A Simple Model of Two-Stage Choice*

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Abstract

I provide choice-theoretic foundations for a simple two-stage model, called *transitive shortlist methods*, where choices are made by sequentially by applying a pair of transitive preferences (or *rationales*) to eliminate inferior alternatives. Despite its simplicity, the model accommodates a wide range of choice phenomena including the *status quo* bias, framing, homophily, compromise, and limited willpower.

I establish that the model can be succinctly characterized in terms of some well-documented *context effects* in choice. I also show that the underlying rationales are straightforward to determine from readily observable *reversals* in choice. Finally, I highlight the usefulness of these results in a variety of applications.

JEL Codes: D01 **Keywords:** Shortlisting; Axiomatization; Revealed Preference; Identification.

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1 Introduction

Since Manzini and Mariotti [2007], the theory literature has witnessed a proliferation of bounded rationality models where the decision maker follows a two-stage choice procedure.[1] Broadly, the purpose of these models is to accommodate *context effects* in choice—related to a variety of psychological, social, or environmental factors—that are inconsistent with preference maximization.[2]

In this paper, I consider a simple two-stage procedure, *transitive shortlist methods* (TSM), which departs only mildly from the standard model. Instead of maximizing a single preference, the decision maker sequentially maximizes a pair (P_1, P_2) of transitive but potentially incomplete *rationales*. From the "shortlist" of feasible alternatives that maximize the first rationale P_1, the decision maker ultimately selects the unique alternative that maximizes the second rationale P_2.

The main results establish solid choice-theoretic foundations for the TSM model. Axiomatically, it is characterized by a *Strong Exclusivity* condition which imposes clear limitations on the possibility of context effects in choice (Theorem 2). This axiom stipulates that behavior consistent with the model cannot simultaneously display two (types of) *choice reversals* frequently observed in the data. In terms of identification, these choice reversals bear a straightforward connection to revealed preference in the model: each type of reversal independently reveals the content of one rationale (Theorem 4). What is more, these *revealed rationales* may be determined from choices on "small" menus (Theorem 3); and, they may be used to characterize the TSM-representations consistent with behavior (Theorem 5).

To put these results in context, it is worth noting that the TSM model has been studied in several papers (Au and Kawai [2011]; Lleras et al. [2011]; and, Yildiz [2015]) since it was suggested by Manzini and Mariotti [2006].[3] As discussed in Section 7, the characterizations provided in these papers rely on a technical acyclicity condition (similar to SARP) that provides little insight into the behavior associated with the model. In addition, the limited work on identification in these papers does not establish the simple connection between the rationales and some readily observable context effects in choice.

Three separate motivations underlie the main results. First and foremost is the view that a clear grasp of foundations can provide insights in applications (Dekel and Lipman [2010]; and Spiegler [2008]). Recently, the literature has displayed a growing interest to integrate bounded rationality considerations into classical theory models of industrial organization (Spiegler [2011]), contracts (Koszegi [2014]), and implementation (Korpela [2012]; and, de Clippel [2014]). Among two-stage procedures, the TSM model seems particularly well-suited to these applications: while it departs only marginally from the standard paradigm, it accommodates a wide range of choice phenomena (see the examples in Section 2). In the

[1]Besides the other papers cited in footnote 2 and the Introduction, see: Alcantud and García-Sanz [2015]; Bajraj and Ülkü [2015]; Manzini and Mariotti [2012]; Houy [2008]; and, Spears [2011].

[2]Some psychological factors that have been considered include: *compromise* (Chandrasekher [2015]); *limited attention/consideration* (Masatlioglu et al. [2012]; Lleras et al. [2011]); *willpower* (Masatlioglu et al. [2011]); *rationalization* (Cherepanov et al. [2013]); and, *status quo bias* (Masatlioglu and Ok [2005]; Apesteguia and Ballester [2013]). Environmental and social factors include *framing* (Salant and Siegel [2013]) and *homophily* (Cuhadaroglu [2015]).

[3]In fact, the origins of the TSM model (and every other *shortlist method* in Table 1 below) can be traced back to some early work by Soviet mathematicians (see Aizerman [1985]). However, this work is not well-known in economics.

hope of encouraging further applications of the TSM model, I devote Section 6 of the paper to illustrate how the main results *are* useful in a variety of natural applications (see Remarks 1-3).

A second motivation is to establish clear connections in the *empirical scope* of various two-stage procedures. Frequently, differences in axiomatic approaches make it difficult to compare these models in terms of behavior.[4] The characterization of the TSM model suggests that *exclusivity* conditions might provide a simple basis to compare *shortlist methods*. To support this intuition, I establish an additional result (Theorem 6): by progressively weakening Strong Exclusivity, one obtains characterizations of related models that dispense with transitivity for the second rationale (Matsuki and Tadenuma [2013]) *or* both rationales (see Manzini and Mariotti [2007]; and, Rubinstein and Salant [2008]). While these models were originally characterized along different lines, the treatment here provides a unified approach for understanding all three models in terms of choice reversals.

A final (but equally significant) motivation relates to the *principle of parsimony*. Given the wealth of recent two-stage models (see Section 7), it is worth understanding whether the TSM model accommodates the kinds of choice anomalies most frequently observed in practice—or whether a more general model is required. The main results furnish the tools necessary to pursue this question. The identification results give a simple way to infer the rationales from choice data that is readily collected in experimental and market settings. What is more, Strong Exclusivity makes it easy to test the model (Theorem 7): for choice data consistent with Manzini and Mariotti's [2007] more general model, Strong Exclusivity holds if and only if there is no "short" cycle of reversals related to choices from "small" menus. By comparison, the acyclicity condition used in prior characterizations of the model is stronger and, consequently, more difficult to test. Even though it is unlikely to be relevant in practice, it is worth noting that this advantage disappears when Strong Exclusivity is tested on its own. In that case, the *computational complexity* of testing Strong Exclusivity is comparable to the acyclicity condition used in prior characterizations.[5]

Layout: Section 2 presents the TSM model, briefly discussing some applications and related models. The next two sections contain the main theoretical results of the paper: Section 3 provides an axiomatic characterization of the model while Section 4 gives identification results. Section 5 takes a step back from the TSM model to highlight the important role played by exclusivity conditions in a broader range of two-stage procedures. In turn, Section 6 illustrates the usefulness of the theoretical results in a variety of applications. Finally, Section 7 discusses the contribution of the paper relative to prior work.

2 Preliminaries

I first present the TSM model more formally before discussing some applications and related models.

[4] In recent work, Tyson [2013] makes clear progress on this question by developing a common framework to characterize a variety of models. One drawback is that his approach relies on acyclicity conditions as an essential ingredient.

[5] Two distinct tasks are involved in testing each of these axioms: *definition* and *evaluation*. The first requires the analyst to identify the relevant reversals by considering every menu in the choice data. When there are n alternatives, this means that up to $O(2^n)$ menus must be considered. In turn, the second task requires the analyst to search for a combination of reversals that is "prohibited" by the axiom. For both axioms, the complexity of this task is $O(n^2)$.

2.1 The TSM Model

Let X denote a finite choice domain. A *rationale* P is an *asymmetric* binary relation on X (i.e., for all $x, y \in X$, xPy precludes yPx). It is *total* if xPy or yPx for all distinct $x, y \in X$; and, *transitive* if xPy and yPz imply xPz for all $x, y, z \in X$. A **transitive shortlist method** (TSM) is a pair (P_1, P_2) of transitive rationales (i.e., *quasi-transitive* preferences) that defines a choice function $c_{(P_1,P_2)} : 2^X \to X$ as follows:

$$c_{(P_1,P_2)}(B) \equiv \max(\max(B; P_1); P_2) \quad \text{for all } B \subseteq X$$

where $\max(A; P) \equiv \{x \in A : \text{no } y \in B \text{ s.t. } yPx\}$ denotes the P-maximal alternatives in A. Conversely, a pair of transitive rationales (P_1, P_2) on X is a **TSM-representation** of the choice function $c : 2^X \to X$ if $c(B) = c_{(P_1,P_2)}(B)$ for every menu $B \subseteq X$. In this case, c is said to be **TSM-representable**.

2.2 Applications

Despite its simplicity, the TSM model accommodates a wide range of choice phenomena. To highlight its flexibility and relevance in applications, consider some examples drawn from the recent literature:

Extreme Status Quo (Apesteguia and Ballester [2013]; see also Masatlioglu and Ok [2005]) *A default (or status quo) option d biases the consumer against choosing products in $A_d \subseteq X$. When her feasible choices B include d, she ignores everything in $A_d \cap B$ and maximizes her total preference P on $B \setminus A_d$.*

Framing (Salant and Siegel [2013]) *Because of framing at the point of sale, the consumer chooses from the seller's menu of products B according to a total preference P_f that differs from her "true" total preference P. After the consumer leaves the store and the framing effect dissipates, she returns the product chosen from B unless it is preferred to her outside option $o \notin B$ according to P.*

Homophily (Cuhadaroglu [2015]) *Two friends influence the choice behavior of one another. Each first maximizes her own preference P before using her friend's preference P' to refine her choices.*

Compromise (Chandrasekher [2015]) *The domain X is partitioned into categories $\{C_1, ..., C_n\}$. The agent consists of conflicting selves: a "planner" with total preference P; and, a "doer" with preference P^{-1}. Given a menu B, the planner selects a (non-empty) category $C_i \cap B$; and, the doer selects an alternative from $C_i \cap B$. When picking a category, the planner accounts for the doer's adverse interests.*

Willpower (Masatlioglu et al. [2011]) *The decision maker relies on a stock of willpower w to resist against the alternative b^* that presents the greatest temptation v in the menu B. Relative to another alternative $x \in B$, she cannot resist b^* when $v(b^*) > v(x) + w$. To select among the alternatives in the menu B that resist against b^* (if any),*[6] *the decision maker maximizes her total preference P.*

In each of these applications, the rationales have very natural interpretations.[7] In the *compromise*

[6]Formally, this first stage involves the maximization of a special transitive rationale $P_{v,w}$ called a *semiorder*.

[7]In each case, behavior is consistent with the TSM model as long as the "preferences" specified are transitive rationales.

model, for instance, they capture strategic considerations in a two-stage game: the first reflects the intra-category preferences of the doer while the second reflects the preferences of the forward-looking planner. In the other applications, the rationales reflect the tension between "true" preference P and another factor (whether psychological, environmental, or social) that influences behavior.[8]

In each case, the interpretation imposes some structure on one of the rationales (beyond transitivity). In the compromise model, for instance, the first rationale prefers x to y precisely when: the second rationale prefers y to x; and, the two alternatives belong to the same category. In the other examples, the rationale with additional structure is the one unrelated to true preference. For instance, *every* comparison in the first rationale of the *status quo model* involves the default; and, *no* comparison in the first rationale of the *framing model* involves the outside option.

Of these models, only the homophily model is well-understood in terms of choice behavior.[9] What is more, Cuhadaroglu's characterization of that model exploits the kinds of insights developed here. In Section 6, I show how the results provide an understanding of behavior in the other applications as well.

2.3 Other Shortlist Methods

The TSM model is related to a variety of other *shortlist methods* which impose more or less structure on the rationales. The most general, Manzini and Mariotti's [2007] *rational shortlist methods* (RSM) (see also Rubinstein and Salant [2008]; and, Dutta and Horan [2015]), places no formal restrictions on either rationale (P_1 and P_2 are asymmetric binary relations). Between the RSM and TSM models are shortlist methods that *only* require one of the rationales to be transitive (P_1 in Matsuki and Tadenuma [2013]; and, P_2 in Houy [2008]). Finally, the *transitive RSM model* (Au and Kawai [2011]; Lleras et al. [2011]; and, Yildiz [2015]) specializes the TSM model by requiring the second rationale to be total. Since the added structure of the second rationale in this model imposes no restrictions on behavior beyond those associated with the TSM model, the two models are equivalent in terms of axiomatics and identification.

The table below summarizes the features of the rationales in each of these models:

To simplify, I ignore the possibility of multi-valued choice (which is ruled out by the TSM model). As stated, choice in the *homophily model* can be multi-valued. While the other examples all rule out this possibility, each can be extended to accommodate multi-valued choice by dropping "total" where it is specified (as is done in some of the cited papers).

[8] In the homophily model, the first rationale reflects true preference. In the other three models, it is the second.

[9] The extreme status quo model has only been characterized jointly with a second status quo model (described in Section 6). In turn, characterizations of the compromise and *willpower* models exist only for the menu preference domain.

Model	P_1 rationale[10]	P_2 rationale[11]
RSM	unrestricted	unrestricted
T_1SM	transitive	unrestricted
T_2SM	unrestricted	transitive
TSM	**transitive**	**transitive**
Transitive RSM	transitive	transitive and total

Table 1: Comparing shortlist methods

Besides the TSM model, I also establish results related to some other shortlist methods in Table 1. To facilitate the discussion, I postpone the comparison to previous work on these models until Section 7.

3 Axiomatic Foundations

I show that the TSM model may be characterized in terms of two prevalent choice reversals.

3.1 Choice Reversals

The standard model of strict preference maximization is characterized by the *Independence of Irrelevant Alternatives* (IIA). Intuitively, this axiom ensures that choice behavior does not depend on the availability of alternatives which are *not* ultimately chosen by the decision maker. Formally:

IIA *If* $c(A) = x$*, then* $c(B) = x$ *for all* B *such that* $\{x\} \subseteq B \subseteq A$.

Intuitively, IIA rules out *context effects* where the choice of one alternative requires the presence of another. When such effects are present, the addition or removal of an unchosen alternative may cause the decision maker to reverse her choice.

The experimental literature points to two *choice reversals* that are particularly prevalent. In the first type of reversal, the addition of an unchosen alternative "directly" prevents the choice of another. Formally, a choice function c displays a **direct** $\langle x, y\rangle$ **reversal** on $B \subseteq X \setminus \{x\}$ if

$$c(B) = y \text{ and } c(B \cup \{x\}) \notin \{x, y\}.$$

A well-documented bias consistent with this behavior is the *attraction effect* (Huber et al. [1982]). Put in terms of this effect, the idea is that the presence of x attracts the decision maker to choose $c(B \cup \{x\})$ over y. Intuitively, this violates IIA because x plays an attracting role unrelated to preference.

[10] To rule out empty choice, P_1 is nonetheless required to be acyclic (see Dutta and Horan [2015]).
[11] Requiring P_2 to be transitive (and total) is equivalent to requiring P_2 to be acyclic (see Lemma 14 of the Appendix).

In the second type of reversal, the removal of an apparently "weak" alternative has an impact on choice. Formally, a choice function c displays a **weak $\langle x, y\rangle$ reversal** on $B \supset \{x, y\}$ if

$$c(x, y) = x \text{ and } c(B) \neq c(B \setminus \{y\}).$$

With this type of behavior, y affects the choice from B despite the availability of an alternative x that is pairwise preferred.[12] When $c(B) = y$, a weak alternative is chosen from B despite the presence of an apparently stronger alternative. When $c(B) \neq y$, the behavior is suggestive of *limited attention* (Masatlioglu et al. [2012]). Here, the idea is that the decision maker only considers (and ultimately chooses) the alternative $c(B)$ because y attracts attention to it. In either case, the behavior violates IIA because y plays a special role that is unrelated to preference.

While this discussion makes it clear that these two choice reversals are inconsistent with IIA, more surprising is that any choice behavior inconsistent with IIA *must* display both types of reversal:

Theorem 1 *For a choice function c, the following are equivalent:*

(**i**) *c satisfies the Independence of Irrelevant Alternatives;*
(**ii**) *c displays no direct reversals for any pair of alternatives; and,*
(**iii**) *c displays no weak reversals for any pair of alternatives.*

3.2 A Simple Axiomatization

It turns out that a single axiom, called *Strong Exclusivity*, characterizes the class of RSMs with TSM-representations. The strength of this axiom is to provide a clear understanding of the context effects consistent with the TSM model. To illustrate, first consider a weaker version of this axiom:

Exclusivity *For every pair of alternatives $\langle x, y\rangle \in X \times X$, either:*

(i) *c displays no weak $\langle x, y\rangle$ reversals; or,*
(ii) *c displays no direct $\langle x, y\rangle$ reversals.*

For any pair of alternatives, this axiom precludes choice behavior which exhibits *both* of the context effects discussed above. Put differently, the possibility of weak reversals for a given pair of alternatives is ruled out by observing a single direct reversal for that pair (and *vice versa*).

In terms of choice, Exclusivity is tantamount to the transitivity of the first rationale:

Lemma 1 *If c is RSM-representable, then: it is T_1SM-representable iff it satisfies Exclusivity.*

[12] For the TSM model, this generalizes to non-binary menus. If $c(A) = x$ for some menu A such that $\{x, y\} \subset A \subset B$, then $c(x, y) = x$ (see Lemma 30 of the Supplemental Appendix; and, Lleras et al. [2011]).

