Agendas in Legislative Decision-Making*

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Abstract

Despite the wide range of agendas used in legislative decision-making, the literature has focused almost exclusively on two stylized formats, the so-called Euro-Latin and Anglo-American agendas. As emphasized by Ordeshook and Schwartz [1987], this focus leaves a sizable gap in our understanding of the legislative process.

To help address the deficiency, I first define a very broad class of agendas (called *simple agendas*) whose features are common among agendas used in legislative settings. I then characterize the *sophisticated* (Farquharson [1969]) voting outcomes implemented by agendas in this class. By establishing a clear connection between the structure of simple agendas and the outcomes associated with them, the characterization extends our understanding of legislative decision-making well beyond the very limited scope of Euro-Latin and Anglo-American agendas.

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1 Introduction

Agenda voting provides a straightforward way to make a group decision. By majority voting on the sequence of "aye" (yes) or "nay" (no) questions set out by the agenda, the group can whittle down the issue gradually; and, ultimately, arrive at a consensus. In legislative settings, almost every issue is decided by agenda; and, a wide range of agenda formats are used (Black [1948, 1958]; Riker [1958]; Ordeshook and Schwartz [1987]; Miller [1995]). Yet, the literature has focused almost exclusively on two stylized formats, the *Euro-Latin* and *Anglo-American* agendas (Miller [1977, 1980]; Shepsle and Weingast [1984]; Banks [1985]; Rasch [2000]; Apesteguia, Ballester, and Masatlioglu [2014]; Gershkov, Moldovanu, and Shi [2015]; Barberà and Gerber [2016]).[1]

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[1] These two agendas are named for their relative prominence in the Euro-Latin and Anglo-American legal traditions. Following Farquharson [1969], some of the cited papers adopt a different nomenclature: Euro-Latin agendas are called *successive agendas*; and, Anglo-American agendas are called *amendment agendas*.

As emphasized by Ordeshook and Schwartz, the focus on Euro-Latin and Anglo-American agendas leaves a significant gap in our understanding of legislative decision-making. These formats are not as prevalent as their place in the literature suggests; and, they can lead to outcomes which are quite different from other agendas used in practice. These observations are troubling. Legislative decision-making plays a crucial role in the behavior of political actors and policy makers; and, without a clear picture of this basic institution, one cannot begin to address some of the most basic questions in political economy and public economics.

The goal of this paper is to extend our understanding of strategic voting behavior (called *sophisticated voting* by Farquharson [1969]) to a much broader range of legislative agendas beyond Euro-Latin and Anglo-America agendas. To do so, I define a new class of agendas, called *simple agendas*, whose structural features are prevalent among agendas used in practice. I also define a special sub-class, called *priority agendas*, whose sequential structure is most closely related to Euro-Latin and Anglo-America agendas.

To get a better sense of these agenda classes, recall that an agenda is a sequence of questions to be decided by simple majority voting. Formally, this corresponds to a binary game tree where each stage game involves a vote between "aye" and "nay" options. Figure 1 illustrates the questions (*top*) and game trees (*bottom*) associated with the Euro-Latin and Anglo-American agendas on three alternatives.

Q1. *Should x_1 be accepted?*
Q2. *If not, should x_2 be accepted?*

Q1. *Should x_2 be eliminated over x_1?*
Q2. *Should x_3 be eliminated over the "winner" of Q1.?*

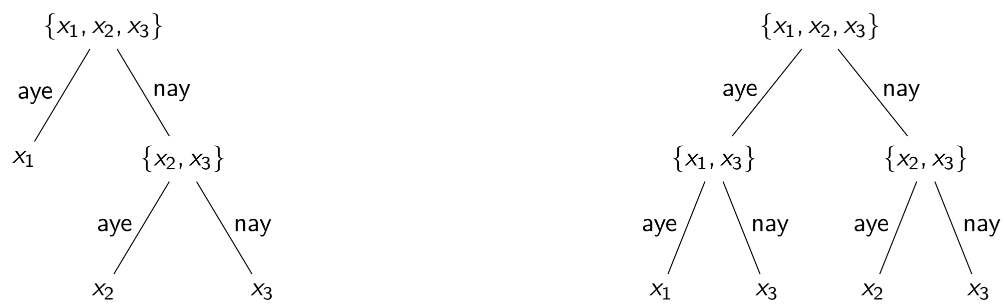


Figure 1: **Left**: a Euro-Latin agenda on three alternatives. **Right**: an Anglo-American agenda on three alternatives.

For a Euro-Latin agenda (*left*), the voting is by *sequential majority approval*: the voters consider one alternative in each stage. Ultimately, the alternative selected is the first one to be approved by majority. In contrast, the voting for an Anglo-American agenda (*right*) is by *sequential majority comparison*: the voters compare two alternatives in each stage with the "loser" being eliminated and the "winner" moving on to the next comparison. Ultimately, the alternative selected is the only one which survives this process.

Despite their differences, these agendas share the same kind of structure. Both can be built using a *priority ordering* which specifies when to add each alternative to the agenda;[2] and, an *amendment rule* which specifies how to add each alternative.[3] For both agendas in Figure 1, x_1 is the first alternative added to the agenda, x_2

[2]The priority is *not* intended to capture the temporal order in which motions are proposed. As illustrated by Figure 3, the agenda may be *backwards-built*, a practice which is relatively common in the Anglo-American legal tradition (see Miller [1995]).

[3]Whether it involves a *perfecting amendment*, a *second-order amendment*, or a *substitute* affects how the alternative is added.

the second, and x_3 the third. The difference is the amendment rule: on the Euro-Latin agenda, the alternative x_3 can only ever face x_2; and, on the Anglo-American agenda, it can face x_1 *or* x_2.

These amendment rules are straightforward to generalize to longer agendas: for a Euro-Latin agenda, each new addition only faces the last alternative added to the agenda; and, for an Anglo-American agenda, it faces every alternative previously added to the agenda. Figure 2 illustrates how each agenda in Figure 1 is extended when a fourth alternative (x_4) is added.[4]

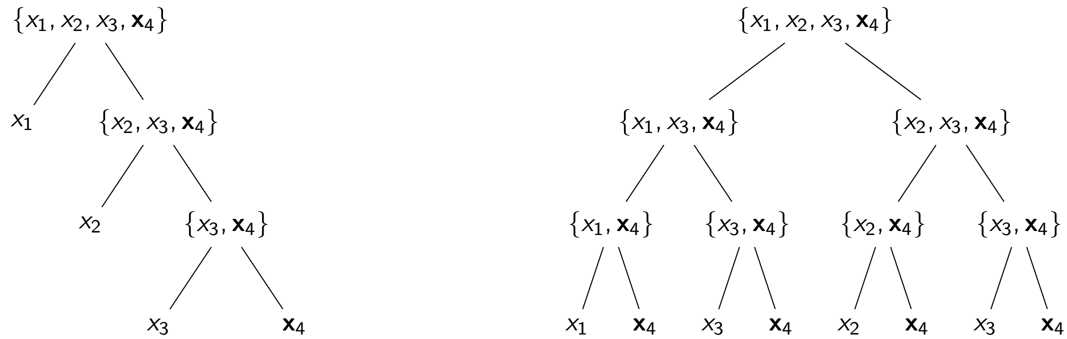


Figure 2: Extensions of the agendas in Figure 1. **Left**: a Euro-Latin agenda. **Right**: an Anglo-American agenda.

Euro-Latin and Anglo-American agendas have the feature that the same kind of amendment occurs at each stage. One way to expand the set of agendas is to allow for a mix of Euro-Latin and Anglo-American amendments. Figure 3 illustrates two such "hybrid" agendas (as they are called by Ordeshook and Schwartz).

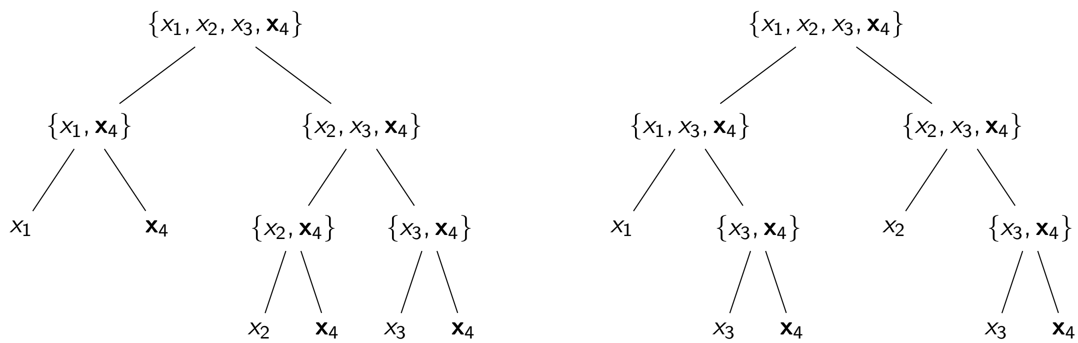


Figure 3: Hybrid extensions of the agendas in Figure 1. **Left**: a Euro-Latin agenda extended by adding x_4 via Anglo-American amendment. **Right**: an Anglo-American agenda extended by adding x_4 via Euro-Latin amendment.

Priority agendas generalize the "mixed" structure of the hybrid agendas in Figure 3. Like Euro-Latin and Anglo-American agendas, they are defined by a priority ordering and an amendment rule. The difference is that they allow for a much wider range of amendments: a new addition z to the agenda can face *any* alternative y that is already on the agenda — provided that it faces every alternative that was added between y and z. Because of their flexible structure, priority agendas include a wide variety of agendas used in practice.

[4]To avoid clutter, I omit the edge labels. Instead, I adopt the convention that left means "aye" and right means "nay".

To illustrate, suppose that a bill b is put up for consideration; and, subsequently, two amendments a' and a'' to b are proposed. If the two amendments are *incompatible* (in the sense that both cannot be applied to b simultaneously), then Congressional procedure could result in the following sequence of questions (see Figure 2 of Schwartz [2008]; or, Figure 5 of Ordeshook and Schwartz [1987]):

Q1. *Should the first amendment a' be accepted?*
Q2. *If not, should the second amendment a'' be accepted?*
Q3. *Should the bill as amended be selected over the status quo?*

This corresponds to the left-hand agenda in Figure 3 where: x_4 is the *status quo* legislation; x_3 the original bill b; x_2 the bill amended by a''; and, x_1 the bill amended by a'. Effectively, this agenda uses Euro-Latin voting to determine which version of the bill (x_1, x_2, or x_3) is put to a final *up or down* vote.

A very different approach would be to first decide on recognizing just one of the amendments (by way of a preliminary motion); and, then use Euro-Latin voting to decide among the remaining legislative options:

Q1. *Should the amendment a' be recognized over the amendment a''?*
Q2. *Should the bill as amended (by the "winner" of Q1.) be accepted?*
Q3. *If not, should the original bill be accepted?*

It is not difficult to see that this sequence of questions corresponds to the right-hand agenda in Figure 3.

Despite their flexibility, priority agendas rule out a variety of agendas used in practice. To illustrate, suppose instead that a' and a'' are *compatible*. Then, Congressional procedure requires an *up or down* vote on a' before a'' can be recognized (see Figure 3 of Ordeshook and Schwartz [1987]). This leads to the agenda in Figure 4.

Q1. *Should the first amendment a' be accepted?*
Q2. *Should the second amendment a'' be accepted?*
Q3. *Should the bill as amended be selected over the status quo?*

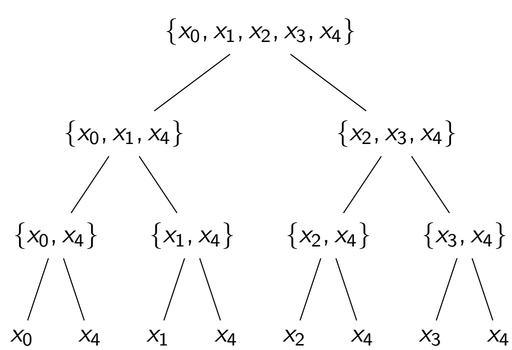


Figure 4: A two-period Congressional agenda.

Compared with the left-hand agenda in Figure 3, the only difference (which stems from Q2.) is the addition of the bill b as amended by a' and a''. In Figure 4, this "doubly amended" bill is represented by x_0. Despite its close relationship to the left-hand agenda in Figure 3, the *two-period* Congressional agenda in Figure 4 is *not* a priority agenda. Intuitively, the problem is that the symmetric structure of the agenda cannot be reconciled with the existence of a priority ordering among the alternatives in $\{x_1, ..., x_4\}$.

To accommodate this agenda (and many others), I set aside the procedure used to build agendas; and, instead concentrate on their structural features. The simple agendas that I consider are defined by two features: *history-independence*, which stipulates that the question posed can only depend on the alternatives not yet eliminated from consideration; and, *persistence*, which requires that the "safe" alternatives (which do not risk elimination) for a given question must be tested as a group by one of the subsequent questions.[5]

Both features appear to be common in practice. Among the legislative agendas discussed in the literature, the only ones which violate history-independence involve *procedural* motions that do not eliminate anything from consideration; and, the only ones which violate persistence involve *substitute* amendments. Stated in terms of the taxonomy introduced by Ordeshook and Schwartz [1987], the procedural questions make an agenda *repetitive*; and, the substitute amendments make it *discontinuous*. In the result, the class of simple agendas appears to include almost all agendas dealing with substantive issues — with the exception of discontinuous agendas like the *two-stage amendment agendas* used in Congress (as in Example 9 of the Supplementary Appendix; the paper by Banks [1989]; or, Figure 2 of Ordeshook and Schwartz [1987]).

1.1 Results

While a contribution in itself, the definition of these agenda classes is only the first step of the analysis. The main contribution is to characterize what *decision rules* can be implemented by simple agendas and priority agendas, respectively: as the voters' preferences *or* the alternatives change, how must the sophisticated outcome change? The point is to clarify how the structure of an agenda relates to its voting outcomes.

A secondary contribution (made possible by the main results) is to compare the sophisticated outcomes for simple agendas and priority agendas to some other agenda classes — namely the broader class of non-repetitive and continuous agendas; and, the narrower classes of Euro-Latin and Anglo-American agendas. Here, the point is to clarify how the scope of outcomes changes as the range of agendas expands.

Before turning to the related literature, I briefly describe these two contributions in greater detail.

Implementation: In Section 3, I first show that two natural conditions, *Issue Splitting* and the *Independence of Losing Alternatives*, characterize sophisticated voting on simple agendas (Theorem 1). The first weakens *Path Independence* (Plott [1973]). It stipulates that the outcome can always be determined by splitting the decision problem into a pair of sub-issues.[6] In turn, the second condition weakens the *Independence of Irrelevant Alternatives*[7] (Chernoff [1954]; Radner and Marschak [1954]). It states that the removal of an alternative with no majority appeal (known as the *Condorcet loser*) cannot affect the outcome.[8]

In Section 5, I show that sophisticated voting on the narrower class of priority agendas satisfies a *Marginalization* property known to hold for Euro-Latin and Anglo-American agendas (Proposition 3): there is some

[5]The agenda in Figure 4 is clearly history-independent (since every node has a unique label). Likewise, it is persistent. To see this, note that x_4 is the only alternative which is ever safe; but, whenever that is the case, it is tested by a subsequent question.

[6]It also weakens a condition called *Division Consistency* (Apesteguia et al. [2014]) or *Weak Separability* (Xu and Zhou [2007]).

[7]This "elimination-based" IIA condition was first proposed in the context of individual choice; and, it is formally distinct from the IIA condition that Arrow [1950] proposed for social choice (Ray [1973]). Having said this, elimination-based independence conditions have been considered in the context of voting (see e.g., Fishburn [1974]; Young [1995]; and, Ching [1996]).

[8]Apesteguia et al. [2104] call this condition *Condorcet Loser Consistency*. That name usually refers to a much weaker property which stipulates that a Condorcet loser cannot be chosen. To avoid potential confusion, I depart from their nomenclature.

alternative which is chosen only if it has majority support over every other alternative (Miller [1977]; Moulin [1991]). (Like Euro-Latin and Anglo-American agendas, the *marginal alternative* is the last one added to the agenda.) Combined with *Issue Splitting* and the *Independence of Losing Alternatives*, this *Marginalization* property characterizes sophisticated voting on priority agendas (Theorem 2).

Comparison: In Section 4, I show that simple agendas have two appealing properties that distinguish them from the broader class of non-repetitive and continuous agendas. First, the outcome is unchanged when it improves in the voters' preferences (Proposition 1).[9] Previously, only knockout and Anglo-American agendas were known to exhibit this *Preference Monotonicity* property (Moulin [1986, 1991]; Altman et al. [2009]). Second, the removal of an alternative yields an agenda which is outcome-equivalent to a simple agenda (Proposition 2). This *Self-Consistency* property limits the ability to manipulate an agenda by expanding it.

In Section 6, I show that every property of Euro-Latin and Anglo-American agendas emphasized in the literature extends to priority agendas. Besides Preference Monotonicity (Proposition 1), Self-Consistency (Proposition 2), and Marginalization (Proposition 3), priority agendas have the property that the outcome is unchanged when it moves up in terms of priority (Proposition 4). Previously, this *Priority Monotonicity* property was only known to hold for Euro-Latin and Anglo-American agendas (Moulin [1986, 1991]; Jung [1990]).

Despite these commonalities, there is a natural way to distinguish Euro-Latin and Anglo-American agendas from other priority agendas. It is tied to the fact that Euro-Latin agendas are less *discriminating* than Anglo-American agendas: while the former can select alternatives throughout the Top Cycle (Miller [1977]), the latter can only select alternatives from a special subset (identified by Banks [1985]). In fact, the discriminability of every priority agenda must be somewhere between these two extremes (Proposition 5).

1.2 Related Literature

At one extreme, much of the literature on sophisticated voting contemplates a very narrow range of agendas. Besides the many papers on Euro-Latin and Anglo-American agendas (cited above), there is some work on specialized agendas that arise in the context of legislative voting (Banks [1989]; Miller [1995]; Schwartz [2008]); or, *tournament solutions* (Coughlan and Le Breton [1999]; Fischer et al. [2011]; Iglesias et al. [2014]).[10]

At the other extreme, a small number of papers are concerned with general conditions which are either necessary (McKelvey and Niemi [1978]; Moulin [1986]; Srivastava and Trick [1996]) or necessary *and* sufficient (Horan [2013]) for implementation by agenda. On the whole, the existence results in these papers are non-constructive. So, while they delimit what outcomes can and cannot be achieved with strategic agenda voting *in general*, they are silent about what outcomes can be achieved with a *particular* agenda.

The current paper bridges the gap between these two strands of the literature — by characterizing the relationship between"form" and "function" for a wide range of agendas used in legislative settings. Since the prior work on this issue is quite limited, the main connection to the existing literature is methodological.

In this regard, two papers merit particular emphasis. First and foremost is Ordeshook and Schwartz [1987], which defines a useful taxonomy of agenda features (later extended by Miller [1995]). While my definition of

[9]This weakens *Strategyproofness* (Sanver and Zwicker [2009]) and May's [1952] *Positive Responsiveness* (Fishburn [1982]).
[10]For a nice overview of the vast literature on tournament solutions, see Laslier [1997] or Brandt et al. [2016].

simple agendas does not rely on the structural features which they identify, it is motivated by a similar desire to abstract away from institutional details and focus on the structural elements of agendas.

No less important is Apesteguia et al. [2014]. The key innovation of that paper, which I exploit, is to adopt a *decision rule* framework to analyze agenda voting.[11] Unlike the conventional approach to implementation, this framework allows both the voter preferences *and* the set of feasible alternatives to vary. For agendas, it turns out that this does not enrich the scope of implementable rules (see Remark 1). However, it dramatically simplifies the statement and the interpretation of the conditions for implementation. As an added benefit, it also provides a natural setting to study problems, like *strategic candidacy* and *agenda-setting* (discussed in Section 7), that depend on how voting outcomes change when alternatives become unavailable.

2 Definitions

In this section, I review the basic definitions used in the paper.[12] There is a finite set N of *voters* and a finite universe X of social *alternatives*. For convenience, I assume that $|N|$ is odd and $|N|, |X| \geq 2$. A *profile* of voter preferences $P \equiv (P_i)_{i \in N}$ is an $|N|$-tuple of (linear) orderings P_i over X; an *issue* A is a non-empty subset of X; and, a *decision problem* is pair (P, A). Let $\mathbf{P}$ denote the collection of all profiles; $\mathbf{X}$ the collection of all (non-empty) issues; and, $\mathbf{X_j} \equiv \{A \in \mathbf{X} : |A| = j\}$ the sub-collection of issues with cardinality j.

Definition 1 *A **decision rule** is a mapping $v : \mathbf{P} \times \mathbf{X} \to X$ such that $v(P, A) \in A$ for all $(P, A) \in \mathbf{P} \times \mathbf{X}$.*

In other words, a decision rule prescribes a social outcome for every possible combination of voter preferences and social alternatives. In the sequel, I study the implementation of decision rules by agenda. Formally:

Definition 2 *An **agenda** T_A on $A \subseteq X$ is a* finite binary tree *such that:*

- **(i)** *each* terminal node *is* labeled *by (a set consisting of) one alternative in A;*
- **(ii)** *each alternative in A labels one or more terminal nodes; and,*
- **(iii)** *each* non-terminal node *is labeled by the set of alternatives that label its two* successors.[13]

An agenda defines a *game tree*: each terminal node (or *outcome*) represents a social alternative; and, each non-terminal node (or *stage game*) represents a majority vote between the two subgames that follow it. In legislative settings, each vote relates to a *question* of policy. By convention, the left subgame represents majority support for "aye" while the right subgame represents majority support for "nay." With this interpretation in mind, the first two features of Definition 2 ensure that: (i) every sequence of questions leads to some social outcome; and, (ii) every social outcome results from some sequence of questions. In turn, feature (iii) amounts to a labeling convention: every node is designated by the outcomes that are *reachable* from it.[14]

[11] The Supplementary Appendix provides a more detailed discussion of the connection to Apesteguia et al. [2014].

[12] Whenever possible, I follow the notation used by Apesteguia et al. [2014] (see Section 2 of their paper).

[13] The Supplementary Appendix gives a primer of the standard graph theory terminology used here and elsewhere in the paper.

[14] This convention follows the Farquharson-Miller definition rather than the Ordeshook-Schwartz definition (see Schwartz [2008]). Since my focus is *sophisticated* rather than *sincere* voting, the labeling convention plays no role in the analysis. I follow it only because it simplifies the definition of the agenda classes considered in the sequel.

To get a sense of these three features, it is worth re-considering Figures 1 to 4 of the Introduction. There, I abuse notation by omitting the set brackets for the labels of the terminal nodes. In the sequel, I perpetuate this abuse by omitting the brackets for singleton sets when there is no danger of causing confusion.

To adapt an agenda T_X to a sub-issue $A \subset X$, one can *prune away* all of the terminal nodes labeled by infeasible alternatives (see Xu and Zhou [2007]; Horan [2011]; and, Bossert and Sprumont [2013]).

Definition 3 *Given an agenda T_X on X, the* ***pruned agenda*** $\mathrm{T}_{X|A}$ *on $A \subseteq X$ is constructed as follows:*

(1) *First, remove every terminal node of T_X that is labeled by an alternative $x \in X \setminus A$.*

(2) *Then, remove every node with one successor, connecting its successor to its predecessor.*[15]

(3) *Finally, relabel every non-terminal node of the resulting tree to conform with Definition 2.*

Like a single-elimination competition in sports, the idea is that the infeasible alternatives (i.e., the alternatives in $X \setminus A$) "forfeit" their position without otherwise modifying the agenda. To illustrate, consider the following agenda T_X on $X = \{a, a', b, c, x\}$ and the associated pruned agenda $\mathrm{T}_{X|A}$ on $A = \{b, c, x\}$:

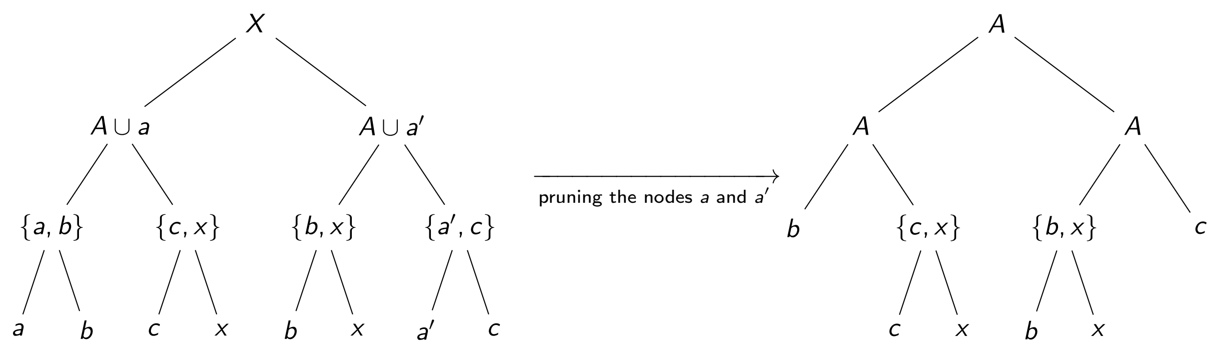


Figure 5: **Left**: an agenda T_X. **Right**: the associated pruned agenda $\mathrm{T}_{X|A}$.

Given an agenda T_A, each pair $(\mathrm{T}_A; P)$ defines a *complete information extensive-form game* with voter preferences given by P and outcomes in A. For games like this, the natural solution concept is Farquharson's [1969] notion of "sophisticated" voting, which is based on the idea that voters are forward-looking. For this solution concept, the (unique) equilibrium outcome only depends on the majority relation.

Definition 4 *For $(P, A) \in \mathbf{P} \times \mathbf{X}$, the* ***majority relation*** M_A^P *between alternatives $x, y \in A$ is given by:*

$$xM_A^P y \quad \textit{if} \quad |\{i \in N : xP_i y\}| > |N|/2.$$

Since $|N|$ is odd, M_A^P is a *total* (for distinct $x, y \in A$, $xM_A^P y$ or $yM_A^P x$) and *asymmetric* (for $x, y \in A$, not $xM_A^P y$ and $yM_A^P x$) binary relation on A. In the literature, such a relation is known as a *tournament*.[16]

[15] In case the root node (which has no predecessor) has a unique successor, its successor becomes the new root node.

