Ordinally consistent tournament solutions

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Abstract

A set ranking method assigns to each tournament on a given set an ordering of the subsets of that set. Such a method is consistent if (i) the items in the set are ranked in the same order as the sets of items they beat and (ii) the ordering of the items fully determines the ordering of the sets of items. We describe two consistent set ranking methods.

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1 Introduction

We reconsider the problem of extracting an ordering from a tournament. If the incidence matrix of a tournament on m items is irreducible, the Perron-Frobenius theorem ensures that it possesses a unique eigenvector in the m-simplex. The eigenvector solution assigns to each item x a rating equal to the value of the xth coordinate of that eigenvector (Landau (1895), Wei (1952), Kendall (1955)). The rating of x is thus proportional to the sum of the ratings of the items that x beats in the tournament. This self-consistency property is what lends appeal to the solution.

Implicitly, the eigenvector solution defines what may be called a *set rating method.* It assigns a rating not only to each item but also to each set of items: the rating of a set is the sum of the ratings of its members.

Of course, as a by-product, the solution delivers a *ranking* of the sets of items –a set is ranked above another if and only if its rating is higher. But the construction of this ranking (hence also the construction of the ranking of items it induces) requires that the strength of an item be *cardinally* measurable. Indeed, the condition that an item's rating be *proportional* to the rating of the set it beats is based on that assumption. Moreover, if the items' ratings have no cardinal meaning, the ordering of two sets of items should not vary with an increasing transformation of

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the ratings of their members –but it clearly does since it depends upon the *sum* of these ratings. The eigenvector solution is inherently cardinal.

This note formulates a version of the consistency property of the eigenvector solution that does not assume cardinal measurability of the strength of the items. We call a set ranking method consistent if (i) the items are ranked in the same order as the sets they beat and (ii) the ordering of the items completely determines the ordering of the sets of items. While the eigenvector set ranking method satisfies the first condition, it violates the second. The question arises whether these conditions are compatible. We prove that they are, and describe two consistent set ranking solutions.

2 Definitions

Let X be a finite set of m items and let $\mathcal{X}$ be the set of nonempty subsets of X. A tournament is a complete and asymmetric binary relation T on X. Let $\mathcal{T}$ denote the set of tournaments. If $T \in \mathcal{T}$ and $x \in X$, let $t(x) = \{y \in X : xTy\}$. Let $\mathcal{R}(X)$ be the set of orderings of X and let $\mathcal{R}(\mathcal{X})$ be the set of orderings of $\mathcal{X}$.

A *set ranking method* is a function $R : \mathcal{T} \to \mathcal{R}(\mathcal{X})$. We interpret $R(T)$ as the ordering of $\mathcal{X}$ recommended by the method R for the tournament T. Let $P(T)$ and $I(T)$ denote, respectively, the strict ordering and the equivalence relation generated by the ordering $R(T)$. Denote by $R_X(T) \in \mathcal{R}(X)$ the ordering of the items induced by $R(T)$: by definition, $xR_X(T)y$ if and only if $\{x\} R(T) \{y\}$. We call $R_X : \mathcal{T} \to \mathcal{R}(X)$ a *ranking method.*

A set ranking method R is *consistent* if it satisfies the following two conditions:

(i) for all $T \in \mathcal{T}$ and $x, y \in X$, $xR_X(T)y \Leftrightarrow t(x)R(T)t(y)$,

(ii) for all $T, T' \in \mathcal{T}$, $R_X(T) = R_X(T') \Rightarrow R(T) = R(T')$.

The first condition says that the ranking of two items should be the same as the ranking of the sets they beat: item x is stronger than y if and only if x beats a stronger set than y does. This is the ordinal version of the self-consistency property of the eigenvector solution. The second condition says that the ranking of the items fully determines the ranking of the sets of items: the *extension rule* for deriving an ordering on $\mathcal{X}$ from one on X is the same in every tournament. The eigenvector solution imposes a cardinal version of this requirement: the rule for extending ratings from items to sets does not vary with T –moreover, it takes the particular form of the summation. Note that in the absence of condition (ii), condition (i) has no bite: the partial ordering on $\mathcal{X}$ derived from $R_X(T)$ and condition (i) can always be completed.

