\ell-1}\Big)\\ &\simeq 0,\end{aligned} \tag{7.6.28}$$

where the last line is obtained in the following way. Start by using proposition 7.4.5 to obtain

$$\mathsf{P}_{p\ell-(\ell-1)}\times_f\mathsf{P}_{\ell-1}\simeq\mathsf{P}_{p\ell}\oplus\mathsf{P}_{(p-2)\ell}\oplus\bigoplus_{\substack{\sigma=(\ell-1)\mod 2\\ step=2}}^{\ell-3}\mathsf{P}_{(p-1)\ell-1+\sigma} \tag{7.6.29}$$

if $\ell\neq 2$, and

$$\mathsf{P}_{p\times 2-(2-1)}\times_f\mathsf{P}_{2-1}\simeq\mathsf{P}_{p\times 2}, \tag{7.6.30}$$

when $\ell=2$. Then, notice that the projective modules $\mathsf{P}_{s\ell-2}$, the only projective module containing $\mathsf{I}_{s\ell-2}$ as a submodule never appears in these fusions for any $p\in[k-r,k-r+2,\dots,k+r-2]$.

However, using the definition of the fusion quotient (see section 7.C) and proposition 7.C.12 also give

$$\begin{aligned}\mathrm{Hom}\big(\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2}\times_f\mathsf{P}_{\ell-1},\mathsf{P}_{p\ell-(\ell-1)}\big)&\simeq\mathrm{Hom}\big(\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2},\mathsf{P}_{p\ell-(\ell-1)}\div_f\mathsf{P}_{\ell-1}\big)\\ &\simeq\mathrm{Hom}\big(\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2},\mathsf{P}_{p\ell-(\ell-1)}\times_f\mathsf{P}_{\ell-1}\big).\end{aligned} \tag{7.6.31}$$

It follows that

$$\mathrm{Hom}\big(\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2},\mathsf{P}_{p\ell-(\ell-1)}\times_f\mathsf{P}_{\ell-1}\big)\simeq 0,$$

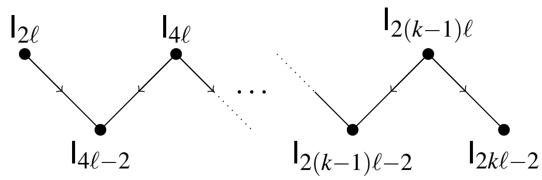
and in particular

$$\mathrm{Hom}\big(\mathsf{I}_{k\ell-2}\times_f\mathsf{I}_{r\ell-2},\mathsf{P}_{p\ell}\big)\simeq 0$$

for all $p\in[k-r,k-r+2,\dots,k+r-2]$. Equation (7.6.26) is thus proved, and the conclusion when $r\neq k$ is obtained.

When $r = k$, the proof above does not work because then the injective hull of $\mathsf{I}_{(k-r)\ell} \simeq \mathsf{I}_0$ is B_0^1 instead of P_0. But the Loewy diagram of $\mathsf{T}_{2\ell}^{2k-3}$, figure 7.6.1, shows that I_0 is not one of its quotient, and thus $\mathsf{I}_0 \subset \ker f$. The same argument as for the case $k \neq r$ can then be used to rule out the appearance of the other irreducible

Figure 7.6.1 – The Loewy diagram of $\mathsf{T}_{2\ell}^{2k-3}$.



modules, and it follows that $\mathsf{I}_0 \simeq \ker f$. However, proposition 7.2.1 shows that the only irreducible module which can be extended by I_0 is $\mathsf{I}_{2\ell-2}$, giving

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{I}_{k\ell-2} \simeq M \oplus \bigoplus_{\substack{p=4\\ step=2}}^{2k-2} \mathsf{I}_{p\ell-2}, \tag{7.6.32}$$

where M satisfy the short exact sequence

$$0 \longrightarrow \mathsf{I}_0 \longrightarrow M \longrightarrow \mathsf{I}_{2\ell-2} \longrightarrow 0.$$

Note that this sequence cannot split since $\mathrm{Hom}\big(\mathsf{B}_0^{2(k-1)}, \mathsf{I}_0\big) \simeq 0$. Comparing this sequence with the definition of the B modules then gives

$$M \simeq \mathsf{B}_0^1.$$

■

Using proposition 7.6.3, this can be used to compute the fusion of the other irreducibles. However to do so requires the fusion of B_0^1 with projective modules, which is obtained in section 7.B.5. Proposition 7.6.9 gives

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{I}_{r\ell-2} \simeq \begin{cases} \displaystyle\bigoplus_{\substack{p=k-r+2\\ step=2}}^{k+r-2} \mathsf{I}_{p\ell-2} & \text{if } r < k \\ \mathsf{B}_0^1 \oplus \displaystyle\bigoplus_{\substack{p=4\\ step=2}}^{2k-2} \mathsf{I}_{p\ell-2} & \text{if } r = k \end{cases},$$

and proposition 7.6.3 gives

$$\mathsf{I}_{k\ell-2} \times_f \mathsf{P}_i \simeq \mathsf{I}_{k\ell-2-i},$$

for all $0 \leq i < \ell - 1$. To obtain $\mathsf{I}_{k\ell-2-i} \times_f \mathsf{I}_{r\ell-2-j}$ we must therefore compute $\mathsf{I}_{p\ell-2} \times_f \mathsf{P}_i \times_f \mathsf{P}_j$ and $\mathsf{B}_0^1 \times_f$

$\mathsf{P}_i \times_f \mathsf{P}_j$. Using propositions 7.4.5, and 7.6.4,

$$\begin{aligned}\mathsf{I}_{p\ell-2} \times_f \mathsf{P}_i \times_f \mathsf{P}_j &\simeq \mathsf{I}_{p\ell-2} \times_f \Big(\bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{P}_\sigma \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)\mathrm{mod}2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{\ell-1+\sigma} \Big) \\ &\simeq \bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{I}_{p\ell-2-\sigma} \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)\mathrm{mod}2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{(p-1)\ell-1+\sigma}.\end{aligned}$$

Similarly, using proposition 7.4.5 with propositions 7.B.15 and 7.B.16 gives

$$\mathsf{B}_0^1 \times_f \mathsf{P}_i \times_f \mathsf{P}_j \simeq \bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{B}_\sigma^1 \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)\mathrm{mod}2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{\ell-1+\sigma}. \tag{7.6.33}$$

These give the final result.

Theorem 7.6.10. *For* $1 < r < k$, $0 < i, j < \ell - 1$,

$$\mathsf{I}_{k\ell-2-i} \times_f \mathsf{I}_{r\ell-2-j} \simeq \bigoplus_{\substack{p=k-r+2 \\ step=2}}^{k+r-2} \Big(\bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{I}_{p\ell-2-\sigma} \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)mod2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{(p-1)\ell-1+\sigma} \Big), \tag{7.6.34}$$

$$\begin{aligned}\mathsf{I}_{k\ell-2-i} \times_f \mathsf{I}_{k\ell-2-j} \simeq & \bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{B}_\sigma^1 \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)mod2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{\ell-1+\sigma} \\ & \oplus \bigoplus_{\substack{p=4 \\ step=2}}^{2k-2} \Big(\bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{I}_{p\ell-2-\sigma} \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)mod2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{(p-1)\ell-1+\sigma} \Big).\end{aligned} \tag{7.6.35}$$

We still need to compute the fusion rules for the irreducibles with $k = 1$. Recall that it was established in proposition 7.6.5 that

$$\mathsf{I}_{\ell-2} \times_f \mathsf{I}_r \simeq 0$$

for all $r \geq \ell$. Using the short-exact sequence

$$0 \longrightarrow \mathsf{I}_\ell \longrightarrow \mathsf{P}_{\ell-2} \longrightarrow \mathsf{I}_{\ell-2} \longrightarrow 0,$$

with the right exactness of fusion, it follows that

$$\mathsf{I}_{\ell-2} \times_f \mathsf{I}_{\ell-2} \simeq \mathsf{P}_{\ell-2} \times_f \mathsf{I}_{\ell-2} \simeq \mathsf{I}_0, \tag{7.6.36}$$

and thus that

$$
\begin{aligned}
\mathsf{I}_{\ell-2-i} \times_f \mathsf{I}_{\ell-2-j} &\simeq \mathsf{I}_{\ell-2} \times_f \mathsf{I}_{\ell-2} \times_f (\mathsf{P}_i \times_f \mathsf{P}_j) \\
&\simeq \mathsf{I}_0 \times_f \left(\bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{P}_\sigma \oplus \bigoplus_{\substack{\sigma=(i+j+\ell+1)\mathrm{mod}2 \\ step=2}}^{i+j-\ell+1} \mathsf{P}_{\ell-1+\sigma} \right) \\
&\simeq \bigoplus_{\substack{\sigma=|i-j| \\ step=2}}^{\min(i+j,2\ell-(i+j)-4)} \mathsf{I}_\sigma,
\end{aligned}
\tag{7.6.37}
$$

where the last line is obtained by proposition 7.6.3. The following theorem is then obtained by changing the indices.

Theorem 7.6.11. *For all* $0 \leq i, j < \ell - 1$*,*

$$
\mathsf{I}_i \times_f \mathsf{I}_j \simeq \bigoplus_{\substack{p=|i-j| \\ step=2}}^{\min\{i+j,2\ell-(i+j)-4\}} \mathsf{I}_p. \tag{7.6.38}
$$

It should be noted that for a minimal model $M(p', p)$ of the Virasoro algebra, the fusion rule between two primary fields is

$$
\phi_{1,1+s} \times_f \phi_{1,1+r} = \sum_{\substack{l=|r-s| \\ step=2}}^{\mathrm{Min}(r+s,2p-(s+r)-4))} \phi_{1,1+l}, \tag{7.6.39}
$$

which is identical to (7.6.38) under the correspondence $\ell \to p$, $\mathsf{I}_i \to \phi_{1,1+i}$. Taking $p' = p+1$ would give a correspondence with the RSOS models with maximum height ℓ [21].

7.7 Conclusion

The main results of the paper are now reviewed. A definition of a fusion product on the Temperley-Lieb family as been proposed by Read and Saleur [61, 62]; in section 7.3, we generalize their definition to more general families of associative algebras, including the dilute Temperley-Lieb algebra. A straightforward consequence of this definition is that the fusion of pairs of projective modules are also projective. In the Temperley-Lieb algebras, when q is not a root of unity, the projective modules $\mathsf{P}_{n,k}$ behaves under fusion like irreducible $su(2)$ representations under tensor product:

$$
\mathsf{P}_{n,k} \times_f \mathsf{P}_{m,r} \simeq \bigoplus_{\substack{p=|k-r| \\ step=2}}^{k+r} \mathsf{P}_{n+m,p}.
$$

When q is a root of unity, they behave like a polynomial ring, with a basis of Chebyshev polynomials of the second kind:

$$\mathsf{P}_{n,i} \to U_i(x), \qquad \mathsf{P}_{n,k_c} \to U_{k_c}(x), \qquad \mathsf{P}_{n,k_c+i} \to U_{k_c-i}(x) + U_{k_c+i}(x).$$

In section 7.5, we use this information to compute fusion products of standard modules $\mathsf{S}_{n,k}$ with projective modules and other standard modules. It is shown that these can once again be interpreted as a polynomial ring with a basis of Chebyshev polynomials, albeit with a different product. The correspondence is

$$\mathsf{S}_{n,k} \to U_k(x),$$

and when taking a product, the result must first be re-written in terms of the polynomials representing projective modules $\mathsf{P}_{n,p}$, starting with the smallest p; the remaining polynomials are then identified with the standard modules.

In section 7.6, it is shown how to use fusion rules obtained previously to construct more complex ones. In particular, we compute the fusion product of an irreducible modules, and a standard modules. This shows the appearance of two other classes of indecomposable modules, the B's and the T's. After computing their fusion rules in section 7.B.1 to 7.B.4, the fusion product of pairs of irreducible modules is computed in section 7.6.3. Here, we use the adjoint of the fusion product, the fusion quotient, which simplifies the proofs greatly. Finally, we find a general formula for the fusion product of pairs of irreducible modules lying on the left of the first critical line.

It is then recognized that the irreducible modules $\mathsf{I}_{n,i}$, with $i \leq \ell - 2$, behave under fusion like primary fields in the first line of the Kacs table of a Virasoro minimal model $M(p', p)$, with $p = \ell$.

A correspondence between the projective, and the standard modules of the Temperley-Lieb algebra with certain fields of a logarithmic comformal field theory was given in [61]. The translation of the fusion rules for these modules in terms of the fields in the LCFT is given in [24]. It is conjectured in [61] that the fusion rules for the dilute case should correspond exactly to those of the regular case, both in the algebra and in the LCFT, except when $\ell = 2$. This is in accordance with the results obtained in sections 7.4 and 7.5. However, these only apply for the projective, and the standard modules; it would be interesting to see how the fusion rules for the other, more complex modules, could be translated in terms of fields in a LCFT.

There are still many fusion rules between indecomposable modules which we have yet to compute. We chose to limit ourselves to the projective, standard and irreducible modules because they are very important in the representation theory of the Temperley-Lieb algebras, but it would be interesting to find out how the other, more exotic, modules behave under this fusion product, as they do appear in physical problems [51]. We believe that the arguments used here could be extended to obtain these fusions.

The appearance of the fusion quotient is a simple consequence of the definition of the fusion product. However, while it is conjectured that the fusion product corresponds to the fusion product on the Virasoro algebra in the limit, the meaning of this fusion quotient is unclear. Is there a corresponding functor on the Virasoro algebra?

Another important question is whether the techniques developed here could be used to compute fusion rules for other generalized Temperley-Lieb algebras, like the blob algebra [31] bTL_n or the periodic Temperley-Lieb algebra pTL_n [18, 27–29]. For the blob algebras, the main problem is that the definition of fusion rules used in section 7.3 relies on morphisms $\mathsf{bTL}_n \otimes \mathsf{bTL}_m \to \mathsf{bTL}_{n+m}$. The existence of such morphism is not obvious; if it does not exist, a new definition of the fusion $\times_f$ will be needed. For the periodic algebras such morphisms are known [47] and fusion can thus be defined. However, these algebras are infinite dimensional and their representation theory is still relatively unknown, so while the techniques used in this paper can be used in this case, carrying out the calculations is much more difficult [12].

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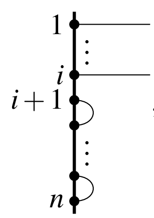
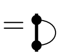
Appendix

7.A Fusion of $\mathsf{P}_{2,0}$ in TL_n

We investigate here the fusion of $\mathsf{P}_{2,0}$ with various modules in the regular family TL_n when $\ell \neq 2$.

Proposition 7.A.1. *If* $\beta \neq 0$*, then* $\mathsf{S}_{n,i} \times_f \mathsf{P}_{2,0} \simeq \mathsf{S}_{n+2,i}$.

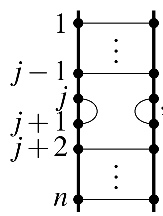
Proof. Pick

$z =$  , $x =$  ,

where z is a generator of $\mathsf{S}_{n,i}$ because $< z, z >= \beta^{\frac{n-i}{2}} \neq 0$, and x is a generator of $\mathsf{P}_{2,0} = \mathbb{C}x$. Note that

$$\begin{aligned} \mathsf{S}_{n,i} \times_f \mathsf{P}_{2,0} &\simeq \mathsf{TL}_{n+2} \otimes (\mathsf{S}_{n,i} \otimes \mathsf{P}_{2,0}) \\ &\simeq \mathsf{TL}_{n+2} \otimes (\mathsf{TL}_n \otimes \mathsf{TL}_2)(z \otimes x) \\ &\simeq \mathsf{TL}_{n+2} \otimes (z \otimes x). \end{aligned} \tag{7.A.1}$$

With the usual generator,

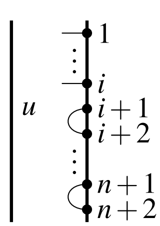
$e_j =$  , (7.A.2)

we note that

$$e_{i+1}e_{i+3}\dots e_{n-1} \otimes (z \otimes x) = \beta^{\frac{n-i}{2}+1}(z \otimes x).$$

It thus follows that

$$\begin{aligned} \mathsf{S}_{n,i} \times_f \mathsf{P}_{2,0} &\simeq \mathsf{TL}_{n+2} \otimes (e_{i+1}e_{i+3}\dots e_{n-1} \otimes e_1)(z \otimes x) \\ &\simeq \mathsf{TL}_{n+2}(e_{i+1}e_{i+3}\dots e_{n-1}e_{n+1}) \otimes (z \otimes x) \\ &\simeq \mathrm{Span}_{\mathbb{C}} \left\{ \; [\text{diagram}] \otimes (z \otimes x) \,|\, \text{where } u \in \mathsf{S}_{n+2,i} \right\} \\ &\simeq \mathsf{S}_{n+2,i}, \end{aligned} \tag{7.A.3}$$



where the last two lines are obtained by straightforward calculations. ∎

Proposition 7.A.2. *If* $\beta \neq 0$, *then* $\mathsf{P}_{n,i} \times_f \mathsf{P}_{2,0} \simeq \mathsf{P}_{n+2,i}$.

Proof. If $i_- < 0$(see definition of $i_\pm$ in section 7.2.1), then $\mathsf{P}_{n,i} \simeq \mathsf{S}_{n,i}$ and the result is given by proposition 7.A.1. If $i_- \geq 0$, there is a short exact sequence

$$0 \longrightarrow \mathsf{S}_{n,i_-} \longrightarrow \mathsf{P}_{n,i} \longrightarrow \mathsf{S}_{n,i} \longrightarrow 0, \tag{7.A.4}$$

which becomes

$$\mathsf{S}_{n+2,i_-} \longrightarrow \mathsf{P}_{n,i} \times_f \mathsf{P}_{2,0} \longrightarrow \mathsf{S}_{n+2,i} \longrightarrow 0, \tag{7.A.5}$$

by fusing it with $\mathsf{P}_{2,0}$ and using proposition 7.A.1. However, since the fusion of two projective modules is projective, it follows that $\mathsf{P}_{n,i} \times_f \mathsf{P}_{2,0}$ is a projective module having $\mathsf{S}_{n+2,i}$ as a quotient, and whose dimension is at most $\dim \mathsf{S}_{n+2,i_-} + \dim \mathsf{S}_{n+2,i} = \dim \mathsf{P}_{n+2,i}$. Since $\mathsf{P}_{n+2,i}$ is the projective cover of $\mathsf{S}_{n+2,i}$, the conclusion follows. ∎

Similar arguments can be used to compute the action of $\mathsf{P}_{2,0}$ on other modules. We simply state the result.

Proposition 7.A.3. *If* $n \geq i_+$,

$$\mathsf{I}_{n,i} \times_f \mathsf{P}_{2,0} \simeq \mathsf{I}_{n+2,i},$$

and if $n \geq k$,

$$\mathsf{B}^j_{n,k} \times_f \mathsf{P}_{2,0} \simeq \mathsf{B}^{2j}_{n+2,k},$$

$$\mathsf{T}^j_{n,k} \times_f \mathsf{P}_{2,0} \simeq \mathsf{T}^j_{n+2,k}.$$

7.B Fusion of the B and T modules

This appendix gathers the computation of some of the fusion products involving modules of type B and T. These results are used in section 7.6.

7.B.1 Fusion of $\mathsf{B}^{2i}_{n,k}$ and projective modules

The rules for the induction of these modules are [10]

$$\mathsf{B}^{2i}_{n,k}\uparrow \simeq \left.\begin{cases} \mathsf{B}^{2i}_{n+1,k}, & \text{in } \mathsf{dTL}_n \\ 0, & \text{in } \mathsf{TL}_n \end{cases}\right\} \oplus \begin{cases} \bigoplus_{p=0}^{i} \mathsf{P}_{n+1,k+2p\ell-1} & \text{if } k=0 \mod \ell \\ \mathsf{B}^{2i}_{n+1,k-1} & \text{otherwise} \end{cases} \oplus \begin{cases} \bigoplus_{p=0}^{i-1} \mathsf{P}_{n+1,k+2p\ell+1} & \text{if } k+2=0 \mod \ell \\ \mathsf{B}^{2i}_{n+1,k+1} & \text{otherwise} \end{cases}. \tag{7.B.1}$$

The usual argument on the parity of the modules gives the following fusion rules.

Proposition 7.B.1. *In the dilute family*

$$\mathsf{B}^{2i}_{n,k} \times_f \mathsf{P}_{1,0} \simeq \mathsf{B}^{2i}_{n+1,k}, \tag{7.B.2}$$

while in both families

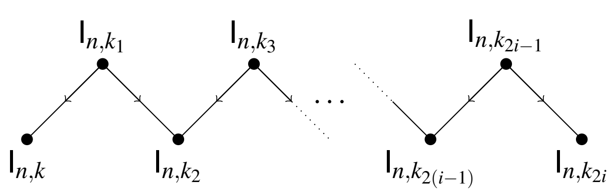
$$\mathsf{B}^{2i}_{n,k} \times_f \mathsf{P}_{1,1} \simeq \begin{cases} \bigoplus_{p=0}^{i} \mathsf{P}_{n+1,k+2p\ell-1} & \textit{if } k=0 \mod \ell \\ \mathsf{B}^{2i}_{n+1,k-1} & \textit{otherwise} \end{cases} \oplus \begin{cases} \bigoplus_{p=0}^{i-1} \mathsf{P}_{n+1,k+2p\ell+1} & \textit{if } k+2=0 \mod \ell \\ \mathsf{B}^{2i}_{n+1,k+1} & \textit{otherwise} \end{cases}. \tag{7.B.3}$$

The first formula shows that the parameter n can be adjusted by simply fusing the module with $\mathsf{P}_{1,0}$. In the regular family, proposition 7.A.3 gives

$$\mathsf{B}^{2i}_{n,k} \times_f \mathsf{P}_{2,0} \simeq \mathsf{B}^{2i}_{n+2,k}.$$

Like for the standard modules, we therefore omit this parameter and simply assume n to be big enough for the modules to exist.

Figure 7.B.1 – The Loewy diagram of $\mathsf{B}^{2i}_{n,k}$, where $1 \le i$, and $0 \le k < k_{2i} \le n$.