[16] McGarvey [1953] was the first to consider the problem of finding an upper bound on the minimum number of voters $|N|$ such that $\{M_A^P : P \in \mathbf{P}\}$ coincides with the set of *all* tournaments on A. He showed that $|N| \leq |A|^2 - |A|$. Subsequently, Stearns [1959] improved the bound to $|N| \leq |A| + 2$ for $|A|$ even; and, $|N| \leq |A| + 1$ for $|A|$ odd. More recently, Fiol [1992] improved it to $|N| \leq |A| - \lfloor \log_2 |A| \rfloor + 1$ for $|A| - \lfloor \log_2 |A| \rfloor$ even; and, $|N| \leq |A| - \lfloor \log_2 |A| \rfloor$ for $|A| - \lfloor \log_2 |A| \rfloor$ odd.

In any vote at the final stage of T_A, each voter has a weakly dominant strategy to endorse her preferred alternative. So, M_A^P determines which alternative wins a majority in any vote at this stage. When deciding how to vote in any subgame at the penultimate stage, the forward-looking voters discount the alternatives that lose at the final stage; and, again perceive the vote as a choice between two alternatives. So, M_A^P again determines which option ("yea" or "nay") wins a majority at this stage. By extending this reasoning back to the root node, one can use M_A^P to determine the sophisticated voting outcome of $(\mathrm{T}_A; P)$.

As shown by McKelvey and Niemi [1978], this "backward induction" reasoning is equivalent to finding the *dominance solvable* (Moulin [1979]) or *undominated Nash equilibrium* outcome of $(\mathrm{T}_A; P)$.[17] To formalize:

Definition 5 *For a game $(\mathrm{T}_A; P)$, the* ***sophisticated voting outcome*** *$UNE[\mathrm{T}_A; P]$ is defined as follows:*

(1) *First, define $\mathrm{T}_A^P(1) \equiv \mathrm{T}_A$ to be the original agenda.*

(2) *Then, for each $j > 1$, define the agenda $\mathrm{T}_A^P(j+1)$ from $\mathrm{T}_A^P(j)$ as follows:*

- *In each terminal subgame of $\mathrm{T}_A^P(j)$, prune away the loser according to M_A^P; and,*
- *Relabel the non-terminal nodes of the resulting binary tree to conform with Definition 2.*

(3) *Finally, define $UNE[\mathrm{T}_A; P] \equiv \mathrm{T}_A^P(K)$ where K is the smallest j such that $\mathrm{T}_A^P(j) = \mathrm{T}_A^P(j+1)$.*

Having defined the game form and solution concept, it remains to formalize the notion of implementation:

Definition 6 *A decision rule v is* ***implementable by agenda*** *if there exists an agenda T_X such that*

$$v(P, A) = UNE[\mathrm{T}_{X|A}; P]$$

for every decision problem $(P, A) \in \mathbf{P} \times \mathbf{X}$. In that case, the agenda T_X ***implements*** *the decision rule v.*

Despite appearances, this is no more general than the usual notion of implementation with a *fixed* issue X. To see this, consider a decision problem (P, A) and let P^A denote a profile that coincides with P on A but, in each voter preference, places the alternatives in $X \setminus A$ (in a specific order) below every alternative in A. For an agenda, this "demotion" amounts to the same as "pruning away" the alternatives in $X \setminus A$:

Remark 1 *If a decision rule v is implementable by agenda, then $v(P, A) = v(P^A, X)$ for all $(P, A) \in \mathbf{P} \times \mathbf{X}$.*

Put differently, the sub-issues play no role in the analysis of agenda implementation. Once the agenda-setter determines what to implement for X, the outcomes for all sub-issues $A \subset X$ are pinned down.

3 Simple Agendas

After defining simple agendas, I characterize the family of decision rules implemented by agendas in this class; and, provide a "recipe" for the unique simple agenda that implements each decision rule in this family.

3.1 Definitions and Examples

Simple agendas are defined by two features. The first stipulates that the label of each node fully determines the structure of the agenda below. For an agenda T_X, let $\ell(q)$ denote the label of node q. By the convention

[17] The quotation marks (which I continue to use throughout the paper) serve as a reminder that this is not technically equivalent to backward induction. The issue is the lack of full information: every stage game involves simultaneous choices by all voters.

in Definition 2(3), $\ell(q)$ is the set of outcomes that are reachable from q. Letting q_1 and q_2 denote the two successors of a non-terminal node q, the first feature may then be stated as follows:

Feature 1 *An agenda is* ***history-independent*** *if, for every pair of non-terminal nodes q and $\tilde{q}$,*

$$\ell(q) = \ell(\tilde{q}) \text{ only if } \{\ell(q_1), \ell(q_2)\} = \{\ell(\tilde{q}_1), \ell(\tilde{q}_2)\}.$$

This condition requires the same question to be posed at any two nodes that ultimately lead to the same set of potential outcomes. Equivalently, the structure of the agenda below each node q can only depend on the outcomes that are reachable from q (and *not* on the history of nodes that led to q).[18]

In turn, the second feature requires that outcomes are contested in an orderly fashion. Given a non-terminal node q, let $u(q) \equiv \ell(q_1) \cap \ell(q_2)$ denote the set of *uncontested* outcomes (that do not risk elimination) at q; and, let $c(q) \equiv \ell(q) \setminus u(q)$ denote the set of *contested* outcomes (that do risk elimination) at q. Using these definitions, the second feature may be stated as follows:

Feature 2 *An agenda is* ***persistent*** *if, for every non-terminal node q such that $u(q) \neq \emptyset$ and each terminal node t below q, there exists a non-terminal node q^t* between *q and t such that $u(q) \in \{\ell(q_1^t), \ell(q_2^t)\}$.*

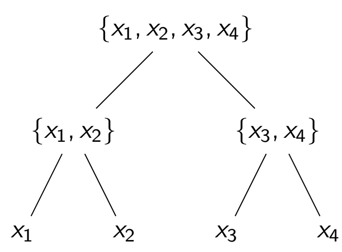
The idea is that $u(q)$ is an issue that must be addressed before making a final choice. Since q does not engage this issue, there must be some subsequent question q^t that does so for each outcome $\ell(t)$ below q.

If an agenda is persistent *and* history-independent, then its structure is particularly simple (see Claim 3 of the Appendix).[19] In that case, the question q^t pits $u(q)$ against (an issue that includes) $c(q) \cap \ell(q^t)$. So, the question q^t either eliminates some of the outcomes uncontested at q; or, all of the remaining outcomes contested at q. In fact, q^t must be the first question after q where any outcome in $u(q)$ is contested. For all questions between q and q^t, the contested outcomes must be ones that were previously contested at q.

Class 1 *An agenda is* ***simple*** *if it is both history-independent and persistent.*

In the sequel, I denote a simple agenda on A by S_A; and, the class of simple agendas (on issues $A \in \mathbf{X}$) by $\mathcal{S}$. This class includes a wide range of agendas used in practice. Some examples will help to illustrate.

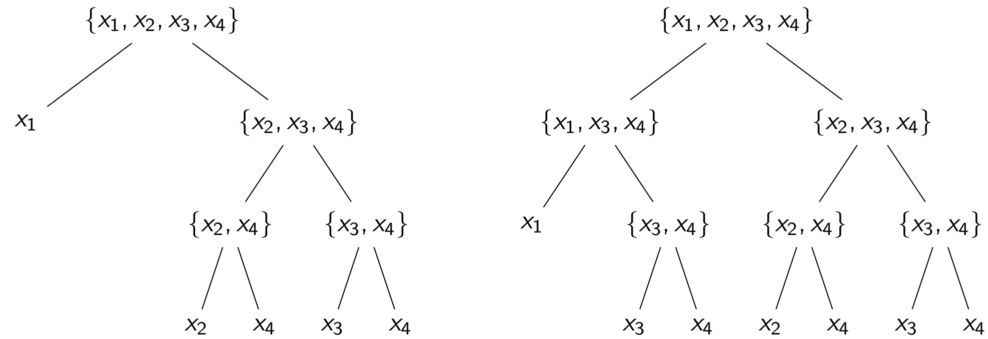
Example 1 *One important sub-class is the class of* ***knockout*** *(or* ***partition****)* ***agendas*** *where each alternative labels exactly one terminal node. Up to permutation of the alternatives, there are two knockout agendas for $|X| = 4$ alternatives: the balanced agenda below (which is sometimes called a* ***Plott-Levine agenda*** *in the literature); and, the unbalanced (Euro-Latin) agenda in Figure 2.*



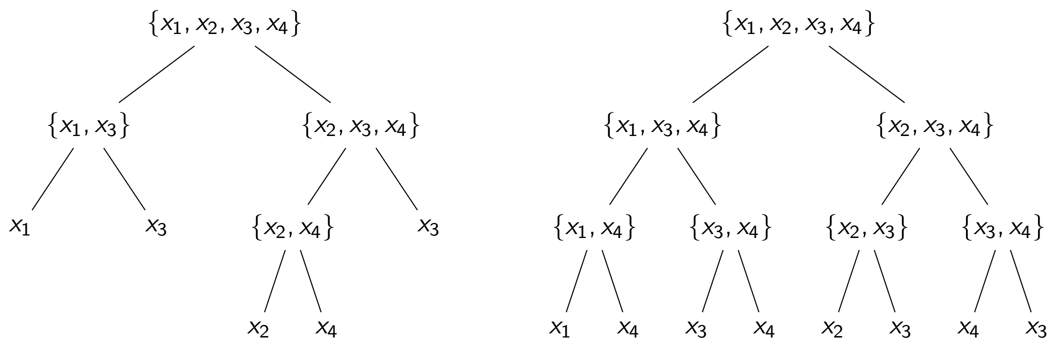
[18]Given the finiteness of the agenda, this implies Farquharson Axiom II: for each non-terminal node q, $\ell(q) \neq \ell(q_1), \ell(q_2)$.

[19]Example 12 of the Supplementary Appendix shows that these two features are independent.

Example 2 *Another important sub-class is the class of* ***priority agendas*** *studied in Section 5. For $|X| = 3$, the only priority agendas are the Euro-Latin and Anglo-American agendas. For $|X| = 4$, there are four additional priority agendas (up to permutation): the two agendas in Figure 3; and, the two agendas below.*



Example 3 *When $|X| = 3$, every simple agenda is a knockout agenda or a priority agenda. As the number of alternatives grows, an increasing variety of simple agendas fall outside these two sub-classes. When $|X| = 4$, there are already two such agendas (up to permutation):*



Examples 1-3 describe all of the simple agendas on $|X| = 4$ alternatives: the six priority agendas; the knockout agenda in Example 1; and, the two agendas in Example 3. One natural way to build larger agendas from these nine agendas is through the formation of compound agendas.

Definition 7 *Given an agenda $\mathtt{T}_A$ on $A = \{a_1, ..., a_m\}$ and a collection of agendas $\{\mathtt{T}^{a_i}_{D_i} : a_i \in A\}$ such that $D_i \cap A = a_i$ for each i, the* ***compound agenda*** *$\mathtt{T}_A \otimes \{\mathtt{T}^{a_i}_{D_i} : a_i \in A\}$ on $D \equiv \bigcup_{a_i \in A} D_i$ is constructed as follows:*

- **(1)** *First, replace each terminal node of $\mathtt{T}_A$ labeled by $a_i \in A$ with the agenda $\mathtt{T}^{a_i}_{D_i}$.*
- **(2)** *Then, relabel the non-terminal nodes of the resulting binary tree to conform with Definition 2.*

The basic idea is that $D \setminus A$ consists of alternatives that were not up for consideration in $\mathtt{T}_A$. Instead of having to settle on $a_i \in A$, the voters have the opportunity to select among the alternatives in D_i. To do so, they follow the agenda $\mathtt{T}^{a_i}_{D_i}$ at any point where they would have otherwise selected a_i from $\mathtt{T}_A$.

It is clear that the simplicity of the "base" agenda $\mathtt{T}_A$ and every "extension" $\mathtt{T}^{a_i}_{D_i}$ is necessary but *not* sufficient for the resulting compound agenda to be simple. Nonetheless, simplicity *is* preserved under a variety of natural compounding procedures — like the procedure used to extend priority agendas (discussed in Example 8 below); or, the procedure used to "standardize" an agenda.

Definition 8 *An agenda is* ***standard*** *if the last question always involves the* ***status quo*** *alternative x^*. Clearly, one can "standardize" any agenda T_A such that $x^* \notin A$ by adding a final question at each terminal node. Formally, this defines a compound agenda $\mathrm{T}^*_A \equiv \mathrm{T}_A \otimes \{\mathrm{S}_{\{a_i,x^*\}} : a_i \in A\}$ on $A \cup x^*$.*

It is not difficult to see that a standardized agenda is simple if and only if the base agenda is simple. In fact, many agendas used in practice are standardized simple agendas. The hybrid agenda on the left-hand side of Figure 3 is a standardized Euro-Latin agenda, for instance. Similarly, the two-period Congressional agenda in Figure 4 is a standardized Plott-Levine agenda.[20] (In both cases, x_4 serves as the status quo alternative).

3.2 Implementation

The necessary and sufficient conditions for implementation by simple agenda are natural conditions related to *Path Independence* and the *Independence of Irrelevant Alternatives* (IIA).

The first condition stipulates that the outcome can be determined by "splitting up" an issue into simpler issues. A pair $(B, C) \in \mathbf{X} \times \mathbf{X}$ defines a **splitting** of $A \in \mathbf{X}$ for a decision rule v if:

- **(i)** $B \cup C = A$ (so that B and C form an *exact bi-cover* of A);
- **(ii)** $B \cap C \neq B, C$ (so that B and C are *non-nested* sets); and,
- **(iii)** $v(P, A) = v(P, \{v(P, B), v(P, C)\})$ for every profile $P \in \mathbf{P}$.

When there exists a splitting of A, v is **splittable** for A. Then, the first condition can be stated as follows:

Issue Splitting **(IS)** *For every issue $A \notin \mathbf{X}_1$, v is splittable.*

By comparison, Path Independence imposes a much stronger requirement: the equality in (iii) must hold for *all* pairs (B, C) that (i) form an exact bi-cover of A even when (ii) the sub-issues B and C are nested.[21]

In turn, the second condition stipulates that outcomes cannot be affected by the presence of a universally unappealing alternative. An alternative $a \in A$ is the *Condorcet loser* for a decision problem $(P, A) \in \mathbf{P} \times \mathbf{X}$ if $xM^P_A a$ for all $x \in A \setminus a$. Then, a decision rule v is **independent of the losers** for $A \in \mathbf{X}$ if: for all $P \in \mathbf{P}$ and $a \in A$, $v(P, A) = v(P, A \setminus a)$ if a is the Condorcet loser for (P, A). The second condition requires this kind of independence for every non-degenerate issue.

Independence of Losing Alternatives **(ILA)** *For every issue $A \notin \mathbf{X}_1$, v is independent of the losers.*

By comparison, IIA imposes a much stronger requirement: the removal of any unchosen alternative must not affect the outcome, *regardless* of whether this alternative is the Condorcet loser.

[20] Ordeshook and Schwartz [1987] discuss some related agendas: the agenda in their Figure 14 is a standardized knockout agenda; and, the agenda in their Figure 5 is a standardized priority agenda (which extends the left-hand agenda in Example 2).

[21] Likewise, IS weakens Apesteguia et al.'s [2014] *Division Consistency*. One difference is that condition (ii) does not require the sub-issues (B, C) of A to be disjoint. Another is that IS does not impose any consistency between the splitting of A and the splitting of its sub-issues. In particular, it does *not* require that $v(P, D) = v(P, \{v(P, B \cap D), v(P, C \cap D)\})$ for all $D \subset A$.

Together, these two conditions characterize implementation by simple agenda. As shown in Section 3.3 below, the structure of the unique implementing agenda is determined by the outcomes of v for profiles, called *Condorcet triples*, where the majority preference relation forms a cycle on three alternatives:

Theorem 1 ***(Necessary and Sufficient Conditions)*** *A decision rule v is implementable by simple agenda if and only if it satisfies Issue Splitting and the Independence of Losing Alternatives.* ***(Uniqueness)*** *For every decision rule v that satisfies these two properties, there is a unique simple agenda S_X^v that implements v; and, the structure of S_X^v is fully determined by the outcomes on Condorcet triples.*

The Supplementary Appendix contains examples showing that IS and ILA are independent. In general, ILA is necessary for implementation by *any* kind of agenda (whether or not the agenda in question is simple). This is a straightforward consequence of Remark 1: "backward induction" is unaffected by pruning the Condorcet loser away from an agenda. Conversely, IS is *not* generally necessary for implementation by agenda, as the following example illustrates. To simplify the presentation, let P_{xyz} denote a Condorcet triple where x is majority preferred to y, y to z, and z to x.[22]

Example 4 *The decision rule implemented by the agenda T_X in Figure 5 violates IS. To see this, it is enough to observe that, for the decision problems (P_{xbc}, A) and (P_{xcb}, A), the outcome is the majority winner between b and c. So, $UNE[\mathrm{T}_{X|A}; P_{xbc}] = b$ cannot be paired with x in any potential splitting of A. Otherwise, b must be eliminated by x, which is majority preferred on (P_{xbc}, A). By the same reasoning, $UNE[\mathrm{T}_{X|A}; P_{xcb}] = c$ cannot be paired with x. This leaves $(x, \{b, c\})$ as the only potential splitting of A. Since that would require x as the outcome for both (P_{xbc}, A) and (P_{xcb}, A) however, there can be no way to split A.*

3.3 A Recipe

For a decision rule v that satisfies IS and ILA, every issue $A \notin \mathbf{X}_1$ has a unique splitting (see Claim 8 of the Appendix). This makes it straightforward to define, by recursion, the unique simple agenda that implements v. To formalize, let $\mathrm{S}_{\{\ell_1,\ell_2\}}$ denote the unique simple agenda with two terminal nodes labeled ℓ_1 and ℓ_2; and, let $\mathrm{S}_{\{\ell\}}$ denote the unique degenerate agenda with a single node labeled ℓ.

Definition 9 *Given a decision rule v that satisfies IS and ILA, the agenda S_X^v is defined as follows:*

- **(1)** *First, define $\mathrm{S}^v(1) \equiv \mathrm{S}_{\{X\}}$ to be the degenerate agenda whose only node is labeled X.*
- **(2)** *Then, for each $j > 1$, define the agenda $\mathrm{S}^v(j+1)$ from $\mathrm{S}^v(j)$ as follows:*
  - *For each terminal node labeled by a non-singleton A (if any), find the splitting (B_A, C_A); and,*
  - *Replace every terminal node of $\mathrm{S}^v(j)$ labeled by A with the simple agenda $\mathrm{S}_{\{B_A,C_A\}}$.*
- **(3)** *Finally, define $\mathrm{S}_X^v \equiv \mathrm{S}^v(|X|-1)$.*

Since the recursion continues for $|X| - 1$ steps, each terminal node of S_X^v must be labeled by a singleton. It follows that S_X^v defines an agenda. What is more, this agenda is history-independent: at any interim stage, the

[22] Since $|N| \geq 3$, it is guaranteed that such a profile P_{xyz} exists (see footnote 16 above).

construction is identical for each terminal node labeled by the issue A. While not immediately apparent from Definition 9, it turns out that S^v_X is also persistent (see Claim 9 of the Appendix).

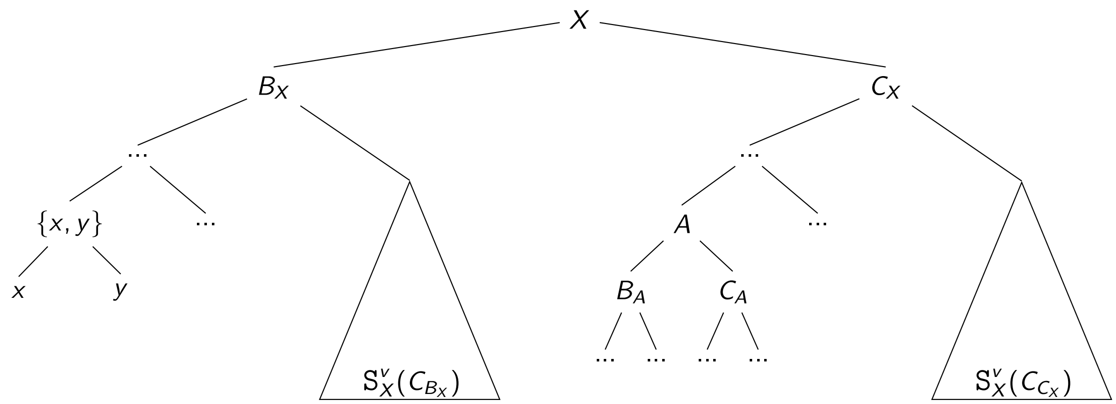


Figure 6: The recursive construction of S^v_X.

In Figure 6, the leftmost nodes below B_X illustrate the construction for $|A| = 2$ while the leftmost nodes below C_X illustrate it for $|A| > 2$. In turn, the two triangles represent the subgames starting from the nodes labeled C_{B_X} and C_{C_X} while the ellipses indicate where details have been omitted.[23]

As Theorem 1 indicates, Condorcet triples may be used to describe S^v_X more explicitly. To see how, fix an issue $\{x, b, c\}$. Clearly, there are three potential splittings where b and c appear in separate sub-issues:

$$(\{b, x\}, \{c, x\}) \quad (b, \{c, x\}) \quad (\{b, x\}, c)$$

Each of these splittings corresponds to the initial stage game of a different simple agenda on $\{x, b, c\}$:

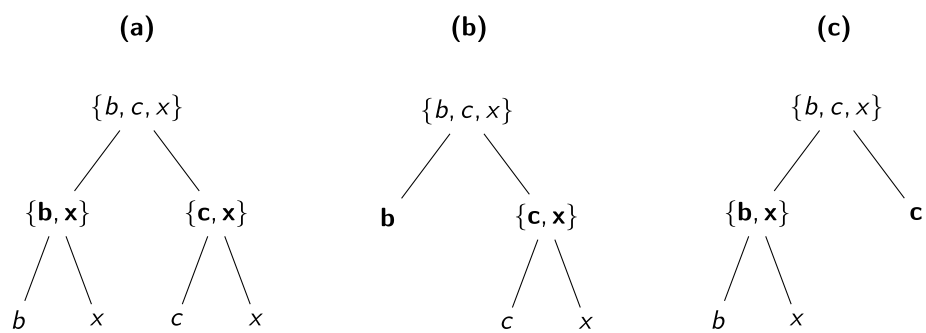


Figure 7: Simple agendas on three alternatives $\{x, b, c\}$.

In turn, each of these agendas implements a different combination of outcomes on the two Condorcet triples:

[23] It is important not to confuse the agenda $\mathrm{S}_X(A)$ with the agenda $\mathrm{S}_{X|A}$. While the former refers to the subgame at node A in S_X, the latter refers to the pruned agenda on A obtained from S_X. As Figure 5 illustrates, these agendas may be quite different.

Profile\Agenda	(a)	(b)	(c)
P_{xbc}	c	b	c
P_{xcb}	b	b	c
Outcomes	Majority loser between b and c	Outcome b for both triples	Outcome c for both triples

Table 1: Outcomes implemented by agendas (a)-(c) on $\{x, b, c\}$.

By construction, there exist alternatives b_A and c_A that appear only on opposite sides of the agenda S^v_X starting at any node labeled A. For these alternatives, the outcomes on each issue $\{x, b_A, c_A\}$ involving an alternative $x \in A$ must coincide with one of the possibilities in Table 1. Using this to pin down alternatives $b_A, c_A \in A$, one can then use Table 1 to describe the unique splitting (B_A, C_A) of A in terms of Condorcet triples:

$$B_A \equiv \{b_A\} \cup \{x \in A \setminus \{b_A, c_A\} : \text{the outcomes on } \{x, b_A, c_A\} \text{ are type-(a) or type-(c)}\}; \quad \text{and}$$
$$C_A \equiv \{c_A\} \cup \{x \in A \setminus \{b_A, c_A\} : \text{the outcomes on } \{x, b_A, c_A\} \text{ are type-(a) or type-(b)}\}.$$

Intuitively, the type-(a) outcomes reveal that x appears in *both* sub-issues of (B_A, C_A) while the type-(c) outcomes (resp. type-(b) outcomes) reveal that x appears *only* in the same sub-issue as b_A (resp. c_A).

4 Comparison with Recursive Agendas

In this section, I consider a class of agendas that is somewhat broader than the class of simple agendas; and, highlight two appealing properties of simple agendas that do not extend to every agenda in this class.

4.1 Definitions

Despite their prevalence, history-independence and persistence do not figure in the agenda taxonomy proposed by Ordeshook and Schwartz. Instead, these authors focus on two other common features of agendas.

The first stipulates that one or more alternatives is eliminated at every non-terminal node. Such an agenda is **non-repetitive**: for every non-terminal node q, $c_i(q) \equiv c(q) \cap \ell(q_i) \neq \emptyset$ for $i = 1, 2$.[24] In turn, the second feature requires that some of the contested outcomes at each node continue to be contested until they are eliminated or ultimately selected. Such an agenda is **continuous**: for every non-terminal node q such that $c_i(q) \neq \emptyset$, there exists some alternative in $c_i(q)$ labels exactly one terminal node below q_i.[25] In combination, these two features can be re-stated more succinctly as follows:[26]

Remark 2 *An agenda is non-repetitive and continuous if and only if, for every non-terminal node q and each successor q_i of q, there exists an alternative $x^q_i \in c_i(q)$ that labels exactly one terminal node below q_i.*

[24] This directly implies Farquharson's Axiom II (see footnote 18 above).

[25] Originally, Ordeshook and Schwartz [1987, p.185] only defined continuity for "Ordeshook-Schwartz" agendas (see footnote 14). Miller [1995, p.26] later adapted the concept to "Farquharson-Miller" agendas by requiring that *every* contested alternative be contested until it is either eliminated or selected. The underlying motivation for the weaker definition given here is to address the problems pointed out by Groseclose and Krehbiel [1993].

[26] Example 10 of the Supplementary Appendix shows that the two features are independent.

Agendas with these two features have an appealing recursive structure. Where $\ell_1(\mathtt{T}_A)$ denotes the alternatives in A that label exactly one terminal node in $\mathtt{T}_A$, $\mathtt{T}_X = \mathtt{S}_{\{b,c\}} \otimes \{\mathtt{T}_B, \mathtt{T}_C\}$ is non-repetitive and continuous if and only if: (i) both $\mathtt{T}_B$ and $\mathtt{T}_C$ are non-repetitive and continuous; and, (ii) $\ell_1(\mathtt{T}_B) \cap \ell_1(\mathtt{T}_C) \neq \ell_1(\mathtt{T}_B), \ell_1(\mathtt{T}_C)$ (so that $\ell_1(\mathtt{T}_B)$ and $\ell_1(\mathtt{T}_C)$ are non-nested). This observation motivates the following definition.