Condition (i) imposes severe restrictions on the extension procedure in condition (ii). We describe two examples of consistent set ranking methods. Characterizing the set of consistent methods is an open problem.

3 Results

A tournament is irreducible if its transitive closure is a complete relation. Every tournament can be decomposed into a collection of uniquely defined irreducible components: the top component is the top cycle, the second is the top cycle of the restriction of the tournament to the remaining items, and so on. The *decomposition ordering* ranks the items according to the irreducible component they belong to.

Formally, for any ordering $R_0 \in \mathcal{R}(X)$ and $Y \in \mathcal{X}$, let $\max_Y R_0$ denote the set of maximal elements of R_0 in Y. Since yI_0y' for all $y, y' \in \max_Y R_0$, we abuse notation and write $(\max_Y R_0)R_0(\max_Z R_0)$ if yR_0z for all $y \in \max_Y R_0$ and $z \in \max_Z R_0$. The *top cycle* of a tournament T is the set $X_1(T) := \max_X \overline{T}$ of maximal elements of the transitive closure $\overline{T}$ of T in X. For any $Y \in \mathcal{X}$, let T_Y denote the restriction of tournament T to the subset of items Y. Define inductively $X_k(T)$ to be the set of maximal elements of $\overline{T_{X \setminus \cup_{h=1}^{k} X_h(T)}}$ in $X \setminus \cup_{h=1}^{k} X_h(T)$. The resulting partition $\{X_1(T), ..., X_K(T)\}$ of X defines the *decomposition ordering* $R_X^*(T)$ of X :

$$xR_X^*(T)y \Leftrightarrow k(x,T) \leq k(y,T),$$

where $k(z,T)$ is the unique integer k such that $z \in X_k(T)$.

Call a set ranking method R' *finer* than R if for all $T \in \mathcal{T}$ and all $Y, Z \in \mathcal{X}$, $YP(T)Z \Rightarrow YP'(T)Z$.

Proposition 1. *There exists a unique finest consistent set ranking method R such that*

$$YR(T)Z \Leftrightarrow (\max_Y R_X(T))\ R_X(T)\ (\max_Y R_X(T)) \tag{1}$$

for all $T \in \mathcal{T}$ and $Y, Z \in \mathcal{X}$. The induced ranking method R_X chooses the decomposition ordering of X in each $T \in \mathcal{T}$.

Like the eigenvector method, the set ranking method in Proposition 1 ranks items according to the strength of the set of items they beat –it satisfies condition (i) in the definition of Consistency. But the method ranks sets of items according to the strength of their *strongest* member, not according to the sum of the strengths of their members. This ensures that it satisfies condition (ii) in the definition of Consistency, contrary to the eigenvector method.

Proof of Proposition 1. For every $a \in \{0, 1, ..., m-1\}^X$, define the ordering $R^a \in \mathcal{R}(\mathcal{X})$ by

$$YR^aZ \Leftrightarrow \max_{y \in Y} a_y \geq \max_{z \in Z} a_z. \tag{2}$$

Call $a, a' \in \{0, 1, ..., m-1\}^X$ *ordinally equivalent* if they generate the same ordering, that is,

$R^a = R^{a'}$. Call them *ordinally compatible* if they generate compatible orderings:

$$YP^aZ \Rightarrow YR^{a'}Z \text{ and } YP^{a'}Z \Rightarrow YR^aZ. \tag{3}$$

Call a' *finer* than a if for all $Y, Z \in \mathcal{X}$, $YP^aZ \Rightarrow YP^{a'}Z$.

For any $T \in \mathcal{T}$, define the function $f^T : \{0, 1, ..., m-1\}^X \to \{0, 1, ..., m-1\}^X$ by

$$f_x^T(a) = \max_{y \in t(x)} a_y \text{ for all } x \in X,$$

where, by convention, $\max_{y \in \emptyset} a_y = 0$. Since $\{0, 1, ..., m-1\}^X$ is a complete lattice and f^T is non-decreasing, Tarski's theorem implies that f^T has a fixed point: there exists $a \in \{0, 1, ..., m-1\}^X$ such that

$$a_x = \max_{y \in t(x)} a_y \text{ for all } x \in X.$$

We claim that all fixed points of f^T are ordinally compatible. To see why, let a, a' be two such fixed points and check first that for any $x, y \in X$,

$$a_x > a_y \Rightarrow a'_x \geq a'_y \text{ and } a'_x > a'_y \Rightarrow a_x \geq a_y. \tag{4}$$