When both rationales are transitive, weak reversals are *also* ruled out "indirectly" by certain pairs of direct reversals.[13] Formally, c displays an **indirect** $\langle x, y\rangle$ **reversal** on $B \subseteq X \setminus \{w, x\}$ if

$$c(B) = y,\ c(B \cup \{x\}) = x,\ c(B \cup \{w, x\}) = w,\ \text{and}\ c(B \cup \{w\}) = z \notin \{y, w\}.$$

Here, the role of x is to "obscure" that y is directly reversed by w. To elaborate, observe that the first three choices are IIA-consistent. In fact, the second and third choices are consistent *because* the direct $\langle x, z\rangle$ reversal on $B \cup \{w\}$ leads to $c(B \cup \{w, x\}) = w$. If it led to any other choice, the attracting role of w would be apparent even without observing $c(B \cup \{w\})$. In this way, the direct $\langle x, z\rangle$ reversal on $B \cup \{w\}$ serves to obscure the direct $\langle w, y\rangle$ reversal on B.

Using this notion of indirect reversals, Strong Exclusivity may be stated as follows:

Strong Exclusivity *For every pair of alternatives* $\langle x, y\rangle \in X \times X$*, either:*

(i) *c displays no weak $\langle x, y\rangle$ reversals; or,*

(ii) *c displays no direct or indirect $\langle x, y\rangle$ reversals.*

This axiom strengthens Exclusivity. For a given pair of alternatives, direct reversals *and* indirect reversals both rule out weak reversals. This restriction on choice captures the transitivity of *both* rationales:

Lemma 2 *If c is RSM-representable, then: it is TSM-representable iff it satisfies Strong Exclusivity.*

Manzini and Mariotti [2007, Theorem 1] identify two axioms, *Expansion* and *Weak WARP*, that characterize every RSM-representable choice function c. To state these for the unfamiliar reader:

Expansion *If $c(A) = x = c(B)$, then $c(A \cup B) = x$.*

Weak WARP *If $c(A) = x = c(x, y)$, then $c(B) \neq y$ for all B such that $\{x, y\} \subset B \subset A$.*

The first axiom (also called Sen's γ) is the standard requirement that choices from larger menus be consistent with choices from smaller menus. In turn, the second axiom weakens IIA (or, equivalently, WARP for choice functions) since it requires $c(B) \neq y$ rather than $c(B) = x$ when $c(A) = x = c(x, y)$. Interestingly, Weak WARP also has a natural interpretation in terms of exclusivity:

Weak Exclusivity *For every pair of alternatives* $\langle x, y\rangle \in X \times X$*, either:*

(i) *c displays no weak $\langle x, y\rangle$ reversal for any menu B such that $c(B) = y$; or,*

(ii) *c displays no direct $\langle x, y\rangle$ reversals.*

This axiom weakens Exclusivity. For a given pair of alternatives, direct reversals continue to rule out weak reversals where the weak alternative is chosen. However, they do *not* rule out weak reversals where the weak alternative merely "attracts" the decision maker to a different choice.

[13] In the standard model of preference maximization, an "indirect" revealed preference $x \succ^c_I y$ is a chain of "connected" pairwise preferences $x \succ^c \ldots \succ^c y$ (see Section 4.1 below). In the TSM model, an indirect $\langle x, y\rangle$ reversal is a chain of three connected *weak* reversals $\langle y, z\rangle$, $\langle z, w\rangle$, and $\langle w, x\rangle$ (see Section 5.2 below).

In the presence of Expansion, Weak Exclusivity is equivalent to Weak WARP.[14]

Lemma 3 *If c satisfies Expansion, then: it satisfies Weak Exclusivity iff it satisfies Weak WARP.*

In light of Manzini and Mariotti's result, Lemmas 2 and 3 characterize the TSM model:

Theorem 2 *c is TSM-representable if and only if it satisfies Expansion and Strong Exclusivity.*

Examples 2-3 of the Supplemental Appendix establish the independence of the two axioms.

The proofs of Lemmas 1-3 are given in the Appendix. For these results, the main challenge is to establish the sufficiency of Strong Exclusivity in Lemma 2. Effectively, this amounts to showing that the revealed preference R_2^c for the second rationale (as defined in Section 4.1 below) is acyclic.

4 Identification

I first show that choice reversals on small menus define *revealed rationales* for the TSM model before showing that these revealed preference definitions extend naturally to larger menus. Finally, I show how revealed preference may be used to characterize the entire class of TSM-representations.

4.1 Revealed Rationales

The proof of Lemma 1 provides an insight that will serve as the foundation for the revealed preference exercise. To formalize, define the (usual) *pairwise preference* $\succ^c$ by $x \succ^c y$ if $c(x, y) = x$; and, define an *n-cycle* to be a sequence $x_0 ... x_i ... x_{n-1}$ such that $x_{i-1} \succ^c x_i$ for $1 \leq i \leq n-1$ and $x_{n-1} \succ^c x_0$.

Corollary 1 *If c and $\tilde{c}$ are T_1SM-representable, then $c(\cdot) = \tilde{c}(\cdot)$ if and only if:*

(**i**) $c(x, y) = \tilde{c}(x, y)$ *for all* $\{x, y\} \subseteq X$*; and,*

(**ii**) $c(x, y, z) = \tilde{c}(x, y, z)$ *for all 3-cycles* xyz*.*

This shows that choices from small menus pin down behavior in the TSM model. The standard model of preference maximization exhibits the same kind of "small menu" feature. As in that model, this feature makes it possible to provide simple revealed preference definitions in the TSM model.[15]

To fix ideas, suppose (P_1, P_2) is a TSM-representation of c. Notice that the two rationales together $(P_1 \cup P_2)$ must contain the pairwise preferences in $\succ^c$. Otherwise, (P_1, P_2) cannot induce choice behavior consistent with c for every menu of two alternatives. This observation shows that identification in the model effectively boils down to assigning the preference pairs in $\succ^c$ to one of the rationales.

First, consider the task of assigning the pairwise preferences of a 3-cycle xyz:

[14] Without Expansion, this equivalence no longer holds. In fact, it can be shown that even Strong Exclusivity and Weak WARP are independent in the presence of Expansion(see Examples 5-4 of the Supplemental Appendix).

[15] In fact, the same is true for the T_1SM model. See Remark 10 of the Supplemental Appendix.

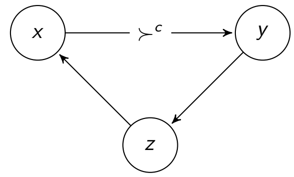

Figure 1: Pairwise preferences of the 3-cycle xyz

In this case, the choice from $\{x, y, z\}$ determines how to divide the preference pairs. If $c(x, y, z) = z$, for instance, x must eliminate y before y eliminates z. So, xP_1y and yP_2z. Since P_1 is transitive, zP_2x as well. Otherwise, zP_1xP_1y so that zP_1y which contradicts $y \succ^c z$. (Since it depends on the transitivity of P_1, this inference about P_2 is *not* justified in the RSM or T_2SM models.)

Next, consider the task of assigning the pairwise preference $x \succ^c y$ when $wxyz$ is a 4-cycle consisting of overlapping 3-cycles wxz and wyz such that $c(w, x, z) = w$ and $c(w, y, z) = z$:

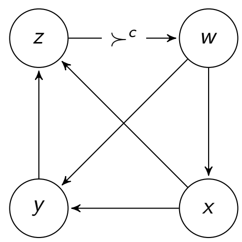

Figure 2: Pairwise preferences of the 4-cycle $wxyz$

Following the same reasoning as in the last paragraph, $yP_2zP_2wP_2x$. Since P_2 is a transitive rationale, xP_1y as well. Otherwise, $yP_2zP_2wP_2xP_2y$ so that P_2 contains a cycle. (Since it depends on the acyclicity of P_2, this inference about P_1 is *not* justified in the RSM or T_1SM models.)

Collecting the observations from the last two paragraphs motivates the following definitions:

Definition 1 *Given a choice function c, first define the binary relations R_1^c and R_2^c by:*

(1) *xR_1^cy if $c(x, y) = x$ and —*

(i) *there exists a 3-cycle xyz such that $c(x, y, z) = z$;* **or**,

(ii) *there exist 3-cycles wxz, wyz such that $c(w, x, z) = w$ and $c(w, y, z) = z$.*

(2) *xR_2^cy if there exists a 3-cycle xyz such that $c(x, y, z) \neq z$.*

For $i = 1, 2$, next define the ***revealed i-rationale*** *$P_i^c \equiv tc(R_i^c)$ to be the transitive closure of R_i^c. (In other words, let xP_i^cy if there exists a sequence $\{z_i\}_{i=1}^n$ in X such that $xR_i^cz_1R_i^c...R_i^cz_n = y$.)*

These definitions reflect *some* features of the representation that are particularly straightforward to infer from behavior. Surprisingly, they capture *everything* that choice reveals about the rationales:

Theorem 3 *If c is TSM-representable, then:*

(1) *$xP_1^c y$ if and only if $xP_1 y$ for every TSM-representation (P_1, P_2) of c; and,*

(2) *$xP_2^c y$ if and only if $xP_2 y$ for every TSM-representation (P_1, P_2) of c.*

This result shows that P_1^c and P_2^c characterize revealed preference in the TSM model. To elaborate, the only features of the rationales with potential implications for choice are those which are common to all TSM-representations of behavior. Since the revealed rationales capture precisely these features, they reflect *all* possible inferences about revealed preference in the model.

4.2 Role of Choice Reversals

Choice reversals are closely related to revealed preference in the TSM model. In fact, one might have defined R_1^c and R_2^c directly in terms of choice reversals on small menus. In particular:

(1) *$xR_1^c y$ if and only if c displays a direct or indirect $\langle x, y\rangle$ reversal on $\{y, z\}$ for some z;*[16] and,

(2) *$xR_2^c y$ if and only if c displays a weak $\langle x, y\rangle$ reversal on $\{x, y, z\}$ for some z.*

In this section, I show that this relationship extends naturally to choice reversals on larger menus.

Effectively, the "small menu" feature of the T_1SM model (Corollary 1) implies that all reversals boil down to reversals on small menus.[17] Both direct and indirect reversals imply the same type of reversal on a single small menu: *if a direct or indirect $\langle x, y\rangle$ reversal occurs on B, then it also occurs on $\{y, z\}$ for some $z \in B$.* Indeed, the same is true for all weak reversals where the "weak" alternative is chosen.[18] However, there are weak reversals which are not reducible to a single small menu reversal.

To illustrate, consider a T_1SM-representable choice function c that exhibits the pairwise preferences in Figure 2. Now, suppose $c(w, x, y, z) = w \neq z = c(w, x, z)$. Since $c(x, y) = x$, this means that c displays a weak $\langle x, y\rangle$ reversal on $\{w, x, y, z\}$. Since Expansion requires $c(w, x, y) = w = c(w, x)$ and $c(x, y, z) = x = c(x, z)$ however, c does not display a weak $\langle x, y\rangle$ reversal on any smaller menu.

This example suggests how to extend R_2^c to weak reversals that are not fully reducible:

Definition 2 *Given a choice function c, define the relation $\widehat{R_2^c}$ by $x\widehat{R_2^c}y$ if $c(x, y) = x$ and —*

(i) *there exists a 3-cycle xyz such that $c(x, y, z) \neq z$ (i.e. $xR_2^c y$);* **or**,

(ii) *there exist 3-cycles wxz, wyz such that $c(w, x, z) = z$ and $c(w, y, z) = w$.*

Part (ii) of the definition reflects the weak $\langle x, y\rangle$ reversal in the example above.[19] As the next result shows, this captures the extent to which weak reversals are irreducible in the T_1SM model:

Theorem 4 *If c is T_1SM-representable, then:*

[16] Since $c(w, x, y, z) = w$ by Expansion, Definition 1(1)(ii) describes an indirect $\langle x, y\rangle$ reversal on $\{y, z\}$.

[17] In the standard model, the revealed preference P^c is defined by $xP^c y$ if $x \in c(B)$ and $y \notin c(B)$ for some menu B such that $y \in B$. The "small menu" feature of that model implies that this boils down to the pairwise preference $\succ^c$.

[18] Formally: *if a weak $\langle x, y\rangle$ reversal where y is chosen occurs on B, then it also occurs on $\{x, y, z\}$ for some $z \in B$.*

[19] To see this, note that Expansion and Weak WARP require $c(w, y, z) = w$ for the choices in this example.

(1) *$xR_1^c y$ if and only if c displays a direct or indirect $\langle x, y\rangle$ reversal; and,*
(2) *$x\widehat{R_2^c}y$ if and only if c displays a weak $\langle x, y\rangle$ reversal.*

This result shows that large menu reversals have the same revealed preference implications as small menu reversals. Although $\widehat{R_2^c}$ is weakly finer than R_2^c, the transitive closure of $\widehat{R_2^c}$ coincides with P_2^c.[20]

4.3 Representations of Behavior

The proof of Lemma 2 uses revealed preference to construct a representation for any choice function c consistent with the model. Specifically, it shows that c is TSM-represented by $(\widehat{Q_1^c}, P_2^c)$ where $\widehat{Q_1^c}$ is a transitive rationale defined by $\widehat{Q_1^c} \equiv \succ^c \setminus \widehat{R_2^c}$. Given Theorem 4, $\widehat{Q_1^c}$ reflects the pairs of alternatives that display no weak reversals (i.e. $x\widehat{Q_1^c}y$ if and only if $c(B) = c(B \setminus \{y\})$ for all $B \supseteq \{x, y\}$).

In general, behavior consistent with the model admits a number of TSM-representations. In this section, I show that revealed preference can be used to describe the class of all such representations.

For the first rationale, the representation $(\widehat{Q_1^c}, P_2^c)$ provides a key insight. Because it excludes only those pairwise preferences which are directly revealed to be in the second rationale, $\widehat{Q_1^c}$ must be the finest first rationale that can be used to represent c (see Lemma 25 of the Appendix). Just as Theorem 3 identifies the revealed rationale P_1^c as the most *conservative* estimate of P_1, this establishes $\widehat{Q_1^c}$ as the most *liberal* estimate. In other words, every P_1 used to represent c must satisfy $P_1^c \subseteq P_1 \subseteq \widehat{Q_1^c}$.

Turning to the second rationale, P_2 must include every pairwise preference not in the first rationale P_1. Otherwise, (P_1, P_2) cannot induce choice behavior consistent with c for every menu of two alternatives. In other words, every P_2 used to represent c must satisfy $P_2 \supseteq tc(\succ^c \setminus P_1)$.[21]

In fact, these two necessary conditions are also sufficient for a TSM-representation:

Theorem 5 *If c is TSM-representable, then (P_1, P_2) represents c if and only if:*

(1) *P_1 is a transitive rationale such that $P_1^c \subseteq P_1 \subseteq \widehat{Q_1^c}$; and,*
(2) *P_2 is a transitive rationale such that $P_2 \supseteq tc(\succ^c \setminus P_1)$.*

This result characterizes the class of TSM-representations in terms of the first rationale. By fixing a transitive P_1 such that $P_1^c \subseteq P_1 \subseteq \widehat{Q_1^c}$, any transitive P_2 finer than $tc(\succ^c \setminus P_1)$ can be used to represent c. Equivalently, the class of TSM-representations can be characterized in terms of the second rationale: *a transitive pair (P_1, P_2) represents c iff (1) $tc(\succ^c \setminus P_2) \subseteq P_1 \subseteq \widehat{Q_1^c}$ and (2) $P_2 \supseteq P_2^c$.*

Theorem 5 reflects an inherent trade-off in TSM-representations. To the extent that the analyst takes a more conservative view about what to include in one rationale, she *must* take a less conservative view about the other. Since it forces the analyst to take a position, this feature might be viewed as a shortcoming of the model. However, it might also be viewed as a strength. Precisely because of the

[20] For any weak $\langle x, y\rangle$ reversal which is not fully reducible (i.e. a pair $(x, y) \in \widehat{R_2^c} \setminus R_2^c$), Definition 2 implies that there exists a chain of three small menu weak reversals $\langle x, z\rangle$, $\langle z, w\rangle$, and $\langle w, y\rangle$. As a result, it follows that $xR_2^c zR_2^c wR_2^c y$.

[21] Since $P_2^c \subseteq tc(\succ^c \setminus P_1^c)$ in the TSM model, this generally rules out (P_1^c, P_2^c) as a possible TSM-representation.

flexibility in TSM-representations, the model accommodates a range of applications where the rationales are given a specific interpretation and some additional structure (as discussed in Section 2).

Even without a particular application in mind, the analyst may wish to rule out certain representations. In general, a TSM-representation (P_1, P_2) may contain *duplication* (xP_1y and xP_2y) or *conflict* (xP_1y and yP_2x) between the rationales. To limit this kind of redundancy, the analyst might focus on minimal representations. Formally, a TSM-representation (P_1, P_2) is *minimal* if $P_1' \subseteq P_1$ and $P_2' \subseteq P_2$ for no other TSM-representation (P_1', P_2'). Intuitively, these representations estimate each rationale P_i as conservatively as possible given the estimate P_{-i} of the other rationale.