Class 2 *An agenda is **recursive** if it is both non-repetitive and continuous.*

In the sequel, I denote a recursive agenda on A by R_A; and, the class of all recursive agendas by $\mathcal{R}$. It is not difficult to show that this class is strictly larger than the class of simple agendas (i.e., $\mathcal{R} \supset \mathcal{S}$):

Remark 3 ***(i)*** *Every simple agenda is recursive.* ***(ii)*** *But, not every recursive agenda is simple.*

While this does not give a sense of the gap between the two classes, there is reason to suspect that it is significant. Many recursive agendas seem to lack one (or both) of the features that define simple agendas (see Example 12 of the Supplementary Appendix); and, their number appears to increase rapidly with $|X|$.[27]

4.2 Two Key Distinctions

Despite the apparent gap between the two classes, few recursive agendas used in practice fail to be simple. None of the agendas discussed by Ordeshook and Schwartz fit into this category, for instance. One plausible explanation is that voting outcomes on recursive agendas (which are not simple) lack the compelling properties of voting outcomes on simple agendas. In this section, I identify two such properties.

The first is a monotonicity property for voter preferences. Given a profile P, let $P^{\uparrow x}$ denote a profile where every voter's preference is identical to her preference in P except for *one* voter, whose preference between x and the immediately preferred alternative (if any) is reversed. Thus, $P^{\uparrow x}$ differs from the original profile P only by improving the alternative x in the eyes of a single voter. Then, a decision rule v is **preference monotonic** at A if, for every $P \in \mathbf{P}$, $v(P, A) = x$ implies $v(P^{\uparrow x}, A) = x$ for every profile $P^{\uparrow x}$.

Preference Monotonicity *At every issue $A \notin \mathbf{X}_1$, v is preference monotonic.*

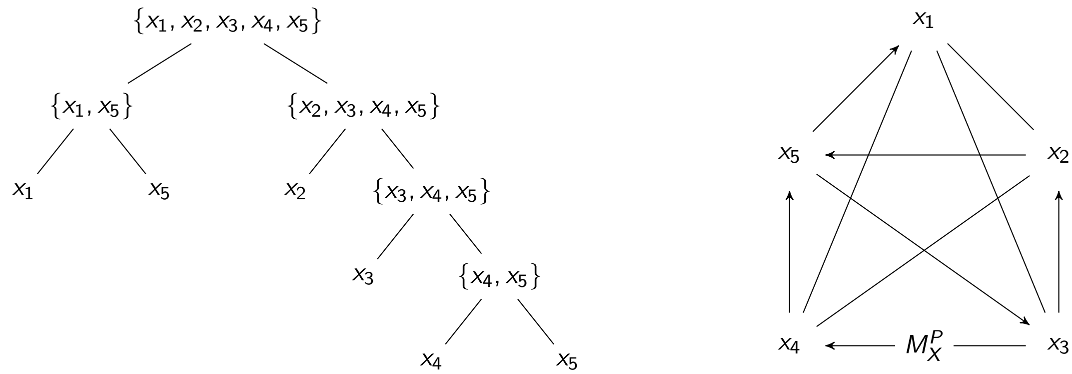
Clearly, every knockout agenda (induces a decision rule that) satisfies Preference Monotonicity (Moulin [1991]; Altman et al. [2009]). To see this, suppose that x is the outcome at P; and, consider the backward induction "path" that leads to x. Since each alternative appears only once, what x faces along this path cannot be affected by improving it. So, x is still the outcome at $P^{\uparrow x}$. The same is true for Anglo-American agendas (Moulin [1986, 1991]). Even though some alternatives now appear more than once, the improvement cannot affect which alternatives the original winner faces. In fact, this argument extends to *all* simple agendas.

Proposition 1 *Every decision rule v implementable by simple agenda satisfies Preference Monotonicity. Conversely, for each X such that $|X| \geq 5$, there are recursive agendas that violate Preference Monotonicity.*

The following example (which extends easily to $|X| \geq 6$) establishes the second part of the statement.

[27] With $|X| = 4$, six of the fifteen recursive agendas are not simple (see Example 11 of the Supplementary Appendix).

Example 5 *Consider the recursive (but non-persistent) agenda on $X = \{x_1, x_2, x_3, x_4, x_5\}$ depicted below.*



To see that it violates Preference Monotonicity, fix a profile P which induces the (partial) majority relation M_X^P shown above.[28] *When the majority relation between x_4 and x_5 is reversed, the outcome switches from x_5 to x_2. Since every recursive agenda on $|X| \leq 4$ alternatives satisfies Preference Monotonicity, this counter-example is minimal (both in terms of the number of alternatives and the number of terminal nodes).*

The second compelling property of simple agendas relates to implementation on restricted domains.

Definition 10 *Given a decision rule v and an issue $A \in \mathbf{X}$, the* ***restricted*** *rule $v_A : \mathbf{P} \times \mathbf{A} \to A$ is defined by*

$$v_A(P, D) \equiv v(P, D) \text{ for each } (P, D) \in \mathbf{P} \times \mathbf{A}.$$

Pruning provides a way to implement restricted rules: if T_X implements v, then the pruned agenda $\mathrm{T}_{X|A}$ implements v_A. One drawback is that pruning tends to destroy the structural features of the original agenda. To illustrate, consider Figure 5. While the agenda on the left is recursive, the pruned version on the right is not. For a class of agendas $\mathcal{T}$ that satisfies the following property, this issue can be avoided.

$\mathcal{T}$-**Consistency** *For every issue $A \in \mathbf{X}$, the restricted rule v_A is implementable by an agenda $\mathrm{T}_A \in \mathcal{T}$.*

Rather than using the pruned agenda $\mathrm{T}_{X|A}$ to implement v_A, one can use an agenda $\mathrm{T}_A \in \mathcal{T}$. One advantage relates to the possibility of simplifying the voting procedure: pruned agendas tend to include redundant questions that cannot affect the outcome for any profile (like the node $\{c, x\}$ for the right-hand agenda in Figure 5).

Another advantage relates to agenda-setting. Imagine that $\mathcal{T}$ represents the agendas permitted by parliamentary procedure; and, $A \subset X$ reflects the "real" issue up for debate. By introducing "dummy" alternatives B, one might generate an agenda $\mathrm{T}_{A \cup B} \in \mathcal{T}$ that differs substantively from the permitted agendas on A (i.e., for each $\mathrm{T}_A \in \mathcal{T}$, there exists a profile $P \in \mathbf{P}$ such that $UNE[\mathrm{T}_{A \cup B|A}; P] \neq UNE[\mathrm{T}_A; P]$). By restricting oneself to a class of agendas $\mathcal{T}'$ that satisfies $\mathcal{T}$-Consistency, this type of manipulation can be ruled out.

On either account, the most appealing agendas classes are the "self-consistent" classes. Formally, an agenda class $\mathcal{T}$ satisfies **Self-Consistency** if the decision rule induced by every agenda $\mathrm{T}_A \in \mathcal{T}$ satisfies $\mathcal{T}$-Consistency.

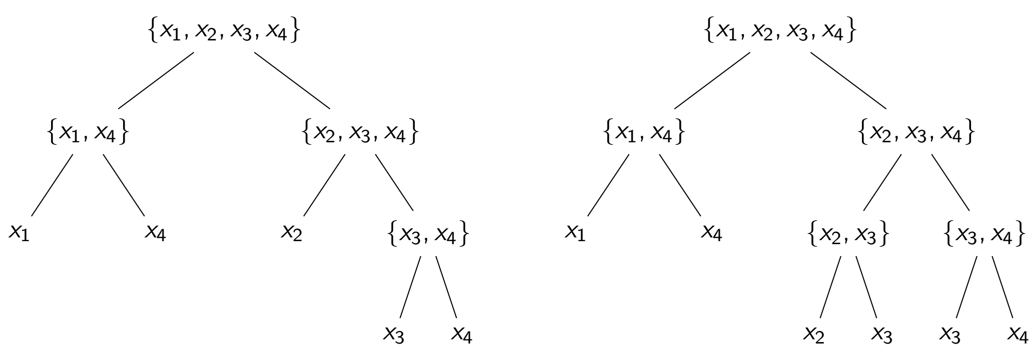
[28] Technically, $|N|$ must be "sufficiently large" relative to $|X|$ (see footnote 16 above). Since $|X| = 5$, it is enough that $|N| \geq 5$.

Trivially, the Euro-Latin agendas exhibit this property: pruning preserves the structure of the agenda. In fact, the Anglo-American agendas do as well (Apesteguia et al. [2014]).[29] More generally, the class of simple agendas satisfies Self-Consistency; but, the class of recursive agendas does not.

Proposition 2 *Every decision rule v implementable by simple agenda satisfies $\mathcal{S}$-Consistency. Conversely, for each X such that $|X| \geq 4$, there are decision rules implementable by recursive agenda that violate $\mathcal{R}$-Consistency.*

The first statement follows from Theorem 1: if v satisfies IS and ILA, then so does the restricted rule v_A on $A \subset X$. (In fact, one can use the recipe in Section 3.3 to define the simple agenda on A that implements v_A.) In turn, the following example (which is easily extended to $|X| \geq 5$) establishes the second statement:

Example 6 *Consider the two recursive (but non-persistent) agendas on $X = \{x_1, x_2, x_3, x_4\}$ depicted below. The left-hand agenda violates $\mathcal{R}$-Consistency. To see this, note that the outcome for each Condorcet triple on $\{x_2, x_3, x_4\}$ is the majority winner between x_2 and x_4 — which violates Issue Splitting by Example 4. Conversely, the right-hand agenda satisfies $\mathcal{R}$-Consistency. To see this, it suffices to check that* ***no*** *pair of Condorcet triples has outcomes that coincide with the majority winner between two alternatives.*



The left-hand agenda shows that $\mathcal{R}$ violates Self-Consistency. In turn, the right-hand agenda shows that $\mathcal{S}$ is not the largest self-consistent sub-class of $\mathcal{R}$: it can be added to $\mathcal{S}$ to make a larger self-consistent sub-class.

5 Priority Agendas

After first defining priority agendas, I then characterize the decision rules implemented by agendas in this class.

5.1 Definitions and Examples

A priority agenda on X is defined by an admissible pair $(\succsim, \alpha)$ consisting of a priority $\succsim$ and an amendment rule α. This pair provides instructions for building the agenda: the priority determines *when* each alternative is added to the agenda; and, the amendment rule determines *how* each is added to the agenda.

Formally, a *priority* $\succsim$ is a weak ordering on X whose indifference classes contain at most two alternatives. If $x \succ y$, then x has *higher* priority and is added to the agenda before y. (If $x \sim y$, then x and y have *equal*

[29] If a decision rule v on X satisfies the axioms listed in their Theorem 2, then so does the restricted rule v_A on $A \subset X$.

priority and are added simultaneously.) Let $\widetilde{X}_j$ denote the j^{th} highest indifference class of $\succsim$ (so that $\widetilde{X}_1$ denotes the set of *top priority* alternative(s) in X and $X \setminus \widetilde{X}_1$ the set of *non-top priority* alternative(s) in X).

In turn, an *amendment rule* α is a mapping $\alpha : X \to \mathbf{X}_2 \cup \mathbf{X}_1 \cup \emptyset$ that assigns at most two alternatives to each alternative $z \in X$. The interpretation is that $\alpha(z)$ is the (set of) highest priority alternative(s) that are faced (or "amended") by z. Formally, one alternative $z \in X$ *amends* another alternative $y \in X$ if

$$y \in \alpha(z) \;\text{ or }\; x \succ y \succ z \text{ for some } x \in \alpha(z).$$

In other words, z amends each $x \in \alpha(z)$ and each y that is added to the agenda between x and z.

Finally, to be *admissible*, a pair $(\succsim, \alpha)$ must satisfy the following conditions for each $z \in X$:

- **(i)** if $z \in \widetilde{X}_1$, then $\alpha(z) = \emptyset$;
- **(ii)** if $z \in \widetilde{X}_j$ and $j \neq 1$, then $\alpha(z) \subseteq \widetilde{X}_k$ for some $k < j$;
- **(iii)** if $z, z' \in \widetilde{X}_j$ (and $j \neq 1$), then $\alpha(z) = \alpha(z')$; and,
- **(iv)** if $z \in \widetilde{X}_{j+1}$ and $|\widetilde{X}_j| = 2$, then $\alpha(z) = \widetilde{X}_j$ or $\alpha(z) \subseteq \widetilde{X}_k$ for some $k < j$.

These conditions ensure that a pair $(\succsim, \alpha)$ is consistent with the desired interpretation of the parameters. In words, they stipulate that: (i) top priority alternatives amend nothing; (ii) non-top priority alternatives amend higher priority alternative(s); (iii) equal priority alternatives amend the same alternative(s); and, (iv) an alternative with priority immediately below two equal priority alternatives amends both.

To re-construct a priority agenda from an admissible pair $(\succsim, \alpha)$, one uses the amendment rule α to add the alternatives in decreasing order of priority $\succsim$. To formalize this recursive construction:

Class 3 *Given an admissible pair $(\succsim, \alpha)$ on X, the **priority agenda** $\mathtt{Pr}_{(\succsim,\alpha)}$ is defined as follows:*

- **(1)** *First, define $\mathtt{Pr}_{(\succsim,\alpha)}(1) \equiv \mathrm{S}_{\widetilde{X}_1}$ to be the simple agenda on the top priority alternative(s) $\widetilde{X}_1$ of $\succsim$.*
- **(2)** *Then, for each $j > 1$, define $\mathtt{Pr}_{(\succsim,\alpha)}(j+1)$ from $\mathtt{Pr}_{(\succsim,\alpha)}(j)$ as follows:*
  - **(i)** *If $x_k \in \alpha(x_{j+1}) \cap \widetilde{X}_k$, replace every terminal node of $\mathtt{Pr}_{(\succsim,\alpha)}(j)$ labeled by:*
    - *x_k with the simple agenda $\mathrm{S}_{\{x_k, \widetilde{X}_{j+1}\}}$; and,*
    - *$x_{k'} \in \widetilde{X}_{k'}$ such that $k < k' \leq j$ with the simple agenda $\mathrm{S}_{\{x_{k'}, \widetilde{X}_{j+1}\}}$.*
  - **(ii)** *If $|\widetilde{X}_{j+1}| \neq 1$, replace every terminal node X_{j+1} resulting from (i) with the simple agenda $\mathrm{S}_{\widetilde{X}_{j+1}}$.*
- **(3)** *Finally, define $\mathtt{Pr}_{(\succsim,\alpha)} \equiv \mathtt{Pr}_{(\succsim,\alpha)}(K)$ where $K \equiv |X/\sim|$ is the number of indifference classes in $\succsim$.*[30]

At any stage $j \leq K$ of this recursion, $\mathtt{Pr}^j_{(\succsim,\alpha)}$ is compounded (see Definition 7) by adding simple agendas (consisting of two or three alternatives) to some of the terminal nodes.[31] To illustrate this construction:

[30] Technically, one must then relabel the non-terminal nodes to conform with Definition 2. Since this step is straightforward but cumbersome, it has been omitted to preserve clarity. For the details, please see Claim 21 of the Appendix.

[31] Technically, it is not necessary to add equal priority alternatives simultaneously. Nothing would change by adding them one at a time (*provided* that one re-interpreted the amendment rule to mean that the second of the two additions only amends the first).

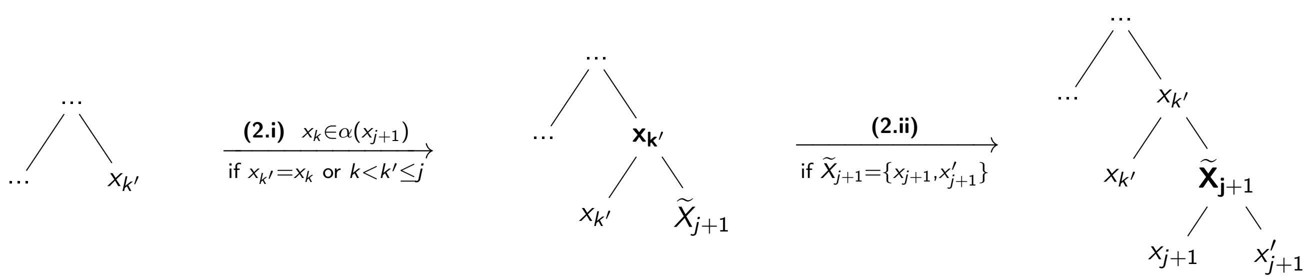


Figure 8: Detail at the terminal node $x_{k'}$ in stage $j \le K$ of the construction.

This diagram makes it clear that the resulting agenda must be history-independent. In fact, it must also be persistent (see Claim 23 of the Appendix). In other words, every priority agenda is simple. At the same time, not every simple agenda is a priority agenda (see Example 3).

Remark 4 ***(i)*** *Every priority agenda is simple.* ***(ii)*** *But, not every simple agenda is a priority agenda.*

For ease of notation, I sometimes suppress the priority-amendment pair $(\succsim, \alpha)$ and simply denote a priority agenda on X by $\mathtt{Pr}_X$. I also denote the class of all priority agendas by $\mathcal{PR}$.

The following examples highlight some important features of priority agendas. The first example shows that Euro-Latin and Anglo-American agendas are straightforward to represent in terms of priority-amendment pairs.

Example 7 *In fact, each of these agendas admits two distinct representations for $X = \{x_1, ..., x_m\}$:*

Agenda	Representation	Priority $\succsim$	Amendment Rule α
Euro-Latin	strict	$x_1 \succ ... \succ x_{m-1} \succ x_m$	$\alpha(x_i) = x_{i-1}$ for $2 \le i \le m$
	weak	$x_1 \succ ... \succ x_{m-1} \sim x_m$	$\alpha(x_i) = \begin{cases} x_{i-1} & \text{for } 2 \le i \le m-2 \\ x_{m-2} & \text{for } i = m-1, m \end{cases}$
Anglo-American	strict	$x_1 \succ x_2 \succ ... \succ x_m$	$\alpha(x_i) = x_1$ for $2 \le i \le m$
	weak	$x_1 \sim x_2 \succ ... \succ x_m$	$\alpha(x_i) = \{x_1, x_2\}$ for $3 \le i \le m$

In either case, the non-uniqueness is due to *symmetry* between alternatives (namely x_{m-1} and x_m in the Euro-Latin agenda; and, x_1 and x_2 in the Anglo-American agenda). For the symmetric alternatives, one can either assign equal priority *or* consecutive priority. In fact, this is true for every priority agenda with symmetric alternatives. Having said this, non-uniqueness is not limited to agendas with symmetric alternatives (see

Example 13 of the Supplementary Appendix which discusses the right-hand agenda in Example 2).[32]

In turn, the second example shows that one can define priority agendas by compounding (see Definition 7):

Example 8 *Up to permutation, the only priority agendas on $\{x_1, x_2, x_3\}$ are the (i) Euro-Latin and (ii) Anglo-American agendas in Figure 1. By starting from a strict priority on $\{x_1, x_2, x_3\}$, there are three distinct ways to extend each of these agendas into a compound agenda on $\{x_1, x_2, x_3, x_4\}$.*

- **Euro-Latin amendment:** *x_4 amends only the lowest priority alternative x_3 in $\{x_1, x_2, x_3\}$. This leads to (i) the Euro-Latin agenda in Figure 2 and (ii) the right-hand agenda in Figure 3.*
- **Anglo-American amendment:** *x_4 amends every alternative in $\{x_1, x_2, x_3\}$. This leads to (i) the left-hand agenda in Figure 3 and (ii) the Anglo-American agenda in Figure 2.*
- **Intermediate amendment:** *x_4 amends the two lowest priority alternatives x_2 and x_3 in $\{x_1, x_2, x_3\}$ (but not x_1). This type of amendment leads to the two agendas depicted in Example 2.*

By starting from a weak priority on $\{x_1, x_2, x_3\}$, one cannot generate any additional agendas.[33] *As a result, the six agendas listed above exhaust the collection of priority agendas on four alternatives (up to permutation).*

5.2 Implementation

Euro-Latin and Anglo-American agendas have the property that the lowest priority alternative is chosen only when it is unambiguously appealing. To formalize, an alternative $a \in A$ is the *Condorcet winner* for the decision problem $(P, A) \in \mathbf{P} \times \mathbf{X}$ if $aM_A^P x$ for all $x \in A \setminus a$. For a decision rule v, an alternative $a^* \in A$ is **marginal** on $A \in \mathbf{X}$ if, for all $P \in \mathbf{P}$, $v(P, A) = a^*$ only if a^* is the Condorcet winner for (P, A). Then, the stated property of Euro-Latin and Anglo-American agendas can be formalized as follows:

***Marginalization* (M)** *On every issue $A \in \mathbf{X}$, v has a marginal alternative.*

The next result shows that this property is satisfied by *every* priority agenda.

Proposition 3 *Every decision rule v that is implementable by priority agenda satisfies Marginalization.*

In a sense, M is the distinguishing property of decision rules implementable by priority agenda.

Theorem 2 ***(Necessary and Sufficient Conditions)*** *A decision rule v is implementable by priority agenda if and only if it satisfies Issue Splitting, the Independence of Losing Alternatives, and Marginalization.* ***(Uniqueness)*** *For every decision rule v satisfying these properties, there is a unique priority agenda $\mathtt{Pr}_X^v$ that implements v; and, the pair $(\succsim_v, \alpha_v)$ that defines $\mathtt{Pr}_X^v$ is fully determined by the outcomes on Condorcet triples.*

To establish sufficiency of the axioms, a key step is to determine the priority structure associated with any decision rule that satisfies the axioms. To do this, I rely on the familiar tool of revealed preference.

[32] It is worth pointing out that one cannot eliminate multiplicity by requiring a representation to use strict priority. The problem is that not every priority agenda can be represented with a strict priority (Example 14 of the Supplementary Appendix). Instead, the way to eliminate multiplicity is to allow strict priority only when it is necessary to represent the agenda.

[33] While it is possible to generate the other five agendas, one cannot generate the Euro-Latin agenda in Figure 2. By condition (iv) on admissible pairs, there is no Euro-Latin amendment for the weak representation of the Euro-Latin agenda on $\{x_1, x_2, x_3\}$.

Definition 11 *For a decision rule v, first define the binary relations $\succ_v$ and $\sim_v$ on X as follows:*

$y \succ_v z$ if there exists an issue $A \in \mathbf{X}$ such that $y \in A$ where z is marginal and y is not; and,

$y \sim_v z$ if, for every issue $A \in \mathbf{X}$ such that $y, z \in A$, y is marginal if and only if z is marginal.

*Then, define the **revealed priority** $\succsim_v$ on X by $y \succsim_v z$ if $y \succ_v z$ or $y \sim_v z$.*

For the Anglo-American and Euro-Latin agendas, it is easy to see that the revealed priority reflects the "true" (underlying) priority. For every issue, the Anglo-American agenda marginalizes the lowest ranked alternative in A according to $\succsim_v$ while the Euro-Latin agenda marginalizes the *two* lowest ranked alternatives. In fact, $\succsim_v$ defines a priority with these kinds of features for any decision rule that satisfies the axioms:

Lemma 1 *If a decision rule v satisfies IS, ILA, and M, then: **(i)** the revealed priority $\succsim_v$ defines a priority (i.e., $\succsim_v$ is a complete and transitive binary relation whose indifference classes contain at most two alternatives); and; **(ii)** for each issue $A \in \mathbf{X}$, v marginalizes the lowest or two lowest alternatives in A according to $\succsim_v$.*

As Theorem 2 indicates, the revealed priority $\succsim_v$ can be re-defined entirely in terms of Condorcet triples. Table 1 suggests how to do this. For each issue $\{x, y, z\}$, there are six potential splittings. By Table 1, each entails a different combination of outcomes on the Condorcet triples P_{xyz} and P_{xzy}. Four of the six combinations directly reveal $y \succ_v z$ or $z \succ_v y$ while two are consistent with every priority ranking of the alternatives y and z.[34] By varying the alternative $x \in X \setminus \{y, z\}$, one can resolve the ambiguous cases and determine the revealed priority between any two alternatives y and z (see Corollary 1 of the Appendix).

In fact, Table 1 also suggests how to define the amendment rule α_v. Intuitively, x is *revealed to amend* a higher priority alternative b if the Condorcet triples P_{xbc} and P_{xcb} for every alternative $c \in X \setminus \{x, b\}$ with intermediate priority yield type-(a) or type-(c) outcomes. Intuitively, this reveals that x must appear in the same sub-issue as b (see Figure 7). Based on this insight, $\alpha_v(x)$ may then be defined as the highest priority alternative(s) that x is revealed to amend (see Definition 14 and Lemma 2 of the Appendix).

6 Comparison with Euro-Latin and Anglo-American Agendas

In this section, I examine the relationship of Euro-Latin and Anglo-American agendas to the broader class of priority agendas. I show that four well-known properties of these agendas extend to *all* priority agendas; and, highlight a natural way to distinguish Euro-Latin and Anglo-American agendas from other priority agendas.

6.1 Four Common Properties

It has already been shown that three well-known properties of Euro-Latin and Anglo-American agendas (namely Preference Monotonicity, Self-Consistency, and Marginalization) extend to priority agendas.[35] Here, I show that the same is true for a fourth property — a monotonicity condition related to priority.

Given a priority agenda $\mathtt{Pr}_X$ defined by $(\succsim, \alpha)$ and an alternative $x \in \widetilde{X}_j$, fix an alternative $y \in X$ (if any) such that: (i) $y \in \widetilde{X}_{j-1}$; or, (ii) $y \in X_j$ and x is amended by every alternative that amends y. Let $\mathtt{Pr}_X^{\uparrow x}$ denote

[34] The two outcome pairs with $v(P_{xyz}, \{x, y, z\}) = y$ reveal $y \succ_v z$; and, the two with $v(P_{xzy}, \{x, y, z\}) = z$ reveal $z \succ_v y$.

[35] Preference Monotonicity and Marginalization follow from Propositions 1 and 3, respectively. In turn, Self-Consistency follows directly from Theorem 2 (rather than Proposition 2): if a rule v satisfies IS, ILA, and M, then so does the restricted rule v_A.

the priority agenda obtained by permuting the labels of the terminal nodes in $\mathtt{Pr}_X$ marked by x and y. The idea is that $\mathtt{Pr}_X^{\uparrow x}$ (weakly) improves the priority of x relative to $\mathtt{Pr}_X$ by swapping it with an alternative having (i) immediately higher priority or (ii) the same priority but weakly fewer amendments.