We start by studying the fusion of B_{23}^{2i} in the $\ell = 5$ case. The preceding proposition gives

$$\mathsf{B}_{23}^{2i} \times_f \mathsf{P}_0 \simeq \mathsf{B}_{23}^{2i}, \tag{7.B.4}$$

$$\mathsf{B}_{23}^{2i} \times_f \mathsf{P}_1 \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(5+2p)5-1} \oplus \mathsf{B}_{22}^{2i}. \tag{7.B.5}$$

Fusing the last equation with P_1 yields

$$\mathsf{B}_{23}^{2i} \times_f (\mathsf{P}_0 \oplus \mathsf{P}_2) \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(5+2p)5-1} \times_f \mathsf{P}_1 \oplus \mathsf{B}_{21}^{2i} \oplus \mathsf{B}_{23}^{2i}. \tag{7.B.6}$$

Comparing this result with proposition 7.B.1, it follows that

$$\mathsf{B}_{23}^{2i} \times_f \mathsf{P}_2 \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(5+2p)5-1} \times_f \mathsf{P}_1 \oplus \mathsf{B}_{21}^{2i}. \tag{7.B.7}$$

Repeating the argument gives

$$\mathsf{B}_{23}^{2i} \times_f \mathsf{P}_3 \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(5+2p)5-1} \times_f \mathsf{P}_2 \oplus \mathsf{B}_{20}^{2i}. \tag{7.B.8}$$

Proposition 7.B.2. *For all* $0 \leq j < \ell - 1$, $\ell \geq 2$, $k > 1$

$$\mathsf{B}_{k\ell-2}^{2i} \times_f \mathsf{P}_j \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{j-1} \oplus \mathsf{B}_{k\ell-2-j}^{2i}, \tag{7.B.9}$$

$$\mathsf{B}_{k\ell-2}^{2i} \times_f \mathsf{P}_{\ell-1} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{P}_{(k-1)\ell-1}. \tag{7.B.10}$$

Proof. If $j = 1, 0$, proposition 7.B.1 already gives the conclusion. Suppose that the result stands for $j-1, j$ with $j < \ell - 2$. Then

$$\mathsf{B}_{k\ell-2}^{2i} \times_f \mathsf{P}_j \times_f \mathsf{P}_1 \simeq \mathsf{B}_{k\ell-2}^{2i} \times_f (\mathsf{P}_{j-1} \oplus \mathsf{P}_{j+1}) \tag{7.B.11}$$

$$\simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f (\mathsf{P}_j \oplus \mathsf{P}_{j-2})\big) \oplus \mathsf{B}_{k\ell-2-j-1}^{2i} \oplus \mathsf{B}_{k\ell-2-j+1}^{2i}. \tag{7.B.12}$$

Comparing the two lines and using the induction hypothesis yields the conclusion for $j+1$. In particular, this yields

$$\mathsf{B}_{k\ell-2}^{2i} \times_f \mathsf{P}_{\ell-2} \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-3} \oplus \mathsf{B}_{(k-1)\ell}^{2i}. \tag{7.B.13}$$

Fusing this result with P_1 gives

$$\mathsf{B}^{2i}_{k\ell-2} \times_f (\mathsf{P}_{\ell-1} \oplus \mathsf{P}_{\ell-3}) \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f (\mathsf{P}_{\ell-4} \oplus \mathsf{P}_{\ell-2})\big) \oplus \mathsf{B}^{2i}_{(k-1)\ell+1} \tag{7.B.14}$$

$$\oplus \bigoplus_{p=0}^{i} \mathsf{P}_{(k-1+2p)\ell-1}. \tag{7.B.15}$$

Comparing the two sides, using the result of the first part and rearranging the terms gives

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell-1} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-2} \oplus \mathsf{P}_{(k+2p+1)\ell-1}\big) \oplus \mathsf{P}_{(k-1)\ell-1} \tag{7.B.16}$$

$$\simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{P}_{(k-1)\ell-1} \tag{7.B.17}$$

where the second line follows from proposition 7.5.3. ∎

The last formula can be used to quickly obtain the fusion rules with the bigger projectives. Thus,

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell-1} \times_f \mathsf{P}_1 \simeq \mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell}\big) \oplus \mathsf{P}_{(k-1)\ell},$$

$$\mathsf{B}^{2i}_{kl-2} \times_f \mathsf{P}_{\ell+1} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell+1}\big) \oplus \mathsf{P}_{(k-1)\ell+1},$$

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell+2} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell+2}\big) \oplus \mathsf{P}_{(k-1)\ell+2}.$$

Continuing in this manner eventually gives

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{2\ell-1} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{2\ell-1}\big) \oplus \mathsf{P}_{(k-2)\ell-1} \oplus \mathsf{P}_{k\ell-1}.$$

Note that if $k=2$, $\mathsf{P}_{(k-2)\ell-1} \simeq 0$. Repeating these arguments, the proof for the general formula is straightforward.

Proposition 7.B.3. *For all $k>1, r>0$, $i>0$, $0 \le j < \ell$,*

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{r\ell-1+j} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{r\ell-1+j}\big) \oplus \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \mathsf{P}_{p\ell-1+j}. \tag{7.B.18}$$

The same method can be used to obtain the formulas for the fusion of $\mathsf{B}^{2i}_{k\ell-t}$. Fusing the formula in the preceding proposition with P_1 yields

$$\mathsf{B}^{2i}_{k\ell-2}\times_f \mathsf{P}_{r\ell-1+j}\times_f \mathsf{P}_1 \simeq \bigoplus_{p=0}^{i-1}\big(\mathsf{P}_{(k+2p)\ell-1}\times_f \mathsf{P}_{r\ell-1+j}\big)\oplus \mathsf{B}^{2i}_{k\ell-3}\times_f \mathsf{P}_{r\ell-1-j} \tag{7.B.19}$$

$$\simeq \bigoplus_{p=0}^{i-1}\Big(\big(\mathsf{S}_{(k+2p)\ell+1}\oplus \mathsf{P}_{(k+2p)\ell-1}\big)\times_f \mathsf{P}_{r\ell-1+j}\Big) \tag{7.B.20}$$

$$\oplus \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2} \mathsf{P}_{p\ell-1+j}\times_f \mathsf{P}_1. \tag{7.B.21}$$

Comparing the two lines yields

$$\mathsf{B}^{2i}_{k\ell-3}\times_f \mathsf{P}_{r\ell-1+j}\simeq \bigoplus_{p=0}^{i-1}\big(\mathsf{S}_{(k+2p)\ell+1}\times_f \mathsf{P}_{r\ell-1+j}\big)\oplus \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2}\big(\mathsf{P}_{p\ell-1+j}\times_f \mathsf{P}_1\big). \tag{7.B.22}$$

Once again, this operation can be repeated and gives the following general formula.

Proposition 7.B.4. *For all $k>1, r, i>0$, $0<t<\ell$, $0\le j<\ell$,*

$$\mathsf{B}^{2i}_{k\ell-1-t}\times_f \mathsf{P}_{r\ell-1+j}\simeq \bigoplus_{p=0}^{i-1}\big(\mathsf{S}_{(k+2p)\ell-1+t}\times_f \mathsf{P}_{r\ell-1+j}\big)\oplus \bigoplus_{\substack{p=\max(k-r,r-k+2)\\ step=2}}^{k+r-2}\big(\mathsf{P}_{p\ell-1+j}\times_f \mathsf{P}_{t-1}\big). \tag{7.B.23}$$

Note that, in this section, the case $\mathsf{B}^{2i}_{\ell-2}$ has been avoided. In this case, there is a short-exact sequence

$$0\longrightarrow \mathsf{I}_{\ell-2}\longrightarrow \mathsf{B}^{2i}_{\ell-2}\longrightarrow \mathsf{T}^{2i-1}_{\ell}\longrightarrow 0,$$

and since $\mathsf{I}_{\ell-2}\times_f \mathsf{P}_j\simeq 0$, for all $j\ge \ell-1$,

$$\mathsf{B}^{2i}_{\ell-2}\times_f \mathsf{P}_j\simeq \mathsf{T}^{2i-1}_{\ell}\times_f \mathsf{P}_j.$$

This case will be treated in section 7.B.3

7.B.2 Fusion of B^{2i}_k and standard modules

We now want to compute the fusion of B^{2i}_k with the standard modules S_q that are not projective. The first step is to find a formula for $\mathsf{B}^{2i}_{k\ell-2}\times_f \mathsf{S}_\ell$.

Using the projective cover of S_ℓ (see section 7.2.1) and the right-exactness of fusion, one can obtain the exact sequence

$$\mathsf{B}^{2i}_{k\ell-2}\times_f \mathsf{P}_{\ell-2}\xrightarrow{f}\mathsf{B}^{2i}_{k\ell-2}\times_f \mathsf{P}_{\ell}\longrightarrow \mathsf{B}^{2i}_{k\ell-2}\times_f \mathsf{S}_{\ell}\longrightarrow 0. \tag{7.B.24}$$

Using propositions 7.B.2, 7.B.3, and 7.5.4 gives

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell-2} \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-3} \oplus \mathsf{B}^{2i}_{(k-1)\ell},$$

and

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell} \simeq \bigoplus_{p=0}^{i-1} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f (\mathsf{P}_{\ell-3} \oplus \mathsf{P}_{\ell-1})\big) \oplus \bigoplus_{p=0}^{i} \mathsf{P}_{(k-1+2p)\ell}.$$

Therefore

$$\big(\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell}\big)/\big(\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell-2}\big) \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{B}^{2(i+1)}_{(k-1)\ell-2}$$

where we used proposition 7.2.2. If it can be proved that f is injective, this will give a formula for $\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_{\ell}$. To do this, we will prove that the dimension of $\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_{\ell}$ is that of $\big(\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell}\big)/\big(\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell-2}\big)$, and this will be done by induction on i.

Note first that by proposition 7.6.6

$$\mathsf{B}^{2\times 0}_{k\ell-2} \times_f \mathsf{S}_{\ell} \simeq \mathsf{I}_{k\ell-2} \times_f \mathsf{S}_{\ell} \simeq \mathsf{B}^{2}_{(k-1)\ell-2} \simeq \big(\mathsf{B}^{0}_{k\ell-2} \times_f \mathsf{P}_{\ell}\big)/\big(\mathsf{B}^{0}_{k\ell-2} \times_f \mathsf{P}_{\ell-2}\big). \tag{7.B.25}$$

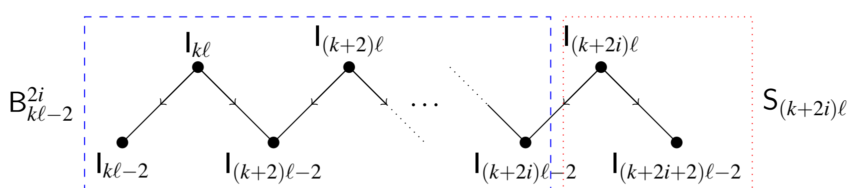
This gives the case $i = 0$ for all $k > 1$. Assume now that

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_{\ell} \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{B}^{2(i+1)}_{(k-1)\ell-2}$$

for a certain i and all $k > 1$. To proceed with the induction, we will use the short exact sequence

$$0 \longrightarrow \mathsf{B}^{2i}_{k\ell-2} \longrightarrow \mathsf{B}^{2(i+1)}_{k\ell-2} \longrightarrow \mathsf{S}_{(k+2i)\ell} \longrightarrow 0. \tag{7.B.26}$$

It can be seen by inspecting the Loewy diagram of the B modules and proved using techniques developed in [10].



The right-exactness of fusion yields the exact sequence

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_{\ell} \xrightarrow{g} \mathsf{B}^{2(i+1)}_{k\ell-2} \times_f \mathsf{S}_{\ell} \longrightarrow \mathsf{S}_{(k+2i)\ell} \times_f \mathsf{S}_{\ell} \longrightarrow 0. \tag{7.B.27}$$

It follows that

$$\dim\Big(\mathsf{B}^{2(i+1)}_{k\ell-2} \times_f \mathsf{S}_{\ell}\Big) \leq \dim\big(\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_{\ell}\big) + \dim\big(\mathsf{S}_{(k+2i)\ell} \times_f \mathsf{S}_{\ell}\big)$$

$$= \dim\left(\bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{B}^{2(i+1)}_{(k-1)\ell-2}\right)$$
$$+ \dim\left(\mathsf{P}_{(k+2i)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{S}_{(k+2i+1)\ell}\right)$$
$$= \dim\left(\bigoplus_{p=0}^{i+1-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{B}^{2(i+2)}_{(k-1)\ell-2}\right)$$

where the equality occurs if and only if g is injective. The exact sequence (7.B.24) gives

$$\begin{aligned}\dim\left(\mathsf{B}^{2(i+1)}_{k\ell-2} \times_f \mathsf{S}_\ell\right) &= \dim\left(\mathsf{B}^{2(i+1)}_{k\ell-2} \times_f \mathsf{P}_\ell\right) - \dim \operatorname{im} f \\ &\geq \dim\left(\mathsf{B}^{2(i+1)}_{k\ell-2} \times_f \mathsf{P}_\ell\right) - \dim\left(\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{\ell-2}\right) \\ &= \dim\left(\bigoplus_{p=0}^{i+1-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{B}^{2(i+2)}_{(k-1)\ell-2}\right).\end{aligned}$$

It follows that $\ker f \simeq 0$, and the following result is thus proved.

Proposition 7.B.5. *For all $i \geq 0$, $k > 1$,*

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_\ell \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{B}^{2(i+1)}_{(k-1)\ell-2}. \tag{7.B.28}$$

Fusion rules for bigger standard modules will not be needed to compute the fusion of irreducible modules but we include them for the sake of completeness.

Proposition 7.B.6. *For all $0 < r < k$,*

$$\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{S}_{r\ell} \simeq \bigoplus_{p=0}^{i-1} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{r\ell-1} \oplus \mathsf{B}^{2(i+r)}_{(k-r)\ell-2}. \tag{7.B.29}$$

Proof. We proceed by induction on r, the case $r = 1$ being given by the previous proposition. Assume the result for some $r < k-1$. Using propositions 7.B.3, 7.5.4, and 7.4.5, we start by noting that

$$\begin{aligned}\mathsf{B}^{2i}_{k\ell-2} \times_f \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1} &\simeq \bigoplus_{p=0}^{i-1} \left(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1}\right) \oplus \bigoplus_{\substack{p=k-r \\ step=2}}^{k+r-2} \left(\mathsf{P}_{p\ell-1} \times_f \mathsf{P}_{\ell-1}\right) \\ &\simeq \bigoplus_{p=0}^{i-1} \left(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-2} \oplus \mathsf{P}_{(k+2p+1)\ell-1} \times_f \mathsf{P}_{r\ell-1}\right) \\ &\quad \oplus \bigoplus_{p=0}^{r-1} \left(\mathsf{P}_{(k-r+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\right),\end{aligned} \tag{7.B.30}$$

and

$$\mathsf{P}_{(k+2p+1)\ell-1}\times_f\mathsf{P}_{r\ell-1}\simeq\mathsf{P}_{(k+2p)\ell-1}\times_f\mathsf{P}_{(r-1)\ell-1}\oplus\mathsf{P}_{(k+r+2p)\ell-1}\times_f\mathsf{P}_{\ell-1}.$$

Next, we fuse the left side of equation (7.B.29) with S_ℓ and use propositions 7.B.5, 7.5.4, and 7.5.7 to obtain

$$\mathsf{B}^{2i}_{k\ell-2}\times_f(\mathsf{S}_{r\ell}\times_f\mathsf{S}_\ell)\simeq\mathsf{B}^{2i}_{k\ell-2}\times_f\big(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-1}\oplus\mathsf{S}_{(r+1)\ell}\big),\tag{7.B.31}$$

while fusing the right side of equation (7.B.29) with S_ℓ gives

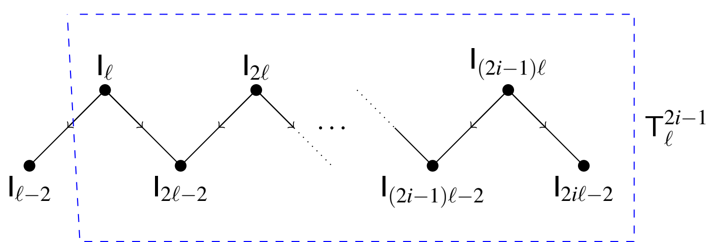
$$\begin{aligned}&\bigoplus_{p=0}^{i-1}\mathsf{P}_{(k+2p)\ell-1}\times_f\mathsf{P}_{r\ell-1}\times_f\mathsf{S}_\ell\oplus\mathsf{B}^{2(i+r)}_{(k-r)\ell-2}\times_f\mathsf{S}_\ell\\&\qquad\simeq\bigoplus_{p=0}^{i-1}\big(\mathsf{P}_{(k+2p)\ell-1}\times_f\big(\mathsf{P}_{r\ell-1}\times_f\mathsf{P}_{\ell-2}\oplus\mathsf{P}_{(r-1)\ell-1}\oplus\mathsf{P}_{(r+1)\ell-1}\big)\big)\\&\qquad\oplus\bigoplus_{p=0}^{i+r-1}\big(\mathsf{P}_{(k-r+2p)\ell-1}\times_f\mathsf{P}_{\ell-1}\big)\oplus\mathsf{B}^{2(i+1+r)}_{(k-r-1)\ell-2}.\end{aligned}\tag{7.B.32}$$

Comparing the two results and using the previous observations gives the conclusion for $r+1$. ■

Note that in all of these calculations, we carefully avoided the case $k=1$ (and $r=k$). There is a short exact sequence

$$0\longrightarrow\mathsf{I}_{\ell-2}\longrightarrow\mathsf{B}^{2i}_{\ell-2}\longrightarrow\mathsf{T}^{2i-1}_\ell\longrightarrow0,\tag{7.B.33}$$

which can be seen by inspecting the Loewy diagram of $\mathsf{B}^{2i}_{\ell-2}$ and proved using techniques developed in [10].



(7.B.34)

Since it was already noted in proposition 7.6.5 that $\mathsf{I}_{\ell-2}\times_f\mathsf{S}_{q\ell}\simeq0$ for all $q\geq1$,

$$\mathsf{B}^{2i}_{\ell-2}\times_f\mathsf{S}_{q\ell}\simeq\mathsf{T}^{2i-1}_\ell\times_f\mathsf{S}_{q\ell}.\tag{7.B.35}$$

Therefore, to compute $\mathsf{I}_{2\ell-2}\times_f\mathsf{S}_{2\ell}$, we will have to compute

$$(\mathsf{I}_{2\ell-2}\times_f\mathsf{S}_\ell)\times_f\mathsf{S}_\ell\simeq\mathsf{B}^2_{\ell-2}\times_f\mathsf{S}_\ell\simeq\mathsf{T}^1_\ell\times_f\mathsf{S}_\ell.\tag{7.B.36}$$

The fusion rules for T^{2i-1}_ℓ will thus be needed to compute the fusion rules of the irreducible modules.

7.B.3 Fusion of T_k^{2i+1} and projective modules

The formulas for the induction of T_k^{2i+1} are [10]:

$$\mathsf{T}_{n,k}^{2i+1}\!\uparrow \simeq \mathsf{T}_{n+1,k-1}^{2i+1} \oplus \mathsf{T}_{n+1,k+1}^{2i+1} \oplus \begin{Bmatrix} \mathsf{T}_{n+1,k}^{2i+1}, & \text{in } \mathsf{dTL}_n \\ 0, & \text{in } \mathsf{TL}_n \end{Bmatrix}, \tag{7.B.37}$$

where

$$\mathsf{T}_{n,k}^{2i+1} \simeq \bigoplus_{p=0}^{i} \mathsf{P}_{n,k+2p\ell} \tag{7.B.38}$$

if k is critical and

$$\mathsf{T}_{n,-1}^{2i+1} \simeq \bigoplus_{p=1}^{i} \mathsf{P}_{n,2p\ell-1}. \tag{7.B.39}$$

Using the parity of the relevant modules gives the following fusion rules.

Proposition 7.B.7. *For all k,i, in the dilute family*

$$\mathsf{T}_{n,k}^{2i+1} \times_f \mathsf{P}_{1,0} \simeq \mathsf{T}_{n+1,k}^{2i+1}, \tag{7.B.40}$$

while in both families

$$\mathsf{T}_{n,k}^{2i+1} \times_f \mathsf{P}_{1,1} \simeq \mathsf{T}_{n+1,k-1}^{2i+1} \oplus \mathsf{T}_{n+1,k+1}^{2i+1}. \tag{7.B.41}$$

Once again, fusing these modules with $\mathsf{P}_{1,0}$ simply increases the parameter n. In the regular case, proposition 7.A.3 gives

$$\mathsf{T}_{n,k}^{2i+1} \times_f \mathsf{P}_{2,0} \simeq \mathsf{T}_{n+2,k}^{2i+1},$$

as long as $\ell \neq 2$. As before, the proofs will be independent of n so we simply omit this parameter and assume n to be big enough for the modules to exist.

We start by studying the modules $\mathsf{T}_{k\ell}^{2i+1}$. Note that

$$\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{P}_1 \simeq \bigoplus_{p=0}^{i} \mathsf{P}_{(k+2p)\ell-1} \oplus \mathsf{T}_{k\ell+1}^{2i+1}.$$

Fusing this expression with P_1 yields

$$\begin{aligned} \mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{P}_1 \times_f \mathsf{P}_1 &\simeq \mathsf{T}_{k\ell}^{2i+1} \times_f (\mathsf{P}_0 \oplus \mathsf{P}_2) \\ &\simeq \bigoplus_{p=0}^{i} \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_1 \oplus \mathsf{T}_{k\ell}^{2i+1} \oplus \mathsf{T}_{k\ell+2}^{2i+1} \end{aligned}$$

Comparing the first and second lines and using proposition 7.B.7 give the fusion rule

$$\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{P}_2 \simeq \bigoplus_{p=0}^{i} \left(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_1\right) \oplus \mathsf{T}_{k\ell+2}^{2i+1}.$$

It is a simple exercise to repeat this argument and obtain the fusion rules for the other small projectives.

Proposition 7.B.8. *For all $i,k \geq 0$, $0 \leq j < \ell - 1$,*

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_j \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{j-1}\big) \oplus \mathsf{T}^{2i+1}_{k\ell+j}. \tag{7.B.42}$$

In particular,

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell-2} \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-3}\big) \oplus \mathsf{T}^{2i+1}_{(k+1)\ell-2}.$$

Fusing this expression with P_1 gives

$$\begin{aligned}\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell-2} \times_f \mathsf{P}_1 &\simeq \mathsf{T}^{2i+1}_{k\ell} \times_f (\mathsf{P}_{\ell-3} \oplus \mathsf{P}_{\ell-1}) \\ &\simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f (\mathsf{P}_{\ell-4} \oplus \mathsf{P}_{\ell-2})\big) \oplus \mathsf{T}^{2i+1}_{(k+1)\ell-3} \\ &\quad \oplus \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p+1)\ell-1}\big).\end{aligned}$$

Comparing the first and second lines gives

$$\begin{aligned}\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell-1} &\simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-2} \oplus \mathsf{P}_{(k+1+2p)\ell-1}\big) \\ &\simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell-1}\big),\end{aligned} \tag{7.B.43}$$

where the known fusion rules for standard modules (proposition 7.5.3) were used in the second line. Fusing this expression with P_1 gives

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell-1} \times_f \mathsf{P}_1 \simeq \mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell} \simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell}\big).$$

Fusing the latter expression again with P_1 gives

$$\begin{aligned}\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell} \times_f \mathsf{P}_1 &\simeq \mathsf{T}^{2i+1}_{k\ell} \times_f (2\mathsf{P}_{\ell-1} \oplus \mathsf{P}_{\ell+1}) \\ &\simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell} \times_f (2\mathsf{P}_{\ell-1} \oplus \mathsf{P}_{\ell+1})\big).\end{aligned}$$

Comparing the two lines yields the fusion rule

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell+1} \simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{\ell+1}\big).$$

The same arguments prove the following proposition.

Proposition 7.B.9. *For all $i,k \geq 0$, $r \geq \ell - 1$,*

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_r \simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_r\big), \tag{7.B.44}$$

The fusion rules for $\mathsf{T}^{2i+1}_{k\ell+i}$ can be obtained from these formulas. We start by fusing (7.B.44) with P_1.

$$\begin{aligned}\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_r \times_f \mathsf{P}_1 &\simeq \left(\bigoplus_{p=0}^{i} \mathsf{P}_{(k+2p)\ell-1} \oplus \mathsf{T}^{2i+1}_{k\ell+1}\right) \times_f \mathsf{P}_r \\ &\simeq \bigoplus_{p=0}^{i} \big(\big(\mathsf{S}_{(k+2p)\ell+1} \oplus \mathsf{P}_{(k+2p)\ell-1}\big) \times_f \mathsf{P}_r\big).\end{aligned}$$

Comparing the two lines yields

$$\mathsf{T}^{2i+1}_{k\ell+1} \times_f \mathsf{P}_r \simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell+1} \times_f \mathsf{P}_r\big). \tag{7.B.45}$$

This argument can be repeated to obtain the following proposition.

Proposition 7.B.10. *For all $i,k \geq 0$, $0 < j < \ell$, $r \geq \ell - 1$,*

$$\mathsf{T}^{2i+1}_{k\ell-1+j} \times_f \mathsf{P}_r \simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell-1+j} \times_f \mathsf{P}_r\big). \tag{7.B.46}$$

7.B.4 Fusion of T^{2i+1}_k and standard modules

We want to compute fusions of T^{2i+1}_k with non-projective standard modules. Proceeding as in the previous sections, we start by computing $\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{S}_\ell$, where $k \neq 0$.