Theorem 5 shows that either of the revealed rationales P_i^c defines a minimal representation when the other rationale is $Q_{-i}^c \equiv tc(\succ^c \setminus P_i^c)$.[22] Intuitively, these minimal representations reflect opposite (but equally conservative) views about how to assign the preferences in $\succ^c$ that are *not* in either revealed rationale: (P_1^c, Q_2^c) assigns them all to the second rationale; and, (Q_1^c, P_2^c) assigns them all to the first. Between these extremes, there is a range of minimal representations that reflect intermediate views about how to assign the preference pairs in $\succ^c \setminus (P_1^c \cup P_2^c)$. For the interested reader, I provide a characterization of minimal representations in Section B.2 of the Supplemental Appendix.

5 Exclusivity in Shortlist Methods

I first show that two generalizations of the TSM model (the T_1SM and RSM models discussed in Section 2 above) may also be characterized in terms of Expansion and an exclusivity condition. I then show how the exclusivity requirements of the TSM and T_1SM models are particularly straightforward to test.

5.1 Empirical Scope

Given Manzini and Mariotti's characterization, Lemmas 1 and 3 establish the following:

Theorem 6 *A choice function c is:*

- (**i**) *T_1SM-representable if and only if it satisfies Expansion and Exclusivity; and,*
- (**ii**) *RSM-representable if and only if it satisfies Expansion and Weak Exclusivity.*

Combined with Theorem 2, this result pinpoints the key similarities and differences in behavior among three shortlist methods. While all three models satisfy Expansion, the two generalizations of the TSM model impose progressively weaker exclusivity restrictions on potential choice reversals.

This suggests a parallel with choice under uncertainty. Just as *independence* conditions differentiate among models in that setting, *exclusivity* conditions provide a basis to distinguish among shortlist methods. The analysis in Section 6 below only serves to reinforce this point. For the specialized TSM models considered there, exclusivity considerations play a key role.

[22] It is worth noting some features of these rationales. First, Q_2^c is total (see Lemma 22 of the Appendix). Second, Q_1^c and $\widehat{Q_1^c}$ are distinct (see Example 1 of the Appendix for a TSM-representable choice function where $Q_1^c \subset \widehat{Q_1^c}$).

To be clear, the point is not to undermine the significance of Weak WARP. To the contrary, this condition describes the exclusivity inherent in *every* shortlist method (Lemma 3). Rather, the point is to suggest that the concept of exclusivity itself provides a powerful tool for analyzing two-stage procedures. Indeed, one can develop key insights into specialized shortlisting models simply by considering how they strengthen the exclusivity requirements associated with the RSM model.

5.2 Testability

Not only do exclusivity conditions help distinguish among shortlist methods, but they provide a practical way to test these models in experimental and market settings. To rule out the possibility that choice data is consistent with a given exclusivity condition, it is sufficient to identify a pair of "incompatible" choice reversals. In the worst case, all of the reversals in the data must be considered.

While it is quite simple, this "naive" approach actually overstates the difficulty of testing axioms like Strong Exclusivity. The intuition comes from observing the connection between indirect and weak reversals: if c is RSM-representable, then an indirect $\langle x, y\rangle$ reversal amounts to a chain of three connected weak reversals $\langle y, z\rangle$, $\langle z, w\rangle$, and $\langle w, x\rangle$.[23] As such, Strong Exclusivity rules out *particular* four-cycles of weak reversals: if c displays a chain of weak reversals (arising from an indirect $\langle x, y\rangle$ reversal), then it cannot display a weak $\langle x, y\rangle$ reversal. In a similar fashion, the incompatibility between direct and weak reversals implied by Strong Exclusivity rules out *particular* three-cycles of weak reversals.[24]

As it turns out, the connection to cycles of weak reversals is even more fundamental than these observations suggest. For RSM-representable behavior, Strong Exclusivity precludes all three- and four-cycles of weak reversals; and, conversely, choice functions without such cycles satisfy Strong Exclusivity. What is more, there is a similar connection between Exclusivity and three-cycles of weak reversals. Given Theorem 4, these observations can be stated more succinctly as follows:

Theorem 7 *If c is RSM-representable, then:*

- (**i**) *it satisfies Strong Exclusivity if and only if $\widehat{R^c_2}$ is quadruple-acyclic; and,*
- (**ii**) *it satisfies Exclusivity if and only if $\widehat{R^c_2}$ is triple-acyclic.*

This result provides a very practical way to test the requirements of (Strong) Exclusivity beyond Expansion and Weak WARP.[25] It is worth emphasizing three features of this test that are particularly appealing: (i) the test is based on one type of choice reversal only; (ii) the behavior of interest is restricted to small menus; and, (iii) the scope of the test is limited to "short" cycles.

[23] An indirect $\langle x, y\rangle$ reversal consists of choices $c(B) = y$, $c(B \cup \{x\}) = x$, $c(B \cup \{w, x\}) = w$, and $c(B \cup \{w\}) = z \notin \{y, w\}$ for some menu B. To see where the stated weak reversals arise, first observe that: $c(y, z) = y$ and $c(z, w) = z$ by Weak WARP; and $c(w, x) = w$ by Expansion. Then, it is easy to see that c displays the following weak reversals: a $\langle y, z\rangle$ reversal at $B \cup \{w\}$; a $\langle z, w\rangle$ reversal at $B \cup \{w, x\}$; and, a $\langle w, x\rangle$ reversal at $B \cup \{w, x\}$.

[24] A direct $\langle x, y\rangle$ reversal consists of choices $c(B) = y$ and $c(B \cup \{x\}) \equiv z \notin \{x, y\}$ for some menu B. If $c(x, z) = z$, then this can be parsed into a pair weak reversals at $B \cup \{x\}$: a $\langle y, z\rangle$ reversal (since $c(y, z) = y$ by WWARP); and a $\langle z, x\rangle$ reversal. So, Strong Exclusivity also rules out the particular three-cycle of weak reversals $\langle x, y\rangle$, $\langle y, z\rangle$, and $\langle z, x\rangle$.

[25] In fact, it also provides another way to characterize the TSM and T_1SM models. In particular: *c is TSM-representable (T_1SM-representable) iff it satisfies Expansion, Weak WARP, and its weak reversals are quadruple-acyclic (triple-acyclic).*

6 Applications of the Results

The theoretical results presented in the previous sections are motivated by the fact that the TSM model is well-suited to applications. In this section, I first show how these results provide insights into the kinds of specialized choice models that are likely to figure in applied work. I then highlight the implications for some classical theory applications where it is natural to incorporate two-stage choice.

6.1 Specialized Models of Two-Stage Choice

The homophily model in Section 2 concerns mutual influence between two agents: the choices c of one agent are determined by the transitive rationale pair (P, P') while the choices c' of the other are determined by (P', P). To axiomatize the model, Cuhadaroglu [2015] relies on a simple observation: for any pair of alternatives, c and c' cannot both display weak reversals. This natural exclusivity condition follows from Theorem 4. Since the "second" rationale of one agent is the "first" rationale of the other, no pairwise preference can be revealed to be in the "second" rationale for both agents.

Cuhadaroglu's analysis highlights how the revealed rationales may yield insights into specialized TSM models. In this section, I show how the other results in Sections 3 and 4 also provide tools to understand these models. To illustrate, I first consider the model of status quo bias described in Section 2 before outlining a general recipe that is suitable for any specialized TSM model.

Status Quo Bias

Formally, the extreme status quo (ESQ) model from Section 2 is parametrized by (d, P, U_d) where: $d \in X$ is the default option; P is a transitive and total rationale on X; and, $U_d \subseteq \{x \in X : xPd\}$ are the alternatives "unaffected" by the status quo bias. The choice from a given menu $B \subseteq X$ is:

$$c_{(d,P,U_d)}(B) \equiv \begin{cases} \max(B; P) & \text{if } d \notin B \\ d & \text{if } d \in B \text{ and } U_d \cap B = \emptyset \\ \max(U_d \cap B; P) & \text{otherwise.} \end{cases}$$

This model amounts to a shortlist method where the first rationale reflects the bias against the "affected" alternatives $A_d \equiv X \setminus U_d$ while the second reflects the "true" preference P. In particular, (d, P, U_d) is TSM-represented by (E, P) where the rationale E is defined by dEa if $a \in A_d$.

This natural representation shows that the model displays a limited range of reversals on small menus. Given parameters (d, P, U_d), every 3-cycle must take the form aud where $a \in A_d$, $u \in U_d$, and $aPuPdEa$. Since $c_{(d,P,U_d)}(a, u, d) = u$, the only direct reversals on small menus are $\langle d, a \rangle$ reversals. By Theorem 4, *every* direct reversal must take this form. So, choice must satisfy the following condition:

ESQ Exclusivity *For all distinct alternatives $x, y \in X$, either:*

(i) *c displays no direct $\langle x, z \rangle$ reversal for any $z \in X$; or,*

(ii) *c displays no direct $\langle y, z\rangle$ reversal for any $z \in X$.*

In other words, at most one alternative (the default) *causes* direct reversals. Effectively, this is the behavioral implication of the observation in Section 2 about the special structure of the rationales.

Because it limits direct reversals in this way, ESQ Exclusivity also rules out indirect reversals. Given Theorem 6, these two observations lead to a very simple characterization of the ESQ model:

Remark 1 *c is ESQ-representable if and only if it satisfies Expansion, Exclusivity, and ESQ Exclusivity.*

Relative to more general shortlist methods, the ESQ model is succinctly characterized by an exclusivity property which reflects the decision maker's "extreme" reaction to the default.

Apesteguia and Ballester [2013] also consider a *weak status quo* (WSQ) model where the default induces a "weak" reaction: when $d \in B$ and $U_d \cap B \neq \emptyset$, the choice is $\max(B \setminus \{d\}; P)$ rather than $\max(U_d \cap B; P)$. For small menus, the only direct reversals in this model are $\langle u, d\rangle$ reversals. Following the same kind of reasoning as above, this observation allows one to characterize the WSQ model. Compared with Remark 1, the only difference is that ESQ Exclusivity is replaced by *WSQ Exclusivity*, a property which states that at most one alternative (the default) *suffers* direct reversals.

A General Recipe

While the goal of the preceding analysis was to illustrate the implications of the results for a particular model, it actually provides a general recipe to understand *any* specialized TSM model. Starting from a "natural" TSM-representation of the model in question, one can first determine the scope of possible small menu reversals. Given Theorem 4, one can then extend these observations to larger menus in a natural way; and, ultimately formulate a variety of necessary conditions on choice reversals.

The basic intuition is to leverage the "small menu" feature of the model (Corollary 1) to gain broader insights into choice behavior. Consider, for instance, the framing and compromise models from Section 2. By applying the recipe outlined above, it is straightforward to establish the following:

Remark 2 *In both the framing and compromise models, choice displays no indirect reversals.*

For the framing model, this follows from a simple observation about small menus: one alternative (the outside option) must appear in every weak reversal.[26] For the compromise model, a different observation about about small menus plays a key role: direct reversals only occur between alternatives from the same category. Of the two, the second observation is much more difficult to state in terms of choice behavior. Instead of tying reversals to a single alternative, it ties them to a group of alternatives.

To formulate it in terms of behavior, one must first define an equivalence relation $\approx^c$ that partitions X into *revealed categories* (of alternatives linked through direct reversals).[27] For $\approx^c$ to be consistent with the natural representation described in Section 2, alternatives from the same revealed category *must*

[26] Using this observation, one can in fact characterize the framing model along the lines of the status quo models.

[27] To formalize, let $x \sim^c y$ if c displays a direct $\langle x, y\rangle$ or $\langle y, x\rangle$ reversal. And, let $x \approx^c y$ if $x \sim^c \ldots \sim^c y$.

be compared by the first rationale. Let S_1^c denote the restriction of the pairwise preference to alternatives in the same revealed categories (i.e. xS_1^cy if $x \succ^c y$ and $x \approx^c y$). Then, for S_1^c to be consistent with some TSM-representation, Theorem 5 requires $P_1^c \subseteq S_1^c \subseteq \widehat{Q_1^c}$. From the definition of $\widehat{Q_1^c}$ and Theorem 4, the last set inclusion entails a necessary condition that strengthens Exclusivity:

Categorical Exclusivity *For every pair of alternatives $\langle x, y\rangle \in X \times X$, either:*

(i) *c displays no weak $\langle x, y\rangle$ reversals; or,* **(ii)** *the relation $x \approx^c y$ does not hold.*

This effectively formulates the simple observation about the compromise model in terms of behavior: alternatives from the same category are not involved in weak reversals (and *vice versa*).

This analysis highlights how Theorem 5 may be used to supplement the recipe outlined above. For some specialized models, one must refine P_1^c to obtain a natural TSM-representation. Since Theorem 5 limits what can be added to P_1^c, it can be used to formulate special exclusivity requirements. Besides the compromise model, the willpower model described in Section 2 also requires this kind of analysis. Without getting into the details, the issue is that the first rationale of the natural TSM-representation has the added structure of a semiorder.[28]

6.2 Two-Stage Choice in Applications

In this section, I first illustrate how the results of the paper provide insights into a simple model of *monopolistic screening* before briefly discussing some other natural applications.

Monopolistic Screening

A monopolist encounters each consumer type $i \in I$ with a given probability p_i. Her objective is to design a menu of products $M \subseteq X$ to maximize expected profits. For a given menu M, *individual rationality* (IR) and *incentive compatibility* (IC) limit the monopolist's ability to sell product $m_i \in M$ to consumer type i. These conditions require i to select m_i above the outside option o as well as every product in M. In the standard setting where the choice behavior c_i of type i is represented by a utility u_i function, IR and IC impose familiar inequality constraints on the profit maximization objective:

$$u_i(m_i) \geq u_i(o) \tag{IR$_i$}$$

$$u_i(m_i) \geq u_i(m) \quad \text{for all } m \in M \tag{IC$_i$}$$

When c_i does not admit a utility representation, IR$_i$ and IC$_i$ reduce to a single constraint:

$$m_i \in c_i(M \cup \{o\}) \tag{IR-IC$_i$}$$

[28] An interval order is a rationale P such that (xPy and wPz) $\Rightarrow$ (xPz or wPy). (Letting $y = w$ shows that this condition strengthens transitivity.) In turn, a semiorder is an interval order such that $xPyPz \Rightarrow$ (xPw or wPz).

Since it precludes the use of basic optimization techniques (like those relying on convexity), this type of constraint can make it significantly more difficult to characterize the optimal menu.

When consumer behavior is TSM-representable however, the results of the paper can be leveraged to characterize key features of the optimal menu. To illustrate, suppose each consumer type c_i is TSM-represented by (P_1, P_2^i) where P_1 is fixed across consumers and P_2^i is a type-specific linear order. The idea is that the monopolist knows the "bias" P_1 of consumers but not their "true" preference P_2^i. This setup captures a private values environment where the behavior of consumers is nonetheless shaped by a common external influence, such as framing at the point of sale or information obtained from a biased media source. Unlike other models in the literature, the private information relates to consumer preferences rather than a psychological parameter (see Koszegi [2014] for a survey). Accordingly, this setup addresses a gap in our understanding of screening with boundedly rational consumers.

To fix ideas, let m_i denote the product that consumer type i selects from $M \cup \{o\}$. Then,

$$\Pi(M) \equiv \sum_{i \in I \text{ s.t. } m_i \neq o} p_i \cdot \pi(m_i) \quad - \sum_{m \in M} \epsilon(m)$$

defines the expected profits for M. In words, the monopolist obtains a marginal profit $\pi(m_i) > 0$ for each consumer type $i \in I$ that purchases a product in M and faces a menu cost $\epsilon(m) > 0$ for each product $m \in M$.[29] In the sequel, I assume that menu costs are sufficiently small that they only serve to discourage the monopolist from offering a product that does not affect consumer purchases.

In the full information benchmark where the monopolist faces a single consumer type (to whom it is profitable to sell), the optimal menu contains at most two alternatives: a *purchase product* that is chosen by the consumer; and, in some cases, a *decoy product* that prevents the consumer from choosing the outside option over the purchase product. Theorem 5 shows that this simple feature extends to the case of asymmetric information about the true preferences of consumers:

Remark 3 *If $P_1 \neq \emptyset$, the optimal menu M^* contains at most one product d that is never chosen. This product acts a decoy in the sense that $c_i(M^* \setminus \{d\} \cup \{o\}) = o$ for some consumer type(s) $i \in I$.*

This contrasts with the result in the standard setting. When consumers are *unbiased* ($P_1 = \emptyset$), every product in the optimal menu $M^*_\emptyset$ is purchased by some type. When consumers are *biased* ($P_1 \neq \emptyset$) however, it may be profitable for the monopolist to introduce a decoy. Intuitively, this allows the optimal menu M^* to violate the IR$_i$ constraint associated with the true preference of some types. Since the decoy eliminates the outside option (dP_1o), a biased consumer type (P_1, P_2^i) may purchase $m_i \in M^*$ even when her unbiased counterpart $(\emptyset, P_2^i)$ would select the outside option o.