If, say, $a_x > a_y$ and $a'_x < a'_y$, then $\max_{z \in t(x)} a_z > \max_{z \in t(y)} a_z$ and $\max_{z \in t(x)} a'_z < \max_{z \in t(y)} a'_z$. But either xTy or yTx. If xTy, then $y \in t(x)$ and

$$a'_y \leq \max_{z \in t(x)} a'_z < \max_{z \in t(y)} a'_z = a'_y,$$

a contradiction. If yTx, a similar contradiction arises. Statements (4) and (2) now imply (3), i.e., a, a' are ordinally compatible.

It follows that the finest fixed points of f^T are all ordinally equivalent. Call $R(T)$ the common ordering they induce on $\mathcal{X}$ through (2). By construction, R is consistent, and it is the finest consistent set ranking method satisfying (1). That $R_X(T)$ coincides with the decomposition ordering of X at T is a matter of checking.■

The method in Proposition 1 is somewhat unsatisfactory because the ordering $R_X(T)$ is typically quite coarse; it ties all items whenever the tournament T is irreducible.

We now turn to a consistent set ranking method inducing on the items in each tournament a refinement of the Copeland ranking. The Copeland score of item x in tournament T is $|t(x)|$, the number of items x beats. The *Copeland ordering* $R_X^C(T)$ of X ranks items according to their Copeland scores: $xR_X^C(T)y \Leftrightarrow |t(x)| \geq |t(y)|$. For each possible Copeland score, consider the restriction of the tournament T to the items having that score. The *decomposition refinement of the Copeland ordering* ranks these items according to the decomposition ordering of that restriction of T.

Proposition 2 states that there is a finest consistent set ranking method that induces the decomposition refinement of the Copeland ordering of the items. It compares any two sets of items by first looking at their size, and breaks ties by comparing the strongest members of these sets according to the decomposition refinement of the Copeland ordering of the items.

Proposition 2. *There exists a unique finest consistent set ranking method R such that*

$$Y R(T) Z \Leftrightarrow \text{(i) } |Y| > |Z| \text{ or (ii) } |Y| = |Z| \text{ and } (\max_Y R_X(T))\ R_X(T)\ (\max_Z R_X(T)) \tag{5}$$

for all $T \in \mathcal{T}$ and $Y, Z \in \mathcal{X}$. The induced ranking method R_X chooses the decomposition refinement of the Copeland ordering in each $T \in \mathcal{T}$.

Example 1. Consider the tournament T on $X = \{1, 2, 3, 4\}$ with incidence matrix

$$M(T) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Up to a permutation of the items, this is the unique irreducible 4-item tournament. The set ranking method R in Proposition 2 delivers the ranking

$$\begin{aligned} &\{1,2,3,4\}\,, \\ &\{1,2,3\}\,,\ \{1,2,4\}\,,\ \{1,3,4\}\,, \\ &\{2,3,4\}\,, \\ &\{1,2\}\,,\ \{1,3\}\,,\ \{1,4\}\,, \\ &\{2,3\}\,,\ \{2,4\}\,, \\ &\{3,4\}\,, \\ &\{1\}\,, \\ &\{2\}\,, \\ &\{4\}\,, \\ &\{3\}\,. \end{aligned}$$

Condition (i) in the definition of Consistency is met since $t(1) = \{2,3\}\ P(T)\ t(2) = \{3,4\}\ P(T)\ t(4) = \{1\}\ P(T)\ t(3) = \{4\}$. In this example, the ranking induced on the items is strict and coincides with the eigenvector ranking.