Priority Monotonicity *For every* $P \in \mathbf{P}$ *and* $x \in \mathbf{X}$, $UNE[\mathtt{Pr}_X; P] = x$ *only if* $UNE[\mathtt{Pr}_X^{\uparrow x}; P] = x$.

This property posits a straightforward relationship between the structure of a priority agenda and its voting outcomes. This relationship is known to hold for Euro-Latin and Anglo-American agendas (Moulin [1986, 1991]; Jung [1990]). In fact, it holds for all priority agendas.

Proposition 4 *Every priority agenda satisfies Priority Monotonicity.*

6.2 A Key Distinction

There is an sizeable literature on neutral and Condorcet-consistent mappings, called *tournament solutions*, that select a subset of the alternatives in A for every issue A and tournament M_A on A.[36] Arguably the best known tournament solutions are the *Top Cycle* and the *Banks Set*:

Definition 12 *The* ***Top Cycle*** $TC(M_A)$ *is the subset of alternatives in* A *that (directly or indirectly) dominate every other alternative according to* M_A. *Formally,* $TC(M_A) \equiv \{x \in A : xM_A...M_Aa \text{ for all } a \in A \setminus x\}$.

To define the Banks Set, some additional notation is required. A sequence $\langle b \rangle = b_1, ..., b_m$ of alternatives in A is an *M_A-transitive chain* if $b_i M_A b_j$ for all $j > i$. In turn, an M_A-transitive chain is *maximal* if there exists no alternative $a \in A \setminus \langle b \rangle$ such that the sequence $\langle a, b \rangle \equiv a, b_1, ..., b_m$ is an M_A-transitive chain.

Definition 13 *The* ***Banks Set*** $BA(M_A)$ *is the subset of alternatives in* A *that are at the top of some maximal* M_A*-transitive chain. Formally,* $BA(M_A) \equiv \{x \in A : x = b_1 \text{ for some maximal } M_A\text{-transitive chain } \langle b \rangle\}$.

Miller [1977] showed that the Top Cycle coincides with the alternatives which are selected from some Euro-Latin agenda; and, Banks [1985] showed that the set bearing his name coincides with the set of alternatives which are selected from some Anglo-American agenda. To formalize, let $\Sigma(A)$ denote the set of permutations (or bijections) on A; let $\mathtt{T}_A^\sigma$ denote the agenda obtained by permuting the labels of the terminal nodes in $\mathtt{T}_A$ using $\sigma \in \Sigma(A)$; and, let

$$V_\Sigma(\mathtt{T}_A; P) \equiv \bigcup_{\sigma \in \Sigma(A)} UNE[\mathtt{T}_A^\sigma; P]$$

denote the collection of outcomes selected at profile P for some *re-labeling* $\mathtt{T}_A^\sigma$ of the agenda $\mathtt{T}_A$. (For a priority agenda, re-labeling amounts to nothing more than *re-prioritizing* the alternatives.) Then, the results of Miller and Banks establish the following for each decision problem (P, A):

(i) $V_\Sigma(\mathtt{EL}_A; P) = TC(M_A^P)$ for every Euro-Latin agenda $\mathtt{EL}_A$; and,

(ii) $V_\Sigma(\mathtt{AA}_A; P) = BA(M_A^P)$ for every Anglo-American agenda $\mathtt{AA}_A$.

The next proposition provides a natural extension of their results to the entire class of priority agendas:

[36] Since the particular profile P that induces the majority relation is not material, I suppress it in the notation.

Proposition 5 *For every decision problem $(P, A) \in \mathbf{P} \times \mathbf{X}$ and priority agenda $\mathtt{Pr}_A$ on A:*

(i') $V_\Sigma(\mathtt{Pr}_A; P) \subseteq TC(M_A^P)$*; and,*

(ii') $V_\Sigma(\mathtt{Pr}_A; P) \supseteq BA(M_A^P)$.

Part (i') follows from McKelvey and Niemi's [1978] insight that, for any agenda, the sophisticated outcome only depends on the Top Cycle. It is included only as a point of contrast to (i). The real contribution is part (ii'). For a given $x \in BA(M_A^P)$, the proof defines a sequence $\langle a \rangle$ on A (called a *universal sequence*) such that x is selected from *every* priority agenda $\mathtt{Pr}_A$ when the alternatives are re-prioritized to conform with $\langle a \rangle$. This extends the work of Banks [1985], who leverages the algorithm of Shepsle and Weingast [1984] to construct a sequence (called a *sophisticated sequence*) such that $x \in BA(M_A^P)$ is selected from the associated Anglo-American agenda. In general, there may be more than one sophisticated sequence for a given $x \in BA(M_A^P)$. The key insight is that one of these sequences must be universal for x (see Claim 26 of the Appendix).

Combined with observations (i) and (ii), Proposition 5 establishes the following relationships:

$$BA(M_A^P) = V_\Sigma(\mathtt{AA}_A; P) \subseteq V_\Sigma(\mathtt{Pr}_A; P) \subseteq V_\Sigma(\mathtt{EL}_A; P) = TC(M_A^P). \qquad (\dagger)$$

Thus, Anglo-American and Euro-Latin agendas reflect the extremes of what can be implemented by priority agenda. Every other priority agenda displays an intermediate ability to "discriminate" among outcomes. Formally, $\mathtt{T}_A$ is **more discriminating** than $\widehat{\mathtt{T}}_A$ if $V_\Sigma(\mathtt{T}_A; P) \subseteq V_\Sigma(\widehat{\mathtt{T}}_A; P)$ for every profile P.

It would appear that discrimination is *negatively* related to marginalization. For every issue (consisting of three or more alternatives), Euro-Latin agendas marginalize two alternatives; and, Anglo-American agendas marginalize one. For every other priority agenda, the number of marginal alternatives depends on the issue:

Theorem 3 *A decision rule v is implementable by Euro-Latin (resp. Anglo-American) agenda if and only if it satisfies Issue Splitting and the Independence of Losing Alternatives, and it marginalizes two alternatives (resp. one alternative) for every issue $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$.*

This result characterizes implementation by Euro-Latin and Anglo-American agenda by emphasizing the similarities between the two procedures. In recent work, Apesteguia et al. [2014] give alternative characterizations that emphasize the differences (as described at greater length in the Supplementary Appendix).

7 Applications and Extensions

In this paper, I characterize sophisticated voting for two broad classes of agendas — classes defined by structural features that are common in legislative settings. For both classes, I establish a close relationship between voting outcomes and the structure of the agenda. While this advances our understanding of sophisticated voting well beyond Euro-Latin and Anglo-American agendas, it also suggests some new avenues of inquiry.

To conclude, I briefly discuss some intriguing possibilities related to manipulation that can arise after the agenda is already set; and, strategic behavior at the earlier stage of agenda-setting. Of course, both of these issues are vast; and, the cursory observations below only scratch the surface.

7.1 Agenda Manipulation

(i) Preference Monotonicity: This property imposes some natural limitations on agenda manipulation. For one, it discourages voters from misrepresenting their preferences by downgrading an alternative. As noted by Altman et al. [2009], it also discourages a natural kind of manipulation by the alternatives — which can be viewed as *candidates* in an election (or contest). Specifically, it discourages them from "throwing" majority comparisons with other candidates in order to gain an advantage. Famously, this type of manipulation occurred during the 2012 Olympic badminton tournament (see Pauly [2013]).

Given its appealing incentive effects, it would be worth identifying the structural features of an agenda that are necessary and sufficient to guarantee Preference Monotonicity. Proposition 1 goes a long way towards this goal by identifying two general features that are sufficient (but not necessary). Having said this, there are agendas that look quite different from simple agendas which nonetheless satisfy Preference Monotonicity.

Remark 5 *There exist discontinuous (and non-repetitive) agendas that satisfy Preference Monotonicity.*

This suggests that it might be difficult to identify agenda features that are necessary *and* sufficient.

(ii) Strategic Candidacy: Besides "throwing" comparisons, Dutta et al. [2001] suggest another way for a candidate to manipulate the outcome: by dropping out of the election. To formalize, let $A = \{a_1, ..., a_n\}$ denote the set of candidates; and, suppose that, in all candidate preference profiles $P^c \equiv (\succ_{a_1}, ..., \succ_{a_n}) \in \mathbf{P^c}$, every candidate $a_i \in A$ ranks herself first ($a_i \succ_{a_i} x$ for all $x \in A \setminus a_i$). Then, a triple $(\mathtt{T}_A; P, P^c)$ defines a two-stage game where: (1) a subset of the candidates $A^* \subseteq A$ independently decides to enter the election; and, (2) the voters then select one of the entering candidates using the pruned agenda $\mathtt{T}_{A|A^*}$. Let $EQ[\mathtt{T}_A; P, P^c]$ denote the equilibrium outcomes (if any) of this game where:

- the outcome of (2) is $UNE[\mathtt{T}_{A|A^*}; P]$ given the set of candidates A^* who decide to enter in (1); and,
- A^* is a pure Nash equilibrium outcome of (1) induced by anticipating $UNE[\mathtt{T}_{A|B}; P]$ for all $B \subseteq A$.

Dutta et al. [2002] consider the special case where the agenda is Anglo-American; and, characterize the set of outcomes that results when the candidate preferences and priority are allowed to vary. Given a pair $(\mathtt{T}_A, P)$, let

$$\widetilde{V}_\Sigma(\mathtt{T}_A; P) \equiv \bigcup_{\substack{\sigma \in \Sigma(A) \\ P^c \in \mathbf{P^c}}} EQ[\mathtt{T}^\sigma_A; P, P^c]$$

denote the outcomes selected for some re-labeling $\sigma \in \Sigma(A)$ and some candidate preferences $P^c \in \mathbf{P^c}$. Dutta et al. show that $\widetilde{V}_\Sigma(\mathtt{AA}_A; P)$ defines a tournament solution called the *Candidate Stable Set*.[37]

Using Proposition 5, it is possible to generalize their result to priority agendas. Where $CS(M_A)$ denotes the Candidate Stable Set associated with M_A, one obtains a natural analog of the relationships in (†) above:

Remark 6 *For every decision problem $(P, A) \in \mathbf{P} \times \mathbf{X}$ and priority agenda $\mathtt{Pr}_A$ on A:*

$$CS(M^P_A) = \widetilde{V}_\Sigma(\mathtt{AA}_A; P) \subseteq \widetilde{V}_\Sigma(\mathtt{Pr}_A; P) \subseteq \widetilde{V}_\Sigma(\mathtt{EL}_A; P) = TC(M^P_A).$$

[37]See Definition 16 of the Supplementary Appendix. This solution is nested between the Banks Set and the *Uncovered Set* (Miller [1980]) — another tournament solution which specializes the Top Cycle (see Definition 17 of the Supplementary Appendix).

This remark suggests that the results of the current paper might be used to pursue the study of strategic candidacy even further. In that regard, the comments made in Section 7.2(i) below apply equally well here: it would be helpful to understand more about how $\widetilde{V}_\Sigma(\mathtt{Pr}_A;\cdot)$ depends on the amendment structure of $\mathtt{Pr}_A$; and, how $\widetilde{V}_\Sigma(\mathtt{T}_A;\cdot)$ is determined in the more general case where $\mathtt{T}_A$ is not a priority agenda.

7.2 Strategic Agenda-Setting

(i) Discriminability: In the literature, $V_\Sigma(\mathtt{T}_A;\cdot)$ is interpreted as a measure of how much control an "all-powerful" chairman can exert by re-arranging the alternatives on the agenda. For this reason, more discriminating agendas (like Anglo-American agendas) are viewed more favorably than less discriminating agendas (like Euro-Latin agendas). Theorem 3 and Proposition 5 together suggest that the ability of a priority agenda to discriminate among outcomes is closely tied to its amendment structure. One natural idea is that a priority agenda becomes more discriminating as it becomes "bushier" (i.e., as $\alpha(x)$ increases in terms of priority $\succsim$ for a given alternative x). For a fixed universe of alternatives X, *bushiness* defines a partial ordering of the priority agendas on X. However, it does not coincide with the partial ordering defined by discriminability:

Remark 7 *For $|X| = 4$ alternatives, the discriminability and bushiness orderings of priority agendas disagree.*

Ideally, one would hope to have a better understanding of the connection between structure and discriminability for priority agendas. Indeed, one might even hope to identify such a connection for the broader classes of simple agendas or recursive agendas (see Tables 3 and 4 of the Supplementary Appendix).

Besides this issue, Proposition 5 raises another interesting question — related to the fact that the *Uncovered Set* (see footnote 37) is the only well-known tournament solution nested between the Banks Set and the Top Cycle. For every issue A, Horan [2013] shows that there exists an agenda $\mathtt{T}_A$ whose outcomes $V_\Sigma(\mathtt{T}_A;P)$ coincide with the Uncovered Set $UC(M_A^P)$ on every profile $P \in \mathbf{P}$. Since his result says nothing about the structure of $\mathtt{T}_A$, it would be worth knowing whether it could be a priority agenda for every issue A.

(ii) Self-Consistency: Another plausible way to manipulate decision rules is by adding "dummy" alternatives to the agenda (Campbell [1979]; Miller [1995]). Provided that the class of *permissible* agendas satisfies Self-Consistency, there is no added benefit to this kind of manipulation: whatever decision rule that can be achieved in this way can also be achieved by restructuring the original agenda.

In light of this, it would be worth identifying some natural agenda classes that satisfy Self-Consistency. Proposition 1 already makes some significant progress — by showing that the simple agendas $\mathcal{S}$ exhibit this property while the recursive agendas $\mathcal{R}$ do not. Example 6 goes even further — by establishing that there are classes of agendas nested between $\mathcal{S}$ and $\mathcal{R}$ which satisfy Self-Consistency. In fact, as the following remark guarantees, there is a maximal sub-class $\mathcal{R}^{sc} \subseteq \mathcal{R}$ with this property:

Remark 8 *For every class of agendas $\mathcal{T}$, there exists a maximal sub-class which satisfies Self-Consistency.*

Along the lines of the current paper, it would be interesting to determine the structural features that define the agendas in $\mathcal{R}^{sc}$; and, to characterize the family of decision rules implemented by these agendas.

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9 Mathematical Appendix

9.1 Primer on Graph Theory

A *graph* $\mathtt{G} = (Q, E)$ is a pair consisting of: a collection of *nodes* Q; and, a collection of (undirected) *edges* E, each of which connects two nodes q and $\tilde{q}$ in Q. It is *labeled* if each node $q \in Q$ has a label $\ell(q)$; *finite* if $|Q|$ is finite; and, *loopless* if there exist no edges in E (called *loops*) that connect a node in Q to itself. A *(simple) path* in $\mathtt{G}$ is a non-repeating sequence of nodes in Q such that each adjacent pair in the sequence is connected by an edge in E.

A *tree* $\mathtt{T} = (Q, E)$ is a loopless graph such that every pair of distinct nodes in Q is connected by a unique simple path. In a tree, nodes can be ordered in a natural way. In particular, a node $q' \in Q$ is *between* two nodes $q, q'' \in Q$ if the path from q to q'' (inclusive of the end nodes) passes through q'.

A tree is *rooted* if one node $r \in Q$ is specially designated as the root. In a rooted tree $\mathtt{T}_r \equiv (Q, E, r)$, betweenness defines a *partial ordering* $\leq_E$ on the nodes where $q \leq_E \tilde{q}$ for $q, \tilde{q} \in Q$ if q is between the root r and the node $\tilde{q}$. (It is easy to see that $\leq_E$ is reflexive, transitive, and antisymmetric.) Equivalently, the nodes exhibit a *strict partial ordering* $<_E$ defined by $q <_E \tilde{q}$ for $q, \tilde{q} \in Q$ if $q \leq_E \tilde{q}$ and $q \neq \tilde{q}$. Alternatively, a rooted tree can be defined directly in terms of the nodes Q and a partial ordering $\leq$ (or a strict partial ordering $<$) on Q. In either case, the collection of edges $E_<$ is defined by $\{q, \tilde{q}\} \in E_<$ if $q < \tilde{q}$ and $q < q' < \tilde{q}$ for no $\tilde{q} \in Q$. In a rooted tree $\mathtt{T}_r$, $\tilde{q} \in Q$ is a *predecessor* (resp. *successor*) of $q \in Q$ if $\{q, \tilde{q}\} \in E_<$ and $\tilde{q} < q$ (resp. $q < \tilde{q}$). In turn, a node $q \in Q$ is said to be *terminal* if it has no successors; and, *non-terminal* otherwise. By construction, every non-root node of a rooted tree has a unique predecessor; and, every non-terminal node has a successor.

Finally, a *binary tree* is a rooted and labeled tree such that every non-terminal node in Q has exactly two successors.

9.2 Proof of Remark 1

Suppose that v is implemented by $\mathtt{T}_X$. Then, $v(P^A, X) \equiv UNE[\mathtt{T}_X; P^A]$ and $v(P, A) \equiv UNE[\mathtt{T}_{X|A}; P]$ by definition. To see that $UNE[\mathtt{T}_X; P^A] = UNE[\mathtt{T}_{X|A}; P^A]$, note that "backward induction" determines the UNE on any agenda. In any terminal subgame, it selects the Condorcet winner. One can then delete the Condorcet loser and repeat the argument on the resulting (smaller) agenda. Since P^A and P coincide on A, $UNE[\mathtt{T}_{X|A}; P^A] = UNE[\mathtt{T}_{X|A}; P]$ as well. Combining these equalities gives $v(P^A, X) = UNE[\mathtt{T}_X; P^A] = UNE[\mathtt{T}_{X|A}; P^A] = UNE[\mathtt{T}_{X|A}; P] = v(P, A)$ as required.

9.3 Proof of Remarks 2 and 3

Proof of Remark 2. Let q denote a non-terminal node of the agenda $\mathtt{T}_X$. ($\Rightarrow$) Since $\mathtt{T}_X$ is non-repetitive, $c_i(q) \neq \emptyset$. Since $\mathtt{T}_X$ is continuous, some $x_i \in c_i(q)$ labels exactly one terminal node below q_i for $i = 1, 2$. ($\Leftarrow$) By assumption, $c_i(q) \neq \emptyset$ for $i = 1, 2$. So, $\mathtt{T}_X$ is non-repetitive. By assumption, some $x_i \in c_i(q)$ also labels exactly one terminal node below q_i for $i = 1, 2$. So, $\mathtt{T}_X$ is also continuous. ■

Claim 1 *Every history-independent agenda is non-repetitive.*

Proof. Fix a history-independent agenda $\mathtt{T}_X$; and, consider a non-terminal node q where $\ell(q) \equiv B$. By way of contradiction, suppose $\ell(q_1) = B$ and $\ell(q_2) \equiv C \subseteq B$. By history-independence, the successors of q_1 are labeled B and C. Repeating this argument, there is an infinite path down the agenda from q where every node is labeled by B. But, this contradicts the fact that $\mathtt{T}_X$ is finite. ■

Claim 2 *Every simple agenda is continuous.*

Proof. Fix a simple agenda $\mathtt{S}_X$. The proof is by strong induction on $|X| = m$.

The base cases $m = 2, 3$ follow from Claim 1. To complete the induction, suppose the claim holds for $m \le n$ and consider the case $m = n + 1$. Let r denote the root; and, r_1 and r_2 its two successors. Since $\mathtt{S}_X$ is simple, the agenda $\mathtt{S}^i_X$ starting from r_i is simple for $i = 1, 2$. Since $\mathtt{S}_X$ is non-repetitive by Claim 1, $\ell(r_i) \subset X$ for $i = 1, 2$. So, each $\mathtt{S}^i_X$ is continuous by the induction hypothesis. By Claim 1, each $\mathtt{S}^i_X$ is also non-repetitive. Let r_{i1} and r_{i2} denote the successors of r_i. By Remark 2, some $x_{ij} \in c_j(r_i)$ labels exactly one terminal node below r_{ij} for $j = 1, 2$.

Given Remark 2, the proof is complete if $x_{i1} \in c_i(r)$ or $x_{i2} \in c_i(r)$ for $i = 1, 2$. By way of contradiction, suppose $x_{i1}, x_{i2} \in u(r)$. Now, consider any terminal node t below r_i. By persistence, there exists a node q^t (between r and t) with a successor labeled $u(r)$. Since $x_{i1} \in c_1(r_i)$ and $x_{i2} \in c_2(r_i)$, it cannot be any node between r_i and t. So, the only possibility is that $q^t = r$. Hence, $u(r) \in \{\ell(r_1), \ell(r_2)\}$. Without loss of generality, suppose $\ell(r_1) = u(r)$. Then, it must be that $\ell(r_2) = c(r) \cup u(r) = \ell(r)$. But, this contradicts the fact that $\mathtt{S}_X$ is non-repetitive (by Claim 1). ∎

Proof of Remark 3. **(i)** By Claims 1 and 2. **(ii)** It is easy to check that the left-hand agenda in Figure 5 is recursive. However, it is not simple. Since the two nodes labeled $\{x, b, c\}$ give rise to distinct sub-agendas, it violates history-independence. (Still, it is persistent: the root r is the only node with uncontested alternatives; and, both of its successors have successors labeled $u(r) = \{x, b, c\}$. ∎

Claim 3 *Given a simple agenda $\mathtt{S}_X$ with a non-terminal node q such that $u(q) \neq \emptyset$ and a terminal node t below q, there exists a unique non-terminal node q^t between q and t such that $u(q) \in \{\ell(q^t_1), \ell(q^t_2)\}$. Consequently:*

(a.i) *$\emptyset \subset \ell(q^t) \cap c(q) \subseteq \ell(q^t_1) \subset \ell(q^t)$ and $u(q) = \ell(q^t_2) \subset \ell(q^t)$ at node q^t; and,*

(a.ii) *$u(q) \subset \ell(\tilde{q}_i) \subset \ell(\tilde{q})$ for $i = 1, 2$ at each node $\tilde{q}$ strictly between q and q^t.*

What is more, the contested outcomes at the nodes between q and q^t satisfy the following:

(b.i) *$\ell(q^t) \cap c(q) \subset c(q^t) \subseteq \ell(q^t)$ at node q^t; and,*

(b.ii) *$\emptyset \subset c(\tilde{q}) \subseteq \ell(\tilde{q}) \cap c(q)$ at each node $\tilde{q}$ strictly between q and q^t.*

Proof. The fact that $\mathtt{S}_X$ is persistent and non-repetitive (by Claim 1) implies the first part of the claim. **(a.i)** Without loss of generality, $u(q) = \ell(q^t_2)$ by the first part of the claim. So, $\ell(q^t_1) \supseteq \ell(q^t) \setminus \ell(q^t_2) = \ell(q^t) \setminus u(q) = \ell(q^t) \cap c(q) \supset \emptyset$ as required. And, since $\mathtt{S}_X$ is non-repetitive, $\ell(\tilde{q}_i) \subset \ell(\tilde{q})$ for $i = 1, 2$. **(a.ii)** For the lower bounds: $u(q) \subset \ell(\tilde{q}_i)$ by the first part of the claim. For the upper bounds: $\ell(\tilde{q}_i) \subset \ell(\tilde{q})$ since $\mathtt{S}_X$ is non-repetitive. **(b.i)** For the lower bound: note that $c_1(q^t) \supseteq \ell(q^t) \cap c(q)$ by (a.i); and, $c_2(q^t) \supset \emptyset$ since $\mathtt{S}_X$ is non-repetitive. For the upper bound: note that $c(q^t) = \ell(q^t)$ if $\ell(q^t_1) = \ell(q^t) \cap c(q)$. **(b.ii)** For the lower bound: note that $c(\tilde{q}) \supset \emptyset$ since $\mathtt{S}_X$ is non-repetitive. For the upper bound: note that $c(\tilde{q}) = \ell(\tilde{q}) \cap c(q)$ if $(\tilde{c}_1, \tilde{c}_2)$ partitions $\ell(\tilde{q}) \cap c(q)$ and $\ell(q^t_i) \equiv u(q) \cup \tilde{c}_i$ for $i = 1, 2$. ∎

9.4 Proof of Theorem 1

Sub-section 1 establishes sufficiency, while sub-sections 2 and 3 establish necessity and uniqueness, respectively.

9.4.1 Sufficiency of IS and ILA

Remark 9 *For $A = B \cup C$, the following are equivalent: (i) $B, C \neq A$; (ii) $B \cap C \neq B, C$; and, (iii) $B \setminus C, C \setminus B \neq \emptyset$.*

Proof. $(i) \Rightarrow (ii)$: Suppose $B \cap C = B$. Then, $C = (B \cap C) \cup C = B \cup C = A$ which is a contradiction. $(ii) \Rightarrow (iii)$: Suppose $B \setminus C = \emptyset$. Then, $B = (B \cap C) \cup (B \setminus C) = (B \cap C) \cup \emptyset = B \cap C$ which is a contradiction. $(iii) \Rightarrow (i)$: Suppose $B = A$. Then, $C \setminus B = C \setminus A = \emptyset$ which is a contradiction. ∎

Claim 4 *If v satisfies IS and ILA, then $v(P, A) = v(P', A)$ for all $P, P' \in \mathbf{P}$ that coincide on $A \in \mathbf{X}$.*

Proof. The proof is by induction on $|A|$. The base case $|A| = 2$ follows from ILA. To complete the induction, suppose the claim holds for $|A| = n$ and consider the case $|A| = n + 1$. Then, IS, the induction hypothesis, and the base case imply $v(P, A) = v(P, \{v(P, B), v(P, C)\}) = v(P, \{v(P', B), v(P', C)\}) = v(P', \{v(P', B), v(P', C)\}) = v(P', A)$. ■

Claim 5 *If v satisfies IS and ILA, then $v(P, A) = v(P^A, X)$ for all $(P, A) \in \mathbf{P} \times \mathbf{X}$.*

Proof. By ILA, $v(P^A, X) = ... = v(P^A, A)$. Since $v(P^A, A) = v(P, A)$ by Claim 4, $v(P^A, X) = v(P, A)$. ■

Claim 6 *Suppose v satisfies IS and ILA. If (B, C) splits $A \notin \mathbf{X}_1$ and $D \subset A$, then:*

(i) *$v(P, D) = v(P, \{v(P, B \cap D), v(P, C \cap D)\})$; and,* ***(ii)*** *$(B \cap D, C \cap D)$ splits D if $D \neq B \cap D, C \cap D$.*

Proof. Fix some $x \in A$ and let P_x coincide with P except that x is demoted to Condorcet loser on A. Then, $v(P, A \setminus x) = v(P_x, A \setminus x) = v(P_x, A) = v(P_x, \{v(P_x, B), v(P_x, C)\}) = ... = v(P, \{v(P, B \setminus x), v(P, C \setminus x)\})$ by Claim 4, ILA, and IS. Part (i) follows by repeated application of this reasoning. For part (ii), observe that $D \neq B \cap D, C \cap D$ implies $B \cap C \cap D \neq B \cap D, C \cap D$ by Remark 9. So, $(B \cap D, C \cap D)$ splits D by part (i). ■

Claim 7 *Suppose v satisfies IS and ILA. If (B, C) splits $A \notin \mathbf{X}_1$ and $\emptyset \subset B \cap C \subset D \subseteq C$, then either:*

(i) *$(B', B \cap C)$ splits D for some $B' \subset D$; or,* ***(ii)*** *(B', C') splits D for some $B', C' \supset B \cap C$.*

Proof. Let (B', C') denote a splitting of D. To establish the result, it suffices to show: (1) $B \cap C \subseteq B'$ or $B \cap C \subseteq C'$; and, (2) $B \cap C = C'$ if $B \cap C \not\subseteq B'$. Given (1) and (2), the only two possibilities for (B', C') are (i) and (ii).