Having said this, the effect on monopoly profits is ambiguous.[30] While bias may help with the IR$_i$ constraints of some types, it can also work against the monopolist. When xP_1m_i for some $x \in M^*_\emptyset \cup \{o\}$,

[29]When the product $m \equiv (x, q) \in \mathbb{R}^2_+$ consists of a price x and a non-price dimension q, it is conventional to assume that $\pi(m) \equiv x - c(q)$ where the cost function c is increasing, convex, and $c(0) = 0$.

[30]When M^* contains no decoy, the monopolist prefers unbiased consumers $(\emptyset, P_2^i)$. Since they choose like their biased counterparts (P_1, P_2^i) from $M^* \cup \{o\}$, the monopolist can achieve the same profits (and potentially more) with $M^*_\emptyset$.

for instance, the biased consumer type (P_1, P_2^i) cannot select the $m_i \in M_\emptyset^*$ purchased by her unbiased counterpart $(\emptyset, P_2^i)$. As a result, monopoly profits may be larger with unbiased consumers.

Perhaps not surprisingly, the effects on consumer welfare are equally ambiguous (see Example 6 of the Supplemental Appendix). Depending on the specific parametrization, a biased consumer type might be better off choosing from the menu $M_\emptyset^*$ offered to unbiased consumers. Likewise, an unbiased consumer type might be better off choosing from the menu M^* offered to biased consumers.

To emphasize that Remark 3 depends on the particular features of the TSM model, notice that the optimal menu M^* may contain *multiple* unchosen products in the more general case where P_1 fails to be transitive. Even in the full information benchmark, it may include a decoy product d_0, a product d_1 to prevent d_0 from being chosen, a product d_2 to prevent d_1 from being chosen, and so on.

Some Additional Applications

It is worth noting that Remark 3 does not depend on any assumptions about the marginal profits π, the type distribution p, or the bias P_1. This begs the following question: *what (other) features of the optimal contract can be determined non-parametrically from the results of the paper?* Indeed, the same might be asked in any contract setting where agents choose according to the TSM model. The question has bearing on recent work by Salant and Siegel [2013], who study monopolistic screening when agents choose according to the framing model. In their setting, some features of the optimal contract are independent of the type distribution. In light of Remark 3, it would be interesting to see what might be learned directly from the insights about the framing model (described in Section 6.1).

Besides contracting, implementation is another area where there have been efforts to incorporate bounded rationality. For Nash implementation, Moore and Repullo's [1990] condition μ is necessary and sufficient in the standard setting where agents are preference maximizers. Recent work shows that part of this condition remains necessary even when no restrictions are placed on agents' behavior (Korpela [2012]; and de Clippel [2014]). In general, this *Weak* μ condition (see Remark 4 of the Supplemental Appendix) can be difficult to check. When agents choose according to the TSM model however, it may be possible to check this condition more systematically. Since the model satisfies Expansion, a result of Korpela [2012, Lemma 1] effectively limits what needs to be checked. It would be interesting to see whether the results of the current paper might be used to impose further limitations.

Another natural application to implementation is dominant-strategy *house allocation*. In the standard setting, *serial dictatorship* and *top trading cycles* each implement the Pareto optimal allocations (Abdulkadiroglu and Sonmez [1998]). In recent work, Bade [2008, 2013] shows that the two mechanisms need not coincide when agents choose according to the TSM model. Based on a few simple inferences about revealed preference (related to Figure 1), she also develops some insights into both mechanisms. In light of Bade's work, it would be interesting to see whether the results of the current paper might be used to characterize the outcomes implemented by these simple mechanisms.

7 Discussion of Related Work

The TSM model is related to a variety of two-stage models proposed in the recent literature. Most closely related are the other shortlist methods discussed in Section 2. Also related are two-stage procedures where the filtering in the first stage results from something more general than preference maximization (see Tyson [2013] for an overview). Some of these procedures, such as *limited attention/consideration* (Masatlioglu et al. [2012]; Lleras et al. [2011]) generalize the TSM model but maintain the transitivity of P_2. Others, like *rationalization* (Spears [2011]; Manzini and Mariotti [2012]; and, Cherepanov et al. [2013]), also generalize the RSM model by imposing no restrictions on P_2.[31]

To conclude, I discuss the contribution of the paper in terms of the related work on these models.

7.1 Axiomatics

The TSM Model. The discussion after Theorem 2 suggests an alternative characterization of the TSM model. In lieu of Strong Exclusivity, one might require R_2^c to be acyclic. In fact, several papers follow this approach (Au and Kawai [2011]; Lleras et al. [2011]; and, Yildiz [2015]).[32] While this approach makes it easier to show the sufficiency of the axioms, it comes at the cost of a "technical" acyclicity condition. Since acyclicity conditions (like SARP) can be difficult to interpret and generally yield few insights into behavior that cannot be determined from the representation directly, they are seldom employed when more straightforward conditions (like WARP) are available.[33]

With this in mind, a key contribution of Theorem 2 is to establish that R_2^c-acyclicity is *not* essential to characterize the TSM model. Indeed, this technical condition can be replaced by one which imposes restrictions on readily observable reversals in choice. What is more, this Strong Exclusivity condition implies that the TSM model is much simpler to test than the previous characterizations might suggest. According to Theorem 7(i), it is *not* necessary to rule out R_2^c cycles of arbitrary length.

Other Models. Regarding the T_1SM model, Matsuki and Tadenuma [2013] provide a characterization using Expansion, Weak WARP, and a condition called *Elimination*. Since their Elimination condition formally captures the additional content of Exclusivity beyond Weak WARP (Remark 9 of the Supplemental Appendix), their approach complements the characterizations in Theorems 6(i) and 7(ii).

Regarding the T_2SM model, Houy's [2008] characterization requires Expansion and an acyclicity condition. Given Theorem 2, one might ask whether this model can also be characterized by Expansion and a "simple" exclusivity requirement. Since exclusivity amounts to no overlap between the revealed rationales, the issue is whether there exists a simple way to define revealed preference in the model.[34] A potential impediment is that the T_2SM model does not possess the "small menu" feature (so integral to

[31] Another generalization of the RSM model allows for more than two rationales (Apesteguia and Ballester [2013]; Manzini and Mariotti [2011]). In turn, the rigid sequential structure of the rationales can also be relaxed (Horan [2013]).

[32] Since Expansion and R_2^c-acyclicity imply Weak WARP, one of Au and Kawai's conditions is redundant.

[33] For broader discussions of the axiomatic method in decision theory, see Dekel and Lipman [2010].

[34] The issue is the first rationale. For the second rationale, the revealed preference is the same as the RSM model.

obtaining simple revealed preference definitions for the TSM model).

7.2 Identification

The TSM Model. The prior literature partly addresses some identification issues considered here.

Arguably the closest point of comparison is that several papers define binary relations which are equivalent to R_2^c or $\widehat{R_2^c}$ (Remark 5 of the Supplemental Appendix). An important difference is that none of these definitions is stated in terms of small menus. What is more, the sole purpose in two of these papers (Au and Kawai [2011]; Yildiz [2015]) is to characterize the TSM model in terms of an acyclicity condition. While a third paper (Lleras et al. [2011]) is concerned with revealed preference, the focus is not the TSM model, but rather the more general model of *limited consideration*.

Of these three papers, only Au and Kawai address the first rationale. While they identify a binary relation equivalent to P_1^c (Remark 7 of the Supplemental Appendix), their definition (which amounts to $\widehat{Q_1^c} \cap (P_2^c)^{-1}$) is not easy to interpret in terms of behavior. In fact, it gives the impression that the first rationale is difficult to determine from choice data. Though they go on to show that any transitive rationale P_1 in the range $P_1^c \subseteq P_1 \subseteq \widehat{Q_1^c}$ may be used in some TSM-representation (an implication of Theorem 5), the practical value of this result is limited by their intricate definition of P_1^c.[35]

Other Models. Dutta and Horan [2015] characterize revealed preference in the RSM model. In terms of choice reversals, they show that the first rationale is identified with direct reversals; and, the second with weak reversals where the "weak" alternative is chosen (Remark 11 of the Supplemental Appendix). Given their result, Theorem 4 shows how the added structure of the TSM model strengthens revealed preference. The basic insight is that the transitivity of *one* rationale sharpens the revealed preference of the *other*: unless P_2 (P_1) is transitive, one cannot draw any inference about P_1 (P_2) from indirect reversals (weak reversals where the "weak" alternative is not chosen).[36]

This insight extends to the T_1SM model. In that model, revealed preference is "half-way" between the RSM and TSM models (Remark 10 of the Supplemental Appendix): like the RSM model, direct reversals identify the first rationale; and, like the TSM model, weak reversals identify the second.

To close, I emphasize the key role that the "small menu" feature of the TSM model (Corollary 1) plays in identification. While the T_1SM model shares this feature, many other generalizations of the TSM model do not. In practical terms, this can make it very difficult to do identification in these models. In the RSM model, for instance, observing choices from arbitrarily large menus may be necessary to infer certain aspects of either rationale (Examples 7 and 8 of the Supplemental Appendix).

[35]While they suggest that $\widehat{Q_1^c}$ might be defined in terms of choice from menus of four or fewer alternatives (see Remark 1 of their paper), they do not actually define it this way (see Remark 6 of the Supplemental Appendix for the details).

[36]The revealed preference analysis in Section 4.1 shows exactly *where* these two inferences break down.

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A Appendix

A.1 Axiomatic Foundations

The proof of Theorem 2 relies on two results: Manzini and Mariotti's (M&M) [2007] axiomatic characterization of the RSM model; and, Dutta and Horan's (D&H) [2015] complete characterization of RSM-representations.

For the sake of convenience, both results are restated below.

Theorem 1 (M&M) *c is RSM-representable if and only if it satisfies Expansion and Weak WARP.*

Dutta and Horan (D&H) [2015] define revealed rationales for any RSM-representable choice function c:

(1) $xP_1^{RSM}y$ if $c(A) = y$ for some $A \subset X$ and $c(A \cup \{x\}) \notin \{x, y\}$; and,

(2) $xP_2^{RSM}y$ if $c(A) = x$ and $c(B) = y$ for some A, B s.t. $\{x, y\} \subseteq A \subset B$.

In turn, they use these definitions to characterize the class of RSM-representations:

Proposition (D&H) *If c is RSM-representable, then (P_1, P_2) represents c if and only if:*

(i) *P_1 is a rationale such that $P_1^{RSM} \subseteq P_1 \subseteq (\succ^c \setminus P_2^{RSM})$; and,*

(ii) *P_2 is a rationale such that $P_2 \supseteq (\succ^c \setminus P_1)$.*

The proof also relies on the following five choice properties. The first is due to Au and Kawai (A&K) [2011]:

Reduction (A&K) *If $c(A) = y$ and $c(B) = x$ for $\{x, y\} \subseteq B \subset A$, then:*

xyz is a 3-cycle s.t. $c(x, y, z) = y$ for some $z \in A \setminus B$.

The four other properties are as follows:

Selective IIA *If $c(A) = y$ and $c(x, y) = x$, then $c(A \setminus \{x\}) = y$.*

3-Acyclicity *If wxz and wyz are 3-cycles, then $c(w, x, z) = x$ if and only if $c(w, y, z) = y$.*

4-Acyclicity *If wxz and wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$, then:*

$c(x, y) = x$ implies $c(x, y, v) = v$ for any 3-cycle xyv.

5-Acyclicity *If wxz and wyz are 3-cycles s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$, then:*

$c(x, y) = x$ implies $c(x, y, v) \neq v$ for any 3-cycle xyv.

Briefly, Selective IIA is a weakening of IIA which states that any alternative chosen pairwise over $c(A)$ can be discarded without affecting choice. In turn, the other properties restrict Strong Exclusivity to 3-cycles.

A.1.1 Proof of Theorem 1

Since $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are obvious, I show $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$.

$(ii) \Rightarrow (i)$: By way of contradiction, suppose $c(A) = x$ and c violates IIA for $\{x, y\} \subseteq B \subset A$. By finiteness, there is a $\{x, y\} \subseteq D \subset A$ s.t. $c(D) = z \neq x$ and $c(D \cup \{a\}) = x$ for all $a \in A \setminus D$. Then, c displays a direct $\langle a, z \rangle$ reversal on D, which violates the assumption of no direct reversals.

$(iii) \Rightarrow (i)$: Fix any A s.t. $c(A) = x$. Then, $c(A) = x = c(x)$. It suffices to show $c(B) = x$ for all $\{x\} \subset B \subset A$. By way of contradiction, suppose $c(B) = y \neq x$ for some such B. If $c(x, y) = x$, then $c(B) = y \neq c(B \setminus \{y\})$ is a weak $\langle x, y \rangle$ reversal. If $c(x, y) = y$, then $c(A) = x \neq c(A \setminus \{x\})$ is a weak $\langle y, x \rangle$ reversal. But both possibilities violate the assumption of no weak reversals.

A.1.2 Preliminary Results

Lemma 4 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then it satisfies Selective IIA.*

Proof. Let $c(A) = y$ and $c(x, y) = x$. By way of contradiction, suppose $c(A \setminus \{x\}) = z \neq y$.

The case $|A| = 2$ cannot arise. For $|A| = 3$, it follows that $c(x, y, z) = y$, $c(x, y) = x$, and $c(y, z) = z$. If $c(x, z) = x$, then $c(x, y, z) = c(\{x, y\} \cup \{x, z\}) = x \neq y$ by Expansion, a contradiction. If $c(x, z) = z$, then $c(x, y, z) = c(\{y, z\} \cup \{x, z\}) = z \neq y$ by Expansion, another contradiction.

The proof that $c(A) = y$, $c(x, y) = x$, and $c(A \setminus \{x\}) = z \neq y$ generate a contradiction for $|A| = n \geq 4$ is by induction. From these choices, $c(x, z) = x$ and $c(y, z) = z$. If $c(x, z) = z$, then $c(A) = c(A \setminus \{x\} \cup \{x, z\}) = z \neq y$ by Expansion, a contradiction. If $c(y, z) = y$, then $c(A) = y$ implies $c(A \setminus \{x\}) \neq z$ by Weak WARP, a contradiction.

Base case: Suppose $A = \{w, x, y, z\}$. First, (i) $c(w, x) = w$. Otherwise, $c(w, x, y, z) = c(\{x, y\} \cup \{x, z\} \cup \{x, w\}) = x \neq y$ by Expansion, a contradiction. Next, (ii) $c(y, w) = y$. Otherwise, $c(w, x, y, z) = c(\{a, y\} \cup \{w, x, z\}) = a \neq y$ by Expansion where $c(w, x, z) \equiv a$, a contradiction. Finally, (iii) $c(w, x, y) = y$. If $c(w, x, y) = x$, then $c(w, x, y, z) = c(\{x, z\} \cup \{w, x, y\}) = x \neq y$ by Expansion. If $c(w, x, y) = w$, then $c(w, x, y, z) = y = c(y, w)$ violates Weak WARP.

By (i)-(ii), wxy is a 3-cycle. If $c(w, z) = w$, wzy is a 3-cycle. Given (iii), $c(A \setminus \{x\}) = c(w, y, z) \neq z$ by 3-Acyclicity, which is a contradiction. If $c(w, z) = z$, wxz is a 3-cycle. Given (iii), $c(w, x, z) = z$ by 3-Acyclicity. So, $c(A) = c(\{y, z\} \cup \{w, x, z\}) = c(w, x, y, z) = z \neq y$ by Expansion, which is another contradiction.

Induction Step: Suppose Selective IIA holds for $|A| = n$ but some A s.t. $|A| = n + 1$ violates it. First, suppose $c(A \setminus \{a'\}) = b = c(A \setminus \{a''\})$ for distinct $a', a'' \in A \setminus \{b\}$ and $b \in A \setminus \{y\}$. By Expansion, $c(A) = c([A \setminus \{a'\}] \cup [A \setminus \{a''\}]) = b \neq y$, which is a contradiction. So, (i) for all $b \in A \setminus \{y\}$, $c(A \setminus \{a\}) = b$ for at most one $a \in A \setminus \{b\}$. Next, consider any $a \in A \setminus \{y\}$ s.t. $c(A \setminus \{a\}) = y$. (By (i) and the pigeonhole principle, it must be that $c(A \setminus \{a\}) = y$ for some $a \in A \setminus \{y\}$.) Since $c(A \setminus \{x\}) = z$, $a \neq x$. By the induction hypothesis, $c(x, y) = x$ implies $c(A \setminus \{x, a\}) = y$. If $a \neq z$, then $c(A \setminus \{x\}) = z = c(y, z)$ and $c(A \setminus \{x, a\}) = y$, which contradicts Weak WARP. So, (ii) $a = z$. Together, (i) and (ii) establish: (iii) for all $b \in A$, $c(A \setminus \{a\}) = b$ for exactly one $a \in A \setminus \{b\}$; and, (iv) $c(A \setminus \{z\}) = y$.