There is a short-exact sequence

$$0 \longrightarrow \mathsf{P}_{\ell-2} \longrightarrow \mathsf{P}_\ell \longrightarrow \mathsf{S}_\ell \longrightarrow 0,$$

which gives the exact sequence

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell-2} \xrightarrow{f} \mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_\ell \longrightarrow \mathsf{T}^{2i+1}_{kl} \times_f \mathsf{S}_\ell \longrightarrow 0 \tag{7.B.47}$$

by using the right-exactness of fusion. Propositions 7.B.8 and equation (7.B.43) give

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{\ell-2} \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-3}\big) \oplus \mathsf{T}^{2i+1}_{(k+1)\ell-2},$$

$$\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{P}_\ell \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f (\mathsf{P}_{\ell-3} \oplus \mathsf{P}_{\ell-1}) \oplus \mathsf{P}_{(k+1+2p)\ell}\big).$$

Therefore

$$\big(\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{P}_\ell\big)/\big(\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{P}_{\ell-2}\big) \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}_{(k+1)\ell}^{2i+1}, \tag{7.B.48}$$

where proposition 7.2.2, which gives $\Big(\bigoplus_{p=0}^{i} \mathsf{P}_{(k+1+2p)\ell}\Big)/\mathsf{T}_{(k+1)\ell-2}^{2i+1} \simeq \mathsf{T}_{(k+1)\ell}^{2i+1}$, was used. The goal is now to prove that

$$\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{S}_l \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}_{(k+1)\ell}^{2i+1}, \tag{7.B.49}$$

which is equivalent to f being injective. Note that for $i=0$, this is just the fusion of two standard modules, and proposition 7.5.7, or 7.5.10 if $k=1$, $\ell=2$ in the regular family, agrees with (7.B.49). We thus proceed by induction on i. Assume that (7.B.49) stands for i and use the short exact sequence

$$0 \longrightarrow \mathsf{T}_{k\ell}^{2i+1} \longrightarrow \mathsf{T}_{k\ell}^{2i+3} \longrightarrow \mathsf{S}_{(k+2i+2)\ell} \longrightarrow 0,$$

to obtain the exact sequence

$$\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{S}_\ell \longrightarrow \mathsf{T}_{k\ell}^{2i+3} \times_f \mathsf{S}_\ell \longrightarrow \mathsf{S}_{(k+2i+2)\ell} \times_f \mathsf{S}_\ell \longrightarrow 0. \tag{7.B.50}$$

It gives the inequality

$$\begin{aligned}
\dim\big(\mathsf{T}_{k\ell}^{2i+3} \times_f \mathsf{S}_\ell\big) &\le \dim\big(\mathsf{T}_{k\ell}^{2i+1} \times_f \mathsf{S}_\ell\big) + \dim\big(\mathsf{S}_{(k+2i+2)\ell} \times_f \mathsf{S}_\ell\big) \\
&= \dim\left(\bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}_{(k+1)\ell}^{2i+1}\right) \\
&\quad + \dim\big(\big(\mathsf{P}_{(k+2i+2)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{S}_{(k+2i+3)\ell}\big) \\
&= \dim\left(\bigoplus_{p=0}^{i+1} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}_{(k+1)\ell}^{2i+3}\right)
\end{aligned}$$

However, the exact sequence (7.B.47) also give the inequality

$$\begin{aligned}
\dim\big(\mathsf{T}_{k\ell}^{2i+3} \times_f \mathsf{S}_\ell\big) &= \dim\big(\mathsf{T}_{k\ell}^{2i+3} \times_f \mathsf{P}_\ell\big) - \dim \operatorname{im} f \\
&\ge \dim\left(\bigoplus_{p=0}^{i+1} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}_{(k+1)\ell}^{2i+3}\right)
\end{aligned}$$

Comparing the two bounds shows that $\dim(\operatorname{im} f) = \dim\big(\mathsf{T}_{k\ell}^{2i+3} \times_f \mathsf{P}_{\ell-2}\big)$ and thus that f is injective. Formula (7.B.49) must therefore stand for $i+1$, proving the following proposition.

Proposition 7.B.11. *For $i \geq 0$, $k > 0$,*

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{S}_\ell \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}^{2i+1}_{(k+1)\ell}. \tag{7.B.51}$$

Fusions with the bigger standard modules and the other $\mathsf{T}^{2i+1}_{k\ell+i}$ will not be needed but are presented for the sake of completeness.

Proposition 7.B.12. *For all $i \geq 0$, $k, r > 0$,*

$$\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{S}_{r\ell} \simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{r\ell-1}\big) \oplus \mathsf{T}^{2i+1}_{(k+r)\ell}. \tag{7.B.52}$$

Proof. We proceed by induction on r. The case $r = 1$ being contained in proposition 7.B.11 , suppose that the result holds for a certain $r > 1$. Then, we start by noticing that by propositions 7.B.9, and 7.5.3,

$$\begin{aligned}\mathsf{T}^{2i+1}_{k\ell} \times_f \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1} &\simeq \bigoplus_{p=0}^{i} \big(\mathsf{S}_{(k+2p)\ell} \times_f \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1}\big)\\ &\simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{l-2} \oplus \mathsf{P}_{(k+2p+1)\ell-1} \times_f \mathsf{P}_{r\ell-1}\big),\end{aligned}$$

and by proposition 7.4.5,

$$\mathsf{P}_{(k+2p+1)\ell-1} \times_f \mathsf{P}_{r\ell-1} \simeq \mathsf{P}_{(k+2p)\ell-1} \times_f \mathsf{P}_{(r-1)\ell-1} \oplus \mathsf{P}_{(k+2p+r)\ell-1} \times_f \mathsf{P}_{\ell-1}. \tag{7.B.53}$$

Then, fuse the left side of (7.B.52) with S_ℓ and use propositions 7.B.9, and 7.5.3, with equation (7.B.35) to obtain

$$\mathsf{T}^{2i+1}_{k\ell} \times_f (\mathsf{S}_{r\ell} \times_f \mathsf{S}_\ell) \simeq \mathsf{T}^{2i+1}_{k\ell} \times_f \big(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-1} \oplus \mathsf{S}_{(r+1)\ell}\big), \tag{7.B.54}$$

while fusing its right side with S_ℓ and using propositions 7.5.4, and 7.B.11 gives

$$\begin{aligned}&\bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f (\mathsf{P}_{r\ell-1} \times_f \mathsf{S}_\ell)\big) \oplus \mathsf{T}^{2i+1}_{(k+r)\ell} \times_f \mathsf{S}_\ell\\ &\qquad\simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p)\ell-1} \times_f \big(\mathsf{P}_{r\ell-1} \times_f \mathsf{P}_{\ell-2} \oplus \mathsf{P}_{(r+1)\ell-1} \oplus \mathsf{P}_{(r-1)\ell-1}\big)\big)\\ &\qquad\simeq \bigoplus_{p=0}^{i} \big(\mathsf{P}_{(k+2p+r)\ell-1} \times_f \mathsf{P}_{\ell-1}\big) \oplus \mathsf{T}^{2i+1}_{(k+r+1)\ell}.\end{aligned} \tag{7.B.55}$$

Comparing the two sides and using the preceding observations gives the conclusion for $i+1$. ■

7.B.5 Fusion of $\mathsf{B}^1_{n,k}$ and projective modules

We start by giving the behaviour of $\mathsf{B}^1_{n,k}$ under induction [10].

Proposition 7.B.13. *For all $n \geq k^1$, and k not critical,*

$$\mathsf{B}^1_{n,k}\uparrow \simeq \mathsf{B}^1_{n+1,k-1} \oplus \mathsf{B}^1_{n+1,k+1} \oplus \begin{Bmatrix} \mathsf{B}^1_{n+1,k}, & \textit{in } \mathsf{dTL}_n \\ 0, & \textit{in } \mathsf{TL}_n \end{Bmatrix}. \tag{7.B.56}$$

where $\mathsf{B}^1_{n,k\pm1} \simeq \mathsf{P}_{n,k\pm1}$ when k is critical.

Using proposition 7.3.4 together with the parity of the modules yields the following fusion rules.

Proposition 7.B.14. *For all non-critical k, in the dilute Temperley-Lieb family*

$$\mathsf{B}^1_{n,k} \times_f \mathsf{P}_{1,0} \simeq \mathsf{B}^1_{n+1,k}, \tag{7.B.57}$$

while in both families

$$\mathsf{B}^1_{n,k} \times_f \mathsf{P}_{1,1} \simeq \mathsf{B}^1_{n+1,k-1} \oplus \mathsf{B}^1_{n+1,k+1}. \tag{7.B.58}$$

As usual, fusing $\mathsf{B}^1_{n,k}$ with $\mathsf{P}_{1,0}$ simply increases the parameter n. We will thus omit this parameter and always assume that it is big enough for the modules to exist.

We now compute the fusion rules for B^1_{kl}, $k \geq 0$. The preceding proposition gives

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_1 \simeq \mathsf{P}_{k\ell-1} \oplus \mathsf{B}^1_{k\ell+1},$$

where it is understood that $\mathsf{P}_{k\ell-1} \simeq 0$ if $k = 0$. Fusing this result with P_1 yields

$$\begin{aligned} \mathsf{B}^1_{k\ell} \times_f \mathsf{P}_1 \times_f \mathsf{P}_1 \quad &\simeq \mathsf{B}^1_{k\ell} \times_f (\mathsf{P}_0 \oplus \mathsf{P}_2) \\ &\simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_1 \oplus \mathsf{B}^1_{k\ell} \oplus \mathsf{B}^1_{k\ell+2}. \end{aligned}$$

Comparing the two lines gives the fusion rule

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_2 \simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_1 \oplus \mathsf{B}^1_{k\ell+2}.$$

Repeating the argument yields

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_3 \simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_2 \oplus \mathsf{B}^1_{k\ell+3},$$

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_4 \simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_3 \oplus \mathsf{B}^1_{k\ell+4}.$$

This arguments can be repeated as needed to obtain the following fusion rules.

Proposition 7.B.15. *For $0 < i \leq \ell - 1$, $k \geq 0$,*

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_i \simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{i-1} \oplus \mathsf{B}^1_{k\ell+i}, \tag{7.B.59}$$

where it is understood that $\mathsf{P}_{-1} \simeq 0$.

If $k \neq 0$, this proposition gives

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_{\ell-1} \simeq \mathsf{P}_{k\ell-1} \times_f \mathsf{P}_{\ell-2} \oplus \mathsf{P}_{(k+1)\ell-1} \simeq \mathsf{S}_{k\ell} \times_f \mathsf{P}_{\ell-1},$$

where we used proposition 7.5.3. Fusing this expression with P_1 gives

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_{\ell-1} \times_f \mathsf{P}_1 \simeq \mathsf{B}^1_{k\ell} \times_f \mathsf{P}_{\ell} \simeq \mathsf{S}_{k\ell} \times_f \mathsf{P}_{\ell}.$$

Fusing this with P_1 again gives

$$\begin{aligned} \mathsf{B}^1_{k\ell} \times_f \mathsf{P}_{\ell} \times_f \mathsf{P}_1 &\simeq \mathsf{B}^1_{k\ell} \times_f (2\mathsf{P}_{\ell-1} \oplus \mathsf{P}_{\ell+1}) \\ &\simeq \mathsf{S}_{k\ell} \times_f (2\mathsf{P}_{\ell-1} \oplus \mathsf{P}_{\ell+1}). \end{aligned}$$

Comparing the two lines gives the fusion rule

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_{\ell+1} \simeq \mathsf{S}_{k\ell} \times_f \mathsf{P}_{\ell+1}.$$

It is simple enough to repeat this argument and obtain the general formula.

Proposition 7.B.16. *For* $k > 0, r \geq \ell - 1$,

$$\mathsf{B}^1_{k\ell} \times_f \mathsf{P}_r \simeq \mathsf{S}_{k\ell} \times_f \mathsf{P}_r. \tag{7.B.60}$$

For $k = 0$, recall the short exact sequence

$$0 \longrightarrow \mathsf{I}_0 \longrightarrow \mathsf{B}^1_0 \longrightarrow \mathsf{I}_{2\ell-2} \longrightarrow 0. \tag{7.B.61}$$

Since $\mathsf{I}_0 \times_f \mathsf{P}_r \simeq 0$ for all $r \geq \ell - 1$ (see propositions 7.6.5 and 7.6.3), the following result is obtained.

Proposition 7.B.17. *For all* $r \geq 1$, $0 \leq j < \ell$,

$$\mathsf{B}^1_0 \times_f \mathsf{P}_{r\ell-1+j} \simeq \mathsf{I}_{2\ell-2} \times_f \mathsf{P}_{r\ell-1+j} \simeq \mathsf{P}_{r\ell-1+j}. \tag{7.B.62}$$

More general results could be easily obtained but we will stop here since we have all we need to finish the computation of fusions of irreducible modules.

7.C Fusion quotient

We present here a brief study of the operator adjoint to the fusion product, the fusion quotient. We begin with the definition then present the basic properties that follows from it. Finally, we give the fusion quotients of a few Temperley-Lieb modules to show that the two operations, while giving similar results, are not equivalent.

7.C.1 Definition of the fusion quotient

Proposition 7.C.1. *Consider a family of algebras $(A_i)_{i\in\mathbb{N}}$ on which fusion is defined (see the beginning of section 7.3), U a A_i-module, V a A_j-module and W a A_{i+j}-module. There is an isomorphism of vector spaces*

$$\mathrm{Hom}_{A_{i+j}}(U\times_f V, W) \simeq \mathrm{Hom}_{A_i}\big(U, Hom_{A_{i+j}}(A_i\times_f V, W)\big) \tag{7.C.1}$$

where $A_i\times_f V$ is seen as a left A_{i+j}-module and a right A_i-module.

Définition 7.C.2. *For U a A_i-module and V a A_{i+j}-module. The* fusion quotient *of V by U, denoted by $V\div_f U$, is the A_j-module*

$$V\div_f U = \mathrm{Hom}_{A_{i+j}}(A_j\times_f U, V) \tag{7.C.2}$$

where the module structure is given by

$$(ag): b\otimes_{A_j\otimes A_i}(c\otimes_{\mathbb{C}} x) \mapsto g\big(b\otimes_{A_j\otimes A_i}(ca\otimes_{\mathbb{C}} x)\big), \tag{7.C.3}$$

where $a,c\in A_j$, $b\in A_{i+j}$, $x\in U, g\in Hom_{A_{i+j}}(A_j\times_f U, V)$.

If the fusion product has additional properties, like linearity, associativity and commutativity the fusion quotient will inherit some of those.

Proposition 7.C.3. *Let Q and $\bar{Q}$ be a pair of A_{i+j+k}-modules, $U,\bar{U}$ two A_j-modules and V a A_k-module,*

$$\big(Q\oplus\bar{Q}\big)\div_f(U\oplus\bar{U}) \simeq (Q\div_f U)\oplus\big(\bar{Q}\div_f U\big)\oplus(Q\div_f\bar{U})\oplus\big(\bar{Q}\div_f\bar{U}\big). \tag{7.C.4}$$

If the fusion product on the family $\{A_i\}$ is associative, then

$$(Q\div_f U)\div_f V \simeq Q\div_f(V\times_f U). \tag{7.C.5}$$

If the fusion product is also commutative, then

$$(Q\div_f U)\div_f V \simeq (Q\div_f V)\div_f U. \tag{7.C.6}$$

Proof. The proof of (7.C.4) follows from the linearity of the fusion product and of the Hom functor. If the fusion product on the family $\{A_i\}$ is associative, then

$$\begin{aligned}(Q\div_f U)\div_f V &= \mathrm{Hom}_{A_{i+k}}(A_i\times_f V, Q\div_f U) && (7.C.7)\\ &\simeq \mathrm{Hom}_{A_{i+k+j}}((A_i\times_f V)\times_f U, Q) && (7.C.8)\\ &\simeq \mathrm{Hom}_{A_{i+j+k}}(A_i\times_f(V\times_f U), Q) && (7.C.9)\end{aligned}$$

$$= Q \div_f (V \times_f U). \tag{7.C.10}$$

The first and last lines are simply the definition of the fusion quotient, while the second is proposition 7.C.1 and the third is obtained by using the associativity of the fusion product. If the product is also commutative, it is clear that

$$(Q \div_f U) \div_f V \simeq Q \div_f (V \times_f U) \simeq Q \div_f (U \times_f V) \simeq (Q \div_f V) \div_f U \tag{7.C.11}$$

by using (7.C.5). ■

The following proposition gives the behaviour of short exact sequences under the fusion quotient.

Proposition 7.C.4. *Let*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

be a short exact sequence of A_i-modules and Q be a A_j-module. If $i > j$, the sequence of A_{i-j}-modules

$$0 \longrightarrow U \div_f Q \longrightarrow V \div_f Q \xrightarrow{f} W \div_f Q \tag{7.C.12}$$

is exact. If Q is projective, then f is surjective. If $j > i$, the sequence of A_{j-i}-module

$$0 \longrightarrow Q \div_f W \longrightarrow Q \div_f V \longrightarrow Q \div_f U \tag{7.C.13}$$

is exact.

Proof. For the case $i > j$ simply use the fact that $\mathrm{Hom}\big(P,-\big)$ is always left-exact for all module P. If moreover Q is projective, proposition 7.3.1 shows that $A_{i-j} \times_f Q$ is projective so that $\mathrm{Hom}_{A_{i+j}}\big\{A_{i-j} \times_f Q, -\big\}$ is also right-exact. For the other case, the right-exactness of the fusion product is used to obtain the exact sequence

$$A_{j-i} \times_f U \longrightarrow A_{j-i} \times_f V \longrightarrow A_{j-i} \times_f W \longrightarrow 0. \tag{7.C.14}$$

The final result is obtained by using the fact that $\mathrm{Hom}\big(-,P\big)$ is always left-exact and contravariant. ■

Note also that the fusion quotient of an A_{i+j}-module U by A_j has the structure of a $A_i \otimes_{\mathbb{C}} A_j$-module. It can be seen that this quotient is in fact isomorphic to the restriction of U to the subalgebra $A_i \otimes_{\mathbb{C}} A_j$. The following proposition relates this structure to the quotient of U by a A_j-module V.

Proposition 7.C.5. *For U a A_{i+j}-module and V a A_j-module,*

$$U \div_f V \simeq \mathrm{Hom}_{A_j}(V, U \div_f A_i) \tag{7.C.15}$$

where the action of A_i on $\mathrm{Hom}_{A_j}(V, U \div_f A_i)$ is given by

$$(a_i g) : v_j \mapsto \big(b_{i+j} \otimes_{A_j \otimes A_i} (c_j \otimes_{\mathbb{C}} d_i) \mapsto g(v_j)\big(b_{i+j} \otimes_{A_j \otimes A_i} (c_j \otimes_{\mathbb{C}} d_i a_i)\big)\big), \tag{7.C.16}$$

where the indices on $a_i, b_{i+j}, \ldots$ refers to which of $A_i, A_{i+j}, \ldots$ they belong.

Proof. The proof proceeds by construction. Define the vector space homomorphism $\phi : U \div_f V \to \mathrm{Hom}_{A_j}(V, U \div_f A_i)$ by

$$\phi(g) = \big(v_j \mapsto \big(b_{i+j} \otimes_{A_j \otimes A_i} (c_j \otimes_{\mathbb{C}} d_i) \mapsto g(b_{i+j} \otimes_{A_j \otimes A_i} (c_j v_j \otimes_{\mathbb{C}} d_i))\big)\big) \quad (7.C.17)$$

and another homomorphism $\psi : \mathrm{Hom}_{A_j}(V, U \div_f A_i) \to U \div_f V$ by

$$\psi(g) = \big(b_{i+j} \otimes_{A_j \otimes A_i} (v_j \otimes_{\mathbb{C}} d_i) \mapsto g(v_j)(b_{i+j} \otimes_{A_j \otimes A_i} (\mathrm{id}_{A_j} \otimes_{\mathbb{C}} d_i))\big). \quad (7.C.18)$$

It is straightforward to verify that these two morphisms are inverse of each other, and that the action of A_i defined in the proposition makes them into A_i-module homomorphisms. ■

7.C.2 Fusion quotient in the Temperley-Lieb families

We present here the fusion quotient of some modules in the TL_n and dTL_n families.

Proposition 7.C.6. *Let A_n be TL_n or dTL_n. For any A_{n+1}-module U,*

$$U \div_f A_1 \simeq U{\downarrow},$$

where the restriction functor is $-{\downarrow} =_{A_n} (A_{n+1})_{A_{n+1}} \otimes_{A_{n+1}} -$.

Proof. The functor $-{\downarrow}$ is the adjoint of the functor $-{\uparrow}$ defined in section 7.2.3. Since $- \times_f A_1$ is equivalent to $-{\uparrow}$, their adjoints must also be equivalent. ■

This restriction functor as also been computed [10, 11, 63].

Proposition 7.C.7. *For $0 \le i < \ell$, $0 < j < \ell$ such that $n+1 \ge k_c + i$,*

$$\mathsf{P}_{n+1,k_c+i}{\downarrow} \simeq \begin{Bmatrix} 2\mathsf{P}_{n,k_c}, & \text{if } i=1 \\ 0, & \text{if } i=0 \\ \mathsf{P}_{n,k_c+i-1}, & \text{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \mathsf{P}_{n,k_c+i}, & \text{in } \mathsf{dTL}_n \text{ and } n \ge k_c+i \\ \mathsf{S}_{n,k_c-i}, & \text{in } \mathsf{dTL}_n \text{ and } n < k_c+i \\ 0, & \text{in } \mathsf{TL}_n \end{Bmatrix}$$
$$\oplus \begin{Bmatrix} \mathsf{P}_{n,k_c-\ell} \oplus \mathsf{P}_{n,k_c+\ell}, & \text{if } i=\ell-1 \text{ and } n \ge k_c+\ell \\ \mathsf{P}_{n,k_c-\ell}, & \text{if } i=\ell-1 \text{ and } n < k_c+\ell \\ \mathsf{S}_{n,k_c-(i+1)}, & \text{if } i \ne \ell-1 \text{ and } n < k_c+i+1 \\ \mathsf{P}_{n,k_c+i+1}, & \text{if } i \ne \ell-1 \text{ and } n \ge k_c+i+1 \end{Bmatrix}, \quad (7.C.19)$$

$$\mathsf{S}_{n+1,k_c+i}{\downarrow} \simeq \begin{Bmatrix} \mathsf{P}_{n,k_c}, & \text{if } i=1 \\ \mathsf{S}_{n,k_c+i-1}, & \text{otherwise} \end{Bmatrix} \oplus \begin{Bmatrix} \mathsf{S}_{n,k_c+i}, & \text{in } \mathsf{dTL}_n \text{ and } n \ge k_c+i \\ 0, & \text{in } \mathsf{dTL}_n \text{ and } n < k_c+i \\ 0, & \text{in } \mathsf{TL}_n \end{Bmatrix}$$

$$\oplus \left\{ \begin{array}{ll} \mathsf{P}_{n,k_c+\ell}, & \textit{if } i=\ell-1 \textit{ and } n\geq k_c+\ell \\ \mathsf{S}_{n,k_c+i+1}, & \textit{if } i\neq\ell-1 \textit{ and } n\geq k_c+i+1 \\ 0, & \textit{if } n<k_c+i+1 \end{array} \right\}. \tag{7.C.20}$$

Corollary 7.C.8. *If $k \leq n$,*

$$\mathsf{P}_{n+2,k}\!\downarrow \simeq \mathsf{P}_{n,k}\!\uparrow, \qquad \mathsf{S}_{n+2,k}\!\downarrow \simeq \mathsf{S}_{n,k}\!\uparrow. \tag{7.C.21}$$

As for the fusion product, we now need to compute the fusion quotient of a standard module by $\mathsf{P}_{2,0}$ in TL_n.