(1) By way of contradiction, suppose $B \cap C \not\subseteq B', C'$. Fix alternatives $b \in (B \cap C) \setminus C'$ and $c \in (B \cap C) \setminus B'$. Since $B \cap C \subset D$, there exists some $d \in D \setminus (B \cap C)$. Without loss of generality: (a) $d \in B' \cap C'$; or, (b) $d \in B' \setminus C'$ (since $d \in C' \setminus B'$ is similar). Since (B, C) splits A, $v(P, \{b, c, d\}) = v(P, \{v(P, \{b, c\}), v(P, \{b, c, d\})\})$ by Claim 6(i).

For cases (a) and (b), I show that this entails a contradiction. **(a)** Since (B', C') splits D,

$$v(P, \{b, c, d\}) = v(P, \{v(P, \{b, d\}), v(P, \{c, d\})\}).$$

So, $v(P_{bcd}, \{b, c, d\}) = c$ for the Condorcet triple P_{bcd}. Combining this with the formula in the last paragraph implies

$$c = v(P_{bcd}, \{b, c, d\}) = v(P_{bcd}, \{v(P_{bcd}, \{b, c\}), v(P_{bcd}, \{b, c, d\})\}) = v(P_{bcd}, \{b, c\}) = b,$$

which is a contradiction. **(b)** Since (B', C') splits D, $v(P, \{b, c, d\}) = v(P, \{v(P, \{b, d\}), c\})$. So, $v(P_{bcd}, \{b, c, d\}) = c$. Then, by the same reasoning as in case (a), this entails the contradiction that $c = v(P_{bcd}, \{b, c, d\}) = b$.

(2) Since $B \cap C \not\subseteq B'$, $B \cap C \subseteq C'$ by (1). By way of contradiction, suppose $B \cap C \subset C'$. Fix alternatives $b \in B' \setminus C'$, $c \in (B \cap C) \setminus B'$, and $d \in C' \setminus (B \cap C)$. Since $B \cap C \subset C'$, $b \in B' \setminus (B \cap C)$. There are two cases: (a) $d \in B' \cap C'$; and, (b) $d \in C' \setminus B'$. Since (B, C) splits A, $v(P, \{b, c, d\}) = v(P, \{c, v(P, \{b, c, d\})\})$ by Claim 6(i).

For cases (a) and (b), I show that this entails a contradiction. **(a)** Since (B', C') splits D,

$$v(P, \{b, c, d\}) = v(P, \{v(P, \{b, d\}), v(P, \{c, d\})\}).$$

So, $v(P_{bdc}, \{b, c, d\}) = b$ for the Condorcet triple P_{bdc}. Combining this with the formula in the last paragraph implies

$$b = v(P_{bdc}, \{b, c, d\}) = v(P_{bdc}, \{c, v(P_{bdc}, \{b, c, d\})\}) = v(P_{bdc}, \{b, c\}) = c,$$

which is a contradiction. **(b)** Since (B', C') splits D, $v(P, \{b, c, d\}) = v(P, \{b, v(P, \{c, d\})\})$. So, $v(P_{bdc}, \{b, c, d\}) = c$. Then, by the same reasoning as in case (a), this entails the contradiction that $b = v(P_{bdc}, \{b, c, d\}) = c$. ■

Claim 8 *If v satisfies IS and ILA, then there exists a unique splitting for all $A \notin \mathbf{X}_1$.*

Proof. The proof is by induction on $|A|$. The case $|A| = 2$ is trivial. For $|A| = 3$, Table 1 shows that every potential splitting of A yields a distinct pair of outcomes on the two Condorcet triples. So, the splitting of A must be unique.

To complete the induction, suppose the claim holds for $|A| = n$ and consider $|A| = n+1$. By way of contradiction, suppose (B, C) and (B', C') are distinct splittings of A. First, suppose (B, C) and (B', C') partition A. Then, there exists some $x \in A$ such that $(B \setminus x, C \setminus x)$ and $(B' \setminus x, C' \setminus x)$ are distinct partitions of $A \setminus x$, which contradicts the induction hypothesis. Next, suppose $x \in B \cap C$ for some $x \in A$. If $x \in B' \cap C'$, then $(B \setminus x, C \setminus x)$ and $(B' \setminus x, C' \setminus x)$ are distinct splittings of $A \setminus x$, which again contradicts the induction hypothesis. Finally, suppose $x \in B' \setminus C'$. There are two cases for $(B \setminus x, C \setminus x)$ and $(B' \setminus x, C' \setminus x)$: either (i) they coincide; or, (ii) they are distinct.

(i) Without loss of generality, $B \setminus x = B' \setminus x$ and $C \setminus x = C'$ (the only other possibility is symmetric). Since (B, C) splits A, there exist alternatives $b \in B \setminus C$ and $c \in C \setminus B$. Since $B \setminus x = B' \setminus x$ and $C \setminus x = C'$, $b \in B' \setminus C'$ and $c \in C' \setminus B'$. Now, consider $D = \{b, c, x\}$. From the observations above: $(B \cap D, C \cap D) = (\{x, b\}, \{x, c\})$ and $(B' \cap D, C' \cap D) = (\{x, b\}, \{c\})$. For the Condorcet triple P_{bxc}: $v(P_{bxc}, \{v(P_{bxc}, B \cap D), v(P_{bxc}, C \cap D)\}) = v(P_{bxc}, \{b, x\}) = b$; and, $v(P_{bxc}, \{v(P_{bxc}, B' \cap D), v(P_{bxc}, C' \cap D)\}) = v(P_{bxc}, \{b, c\}) = c$. But, this contradicts Claim 6(i).

(ii) First, suppose $B' \setminus C' \neq x$. Then $(B \setminus x, C \setminus x)$ and $(B' \setminus x, C' \setminus x)$ are distinct splittings of $A \setminus x$, which contradicts the induction hypothesis. Next, suppose $B' \setminus C' = x$. Since (B, C) splits A, there exist $b \in B \setminus C$ and $c \in C \setminus B$. Now, consider $D = \{b, c, x\}$. From the observations above: $(B \cap D, C \cap D) = (\{x, b\}, \{x, c\})$, $x \in B' \setminus C'$, and $b, c \in C'$. The last two set inclusions leave three possibilities: (a) $(B' \cap D, C' \cap D) = (\{x\}, \{b, c\})$; (b) $(B' \cap D, C' \cap D) = (\{x, b\}, \{b, c\})$; and, (c) $(B' \cap D, C' \cap D) = (\{x, b, c\}, \{b, c\})$. For (a)-(b), $v(P_{bcx}, \{v(P_{bcx}, B \cap D), v(P_{bcx}, C \cap D)\}) = v(P_{bcx}, \{x, c\}) = c$ and $v(P_{bcx}, \{v(P_{bcx}, B' \cap D), v(P_{bcx}, C' \cap D)\}) = v(P_{bcx}, \{x, b\}) = x$. But, this contradicts Claim 6(i). For (c), $v(P_{bcx}, D) = c$ by Claim 6(i); and, $v(P_{bcx}, D) = v(P_{bcx}, \{v(P_{bcx}, B' \cap D), v(P_{bcx}, C' \cap D)\}) = v(P_{bcx}, \{v(P_{bcx}, D), v(P_{bcx}, \{b, c\})\}) = v(P_{bcx}, \{c, b\}) = b$. But, this is a contradiction. ∎

Note: In the proof of Claims 9 and 10, I abuse notation by referring to a node q of $\mathtt{S}^v_X$ by its label $\ell(q)$. By virtue of Claim 8 however, this abuse does not create any possibility for confusion.

Claim 9 *If v satisfies IS and ILA, then $\mathtt{S}^v_X$ defines a simple agenda.*

Proof. By Claim 8, the recursive construction of $\mathtt{S}^v_X$ described in the text is well-defined. Since v satisfies IS, this construction is finitary and, thus, defines an agenda. Clearly, $\mathtt{S}^v_X$ is history-independent by construction. The proof that $\mathtt{S}^v_X$ is persistent is by strong induction on $|X| = m$. The base cases $m = 2, 3$ follow from IS and the definition of $\mathtt{S}^v_X$. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$.

Let B_1 and C_1 denote the successors of the root X in $\mathtt{S}^v_X$. Consider the restricted decision rules v_{B_1} and v_{C_1} on $\mathbf{B}_1$ and $\mathbf{C}_1$ (see Definition 10). Note that v_{B_1} and v_{C_1} inherit IS and ILA from v; and, $|B_1|, |C_1| \leq n$ (since $B_1, C_1 \subset X$ by IS). By the induction hypothesis, it follows that $\mathtt{S}^{v_{B_1}}_{B_1}$ and $\mathtt{S}^{v_{C_1}}_{C_1}$ are persistent. Since $\mathtt{S}^v_X(B_1) = \mathtt{S}^{v_{B_1}}_{B_1}$ and $\mathtt{S}^v_X(C_1) = \mathtt{S}^{v_{C_1}}_{C_1}$ by construction, this establishes that $\mathtt{S}^v_X$ is persistent at every non-terminal node except possibly the root X.

To check persistence at X, suppose $B_1 \cap C_1 \neq \emptyset$ and consider any terminal node x below C_1. Let $X, C_1, ..., C_{n_x}, x$ denote the path from X to x. First, consider the node C_1. By construction of $\mathtt{S}^v_X$, (B_2, C_2) splits $C_1 \supset B_1 \cap C_1$. By Claim 7, there are three possibilities: $B_2 = B_1 \cap C_1$; $C_2 = B_1 \cap C_1$; or, $B_2, C_2 \supset B_1 \cap C_1$. In the first two cases, C_1 is a node between X and x with a successor $B_1 \cap C_1$. In the third case, consider the node C_2. Since $B_1 \cap C_1 \subset C_2 \subset C_1$, Claim 7 applies. So, one can repeat the argument given for C_1. Since $\mathtt{S}^v_X$ is finite, continuing inductively in the same vein establishes that one of the nodes C_i between X and x must have a successor $B_1 \cap C_1$.

By the same argument, the same is true for any terminal node x below B_1. So, $\mathtt{S}^v_X$ is persistent at X. ∎

Claim 10 *If v satisfies IS and ILA, then $\mathtt{S}^v_X$ implements v.*

Proof. By Claim 9, $\mathtt{S}_X^v$ defines an agenda. The proof that it implements v is by strong induction on $|X| = m$. The base cases $m = 2, 3$ follow from IS and the definition of $\mathtt{S}_X^v$. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. It suffices to show that $v(P, X) = UNE[\mathtt{S}_X^v; P]$ for all $P \in \mathbf{P}$. (To see this, fix some $(P, A) \in \mathbf{P} \times \mathbf{X}$. Since $v(P, A) = v(P^A, X)$ (by Claim 5) and $UNE[\mathtt{S}_X^v; P^A] = UNE[\mathtt{S}_{X|A}^v; P]$ (by Remark 1), it follows that $v(P, A) = v(P^A, X) = UNE[\mathtt{S}_X^v; P^A] = UNE[\mathtt{S}_{X|A}^v; P]$. In other words, $\mathtt{S}_X^v$ implements v.)

To show that $v(P, X) = UNE[\mathtt{S}_X^v; P]$, let B and C denote the successors of the root X in $\mathtt{S}_X^v$. Consider the restricted decision rules v_B and v_C on $\mathbf{B}$ and $\mathbf{C}$. By the induction hypothesis, $\mathtt{S}_B^{v_B}$ implements v_B and $\mathtt{S}_C^{v_C}$ implements v_C. Since $\mathtt{S}_X^v(B) = \mathtt{S}_B^{v_B}$ and $\mathtt{S}_X^v(C) = \mathtt{S}_C^{v_C}$ by construction, $v_B(P, B) = UNE[\mathtt{S}_X^v(B); P]$ and $v_C(P, C) = UNE[\mathtt{S}_X^v(C); P]$. Like the UNE, v selects the Condorcet winner between two alternatives (by ILA). So, "backward induction" requires $UNE[\mathtt{S}_X^v; P] = v(P, \{v_B(P, B), v_C(P, C)\})$. By definition of v_B and v_C, it then follows that $UNE[\mathtt{S}_X^v; P] = v(P, \{v(P, B), v(P, C)\})$. And, since (B, C) splits X by construction, $v(P, \{v(P, B), v(P, C)\}) = v(P, X)$. Combining these observations gives $v(P, X) = v(P, \{v(P, B), v(P, C)\}) = UNE[\mathtt{S}_X^v; P]$ as required. ∎

Proof of Theorem 1 (Sufficiency). This follows directly from Claims 9-10 above. ∎

9.4.2 Necessity of IS and ILA

Note: In this section, I frequently abuse notation by referring to a node q of $\mathtt{S}_X$ by its label $\ell(q)$. Since $\mathtt{S}_X$ is simple (and, thus, history-independent by the discussion in the text), this abuse does not create any possibility for confusion.

Claim 11 *If $\mathtt{S}_X$ is a simple agenda where B and C are the successors of the root X, then:*

$$UNE[\mathtt{S}_X; P] = x \in B \cap C \textit{ implies } UNE[\mathtt{S}_X(B); P] = x = UNE[\mathtt{S}_X(C); P].$$

Proof. The proof is by strong induction on $|X| = m$. The base cases $m = 2, 3$ follow from the fact that $\mathtt{S}_X$ is non-repetitive (by Claim 1). To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n+1$. By way of contradiction, suppose the claim is false. Since x must be selected from $\mathtt{S}_X(B)$ or $\mathtt{S}_X(B)$, suppose $UNE[\mathtt{S}_X(B); P] = xM_X^P c = UNE[\mathtt{S}_X(C); P]$.

First, consider $\mathtt{S}_X(B)$. Since $\mathtt{S}_X$ is simple, so is $\mathtt{S}_X(B)$. By persistence, every path to a terminal node t below B passes through a node B_t^* (potentially B) with successors B_t and $B \cap C$. Let $T(B, x) \equiv \{t : x \in B_t\}$. There are two cases: (i) $T(B, x) = \emptyset$; and, (ii) $T(B, x) \neq \emptyset$. For both, $UNE[\mathtt{S}_X(B \cap C); P] = x$. **(i)** Since $UNE[\mathtt{S}_X(B); P] = x$ and $x \notin B_t$ for all t below B, "backward induction" requires $UNE[\mathtt{S}_X(B \cap C); P] = x$. **(ii)** Since $UNE[\mathtt{S}_X(B); P] = x$, "backward induction" requires $UNE[\mathtt{S}_X(B_t^*); P] = x$ for some $t \in T(B, x)$. Since $\mathtt{S}_X$ is non-repetitive by Claim 1, $B_t^* \subset X$ so that $|B_t^*| \leq n$. Since $\mathtt{S}_X(B_t^*)$ is simple, the induction hypothesis implies $UNE[\mathtt{S}_X(B \cap C); P] = x$.

Next, consider $\mathtt{S}_X(C)$. If $c \in B \cap C$, the same reasoning as in the last paragraph establishes $UNE[\mathtt{S}_X(B \cap C); P] = c$, which is a contradiction. So, $c \in C \setminus B$. By persistence, every path to a terminal node t below C passes through a node C_t^* (potentially C) with successors C_t and $B \cap C$. Let $T(C, c) \equiv \{t : c \in C_t\}$. By construction, $T(C, c) \neq \emptyset$. Since $UNE[\mathtt{S}_X(C); P] = c$, "backward induction" requires $UNE[\mathtt{S}_X(C_t); P] = c$ for some $t \in T(C, c)$. Since $UNE[\mathtt{S}_X(B \cap C); P] = x$ and $xM_X^P c$ however, $UNE[\mathtt{S}_X(C_t^*); P] = x$. It then follows from "backward induction" that $UNE[\mathtt{S}_X(C_t); P] \neq c$, which is a contradiction. ∎

Claim 12 *If $\mathtt{S}_X$ is a simple agenda where B and C are the successors of the root X, $UNE[\mathtt{S}_{X|B}; P] = UNE[\mathtt{S}_X(B); P]$.*

Proof. If $B \cap C = \emptyset$, then $\mathtt{S}_{X|B} = \mathtt{S}_X(B)$ by definition and the claim follows. So, suppose $B \cap C \neq \emptyset$. Fix a profile $P \in \mathbf{P}$ such that $UNE[\mathtt{S}_{X|B}; P] = b$. By Remark 1, $UNE[\mathtt{S}_X; P^B] = b$. Since P and P^B coincide on B, the proof is complete if $UNE[\mathtt{S}_X(B); P^B] = b$. (In that case, $UNE[\mathtt{S}_X(B); P] = b = UNE[\mathtt{S}_{X|B}; P]$.) To establish $UNE[\mathtt{S}_X(B); P^B] = b$, there are two cases: (i) $b \in B \setminus C$; and, (ii) $b \in B \cap C$. **(i)** Since $b \in B \setminus C$, "backward induction" requires $UNE[\mathtt{S}_X(B); P^B] = b$. **(ii)** Since $UNE[\mathtt{S}_X; P^B] = b \in B \cap C$, Claim 11 implies $UNE[\mathtt{S}_X(B); P^B] = b$. ∎

Claim 13 *If a decision rule v is implementable by simple agenda, then it satisfies IS.*

Proof. Suppose v is implemented by $\mathtt{S}_X$. The proof is by strong induction on $|X| = m$. Since $\mathtt{S}_X$ is non-repetitive (by Claim 1), the base cases $m = 2, 3$ are trivial. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. Let B and C denote the two successors of the root X. To establish the induction step, fix an issue $A \in \mathbf{X}_{-1}$. Without loss of generality, there are two possibilities: (1) $A \not\subseteq B, C$; or, (2) $A \subseteq B$.

(1) In this case, $(B \cap A, C \cap A)$ splits A. To see this, it suffices to check conditions (i)-(iii) of the definition. **(i)** Since $B \cup C = X$, it follows that $(B \cap A) \cup (C \cap A) = (B \cup C) \cap A = X \cap A = A$. **(ii)** By way of contradiction, suppose that $B \cap A = (B \cap A) \cap (C \cap A)$. Then, $A = (B \cap A) \cup (C \cap A) = [(B \cap A) \cap (C \cap A)] \cup (C \cap A) = C \cap A$. So, $A \subseteq C$ which is a contradiction. **(iii)** Note that $v(P, B) \equiv UNE[\mathtt{S}_{X|B}; P] = UNE[\mathtt{S}_X(B); P]$ and $v(P, C) \equiv UNE[\mathtt{S}_{X|C}; P] = UNE[\mathtt{S}_X(C); P]$ by Claim 12. Since $v(P, X) \equiv UNE[\mathtt{S}_X; P]$ as well, "backward induction" requires $v(P, X) = v(P, \{v(P, B), v(P, C)\})$. Combined with Remark 1, this identity implies the following:

$$\begin{aligned} v(P, A) &= v(P^A, X) = v(P^A, \{v(P^A, B), v(P^A, C)\}) = v(P^A, \{v(P^{B \cap A}, X), v(P^{C \cap A}, X)\}) \\ &= v(P^A, \{v(P, B \cap A), v(P, C \cap A)\}) = v(P^{\{v(P, B \cap A), v(P, C \cap A)\}}, X) \\ &= v(P, \{v(P, B \cap A), v(P, C \cap A)\}). \end{aligned}$$

(2) By Claim 12, $v(P, B) \equiv UNE[\mathtt{S}_{X|B}; P] = UNE[\mathtt{S}_X(B); P]$. By Remark 1, it follows that $\mathtt{S}_X(B)$ implements the restricted decision rule v_B on $\mathbf{B}$. Since $\mathtt{S}_X$ is non-repetitive by Claim 1, $B \subset X$ so that $|B| \leq n$. Since $\mathtt{S}_X$ is simple, so is $\mathtt{S}_X(B)$. By the induction hypothesis, it then follows that A is splittable for v_B. To conclude, suppose (B_A, C_A) splits A for v_B. Since $v_B(P, D) \equiv v(P, D)$ for all $D \in \mathbf{B}$, it follows that (B_A, C_A) also splits A for v. ■

Proof of Theorem 1 (Necessity). Let v denote the decision rule implemented by a simple agenda $\mathtt{S}_X$. By Claim 13, v satisfies IS. By the argument given in the text (after the statement of the theorem), v also satisfies ILA. ■

9.4.3 Proof of Uniqueness

Note: In this section, I abuse notation by referring to a node q of a simple agenda $\mathtt{S}_X$ by its label $\ell(q)$.

As outlined in the text, the outcomes of v on Condorcet triples uniquely identify $\mathtt{S}_X^v$. The proof that $\mathtt{S}_X^v$ is the only simple agenda implementing v by strong induction on $|X| = m$. The base cases $m = 2, 3$ follow from IS. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. Fix a simple agenda $\mathtt{S}_X$ that implements v; and, let B and C denote the two successors of the root X in $\mathtt{S}_X$.

First, note that (B, C) is the unique splitting of X. Given Claim 8, it suffices to check conditions (i)-(iii) of the definition. **(i)** Since $\mathtt{S}_X$ is an agenda, $B \cup C = X$. **(ii)** Since $\mathtt{S}_X$ is non-repetitive (by Claim 1), $B, C \neq B \cap C$ (by Remark 9). **(iii)** By Claim 12, $UNE[\mathtt{S}_{X|B}; P] = UNE[\mathtt{S}_X(B); P]$ and $UNE[\mathtt{S}_{X|C}; P] = UNE[\mathtt{S}_X(C); P]$ for all $P \in \mathbf{P}$. Since $\mathtt{S}_X$ implements v, $v(P, B) = UNE[\mathtt{S}_{X|B}; P]$ and $v(P, C) = UNE[\mathtt{S}_{X|C}; P]$. So, $v(P, B) = UNE[\mathtt{S}_X(B); P]$ and $v(P, C) = UNE[\mathtt{S}_X(C); P]$. Since $v(P, X) = UNE[\mathtt{S}_X; P]$ as well, $v(P, X) = v(P, \{v(P, B), v(P, C)\}$.

To conclude, consider the restricted decision rules v_B and v_C on $\mathbf{B}$ and $\mathbf{C}$. As shown in the last paragraph, $v_B(P, B) = UNE[\mathtt{S}_X(B); P]$ and $v_C(P, C) = UNE[\mathtt{S}_X(C); P]$ for all $P \in \mathbf{P}$. So, $\mathtt{S}_X(B)$ implements v_B (by Remark 1); and, similarly, $\mathtt{S}_X(C)$ implements v_C. Since $\mathtt{S}_X$ is simple, so are $\mathtt{S}_X(B)$ and $\mathtt{S}_X(C)$. By the induction hypothesis, $\mathtt{S}_B^{v_B}$ (resp. $\mathtt{S}_C^{v_C}$) is the unique simple agenda that implements v_B (resp. v_C). Consequently, $\mathtt{S}_B^{v_B} = \mathtt{S}_X(B)$ and $\mathtt{S}_C^{v_C} = \mathtt{S}_X(C)$. Since (B, C) splits X (as shown in the last paragraph), it then follows from the definition of $\mathtt{S}_X^v$ that $\mathtt{S}_X = \mathtt{S}_X^v$.

9.5 Proof of Propositions 1 and 2

Proof of Proposition 1. Consider a decision rule v that is implementable by simple agenda. It suffices to show that v is preference monotonic at X. Since $v(P, A) = v(P^A, X)$ by Remark 1, the result follows for $A \subset X$. The proof is by strong induction on $|X| \equiv m$. The base cases $m = 2, 3$ are trivial.

To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. Let $\mathtt{S}_X$ denote a simple agenda that implements v; and, let B and C denote the successors of the root X. For a profile P such that $v(P, X) \equiv UNE[\mathtt{S}_X; P] = x$, there are two cases: (i) $x \in B \cap C$; and, (ii) $x \in B \setminus C$.

(i) By Claim 11, $UNE[\mathtt{S}_X(B); P] = UNE[\mathtt{S}_X(C); P] = x$. Since $\mathtt{S}_X$ is non-repetitive by Claim 1, $B, C \subset X$ so that $|B|, |C| \leq n$. Since $\mathtt{S}_X$ is simple, so are $\mathtt{S}_X(B)$ and $\mathtt{S}_X(C)$. So, $UNE[\mathtt{S}_X(B); P^{\uparrow x}] = x$ and $UNE[\mathtt{S}_X(C); P^{\uparrow x}] = x$ by the induction hypothesis. Then, "backward induction" requires $v(P^{\uparrow x}, X) \equiv UNE[\mathtt{S}_X; P^{\uparrow x}] = x$. **(ii)** By the same reasoning as case (i), $UNE[\mathtt{S}_X(B); P^{\uparrow x}] = UNE[\mathtt{S}_X(B); P] = x$. Moreover, $UNE[\mathtt{S}_X(C); P^{\uparrow x}] = UNE[\mathtt{S}_X(C); P]$ since $x \notin C$. Then, "backward induction" requires $v(P^{\uparrow x}, X) \equiv UNE[\mathtt{S}_X; P^{\uparrow x}] = x$. ∎

Proof of Proposition 2. This follows directly from the discussion in the text. ∎

9.6 Proof of Theorem 2

Sub-sections 1 and 2 establish sufficiency and necessity, respectively.