By Weak WARP, $c(A \setminus \{a\}) = b$ for $a, b \neq y$ implies $c(b, y) = y$. Setting $w \equiv c(A \setminus \{y\})$, (iii) implies (v) $c(b, y) = y$ for all $b \in A \setminus \{y, w\}$. Moreover, (vi) $c(y, w) = y$. Otherwise, $c(A) = c([A \setminus \{y\}] \cup \{y, w\}) = w \neq y$ by Expansion, which is a contradiction. By the induction hypothesis, (iv)-(vi) imply (vii) $c(w, y, b) = y$ for all $b \in A \setminus \{y, w, z\}$. Finally, fix a $b \in A \setminus \{y, w, z\}$ s.t. $c(A \setminus \{a\}) = b$ and $a \neq y, w$. Since $|A| \geq 5$, there exists such a b. (In particular, the only $\langle b, a \rangle$ pairs ruled out are: $\langle y, z \rangle$, $\langle w, y \rangle$, $\langle z, x \rangle$, and $\langle v, w \rangle$ where $v \equiv c(A \setminus \{w\})$.) Since $c(A \setminus \{a\}) = b = c(b, y)$ and $\{b, y\} \subset \{b, w, y\} \subset A \setminus \{a\}$, (vii) contradicts Weak WARP, which establishes the induction step. ■

Lemma 5 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then it satisfies Reduction.*

Proof. Suppose $c(A) = y$ and $c(B) = x$ for distinct $x, y \in B \subset A$. Then, $c(x, y) = x$. Otherwise, $c(A) = y = c(x, y)$ and $c(B) = x$ violate Weak WARP. First, define $L \equiv \{l \in A \setminus B : c(x, l) = x\}$ and $B' \equiv B \cup L$. By Expansion, $c(B') = c(B \bigcup_{l \in L} \{x, l\}) = x$. Next, define $W \equiv \{w \in A \setminus B' : c(y, w) = w\}$ and $A' \equiv A \setminus W$. By Selective IIA, $c(A') = y$. If $B' = A'$, then $x = c(A') = y$. So, $B' \subset A'$. By construction, xyz is a 3-cycle for any $z \in A' \setminus B'$. Moreover, $c(x, y, z) \neq z$. Otherwise, $c(A) = y = c(y, z)$ and $c(x, y, z) = z$ violate Weak WARP. If $c(x, y, z) = x$ for all $z \in A' \setminus B'$, then $c(A') = c(B' \bigcup_{z \in A' \setminus B'} \{x, y, z\}) = x \neq y$. So, $c(x, y, z) = y$ for some $z \in A' \setminus B'$. ■

Lemma 6 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then: $x\widehat{R_2^c}y$ iff c displays a weak $\langle x, y \rangle$ reversal.*

Proof. (⇒) From Definition 2: (i) there exists a 3-cycle xyz s.t. $c(x, y, z) \neq z$; or, (ii) $c(x, y) = x$ and there exist 3-cycles wxz, wyz s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$. In case (i), c displays a weak $\langle x, y \rangle$ reversal on $\{x, y, z\}$. In case (ii), Expansion implies $c(w, x, y, z) = c(\{w, y, z\} \cup \{w, x\}) = w$. So, c displays a weak $\langle x, y \rangle$ on $\{w, x, y, z\}$.

(⇐) By definition, $c(x,y)=x$ and $c(A)\neq c(A\setminus\{y\})$ for some $A\supset\{x,y\}$. If $c(x,y,z)\neq z$ for some 3-cycle xyz, then $x\widehat{R_2^c}y$ follows by definition. So, suppose $c(x,y,z)=z$ for every 3-cycle xyz.

By Lemma 5, c satisfies Reduction. First, suppose $c(A)=x$. Since $c(A\setminus\{y\})=b\neq x$, Reduction implies xyb is a 3-cycle s.t $c(b,x,y)=x$. But this contradicts $c(x,y,z)=z$ for any 3-cycle xyz. Next, suppose $c(A)=y$. Since $c(x,y)=x$, Reduction implies there exists a 3-cycle xyz with $z\in A\setminus\{x,y\}$ s.t. $c(x,y,z)=y$. This also contradicts $c(x,y,z)=z$ for any 3-cycle xyz. So, $c(A)=a\neq x,y$. Moreover, $c(A\setminus\{y\})\neq x$. Otherwise, Expansion requires $c(A)=c(\{x,y\}\cup(A\setminus\{y\}))=x\neq a$. So, $c(A\setminus\{y\})\equiv b\neq x$. Since the elements of $\{a,b,x,y\}$ are distinct, $|A|\geq 4$.

Next, observe: **(I)** bay is a 3-cycle s.t $c(a,b,y)=a$; and, **(II)** bdx is a 3-cycle s.t $c(b,d,x)=b$ for $d\in A\setminus\{b,x,y\}$.

Proof of (I): By Reduction on $c(A)=a$ and $c(A\setminus\{y\})=b$. □

Proof of (II): If $c(b,x)=x$, Reduction implies the desired result since $c(A\setminus y)=b$ and $c(b,x)=x$. To see that $c(b,x)=x$, suppose otherwise. If $c(b,x)=b$, xyb is a 3-cycle. Since $c(b,y)=y$ by (I) and $c(x,y)=x$, $c(b,x)=b$ by assumption, xyb is a 3-cycle. By assumption, $c(b,x,y)=b$. By Expansion, $c(A)=c(\{b,x,y\}\cup[A\setminus\{y\}])=b\neq a$, which is a contradiction. □

To complete the proof, there are three cases to consider: (1) $c(x,a)=a$; (2) $c(x,a)=x$, $c(y,d)=d$; and, (3) $c(x,a)=x$, $c(y,d)=y$. (If $|A|=4$, case (1) obtains automatically because $d=a$.) I show that cases (1) and (2) imply $x\widehat{R_2^c}y$ (under branch (ii) of the definition) while case (3) implies a contradiction.

(1) Since $c(a,b)=b$ and $c(b,x)=x$ by (I) and (II), bax is a 3-cycle. Moreover, $c(a,b,x)=b$. To see this, note that $c(a,b,x)=x$ contradicts Weak WARP since $c(a,x)=a=c(A)$. Similarly, $c(a,b,x)=a$ contradicts Weak WARP since $c(a,b)=b=c(A\setminus\{y\})$. Combined with (I), this gives $x\widehat{R_2^c}y$ with $w=a$ and $z=b$. **(2)** Since $c(b,y)=y$ and $c(b,d)=b$ by (I) and (II), bdy is a 3-cycle. Since bay is a 3-cycle s.t. $c(a,b,y)=a$ by (I), 3-Acyclicity implies $c(b,d,y)=d$. Combined with (II), this gives $x\widehat{R_2^c}y$ with $w=d$ and $z=b$. **(3)** Since $c(x,y)=x$ and $c(d,x)=d$ by (II), xyd is a 3-cycle. Since xbd is a 3-cycle and $c(b,x,d)=b$ by (II), 3-Acyclicity implies $c(x,y,d)=y$. But, this contradicts the assumption that $c(x,y,z)=z$ for any 3-cycle xyz. ■

Lemma 7 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then: $x\widehat{Q_1^c}y$ iff $c(A)=c(A\setminus\{y\})$ for all $A\supset\{x,y\}$.*

Proof. Since $\widehat{Q_1^c}\equiv(\succ^c\setminus\widehat{R_2^c})$ by definition, the result follows immediately from Lemma 6. ■

Lemma 8 *If c satisfies Weak WARP, Expansion, and $\{3,4,5\}$-Acyclicity, then R_2^c is acyclic.*[37]

Proof. Suppose there exists an R_2^c-cycle $x_0...x_i...x_{n-1}$ (i.e. $x_0R_2^c...R_2^cx_iR_2^c...R_2^cx_{n-1}R_2^cx_0$). Notice that R_2^c is asymmetric by definition. Thus, $x_0R_2^cx_0$ and $x_0R_2^cx_1R_2^cx_0$ both entail a contradiction. The proof of a contradiction for $n\geq 3$ is by strong induction on the length of the R_2^c-cycle.

Base case $n=3$**:** Note that $x_0x_1x_2$ is a 3-cycle. Without loss of generality, suppose $c(x_0,x_1,x_2)=x_0$. Then, $x_2R_2^cx_0R_2^cx_1$ by definition of R_2^c. Since $x_1R_2^cx_2$, there exists an $a\notin\{x_0,x_1,x_2\}$ s.t. ax_1x_2 is a 3-cycle and $c(a,x_1,x_2)\neq a$. But, by 3-Acyclicity, this is a contradicts $c(x_0,x_1,x_2)=x_0$.

Induction step: Without loss of generality, suppose $c(x_0,...,x_{n-1})=x_0$. First, observe that:

(I) $x_{i-1}x_ix_{i+1}$ is a 3-cycle for $i\neq 1,2$ [to be understood (mod n) when $i=0,n-1$];

(II) there are no other 3-cycles $x_jx_ix_k$; and,

(III) $c(x_{i-1},x_i,x_{i+1})=x_i$.

These observations, in turn, follow from: **(i)** $c(x_0,x_2)=x_0$; and, **(ii)** $c(x_1,x_3)=x_1$.

Proof of (i): By way of contradiction, suppose $c(x_0,x_2)=x_2$. Then, $x_0x_1x_2$ is a 3-cycle. From the base case of the induction, $c(x_0,x_1,x_2)=x_1$. Otherwise, $x_2R_2^cx_0$ so that $x_0x_1x_2$ is an R_2^c-cycle of length 3. Since $\{x_0,x_1\}\subset\{x_0,x_1,x_2\}\subset$

[37] The notation I-Acyclicity for the index set I refers to i-Acyclicity for all $i\in I$.

$\{x_0, ..., x_{n-1}\}$ and $c(x_0, x_1) = x_0 = c(x_0, ..., x_{n-1})$ however, $c(x_0, x_1, x_2) = x_1$ violates Weak WARP. This is the desired contradiction. □

To establish **(ii)**, first observe that, for $n \geq 5$, **(i)** implies:

$$c(x_i, x_j) = x_i \text{ for } [i = 0 \text{ and } 2 \leq j \leq n-2] \text{ or } [i \geq 4 \text{ and } 2 \leq j \leq i-2] \quad (1)$$

To see this, first observe that $c(x_0, x_2) = x_0$ implies $c(x_0, x_3) = x_0$. Otherwise, $x_0x_2x_3$ is a 3-cycle so that $x_0R_2^cx_2$ or $x_3R_2^cx_0$. Both contradict the induction hypothesis: either $x_0x_2...x_{n-1}$ is an R_2^c-cycle of length $n-1$; or, $x_0x_1x_2x_3$ is an R_2^c-cycle of length 4. By a simple induction argument, it then follows that $c(x_0, x_j) = x_0$ for any j s.t. $2 \leq j \leq n-2$.

By the same kind of reasoning as the last paragraph, $c(x_0, x_2) = x_0$ also implies $c(x_{n-1}, x_2) = x_{n-1}$. By a simple induction argument, it then follows that $c(x_i, x_2) = x_i$ for any $i \geq 4$. Applied to each $i \geq 4$, the same kind of induction argument gives $c(x_i, x_j) = x_i$ for any $2 \leq j \leq i-2$.

Proof of (ii): Suppose $c(x_1, x_3) = x_3$. Consider the cases $n = 4$, $n = 5$, and $n > 5$ separately:

$n = 4$: Since $c(x_0, x_2) = x_0$, $x_2x_3x_0$ is a 3-cycle. From the base case, $c(x_0, x_2, x_3) = x_3$. Otherwise, $x_0R_2^cx_2$ so that $x_0x_2x_3$ is an R_2^c-cycle of length 3. Since $c(x_1, x_3) = x_3$, Expansion implies $c(x_0, x_1, x_2, x_3) = c(\{x_0, x_2, x_3\} \cup \{x_1, x_3\}) = x_3 \neq x_0$, a contradiction. So, $c(x_1, x_3) = x_1$.

For $n \geq 5$, $x_{n-1}x_0x_1$ is a 3-cycle s.t. $c(x_0, x_1, x_{n-1}) = x_0$. To see that $c(x_1, x_{n-1}) = x_1$, suppose otherwise. Since $c(x_i, x_{n-1}) = x_{n-1}$ for $i \neq 1$ by observation (1) and $c(x_1, x_{n-1}) = x_{n-1}$ by assumption, repeated application of Expansion on $\{x_1, x_{n-1}\}$ implies $c(x_0, ..., x_{n-1}) = x_{n-1}$. Since this contradicts $c(x_0, ..., x_{n-1}) = x_0$, $c(x_1, x_{n-1}) = x_1$. By the base case, $c(x_0, x_1, x_{n-1}) = x_0$. Otherwise, $x_1R_2^cx_{n-1}$ so that $x_{n-1}x_0x_1$ is an R_2^c-cycle of length 3.

$n = 5$: Since $c(x_4, x_2) = x_4$ by (1), $x_2x_3x_4$ is a 3-cycle. By the base case, $c(x_2, x_3, x_4) = x_3$. Otherwise, $x_2x_3x_4$ is an R_2^c-cycle of length 3. Since $c(x_1, x_3) = x_3$ by assumption, $x_1x_2x_3$ is a 3-cycle. By the base case, $c(x_1, x_2, x_3) = x_2$. Otherwise, $x_3R_2^cx_1$ so that $x_1x_2x_3$ is an R_2^c-cycle of length 3. Since $c(x_1, x_2, x_3) = x_2$, $c(x_2, x_3, x_4) = x_3$, and $c(x_1, x_4) = x_1$, 5-Acyclicity implies $c(x_1, x_4, v) \neq v$ for any 3-cycle x_1x_4v. Since this contradicts $c(x_0, x_1, x_4) = x_0$, $c(x_1, x_3) = x_1$.

$n > 5$: Since $c(x_3, x_{n-1}) = x_{n-1}$ by (1), $x_1x_{n-1}x_3$ is a 3-cycle. Since $x_0x_1x_{n-1}$ is a 3-cycle s.t. $c(x_0, x_1, x_{n-1}) = x_0$, 3-Acyclicity implies $c(x_1, x_3, x_{n-1}) = x_3$ so that $x_3R_2^cx_1$ and $x_1x_2x_3$ is an R_2^c-cycle of length 3. Since this contradicts the base case, $c(x_1, x_3) = x_1$. □

For $n = 4$, (i)-(ii) and $c(x_i, x_{i+1}) = x_i$ directly imply (I)-(III). For $n \geq 5$, (ii) implies $c(x_1, x_i) = x_1$ for any $i \neq 0, 1$ by a simple induction argument along the same lines as (1). Together with (1) and the fact that $c(x_i, x_{i+1}) = x_i$, this establishes (I)-(III) for $n \geq 5$. Since choice from all pairs $\{x_i, x_j\}$ are identified, (I) and (II) are immediate. In turn, (III) follows from the base case. If $c(x_{i-1}, x_i, x_{i+1}) \neq x_i$, $x_{i+1}R_2^cx_{i-1}$ so that $x_{i-1}x_ix_{i+1}$ is an R_2^c-cycle of length 3.

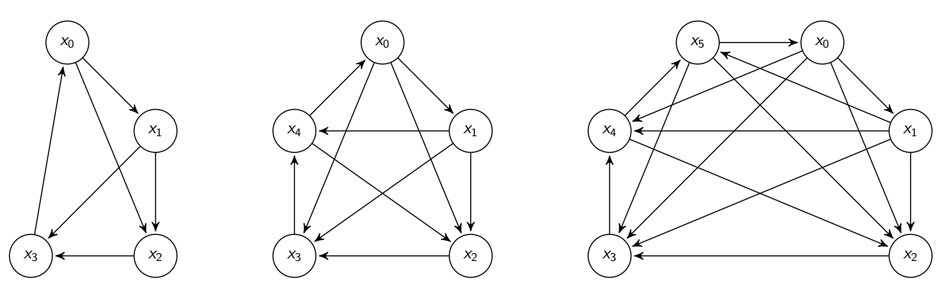

Figure 3: Preference pairs for the cases $n = 4$, $n = 5$, and $n = 6$.

(I) and (II) imply that there exists no 3-cycle $x_1x_2x_i$. Since $x_1R_2^cx_2$, there exists an $a \notin \{x_0, ..., x_{n-1}\}$ s.t. ax_1x_2 is a 3-cycle and $c(a, x_1, x_2) \neq a$. Consider any such a.

To complete the proof, I show that each of the cases $n=4$, $n=5$, and $n>5$ entails a contradiction.

$n=4$: Here, $x_0x_1x_3$ and $x_0x_2x_3$ are 3-cycles s.t. $c(x_0,x_1,x_3)=x_0$ and $c(x_0,x_2,x_3)=x_3$ (see Figure 3). Since $c(x_1,x_2)=x_1$, 4-Acyclicity implies $c(x_1,x_2,v)=v$ for any 3-cycle x_1x_2v. But, this contradicts $c(a,x_1,x_2)\neq a$.