Proposition 7.C.9. *In the regular TL_n family, if $\ell \neq 2$, and $n-2m \geq q$, then*

$$\mathsf{S}_{n,q} \div_f \mathsf{P}_{2m,0} \simeq \mathsf{S}_{n-2m,q}. \tag{7.C.22}$$

If $n-2m<q$, then $\mathsf{S}_{n,q} \div_f \mathsf{P}_{2m,0} \simeq 0$.

Proof. Start with the case $m=1$. The first step is to prove that the two modules have the same dimension. For this, note that

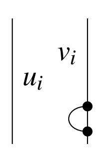
$$\begin{aligned} \mathsf{S}_{n,q} \div_f \mathsf{P}_{2,0} &\overset{1}{\simeq} \mathrm{Hom}\big(\mathsf{TL}_{n-2} \times_f \mathsf{P}_{2,0}, \mathsf{S}_{n,q}\big) \\ &\overset{2}{\simeq} \bigoplus_{\substack{i=n \mod 2 \\ step=2}}^{n-2} \big((\dim \mathsf{I}_{n-2,i})\mathrm{Hom}\big(\mathsf{P}_{n-2,i} \times_f \mathsf{P}_{2,0}, \mathsf{S}_{n,q}\big)\big) \\ &\overset{3}{\simeq} \bigoplus_{\substack{i=n \mod 2 \\ step=2}}^{n-2} \big((\dim \mathsf{I}_{n-2,i})\mathrm{Hom}\big(\mathsf{P}_{n,i}, \mathsf{S}_{n,q}\big)\big) \\ &\overset{4}{\simeq} \mathbb{C}(\dim \mathsf{I}_{n-2,q} + \dim \mathsf{I}_{n-2,q_1}), \end{aligned} \tag{7.C.23}$$

where the isomorphism are morphism of vector spaces. Here, 1 is simply the definition of the fusion quotient while 2 is Wedderburn's theorem with linearity of Hom. The morphism 3 is obtained by using proposition 7.4.5 while 4 is obtained by inspecting the Loewy diagrams of the projective modules to find the morphism from the Ps to $\mathsf{S}_{n,q}$. It follows that

$$\dim(\mathsf{S}_{n,q} \div_f \mathsf{P}_{2,0}) = \dim \mathsf{I}_{n-2,q} + \dim \mathsf{I}_{n-2,q_1} = \dim \mathsf{S}_{n-2,q}.$$

Note that one or both of these irreducible modules may not be defined, in which case we simply set their dimension to zero. In particular if both $q,q_1>n-2$, then $\mathsf{S}_{n,q} \div_f \mathsf{P}_{2,0} \simeq 0$.

To identify the action of TL_{n-2} on $\mathsf{S}_{n,q} \div_f \mathsf{P}_{2,0}$, we proceed as follows. Note that $\mathsf{TL}_{n-2} \times_f \mathsf{P}_{2,0}$ is isomorphic as a left TL_n-module and as a right TL_{n-2}-module to J, the left ideal of TL_n spanned by diagrams where the bottom two nodes on their right side are linked together, i.e. those of the form



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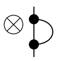
where $u_i \in \mathsf{S}_{n,i}$, $v_i \in \mathsf{S}_{n-2,i}$ for some $0 \le i \le n-2$, and where the action of TL_{n-2} on J is obtained by adding two straight lines at the bottom of every diagram. To see this, verify that $\phi : a \mapsto az$, defines a bi-module isomorphism between the two, where

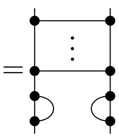
$$z = \mathrm{id}_{\mathsf{TL}_n} \otimes_{\mathsf{TL}_{n-2} \times_f \mathsf{TL}_2} \left(\mathrm{id}_{\mathsf{TL}_{n-2}} \otimes \right).$$



Next, notice that g is an homomorphism from TL_n to $\mathsf{S}_{n,q}$ if and only if there exists a unique x in $\mathsf{S}_{n,q}$ such that $g \equiv g_x : a \mapsto ax$. Furthermore, since J is isomorphic to $\mathsf{TL}_{n-2} \times_f \mathsf{P}_{2,0}$, it is a direct summand of TL_n, and thus every morphism from J to $\mathsf{S}_{n,q}$ must be of the form $g_x \circ i$ for some x, where i is the canonical injection.

Now, consider the diagram

$e =$



in TL_n and notice that for any $a \in J$, $ae\frac{1}{\beta} = a$. It follows that

$$\mathrm{Hom}\big(J, \mathsf{S}_{n,q}\big) \simeq \big\{(a \mapsto ax) | x \in \mathsf{S}_{n,q} \text{ such that } ex = \beta x\big\}.$$

Note now that any link diagram in $\mathsf{S}_{n,q}$ where the two bottom nodes are linked together will define such a morphism. These span a vector space of dimension $\dim \mathsf{S}_{n-2,q}$.

Using the action of TL_{n-2} defined on $\mathsf{S}_{n,q}$ by adding two straight lines at the bottom of every diagram, it can be directly verified that for any $b \in \mathsf{TL}_{n-2}$,

$$(bg_x \circ i) : a \mapsto abx = g_{bx} \circ i,$$

and thus, that $\mathrm{Hom}\big(J, \mathsf{S}_{n,q}\big)$ is isomorphic, as a left TL_{n-2}-module, to the submodule of $\mathsf{S}_{n,q}$ spanned by diagrams where the two bottom nodes are linked together. Comparing these link diagrams with a basis of $\mathsf{S}_{n-2,q}$ gives the conclusion.

The proof then proceed by induction on m. The case $m = 1$ is proved so assume that the result stands for some m. Then

$$(\mathsf{S}_{n,q} \div_f \mathsf{P}_{2m,0}) \div_f \mathsf{P}_{2,0} \simeq \mathsf{S}_{n,q} \div_f (\mathsf{P}_{2m,0} \times_f \mathsf{P}_{2,0}) \simeq \mathsf{S}_{n,q} \div_f \mathsf{P}_{2(m+1),0} \simeq \mathsf{S}_{n-2(m+1),q}, \tag{7.C.24}$$

where we simply used propositions 7.4.5 and 7.C.3. ■

Corollary 7.C.10. *In TL_n, if $\ell \neq 2$ and $n - 2m \geq q$,*

$$\mathsf{P}_{n,q} \div_f \mathsf{P}_{2m,0} \simeq \mathsf{P}_{n-2m,q}. \tag{7.C.25}$$

Proof. If $q < \ell - 1$ or if q is critical, this is trivial. If $q > \ell - 1$ is not critical, there is a short-exact sequence

$$0 \longrightarrow \mathsf{S}_{n,q_{-1}} \longrightarrow \mathsf{P}_{n,q} \longrightarrow \mathsf{S}_{n,q} \longrightarrow 0,$$

which gives the short-exact sequence of TL_{n-m}-modules

$$0 \longrightarrow \mathsf{S}_{n-2m,q_{-1}} \longrightarrow \mathsf{P}_{n,q} \div_f \mathsf{P}_{2m,0} \longrightarrow \mathsf{S}_{n-2m,q} \longrightarrow 0,$$

by using proposition 7.C.4. Since

$$\mathrm{Hom}\big(\mathsf{S}_{n-2m,q}, \mathsf{P}_{n,q} \div_f \mathsf{P}_{2m,0}\big) \simeq \mathrm{Hom}\big(\mathsf{S}_{n,q}, \mathsf{P}_{n,q}\big) \simeq \mathbb{C},$$

the only morphism from $\mathsf{S}_{n-2m,q}$ to $\mathsf{P}_{n,q} \div_f \mathsf{P}_{2m,0}$ must be the one which goes through $\mathsf{S}_{n-2m,q_{-1}}$, and thus this sequence does not split. Comparing this sequence with the definition of $\mathsf{P}_{n-2m,q}$ gives the conclusion. ■

Note that a consequence of this is that $\mathsf{P}_{n,q} \div_f \mathsf{P}_{2m,0} \simeq \mathsf{S}_{n-2m,q_{-1}}$ if $q_{-1} \leq n < q$, and $\mathsf{P}_{n,q} \div_f \mathsf{P}_{2m,0} \simeq 0$ if $q_{-1} > n$.

Proposition 7.C.11. *If U is a dTL_{n+m}-module, V a TL_m-module, both with well-defined parity, then $U \div_f V$ is even if they are both of the same parity and odd otherwise.*

Proof. It was argued in a comment preceding proposition 7.3.4 that for W, V, two modules with well-defined parities, $W \times_f V$ is even if they are both of the same parity and odd otherwise. In particular, take $W = \mathsf{edTL}_n$, the even ideal of dTL_n. Then $W \times_f V$ is even (odd), if and only if V is even (odd). But, by definition,

$$\mathrm{Hom}\big(W \times_f V, U\big) \simeq \mathrm{Hom}\big(W, U \div_f V\big).$$

The right side of this equality is non-zero if and only if $U \div_f V$ is even, while the left side vanishes unless U is of the same parity as $W \times_f V$. It follows that $U \div_f V$ is even if and only if U is of the same parity as V. ■

Proposition 7.C.12. *Unless $\ell = 2$ in the regular family, for all $0 \leq i \leq n$ and $0 \leq j \leq m$,*

$$\mathsf{P}_{m+2n,j} \div_f \mathsf{P}_{n,i} \simeq \mathsf{P}_{m,j} \times_f \mathsf{P}_{n,i}, \tag{7.C.26}$$

$$\mathsf{S}_{m+2n,j} \div_f \mathsf{P}_{n,i} \simeq \mathsf{S}_{m,j} \times_f \mathsf{P}_{n,i}. \tag{7.C.27}$$

If $\ell = 2$ in the regular family, the statement is still true for $i = n = 1$.

Proof. We do the proof for the first equality as that of the second is identical. Using the restriction of $\mathsf{P}_{m+2n,j}$ with the preceding proposition gives the conclusion for $i = 1$ in both families, and $i = 0$ in the dilute family. The case $i = 0$ in the regular family is contained in corollary 7.C.10. We thus proceed by induction on i. If the result stands for i, then

$$\big(\mathsf{P}_{m+2n,j} \div_f \mathsf{P}_{n-1,i}\big) \div_f \mathsf{P}_{1,1} \overset{1}{\simeq} \mathsf{P}_{m+2n,j} \div_f \big(\mathsf{P}_{n-1,i} \times_f \mathsf{P}_{1,1}\big)$$

$$\overset{2}{\simeq} \mathsf{P}_{m+2n,j} \div_f (\mathsf{P}_{n,i-1} \oplus \mathsf{P}_{n,i+1}),$$

and

$$\begin{aligned}(\mathsf{P}_{m+2n,j} \div_f \mathsf{P}_{n-1,i}) \div_f \mathsf{P}_{1,1} &\overset{3}{\simeq} (\mathsf{P}_{m+2,j} \times_f \mathsf{P}_{n-1,i}) \div_f \mathsf{P}_{1,1} \\ &\overset{4}{\simeq} \bigoplus_{\lambda\in\Lambda} \mathsf{P}_{m+n+1,\lambda} \div_f \mathsf{P}_{1,1} \\ &\overset{5}{\simeq} \bigoplus_{\lambda\in\Lambda} \mathsf{P}_{m+n-1,\lambda} \times_f \mathsf{P}_{1,1} \\ &\overset{6}{\simeq} (\mathsf{P}_{m,j} \times_f \mathsf{P}_{n-1,i}) \times_f \mathsf{P}_{1,1} \\ &\overset{7}{\simeq} \mathsf{P}_{m,j} \times_f (\mathsf{P}_{n,i-1} \oplus \mathsf{P}_{n,i+1}),\end{aligned}$$

where we assumed, for simplicity, that $i, i\pm 1$ were not critical. These cases are simple generalizations of the same arguments. The isomorphism 1 is simply proposition 7.C.3, while 2 is proposition 7.4.5. The isomorphism 3 is obtained by applying $-\div_f \mathsf{P}_{1,1}$ on the right side of (7.C.26), and 4 is obtained by applying proposition 7.4.5, where all the index appearing in the projective modules were grouped in the family Λ. Noting that $\lambda \leq j+i \leq m+n$, for all $\lambda \in \Lambda$, proposition 7.C.12 with $i=1$ can be used, obtaining 5. Finally, use again proposition 7.4.5 to obtain 6, and use the associativity of the fusion product with, again proposition 7.4.5, to obtain 7. Comparing 2 and 7 and using the induction hypothesis gives the conclusion. ■

What happens when we take a quotient of the form $\mathsf{P}_{n+2m,j} \div_f \mathsf{P}_{m,i}$, but $j>n$? It can be seen that

$$\begin{aligned}\mathsf{P}_{n+2m,j} \div_f \mathsf{P}_{m,i} &\simeq \big(\mathsf{P}_{n+2m+(j-n),j} \div_f \mathsf{P}_{j-n,0}\big) \div_f \mathsf{P}_{m,i} \\ &\simeq (\mathsf{P}_{j+2m,j} \div_f \mathsf{P}_{m,i}) \div_f \mathsf{P}_{(j-n),0} \simeq (\mathsf{P}_{j,j} \times_f \mathsf{P}_{m,i}) \div_f \mathsf{P}_{j-n,0}, \end{aligned} \tag{7.C.28}$$

where we simply used propositions 7.C.3 and 7.C.12. There is thus the following “recipe": start by computing $\mathsf{P}_{n,j} \times_f \mathsf{P}_{m,i}$, by applying proposition 7.4.5, ignoring the fact that $\mathsf{P}_{n,j}$ is not well-defined. Then, use the fact that, by definition

$$\mathsf{P}_{n,k} \equiv \begin{cases} \mathsf{S}_{n,k_-}, & \text{when } k_- \leq n < j \\ 0, & \text{when } k_- > n \end{cases}.$$

For instance, in $\ell = 5$,

$$\mathsf{P}_{10,9} \div_f \mathsf{P}_{4,4} \simeq \underbrace{\mathsf{P}_{6,9}}_{0} \oplus \underbrace{\mathsf{P}_{6,11}}_{0} \oplus \underbrace{\mathsf{P}_{6,13}}_{\mathsf{S}_{6,5}} \simeq \mathsf{S}_{6,5}.$$

More complex fusion quotients could be computed by using arguments similar to those we used to compute fusion products. However, the focus of this paper is on the fusion product, we only give one fairly simple case to show that the two operations are distinct.

Proposition 7.C.13. *For $n \geq \ell$,*

$$\mathsf{P}_{m+2n,j} \div_f \mathsf{S}_{n,\ell} \simeq \begin{cases} \mathsf{P}_{m,j-1} \times_f \mathsf{P}_{n,\ell-1} \oplus \mathsf{T}^2_{m+n,\ell-2-j}, & 0 \leq j < \ell - 1 \\ \mathsf{P}_{m,j} \times_f \mathsf{S}_{n,\ell}, & \textit{otherwise} \end{cases}. \tag{7.C.29}$$

Quatrième partie

Tressage dans les catégories de Temperley-Lieb

Chapitre 8

Présentation

8.1 Objectifs et méthodologie

Graham et Lehrer ont introduit les catégories de Temperley-Lieb [27] : les objets de ces catégories sont les entiers non négatifs, et les morphismes entre les entiers n et m sont définis comme étant les (n,m)-diagrammes. L'algèbre $\mathsf{TL}_n(\beta)$ peut alors être identifiée à l'anneau d'endomorphismes de n, où le paramètre q n'intervient que dans la règle de composition des morphismes. Dans cet article, nous utilisons la convention que $\beta = -q - q^{-1}$. La définition du produit de fusion sur les algèbres de Temperley-Lieb permet d'enrichir ces catégories d'une structure monoïdale, et la commutativité de la fusion d'une structure tressée. Ces structures pourraient être importantes, car les catégories tressées peuvent être utilisées pour construire des solutions aux équations de Yang-Baxter, et donc de construire des modèles intégrables.

Notre but est ici de définir et d'explorer la structure tressée des catégories de Temperley-Lieb introduites par Graham et Lehrer, ainsi que sa généralisation naturelle à leur version diluée. Nous commençons alors par introduire la structure monoïdale, puis nous calculons les commuteurs qui engendrent la structure tressée. Avec ces résultats en main, on introduit une méthode pour induire une structure tressée sur la catégorie des modules sur ces catégories. Cette structure permet de prouver la commutativité du produit de fusion en construisant explicitement un isomorphisme entre les produits $M \times_f N$ et $N \times_f M$ de deux modules. La commutativité du produit avait été obtenue auparavant dans [9], mais sans construire l'isomorphisme.

Elle permet également d'introduire une série d'endomorphismes non triviaux des produits de fusions. Ces derniers ont en général une structure très complexe ; ils admettent de nombreux blocs de Jordan de dimension élevée et permettent dans certains cas de briser explicitement un produit de fusion en somme directe.

Finalement, les commuteurs permettent de construire des solutions aux équations de Yang-Baxter spectrales pour les algèbres TL_n et dTL_n. Dans le cas régulier, le résultat coïncide avec la solution utilisée pour construire la matrice de transfert sur Temperley-Lieb (voir section 1.4), mais la solution obtenue dans le cas dilué est distincte de celle obtenue dans [37, 56].

Contributions

J'ai obtenu l'expression des commuteurs pour les quatre catégories considérés et proposé la définition de la catégorie disjointe. J'ai également proposé la construction des matrices double-ligne à partir des commuteurs. La rédaction est faite à parts égales.

8.2 Outils algébriques

Dans cette section, R est un anneau commutatif quelconque. Ces définitions sont tirées de [65].

Catégories monoidales

Une petite catégorie $\mathfrak{C}$ est dite *monoïdale*, ou *tensorielle*, si elle est munie d'un bifoncteur $-\otimes_{\mathfrak{C}}- : \mathfrak{C}\times\mathfrak{C}\to\mathfrak{C}$, le *produit tensoriel*, d'un objet I appelé *identité*, ainsi que de trois isomorphismes naturels :

$$\alpha_{U,V,W} : (U\otimes_{\mathfrak{C}} V)\otimes_{\mathfrak{C}} \to U\otimes_{\mathfrak{C}}(V\otimes_{\mathfrak{C}}), \qquad \lambda_U : \mathsf{I}\otimes_{\mathfrak{C}} U\to U, \qquad \rho_U : U\otimes_{\mathfrak{C}}\mathsf{I}\to U,$$

appelée respectivement l'*associateur*, l'*identiteur à gauche*, et l'*identiteur à droite*. Ceux-ci doivent de plus respecter trois identités qui s'expriment par la commutativité des diagrammes suivants :

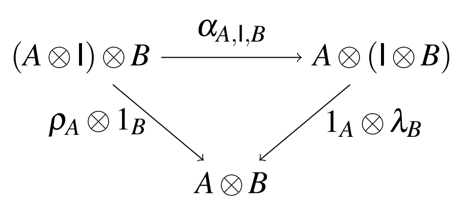


FIGURE 8.2.1 – Le diagramme du triangle

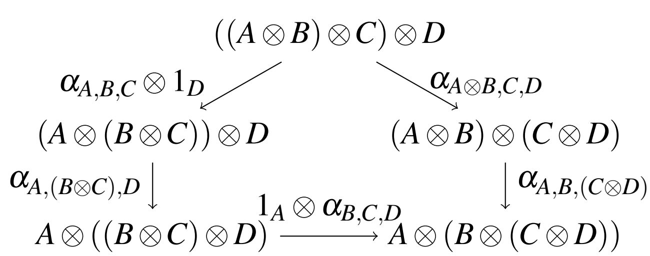


FIGURE 8.2.2 – Le diagramme du pentagone

Voici quelques exemples simples de catégorie monoïdale. Soit $\mathfrak{R}$ la catégorie dont les objets sont les nombres réels, et $\mathsf{Hom}_{\mathfrak{R}}(m,n) = \{(n-m)\}$ avec la règle de composition $(k-n)\circ(n-m) = (k-m)$ et le morphisme identité est (0). Le foncteur produit tensoriel est défini sur les objets par $n \otimes_{\mathfrak{N}} m = n+m$, et sur les morphismes par $(r-n) \otimes_{\mathfrak{N}} (t-m) = (r+t-n-m)$. L'objet identité est 0, l'associateur et les deux identiteurs sont tous le foncteur identité. Une catégorie où tous les isomorphismes naturels sont l'identité est dite *stricte*.

Soit maintenant mod$R$, la catégorie dont les objets sont les R-modules et $\mathsf{Hom}_{\mathrm{mod}R}(U,V)$ est l'espace vectoriel engendré par les morphismes de R-modules avec les règles de composition naturelles (Puisque R est commutatif, les modules sont tous naturellement des modules à gauche et à droite.). Le foncteur produit tensoriel est défini sur les objets par $U \otimes_{\mathrm{mod}R} V = U \otimes_R V$ et de façon naturelle sur les morphismes, alors que l'identité est R vu comme un module sur lui-même. L'associateur est défini par les morphismes $\alpha_{U,V,W}((u \otimes_R v) \otimes_R w) = u \otimes_R (v \otimes_R w)$, et les identiteurs sont définis par les morphismes $\lambda_U(r \otimes_R u) = ru$, $\rho_U(u \otimes_R r) = ru$, où $u \in U, v \in V, w \in W, r \in R$. On vérifie aisément que ces définitions respectent les identités requises.

Catégories tressées

Soit $\mathfrak{C}$ une catégorie monoïdale avec ses isomorphismes naturels (α, λ, ρ). Le produit tensoriel opposé $- \otimes_{\mathfrak{C}}^{\mathrm{op}} -$, est un bifoncteur défini par $U \otimes_{\mathfrak{C}}^{\mathrm{op}} V = V \otimes_{\mathfrak{C}} U$ sur les objets et de façon similaire sur les morphismes. On dit que la catégorie $\mathfrak{C}$ est *tressée* si il existe un isomorphisme naturel η entre le produit tensoriel ordinaire et le produit tensoriel opposé et si celui-ci, appelé *commuteur*, satisfait deux identités qui s'expriment par la commutativité des diagrammes 8.2.3 et 8.2.4

$$\begin{array}{ccccc}
 & A \otimes (B \otimes C) & \xrightarrow{\eta_{A,B\otimes C}} & (B \otimes C) \otimes A & \\
\alpha_{A,B,C} \nearrow & & & & \searrow \alpha_{B,C,A} \\
(A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\
\eta_{A,B} \otimes 1_C \searrow & & & & \nearrow 1_B \otimes \eta_{A,C} \\
 & (B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) &
\end{array}$$

FIGURE 8.2.3 – Le premier diagramme de l'hexagone.

Par exemple, la catégorie $\mathfrak{N}$ utilisée dans la section précédente est évidemment tressée puisque pour les objets $n \otimes_{\mathfrak{N}} m = n+m = m \otimes_{\mathfrak{N}}^{\mathrm{op}} n$ et pour les morphismes $(r-n) \otimes_{\mathfrak{N}} (s-m) = (r+s-n-m) = (s-m) \otimes_{\mathfrak{N}}^{\mathrm{op}} (r-n)$. Une catégorie où le commuteur est l'identité est dite *symétrique*.