9.6.1 Sufficiency of IS, ILA, and M

Claim 14 *Suppose v satisfies IS and ILA. If a^* is marginal for $A \notin \mathbf{X}_1$ and $x \in A \setminus a^*$, then a^* is marginal for $A \setminus x$.*

Proof. Fix some $x \in A \setminus a^*$. By way of contradiction, suppose $v(P, A \setminus x) = a^*$ for some profile P where a^* is not the Condorcet winner in $A \setminus x$. By Claim 4, $v(P_x, A \setminus x) = v(P, A \setminus x)$ for any profile P_x that coincides with P except x is demoted to Condorcet loser in A. Moreover, $v(P_x, A) = v(P_x, A \setminus x)$ by ILA. So, $v(P_x, A) = a^*$, which contradicts the assumption that a^* is marginal for A. ∎

Claim 15 *If v satisfies IS and ILA, then every $A \in \mathbf{X}$ has at most two marginal alternatives.*

Proof. Suppose otherwise. Denote any three marginal alternatives in A by x, y, and z and consider the Condorcet triple P_{xyz} (as defined in the text). Then, $v(P_{xyz}, \{x, y, z\}) \notin \{x, y, z\}$ by Claim 14, which is a contradiction. ∎

Claim 16 *Suppose v satisfies IS and ILA. If (B, C) splits $A \notin \mathbf{X}_1$ and a^* is marginal for A, then:*

(i) *if $a^* \in C \setminus B$, then $(B, C) = (b, A \setminus b)$; and,* ***(ii)*** *if $a^* \in B \cap C$ and a^{**} is also marginal for A, then $a^{**} \in B \cap C$.*

Proof. **(i)** By way of contradiction, suppose $|B \setminus C| \geq 2$. Fix $b, b' \in B \setminus C$ and consider the triple $P_{a^* bb'}$. By Claim 6, $v(P_{a^* bb'}, \{a^*, b, b'\}) = v(P_{a^* bb'}, \{v(P_{a^* bb'}, \{b, b'\}), a^*\}) = a^*$. By Claim 14, this contradicts the assumption that a^* is marginal for A. **(ii)** By way of contradiction, suppose $a^{**} \in C \setminus B$. Fix some $b \in B \setminus C$ and consider the triple $P_{a^* ba^{**}}$. Then, a contradiction obtains along the same lines as (i). ∎

(a) Proof of Lemma 1 – The Revealed Priority

Claim 17 *If v satisfies IS, ILA, and M, then $\succ_v$ is asymmetric and $\succsim_v$ is complete.*

Proof. By definition of $\sim_v$, the completeness of $\succsim_v$ follows from the asymmetry of $\succ_v$. To see that $\succ_v$ is asymmetric, suppose $y \succ_v z$ and $z \succ_v y$ for some $y, z \in X$; and, let Y and Z denote issues that reveal $y \succ_v z$ and $z \succ_v y$, respectively. The proof that this amounts to a contradiction is by induction on $|Y \cup Z|$.

For the base case $|Y \cup Z| = 4$, it must be that $Y = \{a^*, y, z\}$ and $Z = \{x, y, z\}$ for alternatives $a^*, x \in X$. (The possibility that $Y = \{y, z\}$ is ruled out by ILA; and, the possibility that $Z \subset Y$ is ruled out by Claim 14.) By Claim 14, neither y nor z can be marginal for $Y \cup Z$. Since some alternative must be marginal for $Y \cup Z$ by M, suppose, without loss of generality, that a^* is marginal. Then, a^* is also marginal for Y by Claim 14. Now, consider the splitting (B, C) of $\{a^*, x, y, z\}$. There are two cases: (i) $a^* \in C \setminus B$; and, (ii) $a^* \in B \cap C$.

(i) By Claim 16(i), $(B, C) = (b, \{a^*, x, y, z\} \setminus b)$ with $b \neq a^*$. By Claim 6, this is a contradiction: if $b = x$, then z is marginal for Z; if $b = y$, then y is not marginal for Z; and, if $b = z$, then z is not marginal for Y.

(ii) Since a^* and z are marginal for Y, $v(P_{a^*yz}, Y) = y$. There are two cases: (a) $z \in C \setminus B$; and, (b) $z \in B \cap C$. **(a)** There are two sub-cases: $y \in B \setminus C$; and, $y \in C$. In the first case, $(\{a^*, z\}, \{a^*, y\})$ splits Y by Claim 6. So, z is not marginal for Y, a contradiction. In the second case, $v(P_{a^*yz}, Y) = v(P_{a^*yz}, \{v(P_{a^*yz}, B \cap Y), v(P_{a^*yz}, Y)\}) = v(P_{a^*yz}, \{a^*, y\}) = a^* \neq y$ by Claim 6, a contradiction. **(b)** In this case, $(B, C) = (\{a^*, y, z\}, \{a^*, x, z\})$. By Claim 6, $(\{y, z\}, \{x, z\})$ splits Z. So, y is not marginal for Z, which is a contradiction.

To complete the induction, suppose the claim holds for $|Y \cup Z| = n$ and consider the case $|Y \cup Z| = n+1$. By M and Claim 15, Y and Z each have one or two marginal alternatives. If both have two, this reduces to the case $|Y \cup Z| = n$ by Claim 14. (Without loss of generality, $|Y| \geq 4$. So, Y must contain some non-marginal alternative that is not marginal in Z.) If both sets have one marginal alternative, then either y or z is marginal for $Y \cup Z$ by M and Claims 14-15. So, either y is marginal for Y or z is marginal for Z by Claim 14, both contradictions. So, suppose a^* and z are marginal for Y; and, y is marginal for Z. By M and Claim 14, $a^* \notin Z$ is the only marginal alternative in $Y \cup Z$. By Claim 14: a^* and z are marginal for $\{a^*, y, z\}$; and, y is marginal for $\{x, y, z\}$ for any $x \in Z$.

Now, consider the splitting (B, C) of $Z^* \equiv Z \cup a^*$. There are two cases: (i) $(B, C) = (b, Z^* \setminus b)$ with $b \neq a^*$; and, (ii) $a^* \in B \cap C$. For both, y and z must appear in the same sub-issues as a^*. **(i)** By the argument in (i) of the base case, $b \neq y, z$. So, $\{y, z\} \subseteq C$ as claimed. **(ii)** By the argument in (ii)(a) of the base case, $z \in B \cap C$. In turn, this implies $y \in B \cap C$. To see why, suppose $y \in B \setminus C$ and fix some $x \in C \setminus B$. Then, $(\{y, z\}, \{x, z\})$ splits $\{x, y, z\}$ by Claim 6. By the argument in (ii)(b) of the base case, this is a contradiction. So, $\{y, z\} \subseteq B \cap C$ as claimed.

Continuing in this vein on the sub-issues B and C, y and z always appear in the same sub-issues (up to the splitting of $\{a^*, y, z\}$). Accordingly, y and z appear in exactly the same subgames of $\mathbb{S}^v_{Z^*|Z}$ (i.e., $\mathbb{S}^v_{Z^*}$ after a^* is pruned). By assumption, $v(P, Z) = y$ for some profile P where y is not the Condorcet winner in Z. Since $\mathbb{S}^v_{Z^*}$ implements v on Z^* by Theorem 1, $v(P, Z) = UNE[\mathbb{S}^v_{Z^*|Z}; P] = y$. To see the contradiction, consider the profile P^σ that permutes z and y in every voter's preference. From the symmetry of $\mathbb{S}^v_{Z^*|Z}$, $v(P^\sigma, Z) = UNE[\mathbb{S}^v_{Z^*|Z}; P^\sigma] = z$. But, this contradicts that z is marginal for Z and establishes that $\succ_v$ is asymmetric. ■

Claim 18 *If v satisfies IS, ILA, and M, then $\succsim_v$ defines a priority.*

Proof. Given Claim 17, it suffices to show that: (i) the indifference classes of $\succsim_v$ contain one or two alternatives; and, (ii) $\succsim_v$ is transitive. **(i)** By way of contradiction, suppose $x \sim_v y \sim_v z$. By M, $\{x, y, z\}$ has a marginal alternative. By definition of $\sim_v$, it follows that $\{x, y, z\}$ has three marginal alternatives, which contradicts Claim 15. **(ii)** Given (i), it suffices to rule out the possibility that $x \succsim_v y \succ_v z \succ_v x$. Since $\succ_v$ is asymmetric (by Claim 17) and $\{x, y, z\}$ has a marginal alternative (by M), this leads to a contradiction along the lines of (i). ■

Proof of Lemma 1. Claim 18 establishes (i). To establish (ii), fix an issue A. By M, some $x \in A$ is marginal. Let $z \in \min_{\succsim_v} A$ and $y \equiv \min_{\succsim_v} A \setminus z$. If $x \in A \setminus \{y, z\}$, one obtains a contradiction along the lines of Claim 18. ■

(b) Proof of Corollary 1 – Revealed Priority via Condorcet Triples

Claim 19 *Suppose v satisfies IS, ILA, and M. If v (B, C) splits $A \notin \mathbf{X}_1$ and $|C \setminus B| \geq 2$, then $|B \setminus C| = 1$.*

Proof. By way of contradiction, suppose $|C \setminus B|, |B \setminus C| \geq 2$. Fix any $b, b' \in B \setminus C$ and $c, c' \in C \setminus B$. By Claim 6, $(\{b, b'\}, \{c, c'\})$ splits $A' \equiv \{b, b', c, c'\}$. Given this splitting, there exists some profile P_a where $a \in A'$ is not the Condorcet winner such that $v(P_a, A') = a$. So, A' has no marginal alternatives, which contradicts M. ■

For a weak ordering $\succsim$ on X, let $\succ^*$ denote any linear ordering such that $y \succ z$ implies $y \succ^* z$ for all $y, z \in X$. Given an issue $A = \{a_1, ..., a_K\} \in \mathbf{X}$ labeled according to $\succ^*$ (i.e., $a_i \succ^* a_k$ if $i < k$), let

$$A_j^k \equiv \begin{cases} \{a_j, ..., a_k\} & \text{if } j \leq k \leq K \\ \emptyset & \text{otherwise.} \end{cases}$$

Claim 20 *Suppose v satisfies IS, ILA, and M. If $A = \{a_1, ..., a_K\}$ is labeled according to $\succ_v^*$ for $K \geq 2$, then:*

$$(a_1 \cup A_j^K, A_2^K) \text{ splits } A \text{ for some } j \text{ such that } j \in \{3, ..., K+1\}.$$

Proof. By Claim 16(i) and Lemma 1, there are two possibilities for the splitting (B, C) of A: (i) $(B, C) = (b, A \setminus b)$ with $b \neq a_K$; or, (ii) $a_K \in B \cap C$. In either case, I show that (B, C) has the desired structure.

(i) I show $b = a_1$. By way of contradiction, suppose $b = a_k$ for some $k \neq 1, K$. Then, $v(P_{a_K a_k a_1}, \{a_1, a_k, a_K\}) = a_k$ by Claim 6 so that $a_k \succ_v a_1$ by Lemma 1. Since $a_1 \succsim_v a_k$ by assumption, $a_k \succ_v a_1$ contradicts the fact that $\succsim_v$ is a weak ordering (by Claim 17). So, $b = a_1$ as required.

(ii) By Claim 19, there are two possibilities: (1) $(B, C) = (A \setminus c, A \setminus b)$; and, (2) $(B, C) = (b \cup B', A \setminus b)$ for $B' \subset A \setminus b$ and $|A \setminus B'| \geq 3$. **(1)** It suffices to show $b = a_1$ and $c = a_2$. If $a_1 \neq b, c$, then the outcomes on $\{a_1, b, c\}$ lead to the contradictions $b, c \succ_v a_1$ following the same kind of reasoning as in case (i). So, $b = a_1$ without loss of generality. If $a_2 \neq c$, then the outcomes on $\{a_1, a_2, c\}$ lead to the contradiction $c \succ_v a_2$. So, $c = a_2$. **(2)** It suffices to show: (a) $b = a_1$; (b) $a_2 \notin B'$; and, (c) $a_k \in B'$ implies $a_{k+1} \in B'$. **(a)** By the same reasoning as (1), $a_1 \notin B \cap C = B'$. If $a_1 \neq b$, then $\{a_1, b, c\}$ leads to the contradiction $b \succ_v a_1$ for $c \notin b \cup B'$. So, $b = a_1$. **(b)** If $a_2 \in B'$, then $\{a_1, a_2, c\}$ leads to the contradiction $c \succ_v a_2$ for $c \notin a_1 \cup B'$ given (a). So, $a_2 \notin B'$. **(c)** If $a_k \in B'$ and $a_{k+1} \notin B'$, then $\{a_1, a_k, a_{k+1}\}$ leads to the contradiction $a_{k+1} \succ_v a_k$ given (a). ■

Corollary 1 *If v satisfies IS, ILA, and M, then:*

(i) *$y \succ_v z$ if and only if there exists an $x \in X$ such that $v(P_{xyz}, \{x, y, z\}) = y$; and,*

(ii) *$y \sim_v z$ if and only if, for every $x \in X \setminus \{y, z\}$,* $\left\{ \begin{array}{c} v(P_{xyz}, \{x, y, z\}) = z \;\; \textit{and} \;\; v(P_{xzy}, \{x, y, z\}) = y \\ \textbf{\textit{or}} \\ v(P_{xyz}, \{x, y, z\}) = x \;\; \textit{and} \;\; v(P_{xzy}, \{x, y, z\}) = x \end{array} \right\}$.

Proof. (i) ($\Leftarrow$) From Table 1, there are six different potential splittings of $\{x, y, z\}$. For each, $v(P_{xyz}, \{x, y, z\}) = y$ implies $v(P_{xzy}, \{x, y, z\}) \in \{x, y\}$. This establishes that z is marginal for $\{x, y, z\}$ and y is not. So, $y \succ_v z$.

($\Rightarrow$) For $|X| = 3$, the claim is trivial. The claim is also trivial if $|X| \geq 4$ and: (a) $y \succsim_v x \succsim z$ for some $x \in X \setminus \{y, z\}$; or, (b) z and w are marginal for some Y such that $y \in Y$. **(a)** By Lemma 1, y cannot be marginal for $\{x, y, z\}$. Then, $v(P_{xyz}, \{x, y, z\}) = y$ from Table 1. **(b)** By Claim 14, z and w are marginal for $\{y, z, w\}$. So, y cannot be marginal for $\{x, y, z\}$ by Claim 15. Then, $v(P_{xyz}, \{x, y, z\}) = y$ from Table 1.

So, suppose $|X| \geq 4$ and conditions (a)-(b) do not hold. By way of contradiction, suppose $v(P_{xyz}, \{x, y, z\}) \in \{x, z\}$ for all $x \in X$. Now, fix an $A \equiv \{x, x', y, z\}$ where $x, x' \succ_v y$; and, consider the splitting (B, C). By Claim 20 $(B, C) = (x \cup B', \{x', y, z\})$ for $B' \subseteq \{y, z\}$ (where $y \in B'$ implies $z \in B'$). If $z \in B'$ and $y \notin B'$, then $(\{x, z\}, \{y, z\})$

splits $\{x, y, z\}$ by Claim 6. So, $v(P_{xyz}, \{x, y, z\}) = y$, a contradiction. Thus, $B' = \emptyset$ or $B' = \{y, z\}$. In the first case: $v(P_{xx'y}, \{x, x', y\}) = x = v(P_{xx'z}, \{x, x', z\})$ and $v(P_{x'xy}, \{x, x', y\}) = x = v(P_{x'xz}, \{x, x', z\})$. In the second case: $v(P_{xx'y}, \{x, x', y\}) = x' = v(P_{xx'z}, \{x, x', z\})$ and $v(P_{x'xy}, \{x, x', y\}) = x = v(P_{x'xz}, \{x, x', z\})$.

To complete the proof, consider the agenda $\mathtt{S}^v_U$ on $U \equiv \{x : x \succ_v y\} \cup \{y, z\}$. From the observations in the previous paragraph, y and z must appear in the same subgames of $\mathtt{S}^v_U$. By assumption, there is some $Y \subseteq X$ that leads to the inference $y \succ_v z$. Since conditions (a)-(b) do not hold, Lemma 1 implies that $Y \subseteq U$. Then, a contradiction obtains by the symmetry argument given at the end of Claim 17. In particular, $v(P, Y) = UNE[\mathtt{S}^v_{U|Z}; P] \neq z$ iff $v(P^\sigma, Y) = UNE[\mathtt{S}^v_{U|Z}; P^\sigma] \neq y$. Since z is marginal on Y, it follows that y is marginal on Y.

(ii) The result follows from (i), the potential splittings of $\{x, y, z\}$, and the fact that $\succsim_v$ is a weak ordering. ■

(c) Proof of Lemma 2 – The Revealed Amendment Rule

Definition 14 *Given a decision rule v with revealed priority $\succsim_v$, $x \in X$ is **revealed to amend** $b \in X$ if:*

(1) $b \succ_v x$; and, (2) $v(P_{xbc}, \{b, c, x\}) = c$ for all $c \in X$ such that $b \succsim_v c \succ_v x$.

Then, define α_v as follows: $b \in \alpha_v(x)$ if x is revealed to amend b and x is not revealed to amend any $a \succ_v b$.

Lemma 2 *If v satisfies IS, ILA, and M, then $(\succsim_v, \alpha_v)$ defines an admissible pair.*

Proof. It suffices to check that $(\succsim_v, \alpha_v)$ is admissible.

(i) if $z \in X_1$, then $\alpha(z) = \emptyset$: By Definition 14.

(ii) if $z \in X_j$ for $j \neq 1$, then $\alpha(z) \subseteq X_k$ for some $k < j$: By Definition 14, it suffices to check that z amends some alternative. By Lemma 2, there are two possibilities: a unique b has immediately higher priority than z; or, distinct b and c have immediately higher priority than z. In the first case, $b \succ_v z$ implies that z amends b (since (2) is vacuous). In the second, $b \sim_v c \succ_v z$ implies $v(P_{zbc}, \{b, c, z\}) = c$ by Corollary 1. So, again, z amends b.

(iii) if $z, z' \in X_j$ for $j \neq 1$, then $\alpha(z) = \alpha(z')$: By Definition 14, it suffices to show that z' is revealed to amend b if z is revealed to amend b. By way of contradiction, suppose: z is revealed to amend b; and, z' is not. By Definition 14, there exists some c such that $b \succsim_v c \succ_v z' \sim_v z$ and $v(P_{z'bc}, \{b, c, z'\}) \neq c$. By Corollary 1, $v(P_{z'bc}, \{b, c, z'\}) \neq z'$. Otherwise, $z' \succ_v b$ which contradicts that $\succsim_v$ is a weak ordering (by Claim 17). So, $v(P_{z'bc}, \{b, c, y\}) = b$. By Corollary 1, $b \succ_v c$. Since z is revealed to amend b, Definition 14 implies $v(P_{zbc}, \{b, c, z\}) = c$. To summarize, $v(P_{zbc}, \{b, c, z\}) = c$ and $v(P_{z'bc}, \{b, c, z\}) = b$ for some c such that $b \succ_v c \succ_v z' \sim_v z$.

By Claim 20, the splitting of $\{b, c, z, z'\}$ is $(b \cup B', \{c, z, z'\})$ for $B' \subseteq \{z, z'\}$. By Claim 6, $v(P_{zbc}, \{b, c, z\}) = c$ implies $z \in B'$ and $v(P_{z'bc}, \{b, c, z'\}) = b$ implies $y \notin B'$. So, the splitting of $\{b, c, z, z'\}$ is $(\{b, z\}, \{c, z, z'\})$. By Claim 6, this implies $v(P_{bz'z}, \{b, z, z'\}) = y$ so that $z' \succ_v z$ by Corollary 1, which is a contradiction.

(iv) if $z \in X_{j+1}$ and $|X_j| = 2$, then $\alpha(z) = X_j$ or $\alpha(z) \subseteq X_k$ for some $k < j$: By Definition 14 and (i) above. ■

(d) Proof of Sufficiency

Note: In this section, I refer to a node q of a priority (resp. simple) agenda $\mathtt{Pr}_X$ (resp. $\mathtt{S}_X$) by its label $\ell(q)$.

For a weak ordering $\succsim$ on X, let $L_\succsim(a) \equiv \{x \in X : a \succ x\}$ denote the strict lower contour set of $a \in X$.

Claim 21 *Fix a priority agenda $\mathtt{Pr}_{(\succsim,\alpha)}$ and a non-terminal node $A = \{a_1, ..., a_K\}$ labeled according to $\succ^*$. Then, the successors of A are $a_1 \cup A^K_j$ and A^K_2 for some $j \in \{3, ..., K+1\}$. What is more:*

(i) *$A^K_2 = X^m_{m-K+2}$ where $X = \{x_1, ..., x_m\}$ is labeled according to $\succ^*$; and,*

(ii) *$a_2 \succ a_j$ and a_j is a highest priority alternative in A^K_3 that amends a_1 (if such an alternative exists).*

Proof. Let B and C denote the the two successors of node A in $\texttt{Pr}_{(\succsim,\alpha)}$. Since $\succsim$ is a priority, there are two cases: **(1)** $\max_{\succsim} A = \{a_1, a_2\}$; or, **(2)** $\max_{\succsim} A = \{a_1\}$. Since the claim is trivial if $|A| = 2$, suppose $|A| \geq 3$.

(1) By construction (step (ii) in Class 3), a_1 and a_2 must have been added at A. Since they must be added to different successors of A, $a_1 \in B \setminus C$ and $a_2 \in C \setminus B$. By definition of α, the next highest priority alternative(s) in X are added under the successors identified with a_1 and a_2. By a straightforward induction argument, it follows that $B = a_1 \cup L_{\succsim}(a_1)$ and $C = a_2 \cup L_{\succsim}(a_2)$. So, $B = a_1 \cup A_3^K$, $C = A_2^K = X_{m-K+2}^m$, and $a_2 \succ a_3$.

(2) By construction (step (i) in Class 3), a_1 and the next highest priority alternative(s) at A must be added to different successors of A. Letting $A_{-1} \equiv \max_{\succsim}(A \setminus a_1)$, it follows that $a_1 \in B \setminus C$ and $A_{-1} \subseteq C \setminus B$. So, $C = A_{-1} \cup L_{\succsim}(a_2) = A_2^K = X_{m-K+2}^m$ by the same argument as (1). Similarly, $B = a_1 \cup A_j^K$ where a_j is a highest priority in $C = A_2^K = X_{m-K+2}^m$ that amends a_1 besides $a_2 \succ a_j$ (if such an alternative exists). ■

Proof of Theorem 2 (Sufficiency). Fix a decision rule v that satisfies IS, ILA, and M.

First, note that $\succsim_v$ defines a priority (by Lemma 1) and α_v defines an amendment rule consistent with $\succsim_v$ (by Lemma 2). So, $(\succsim_v, \alpha_v)$ defines a priority agenda $\texttt{Pr}_X^v$. By the sufficiency portion of Theorem 1, $\texttt{Pr}_X^v$ implements v if $\texttt{Pr}_X^v = \texttt{S}_X^v$. To establish this, I show that the successors at the root nodes of $\texttt{Pr}_X^v$ and $\texttt{S}_X^v$ coincide. Extending this line of reasoning by a simple induction argument, it follows that $\texttt{Pr}_X^v = \texttt{S}_X^v$.

To see that the successors at the (common) root X of $\texttt{Pr}_X^v$ and $\texttt{S}_X^v$ coincide, label $X = \{x_1, ..., x_m\}$ according to $\succ_v^*$. By Claim 20, the successors of X in $\texttt{S}_X^v$ are $x_1 \cup X_j^m$ and X_2^m for some j such that $j \in \{3, ..., m+1\}$. There are two cases to consider: (i) $j = m+1$ (so that $X_j^m = \emptyset$); and, (ii) $j \in \{3, ..., m\}$ (so that $X_j^m \neq \emptyset$).

In both cases, the fact that the successors of X on $\texttt{Pr}_X^v$ coincide with the successors on $\texttt{S}_X^v$ follows from Claim 21: **(i)** Claim 6 applied to (x_1, X_2^m) gives $v(P_{x_j x_1 c}, \{x_1, x_j, c\}) = x_1$ for all x_j, c such that $x_1 \succsim_v c \succ_v x_j$. So, no x_j such that $x_2 \succ_v x_j$ is revealed to amend x_1 (by Definition 14). **(ii)** Claim 6 applied to $(x_1 \cup X_j^m, X_2^m)$ gives $v(P_{x_j x_k x_1}, \{x_1, x_k, x_j\}) = x_1$ and $v(P_{x_j x_1 x_k}, \{x_1, x_k, x_j\}) = x_k$ for $x_k \in X_2^{j-1}$. So, $x_k \succ_v x_j$ (by Corollary 1) and x_j is revealed to amend x_1. To see that no x_k such that $x_2 \succ_v x_k \succ_v x_j$ is revealed to amend x_1, observe that $v(P_{x_k x_1 x_2}, \{x_1, x_2, x_k\}) = x_1$ for all $x_k \in X_3^{j-1}$. ■

9.6.2 Necessity of IS, ILA, and M

Note: In this section, I abuse notation by referring to a node q of a priority agenda by its label $\ell(q)$.

Claim 22 *Fix a priority agenda $\texttt{Pr}_{(\succsim,\alpha)}$ on $X = \{x_1, ..., x_m\}$ (labeled according to $\succ^*$) such that x_j amends x_1 and x_2 (according to α). Then, every path from X to a terminal node passes through a non-terminal node with a successor X_j^m.*

Proof. The proof is by strong induction on $|X| \equiv m$. The base case $m = 3$ is trivial.

To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n+1$. By Claim 21, the two successors of the root X are $x_1 \cup X_k^m$ and X_2^m for $k \in \{3, ..., m+1\}$. Moreover, $k \neq m+1$. Otherwise, no alternative amends x_1 and x_2. Since path from X to a terminal node passes through $x_1 \cup X_k^m$ or X_2^m, it suffices to show that every path from $x_1 \cup X_k^m$ or X_2^m to a terminal node passes through a non-terminal node with the desired characteristics. There are three cases to consider: (i) $j = k = 3$; (ii) $j = k > 3$; and, (iii) $j \in \{k+1, ..., m\}$.

(i) By Claim 21, X_3^m is a successor of $x_1 \cup X_3^m$ and X_2^m, making these the nodes with the desired characteristics.

(ii) By the same reasoning as case (i), every path from $x_1 \cup X_k^m$ to a terminal node passes through a non-terminal node with the desired characteristics. Next, consider X_2^m. By definition of α and the assumption that x_j amends x_1 and x_2, it follows that x_j also amends x_3. By construction, $\texttt{Pr}_X(X_2^m)$ is a priority agenda on n or fewer alternatives where x_j amends the two top-priority alternatives. By the induction hypothesis, every path from X_2^m to a terminal node passes through a non-terminal node with the desired characteristics.