Proof of Proposition 2. For every $a \in [0, m-1]^X$, define the ordering $R^a \in \mathcal{R}(\mathcal{X})$ by

$$YR^aZ \Leftrightarrow \frac{m}{m+1}|Y| + \frac{1}{m+1}\max_{y\in Y} a_y \geq \frac{m}{m+1}|Z| + \frac{1}{m+1}\max_{z\in Z} a_z. \tag{6}$$

The ordering R^a compares sets on the basis of (i) their size and (ii) the rating of their strongest member. Despite its weighted average representation, R^a is a lexicographic ordering where size comes first:

$$|Y| > |Z| \Rightarrow YP^aZ. \tag{7}$$

Indeed, if $|Y| > |Z|$, then $\frac{m}{m+1}|Y| + \frac{1}{m+1}\max_{y\in Y} a_y \geq \frac{m}{m+1}(|Z|+1) + \frac{1}{m+1}\max_{y\in Y} a_y \geq \frac{m}{m+1}|Z| + \frac{m}{m+1} > \frac{m}{m+1}|Z| + \frac{1}{m+1}\max_{z\in Z} a_z$, hence YP^aZ. The ratings of the strongest members of X and Y matter only if $|X| = |Y|$.

For any $T \in \mathcal{T}$, define the function $f^T : [0, m-1]^X \to [0, m-1]^X$ by

$$f_x^T(a) = \frac{m}{m+1}|t(x)| + \frac{1}{m+1}\max_{y\in t(x)} a_y.$$

The range of f^T is included in $[0, m-1]^X$ because $0 \leq f_x^T(a) \leq \frac{m}{m+1}(m-1) + \frac{1}{m+1}(m-1) = m-1$ for all $a \in [1, m-1]^X$ and $x \in X$. Since $[1, m-1]^X$ is a complete lattice and f^T is nondecreasing, Tarski's theorem implies that f^T has a fixed point: there exists $a \in [1, m-1]^X$ such that

$$a_x = \frac{m}{m+1}|t(x)| + \frac{1}{m+1}\max_{y\in t(x)} a_y \text{ for all } x \in X.$$

As in the proof of Proposition 1, statement (4) is true for all $x, y \in X$ and any two fixed points a, a' of f^T. To check this claim, suppose again that $a_x > a_y$ and $a'_x < a'_y$. Then

$$\frac{m}{m+1}|t(x)| + \frac{1}{m+1}\max_{z\in t(x)} a_z > \frac{m}{m+1}|t(y)| + \frac{1}{m+1}\max_{z\in t(y)} a_z$$

and

$$\frac{m}{m+1}|t(x)| + \frac{1}{m+1}\max_{z\in t(x)} a'_z < \frac{m}{m+1}|t(y)| + \frac{1}{m+1}\max_{z\in t(y)} a'_z.$$

By (6) and (7), this implies that $|t(x)| = |t(y)|$ and

$$\max_{z\in t(x)} a_z > \max_{z\in t(y)} a_z \text{ and } \max_{z\in t(x)} a'_z < \max_{z\in t(y)} a'_z,$$

leading to the same contradiction as in the proof of Proposition 1.

The rest of the argument is the same as before. The finest fixed points of f^T are all ordinally equivalent. If $R(T)$ is the common ordering they induce on $\mathcal{X}$, then R is consistent, and it is the finest consistent set ranking method satisfying (5). It is straightforward to check that $R_X(T)$ coincides with the decomposition refinement of the Copeland ordering of X at T.■

4 Discussion

The consistent set ranking methods described in Propositions 1 and 2 are based on particular rules for extending an order from a set to the set of its subsets: the first method ranks sets according to the strength of their strongest member, the second ranks them by comparing their size first and, in case of a tie, the strength of their strongest member.

Determining which of the many possible extension rules (see Barberà, Bossert and Pattanaik (2004)) are compatible with Consistency is an open problem[1]. It seems that many popular rules are not: here are two examples.

Example 2. The *rank* of item x in an ordering $R_0 \in \mathcal{R}(X)$ is the number $r(x, R_0) = |\{y \in X : xRy\}|$. Note that $r(x, R_0) > 0$ for all $x \in X$. A natural extension rule consists in ranking sets according to the sum of the ranks of their members. This extension rule is incompatible with Consistency: *there is no consistent set ranking method R such that*

$$YR(T)Z \Leftrightarrow \sum_{y \in Y} r(y, R_X(T)) \geq \sum_{z \in Z} r(z, R_X(T)) \tag{8}$$

for all $T \in \mathcal{T}$ and $Y, Z \in \mathcal{X}$.