$n=5$: Since $c(x_1,x_4)=x_1$ and $c(x_2,x_4)=x_4$, $ax_1x_4x_2$ is a 4-cycle (see Figure 3). There are two cases to consider: (5.1) $c(x_4,a)=x_4$; and, (5.2) $c(x_4,a)=a$. **(5.1)** Here, x_1x_4a is a 3-cycle. Since $x_0x_1x_4$ is a 3-cycle s.t. $c(x_0,x_1,x_4)=x_0$, $c(a,x_1,x_4)=a$ by 3-Acyclicity. So, $c(a,x_1,x_2)\neq x_2$ by 3-Acyclicity. Since $c(a,x_1,x_2)\neq a$ by assumption, $c(a,x_1,x_2)=x_1$. So, x_1x_4a and x_1x_2a are 3-cycles s.t. $c(a,x_1,x_4)=a$ and $c(a,x_1,x_2)=x_1$. Since $c(x_2,x_4)=x_4$, 5-Acyclicity implies $c(x_2,x_4,v)\neq v$ for any 3-cycle x_4x_2v. But, this contradicts the fact that $x_2x_3x_4$ is a 3-cycle s.t. $c(x_2,x_3,x_4)=x_3$. **(5.2)** By reasoning similar to (5.1), ax_1x_2 and ax_4x_2 are 3-cycles s.t. $c(a,x_1,x_2)=x_2$ and $c(a,x_2,x_4)=a$. Since $c(x_1,x_4)=x_1$, 5-Acyclicity implies $c(x_1,x_4,v)\neq v$ for any 3-cycle x_1x_4v. But, this contradicts $c(x_0,x_1,x_4)=x_0$.

$n>5$: First, observe that (n.1) $c(a,x_{n-1})=x_{n-1}$ and (n.2) $c(a,x_{n-2})=a$ (see Figure 3). **(n.1)** Here, $ax_{n-1}x_2$ is a 3-cycle. By the induction hypothesis, $c(a,x_2,x_{n-1})=a$. Otherwise $x_{n-1}R_2^c x_2$ so that $x_2...x_{n-1}$ is an R_2^c-cycle of length $n-2$. Since $c(a,x_2,x_{n-1})=a$, $x_2R_2^c aR_2^c x_{n-1}$. So, $x_0x_1x_2ax_{n-1}$ is an R_2^c-cycle of length 5, which contradicts the induction hypothesis. **(n.2)** The reasoning is similar to (n.1). Here, ax_1x_{n-2} is a 3-cycle. By the induction hypothesis, $c(a,x_1,x_{n-2})=a$. Otherwise $x_1R_2^c x_{n-2}$ so that $x_{n-2}...x_1$ is an R_2^c-cycle of length 4. Since $c(a,x_1,x_{n-2})=a$, $x_{n-2}R_2^c aR_2^c x_1$ so that $x_1...x_{n-2}a$ is an R_2^c-cycle of length $n-1$, which contradicts the induction hypothesis.

Given (n.1) and (n.2), ax_1x_{n-1} and $ax_{n-2}x_2$ are 3-cycles. Since $x_{n-1}x_0x_1$ is a 3-cycle s.t. $c(x_{n-1},x_0,x_1)=x_0$, $c(x_{n-1},a,x_1)=a$ by 3-Acyclicity. So, $c(a,x_1,x_2)\neq x_2$ by 3-Acyclicity. Since $c(a,x_1,x_2)\neq a$ by assumption, $c(a,x_1,x_2)=x_1$. So, $c(a,x_2,x_{n-2})=x_{n-2}$ by 3-Acyclicity. Then, $x_{n-2}R_2^c x_2$ so that $x_2...x_{n-2}$ is an R_2^c-cycle of length $n-3$, contradicting the induction hypothesis. ■

A.1.3 The Exclusivity Axioms (and Proof of Lemma 3)

Proof of Lemma 3. ($\Rightarrow$) By way of contradiction, suppose $c(A)=x=c(x,y)$ and $c(B)=y$ for $\{x,y\}\subset B\subset A$. Given $c(A)=x$, Weak Exclusivity ensures that (o) c displays no direct $\langle w,x\rangle$ reversals for any $w\in A$ s.t. $c(x,w)=w$. To derive a contradiction, first define $L\equiv\{l\in B: c(x,l)=x\}$ and $B_1\equiv\{x\}\cup L$. Then, $c(B_1)=c(\bigcup_{l\in L}\{x,l\})=x$ by Expansion. Moreover, $c(x,w)=w$ for all $w\in B\setminus B_1$. Given (o), $c(B_1\cup\{w\})\in\{x,w\}$ for all $w\in B\setminus B_1$. Next, define the following menus by recursion: $W_i\equiv\{w\in B\setminus B_i: c(B_i\cup\{w\})=x\}$; and, $B_{i+1}\equiv B_i\cup W_i$. By a simple induction using Expansion:

$$c(B_i)=c(\bigcup_{w\in W_{i-1}}[B_{i-1}\cup\{w\}])=x \quad \text{for all } i\geq 1.$$

Since $B\setminus B_1$ is finite, $B_{i+1}=B_i$ for some $i\geq 1$. Let i^* be the smallest such i. There are two cases. If $B_{i^*}=B$, then $c(B)=x$, which contradicts $c(B)=y$. If $B_{i^*}\subset B$, then (o) implies $c(B_{i^*}\cup\{w\})=w$ for all $w\in B\setminus B_{i^*}$. Define $w^*\equiv c(B\setminus B_{i^*})$. By Expansion, $c(B)=c([B_{i^*}\cup\{w^*\}]\cup[B\setminus B_{i^*}])=w^*$, which contradicts $c(B)=y$. Since there are contradictions in both cases, the result follows.

($\Leftarrow$) By way of contradiction, suppose: (i) $c(B)=y$ for some $B\supseteq\{x,y\}$ s.t. $c(x,y)=x$; and, (ii) c displays a direct $\langle x,y\rangle$ reversal on D so that $c(D)=y$ and $c(D\cup\{x\})=z\notin\{x,y\}$. By Expansion, $c(B\cup D)=y$. Since $c(D\cup\{x\})=z$ and $c(B\cup D)=y$, Weak WARP implies $c(y,z)=z$. But this contradicts Weak WARP given (ii). ■

Lemma 9 *If c satisfies Exclusivity, then it satisfies 3-Acyclicity.*

Proof. By contradiction, suppose wxz, wyz are 3-cycles s.t. $c(w,x,z)=x$ and $c(w,y,z)\neq y$. Contradicting Exclusivity, c displays: a direct $\langle z,w\rangle$ reversal on $\{w,x\}$; and, a weak $\langle z,w\rangle$ reversal on $\{w,y,z\}$. ■

Lemma 10 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then: it satisfies 5-Acyclicity iff it satisfies Exclusivity.*

Proof. ($\Rightarrow$) By way of contradiction, suppose: (a) $c(A) = y$ and $c(A \cup \{x\}) = v \notin \{x, y\}$; and, (b) $c(B) \neq c(B \setminus \{y\})$ for some $B \supseteq \{x, y\}$. By Lemma 5, c satisfies Reduction. By Reduction, (a) implies (a$'$) xyv is a 3-cycle s.t. $c(x, y, v) = v$. By Lemma 6, (b) implies $x\widehat{R_2^c}y$. There are two cases: (i) there is a 3-cycle xyz s.t. $c(x, y, z) \neq z$; or, (ii) there are 3-cycles wxz and wyz s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$. Given (a$'$), (i) contradicts 3-Acyclicity while (ii) contradicts 5-Acyclicity. ($\Leftarrow$)[38] Suppose wxz and wyz are 3-cycles s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$. By Expansion, $c(w, x, y, z) = c(\{w, y, z\} \cup \{w, x\}) = w$. So, c displays a weak $\langle x, y\rangle$ reversal on $\{w, x, y, z\}$. By Exclusivity, c displays no direct $\langle x, y\rangle$ reversals. Now, consider any 3-cycle xyv. Then, $c(x, y, v) \neq v$ as required. ■

Lemma 11 *If c satisfies Weak WARP, Expansion, and $\{3, 5\}$-Acyclicity, then it satisfies 4-Acyclicity iff the following:*

If c displays an indirect $\langle x, y\rangle$ reversal, then c displays no weak $\langle x, y\rangle$ reversals.

Proof. ($\Rightarrow$) By way of contradiction, suppose c displays both indirect and weak $\langle x, y\rangle$ reversals. In particular: (i) $c(B) = y$, $c(B \cup \{x\}) = x$, $c(B \cup \{w\}) = z \notin \{y, w\}$ and $c(B \cup \{w, x\}) = w$ for some $B \subseteq X \setminus \{w, x\}$; and, (ii) $c(x, y) = x$ and $c(D) \neq c(D \setminus \{y\}$ for some $D \supset \{x, y\}$. By Reduction, (i) implies that wxz and wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$. So, (i$'$) $yR_2^c zR_2^c wR_2^c x$. By Lemma 6, (ii) implies (ii$'$) $x\widehat{R_2^c}y$. By (i$'$)-(ii$'$), R_2^c contains a cycle, which contradicts Lemma 8. ($\Leftarrow$)[39] Suppose wxz and wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$. Then, $c(w, x, y, z) = c(\{w, x, z\} \cup \{w, x\}) = w$ by Expansion. So, c displays an indirect $\langle x, y\rangle$ reversal on $\{y, z\}$. By the stated property, c displays no weak $\langle x, y\rangle$ reversals. Now, consider any 3-cycle xyv. If $c(x, y, v) \neq v$, then c displays a weak $\langle x, y\rangle$ reversal on $\{v, x, y\}$. So, $c(x, y, v) = v$. ■

Lemma 12 *If c satisfies Expansion, then:*
(i) it satisfies $\{3, 5\}$-Acyclicity and Weak WARP iff it satisfies Exclusivity; and,
(ii) it satisfies $\{3, 4, 5\}$-Acyclicity and Weak WARP iff it satisfies Strong Exclusivity.

Proof. (i) This follows from Lemmas 3, 9, and 10. **(ii)** Given (i), this follows from Lemma 11. ■

A.1.4 Proof of Lemmas 1 and 2

Lemma 13 *If c satisfies Weak WARP, Expansion, and $\{3, 5\}$-Acyclicity, then: $\widehat{Q_1^c} \supseteq P_1^{RSM}$ and $\widehat{R_2^c} \supseteq P_2^{RSM}$.*

Proof. First, observe that c satisfies Reduction by Lemma 5 and Exclusivity by Lemma 10.

$\widehat{Q_1^c} \supseteq P_1^{RSM}$: If $xP_1^{RSM}y$, then $c(A \cup \{x\}) \notin \{x, y\}$ for some A s.t. $c(A) = y$. Thus, c displays a direct $\langle x, y\rangle$ reversal. So, c displays no weak $\langle x, y\rangle$ reversals by Exclusivity. Since $c(x, y) = x$ by Expansion, $c(B) = c(B \setminus \{y\})$ for all $B \supseteq \{x, y\}$. Then, $x\widehat{Q_1^c}y$ by Lemma 7.

$\widehat{R_2^c} \supseteq P_2^{RSM}$: If $xP_2^{RSM}y$, then $c(A) = x$ and $c(B) = y$ for some A and B s.t. $B \supset A \supseteq \{x, y\}$. By Reduction, xyz is a 3-cycle s.t. $c(x, y, z) = y$ for some $z \in B \setminus A$. By definition, $x\widehat{R_2^c}y$. ■

Lemma 14 *If (P_1, P_2) RSM-represents c with P_2 acyclic, then:*
(i) (P_1, P_2^*) RSM-represents c where $P_2^* \equiv tc(P_2)$; and,
(ii) (P_1, P_2^{**}) RSM-represents c where P_2^{**} is a completion of P_2^* (as guaranteed to exist by the Szpilrajn theorem).

[38] In fact, the proof given here only depends on the fact that c satisfies Expansion.
[39] In fact, the proof given here only depends on the fact that c satisfies Expansion.

Proof. (i) Consider any $A \subseteq X$ and suppose $c_{(P_1,P_2)}(A) = x$. Let $B \equiv \max(A; P_1)$. Since $\max(B; P_2) = x$, $\neg[bP_2x]$ for all $b \in B \setminus \{x\}$. Hence, $\neg[bP_2^*x]$ for all $b \in B \setminus \{x\}$. So, $x \in \max(B; P_2^*)$. Since $P_2 \subseteq P_2^*$, $\max(B; P_2^*) \subseteq \max(B; P_2)$. And, since $\max(B; P_2) = x$, $\max(B; P_2^*) = x$. So, $c_{(P_1,P_2^*)}(A) = x$. **(ii)** The reasoning is identical with P_2^* in place of P_2 and P_2^{**} in place of P_2^*. ■

Proof of Lemma 2. (⇒) Suppose the following about c: (i) it is RSM-representable; and, (ii) it satisfies Strong Exclusivity. Given (i), c satisfies Weak WARP and Expansion by Theorem 1 of M&M. Given (ii), c satisfies Exclusivity and, hence, $\{3, 5\}$-Acyclicity by Lemma 12(i). Likewise, c satisfies $\{3, 4, 5\}$-Acyclicity by Lemma 12(ii).

For the result, I show that: (a) $\widehat{Q_1^c}$ is transitive given assumption (i) and Exclusivity; (b) $(\widehat{Q_1^c}, \widehat{R_2^c})$ RSM-represents c given assumption (i) and Exclusivity; and, (c) $(\widehat{Q_1^c}, P_2^c)$ TSM-represents c given assumptions (i) and (ii).

(a) By Lemma 7, $a\widehat{Q_1^c}b$ iff $c(A) = c(A \setminus \{b\})$ for all $A \supseteq \{a, b\}$. So, suppose $x\widehat{Q_1^c}y\widehat{Q_1^c}z$ and fix $A \supseteq \{x, z\}$. Then, $c(A) = c(A \cup \{y\}) = c(A \cup \{y\} \setminus \{z\}) = c(A \setminus \{z\})$ so $x\widehat{Q_1^c}z$.[40] **(b)** By Lemma 13, $\widehat{Q_1^c} \supseteq P_1^{RSM}$ and $\widehat{R_2^c} \supseteq P_2^{RSM}$. Moreover, $\widehat{Q_1^c} \equiv \succ^c \setminus \widehat{R_2^c}$ and $\widehat{R_2^c} \subseteq \succ^c$. So: $P_1^{RSM} \subseteq \widehat{Q_1^c} \subseteq (\succ^c \setminus P_2^{RSM})$; and, $\widehat{R_2^c}$ is a rationale s.t. $(\succ^c \setminus \widehat{Q_1^c}) \subseteq \widehat{R_2^c}$. By the Proposition of D&H, $(\widehat{Q_1^c}, \widehat{R_2^c})$ RSM-represents c. **(c)** By Lemma 8, R_2^c is acyclic. So, $\widehat{R_2^c}$ is as well. Moreover, $tc(\widehat{R_2^c}) = tc(R_2^c) \equiv P_2^c$. Given (b), $(\widehat{Q_1^c}, P_2^c)$ TSM-represents c by Lemma 14.

(⇐) Suppose c is TSM-represented by (P_1, P_2). By Theorem 1 of M&M, c satisfies Weak WARP and Expansion. Then, by Lemma 12, it suffices to show $\{3, 4, 5\}$-Acyclicity:

3-Acyclicity: By way of contradiction, suppose $c(w, x, z) = x$ and $c(w, y, z) \neq y$ for 3-cycles wxz and wyz. Since $c(w, x, z) = x$, zP_1w. So, $c(w, y, z) \neq w$. Since $c(w, y, z) \neq y$, $c(w, y, z) = z$. So, wP_1y. Since zP_1wP_1y, zP_1y by transitivity of P_1. But, this contradicts $c(y, z) = y$.

5-Acyclicity: Suppose wxz and wyz are 3-cycles s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$. Then, wP_1x and yP_1z. By way of contradiction, suppose $c(x, y, v) = v$ for some 3-cycle xyv. So, xP_1y. Since $wP_1xP_1yP_1z$, wP_1z by transitivity of P_1. But, this contradicts $c(w, z) = z$.

4-Acyclicity: Suppose wxz and wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$. Then, $yP_2zP_2wP_2x$. By transitivity of P_2, yP_2x. By way of contradiction, suppose $c(x, y, v) \neq v$ for some 3-cycle xyv. Then, xP_2y, which contradicts yP_2x by the asymmetry of P_2. ■

Inspection of the proof above shows that it also establishes Lemma 1:

Proof of Lemma 1. (⇒) Points (a) and (b) in the proof of Lemma 2 establish sufficiency. **(⇐)** By Lemma 12(i), it suffices to show that c satisfies $\{3, 5\}$-Acyclicity. The proof of these properties is the same as in the proof of Lemma 2 (since the proofs only rely on the transitivity of P_1). ■

A.2 Identification

A.2.1 Proof of Corollary 1

Proof of Corollary 1. (⇒) Trivial. **(⇐)** By Lemma 1, $(\widehat{Q_1^c}, \widehat{R_2^c})$ T_1SM-represents c. By definition, $\widehat{Q_1^c}$ and $\widehat{R_2^c}$ are determined by choice on pairs and 3-cycles. Since c and $\tilde{c}$ coincide on pairs and 3-cycles, $(\widehat{Q_1^c}, \widehat{R_2^c})$ also T_1SM-represents $\tilde{c}$. Thus, $c(A) = c_{(\widehat{Q_1^c}, \widehat{R_2^c})}(A) = \tilde{c}(A)$ for all $A \subseteq X$. ■

Lemma 15 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then $Q_1^c \subseteq \widehat{Q_1^c}$.*

Proof. Since $\widehat{Q_1^c} \equiv (\succ^c \setminus \widehat{R_2^c})$ and $\widehat{R_2^c} \subseteq P_2^c$, $(\succ^c \setminus P_2^c) \subseteq \widehat{Q_1^c}$. So, $Q_1^c \equiv tc(\succ^c \setminus P_2^c) \subseteq tc(\widehat{Q_1^c})$. Since $\widehat{Q_1^c}$ is transitive by Lemma 2, $tc(\widehat{Q_1^c}) = \widehat{Q_1^c}$. Consequently, $Q_1^c \subseteq \widehat{Q_1^c}$ as required. ■

[40] The same argument is found in the proof of A&K's Theorem 1.