Le nom *tressé* est tiré de la construction suivante. On choisi un objet quelconque $U \in \mathfrak{C}$ et on construit sa n-ième puissance tensorielle $U_n = \underbrace{U \otimes U \otimes \ldots \otimes U}_{n \text{ copies}}$. On suppose que la catégorie est stricte afin d'éviter de poser les parenthèses et on a supprimé l'étiquette sur les produits tensoriels pour alléger la notation. Les

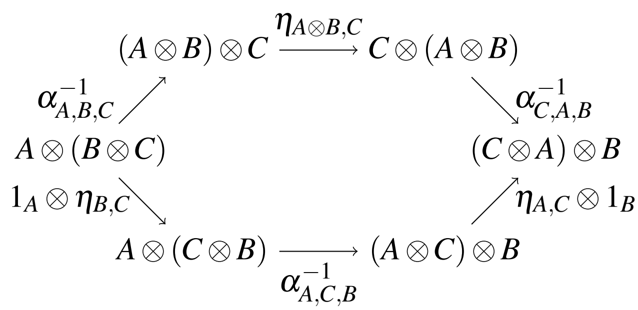


FIGURE 8.2.4 – Le second diagramme de l'hexagone.

commuteurs sont alors utilisés pour construire les isomorphismes

$$t_i = \underbrace{\mathrm{id}_U \otimes \mathrm{id}_U \otimes \ldots \otimes \mathrm{id}_U}_{(i-1)\text{ copies}} \otimes \eta_{U,U} \otimes \underbrace{\mathrm{id}_U \otimes \ldots \mathrm{id}_U}_{n-i-1\text{ copies}},$$

pour tout entier $1 \leq i < n$. On peut alors montrer que les objets t_i engendrent une représentation de l'algèbre de groupe de T_n, le groupe des tresses à n brins, qui est défini par les relations

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad \text{pour tout } 1 \leq i < n-2, \tag{8.2.1}$$

$$t_i t_j = t_j t_i, \qquad |i-j| \geq 2. \tag{8.2.2}$$

La première relation est souvent appelée *équation de Yang-Baxter*. On remarque que le groupe de permutation est obtenu en rajoutant la relation $t_i t_i = \mathrm{id}_{U_n}$. Cette construction peut être répétée en prenant n objets dans $\mathfrak{C}$, et on peut ainsi construire de nombreuses représentations du groupe de tresses et donc autant de solutions aux équations de Yang-Baxter.

Il est toutefois important de remarquer que l'équation (8.2.1) n'est qu'un cas particulier de l'équation de Yang-Baxter spectrale utilisée par les physiciens :

$$X_i(u) X_{i+1}(v) X_i(u/v) = X_{i+1}(u/v) X_i(v) X_{i+1}(u), \tag{8.2.3}$$

où u, v sont des éléments centraux inversibles de $\mathrm{End}\, U_n$, et $X_i(-)$ est une application (pas nécessairement linéaire) de l'ensemble des éléments centraux inversibles [1] de $\mathrm{End}\, U_n$ vers $\mathrm{End}\, U_n$. Bien qu'il soit toujours possible de construire une solution de l'équation de Yang-Baxter spectrale à partir d'une solution de l'équation (8.2.1), en prenant par exemple $X_i(u) = u t_i$, des solutions non triviales, quand elles existent, peuvent être très difficiles à trouver. Un ansatz simple est de prendre $X_i(u) = u t_i^{-1} - u^{-1} t_i$, mais ceci ne fonctionne pas toujours ; nous verrons par contre qu'une solution de (8.2.3) pour l'algèbre de Temperley-Lieb peut être mise sous cette forme.

1. Si u n'est pas inversible alors évidemment $X_i(v/u)$ n'a pas de sens. De même si u^{-1},v ne sont pas centraux, il faudrait distinguer $X_i(v(u)^{-1})$ et $X_i((u)^{-1}v)$; il est donc plus simple de demander que l'argument de $X_i(-)$ soit central.

Chapitre 9

Temperley-Lieb algebras as a braided category

Temperley-Lieb algebras as a braided category

Jonathan Belletête and Yvan Saint-Aubin

ABSTRACT:Graham and Lehrer (1998) introduced a Temperley-Lieb category $\widetilde{\mathsf{TL}}$ whose objects are the non-negative integers and the morphisms in $\mathsf{Hom}(n,m)$ are the link diagrams from n to m nodes. The Temperley-Lieb algebra TL_n is identified with $\mathsf{Hom}(n,n)$. The category $\widetilde{\mathsf{TL}}$ is monoidal. We show that it is also a braided category by constructing explicitly a commutor. We introduce a disjoint module category $\mathcal{M}(\widetilde{\mathsf{TL}})$ whose objects are modules of one of the algebras TL_n, define on it the fusion bifunctor, extending the one introduced by Read and Saleur (2007), and show that the braiding on $\widetilde{\mathsf{TL}}$ induces a natural braiding that makes $\mathcal{M}(\widetilde{\mathsf{TL}})$ into a braided category. We discuss how the braiding on $\widetilde{\mathsf{TL}}$ and integrability of statistical models are related. The extension of these structures to the family of dilute Temperley-Lieb algebras is also discussed.

Keywords Temperley-Lieb algebra, dilute Temperley-Lieb algebra, category of Temperley-Lieb algebra, monoidal category, braided category, fusion ring, fusion product.

9.1 Introduction

The goal of this paper is to introduce a natural braiding of the fusion product of modules over the Temperley-Lieb algebras. The existence of such a braiding has emerged as one of the many conditions that should satisfy fusion in conformal field theories (CFTs) and thus in their finite counterparts, for example statistical models on a finite two-dimensional lattices. (See, for example, the review of Huang and Lepowsky [35] on the construction of braided tensor categories and modular ones for CFTs or, more precisely, vertex operator algebras.)

Statistical models in two dimensions are defined by an evolution operator or a transfer matrix acting on finite-dimensional vector spaces. The sizes of both the lattice and the vector spaces are parameters of the formulation. The limit when these parameters go to infinity is known in some cases (numerically or rigorously) to be a conformal field theory. For several XXZ and loop models [58, 60], the Hamiltonian is first defined as an element of a Temperley-Lieb algebra TL_n, or one of its generalizations, and the actual linear operator is obtained as the representative of this element in some representations over the algebra. The fusion product is an algebraic construction, actually a bifunctor, that associates to two modules M and N over TL_m and TL_n respectively a module over TL_{m+n}. For the Temperley-Lieb algebra, such a fusion product $-_1 \times_f -_2$ was introduced by Read and Saleur [61] and computed in many cases by Gainutdinov and Vasseur [24] and Belletête [9]. It is associative and commutative, and the braiding gives the isomorphism between $\mathsf{M} \times_f \mathsf{N}$ and $\mathsf{N} \times_f \mathsf{M}$.

There are reasons to believe that algebraic information obtained from the finite algebras, either the Temperley-Lieb family, its dilute counterpart or any other one, is intimately related to analogous structures of the CFTs and should help understand them. First there is compelling evidence that, in the limit when the size of the lattice goes to infinity, the spectrum of the Hamiltonian, properly scaled, reproduces characters of the Virasoro algebra. Second, in some representations of the TL family, the Hamiltonian has Jordan blocks (of size 2×2) [53,60], indicating a possible link to logarithmic CFTs. Third, when restricted to TL-modules that are known to give rise to the Virasoro modules appearing in minimal CFTs, the highly non-trivial fusion product defined between TL modules does reproduce the simple fusion rules of these minimal CFTs.

The (original) family of the Temperley-Lieb algebras was cast into a categorical framework by Graham and Lehrer [27] while they were actually studying another family, the periodic (or affine) Temperley-Lieb algebras. The construction brings together all algebras $\mathsf{TL}_n(\beta), n\geq 0$, in the same category $\widetilde{\mathsf{TL}}(\beta)$. Their formulation will be the starting point of section 9.2 where the requirements for a category to be monoidal and then braided will be fulfilled for the TL family. This section also shows that the braiding defined on $\widetilde{\mathsf{TL}}$ induces naturally a braiding on the category of modules over the TL algebras. Section 9.3 then shows how the integrability of two-dimensional statistical models and the components of the commutor $\eta_{r,s}$ defining the braiding are related. Section 9.4 extends the results to the family of dilute Temperley-Lieb algebras dTL_n. A short conclusion follows.

9.2 The Temperley-Lieb category

Graham and Lehrer [27] showed that the algebras $\mathsf{TL}_n(\beta)$, $n\geq 0$, can be studied as a whole and given the structure of a category $\widetilde{\mathsf{TL}}$. The goal of this section is to recall the definitions of monoidal and braided categories and show that the Temperley-Lieb category $\widetilde{\mathsf{TL}}$ is braided.

9.2.1 $\widetilde{\mathsf{TL}}(\beta)$ as a monoidal category

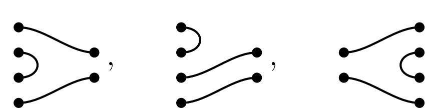
The first step is to cast the family of algebras $\mathsf{TL}_n, n\geq 0$, into a category and show that the additional requirements of a monoidal category are easily fulfilled.

We take the convention that morphisms and functors acts on the left, *so that* $(FG)(X)\equiv F(G(X))$. *We will also assume that the categories appearing in this section are* small.

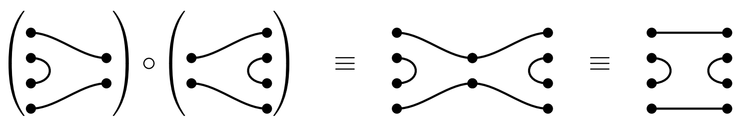
The *Temperley-Lieb category* $\widetilde{\mathsf{TL}}$ is defined as follows. The objects of the category are the non-negative integers:

$$\mathrm{Ob}\,\widetilde{\mathsf{TL}}=\mathbb{N}^0=\{0,1,2,\ldots\}.$$

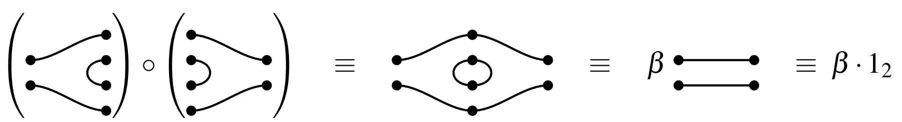
The sets of morphisms $\mathsf{Hom}(n,m)$ from n to m is empty if n and m do not have the same parity and, if they do, are defined as the sets of formal $\mathbb{C}$-linear combinations of (m,n)-diagrams. A (m,n)-diagram $\alpha\in\mathsf{Hom}(n,m)$ is composed of two vertical columns of m nodes on the left, and n nodes on the right, linked pairwise by non-intersecting strings. For instance, here are a pair of $(2,4)$-diagrams and one $(4,2)$-diagram:



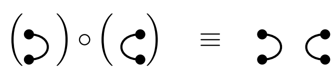
Note that the third diagram can be obtained from the first by reflection through a vertical line midway between the two columns of nodes. We will call the result of this reflection the *transpose* of the diagram. The identity morphism $1_n \in \mathsf{Hom}(n,n)$ is the (n,n)-diagram where every point on the left is linked to the one at the same height on the right. (The identity morphism $1_0 \in \mathsf{Hom}(0,0)$ exists (by definition), but it represented graphically by an empty space.) Compositions of morphisms are defined by linearly expanding the composition rules for diagrams. For an (m,n)-diagram b and a (k,m)-diagram c, the composition $c \circ b$ is a (k,n)-diagram defined by first putting c on the left of b, identifying the m points on the neighboring sites, joining the strings that meets there, and then removing these m nodes. If there is a string no longer attached to any points, it is removed and replaced by a factor $\beta \in \mathbb{C}$. Here is an example of the composition of $(2,4)$- and a $(4,2)$-diagrams:



and of the same diagrams in the other order:



The associativity of the composition of diagrams is easily verified. The depiction of 1_0 by a simple space is consistent with the depiction of the composition by concatenation of diagrams. For example the following product of the $(2,0)$-diagram d and $(0,2)$-diagram e



could equally be understood as $d \circ 1_0 \circ e$. Note finally that $\mathrm{End}(n) \equiv \mathsf{TL}_n(\beta)$ is the usual Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$. The category $\widetilde{\mathsf{TL}}$ can be easily enriched to become a monoidal one.

A category $\mathcal{C}$ is said to be *monoidal* if it is equipped with the following structures [19, 65]:

(i) A bi-functor $- \otimes - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the *tensor product*;

(ii) An object $\mathsf{I} \in \mathrm{Ob}(\mathcal{C})$ called the *identity*;

(iii) Three natural isomorphisms:

- $\alpha : (-_1 \otimes -_2) \otimes -_3 \longrightarrow -_1 \otimes (-_2 \otimes -_3)$, called the *associator*.
- $\lambda : \mathsf{I} \otimes - \longrightarrow -$, the *left unitor*.
- $\rho : - \otimes \mathsf{I} \longrightarrow -$, the *right unitor*.

Moreover these structures have to satisfy the *triangle* and the *pentagon axioms*. These axioms require that the diagrams in Figures 9.2.1 and 9.2.2 commute for all $A, B, C, D \in \mathrm{Ob}\,\mathcal{C}$. Finally, if the associator, the left and right unitors are all identity isomorphisms, the category is said to be *strict*.

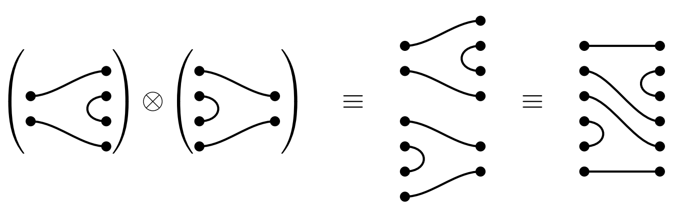
$$\begin{array}{ccc} (A\otimes \mathsf{I})\otimes B & \xrightarrow{\alpha_{A,\mathsf{I},B}} & A\otimes(\mathsf{I}\otimes B)\\ & \rho_A\otimes 1_B \searrow \qquad \swarrow 1_A\otimes\lambda_B & \\ & A\otimes B & \end{array}$$

Figure 9.2.1 – The triangle diagram

$$\begin{array}{ccc} & ((A\otimes B)\otimes C)\otimes D & \\ \alpha_{A,B,C}\otimes 1_D \swarrow & & \searrow \alpha_{A\otimes B,C,D} \\ (A\otimes(B\otimes C))\otimes D & & (A\otimes B)\otimes(C\otimes D) \\ \alpha_{A,(B\otimes C),D}\downarrow & & \downarrow \alpha_{A,B,(C\otimes D)} \\ A\otimes((B\otimes C)\otimes D) & \xrightarrow{1_A\otimes\alpha_{B,C,D}} & A\otimes(B\otimes(C\otimes D)) \end{array}$$

Figure 9.2.2 – The pentagon diagram

Définition 9.2.1. *Let $\mathcal{C}$ be the Temperley-Lieb category $\widetilde{\mathsf{TL}}$. Define the bi-functor $-\otimes-$ in the following way. For objects $n,m \in \mathsf{Ob}\,\widetilde{\mathsf{TL}}$, simply set $n\otimes m \equiv n+m$ where "$+$" stands for the addition in $\mathbb{N}^0$ and, thus, the identity object is $\mathsf{I}=0\in \mathsf{Ob}\,\widetilde{\mathsf{TL}}$. For a (k,n)-diagram b and a (t,m)-diagram c, the $(k+t,n+m)$-diagram $b\otimes c$ is obtained by simply drawing b on top of c. For example, taking b,c as in the previous example gives*



This is then expanded bilinearly to all morphisms. The associator $\alpha_{m,n,k}$ is the isomorphism $(m+n)+k \mapsto m+(n+k)$ and the unitors are $0+m\mapsto m$ and $m+0\mapsto m$ respectively.

Since $(\mathbb{N}^0,+)$ is a monoid, the axioms are trivially verified for the objects. It is easy to verify that the axioms also hold for the morphisms and, thus, $\widetilde{\mathsf{TL}}$ is a strict monoidal category.

9.2.2 $\widetilde{\mathsf{TL}}(\beta)$ as a braided category

Let $\mathcal{C}$ be a monoidal category and let the *opposite tensor product* between two objects $A,B\in \mathsf{Ob}\,\mathcal{C}$ be defined as $A\otimes^{\mathrm{op}} B \equiv B\otimes A$. The category $\mathcal{C}$ is *braided* if there is a natural isomorphism $\eta : -\otimes- \to -\otimes^{\mathrm{op}}-$ such that the two *hexagon* diagrams in Figures 9.2.3 and 9.2.4 commute for all $A,B,C\in \mathsf{Ob}\,\mathcal{C}$. We recall that, for η to be a natural isomorphism, there must exist, for all objects $A,B\in\mathsf{Ob}\,\mathcal{C}$, an isomorphism $\eta_{A,B} : A\otimes B \longrightarrow B\otimes A$ such that for all morphisms $f\in \mathsf{Hom}_{\mathcal{C}}(A,A'), g\in \mathsf{Hom}_{\mathcal{C}}(B,B')$, $\eta_{A',B'}\circ(f\otimes g) = (g\otimes f)\circ\eta_{A,B}$.

If, for all $A, B \in \mathrm{Ob}\, \mathcal{C}$, $\eta_{A,B} \circ \eta_{B,A} = 1_{B \otimes A}$, the category is said to be *symmetric*.

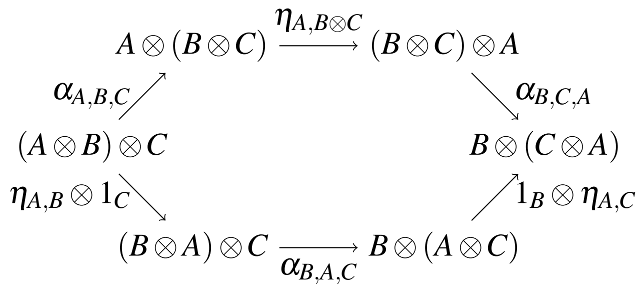


Figure 9.2.3 – The first hexagon diagram

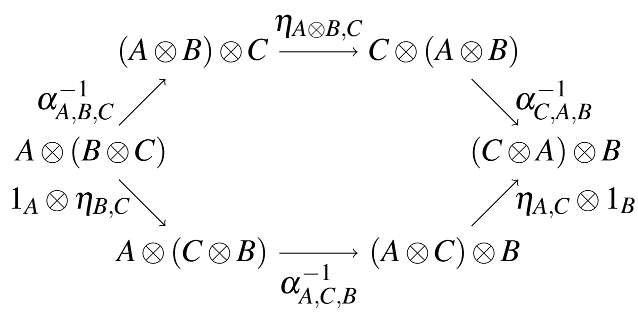


Figure 9.2.4 – The second hexagon diagram

In a strict braided category, the hexagon diagrams are equivalent to the two following identities:

$$\eta_{A,B\otimes C} = (1_B \otimes \eta_{A,C}) \circ (\eta_{A,B} \otimes 1_C), \tag{9.2.1}$$

$$\eta_{B\otimes C,A} = (\eta_{B,A} \otimes 1_C) \circ (1_B \otimes \eta_{C,A}). \tag{9.2.2}$$

To endow $\widetilde{\mathsf{TL}}$ with a braiding requires more work than to define its monoidal structure. We start by outlining the strategy. Since $\widetilde{\mathsf{TL}}$ is strict, the hexagon diagrams are equivalent to

$$\eta_{n,m+k} = (1_m \otimes \eta_{n,k}) \circ (\eta_{n,m} \otimes 1_k), \tag{9.2.3}$$

$$\eta_{u+v,w} = (\eta_{u,w} \otimes 1_v) \circ (1_u \otimes \eta_{v,w}). \tag{9.2.4}$$

It follows that, if we can find $\eta_{1,1}$, the other $\eta_{n,m}$, $n, m \geq 1$, will be uniquely defined by these two conditions, provided that they are consistent, that is, if $\eta_{n,m}$ satisfy the above two conditions, then so does $\eta_{n+1,m} \equiv (\eta_{n,m} \otimes 1_1) \circ (1_n \otimes \eta_{1,m})$, for instance. Proposition 9.2.2 will establish this consistency. We shall then build the $\eta_{n,m}$ recursively. It will then remain to prove that these $\eta_{m,n}$ define natural isomorphisms. This will require several steps: Lemma 9.2.3 will express the morphisms $\eta_{r,s}$ in terms of $\eta_{1,1}$ only and a short computation will express $\eta_{1,1}$ in terms of the generators e_i of Temperley-Lieb algebras. Lemmas 9.2.4 to 9.2.6 show how the $\eta_{r,s}$ braid with the e_i and some diagrams in $\mathsf{Hom}(n,0)$ and $\mathsf{Hom}(0,n)$. Then Proposition 9.2.7 proves that the $\eta_{r,s}$ form together a commutor for the category $\widetilde{\mathsf{TL}}$.

The hexagon axioms fix the isomorphisms $\eta_{n,0}$ and $\eta_{0,w}$. Indeed, when all integers are set to 0, (9.2.3) gives $\eta_{0,0} = (1_0 \otimes \eta_{0,0}) \circ (\eta_{0,0} \otimes 1_0)$ and thus $\eta_{0,0} = 1_0$. Similarly the same equation for $\eta_{n,0+1}$ leads to $\eta_{n,0} = 1_n$. Hence $\eta_{n,0} = \eta_{0,n} = 1_n$ for all $n \geq 0$.

Proposition 9.2.2. *If the morphisms $\{\eta_{i,j}\}_{i\in\mathbb{N}^0_{\leq r}, j\in\mathbb{N}^0_{\leq s}}$ satisfy equations (9.2.3) and (9.2.4) for all $0 \leq n, u+v \leq r$ and $0 \leq m+k, w \leq s$, then so do $\eta_{r+1,s}$ and $\eta_{r,s+1}$ defined as*

$$\eta_{r+1,s} \equiv (\eta_{r,s} \otimes 1_1) \circ (1_r \otimes \eta_{1,s}) \qquad \text{and} \qquad \eta_{r,1+s} \equiv (1_1 \otimes \eta_{r,s}) \circ (\eta_{r,1} \otimes 1_s). \tag{9.2.5}$$

Proof. We only give the proof for $\eta_{r+1,s}$ and equation (9.2.4), as the other checks are similar. Suppose that $n+m = r+1$ with $0 \leq n \leq r$ and $1 \leq m \leq s$. The steps are the following:

$$\begin{aligned}
(\eta_{n,s} \otimes 1_m) \circ (1_n \otimes \eta_{m,s}) &\overset{1}{=} (\eta_{n,s} \otimes 1_m) \circ (1_n \otimes \eta_{m-1,s} \otimes 1_1) \circ (1_n \otimes 1_{m-1} \otimes \eta_{1,s}) \\
&\overset{2}{=} (\eta_{n,s} \otimes 1_{m-1} \otimes 1_1) \circ (1_n \otimes \eta_{m-1,s} \otimes 1_1) \circ (1_{n+m-1=r} \otimes \eta_{1,s}) \\
&\overset{3}{=} (((\eta_{n,s} \otimes 1_{m-1}) \circ (1_n \otimes \eta_{m-1,s})) \otimes 1_1) \circ (1_r \otimes \eta_{1,s}) \\
&\overset{4}{=} (\eta_{r,s} \otimes 1_1) \circ (1_r \otimes \eta_{1,s}) \\
&\overset{5}{=} \eta_{r+1,s}.
\end{aligned}$$

Steps 1 and 4 are obtained by using the fact that $\eta_{m,s}$ and $\eta_{r,s}$ satisfy the hexagon identity (9.2.4). Steps 2 and 3 use the property $1_i \otimes 1_j = 1_{i+j}$ of identity morphisms that holds for all non-negative integers i, j. Finally step 5 is the proposed definition of $\eta_{r+1,s}$. ■

The next lemma solves the recursive expressions (9.2.5) in terms of the "elementary component" $\eta_{1,1}$ only.

Lemma 9.2.3. *The morphisms $\eta_{r,s}$, with $r, s \geq 1$, satisfy equations (9.2.3) and (9.2.4) if and only if they are given by*

$$\eta_{r,s} = \prod_{i=1}^{s} \Big(\prod_{j=r-1}^{0} t_{i+j}(r+s) \Big) = \prod_{i=r}^{1} \Big(\prod_{j=0}^{s-1} t_{i+j}(r+s) \Big), \tag{9.2.6}$$

where $t_i(n) \equiv 1_{i-1} \otimes \eta_{1,1} \otimes 1_{n-i-1} \in \mathsf{Hom}(n,n)$ and the factors in a product are listed starting from the right, that is, $\prod_{i=1}^{s} t_i = t_s t_{s-1} \ldots t_2 t_1$ and $\prod_{i=s}^{1} \equiv t_1 t_2 \ldots t_{s-1} t_s$.