(iii) As in case (ii), x_j amends x_3. So, x_j also amends x_k by definition of α. As in case (ii), every path from X_2^m to a terminal node passes through a non-terminal node with the desired characteristics. By similar reasoning, every path from $x_1 \cup X_k^m$ to a terminal node also passes through a non-terminal node with the desired characteristics. ■

Claim 23 *Every priority agenda is persistent.*

Proof. The proof is by strong induction on $|X| \equiv m$. The base cases $m = 2, 3$ are straightforward. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$.

Fix a priority agenda $\mathtt{Pr}_X$ defined by $(\succsim, \alpha)$; and, label $X = \{x_1, ..., x_m\}$ according to $\succ^*$. By Claim 21, the two successors of the root X are $x_1 \cup X_k^m$ and X_2^m for $k \in \{3, ..., m+1\}$. By construction, $\mathtt{Pr}_X(x_1 \cup X_k^m)$ and $\mathtt{Pr}_X(X_2^m)$ are priority agendas on n or fewer alternatives. From the induction hypothesis, it follows that these agendas are persistent. So, $\mathtt{Pr}_X$ is persistent at every non-terminal node except possibly X. To check persistence at X, there are two cases to consider: (i) $k \in \{3, ..., m\}$; and, (ii) $k = m + 1$.

(i) The uncontested alternatives at X are X_k^m. By definition, x_k amends x_1 and x_2. By Claim 22, it then follows that every path from X to a terminal node in $\mathtt{Pr}_X$ passes through a non-terminal node with successor X_k^m. So, $\mathtt{Pr}_X$ is persistent at X. **(ii)** There are no uncontested alternatives at X. So, $\mathtt{Pr}_X$ is trivially persistent at X. ■

Proof of Remark 4. **(i)** History-independence follows by construction; and, persistence by Claim 23.

(ii) By way of contradiction, suppose that the right-hand agenda in Example 3 is a priority agenda. (In fact, the argument is identical for the left-hand agenda.) Since each only appears at one terminal node, x_1 and x_2 must have the highest priority (requirement (i) of admissibility). Since x_3 and x_4 amend different alternatives, they cannot have equal priority (requirement (iii) of admissibility). From these two observations, x_3 and x_4 must then have consecutive priority. But this is a contradiction (requirement (ii) of admissibility): x_3 does not amend x_4; and, x_4 does not amend x_3. ■

Proof of Proposition 3. Suppose $\mathtt{Pr}_X$ implements v where $\mathtt{Pr}_X$ is defined by $(\succsim, \alpha)$ and $X = \{x_1, ..., x_m\}$ is labeled according to $\succ^*$. To establish the result, it suffices to show that x_m is marginal for X. By Remark 1, it then follows that x_m is marginal for any issue $A \subseteq X$ such that $x_m \in A$. Since the pruned agenda $\mathtt{Pr}_{X|\{x_1,...,x_{m-1}\}}$ is a priority agenda on $X \setminus x_m$ such that $x \succsim x_{m-1}$ for all $x \in X \setminus x_m$, the same argument establishes that x_{m-1} is marginal for any issue $A \subseteq X \setminus x_m$ such that $x_{m-1} \in A$. By progressively deleting the highest index alternative (among those remaining) and repeating this argument, it follows that there exists a marginal alternative for all $A \in \mathbf{X}_{-1}$.

The proof that x_m is marginal for $X = \{x_1, ..., x_m\}$ is by strong induction on $|X| \equiv m$. The base cases $m = 2, 3$ are trivial. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$. By Claim 21, the successors of the root X are $x_1 \cup X_k^m$ and X_2^m. There are two cases: (i) $k = m + 1$; and, (ii) $k \in \{3, ..., m\}$.

(i) By construction, the root splits $\mathtt{Pr}_X$ into x_1 and a priority agenda $\mathtt{Pr}_X(X_2^m)$ on n alternatives. Since $x_i \succsim x_m$ for all $x_i \in X_2^m$, the induction hypothesis implies $UNE[\mathtt{Pr}_X(X_2^m); P] = x_m$ only if x_m is the Condorcet winner on (P, X_2^m). Then, "backward induction" implies that $UNE[\mathtt{Pr}_X; P] = x_m$ only if x_m is the Condorcet winner on $(P, \{x_1, x_m\})$ as well. Combining the last two observations gives the result.

(ii) By construction, the agendas $\mathtt{Pr}_X(x_1 \cup X_k^m)$ and $\mathtt{Pr}_X(X_2^m)$ are priority agendas on n or fewer alternatives. By Claim 11, $UNE[\mathtt{Pr}_X; P] = x_m$ only if $UNE[\mathtt{Pr}_X(x_1 \cup X_k^m); P] = x_m$ and $UNE[\mathtt{Pr}_X(X_2^m); P] = x_m$. Since $x_i \succsim x_m$ for all $x_i \in X$, the induction hypothesis implies that this is the case only if x_m is the Condorcet winner on $(P, x_1 \cup X_k^m)$ and (P, X_2^m). Combining the last two observations gives the result. ■

Proof of Theorem 2 (Necessity). Fix a decision rule v implemented by a priority agenda $\mathtt{Pr}_X$. Given Remark 4, the necessity portion of Theorem 1 establishes that v satisfies IS and ILA. Moreover, v satisfies M by Proposition 3. ■

9.7 Proof of Proposition 4

Note: In this section, I abuse notation by referring to a node q of a priority agenda $\mathtt{Pr}_X$ by its label $\ell(q)$.

Suppose $\mathtt{Pr}_X$ is defined by $(\succsim, \alpha)$ where $X = \{x_1, ..., x_m\}$ is labeled according to $\succ^*$. The proof is by strong induction on $|X| \equiv m$. The base cases $m = 2, 3$ are trivial. To complete the induction, suppose the claim holds for $m \leq n$ and consider the case $m = n + 1$.

Fix a profile P such that $UNE[\mathtt{Pr}_X; P] = x_k$. (Where $x_{k-1} \sim x_k$, suppose that every alternative that amends x_{k-1} also amends x_k.) By Claim 21, the successors of the root X are: $B \equiv x_1 \cup X_j^m$ for some $j \in \{3, ..., m+1\}$; and, $C \equiv X_2^m$. So, there are four cases to consider: (i) $k \in \{j+1, ..., m\}$; (ii) $k \in \{3, ..., j-1\}$; (iii) $k = j$; and, (iv) $k = 2$. To establish the result, I show that $UNE[\mathtt{Pr}_X^k; P] = x_k$ in each case (where $\mathtt{Pr}_X^k \equiv \mathtt{Pr}_X^{x_k}$ to simplify the notation):

(i) Since $x_k \in B \cap C$ and $UNE[\mathtt{Pr}_X; P] = x_k$, Claim 11 implies $UNE[\mathtt{Pr}_X(B); P] = UNE[\mathtt{Pr}_X(C); P] = x_k$. Since $\mathtt{Pr}_X(B)$ and $\mathtt{Pr}_X(C)$ are priority agendas on n or fewer alternatives and $k \neq j$, the induction hypothesis implies $UNE[\mathtt{Pr}_X^k(B); P] = UNE[\mathtt{Pr}_X^k(C); P] = x_k$. So, $UNE[\mathtt{Pr}_X^k; P] = x_k$ as required.

(ii) Since $x_k \notin B$ and $UNE[\mathtt{Pr}_X; P] = x_k$, "backward induction" implies $UNE[\mathtt{Pr}_X(C); P] = x_k$. Since $\mathtt{Pr}_X(C)$ is a priority agenda on n or fewer alternatives, the induction hypothesis implies $UNE[\mathtt{Pr}_X^k(C); P] = UNE[\mathtt{Pr}_X(C); P] = x_k$. Since $x_k \notin B$, $\mathtt{Pr}_X^k(B) = \mathtt{Pr}_X(B)$. So, "backward induction" implies $UNE[\mathtt{Pr}_X^k; P] = x_k$ as required.

(iii) By the same reasoning as in case (i), $UNE[\mathtt{Pr}_X(B); P] = UNE[\mathtt{Pr}_X(C); P] = x_j$ and $UNE[\mathtt{Pr}_X^k(C); P] = x_j$. By way of contradiction, suppose $UNE[\mathtt{Pr}_X^k; P] \neq x_j$. Then, $UNE[\mathtt{Pr}_X^k; P] = x_1$ by Claim 11. Since $UNE[\mathtt{Pr}_X^k(C); P] = x_j$, "backward induction" implies that $x_1 M_X^P x_j$.

Since $UNE[\mathtt{Pr}_X^k; P] = x_1$, "backward induction" implies $UNE[\mathtt{Pr}_X^k(B_j); P] = x_1$ where $B_j \equiv \{x_1, x_{j-1}\} \cup X_{j+1}^m$ is the "left" successor of the root X in $\mathtt{Pr}_X^k$. Since the agenda at the "left" successors $B' \equiv x_1 \cup X_{j'}^m$ of B and B_j coincide and $UNE[\mathtt{Pr}_X^k(B_j); P] = x_1$, "backward induction" implies $UNE[\mathtt{Pr}_X^k(B'); P] = UNE[\mathtt{Pr}_X(B'); P] = x_1$. Since $UNE[\mathtt{Pr}_X(B); P] = x_j$ however, "backward induction" implies that $x_j M_X^P x_1$. Since this contradicts the inference $x_1 M_X^P x_j$ drawn in the last paragraph, $UNE[\mathtt{Pr}_X^k; P] = x_j$.

(iv) This case is similar to (iii) but more involved. Since $UNE[\mathtt{Pr}_X; P] = x_2$ and $x_2 \notin B$, "backward induction" implies $UNE[\mathtt{Pr}_X(C); P] = x_2$. Let $B' \equiv x_2 \cup X_{j'}^m$ and $C' \equiv X_3^m$ denote the successors of C in $\mathtt{Pr}_X$. Since x_2 only appears once below B', it is majority preferred to $UNE[\mathtt{Pr}_X(C'); P]$ and every alternative it meets along the "backward induction" path in $\mathtt{Pr}_X(B')$. Now, consider the re-prioritized agenda $\mathtt{Pr}_X^2$, letting $B_2 \equiv x_2 \cup X_j^m$ and $C_2 \equiv x_1 \cup X_3^m$ denote the successors of X. By construction, $B_2' \equiv x_1 \cup X_{j'}^m$ and $C_2' \equiv C'$ are the successors of C_2 in $\mathtt{Pr}_X^2$.

Since every alternative that amends x_1 in $\mathtt{Pr}_X$ also amends x_2, everything that x_2 meets along the "backward induction path" in $\mathtt{Pr}_X(B_2)$ is an alternative that it meets in $\mathtt{Pr}_X(B')$. Since x_2 is majority preferred to all of these alternatives by the first observation in the last paragraph, $UNE[\mathtt{Pr}_X^2(B_2); P] = x_2$. Moreover, $UNE[\mathtt{Pr}_X^2(C_2'); P] = UNE[\mathtt{Pr}_X(C'); P]$ by the second observation in the last paragraph.

By way of contradiction, suppose $UNE[\mathtt{Pr}_X^2; P] \neq x_2$. Since $UNE[\mathtt{Pr}_X^2(B_2); P] = x_2$ and $x_2 M_X^P UNE[\mathtt{Pr}_X^2(C_2'); P] = UNE[\mathtt{Pr}_X(C'); P]$, "backward induction" implies $UNE[\mathtt{Pr}_X^2; P] = UNE[\mathtt{Pr}_X^2(B_2'); P]$. Since x_2 is majority preferred to everything that x_1 meets along the "backward induction" path in $\mathtt{Pr}_X(B_2')$, $UNE[\mathtt{Pr}_X^2; P] = UNE[\mathtt{Pr}_X^2(B_2'); P] = x_1$. Since $UNE[\mathtt{Pr}_X^2(B_2); P] = x_2$, $x_1 M_X^P x_2$.

Since $UNE[\mathtt{Pr}_X^2(B_2'); P] = x_1$, the same reasoning as above establishes $UNE[\mathtt{Pr}_X(B); P] = x_1$. Since $UNE[\mathtt{Pr}_X; P] = x_2$ however, $x_2 M_X^P x_1$. Since this contradicts the last paragraph, $UNE[\mathtt{Pr}_X^2; P] = x_2$.

9.8 Proof of Proposition 5 and Theorem 3

Definition 15 *A sequence $\langle a \rangle \equiv a_1, ..., a_n$ is **universal** for $x \in A$ on (P, A) if:*

(1) every alternative in A appears exactly once in $\langle a \rangle$;

(2) $\langle a\rangle$ contains an M_A^P-transitive subsequence $\langle b\rangle \equiv b_1, ..., b_m$ such that $x = b_1$; and,

(3) every alternative $a_i \in \langle a\rangle \setminus \langle b\rangle$ satisfies the following two conditions:

(i) a_i appears before some alternative $b \in \langle b\rangle$ in the sequence $\langle a\rangle$; and,

(ii) $b_j M_A^P a_i$ for the first $b_j \in \langle b\rangle$ after a_i in the sequence $\langle a\rangle$.

Claim 24 *A sequence $\langle a\rangle$ in A is universal on (P, A) for at most one alternative in A.*

Proof. By way of contradiction, suppose $\langle a\rangle = a_1, ..., a_n$ is universal for distinct alternatives a_i and a_j. Without loss of generality, suppose $i < j$. Let $\langle b^i\rangle$ and $\langle b^j\rangle$ denote the transitive subsequences associated with a_i and a_j, respectively. By condition (3.i), $b_1^i = a_i$ and $b_1^j = a_j$. So, $a_i \notin \langle b^j\rangle$. Hence, $a_j M_A^P a_i$ by condition (3.ii). Since $\langle b^i\rangle$ is a transitive sequence with a_i maximal by condition (2), $a_j M_A^P a_i$ implies $a_j \notin \langle b^i\rangle$.

By condition (3.ii), $a_j \notin \langle b^i\rangle$ implies $\tilde{b}^i M_A^P a_j$ for some $\tilde{b}^i \in \langle b^i\rangle$ such that $\tilde{b}^i = a_{k(i)}$ and $k(i) > j$. Since $\langle b^j\rangle$ is a transitive subsequence by condition (2), $\tilde{b}^i M_A^P a_j$ implies $\tilde{b}^i \notin \langle b^j\rangle$. By condition (3.ii), $\tilde{b}^i \notin \langle b^j\rangle$ implies $\tilde{b}^j M_A^P \tilde{b}^i$ for some $\tilde{b}^j \in \langle b^j\rangle$ such that $\tilde{b}^j = a_{k(j)}$ and $k(j) > k(i)$. Since $\langle b^i\rangle$ is a transitive subsequence by condition (2), $\tilde{b}^j M_A^P \tilde{b}^i$ implies $\tilde{b}^j \notin \langle b^i\rangle$. Continuing in this vein leads to the contradiction that $\langle a\rangle$ does not terminate. ■

Claim 25 *There exists a universal sequence for x on (P, A) iff $x \in BA(M_A^P)$.*

Proof. ($\Rightarrow$) Fix some $y \in A \setminus BA(M_A^P)$. By way of contradiction, suppose $\langle a\rangle$ is universal for y on (P, A). Consider the transitive subsequence $\langle b\rangle = b_1, ..., b_m$ with $b_1 = y$. Since $y \in A \setminus BA(M_A^P)$, $\langle b\rangle$ is not a maximal transitive sequence. In other words, there exists some $a \in A \setminus \langle b\rangle$ such that $a, b_1, ..., b_m$ is a transitive sequence. So, there exists no $b_j \in \langle b\rangle$ such that $b_j M_A^P a$. Since a must appear before some $b_j \in \langle b\rangle$ by condition (3.i), this contradicts condition (3.ii). In particular, no $b_j \in \langle b\rangle$ may comes after a in $\langle a\rangle$. ($\Leftarrow$) Since $x \in BA(M_A^P)$, there exists a maximal transitive sequence $\langle b\rangle$ in A with $b_1 = x$. For every $a \in A \setminus \langle b\rangle$, there exists some $b \in \langle b\rangle$ such that $bM_A^P a$. Otherwise, $a, b_1, ..., b_m$ is a transitive sequence, which contradicts the maximality of $\langle b\rangle$. Let $b_{i(a)}$ denote the last alternative in $\langle b\rangle$ such that $b_{i(a)} M_A^P a$. Extend $\langle b\rangle$ into a sequence $\langle a\rangle$ on A by inserting each $a \in A \setminus \langle b\rangle$ between the alternatives $b_{i(a)-1}$ and $b_{i(a)}$. By construction, $\langle a\rangle$ satisfies conditions (1)-(3). So, $\langle a\rangle$ is a universal sequence for x on (P, A). ■

Given a priority agenda $\mathtt{Pr}_A$ on A defined by $(\succsim, \alpha)$, define a strict ordering $\succ^*$ from $\succsim$. Let $\langle a\rangle^* = a_1^*, ..., a_n^*$ denote the sequence defined by taking the alternatives in A in the order associated with $\succ^*$. Given another sequence $\langle a\rangle = a_1, ..., a_n$ of the alternatives in A, define the permutation $\sigma^*_{\langle a\rangle} : A \to A$ by $\sigma^*_{\langle a\rangle}(a_i) = a_i^*$. To simplify the notation, let $\mathtt{Pr}_A^{\langle a\rangle} \equiv \mathtt{Pr}_A^{\sigma^*_{\langle a\rangle}}$ denote the agenda where the i^{th} alternative in the sequence $\langle a\rangle$ occupies the i^{th} highest priority position in the agenda.

Claim 26 *Given a priority agenda $\mathtt{Pr}_A$ on A and a universal sequence $\langle a\rangle$ for x on (P, A), $UNE[\mathtt{Pr}_A^{\langle a\rangle}; P] = x$.*

Proof. The proof is by strong induction on $|A|$. The base case $|A| = 2$ is trivial. To complete the induction, suppose the claim holds for $|A| = n$ and consider the case $|A| = n + 1$. Fix a universal sequence $\langle a\rangle$ for x on (P, A). By Claim 24, $\langle a\rangle$ is not universal for any other alternative on (P, A). By definition, $\langle a\rangle$ contains a transitive subsequence $\langle b\rangle$ satisfying conditions (2)-(3) with $x = b_1$. Given Claim 21, let $C \equiv a_1 \cup A_k^{n+1}$ and $C' \equiv A_2^{n+1}$ for $2 < k < n + 2$ denote the successors of the root node in $\mathtt{Pr}_A^{\langle a\rangle}$; and, let $\langle c\rangle$ and $\langle c'\rangle$ denote the subsequences of $\langle a\rangle$ associated with C and C'. There are three cases to consider for $x = b_1$: (i) $x \in C \setminus C'$; (ii) $x \in C \cap C'$; and, (iii) $x \in C' \setminus C$.

(i) Since $C \setminus C' = \{a_1\}$, $x = a_1$. It is easy to verify that $\langle c\rangle = x, a_k, ..., a_{n+1}$ is universal for x on (P, C). Since the last alternative in $\langle a\rangle$ must be in $\langle b\rangle$ by condition (3.i), it is also easy to verify that $\langle c'\rangle = a_2, ..., b_2, ..., a_{n+1}$ is universal for b_2 on (P, C'). Since $\mathtt{Pr}_A^{\langle a\rangle}(C)$ and $\mathtt{Pr}_A^{\langle a\rangle}(C')$ are both priority agendas, $UNE[\mathtt{Pr}_A^{\langle a\rangle}(C); P] = x$ and $UNE[\mathtt{Pr}_A^{\langle a\rangle}(C); P] = b_2$ by the induction hypothesis. Since $xM_A^P b_2$ by construction, $UNE[\mathtt{Pr}_A^{\langle a\rangle}; P] = x$.

(ii) Since $C \cap C' = A_k^n$, $x = a_j$ for some $j \geq k$. As in case (i), it is easy to verify that $\langle c \rangle = a_1, a_k, ..., x, ..., a_{n+1}$ is universal for x on (P, C). Likewise, it is easy to verify that $\langle c' \rangle = a_2, ..., x, ..., a_{n+1}$ is universal for x on (P, C'). So, $UNE[\mathtt{Pr}_A^{\langle a \rangle}(C); P] = x = UNE[\mathtt{Pr}_A^{\langle a \rangle}(C'); P]$ by the induction hypothesis, which implies $UNE[\mathtt{Pr}_A^{\langle a \rangle}; P] = x$.

(iii) Since $C' \setminus C = A_2^{k-1}$, $x = a_j$ for some $2 \leq j < k$. As in case (ii), it is easy to verify that $\langle c' \rangle = a_2, ..., x, ..., a_{n+1}$ is universal for x on (P, C'). So, $UNE[\mathtt{Pr}_A^{\langle a \rangle}(C'); P] = x$ by the induction hypothesis. By way of contradiction, suppose $UNE[\mathtt{Pr}_A^{\langle a \rangle}; P] \neq x$. Since $C \setminus C' = \{a_1\}$, Claim 11 implies $UNE[\mathtt{Pr}_A^{\langle a \rangle}; P] = a_1$. So, $UNE[\mathtt{Pr}_A^{\langle a \rangle}(C); P] = a_1$. Since $UNE[\mathtt{Pr}_A^{\langle a \rangle}(C'); P] = x$, it follows that $a_1 M_A^P x$, which contradicts condition (3.ii). ■

Proof of Proposition 5. Fix a priority agenda $\mathtt{Pr}_A$ and a profile P on A. **(i')** By McKelvey and Niemi's [1978] observation that no agenda can select an alternative outside the Top Cycle of (P, A), it follows that $V_\Sigma(\mathtt{Pr}_A; P) \subseteq TC(M_A^P)$. **(ii')** Consider an $x \in BA(M_A^P)$. By Claim 25, there exists a universal sequence $\langle a \rangle$ for x on (P, A). Then, $UNE[\mathtt{Pr}_A^{\langle a \rangle}; P] = x$ by Claim 26. So, $x \in V_\Sigma(\mathtt{Pr}_A; P)$. Equivalently, $BA(M_A^P) \subseteq V_\Sigma(\mathtt{Pr}_A; P)$. ■

Claim 27 ***(i)*** *If v satisfies IS and ILA and every $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$ has two marginal alternatives, then $\mathtt{S}_X^v$ is Euro-Latin.* ***(ii)*** *If v satisfies IS and ILA and every $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$ has one marginal alternative, then $\mathtt{S}_X^v$ is Anglo-American.*

Proof. (i) Consider the splitting (B_1, X_1) of X and let a^* denote a marginal alternative in X. By Claim 16(i), there are two potential splittings: (1) $a^* \in B_1 \cap X_1$ with $b_1 \in B_1 \setminus X_1$ and $x_1 \in X_1 \setminus B_1$; and, (2) $(B_1, X_1) = (b_1, X \setminus b_1)$. For (1), Claim 6 implies that $\{a^*, b_1, x_1\}$ has one marginal alternative a^*, a contradiction. So, the splitting must be (2). Continuing in the same vein on X_1 establishes that $\mathtt{S}_X^v$ is a Euro-Latin agenda.