This may be proved by using the irreducible 4-item tournament of Example 1. Suppose R is a consistent set ranking method satisfying (8) for all $Y, Z \in \mathcal{X}$. To simplify notation, write $r(x)$ instead of $r(x, R_X(T))$, where T is the tournament of Example 1. We derive a contradiction in four steps:

Step 1: $4P_X(T)3$.

If $3R_X(T)4$, then by Consistency $\{4\} = t(3)R(T)t(4) = \{1\}$, hence $4R_X(T)1$ and by transitivity $3R_X(T)1$. By Consistency again, this implies $\{4\} = t(3)R(T)t(1) = \{2, 3\}$. Since $\{3\}\, R(T)\, \{4\}$, it follows by transitivity that $\{3\}\, R(T)\, \{2, 3\}$, which contradicts (8) since $r(2) > 0$.

Step 2: $2P_X(T)4$.

If $4R_X(T)2$, (8) implies $r(4) \geq r(2)$. It follows that $r(3) + r(4) \geq r(2) + r(3)$, that is, $\{3, 4\}\, R(T)\, \{2, 3\}$. Since $\{3, 4\} = t(2)$ and $\{2, 3\} = t(1)$, Consistency implies $2R_X(T)1$.

On the other hand, $4R_X(T)2$ implies, by Consistency, $\{1\} = t(4)R(T)t(2) = \{3, 4\}$. Since (8) implies $\{3, 4\}\, P(T)\, \{4\}$, it follows that $\{1\}\, P(T)\, \{4\}$, hence $1P_X(T)4$ and, by transitivity, $1P_X(T)2$, a contradiction.

Step 3: $1P_X(T)2$.

If $2R_X(T)1$, then by Consistency $\{3, 4\} = t(2)R(T)t(1) = \{2, 3\}$, and it follows from (8) that $r(4) \geq r(2)$, a contradiction to Step 2.

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[1] The corresponding problem also arises for set *rating* methods. Call such a method cardinally consistent if (i) the ratings of the items are proportional to the ratings of the sets of items they beat and (ii) the ratings of the items fully determine the ratings of the sets of items. The eigenvector set rating method is cardinally consistent. Are there cardinally consistent methods where the rating of a set is not the sum of the ratings of its members?

Step 4: From Steps 1, 2, 3, $1P_X(T)2P_X(T)4P_X(T)3$. Therefore $r(1) = 4 = 3 + 1 = r(2) + r(3)$. Since $\{1\} = t(4)$ and $\{2, 3\} = t(1)$, it follows that $t(4)I(T)t(1)$, violating Consistency since $1P_X4$.

Example 3. Given an ordering $R_0 \in \mathcal{R}(X)$, the *rank vector* of a set $Y \in \mathcal{X}$, is the m-dimensional vector $\mathbf{r}(Y, R_0)$ whose first $|Y|$ coordinates are the ranks of the items in Y listed in non-increasing order, and whose remaining $m - |Y|$ coordinates are all zero. The leximax extension rule ranks sets by applying the lexicographic ordering $\geq_L$ to their rank vectors. Thus, sets are ranked by comparing their strongest members first, their second strongest members second, and so on. This rule too is incompatible with Consistency: *there is no consistent set ranking method R such that*

$$YR(T)Z \Leftrightarrow \mathbf{r}(Y, R_X(T)) \geq_L \mathbf{r}(Z, R_X(T)).$$

for all $T \in \mathcal{T}$ and $Y, Z \in \mathcal{X}$. This may again be proved by using the tournament of Example 1.

5 References

Barberà, S., Bossert, W. and Pattanaik, P.K. (2004), "Ranking Sets of Objects," in S. Barberà, P. J. Hammond and C. Seidl (eds.) *Handbook of Utility Theory,* Volume 2, Dordrecht: Kluwer Academic Publishers, 893–977.

Kendall, M.G. (1955), "Further contributions to the theory of paired comparisons," *Biometrics*, **11,** 43–62.

Landau, E. (1895), "Zur relativen Wertbemessung der Turnierresultate," *Deusches Wochenschach*, **11**, 366–369.

Wei, T.H. (1952), "The algebraic foundations of ranking theory," Ph.D. Thesis, University of Cambridge, UK.