Proof. The proof of the first part is obtained by induction on r and s. Taking the induction first on r, then on s gives the first expression, while doing the inductions in the reverse order yields the second. The proof of the former is given as example. When $r = s = 1$, the first expression is simply $\eta_{1,1}$ and the statement is trivially true. Assume therefore that the result stands for $\eta_{r,1}$. If $\eta_{r+1,1}$ satisfies equation (9.2.4), then in particular

$$\eta_{r+1,1} = (\eta_{r,1} \otimes 1_1) \circ \underbrace{(1_r \otimes \eta_{1,1})}_{t_{r+1}(r+2)},$$

which is $\eta_{r+1,1}$ as given by the first expression in (9.2.6). Assume then that the result stands for some $r, s \geq 1$. Then (9.2.3) gives

$$\begin{aligned}\eta_{r,s+1} &= (1_s \otimes \eta_{r,1}) \circ (\eta_{r,s} \otimes 1_1) \\ &= \Big(\prod_{j=r-1}^{0} t_{j+s+1}(r+s+1)\Big) \circ \prod_{i=1}^{s} \Big(\prod_{j=r-1}^{0} t_{i+j}(r+s+1)\Big)\end{aligned}$$

which is the expression for $\eta_{r,s+1}$ given in (9.2.6).

The converse can be obtained as follows. The first expression gives

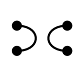
$$\begin{aligned}(1_m \otimes \eta_{n,k}) \circ (\eta_{n,m} \otimes 1_k) &= \Big(\prod_{i=1}^{k} \prod_{j=n-1}^{0} t_{i+j+m}\Big) \circ \Big(\prod_{i=1}^{m} \prod_{j=n-1}^{0} t_{i+j}\Big) \\ &= \Big(\prod_{i=m+1}^{m+k} \prod_{j=n-1}^{0} t_{i+j}\Big) \circ \Big(\prod_{i=1}^{m} \prod_{j=n-1}^{0} t_{i+j}\Big) \\ &= \Big(\prod_{i=1}^{m+k} \prod_{j=n-1}^{0} t_{i+j}\Big) = \eta_{n,m+k}\end{aligned}$$

and, thus, satisfies (9.2.3). The second expression is shown similarly to satify (9.2.4). The proof that the first expression satisfies (9.2.4) is harder and it is then easier, though tedious, to prove that the two expressions are equal. It is done using the identity $t_i t_j = t_j t_i$ for $|i-j| > 1$, that follows from the definition of the $t_i(n)$. Here is an example. The two expressions for $\eta_{2,3}$ are $(t_3t_4)(t_2t_3)(t_1t_2)$ and $(t_3t_2t_1)(t_4t_3t_2)$ and those for $\eta_{2,2}$ are $(t_2t_3)(t_1t_2)$ and $(t_2t_1)(t_3t_2)$. Assuming that the latter are equal, the former are shown to be equal by

$$(t_3t_4)\big((t_2t_3)(t_1t_2)\big) = (t_3t_4)\big((t_2t_1)(t_3t_2)\big) = (t_3t_2t_1)(t_4t_3t_2)$$

where the two expressions for $\eta_{2,2}$ gives the first equatlity while the commutativity of t_4 with t_2 and t_1 gives the second. The argument can be extended into a proof by induction on the sum $r+s$ of the indices of $\eta_{r,s}$. ■

The next step is to find an expression for $\eta_{1,1}$. Since $\eta_{1,1} : 1 \otimes 1 \to 1 \otimes 1$ is an element of $\mathrm{End}(2) \simeq \mathsf{TL}_2(\beta)$, which is two-dimensional, there exists $\alpha, \gamma \in \mathbb{C}$ such that $\eta_{1,1} = \alpha 1_2 + \gamma e_1(2)$, where the notation

$e_i(n) = 1_{i-1} \otimes$  $\otimes 1_{n-(i+1)}$

is used. It can be checked directly from this definition that the e_i satisfy the Temperley-Lieb defining relations:

$$e_i(n)e_i(n) = \beta e_i(n), \qquad e_i(n)e_{i\pm1}(n)e_i(n) = e_i(n), \tag{9.2.7}$$

$$e_i(n)e_j(n) = e_j(n)e_i(n), \qquad \text{if } |i-j| > 1. \tag{9.2.8}$$

In fact, it can be proved that the set $\{e_i(n)\}_{1\le i<n}$ generates $\mathrm{End}(n) = \mathsf{TL}_n(\beta)$. Using these relations, it can be seen that $\eta_{1,1}$ is invertible provided that $\alpha \neq 0$. Now, if the family of $\eta_{r,s}$ is to define a commutor then, in particular, it must verify

$$\eta_{1,2}e_2(3) = e_1(3)\eta_{1,2} \qquad \text{and} \qquad \eta_{1,2}(1_1 \otimes z) = (1_1 \otimes z)\eta_{1,0} \tag{9.2.9}$$

where

$z =$  $\in \mathsf{Hom}_{\widetilde{\mathsf{TL}}}(0,2)$ (9.2.10)

and $\eta_{1,2} = (1_1 \otimes \eta_{1,1}) \circ (\eta_{1,1} \otimes 1_1) = \alpha^2 1_3 + \alpha\gamma(e_1(3)+e_2(3)) + \gamma^2 e_2(3)e_1(3)$. The first equation of (9.2.9) will be satisfied if and only if $\alpha^2 + \beta\alpha\gamma + \gamma^2 = 0$, while the second will be if and only if $\alpha\gamma = 1$. Solving these equations yields

$$\alpha = \pm q^{\pm 1/2}, \qquad \gamma = 1/\alpha,$$

where $q \in \mathbb{C}^\times$ is such that $\beta = -q - q^{-1}$ and the two $\pm$ signs are independent. There are thus four solutions. Note that one of the $\pm$ is responsible for an overal sign on $\eta_{1,1}$ while the remaining one mirrors the invariance of β under $q \mapsto q^{-1}$. Without loss of generalities, we shall concentrate on the following choice:

$$t_i(n) = q^{1/2}(1_n + q^{-1}e_i(n)) \qquad \text{and} \qquad t_i(n)^{-1} = q^{-1/2}(1_n + qe_i(n)) \tag{9.2.11}$$

and $\eta_{1,1} : 1\otimes 1 \to 1 \otimes 1$ is simply $\eta_{1,1} = t_1(2)$. These building blocks $t_i(n)$ of the $\eta_{r,s}$ have appeared numerous times in the literature. The identity (9.2.14) below was recognized by Chow [16] as crucial to identify the center of braid groups. Much later Martin [45] used the t_i (up to a factor) to construct central elements of the Temperley-Lieb algebra.

It now remains to show that this choice does defines a braiding on $\widetilde{\mathsf{TL}}$, but doing so requires a few lemmas. From now on, we shall omit the arguments specifying the Hom-space, unless they are needed to avoid confusion, and assume that these arguments are large enough for the expressions to make sense. For example the next lemma proves that $t_it_{i+1}e_i = e_{i+1}e_i$. The statement stands for $t_i(n)t_{i+1}(n)e_i(n) = e_{i+1}(n)e_i(n)$ for all $i+2 \le n$ as $t_{i+1}(n)$ and $e_{i+1}(n)$ act non-trivially on nodes $i+1$ and $i+2$ of the elements of $\mathsf{Hom}(n,n)$. The next three lemmas prepare the proof that the $\eta_{r,s}$'s are natural isomorphisms and thus define a braiding on $\widetilde{\mathsf{TL}}$. The first is obtained by direct computation.

Lemma 9.2.4. *The morphisms t_i and e_i satisfy*

$$t_it_{i+1}e_i = e_{i+1}e_i = e_{i+1}t_it_{i+1}, \tag{9.2.12}$$

$$t_{i+1}t_ie_{i+1} = e_ie_{i+1} = e_it_{i+1}t_i, \tag{9.2.13}$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}. \tag{9.2.14}$$

The next one is almost as easy.

Lemma 9.2.5. *For all* $1 \le i \le n-1$, $1 \le j \le m-1$,

$$\eta_{n,m} e_i = e_{m+i} \eta_{n,m} \qquad \text{and} \qquad \eta_{n,m} e_{n+j} = e_j \eta_{n,m}. \tag{9.2.15}$$

Thus, for all $f \in \mathrm{End}(n)$ *and* $g \in \mathrm{End}(m)$,

$$\eta_{n,m}(f \otimes g) = (g \otimes f)\eta_{n,m}. \tag{9.2.16}$$

Proof. If $1 \le k \le i$ and thus $k \le i < k+n-1$, the preceding lemma and equation (9.2.8) give

$$t_k t_{k+1} \dots t_{k+n-1} e_i = t_k t_{k+1} \dots \underbrace{t_i t_{i+1} e_i}_{e_{i+1} t_i t_{i+1}} t_{i+2} \dots t_{k+n-1} = e_{i+1} t_k t_{k+1} \dots t_{k+n-1}.$$

It follows that

$$\begin{aligned}
\eta_{n,m} e_i &= \prod_{k=1}^{m} (t_k t_{k+1} \dots t_{k+n-1}) e_i \\
&= \prod_{k=2}^{m} (t_k t_{k+1} \dots t_{k+n-1}) \left(e_{i+1} \prod_{k=1}^{1} (t_k t_{k+1} \dots t_{k+n-1}) \right) \\
&= \prod_{k=3}^{m} (t_k t_{k+1} \dots t_{k+n-1}) \left(e_{i+2} \prod_{k=1}^{2} (t_k t_{k+1} \dots t_{k+n-1}) \right) = \dots \\
&= e_{m+i} \eta_{n,m}.
\end{aligned}$$

The second identity in (9.2.15) is proved similarly using the second expression of (9.2.6). Finally, (9.2.16) follows from the fact that $\mathrm{End}(n) \simeq \mathsf{TL}_n$ is generated by the e_i. ■

Lemma 9.2.6. *For positive integers* p *and* n

$$\eta_{n,2p}(1_n \otimes z^{\otimes p}) = (z^{\otimes p} \otimes 1_n)\eta_{n,0} \quad \text{and} \quad \eta_{0,n}((z^t)^{\otimes p} \otimes 1_n) = (1_n \otimes (z^t)^{\otimes p})\eta_{2p,n} \tag{9.2.17}$$

where $\eta_{0,n} = \eta_{n,0} = 1_n$, z *is defined in equation* (9.2.10), $(z)^t$ *is its transpose, and* $z^{\otimes p} \equiv \underbrace{z \otimes z \otimes \dots \otimes z}_{p \text{ times}}$.

Proof. We prove the first identity only as both proofs are nearly identical. We proceed first by induction on n and then on p. If $p = n = 1$, the equation is the second of the two equations in (9.2.9) that were solved to construct the t_i and obtain (9.2.11). Suppose therefore that the result stands for $p = 1$ and some $n \ge 1$. The hexagon identity (9.2.4) gives

$$\eta_{n+1,2}(1_{n+1} \otimes z) = (\eta_{1,2} \otimes 1_n)(1_1 \otimes \eta_{n,2})(1_1 \otimes 1_n \otimes z)$$

$$\begin{aligned}
&= (\eta_{1,2}\otimes 1_n)(1_1\otimes(\eta_{n,2}(1_n\otimes z)))\\
&= (\eta_{1,2}\otimes 1_n)(1_1\otimes z\otimes 1_n)\\
&= (\eta_{1,2}(1_1\otimes z))\otimes 1_n\\
&= z\otimes 1_{n+1}.
\end{aligned}$$

Assume then that the result stands for some $p \geq 1$ and all $n \geq 1$. The hexagon identity (9.2.3) gives

$$\begin{aligned}
\eta_{n,2p+2}(1_n\otimes z^{\otimes p+1}) &= (1_2\otimes \eta_{n,2p})(\eta_{n,2}\otimes 1_{2p})(1_n\otimes z\otimes z^{\otimes p})\\
&= (1_2\otimes \eta_{n,2p})(\eta_{n,2}(1_n\otimes z)\otimes z^{\otimes p})\\
&= z\otimes(\eta_{n,2p}(1_n\otimes z^{\otimes p}))\\
&= z^{\otimes p+1}\otimes 1_n
\end{aligned}$$

which ends the proof. ■

With these three lemmas, we are now ready to prove the main result of this section.

Proposition 9.2.7. *The category $\widetilde{\mathsf{TL}}$ is braided with a commutor having components*

$$\eta_{r,s} = \prod_{i=1}^{s}\Big(\prod_{j=r-1}^{0} t_{i+j}(r+s)\Big) = \prod_{i=r}^{1}\Big(\prod_{j=0}^{s-1} t_{i+j}(r+s)\Big) \tag{9.2.18}$$

and $t_i(n) = q^{1/2}(1_n + q^{-1}e_i(n))$.

Proof. The category $\widetilde{\mathsf{TL}}$ will be braided if the components $\eta_{r,s}$ are natural isomorphisms satisfying the hexagon axioms. Lemma 9.2.3 has already showed that the proposed expressions for the components $\eta_{r,s}$ satisfy the hexagon axioms. Moreover, since $t_i(n)$ is invertible, so are the morphisms $\eta_{r,s}$. There remains only the naturality condition to prove. It states the following: For all pairs (n,m) and (r,s) in $\mathrm{Ob}\,\widetilde{\mathsf{TL}}\times\widetilde{\mathsf{TL}}$ and all pairs of morphisms $(c,d)\in \mathsf{Hom}(n,r)\times\mathsf{Hom}(m,s)$, the following diagram commutes

$$\begin{array}{ccc}
n\otimes m & \xrightarrow{\;c\otimes d\;} & r\otimes s\\
\Big\downarrow{\scriptstyle \eta_{n,m}} & & \Big\downarrow{\scriptstyle \eta_{r,s}}\\
m\otimes n & \xrightarrow{\;d\otimes c\;} & s\otimes r
\end{array}$$

Since the Hom-spaces are spanned by diagrams and that the $\eta_{r,s}$ are bilinear, it is sufficient to prove that

$$(d\otimes c)\eta_{n,m} = \eta_{r,s}(c\otimes d)$$

for any (n,r)-diagram c and (m,s)-diagram d.

Consider then $c\in\mathsf{Hom}_{\widetilde{\mathsf{TL}}}(n,r)$ a diagram having k through lines, that is, precisely k nodes on the left side of c are connected to k nodes on its right side. Any such diagram can be expressed as

$$c = a(1_k \otimes z^{\otimes \frac{r-k}{2}})(1_k \otimes (z^t)^{\otimes \frac{n-k}{2}})b, \tag{9.2.19}$$

where $a \in \operatorname{End} r$ and $b \in \operatorname{End} n$. The hexagon identities and lemma 9.2.6 give

$$\begin{aligned}
\eta_{r,s}(1_k \otimes z^{\otimes \frac{r-k}{2}} \otimes 1_s) &= (\eta_{k,s} \otimes 1_{r-k})(1_k \otimes \eta_{r-k,s})(1_k \otimes z^{\otimes \frac{r-k}{2}} \otimes 1_s) \\
&= (\eta_{k,s} \otimes 1_{r-k})(1_k \otimes \eta_{r-k,s}(z^{\otimes \frac{r-k}{2}} \otimes 1_s)) \\
&= (\eta_{k,s} \otimes 1_{r-k})(1_k \otimes 1_s \otimes z^{\otimes \frac{r-k}{2}}) \\
&= (1_s \otimes 1_k \otimes z^{\otimes \frac{r-k}{2}})\eta_{k,s}.
\end{aligned}$$

The same arguments also give

$$\eta_{k,s}(1_k \otimes (z^t)^{\otimes \frac{n-k}{2}} \otimes 1_s) = (1_s \otimes 1_k \otimes (z^t)^{\otimes \frac{n-k}{2}})\eta_{n,s}.$$

Using lemma 9.2.5, it follows that

$$\begin{aligned}
\eta_{r,s}(c \otimes 1_s) &= \eta_{r,s}(a \otimes 1_s)(1_k \otimes z^{\otimes \frac{r-k}{2}} \otimes 1_s)(1_k \otimes (z^t)^{\otimes \frac{n-k}{2}} \otimes 1_s)(b \otimes 1_s) \\
&= (1_s \otimes a)\eta_{r,s}(1_k \otimes z^{\otimes \frac{r-k}{2}} \otimes 1_s)(1_k \otimes (z^t)^{\otimes \frac{n-k}{2}} \otimes 1_s)(b \otimes 1_s) \\
&= (1_s \otimes a)(1_s \otimes 1_k \otimes z^{\otimes \frac{r-k}{2}})\eta_{k,s}(1_k \otimes (z^t)^{\otimes \frac{n-k}{2}} \otimes 1_s)(b \otimes 1_s) \\
&= (1_s \otimes a)(1_s \otimes 1_k \otimes z^{\otimes \frac{r-k}{2}})(1_s \otimes 1_k \otimes (z^t)^{\otimes \frac{n-k}{2}})\eta_{n,s}(b \otimes 1_s) \\
&= (1_s \otimes c)\eta_{n,s}.
\end{aligned}$$

The same steps are used to prove that any diagram $d \in \mathsf{Hom}(m,s)$ with ℓ through lines satisfies $\eta_{n,s}(1_n \otimes d) = (d \otimes 1_n)\eta_{n,m}$. Then, for any (n,r)-diagram c and (m,s)-diagram d, these identities give

$$\begin{aligned}
\eta_{r,s}(c \otimes d) &= \eta_{r,s}(c \otimes 1_s)(1_n \otimes d) = (1_s \otimes c)\eta_{n,s}(1_n \otimes d) \\
&= (1_s \otimes c)(d \otimes 1_n)\eta_{n,m} = (d \otimes c)\eta_{n,m}
\end{aligned}$$

which closes the proof. ■

Note that with this braiding, $\widetilde{\mathsf{TL}}$ is not symmetric. In general, the element $\eta_{n,m} \circ \eta_{m,n} \in \operatorname{End}(n+m)$ is not even central. For instance, $\eta_{1,2} = q \cdot 1_3 + (e_1 + e_2) + q^{-1} e_2 e_1$ and $\eta_{2,1} = q \cdot 1_3 + (e_1 + e_2) + q^{-1} e_1 e_2$ and thus

$$\eta_{2,1} \circ \eta_{1,2} e_1 - e_1 \eta_{2,1} \circ \eta_{1,2} = q^{-2}(q - q^{-1})(e_1 e_2 - e_2 e_1) \neq 0.$$

We shall come back to the morphism $\eta_{r,s} \circ \eta_{s,r}$ in section 9.2.4.

9.2.3 Braiding modules

The representation theory of the family of $\mathsf{TL}_n, n \in \mathbb{N}^0$, was cast into a categorical framework by Graham and Lehrer [27] as follows. Let $F \in \mathrm{Funct}(\widetilde{\mathsf{TL}}, \mathsf{Vect}_{\mathbb{C}})$ be a $\mathbb{C}$-linear functor from the category $\widetilde{\mathsf{TL}}$ to that of finite-dimensional vector spaces over $\mathbb{C}$. Then each $F(n), n \in \mathbb{N}^0$, is a vector space and each $F(\alpha)$, for $\alpha \in \mathsf{Hom}(n,n) \simeq \mathsf{TL}_n$ is a linear map $F(n) \to F(n)$. Since F preserves composition, $F(n)$ is naturally a TL_n-module. But the functor F is somewhat richer than a choice of a TL_n-module for each $n \geq 0$. Indeed the functor F also gives linear maps $F(\gamma) : F(n) \to F(m)$ for all $\gamma \in \mathsf{Hom}(n,m)$ between modules over distinct algebras of the Temperley-Lieb family. These linear maps must also preserve the composition of diagrams. We give examples of such functors taken from [27].

Let $k \in \mathbb{N}^0$. A functor $S_k \in \mathrm{Funct}(\widetilde{\mathsf{TL}}, \mathsf{Vect}_{\mathbb{C}})$ is defined as follows. If the parities of n and k are distinct, then the vector space $S_k(n)$ is set to 0. If their parities coincide, then $S_k(n)$ is the formal span of (k,n)-diagrams with exactly k through lines. If $\alpha \in \mathsf{Hom}(n,m)$ with m,n and k having the same parity, then $S_k(\alpha) : S_k(n) \to S_k(m)$ is the linear map defined by its action on (k,n)-diagrams with k through lines. If $\gamma \in S_k(n)$ is such diagram, then $S_k(\alpha)\gamma \in \mathsf{Hom}(k,m)$ is $\alpha \circ \gamma$ if $\alpha \circ \gamma$ has k through lines and 0 otherwise. For all other $\alpha \in \mathsf{Hom}(m',n')$, that is with m' or n' not sharing the parity of k, the linear map $S_k(\alpha)$ is zero. It is straightforward to check that S_k is a functor and that $S_k(n)$ is the usual *standard* or *cellular* TL_n-modules $\mathsf{S}_{n,k}$.

Let I be the module $\mathrm{End}\,0 = \mathsf{Hom}(0,-)(0)$ or, equivalently, $S_0(0)$. (Recall that $\mathsf{Hom}(0,0) = \mathbb{C}$.) Any module M over TL_m is the evaluation of a certain functor F at m. For example, one could take the functor $\mathsf{Hom}(0+m,-) \otimes_{\mathrm{End}\,0 \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}\,m} (\mathsf{I} \otimes_{\mathbb{C}} \mathsf{M}) \simeq \mathsf{Hom}(m,-) \otimes_{\mathrm{End}\,m} \mathsf{M}$.

One could try to endow the set $\mathrm{Funct}(\widetilde{\mathsf{TL}}, \mathsf{Vect}_{\mathbb{C}})$ of such functors with the structure of a monoidal and/or braided category but, for the purpose of studying the braiding of the fusion product introduced by Read and Saleur [61] and studied by Gainutdinov and Vasseur [24] and Belletête [9], we chose to restrict the scope to a simpler category. [1]

Définition 9.2.8. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. The* disjoint module category $\mathcal{M}(\mathcal{C})$ *is defined in the following way.*

(i) $\mathrm{Ob}(\mathcal{M}(\mathcal{C})) = \{F(A) \mid A \in \mathrm{Ob}(\mathcal{C})\,,\, F \in \mathrm{Funct}(\mathcal{C}, \mathsf{Vect}_{\mathbb{C}})\}$;

(ii) *the morphisms from $F(A)$ to $G(B)$ are obtained by considering them as* $\mathrm{End}A$*-modules if $A \simeq B$ in $\mathcal{C}$ and, if not, the only morphism is the zero one. Thus*

$$\mathsf{Hom}_{\mathcal{M}(\mathcal{C})}(F(A), G(B)) = \begin{cases} \mathsf{Hom}_{\mathrm{End}A}(F(A), G(A)), & \text{if } A \simeq B, \\ \{0\}, & \text{otherwise.} \end{cases}$$

If the category $\mathcal{C}$ has additional properties, then $\mathcal{M}(\mathcal{C})$ might inherit some of them in a relatively natural

1. The concept of *fusion category* exists in the literature (see, for example [19]). Even though it describes categories equipped with a bifunctor $-_1 \otimes -_2$, the categories of modules that are under study are not fusion categories, as the latter contain only semisimple modules.

manners. For example, if the category $\mathcal{C}$ is monoidal or braided, so will be $\mathcal{M}(\mathcal{C})$. While "guessing" the right definition for the tensor product, associator, unitors and commutors is relatively straightforward, proving that these satisfy the required properties is rather tedious. We choose to particularize the construction for the case $\mathcal{C} = \widetilde{\mathsf{TL}}$ and only prove the first hexagon axiom for the commutor.