(ii) Consider the splitting (B, C) of X. If $|C \setminus B| \geq 1$ (with $b \in B \setminus C$ and $c, c' \in C \setminus B$), then Claim 6 implies that $\{b, c, c'\}$ has two marginal alternatives c and c', a contradiction. This shows that $|C \setminus B| = |B \setminus C| = 1$. In other words, $(B, C) = (X \setminus c_1, X \setminus b_1)$ for some $b_1 \in B$ and $c_1 \in C$. Continuing in the same vein on $X \setminus c_1$ and $X \setminus b_1$ establishes that $\mathtt{S}_X^v$ is an Anglo-American agenda. ■

Proof of Theorem 3. Sufficiency: By Claim 27(i) above (resp. Claim 27(ii) above). **Necessity:** Theorem 1 ensures that every decision rule implementable by Euro-Latin agenda (resp. Anglo-American agenda) satisfies IS and ILA. In turn, it is easy to check that every decision rule implementable by Euro-Latin agenda (resp. Anglo-American agenda) has two marginal alternatives (resp. a unique marginal alternative) for all A such that $|A| \neq 1$. ■

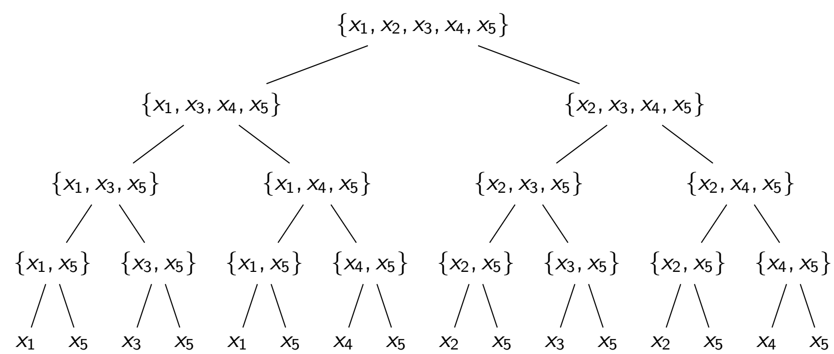
10 Supplementary Appendix

10.1 Examples

Example 9 *In Congress, a* ***two-stage amendment agenda*** *comes about when a substitute to the original bill (or an amendment to the bill) is proposed. In that case, procedure requires voting to determine the features of the "original" and "substitute" versions of the legislation before deciding whether either of these versions should replace the status quo. When there is a bill b, a substitute b′, and amendments (a and a′) to each, for example, the sequence of questions is:*

- ***Q1.*** *Should the amendment a to the original bill b be accepted?*
- ***Q2.*** *Should the amendment a′ to the substitute bill b′ be accepted?*
- ***Q3.*** *Should the substitute bill (as amended) be selected over the original bill (as amended)?*
- ***Q4.*** *Should the surviving bill be selected over the status quo?*

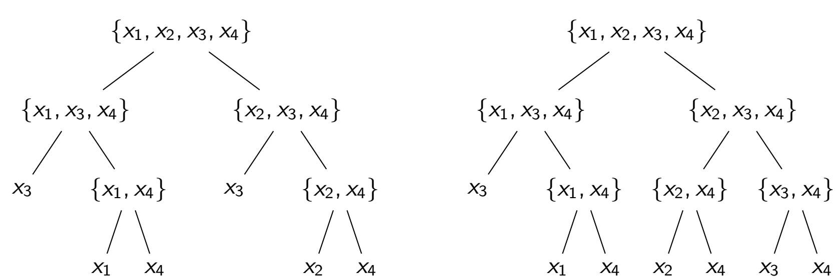
This leads to the agenda below where: x_1 *(resp.* x_3*) represents b (resp. b′);* x_2 *(resp.* x_4*) represents* $b+a$ *(resp.* $b'+a'$*); and,* x_5 *represents the status quo (see also Schwartz [2008, Figure 2]; and, Miller [1995, Figure 4]).*

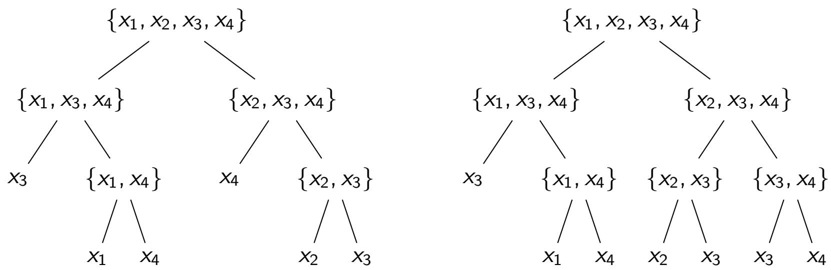


Example 10 *The following agendas show that the two structural features of recursive agendas are independent.*

- ***Continuous and repetitive:*** *Consider the right-hand agenda in Figure 5.*
- ***Discontinuous and non-repetitive:*** *Consider the two-stage amendment agenda depicted in Example 9.*

Example 11 *Up to permutation of the alternatives, there are fifteen recursive agendas for* $|X| = 4$*: the nine simple agendas (see Example 3); the two agendas in Example 6; and, the four agendas depicted below.*





Example 12 *The following example show that recursive agendas can violate one or both properties of simple agendas.*

*- **History-dependent and persistent:** Following Remark 3, consider the left-hand agenda in Figure 5.*

*- **History-independent and non-persistent:** Consider the left-hand agenda in Example 6. It is easy to check that this agenda is recursive. Since the labels of the non-terminal nodes are distinct, it is also history-independent. However, it is not persistent: no node between the root r and the terminal node labeled x_2 has a successor labeled $u(r) = x_4$.*

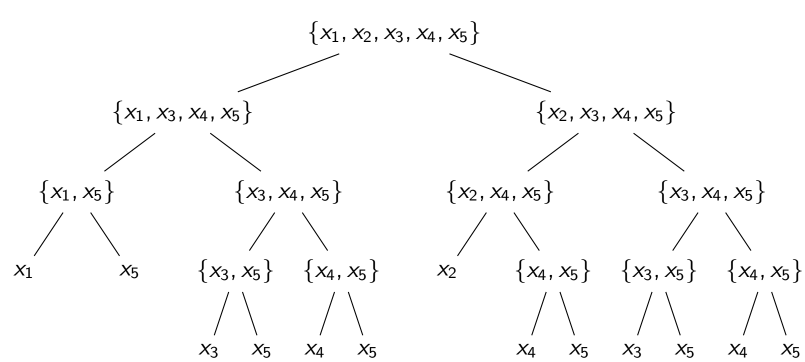
*- **History-dependent and non-persistent:** Starting with the left-hand agenda in Example 6, replace the left (resp. right) terminal node labeled x_4 with the agenda under the left (resp. right) node labeled $\{x, b, c\}$ in Figure 5. The resulting agenda on six alternatives is recursive; but, it clearly violates both properties of simple agendas.*

Example 13 *The right-hand agenda in Example 2 admits "strict" and "weak" priority representations:*

Representation	Priority $\succsim$	Amendment Rule α
strict	$x_1 \succ x_2 \succ x_3 \succ x_4$	$\alpha(x_i) = \begin{cases} x_1 & \text{for } i = 2, 3 \\ x_2 & \text{for } i = 4 \end{cases}$
weak	$x_1 \sim x_2 \succ x_3 \succ x_4$	$\alpha(x_i) = \begin{cases} \{x_1, x_2\} & \text{for } i = 3 \\ x_2 & \text{for } i = 4 \end{cases}$

Although x_1 and x_2 are not symmetric alternatives, they have equal priority in the "weak" representation.

Example 14 *Consider the following extension of the right-hand agenda in Example 2:*



It is easy to see that this agenda is a priority agenda: starting from the weak representation in Example 13, simply define $x_4 \succ x_5$ and $\alpha(x_5) = x_1$. However, there is no way to define both $\alpha(x_4)$ and $\alpha(x_5)$ when one assigns a strict priority between x_1 and x_2. The problem is that x_4 does not amend x_1; and, x_5 does not amend x_2.

10.2 Independence of the Axioms

- **IS and ILA but not M:** The decision rule on $\mathbf{X}$ induced by the knockout agenda in Example 1 clearly satisfies IS and ILA. To see that it violates M, note that x_3 is the outcome for any profile P that gives the majority relation in Figure 9 below; but, x_3 is not the Condorcet winner on (P, X). By symmetry, there are profiles where each of the alternatives is selected without being the Condorcet winner.

- **ILA and M but not IS:** The decision rule on $\mathbf{A}$ induced by the right-hand agenda $\mathtt{T}_{X|A}$ in Figure 5 trivially satisfies ILA. To see that it satisfies M, note that $UNE[\mathtt{T}_{X|A}; P] = x$ only if x is the Condorcet winner on A. As shown in Example 4 however, it violates IS (because there is no way to split A).

- **IS and M but not ILA:** Consider the decision rule v_3 on $\{x_1, x_2, x_3\}$ that selects: the majority preferred alternative between x_1 and x_2 when both are available; x_i on $\{x_i, x_3\}$; and, x_i on $\{x_i\}$. Since $(\{x_1, x_3\}, \{x_2, x_3\})$ splits $\{x_1, x_2, x_3\}$, v_3 satisfies IS. Since x_3 is trivially marginal, v_3 also satisfies M. To see that it violates ILA, consider a profile P_3 where x_3 is the Condorcet winner on $\{x_1, x_2, x_3\}$. If v_3 satisfies ILA, then $v_3(P_3, \{x_1, x_2, x_3\}) = ... = x_3$. But, this contradicts the assumption that $v_3(P_3, \{x_1, x_2, x_3\}) = v_3(P_3, \{x_1, x_2\}) \neq x_3$.

10.3 Connection to Apesteguia et al. [2014]

Sub-section (a) describes the formal connection while sub-section (b) establishes the proofs.

(a) Discussion

The following stipulates that a decision rule marginalizes the same number of alternatives for all issues of a given size.

Neutral Marginalization **(NM)** *For each j, v marginalizes the same number of alternatives for all $A \in \mathbf{X_j}$.*

This property is sufficient to distinguish Euro-Latin and Anglo-American agendas from *all* other priority agendas:

Proposition 6 *Among the decision rules implementable by priority agenda, the only ones that satisfy Neutral Marginalization are the decision rules implementable by Euro-Latin agenda or Anglo-American agenda.*

Given Theorem 2, this provides a *joint* characterization for Euro-Latin and Anglo-American agendas: *a decision rule v is implementable by one of these agendas if and only if it satisfies IS, ILA, M, and NM.* For *separate* characterizations, recall that each of these agenda types marginalizes the same number of alternatives for *every* issue of three or more alternatives (see Section 6.2 of the paper). Given Proposition 6, Theorem 3 then follows immediately.

Apesteguia et al. also give separate characterizations of implementation by Euro-Latin and Anglo-American agenda. Unlike Theorem 3, their approach emphasizes the differences between the resulting decision rules rather than the fundamental similarities. Specifically, they rely on the following two properties:

Condorcet Priority **(CP)** *For each $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$, there exists a **prioritarian** alternative $p^* \in A$ such that, for every Condorcet triple P_{p^*xy} involving alternatives $x, y \in A$, $v(P_{p^*xy}, \{p^*, x, y\}) = p^*$.*

Condorcet Anti-Priority **(CA)** *For each $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$, there exists an **anti-prioritarian** alternative $p_* \in A$ such that, for every Condorcet triple P_{p^*xy} involving alternatives $x, y \in A$, $v(P_{p_*xy}, \{p_*, x, y\}) = y$.*

To characterize the Euro-Latin rule, they require CP, ILA, and *Division Consistency* (DC) (see footnote 21). For the Anglo-American rule, they require CA, ILA, and a property called *Elimination Consistency* (EC).

By Claim 28, CP marginalizes two alternatives for every issue (with three or more alternatives) while CA marginalizes one. Given Corollary 3, this implies that one can replace DC and EC in their characterizations by IS:

Corollary 2 *A decision rule v is implementable by:*

(i) *Euro-Latin agenda if and only if it satisfies IS, ILA, and CP;*

(ii) *Anglo-American agenda if and only if it satisfies IS, ILA, and CA.*

Unlike Apesteguia et al., this result emphasizes the close connection between Euro-Latin and Anglo-American agendas.

(b) Proofs

Proof of Proposition 6. For the necessity of NM: See the necessity proof of Corollary 3.

For the sufficiency of NM: By Theorem 2, every decision rule implementable by priority agenda has a marginal alternative on X. By Claim 15, there are two cases: (i) X has two marginal alternatives (denoted a_1^* and a_2^*); or, (ii) X has a unique marginal alternative. I show that $\mathbb{S}_X^v$ is Euro-Latin for (i) and Anglo-American for (ii).

(i) By Claim 27(i), it suffices to establish that all A such that $|A| \neq 1$ have two marginal alternatives. By Claim 14, a_1^* and a_2^* are marginal for $X \setminus x$ for all $x \neq a_1^*, a_2^*$. By NM, this implies that $X \setminus y$ has two marginal alternatives for all $y \in X$. Continuing in the same vein, the desired result follows by a simple inductive argument.

(ii) By Claim 27(ii), it suffices to establish that all A such that $|A| \neq 2$ have one marginal alternative. By way of contradiction, suppose $|X| \geq 4$ and some $X \setminus x$ has two marginal alternatives. Then, by the argument in case (i), every $A \neq X$ such that $|A| \neq 1$ has two marginal alternatives. To establish the contradiction, consider the splitting (B_1, X_1) of X. By the argument in Claim 27(i), $\mathbb{S}_{X_1}^v$ is Euro-Latin and $(B_1, X_1) = (b_1, X \setminus b_1)$. This establishes that $\mathbb{S}_X^v$ is also Euro-Latin. It then follows that X has two marginal alternatives, which is a contradiction. ■

Claim 28 *Suppose v satisfies IS and ILA.* ***(i)*** *If v satisfies CP, then every $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$ has two marginal alternatives.* ***(ii)*** *If v satisfies CA, then every $A \notin \mathbf{X}_1 \cup \mathbf{X}_2$ has a unique marginal alternative (the unique anti-prioritarian alternative).*

Proof. The proof of (i) [resp. (ii)] is by induction on $|X|$. For $|X| = 3$, the claim follows from ILA and CP [resp. CA]. For the induction step, note that all $A \subset X$ satisfy the claim by the induction hypothesis. To see that X also satisfies the claim, consider the splitting (B, C) of X. There are two possibilities for the prioritarian [resp. anti-prioritarian] p in X: (a) $p \in B \cap C$; and, (b) $p \in B \setminus C$. I address these possibilities separately for (i) and (ii).

(i) Consider $b \in B \setminus C$ and $c \in C \setminus B$. By Claim 6, (a) leads to the contradiction that p is not prioritarian in $\{b, c, p\}$ (let alone X). So, (b) must hold. Using the same kind of reasoning, it can be shown that $B = \{p\}$. (The idea is to suppose that there exists some $b' \in B \setminus p$. Then, consider an issue $\{b', c, p\}$ such that $c \in C \setminus B$. While there are several cases, a contradiction obtains for each.) By the induction hypothesis, $X \setminus p$ has two marginal alternatives. Since the splitting of X is $(p, X \setminus p)$, IS implies that these alternatives are marginal for X as well.

(ii) Consider $b \in B$ and $c \in C \setminus B$. By Claim 6, (b) leads to the contradiction that p is not anti-prioritarian in $\{b, c, p\}$ (let alone X). So, (a) must hold. By the induction hypothesis, p is marginal for B and C (since it is anti-prioritarian for these issues). By IS, it then follows that p is marginal for X. Finally, by Claim 14 and the induction hypothesis, there can be no other marginal alternative in X. ■

Proof of Corollary 2. For the class of Euro-Latin (resp. Anglo-American) agendas: ($\Rightarrow$) IS and ILA are necessary by Theorem 2; and, CP (resp. CA) is necessary by Theorem 1 (resp. 2) of Apesteguia et al. ($\Leftarrow$) By Claims 27 and 28. ■

10.4 Applications and Extensions

Proof of Remark 5. The agenda $\mathtt{A}$ in Example 9 is discontinuous. To see that it satisfies Preference Monotonicity, first observe that the two sub-agendas $\mathtt{A}'$ and $\mathtt{A}''$ below the root node are simple. So, $\mathtt{A}'$ and $\mathtt{A}''$ satisfy Preference Monotonicity by Proposition 1. Letting $UNE[\mathtt{A};P] \equiv x_i$ for $i \in \{1, ..., 5\}$, there are five cases to consider:

i $= 1, 2$: Without loss of generality, let $i = 1$ (since the reasoning for $i = 2$ is similar). Since x_1 only appears in one sub-agenda at the root (say $\mathtt{A}'$), $UNE[\mathtt{A}';P] = x_1$. By Preference Monotonicity of $\mathtt{A}'$, $UNE[\mathtt{S}';P^{x_1\uparrow}] = x_1$. Since $UNE[\mathtt{A}'';P] = UNE[\mathtt{A}'';P^{x_i\uparrow}]$ as well, it then follows that $UNE[\mathtt{A};P^{x_1\uparrow}] = x_1$.

i $= 3, 4$: Without loss of generality, let $i = 3$ (since the reasoning for $i = 4$ is similar). Since x_3 appears in both sub-agendas at the root, there are two cases: (i) $UNE[\mathtt{A}';P] = x_3 = UNE[\mathtt{A}'';P]$; and, (ii) $UNE[\mathtt{A}';P] = x_3 \neq UNE[\mathtt{A}'';P]$.

(i) By Preference Monotonicity of $\mathtt{A}'$ and $\mathtt{A}''$, $UNE[\mathtt{A}';P^{x_3\uparrow}] = x_3 = UNE[\mathtt{A}'';P^{x_3\uparrow}]$. So, $UNE[\mathtt{A};P^{x_3\uparrow}] = x_3$.

(ii) Note that $UNE[\mathtt{A}''(\{x_1, x_3, x_5\});P] = x_3$ so that $x_3 M_A^P x_5$. Since $\mathtt{A}''$ is simple, $UNE[\mathtt{A}'';P] = x_2$ (resp. x_5) implies $UNE[\mathtt{A}''(\{x_2, x_3, x_5\});P] = x_2$ (resp. x_5) by Claim 11. So, $UNE[\mathtt{A}'';P] = x_4$. (If $UNE[\mathtt{A}'';P] = x_2$, then $UNE[\mathtt{A}''(\{x_2, x_3, x_5\});P] = x_2$ so that $x_2 M_A^P x_3$, which contradicts $UNE[\mathtt{A};P] = x_3$. If $UNE[\mathtt{A}'';P] = x_5$, then $x_5 M_A^P x_3$, which is a contradiction.)

Since $UNE[\mathtt{A}'';P] = x_4$, $UNE[\mathtt{A}''(\{x_2, x_4, x_5\});P] = x_4$. In turn, this implies $UNE[\mathtt{A}''(\{x_2, x_3, x_5\});P] = x_2$ (since $x_3 M_A^P x_4$ and x_5). So, $x_2 M_A^P x_3$ and $x_2 M_A^P x_5$. Consequently, $UNE[\mathtt{A}''(\{x_2, x_3, x_5\});P^{x_3\uparrow}] = x_3$ (since $x_3 M_A^P x_5$); and, $UNE[\mathtt{A}'';P^{x_3\uparrow}] = x_3$ (since $UNE[\mathtt{A}''(\{x_2, x_4, x_5\});P^{x_3\uparrow}] = x_4$ and $x_3 M_A^P x_4$). Since $UNE[\mathtt{A}';P^{x_3\uparrow}] = x_3$ by Preference Monotonicity of $\mathtt{A}'$, $UNE[\mathtt{A};P^{x_3\uparrow}] = x_3$.

i $= 5$: Since x_5 is in every terminal sub-game, it must be the Condorcet winner on P. So, $UNE[\mathtt{A};P^{x_5\uparrow}] = x_5$. ∎

Given a tournament M_A, an alternative a covers another alternative b if: aM_Ab; and, for all $c \in A \setminus \{a, b\}$, bM_Ac implies aM_Ac. In turn, a subset $B \subseteq A$ is A-uncovered if, for each $a \in A \setminus B$, there exists a $b \in B$ such that a does not cover b.

Definition 16 *The* ***Candidate Stable Set*** $CS(M_A)$ *is the subset of alternatives in* A *that are at the top of some uncovered* M_A*-transitive chain. Formally,* $CS(M_A) \equiv \{x \in A : x = b_1$ *for some* A*-uncovered* M_A*-transitive chain* $\langle b \rangle\}$.

It is clear that a maximal M_A-transitive chain $\langle b \rangle$ must be A-uncovered: for each $a \in A \setminus \langle b \rangle$, there exists a $b \in \langle b \rangle$ such that bM_Aa. This shows that $CS(M_A) \supseteq BA(M_A)$. Using the next definition, one can show much more:

Definition 17 *The* k*-**Dominating Set*** $D^k(M_A)$ *is the subset of alternatives in* A *that beat every other alternative in* k *steps or less. Formally,* $D^k(M_A) \equiv \{x \in A : x = b_1 M_A ... M_A b_{n+1} = a$ *with* $n \leq k$ *for each* $a \in A \setminus x\}$.

The tournament solution associated with $k = 2$ is called the **Uncovered Set** (i.e., $UC(M_A) \equiv D^2(M_A)$ for all A and every tournament M_A); and, the tournament solution associated with $k = |A| - 1$ is the Top Cycle. Proposition 6 of Dutta et al. [2002] establishes that $UC(M_A) \subseteq CS(M_A) \subseteq D^3(M_A)$ for all A and every tournament M_A.

Proof of Remark 6. Dutta et al. [2002] establish the following results: $\widetilde{V}_\Sigma(\mathtt{AA}_A;P) = CS(M_A^P)$ for every profile (P, A) and Anglo-American agenda $\mathtt{AA}_A$ (Proposition 5); and, $\widetilde{V}_\Sigma(\mathtt{T}_A;P) \subseteq TC(M_A^P)$ for every profile (P, A) and agenda $\mathtt{T}_A$ (Proposition 3). To complete the proof, it suffices to show that: (a) $CS(M_A^P) \subseteq \widetilde{V}_\Sigma(\mathtt{P}_A;P)$ for every profile (P, A) and priority agenda $\mathtt{P}_A$; and, (b) $TC(M_A^P) \subseteq \widetilde{V}_\Sigma(\mathtt{EL}_A;P)$ for every profile (P, A) and Euro-Latin agenda $\mathtt{EL}_A$.

(a) Fix $x \in CS(M_A^P)$. Let $\langle b \rangle$ denote the associated A-uncovered M_A-transitive chain; and, B the set of alternatives in $\langle b \rangle$. Define $B^- \equiv \{x \in A \setminus B : bM_Ax$ for some $b \in B\}$. One can use $\langle b \rangle$ to construct a universal sequence $\langle a^- \rangle$ for x on $A^- \equiv B \cup B^-$ by inserting each $a \in B^-$ immediately before the last $b_i \in \langle b \rangle$ such that $b_i M_A a$. To extend $\langle a^- \rangle$

into a sequence $\langle a \rangle$ on A, insert each $a \in A^+ \equiv A \setminus A^-$ at the end of $\langle a^- \rangle$.[38] By Claim 26 (and the construction of priority agendas), it follows that $UNE[\mathrm{P}^{\langle a \rangle}_{A|A^-}; P] = UNE[\mathrm{P}^{\langle a^- \rangle}_{A^-}; P] = x$.

To establish the result, consider a candidate profile P^c such that, for each $c \in A \setminus x$, $x \succ_c y$ for all $y \in A \setminus c$. I show that A^- is a Nash equilibrium of the first-stage entry game $(\mathrm{P}^{\langle a \rangle}_A; P, P^c)$. In other words, $x \in EQ[\mathrm{P}^{\langle a \rangle}_A; P, P^c]$ so that $x \in \widetilde{V}(M^P_A)$ as required. There are two cases: (i) $c \in A^-$; and, (ii) $c \in A^+$. **(i)** Since $c = x$ has no incentive to exit, suppose $c \in A^- \setminus x$. Since $\langle a^- \rangle \setminus c$ is universal for x on $A^- \setminus c$, Theorem 2 and Claim 26 imply that $UNE[\mathrm{P}^{\langle a \rangle}_{A|A^- \setminus c}; P] = x$. So, $c \in A^- \setminus x$ has no incentive to exit. **(ii)** By construction, $cM^P_A b$ for all $b \in B$. So, by definition of $CS(M^P_A)$, there exists some $b^- \in B^-$ such that $b^- M^P_A c$. Since c has lowest priority on $(P, A^- \cup c)$ but is not the Condorcet winner, Theorem 2 implies that $UNE[\mathrm{P}^{\langle a \rangle}_{A|(A^- \cup c)}; P] \neq c$. So, $c \in A^+$ has no incentive to enter given P^c.

(b) Fix $x \in TC(M^P_A)$. Then, there is a sequence $\langle a \rangle = a_1, ..., a_n$ such that $a_1 = x$ and $a_i M^P_A a_{i+1}$ for all $1 \leq i \leq n$. Clearly, $UNE[\mathrm{EL}^{\langle a \rangle}_A; P] = x$. To establish the result, consider a candidate profile P^c such that, for each $c \in A \setminus x$, $x \succ_c y$ for all $y \in A \setminus c$. Notice that $UNE[\mathrm{EL}^{\langle a \rangle}_{A|A \setminus c}; P, P^c] \neq c$. So, c has no incentive to exit given P^c. So, A is a Nash equilibrium of the first-stage entry game. In other words, $x \in EQ[\mathrm{EL}^{\langle a \rangle}_A; P, P^c]$ so that $x \in \widetilde{V}(M^P_A)$ as required. ■

Proof of Remark 7. Fix a profile P on $X = \{x_1, x_2, x_3, x_4\}$ that induces the following majority relation:

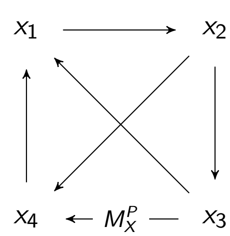


Figure 9: The majority relation M^P_X induced by the profile P.

For this profile, the priority agendas (in Example 8) implement the following outcomes up to re-prioritization:

i	**Description**	**Game Form** Pr^i_X	$V_\Sigma(\mathrm{Pr}^i_X; P)$	**Tournament Solution**
1	Euro-Latin (E-L)	Figure 2, left	X	Top Cycle
2	E-L with intermediate amendment	Example 2, left	X	Top Cycle
3	E-L with Anglo-American (A-A) amendment	Figure 3, left	$X \setminus x_4$	Banks Set
4	A-A with E-L amendment	Figure 3, right	$X \setminus x_4$	Banks Set
5	A-A with intermediate amendment	Example 2, right	X	Top Cycle
6	A-A	Figure 2, right	$X \setminus x_4$	Banks Set

Table 2: Discrimination by the six priority agendas on $\{x_1, x_2, x_3, x_4\}$.

For every majority relation not equivalent to M^X_P (up to permutation), each agenda in Table 2 can yield any outcome in the Top Cycle. Consequently, Pr^3_X, Pr^4_X, and Pr^6_X are more discriminating than the other priority agendas.

In terms of bushiness, the priority agendas in Table 2 can be ordered as follows:

[38]It is worth pointing out that this sequence is different from the one Dutta et al. define in their proof of Proposition 5.

$$\begin{tikzcd}
\mathtt{AA}_X \arrow[r, equal] & \mathtt{Pr}^6_X \arrow[r] & \mathtt{Pr}^5_X \arrow[r] \arrow[d] & \mathtt{Pr}^4_X \arrow[dr] & & \\
& & \mathtt{Pr}^3_X \arrow[r] & \mathtt{Pr}^2_X \arrow[r] & \mathtt{Pr}^1_X \arrow[r, equal] & \mathtt{EL}_X
\end{tikzcd}$$

Figure 10: Bushiness ordering of the priority agendas on X.

This shows that $\mathtt{Pr}^5_X$ is at once strictly bushier than $\mathtt{Pr}^3_X$ and $\mathtt{Pr}^4_X$ and strictly less discriminating. ■

Incidentally, the outcomes associated with the three remaining simple agendas (from Examples 1 and 3) are:

i	**Game Form** $\mathtt{S}^i_X$	$V_\Sigma(\mathtt{S}^i_X; P)$	**Tournament Solution**
1	Example 1	$\{x_2, x_3\}$	Copeland Set
2	Example 3, left	$X \setminus x_4$	Banks Set
3	Example 3, right	$X \setminus x_4$	Banks Set

Table 3: Discrimination by the remaining simple agendas on $\{x_1, x_2, x_3, x_4\}$.

As in the proof of Remark 7, the agendas in Table 3 implement the Top Cycle for any majority relation which is not a four-cycle. So, the knockout agenda $\mathtt{S}^1_X$ is the most discriminating simple agenda on four alternatives.

Finally, the outcomes associated with the six remaining recursive agendas (from Examples 6 and 11) are:

i	**Game Form** $\mathtt{R}^i_X$	$V_\Sigma(\mathtt{R}^i_X; P)$	**Tournament Solution**
1	Example 6, left	$X \setminus x_4$	Banks Set
2	Example 6, right	X	Top Cycle
3	Example 11, top left	$X \setminus x_4$	Banks Set
4	Example 11, top right	$X \setminus x_4$	Banks Set
5	Example 11, bottom left	$X \setminus x_4$	Banks Set
6	Example 11, bottom right	X	Top Cycle

Table 4: Discrimination by the remaining recursive agendas on $\{x_1, x_2, x_3, x_4\}$.

Since the agendas in Table 4 likewise implement the Top Cycle for any majority relation that does not include a four-cycle, the knockout agenda is *also* the most discriminating recursive agenda on $\{x_1, x_2, x_3, x_4\}$.

Proof of Remark 8. Fix a collection of agendas $\mathcal{T}$; and, suppose two sub-collections $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ satisfy Self-Consistency. Then, it is straightforward to see that: $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{T}$; and, moreover, $\mathcal{T}_1 \cup \mathcal{T}_2$ satisfies Self-Consistency. ■