Example 1 *Consider a choice function c on $X = \{v, w, x, y, z\}$ with pairwise choice defined by Figure 4 below and $c(v, w, x) = c(w, x, y) = c(x, y, z) = x$.*

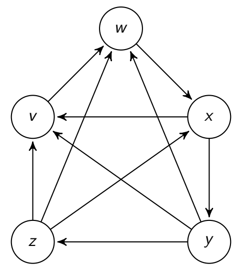

Figure 4: Preference pairs on the 5-cycle $vwxyz$.

For c to be TSM-representable, $c(i, j, k) = i$ for any $\{i, j, k\} \subset X$ s.t. $i \succ^c j \succ^c k$ and $i \succ^c k$. Consistency also requires $c(v, x, y, z) = c(v, w, x, y) = c(w, x, y, z) = c(v, w, x, y, z) = x$, $c(v, w, y, z) = y$, and $c(v, w, x, z) = z$. This pins down c for every subset of X. From these choices:

$$\widehat{R_2^c} = R_2^c = \{(w, x), (x, v), (x, y), (z, x)\} \text{ and } P_2^c = R_2^c \cup \{(w, y), (w, v), (\mathbf{z}, \mathbf{v}), (z, y)\}.$$

So, $\widehat{Q_1^c}$ is the strict weak order $y\widehat{Q_1^c}z\widehat{Q_1^c}v\widehat{Q_1^c}w$ and $Q_1^c \equiv tc(\succ^c \setminus P_2^c) = \widehat{Q_1^c} \setminus (\mathbf{z}, \mathbf{v})$. Since $(\widehat{Q_1^c}, P_2^c)$ and (Q_1^c, P_2^c) both represent c, $Q_1^c \neq \widehat{Q_1^c}$ for some TSM-representable choice functions.[41]

A.2.2 Proof of Theorem 3

Lemma 16 *If (P_1, P_2) TSM-represents c, then $P_1 \subseteq \succ^c$.*

Proof. By way of contradiction, suppose xP_1y and $y \succ^c x$. Then, $c_{(P_1,P_2)}(x, y) \neq y = c(x, y)$. ■

Lemma 17 *If (P_1, P_2) TSM-represents c, then $P_1 \supseteq P_1^c$ and $P_2 \supseteq P_2^c$.*

Proof. From the discussion in Section 3.1 of the text, $R_1^c \subseteq P_1$ and $R_2^c \subseteq P_2$. So, $P_1^c \subseteq P_1$ and $P_2^c \subseteq P_2$. ■

Lemma 18 *If c is TSM-representable, then (a) $R_1^c \subseteq (P_2^c)^{-1}$ and (b) $R_1^c \subseteq (\succ^c \setminus P_2^c)$.*

Proof. Suppose xR_1^cy. In turn, I establish (a) yP_2^cx and (b) $x(\succ^c \setminus P_2^c)y$.

(a) Under branch (i) of R_1^c, there exists a 3-cycle xyz s.t. $c(x, y, z) = z$. By definition, $yR_2^czR_2^cx$ so that yP_2^cx. Under branch (ii), there exist 3-cycles wxz and wyz s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$. By definition, $yR_2^czR_2^cwR_2^cx$ so that yP_2^cx. **(b)** Since xR_1^cy, $x \succ^c y$. By way of contradiction, suppose $\neg[x(\succ^c \setminus P_2^c)y]$. Since $x \succ^c y$, xP_2^cy. Since yP_2^cx (by claim (a) of the lemma), this contradicts the fact that R_2^c is acyclic by Lemma 8. So, $x(\succ^c \setminus P_2^c)y$. ■

Lemma 19 *If c is TSM-representable, then $P_1^c \subseteq (P_2^c)^{-1}$.*

Proof. Suppose xP_1^cy. By definition, there exists an R_1^c-chain $z_0...z_n$ s.t. $x = z_0$ and $y = z_n$. By Lemma 18, $z_{i+1}P_2^cz_i$ for all $0 \leq i \leq n$. Thus, $yP_2^c...P_2^cx$. By transitivity, yP_2^cx. ■

[41]One could equally construct examples where $c_3 = \langle v, y, x\rangle$ or $c_3 = \langle x, w, z\rangle$.

Lemma 20 *If c is TSM-representable, then (a) $R_2^c \cap Q_1^c = \emptyset$ and $R_2^c \subseteq (\succ^c \setminus P_1^c)$.*

Proof. Suppose $xR_2^c y$. In turn, I establish (a) $\neg[xQ_1^c y]$ and (b) $x(\succ^c \setminus P_1^c)y$.

(a) Since $xR_2^c y$, there exists a 3-cycle xyz s.t. $c(x,y,z) \neq z$. By way of contradiction, suppose $xQ_1^c y$. If $c(x,y,z) = x$, then $zR_2^c xR_2^c y$. Since R_2^c is acyclic by Lemma 8, $\neg[yP_2^c z]$. Since $y \succ^c z$, $y(\succ^c \setminus P_2^c)z$ so that $yQ_1^c z$. Since $xQ_1^c y$, $xQ_1^c z$ by transitivity. Since $\widehat{Q_1^c} \subseteq \succ^c$ by Lemma 7 and $Q_1^c \subseteq \widehat{Q_1^c}$ by Lemma 15, $x \succ^c z$ which contradicts $z \succ^c x$. If $c(x,y,z) = y$, a similar argument establishes $z \succ^c y$, which contradicts $y \succ^c z$. **(b)** Since $R_2^c \subseteq \succ^c$ by definition and $Q_1^c \subseteq \succ^c$ by Lemmas 7 and 15, $R_2^c \cup Q_1^c \subseteq \succ^c$. Since $R_2^c \cap Q_1^c = \emptyset$ (by claim (a) of the lemma), (i) $R_2^c \subseteq (\succ^c \setminus Q_1^c)$. By Lemma 18, $P_1^c \subseteq Q_1^c$. So, (ii) $(\succ^c \setminus Q_1^c) \subseteq (\succ^c \setminus P_1^c)$. Combining (i) and (ii), $R_2^c \subseteq (\succ^c \setminus P_1^c)$. ■

Lemma 21 *If c is TSM-representable, then the following relations are acyclic: (i) $\succ^c \setminus P_2^c$; (ii) R_1^c; and, (iii) $\succ^c \setminus P_1^c$.*

Proof. (i) Since $\widehat{R_2^c} \subseteq P_2^c$ and $(\succ^c \setminus \widehat{R_2^c}) \equiv \widehat{Q_1^c}$, $(\succ^c \setminus P_2^c) \subseteq \widehat{Q_1^c}$. Since $\widehat{Q_1^c}$ is acyclic by Theorem 2, so is $\succ^c \setminus P_2^c$.

(ii) Since $R_1^c \subseteq (\succ^c \setminus P_2^c)$ by Lemma 18 and $\succ^c \setminus P_2^c$ is acyclic by (i), so is R_1^c.

(iii) Let $\widetilde{R_2^c} \equiv (\succ^c \setminus P_1^c)$. By way of contradiction, suppose there exists an $\widetilde{R_2^c}$-cycle. Since X is finite, there is an $\widetilde{R_2^c}$-cycle $x_0 ... x_{n-1}$ of minimal length. To establish the result, I show that no such minimal $\widetilde{R_2^c}$-cycle exists. For $n = 3$, $x_0 x_1 x_2$ is a 3-cycle. Without loss of generality, let $c(x_0, x_1, x_2) = x_0$. By definition, $x_1 R_1^c x_2$. So, $\neg[x_1 \widetilde{R_2^c} x_2]$, a contradiction. For $n \geq 4$, the proof is by strong induction.

For the base case $n = 4$, $x_0 x_1 x_2 x_3$ is a 4-cycle. Without loss of generality, suppose $x_2 x_0 x_1$ and $x_0 x_1 x_3$ are 3-cycles. Since there are no $\widetilde{R_2^c}$-cycles of length $n = 3$, $\neg[x_2 \widetilde{R_2^c} x_0]$ and $\neg[x_1 \widetilde{R_2^c} x_3]$. By definition, $x_2 P_1^c x_0$ and $x_1 P_1^c x_3$. As a result, (o) $c(x_0, x_1, x_2) = x_1$ and $c(x_0, x_1, x_3) = x_0$. Since $c(x_2, x_3) = x_2$, (o) implies $x_2 R_1^c x_3$. But this contradicts $x_2 \widetilde{R_2^c} x_3$. To establish (o), suppose $c(x_0, x_1, x_2) = x_2$. Then, $x_0 R_1^c x_1$ so that $x_2 P_1^c x_1$ by transitivity. Since $P_1^c \subseteq \succ^c$,[42] $x_2 \succ^c x_1$ which contradicts $x_1 \succ^c x_2$. Similar contradictions arise if $c(x_0, x_1, x_2) = x_0$ or $c(x_0, x_2, x_3) \neq x_0$.

For the induction step, suppose there are no $\widetilde{R_2^c}$-cycles of length $3 \leq i \leq n$. By way of contradiction, suppose $x_0 ... x_n$ is an $\widetilde{R_2^c}$-cycle of length $n+1$. First, fix an x_i and any x_j s.t. $j \neq i \pm 1 \pmod{n}$. Then: (a) $x_i P_1^c x_j$ if $x_i \succ^c x_j$; and, (b) $x_j P_1^c x_i$ if $x_j \succ^c x_i$. To show (a), suppose otherwise. Then, $x_i \widetilde{R_2^c} x_j$ so that $x_j ... x_i$ is a $\widetilde{R_2^c}$-cycle of length $4 \leq i \leq n$, which contradicts the induction hypothesis. The proof of (b) is analogous. Next, fix x_i, x_{i+1}, and any x_j s.t. $j \notin \{i-1, i, i+1, i+2\}$. Then: (i) $x_i \succ^c x_j$ and $x_{i+1} \succ^c x_j$; or, (ii) $x_j \succ^c x_i$ and $x_j \succ^c x_{i+1}$. In every other case, (a) and (b) imply $x_i P_1^c x_j P_1^c x_{i+1}$ or $x_{i+1} P_1^c x_j P_1^c x_i$. By transitivity, $x_i P_1^c x_{i+1}$ or $x_{i+1} P_1^c x_i$. Since both possibilities contradict $x_i \widetilde{R_2^c} x_{i+1}$, either (i) or (ii) must obtain. Finally, define $j^* \equiv \lceil \frac{n}{2} \rceil$ and suppose $x_0 \succ^c x_{j^*}$. (The case where $x_{j^*} \succ^c x_0$ is analogous.) By (i)-(ii), $x_1 \succ^c x_{j^*}$, $x_0 \succ^c x_{j^*+1}$, and $x_1 \succ^c x_{j^*+1}$. Summarize these choices by $\{x_0, x_1\} \succ^c \{x_{j^*}, x_{j^*+1}\}$. By a simple induction argument, $\{x_i, x_{i+1}\} \succ^c \{x_{j^*+i}, x_{j^*+i+1}\}$ for all $0 \leq i \leq n-1$. In particular, $\{x_{j^*}, x_{j^*+1}\} \succ^c \{x_{2j^*}, x_{2j^*+1}\}$. If n is even, $2j^* = 0 \pmod{n}$ so $x_{j^*} \succ^c x_0$. If n is odd, $2j^* = 1 \pmod{n}$ so $x_{j^*} \succ^c x_1$. In either case, $\{x_{j^*}, x_{j^*+1}\} \succ^c \{x_{2j^*}, x_{2j^*+1}\}$ contradicts $\{x_0, x_1\} \succ^c \{x_{j^*}, x_{j^*+1}\}$, which gives the result. ■

Lemma 22 *If c is TSM-representable, then Q_2^c is a strict weak order.*

Proof. By Lemma 21, Q_2^c is asymmetric and transitive. To show that Q_2^c is total, fix any x, y s.t. $x \succ^c y$. If $\neg[xP_1^c y]$, then $\succ^c \subseteq P_1^c \cup Q_2^c$ implies $xQ_2^c y$. If $xP_1^c y$, then $yP_2^c x$ by Lemma 19. Since $P_2^c \subseteq Q_2^c$ by Lemma 20, $yQ_2^c x$. Thus, $x \succ^c y$ implies $xQ_2^c y$ or $yQ_2^c x$. Since $\succ^c$ is total, so is Q_2^c. ■

Lemma 23 *Given a choice function c and two rationales P and $\widetilde{P}$, $P \subseteq \widetilde{P}$ implies $tc(\succ^c \setminus \widetilde{P}) \subseteq tc(\succ^c \setminus P)$.*

Proof. Since $P \subseteq \widetilde{P}$, it follows that $(\succ^c \setminus \widetilde{P}) \subseteq (\succ^c \setminus P)$. The result then follows by transitive closure. ■

[42] In particular: $P_1^c \subseteq Q_1^c$ by Lemma 18; $Q_1^c \subseteq \widehat{Q_1^c}$ by Lemma 15; and, $\widehat{Q_1^c} \subseteq \succ^c$ by Lemma 7.

Lemma 24 *If c is TSM-representable, then (P_1^c, Q_2^c) and (Q_1^c, P_2^c) are TSM-representations of c.*

Proof. By Lemmas 8 and 21, R_2^c, R_1^c, and $(\succ^c \setminus P_2^c)$ are acyclic. So, P_1^c, P_2^c, and Q_1^c are transitive rationales. By Lemma 22, Q_2^c is a strict weak order. To establish the result, it suffices to show that (P_1^c, Q_2^c) and (Q_1^c, P_2^c) satisfy the two conditions in the Proposition of D&H (and, thus, RSM-represent c).

For the first condition, note the following set inclusions: $P_1^c \subseteq Q_1^c$ by Lemma 18; $Q_1^c \subseteq \widehat{Q_1^c}$ by Lemma 15; and, $\widehat{Q_1^c} \subseteq (\succ^c \setminus P_2^{RSM})$ by Lemma 2. To see that $P_1^{RSM} \subseteq P_1^c$, suppose $xP_1^{RSM}y$. Then, xR_1^cy by Reduction and hence xP_1^cy. Combining these set inclusions gives $P_1^{RSM} \subseteq P_1^c \subseteq Q_1^c \subseteq (\succ^c \setminus P_2^{RSM})$ as required.

For the second condition, note that $Q_i^c \equiv tc(\succ^c \setminus P_{-i}^c) \supseteq (\succ^c \setminus P_{-i}^c)$ for $i = 1, 2$. So, $Q_2^c \supseteq (\succ^c \setminus P_1^c)$ and $P_2^c \supseteq (\succ^c \setminus Q_1^c)$ as required (where the second inclusion follows from $Q_1^c \supseteq (\succ^c \setminus P_2^c)$ and Lemma 23). ■

Proof of Theorem 3. Let $\{(P_1^j, P_2^j)\}_{j=1}^n$ denote the collection of all TSM-representations for c. By Lemma 17, $P_i^c \subseteq \cap_{j=1}^n P_i^j$ for $i = 1, 2$. Since (P_1^c, Q_2^c) and (Q_1^c, P_2^c) are TSM-representations of c by Lemma 24, $P_i^c \supseteq \cap_{j=1}^n P_i^j$ for $i = 1, 2$ as well. Combining these set inclusions gives $P_i^c = \cap_{j=1}^n P_i^j$ for $i = 1, 2$. This establishes (1) and (2). ■

A.2.3 Proof of Theorem 4

First, note that c satisfies Expansion, Weak WARP, and 3-acyclicity by Theorem 6 and Lemma 12.

Part (2) follows from Lemma 6. For part (1): **(⇒)** Suppose xR_1^cy. If xyz is a 3-cycle s.t. $c(x, y, z) = z$, c displays a direct $\langle x, y\rangle$ reversal on $\{y, z\}$. If $c(x, y) = x$ and wxz, wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$, c displays an indirect $\langle x, y\rangle$ reversal on $\{y, z\}$. **(⇐)** If c displays a direct $\langle x, y\rangle$ reversal, there is some B s.t. $c(B) = y$ and $c(B \cup \{x\}) \equiv z \notin \{x, y\}$. By Reduction, xyz is a 3-cycle s.t. $c(x, y, z) = z$. So, xR_1^cy by part (i) of the definition. If c displays an indirect $\langle x, y\rangle$ reversal, there is some B s.t. $c(B) = y$, $c(B \cup \{x\}) = x$, $c(B \cup \{w\}) = z \notin \{y, w\}$, and $c(B \cup \{x, w\}) = w$. By Reduction, wxz and wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$. Moreover, $c(x, y) = x$. Otherwise, $c(B \cup \{x\}) = c(B \cup \{x, y\}) = y \neq x$ by Expansion. So, xR_1^cy by part (ii) of the definition.