Proposition 9.2.9. *Let $\mathcal{M}(\widetilde{\mathsf{TL}})$ be the disjoint module category with objects $F(n)$ for $F \in \mathsf{Funct}(\widetilde{\mathsf{TL}}, \mathsf{Vect}_{\mathbb{C}})$ and morphisms*

$$\mathsf{Hom}_{\mathcal{M}(\widetilde{\mathsf{TL}})}(F(n), G(m)) = \begin{cases} \mathsf{Hom}_{\mathrm{End}\, n}(F(n), G(n)), & \text{if } n = m, \\ \{0\}, & \text{otherwise}, \end{cases}$$

for $n, m \in \mathrm{Ob}\,\widetilde{\mathsf{TL}} = \mathbb{N}^0$. Define, on this category, the bifunctor $\otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})}$, the identity object I, and the natural isomorphisms α, λ and ρ as follows. The tensor product of objects $F(n)$ and $G(m)$ in $\mathcal{M}(\mathcal{C})$ is

$$F(n) \otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})} G(m) \equiv \mathsf{TL}(n+m) \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(m)} (F(n) \otimes_{\mathbb{C}} G(m)) \tag{9.2.20}$$

$$= [\mathsf{Hom}_{\widetilde{\mathsf{TL}}}(n \otimes_{\widetilde{\mathsf{TL}}} m, -) \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(m)} (F(n) \otimes_{\mathbb{C}} G(m))](n \otimes_{\widetilde{\mathsf{TL}}} m) \tag{9.2.21}$$

and that of morphisms $\phi : F \to R$ and $\psi : G \to S$

$$\phi \otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})} \psi \equiv 1_{n \otimes_{\widetilde{\mathsf{TL}}} m} \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(m)} (\phi \otimes_{\mathbb{C}} \psi). \tag{9.2.22}$$

The associator α has components defined by

$$\begin{aligned} &\alpha_{F(n),G(m),H(k)} \Big(f \otimes_{\mathrm{End}(n \otimes_{\widetilde{\mathsf{TL}}} m) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(k)} \big((1_{n \otimes_{\widetilde{\mathsf{TL}}} m} \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(m)} (x \otimes_{\mathbb{C}} y)) \otimes_{\mathbb{C}} z \big) \Big) \\ &\quad = f \circ \alpha^{-1}_{n,m,k} \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(m \otimes_{\widetilde{\mathsf{TL}}} k)} \big(x \otimes_{\mathbb{C}} (1_{m \otimes_{\widetilde{\mathsf{TL}}} k} \otimes_{\mathrm{End}(m) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(k)} (y \otimes_{\mathbb{C}} z)) \big) \end{aligned} \tag{9.2.23}$$

where $f \in \mathsf{TL}_{n+m+k}$, $x \in F(n)$, $y \in G(m)$ and $z \in H(k)$. The identity object I is $\mathrm{End}(0) = \mathsf{Hom}_{\widetilde{\mathsf{TL}}}(0, -)(0) \in \mathcal{M}(\widetilde{\mathsf{TL}})$. Finally the unitors have components

$$\lambda_{F(n)}(f \otimes_{\mathrm{End}(0) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(n)} (g \otimes_{\mathbb{C}} x)) = F(f \circ (g \otimes_{\widetilde{\mathsf{TL}}} 1_n))x, \tag{9.2.24}$$

$$\sigma_{F(n)}(f \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(0)} (x \otimes_{\mathbb{C}} g)) = F(f \circ (1_n \otimes_{\widetilde{\mathsf{TL}}} g))x \tag{9.2.25}$$

for $f \in \mathsf{TL}_n, g \in \mathsf{TL}_0 \simeq \mathbb{C}$ and $x \in F(n)$. The data $(\mathcal{M}(\widetilde{\mathsf{TL}}), \otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})}, \alpha, \mathsf{I}, \lambda, \rho)$ form a monoidal category. Moreover, together with the commutor defined for $f \in \mathsf{TL}_{n+m}, x \in F(n)$ and $y \in G(m)$ by

$$\eta_{F(n),G(m)}\big(f \otimes_{\mathrm{End}(n) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(m)} (x \otimes_{\mathbb{C}} y)\big) = (f \circ \eta_{m,n}) \otimes_{\mathrm{End}(m) \otimes_{\widetilde{\mathsf{TL}}} \mathrm{End}(n)} (y \otimes_{\mathbb{C}} x), \tag{9.2.26}$$

it is a braided category.

Proof. We present a partial proof of these statements, namely we check that the commutor is well-defined and verify the first hexagon axiom. The rest of the proof is as direct (and tedious). Here is the notation used: The n, m, k are non-negative integers, U, V and W are respectively TL_n-, TL_m- and TL_k-modules, u, v and w

are elements of U, V and W respectively and, finally, $a \in \mathsf{TL}_{n+m+k}$, $b \in \mathsf{TL}_{n+m}$, $c \in \mathsf{TL}_n$ and $d \in \mathsf{TL}_m$. We also use the shorthand notation $\otimes \equiv \otimes_{\mathbb{C}}$ and $\otimes_{i,j} \equiv \otimes_{\mathsf{TL}_i \otimes_{\widetilde{\mathsf{TL}}} \mathsf{TL}_j}$, for any integers i and j.

The commutor $\eta_{\mathsf{U},\mathsf{V}}$ is well-defined. Indeed

$$\begin{aligned}
\eta_{\mathsf{U},\mathsf{V}}[b \otimes_{n,m} (cu \otimes dv)] &= (b \circ \eta_{m,n}) \otimes_{m,n} (dv \otimes cu) \\
&= (b \circ \eta_{m,n} \circ (d \otimes_{\widetilde{\mathsf{TL}}} c)) \otimes_{m,n} (v \otimes u) \\
&= (b \circ (c \otimes_{\widetilde{\mathsf{TL}}} d) \circ \eta_{m,n}) \otimes_{m,n} (v \otimes u) \quad \text{because of Lemma 9.2.5} \\
&= \eta_{\mathsf{U},\mathsf{V}}[(b \circ (c \otimes_{\widetilde{\mathsf{TL}}} d)) \otimes_{n,m} (u \otimes v)].
\end{aligned}$$

Note that it is also an isomorphism since $\eta_{m,n}$ is invertible.

A general element of $(\mathsf{U} \otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})} \mathsf{V}) \otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})} \mathsf{W}$ is of the form

$$a \otimes_{n+m,k} [(b \otimes_{n,m} (u \otimes v)) \otimes w].$$

We apply to this element the natural isomorphisms of the upper part of the hexagon diagram and then those of the lower. The notation $\mathsf{U} \otimes_{\mathcal{M}(\widetilde{\mathsf{TL}})} \mathsf{V}$ is shortened to $\mathsf{U} \otimes \mathsf{V}$. (Since the use of capital letters is reserved to modules, this should not lead to any confusion.) The upper part of the hexagon gives

$$\begin{aligned}
&\alpha_{\mathsf{V},\mathsf{W},\mathsf{U}} \circ \eta_{\mathsf{U},\mathsf{V}\otimes\mathsf{W}} \circ \alpha_{\mathsf{U},\mathsf{V},\mathsf{W}} \big(a \otimes_{n+m,k} [(b \otimes_{n,m} (u \otimes v)) \otimes w]\big) \\
&\quad = \alpha_{\mathsf{V},\mathsf{W},\mathsf{U}} \circ \eta_{\mathsf{U},\mathsf{V}\otimes\mathsf{W}} \big(a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \otimes_{n,m+k} [u \otimes (1_{m+k} \otimes_{m,k} (v \otimes w))]\big) \\
&\quad = \alpha_{\mathsf{V},\mathsf{W},\mathsf{U}} \big(a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ \eta_{m+k,n} \otimes_{m+k,n} [(1_{m+k} \otimes_{m,k} (v \otimes w)) \otimes u]\big) \\
&\quad = a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ \eta_{m+k,n} \otimes_{m,k+n} \big(v \otimes (1_{k+n} \otimes_{k,n} (w \otimes u))\big)
\end{aligned}$$

and the lower one

$$\begin{aligned}
&(1_{\mathsf{V}} \otimes_{\widetilde{\mathsf{TL}}} \eta_{\mathsf{U},\mathsf{W}}) \circ \alpha_{\mathsf{V},\mathsf{U},\mathsf{W}} \circ (\eta_{\mathsf{U},\mathsf{V}} \otimes_{\widetilde{\mathsf{TL}}} 1_{\mathsf{W}}) \big(a \otimes_{n+m,k} [(b \otimes_{n,m} (u \otimes v)) \otimes w]\big) \\
&\quad = (1_{\mathsf{V}} \otimes_{\widetilde{\mathsf{TL}}} \eta_{\mathsf{U},\mathsf{W}}) \circ \alpha_{\mathsf{V},\mathsf{U},\mathsf{W}} \big(a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ (\eta_{m,n} \otimes_{\widetilde{\mathsf{TL}}} 1_k) \otimes_{m+n,k} [(1_{m+n} \otimes_{m,n} (v \otimes u)) \otimes w]\big) \\
&\quad = (1_{\mathsf{V}} \otimes_{\widetilde{\mathsf{TL}}} \eta_{\mathsf{U},\mathsf{W}}) \big(a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ (\eta_{m,n} \otimes_{\widetilde{\mathsf{TL}}} 1_k) \otimes_{m,n+k} [v \otimes (1_{n+k} \otimes_{n,k} (u \otimes w))]\big) \\
&\quad = a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ (\eta_{m,n} \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ (1_m \otimes_{\widetilde{\mathsf{TL}}} \eta_{k,n}) \otimes_{m,k+n} \big[v \otimes (1_{k+n} \otimes_{k,n} (w \otimes u))\big] \\
&\quad = a \circ (b \otimes_{\widetilde{\mathsf{TL}}} 1_k) \circ \eta_{m+k,n} \otimes_{m,k+n} \big[v \otimes (1_{k+n} \otimes_{k,n} (w \otimes u))\big] \quad \text{because of (9.2.1).}
\end{aligned}$$

Since the two expressions coincide, the family of commutors $\eta_{\mathsf{U},\mathsf{V}}$, for $\mathsf{U}, \mathsf{V} \in \mathrm{Ob}\,\mathcal{M}(\widetilde{\mathsf{TL}})$, satisfies the first hexagon axiom. ■

The disjoint module category $\mathcal{M}(\widetilde{\mathsf{TL}})$ corresponds to the disjoint union of the categories $\mathrm{mod}\,\mathsf{TL}_n$ for $n \geq 0$. This can be directly seen by using the standard identification of modules over an algebra with functors. The monoidal structure thus induced corresponds to what physicists call the *fusion product*, or *fusion rules* [9, 24, 61] and is there denoted by $- \times_f -$. Using different arguments, it is relatively easy to prove that this fusion product is naturally commutative (see for example [9]), but these arguments do not give the isomorphism between, say, $\mathsf{U}\times_f\mathsf{V}$ and $\mathsf{V}\times_f\mathsf{U}$. The braiding $\eta_{\mathsf{U},\mathsf{V}}$ does.

Corollary 9.2.10. *Let n,m be positive integers and U,V be modules over $\mathrm{End}\,n\simeq\mathsf{TL}_n$ and $\mathrm{End}\,m\simeq\mathsf{TL}_m$, respectively. Then the isomorphism $\mathsf{U}\times_f\mathsf{V}\simeq\mathsf{V}\times_f\mathsf{U}$ is given by*

$$\eta_{\mathsf{U},\mathsf{V}}(f\otimes_{\mathsf{TL}_n\otimes_{\widetilde{\mathsf{TL}}}\mathsf{TL}_m}(u\otimes_{\mathbb{C}}v))=(f\circ\eta_{m,n})\otimes_{\mathsf{TL}_m\otimes_{\widetilde{\mathsf{TL}}}\mathsf{TL}_n}(v\otimes_{\mathbb{C}}u),$$

where $\eta_{m,n}$ is given in proposition 9.2.7, $f\in\mathsf{TL}_{n+m}, u\in\mathsf{U}$ and $v\in\mathsf{V}$.

It follows then that $\eta_{\mathsf{V},\mathsf{U}}\circ\eta_{\mathsf{U},\mathsf{V}}$ is a non-trivial endomorphism of $\mathsf{U}\times_f\mathsf{V}$.

9.2.4 The morphisms $\eta_{r,s}\circ\eta_{s,r}$

If M and N are respectively a TL_m- and a TL_n-module, then the map $\eta_{n,m}\circ\eta_{m,n}:\mathsf{M}\otimes\mathsf{N}\to\mathsf{M}\otimes\mathsf{N}$ is an isomorphism. The only goal of this short section is to stress that this isomorphism is highly non-trivial. We do not understand its structure in full generality. In the following we use fairly standard notation, writing $\mathsf{I}_{n,k},\mathsf{S}_{n,k}$ and $\mathsf{P}_{n,k}$ for the irreducible, standard and projective modules over TL_n. The standard $\mathsf{S}_{n,k}$ (denoted above by $S_k(n)$) was described at the beginning of section 9.2.3, the irreducible $\mathsf{I}_{n,k}$ is its irreducible quotient and $\mathsf{P}_{n,k}$ the projective cover of $\mathsf{I}_{n,k}$. (See [9, 63], and also [27] where $\mathsf{S}_{n,k}$ is denoted by $W_k(n)$.)

One example will suffice and we choose one of the simplest possible, namely $\mathsf{M}=\mathsf{N}=\mathsf{TL}_2$. Then, in the generic case, that is when q is not a root of the unity, the fusion product is [9, 24]

$$\mathsf{TL}_2\otimes\mathsf{TL}_2\simeq\mathsf{TL}_4\simeq\mathsf{S}_{4,0}\oplus\mathsf{S}_{4,0}\oplus\mathsf{S}_{4,2}\oplus\mathsf{S}_{4,2}\oplus\mathsf{S}_{4,2}\oplus\mathsf{S}_{4,4}.$$

Then $\eta_{2,2}\circ\eta_{2,2}$ is diagonalisable, but it does not take the same eigenvalues on isomorphic copies of the standard modules in TL_4. The eigenvalues on the two copies isomorphic to $\mathsf{S}_{4,0}$ are q^{-8} and 1, on $\mathsf{S}_{4,2}$ they are q^{-4}, 1 and 1, and on $\mathsf{S}_{4,4}$ it is q^4.

When q is a root of unity, the fusion product $\mathsf{TL}_2\times_f\mathsf{TL}_2$ may differ from the one above. For example, at $q=\sqrt{-1}$ and thus $\beta=0$, this product is then

$$\mathsf{TL}_2\otimes\mathsf{TL}_2\simeq\mathsf{TL}_4\simeq\mathsf{P}_{4,2}\oplus\mathsf{P}_{4,2}\oplus\mathsf{P}_{4,4}.$$

At this value of q, the isomorphism $\eta_{2,2}\circ\eta_{2,2}$ is even more complicated. It is not anymore diagonalisable, though it has a single eigenvalue equal to 1. The Hom-group $\mathrm{Hom}(\mathsf{P}_{4,i},\mathsf{P}_{4,i})$ for $i=2,4$ is two-dimensional. Beside the identity id, there is a map sending the head of $\mathsf{P}_{4,i}$ to its socle, both being isomorphic to $\mathsf{I}_{4,i}$. Let f be this map. While $\eta_{2,2}\circ\eta_{2,2}$ is non-diagonalisable, it is possible to find two subspaces A and B, both isomorphic to $\mathsf{P}_{4,2}$ that allows for an easy description of the morphism. Let C be the other summand $\mathsf{P}_{4,4}$.

Then $\eta = \eta_{2,2} \circ \eta_{2,2}$ can be broken down into its action on each summand as

$$\begin{bmatrix} \eta_{\mathsf{A},\mathsf{A}} \\ \eta_{\mathsf{A},\mathsf{B}} \\ \eta_{\mathsf{A},\mathsf{C}} \end{bmatrix} : \mathsf{A} \longrightarrow \mathsf{TL}_4, \qquad \begin{bmatrix} \eta_{\mathsf{B},\mathsf{A}} \\ \eta_{\mathsf{B},\mathsf{B}} \\ \eta_{\mathsf{B},\mathsf{C}} \end{bmatrix} : \mathsf{B} \longrightarrow \mathsf{TL}_4, \qquad \begin{bmatrix} \eta_{\mathsf{C},\mathsf{A}} \\ \eta_{\mathsf{C},\mathsf{B}} \\ \eta_{\mathsf{C},\mathsf{C}} \end{bmatrix} : \mathsf{C} \longrightarrow \mathsf{TL}_4.$$

They are

$$\begin{aligned} &\eta_{\mathsf{A},\mathsf{A}} = \mathrm{id} + \mu f, && \eta_{\mathsf{A},\mathsf{B}} = \mathrm{id}_{\mathsf{A},\mathsf{B}}, && \eta_{\mathsf{A},\mathsf{C}} = 0, \\ &\eta_{\mathsf{B},\mathsf{A}} = 0, && \eta_{\mathsf{B},\mathsf{B}} = \mathrm{id} + \nu f, && \eta_{\mathsf{B},\mathsf{C}} = 0, \\ &\eta_{\mathsf{C},\mathsf{A}} = 0, && \eta_{\mathsf{C},\mathsf{B}} = 0, && \eta_{\mathsf{C},\mathsf{C}} = \mathrm{id} + \rho f, \end{aligned}$$

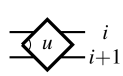
where μ, ν, ρ are non-zero constants and $\mathrm{id}_{\mathsf{A},\mathsf{B}}$ stands for the isomorphism between A and B. From these maps, it is straighforward to compute the Jordan form of η. Its non-trivial Jordan blocks are 2 blocks 3×3 and 2 blocks 2×2.

The root $q = \sqrt{-1}$ is somewhat special in the representation theory of the algebra TL_n: It is the only value for which the semisimplicity of TL_n varies with the parity of n. (For all other roots $q^{2\ell} = 1$ with $\ell \geq 3$, the algebra $\mathsf{TL}_n(\beta = -q - q^{-1}), n \geq \ell$, is never semisimple.) Although the example above was given at this particular value $q = \sqrt{-1}$, it seems to be representative of what happens at other values of q.

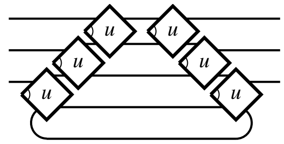
9.3 Braiding and integrability

One of the most profound uses of Temperley-Lieb algebras in physics is in the study of solvable models, like the XXZ Hamiltonians or loop models on two-dimensional lattices. The goal of the present section is to tie braiding and integrability in some of these statistical models. The former will appear through the *elementary brainding* $\eta_{1,1}$ (or $t_i(n)$) that was used in proposition 9.2.7 to write all other components $\eta_{r,s}$ of the braiding natural isomorphism. The latter will also be cast in terms of a fundamental “face operator" that must satisfy three identities. The physical object, that is, the Hamiltonian or the transfer matrix, is then defined in terms of several copies of this face operator.

In the literature on statistical models, the *face operator* $X_i(q,u)$ is also an element of one of the algebras $\mathsf{TL}_n(\beta)$. It depends on several parameters: The *spectral parameter* λ, tied to β by $\beta = -q - q^{-1}$ and $q = e^{i\lambda}$, and the *anisotropy parameter* u that measures the ratio of the interaction constants along two linearly independent vectors spanning the lattice. As for the t_i, the X_i is usually a linear combination of TL_n generators and it is represented graphically by



Since all faces will be evaluated at the same value of the parameter q (or λ), this parameter is often omitted. In terms of the face $X_i(q,u)$, the transfer matrix $D_n(\lambda,u) \in \mathsf{Hom}(n,n)$ on n sites is constructed out of $2n$ tiles organized in diagonal lines. For example the case $n = 3$ is depicted as follows:

$D_3(\lambda, u) =$  .

In the notation of the previous section, it is thus

$$D_n(\lambda, u) = (1_n \otimes z^t) \circ \Big(\prod_{i=1}^{n} X_i(q,u) \prod_{i=n}^{1} X_i(q,u) \Big) \circ (1_n \otimes z) \in \mathsf{Hom}(n,n). \tag{9.3.1}$$

Its physical properties are revealed through its spectrum in some representations. It was recognized by Behrend, Pearce and O'Brien [6] that some algebraic conditions on the face operator $X_i(q,u)$ ensure that the transfer matrix, constructed from it, will have the properties that $D_n(\lambda,u) \circ D_n(\lambda,v) - D_n(\lambda,v) \circ D_n(\lambda,u) = 0$ in TL_n. This means that, in any representation $\phi : \mathsf{TL}_n \to gl(\mathsf{V})$ with V some vector space, the matrices $\phi(D_n(\lambda,u))$ and $\phi(D_n(\lambda,v))$ will commute for all values u and v. The modes $\phi(D_n(\lambda,u))$ in any expansion with respect to u (Taylor's expansion, Fourier's, ...) will commute, that is, they will be integrals of motions. Thus the integrability of the models based on such a transfer matrix D_n follows from these algebraic conditions. Here they are.

Proposition 9.3.1 (section 3.4 of [6]). *If $X_i(q,u)$ verifies the following three conditions, then $D_n(\lambda,u) \circ D_n(\lambda,v) = D_n(\lambda,v) \circ D_n(\lambda,u)$, for all $u, v \in \mathbb{C}$:*

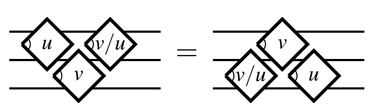
$$\text{(Yang-Baxter equation)} \quad X_i(q,u)X_{i+1}(q,v)X_i(q,v/u) = X_{i+1}(q,v/u)X_i(q,v)X_{i+1}(q,u), \tag{9.3.2}$$

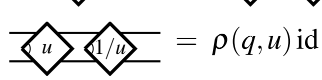
$$\text{(inversion relation)} \quad X_i(q,u)X_i(q,u^{-1}) = \rho(q,u)\,\mathrm{id}, \tag{9.3.3}$$

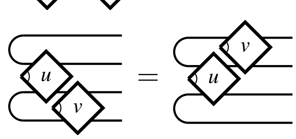
$$\begin{aligned} \text{(boundary Yang-Baxter)} \quad & X_i(q,u)X_{i+1}(q,v) \circ (z \otimes z) \\ & = X_i(q,u)X_{i-1}(q,v) \circ (z \otimes z) \end{aligned} \tag{9.3.4}$$

for some non-identically zero function $\rho(q,u)$.

These conditions are found in the literature drawn as follows:

(Yang-Baxter equation) 

(inversion relation) 

(boundary Yang-Baxter) 

It is not too difficult to construct such a face operator X_i out of the elementary braiding element $\eta_{1,1} = t_i$.

As an intermediary step, consider

$$y_i(u) = u^{-1}t_i - ut_i^{-1}.$$

Both products $y_i(u)y_{i+1}(v)y_i(w)$ and $y_{i+1}(w)y_i(v)y_{i+1}(u)$ contain eight terms, each cubic in the generators t_i, t_{i+1} and their inverses. The identity (9.2.14) gives rise to the six following ones:

$$\begin{aligned} t_it_{i+1}t_i &= t_{i+1}t_it_{i+1}, & t_{i+1}t_it_{i+1}^{-1} &= t_i^{-1}t_{i+1}t_i, & t_it_{i+1}t_i^{-1} &= t_{i+1}^{-1}t_it_{i+1},\\ t_{i+1}t_i^{-1}t_{i+1}^{-1} &= t_i^{-1}t_{i+1}^{-1}t_i, & t_it_{i+1}^{-1}t_i^{-1} &= t_{i+1}^{-1}t_i^{-1}t_{i+1}, & t_i^{-1}t_{i+1}^{-1}t_i^{-1} &= t_{i+1}^{-1}t_i^{-1}t_{i+1}^{-1}. \end{aligned}$$

Thanks to these identities, all sixteen terms of the difference of triple products of the y_i cancel pairwise, but four:

$$\begin{aligned} &y_i(u)y_{i+1}(v)y_i(w) - y_{i+1}(w)y_i(v)y_{i+1}(u) \\ &\qquad = uv^{-1}w(t_i^{-1}t_{i+1}t_i^{-1} - t_{i+1}^{-1}t_it_{i+1}^{-1}) - u^{-1}vw^{-1}(t_it_{i+1}^{-1}t_i - t_{i+1}t_i^{-1}t_{i+1}). \end{aligned} \tag{9.3.5}$$

Moreover it is easily verified that

$$(t_i^{-1}t_{i+1}t_i^{-1} - t_{i+1}^{-1}t_it_{i+1}^{-1}) = q(t_it_{i+1}^{-1}t_i - t_{i+1}t_i^{-1}t_{i+1}). \tag{9.3.6}$$

So the difference (9.3.5) will be zero if $quv^{-1}w = u^{-1}vw^{-1}$. This is easily achieved with the following definition of the x_i.

Proposition 9.3.2. *Let $n \geq 2$. The $x_i(q,u)$ defined by*

$$x_i(q,u) = \frac{\sqrt{q}}{u}t_i - \frac{u}{\sqrt{q}}t_i^{-1}, \qquad \textit{for } i < n, \tag{9.3.7}$$

satisfy the three conditions (9.3.2)–(9.3.4) with $\rho(q,u) = ((q^2+q^{-2}) - (u^2+u^{-2}))$.

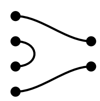
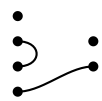
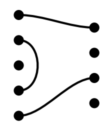
Proof. With the new weights $u \mapsto u' = u/\sqrt{q}$, the relation $qu'v'^{-1}w' = u'^{-1}v'w'^{-1}$ with $w' = v'/u'$ is true and the Yang-Baxter is verified. The other two equations are obtained by expanding the x_i. ■

The solution x_i of the three conditions in theorem 9.3.1 is well-known. For example, Section 3 of [60] is devoted to this solution and its relationship with the Temperley-Lieb algebra. (Note that their λ and our q is related by $q = e^{i\lambda}$ and their u and ours is also related by an exponential. Their β is $q+q^{-1}$ while ours is $-q-q^{-1}$. Finally they consider a larger class of boundary conditions that those above.) However the above discussion shows how the braiding of the Temperley-Lieb category $\widetilde{\mathsf{TL}}$ and integrability of statistical models are intimately related.

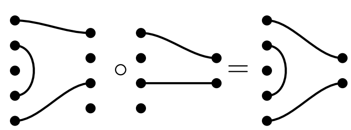
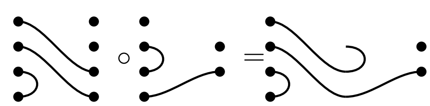
9.4 The dilute category $\mathsf{d\tilde{T}L}$

The dilute Temperley-Lieb algebras $\mathsf{dTL}_n(\beta)$ are a family of algebras defined through diagrams similar to those appearing in the original algebras $\mathsf{TL}_n(\beta)$. This family can be cast into a category $\widetilde{\mathsf{dTL}}$ similar to the category $\widetilde{\mathsf{TL}}$ introduced in section 9.2.2. This new category can also be given a braided structure. This section introduces this structure and discusses the relationship between the braiding on $\widetilde{\mathsf{dTL}}$ and the integrability of dilute statistical models.

We start by giving the definition of the category $\widetilde{\mathsf{dTL}}$, while recalling the definitions of the algebras dTL_n themselves. (See [11] for further details on the dilute family.) The objects of the *dilute Temperley-Lieb category* $\widetilde{\mathsf{dTL}}$ are the non-negative integers. The morphisms between two integers n and m are defined as linear combinations of *dilute* (n,m)-diagrams. These dilute diagrams are defined in the same way as the (n,m)-diagrams appearing in $\widetilde{\mathsf{TL}}$ except that nodes on either sides of the diagrams are now allowed to be free of strings; a node without a string is called a *vacancy*. For example, the following are all acceptable dilute diagrams:

 ,  and  .

The first two are elements of $\mathsf{Hom}(2,4)$ and the last of $\mathsf{Hom}(4,5)$. Composition of morphisms is defined by extending bilinearly the following composition rule. For b and c dilute (m,n)- and (k,m)-diagrams, the composition $b\circ c$ is a dilute (k,n)-diagram defined by first drawing b on the left of c, identifying the points on the m neighbouring sites, joining the strings that meets there, and then removing the points on this side. If a string is closed in this process, it is removed and the diagram obtained is multiplied by $\beta=-q-q^{-1}\in\mathbb{C}$. If a string is attached to only one of its extremities (because it was joined to a vacancy during the composition), the result $b\circ c$ is the zero morphism [2]. Here are few examples of these compositions. In the first $b\in\mathsf{Hom}(4,5), c\in\mathsf{Hom}(2,4)$ and $b\circ c\in\mathsf{Hom}(2,5)$.

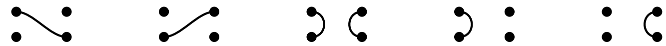
 and  $=0$

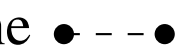
For a stricly positive integer n, the algebra $\mathsf{dTL}_n(\beta)$ is identified to the set $\mathsf{Hom}(n,n)$ with the product being the composition just defined.

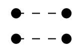
Endowing this category with a monoidal structure in a straigthforward generalisation of the one on $\widetilde{\mathsf{TL}}$. Again define $n\otimes m\equiv n+m$. For morphisms, if a and b are dilute (n,m)- and (r,s)-diagrams, define their tensor product $a\otimes b$ as the $(n+r,m+s)$-diagram obtained by simply putting a on top of b, as in $\widetilde{\mathsf{TL}}$, and extend this bilinearly to all morphisms. With the associator and the unitors as identities, this tensor product makes $\widetilde{\mathsf{dTL}}$ into a strict monoidal category.

The commutor for $\widetilde{\mathsf{dTL}}$ is obtained similarly to that of $\widetilde{\mathsf{TL}}$. We only outline the computation of $\eta_{1,1}$. The space End 2 is spanned by the following 9 diagrams:

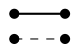
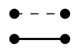
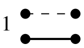
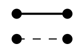
2. Note that the case of a string ending at a vacancy can also be resolved by simply removing it and replacing its two ends by vacancies. This then yields a different structure, the planar rook algebras (see, for example, [20]).



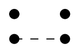
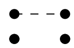
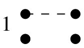
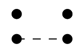
and the elementary commutor $\eta_{1,1}$ is a linear combination of these nine diagrams. A line  in any diagram will mean the sum of two diagrams, the first with a straight line between the two nodes, the second with nothing between these nodes that are then vacancies. The identity $1_1 \in \mathsf{Hom}(1,1)$ is thus such a dashed line and the identity in End 2

$1_2 =$ 

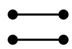
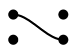
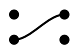
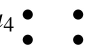
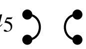
is the sum of the first four of the nine diagrams above. Four of the coefficients of $\eta_{1,1}$ are easily set to zero by the following requirements:

$\eta_{1,1}$  $=$  $\eta_{1,1}$ $\qquad$ $\eta_{1,1}$  $=$  $\eta_{1,1}$

and

$\eta_{1,1}$  $=$  $\eta_{1,1}$ $\qquad$ $\eta_{1,1}$  $=$  $\eta_{1,1}$.

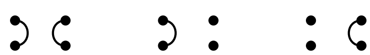
The commutor $\eta_{1,1}$ is thus found to be a sum of the remaining five diagrams:

$\eta_{1,1} = a_1$  $+a_2$  $+a_3$  $+a_4$  $+a_5$  .

As in section 9.2.2, we define $t_i(n) \equiv 1_{i-1} \otimes \eta_{1,1} \otimes 1_{n-i-1} \in \mathsf{Hom}(n,n)$ and write $\eta_{1,2} = t_2 t_1$ and $\eta_{2,1} = t_1 t_2$. The conditions are then

$$\eta_{1,2}(a \otimes b) = (b \otimes a)\eta_{1,2} \qquad \text{and} \qquad \eta_{2,1}(b \otimes a) = (a \otimes b)\eta_{2,1}$$

for all $a \in \mathsf{dTL}_1$ and $b \in \mathsf{dTL}_2$. Choosing a as 1_1 and b as one of the following ones



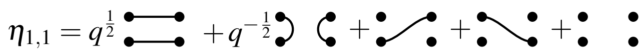
gives the algebraic equations

$$a_1^2 + a_1 a_5 \beta + a_5^2 = 0 \qquad \text{and} \qquad a_2^2 = a_3^2 = a_4^2 = a_1 a_5.$$

Finally the conditions $\eta_{1,1}(a \otimes b) = (b \otimes a)\eta_{1,0} = b \otimes a$ and $\eta_{1,1}(b \otimes a) = (a \otimes b)\eta_{0,1} = a \otimes b$ with $a \in \mathsf{dTL}_1$ and $b \in \mathsf{Hom}(0,1)$ give $a_2 = a_3 = a_4 = 1$. The first equation above becomes $a_1^2 + \beta + a_1^{-1} = 0$ whose solutions are $a_1 = \pm q^{\pm\frac{1}{2}}$, the two $\pm$ being independent. We shall choose both upper signs. This determines completely $\eta_{1,1}$ and it is possible to check that all other conditions on $\eta_{1,1}$, $\eta_{1,2} = t_2 t_1$ and $\eta_{2,1} = t_1 t_2$ are satisfied.

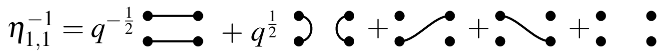
Proposition 9.4.1. *The category $\widetilde{\mathsf{dTL}}$ is braided for a commutor with components $\eta_{r,s}$ given by* (9.2.18)*,*

$t_i(n) \equiv 1_{i-1} \otimes \eta_{1,1} \otimes 1_{n-i-1}$ and $\eta_{1,1}$ and $\eta_{1,1}^{-1}$ now given by



(9.4.1)

and



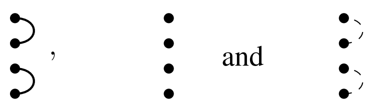
(9.4.2)

It follows that a disjoint module categody $\mathcal{M}(\widetilde{\mathsf{dTL}})$ can be defined along the lines introduced in section 9.2.3 and that it is also braided.

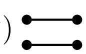
In the case of the original Temperley-Lieb algebras, the construction of $x_i(q,u)$ satisfying the three conditions (9.3.2)–(9.3.4) rested on the identities (9.2.14) and (9.3.6). These can be shown to be satisfied by the t_i defined with $\eta_{1,1}$ in (9.4.1). The elementary braiding (9.4.1) thus leads again to the following non-trivial solution of the Yang-Baxter equation (9.3.2):

$$x_i(q,u) = \frac{\sqrt{q}}{u} t_i - \frac{u}{\sqrt{q}} t_i^{-1}, \qquad \text{for } i < n,$$

where $\eta_{1,1}$ is now given by (9.4.1) and the x_i are understood as elements of $\mathsf{Hom}(n,n)$. Does this solution also satisfy the two other conditions (9.3.3) and (9.3.4)? For the latter, one has first to decide what is to replace the “boundary terms" $(z \otimes z)$. A direct calculation shows that (9.3.4) is satisfied by the dilute x_i for only three boundary conditions, namely



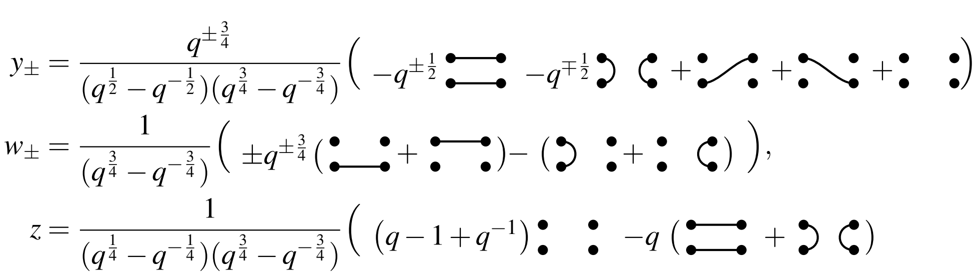
Finally the dilute x_i does not satisfy the inversion relation (9.3.3):

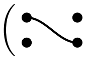
$$x_i(q,u)x_i(q,u^{-1}) = ((q+q^{-1}) - (u^2+u^{-2}))1_2 + (q^2 - q - q^{-1} + q^{-2})$$  .

While this non-trivial solution only partially satisfies (9.3.2)–(9.3.4), there is another one that solves the three conditions. It uses Boltzmann weights discovered by Izergin and Korepin [37], and Nienhuis [56]. It is

$$\hat{x}_i(q,u) = u^{-2} \cdot y_+ + u^{-1} \cdot w_+ + z + u \cdot w_- + u^2 \cdot y_-, \tag{9.4.3}$$

where



$$+(q^{\frac{1}{2}}-1+q^{-\frac{1}{2}})\Big($$  $$+$$  $$\Big)\Big).$$

Like x_i, this $\hat{x}_i$ solves the Yang-Baxter equation. But it also satisfies the inversion relation (with a new function $\hat{\rho}$) and the boundary Yang-Baxter equation with particular boundary conditions [17, 71]. Notice that, up to a global factor, the $y_\pm$ are the commutors $\eta_{1,1}^{\pm 1}$ that would have been obtained if a_1 would have been chosen as $-q^{\frac{1}{2}}$. The other three terms w_+, w_- and z are however completely different. It is not clear whether the integrable model it defines is related to a braiding for a different bifunctor $-\otimes' -$.

9.5 Conclusion

The construction in proposition 9.2.7 of the natural isomorphisms $\eta_{r,s}$ of the commutor is one of the main results of this article. The commutor exists at the level of the braided category $\widetilde{\mathsf{TL}}$ and does not require the definition of the bifunctor $-_1 \otimes -_2 \simeq -_1 \times_f -_2$ between modules. The fact that it is naturally extended to this bifunctor shows that, even though it might not be the only product having this property, the fusion product defined in [61] is “natural". The examples of section 9.2.4 also show that its structure as an isomorphism is highly non-trivial, even on a pair of projective modules. Understanding it better and comparing it to its counterpart in CFT, say for Verma modules over the Virasoro algebra, is natural question.

The question also arises of the existence of a commutor for other family of algebras and its eventual link to integrable models defined using them. Section 9.4 showed that the “elementary brainding $\eta_{1,1}$” does not reproduce the transfer matrix defining dilute loop models. Is it possible to understand better the link between this $\eta_{1,1}$ and the integrability of the dilute models? There are other algebras physically relevant in statistical physics, for example the one-boundary TL family (also known as the *blob algebra* [48]) and the affine (periodic) TL family. It is not known whether one can define a fusion product between their modules or even make a braided category out of their link diagrams.

Acknowledgements

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(Jonathan Belletête) CRM and Département de physique, Université de Montréal, Montréal, QC, Canada, H3C 3J7
(Yvan Saint-Aubin) Département de mathématiques et de statistique, Université de Montréal Montréal, QC, Canada, H3C 3J7.

Chapitre 10

Conclusion

La thèse, avec ses quatre articles, a fait progresser de beaucoup la théorie des représentations des algèbres de Temperley-Lieb régulière et diluée, particulièrement sur des aspects pertinents en physique. Nos résultats permettent d'attaquer de nouveaux problèmes ; nous en discutons quelques-uns.

La théorie des représentations des algèbres diluées de Temperley-Lieb, qui était relativement peu développée, est maintenant bien avancée. En particulier, pour autant que le paramètre β soit non nul, l'algèbre dTL_n est équivalente au sens de Morita à l'algèbre $\mathsf{TL}_n \oplus \mathsf{TL}_{n-1}$. Ceci signifie qu'il existe une correspondance biunivoque entre leurs modules projectifs, et que cette correspondance est compatible avec les morphismes. Autrement dit, les théories de leurs représentations sont à toute fin pratique identiques. Le cas $\beta = 0$ est singulier, mais intéressant ; il correspond au modèle des polymères dilués. Le modèle des polymères denses [59] est un laboratoire théorique important car il est un des rares cas, avec le modèle d'Ising, où il est possible de trouver explicitement les valeurs propres de la matrice de transfert ; il est alors plausible que la version diluée du modèle soit aussi accessible. Si c'était le cas, l'équivalence entre les deux algèbres pourrait être une excellente opportunité pour mieux comprendre le passage à la limite dans les modèles sur réseaux. En effet, si cette limite existe et qu'elle est bien définie, à quel point respecte-t-elle la structure des modules finis ? Si la limite se comporte bien, elle devrait commuter avec l'équivalence de Morita et les modèles denses et dilués devraient donner des théories conformes équivalentes. Par exemple la version diluée du modèle d'Ising est le modèle d'Ising tri-critique [22]. La limite continue de ce dernier est supersymétrique, alors que le modèle original ne l'est pas. Dans cet exemple, alors que la relation entre le modèle d'Ising et les algèbres de Temperley-Lieb régulières est bien connue, la relation entre leurs versions diluées respectives reste inconnue. Si une telle relation existe bien, elle pourrait être utilisée pour mieux comprendre comment l'équivalence de Morita se comporte dans le passage à la limite.

Les règles de fusions sont également identiques pour le cas dilué et le cas régulier (quand $\beta \neq 0$). La méthode que nous avons développée a permis de calculer les règles de fusion pour de vastes classes de modules indécomposables, incluant des modules qui ne pouvaient pas être traités par la dualité quantique de Schur-Weyl [24, 44, 46, 61]. En particulier, nous sommes parvenus à montrer qu'un ensemble de modules se comportait sous la fusion comme des modules irréductibles de Virasoro [21] ; la correspondance que l'on en tire est celle obtenue en comparant les caractères des modules [24], mais a l'avantage de ne pas dépendre du choix d'un hamiltonien précis. D'un point de vue algébrique, l'hamiltonien de Virasoro joue un rôle fondamental dans la structure de ses représentations, mais l'hamiltonien de TL_n ne joue aucun rôle particulier.

L'article sur la fusion a laissé plusieurs questions ouvertes et en soulève de nouvelles. Il reste quelques classes de modules pour lesquelles la fusion n'a pas été calculée mais elle peut l'être directement en utilisant les résultats de [9]. Par exemple, les modules B_{2i}^k sont isomorphes à la fusion d'un module standard et d'un module irréductible. La fusion de deux modules de cette famille peut donc être obtenue en fusionnant d'abord les deux modules standards et les deux modules irréductibles puis en appliquant les résultats connus.

L'article introduit également le quotient de fusion qui, à notre connaissance, n'a jamais été discuté auparavant. Nous n'avons calculé que quelques exemples, mais il semble évident que cette construction est intéressante en soi. Qui plus est, elle est également utile pour décomposer des espaces d'Hilbert, au même titre que le produit de fusion. Rappelons que l'espace d'Hilbert de la chaîne quantique XXZ peut être brisé en somme directe en utilisant le produit de fusion [24] (voir section 1.4.1). La chaîne des dimères est une autre chaîne quantique construite à partir de TL_n en $\beta = 0$ (voir [52] pour un traitement détaillé). L'hamiltonien du modèle est

$$H_d = -\sum_{i=1}^{n-2}\left(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right) = -\sum_{i=1}^{n-1} e_i, \tag{10.0.1}$$

où $\sigma_j^\pm = \sigma_j^x \mp i\sigma_j^y$, $\sigma_0^\pm \equiv \sigma_n^\pm \equiv 0$ et

$$e_j = \sigma_{j-1}^-\sigma_j^+ + \sigma_j^+\sigma_{j+1}^-, \tag{10.0.2}$$

engendrent une représentation de $\mathsf{TL}_n(0)$. Comme pour la chaîne XXZ, $[e_i, \sum_j^{n-1}\sigma_j^z] = 0$ et l'espace d'Hilbert du modèle, en tant que $\mathsf{TL}_n(0)$-module, se décompose en somme directe de niveaux E_{n-1}^k, engendrés par les états ayant exactement k spins $+$. La décomposition de ces niveaux en modules indécomposables, qui fut faite par [52], peut être exprimée à partir des résultats de notre article sur la fusion

$$\mathsf{E}_{n-1}^k \simeq (\mathsf{I}_{k+3,k+1} \times_f \mathsf{S}_{n-k+1,n-1-k}) \div_f \mathsf{I}_{4,2}. \tag{10.0.3}$$

Cette expression se généralise aisément aux autres valeurs de β ; est-il alors possible de construire une généralisation du modèle des dimères où le paramètre β serait non nul, et dont la décomposition des niveaux serait donnée par (10.0.3) ? De plus, puisque $\mathsf{I}_{k+3,k+1}$ est un quotient de $\mathsf{S}_{k+3,k+1}$ et que la fusion est exacte à droite,

$$\mathsf{E}_{n-1}^k \simeq (\mathsf{I}_{k+3,k+1} \times_f \mathsf{S}_{n-k+1,n-1-k}) \div_f \mathsf{I}_{4,2} \subset (\mathsf{S}_{k+3,k+1} \times_f \mathsf{S}_{n-k+1,n-1-k}) \div_f \mathsf{I}_{4,2} \simeq M_{k+1}(n). \tag{10.0.4}$$

En d'autres mots, l'espace d'Hilbert du modèle des dimères est isomorphe à un sous-espace de l'espace d'Hilbert de la chaîne XXZ en $\beta = 0$. Est-ce que l'hamiltonien XXZ restreint à ce sous-espace correspond à l'hamiltonien des dimères ? Si oui, est-ce que le modèle obtenu en quotientant cette partie de la chaîne XXZ est intéressant ?

Alors qu'il est relativement facile de se convaincre que le produit de fusion sur les algèbres de Temperley-

Lieb est commutatif, à isomorphisme près, il n'est pas du tout trivial de trouver quel est cet isomorphisme. Ce dernier découle simplement de la structure tressée de la catégorie de Temperley-Lieb. De plus, ce tressage induit de nombreux automorphismes de modules obtenus par la fusion d'autres modules. Les quelques exemples que nous sommes parvenus à calculer montrent que ces automorphismes sont en général non triviaux, et dans certains cas leur diagonalisation induit une base adaptée au produit de fusion. Les propriétés générales de ces morphismes restent toutefois bien mystérieuses.

Le tressage permet également de construire des éléments centraux pour TL_n et dTL_n. La construction de ces derniers ressemble à la construction des matrices de transfert au bord fusionnés de [7, 60]. Ces matrices de transfert peuvent être vues comme des généralisations de la matrice (1.4.11) pour des conditions aux limites différentes et n'ont à première vue aucun lien avec la fusion décrite dans cette thèse. Par contre, s'il est possible de les construire avec des commuteurs de la catégorie de Temperley-Lieb, peut-être ont-elles effectivement un lien avec la fusion. De plus, il pourrait alors être possible de généraliser ces matrices à d'autres algèbres, comme dTL_n, en exploitant cette construction.

La catégorie disjointe que nous avons introduite porte, par son tressage, une représentation du groupe de tresses. La catégorie des modules de Virasoro porte aussi une représentation du même groupe. Puisque les algèbres TL_n et Virasoro sont très différentes, il est difficile de construire une correspondance entre les deux, et encore plus de le faire dans une hypothétique limite où $n \to \infty$. Par contre, les structure tressées correspondent à des représentations du même groupe. La correspondance entre les deux est donc une question claire, mais surtout c'est une question bien posée : étant donné deux représentations d'un groupe, il existe de nombreuses techniques pour savoir si elles sont équivalentes, si une est une sous-représentation de l'autre ou encore un quotient de l'autre. Comparer ces représentations serait donc une façon simple et efficace d'étudier la limite continue. La représentation du groupe de tresses de Temperley-Lieb pourrait être étudiée de façon relativement directe : il faut comprendre la structure des commuteurs $\eta_{n,k}$, des automorphismes $\eta_{n,k}\eta_{k,n}$, etc. Par contre, la représentation du groupe de tresses sur Virasoro serait un travail beaucoup plus ambitieux ; non seulement les commuteurs ne sont pas connus, mais il y a beaucoup de produits de fusion qui n'ont pas encore été calculés. De plus, il n'a pas encore été prouvé que la structure tressée de Virasoro correspondait bien à la fusion physique dans les CFT. Certains exemples simples pourraient toutefois être approchables. Les modèles minimaux ont des règles de fusion relativement simples qui sont bien connues [21] ; comme notre article sur la fusion a montré que certains champs dans ces modèles se comportent comme des modules sur une algèbre de Temperley-Lieb sous le produit de fusion, comparer les représentations du groupe de tresses correspondant à ces modèles serait un projet naturel et très excitant.

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