A.2.4 Proof of Theorem 5

Lemma 25 *If (P_1, P_2) TSM-represents c, then $P_1 \subseteq \widehat{Q_1^c}$.*

Proof. By way of contradiction, suppose xP_1y and $\neg(x\widehat{Q_1^c}y)$. If $y \succ^c x$, then $c_{(P_1,P_2)}(x, y) = y \neq x = c(x, y)$, a contradiction. If $x \succ^c y$, then $\neg(x\widehat{Q_1^c}y)$ implies $x\widehat{R_2^c}y$. By definition, either: (i) there is a 3-cycle xyz s.t. $c(x, y, z) \neq z$; or, (ii) there are 3-cycles wxz, wyz s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$. To see that both entail contradictions: **(i)** If $c(x, y, z) = x$, then yP_1^cz. Since $P_1 \supseteq P_1^c$ by Theorem 3, xP_1yP_1z. By transitivity, xP_1z. But, then $c_{(P_1,P_2)}(x, z) = x \neq z = c(x, z)$, a contradiction. If $c(x, y, z) = y$, a similar contradiction arises for $\{y, z\}$. **(ii)** In this case, wP_1^cx and yP_1^cz. Since $P_1 \supseteq P_1^c$, $wP_1xP_1yP_1z$. By transitivity, wP_1z. But, then $c_{(P_1,P_2)}(w, z) = w \neq z = c(w, z)$, a contradiction. ■

Lemma 26 *If (P_1, P_2) TSM-represents c, then $P_i' \subseteq P_{-i}$ for $i = 1, 2$.*

Proof. Suppose $xP_i'y$. By definition, there exists a $(\succ^c \setminus P_i)$-chain $x_0 ... x_n$ s.t. $x_0 = x$ and $x_n = y$. Consider any link $x_j(\succ^c \setminus P_i)x_{j+1}$ of this chain. Since $x_j \succ^c x_{j+1}$ and $\neg[x_jP_ix_{j+1}]$, it follows that $x_jP_{-i}x_{j+1}$. Otherwise, (P_1, P_2) cannot TSM-represent c. So, $xP_{-i} ... P_{-i}y$. By transitivity, it follows that $xP_{-i}y$. ■

Proof of Theorem 5. (⇒) Suppose (P_1, P_2) represents c. By Theorem 3, $P_1^c \subseteq P_1$. By Lemma 25, $P_1 \subseteq \widehat{Q_1^c}$. So, $P_1^c \subseteq P_1 \subseteq \widehat{Q_1^c}$ as required. By Lemma 26, $P_1' \subseteq P_2$ as required. (⇐) Suppose P_1, P_2 are transitive rationales s.t. $P_1^c \subseteq P_1 \subseteq \widehat{Q_1^c}$ and $P_1' \subseteq P_2$. Since $P_1^{RSM} \subseteq P_1^c$ by Theorem 3 and $\widehat{Q_1^c} \subseteq (\succ^c \setminus P_2^{RSM})$ by Lemma 2, (P_1, P_2) TSM-represents c by the Proposition of D&H. ■

A.3 Exclusivity in Shortlist Methods

Proof of Theorem 6. **(i)** By Lemma 1 and part (ii). **(ii)** By Lemma 3. ■

Proof of Theorem 7. Suppose c is RSM-representable. By Lemma 12, it suffices to show that:

- **(a)** c satisfies $\{3, 5\}$-acyclicity iff $\widehat{R^c_2}$ is triple-acyclic; and
- **(b)** c satisfies $\{3, 4, 5\}$-acyclicity iff $\widehat{R^c_2}$ is quadruple-acyclic.

(a) ($\Rightarrow$) By way of contradiction, suppose $x\widehat{R^c_2}y\widehat{R^c_2}v\widehat{R^c_2}x$. By definition of $\widehat{R^c_2}$, xyv is a 3-cycle. Without loss of generality, suppose $c(x, y, z) = v$. Then, $y\widehat{R^c_2}v\widehat{R^c_2}x$ by definition. Since $x\widehat{R^c_2}y$ as well: (i) there is a 3-cycle xyz s.t. $c(x, y, v) \neq z$; or, (ii) there are 3-cycles wxz and wyz s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$. Since xyv is a 3-cycle s.t. $c(x, y, z) = v$: (i) contradicts 3-acyclicity; and, (ii) contradicts 5-acyclicity. So, $\widehat{R^c_2}$ is triple-acyclic. ($\Leftarrow$) To establish 3-acyclicity, suppose wxz is a 3-cycle s.t. that $c(w, x, z) = x$. By definition, $w\widehat{R^c_2}x\widehat{R^c_2}z$. If wyz is a 3-cycle such that $c(w, y, z) \neq y$, then $z\widehat{R^c_2}w$ by definition, which contradicts the assumption that $\widehat{R^c_2}$ is triple-acyclic. To establish 5-acyclicity, suppose wxz and wyz are 3-cycles s.t. $c(w, x, z) = z$ and $c(w, y, z) = w$; and, suppose $c(x, y) = x$. By definition, $x\widehat{R^c_2}y$. If xyv is a 3-cycle such that$c(x, y, v) = v$, then $y\widehat{R^c_2}v\widehat{R^c_2}x$ by definition, which contradicts the assumption that $\widehat{R^c_2}$ is triple-acyclic.

(b) ($\Rightarrow$) By Lemma 8, R^c_2 is acyclic. Then, by definition, $\widehat{R^c_2}$ is also acyclic. So, $\widehat{R^c_2}$ is quadruple-acyclic. ($\Leftarrow$) Since $\widehat{R^c_2}$ is triple-acyclic, c satisfies $\{3, 5\}$-acyclicity by part (a) above. To establish 4-acyclicity, suppose wxz and wyz are 3-cycles s.t. $c(w, x, z) = w$ and $c(w, y, z) = z$; and, suppose $c(x, y) = x$. By definition, $y\widehat{R^c_2}z\widehat{R^c_2}w\widehat{R^c_2}x$. If xyv is a 3-cycle such that$c(x, y, v) \neq v$, then $x\widehat{R^c_2}y$ by definition, which contradicts the assumption that $\widehat{R^c_2}$ is quadruple-acyclic. ■

B Supplemental Appendix

Section B.1 shows the independence of the axioms, Section B.2 establishes the results mentioned in Section 4.3, Section B.3 establishes the remarks in Section 6, and Section B.4 provides results formalizing the connection to related models.

B.1 Independence of the Axioms

The first two examples describe choice functions that satisfy exactly one of the two axioms for the TSM model:

Example 2 *Consider a choice function c on $\{w, x, y, z\}$ with the pairwise choices given by Figure 2. In addition, suppose $c(w, x, z) = z$, $c(w, y, z) = y$, $c(w, x, y) = w$, $c(x, y, z) = x$, and $c(w, x, y, z) = y$.*

It is straightforward to check that c satisfies Expansion. However, it violates (Strong) Exclusivity because it displays a direct $\langle z, w\rangle$ reversal on $\{w, y\}$ and a weak $\langle z, w\rangle$ reversal on $\{w, x, z\}$. (Still, it does satisfy Weak Exclusivity. Indeed, c is RSM-represented by the pair (P_1, P_2) where $P_1 \equiv \{(z, w), (w, x)\}$ and $P_2 \equiv \{(y, z), (w, y), (x, z), (x, y)\}$.)

Example 3 *Consider a choice function c on $\{x, y, z\}$ such that $c(x, y) = c(x, z) = x$, $c(y, z) = y$, and $c(x, y, z) = y$.*

It is easy to check that c satisfies (Strong) Exclusivity: it displays a weak $\langle x, y\rangle$ reversal on $\{x, y, z\}$; and, a direct $\langle z, x\rangle$ reversal on $\{x, y\}$. However, it violates Expansion because $c(\{x, y\} \cup \{x, z\}) = y \neq x$.

The last two examples shows that Strong Exclusivity is independent from Weak WARP:

Example 4 *Consider a choice function c on $\{x, y, z\}$ such that $c(x, y) = c(x, z) = x$, $c(y, z) = y$, and $c(x, y, z) = z$.*

Clearly, c satisfies Weak WARP (since this condition imposes no restrictions on the domain $\{x, y, z\}$). However, it violates (Strong) Exclusivity because it displays both weak and direct $\langle x, y\rangle$ reversals (on $\{x, y, z\}$ and $\{x, z\}$, respectively).

Example 5 *Consider a choice function c on $\{w, x, y, z\}$ with pairwise choice given by the linear order $w > x > y > z$. In addition, suppose $c(w, x, z) = c(w, y, z) = c(w, x, y) = w$, $c(x, y, z) = y$, and $c(w, x, y, z) = x$.*

It is straightforward (but computationally intensive) to check that c satisfies (Strong) Exclusivity: the weak reversals are $\langle a, b\rangle$ for $a, b \in \{w, x, y, z\}$ and $a > b$; the direct reversals are $\langle y, w\rangle$, $\langle z, w\rangle$, and $\langle z, x\rangle$; and, there are no indirect reversals. However, it violates Weak WARP because $c(w, x, y, z) = x = c(x, y)$ and $c(x, y, z) = y$.

B.2 Minimal Representations

In this section, I use the additional notation that $P' \equiv tc(\succ^c \setminus P)$.

Lemma 27 *If (P_1, P_2) TSM-represents c, then $R_i^c \cap P_{-i} = \emptyset$ for $i = 1, 2$.*

Proof. ($i = 1$) By way of contradiction, suppose (i) $xR_1^c y$ and (ii) $xP_2 y$. Given (i), $yP_2^c x$ by Lemma 19. Since $P_2^c \subseteq P_2$ by Theorem 3, $yP_2 x$. Given (ii), this contradicts the asymmetry of P_2. ($i = 2$) By way of contradiction, suppose (i) $xR_2^c y$ and (ii) $xP_1 y$. Given (i), there is a 3-cycle xyz s.t. $c(x, y, z) \neq z$. If $c(x, y, z) = x$, $yR_1^c z$. Since $P_1^c \subseteq P_1$ by Theorem 3, $yP_1 z$. Given (ii), $xP_1 z$ by transitivity. So, $c(x, z) \neq z$, a contradiction. If $c(x, y, z) = y$, a similar contradiction arises. ■

Definition 3 *Given a choice function c, let $\mathcal{P}_i(c) \equiv \{P : P_i^c \subseteq P \subseteq Q_i^c\}$ and $\mathcal{P}_i''(c) \equiv \{P_i'' : P_i \in \mathcal{P}_i(c)\}$.*

Definition 4 *A rationale P_1 is P_2-**minimal** if (i) (P_1, P_2) TSM-represents c and (ii) there exists no transitive rationale $\widetilde{P}_1 \subset P_1$ such that $(\widetilde{P}_1, P_2)$ TSM-represents c. The notion of P_1-**minimality** is defined analogously.*

Proposition 1 *If (P_1, P_2) TSM-represents c, then $P_i' \in \mathcal{P}_{-i}(c)$ is the unique P_i-minimal rationale.*

Proof. I show: (i) $P_i' \in \mathcal{P}_{-i}(c)$; (ii) $P_i' \subseteq P_{-i}$; (iii.a) (P_1, P_1') represents c; and, (iii.b) (P_2', P_1) represents c.

(i) By Theorem 3, $P_i^c \subseteq P_i$. So, (a) $P_i' \subseteq Q_{-i}^c$ by Lemma 23. By Lemma 27, $R_i^c \cap P_{-i} = \emptyset$. So, $R_i^c \subseteq (\succ^c \setminus P_{-i})$. By transitive closure, (b) $P_i^c \subseteq P_{-i}'$. Given (a)-(b), $P_i' \in \mathcal{P}_{-i}(c)$. **(ii)** By Lemma 26.

(iii.a) Since (P_1, P_2) RSM-represents c, $P_1^{RSM} \subseteq P_1 \subseteq (\succ^c \setminus P_2^{RSM})$ by the Proposition of D&H. Moreover, $P_1' \equiv tc(\succ^c \setminus P_1) \supseteq (\succ^c \setminus P_1)$. Since $P_1' \in \mathcal{P}_2(c)$ by (i) and P_2^c, Q_2^c are asymmetric by Lemma 24, so is P_1'. Since P_1, P_1' are transitive, (P_1, P_1') represents c by the Proposition of D&H. **(iii.b)** Given (i), $P_2' \in \mathcal{P}_1(c)$. So, $P_1^{RSM} \subseteq P_2' \subseteq (\succ^c \setminus P_2^{RSM})$ by Theorem 3. Since $P_2' \supseteq (\succ^c \setminus P_2)$, $P_2 \supseteq (\succ^c \setminus P_2')$ by Lemma 23. Since $P_2' \in \mathcal{P}_1(c)$ and P_1^c, Q_1^c are asymmetric by Lemma 24, P_1' is as well. Since P_2' and P_2 are transitive, (P_2', P_2) TSM-represents c by the Proposition of D&H. ■

Lemma 28 *Given a choice function c and a transitive rationale P, $xP''y$ implies xPy.*

Proof. Suppose $xP''y$. First, suppose $x(\succ^c \setminus P')y$. Then, $\neg[xP'y]$. By definition, $\neg[x(tc(\succ^c \setminus P))y]$. Since $x \succ^c y$, xPy. Next, suppose there is a $(\succ^c \setminus P')$-chain $x_0 \ldots x_n$ with $x_0 = x$ and $x_n = y$. From the first case, $x_i P x_{i+1}$ for any link in the chain. So, $xPx_1P \ldots Px_{n-1}Py$. By transitivity, xPy. ■

Definition 5 *A TSM-representation (P_1, P_2) of c is* ***minimal*** *(i) P_1 is P_2-minimal and (i) P_2 is P_1-minimal.*

Lemma 29 *If (P_1, P_2) TSM-represents c, then (P_1'', P_1') and (P_2', P_2'') are minimal.*

Proof. I show that (P_1'', P_1') is minimal. The proof for (P_2', P_2'') is analogous. By Proposition 1, (P_1, P_1') TSM-represents c and P_1' is P_1-minimal. So: (i) (P_1'', P_1') TSM-represents c and P_1'' is P_1'-minimal; and, (ii) (P_1'', P_1''') TSM-represents c and P_1''' is P_1''-minimal. Given (i)-(ii), it suffices to show $P_1''' = P_1'$. By Lemma 28, (1) $P_1'' \subseteq P_1$ (using $P = P_1$) and (2) $P_1''' \subseteq P_1'$ (using $P = P_1'$). By Lemma 23, (1) implies $P_1' \subseteq P_1'''$. Given (2), it follows that $P_1''' = P_1'$. ■

Proposition 2 *If (P_1, P_2) TSM-represents c, then: (P_1, P_2) is minimal iff $P_i \in \mathcal{P}_i''(c)$ and $P_{-i} = P_i'$.*

Proof. I show that (P_1, P_2) is minimal iff $P_1 \in \mathcal{P}_1''(c)$ and $P_2 = P_1'$. The proof for $P_2 \in \mathcal{P}_2''(c)$ and $P_1 = P_2'$ is similar. ($\Rightarrow$) Suppose (P_1, P_2) is minimal. By Proposition 1, $P_2 = P_1'$ as required. Similarly, $P_1 = P_2' = P_1''$. So, $P_1 = P_1'' \in \mathcal{P}_1''(c)$ as required. ($\Leftarrow$) Suppose (P_1, P_2) TSM-represents c and $\widetilde{P}_1 \in \mathcal{P}_1(c)$ is a transitive rationale s.t. $P_1 = \widetilde{P}_1''$ and $P_2 = \widetilde{P}_1'''$. It suffices to show that $(P_1, \widetilde{P}_1')$ TSM-represents c. Then, $(P_1, P_2) = (\widetilde{P}_1'', \widetilde{P}_1''')$ is minimal by Lemma 29. To see this, note: $P_1^{RSM} \subseteq P_1 \subseteq (\succ^c \setminus P_2^{RSM})$ because (P_1, P_2) RSM-represents c; and, $P_2 = \widetilde{P}_1''' \subseteq \widetilde{P}_1'$ by Lemma 28. So, $(P_1, \widetilde{P}_1')$ RSM-represents c by the Proposition of D&H. Since P_1 and $\widetilde{P}_1'$ are transitive rationales, $(P_1, \widetilde{P}_1')$ TSM-represents c. ■

B.3 Proof of Remarks 1 to 3

Proof of Remark 1. **($\Rightarrow$)** This follows from the discussion in the text. **($\Leftarrow$)** By ESQ Exclusivity, there exists at most one alternative d that creates direct reversals. It suffices to show that (E^c, P^c) satisfies the conditions of Theorem 5 where: E^c is defined by $dE^c a$ if $d \succ^c a$; and, $P^c \equiv Q_2^c$. By definition of R_1^c and ESQ Exclusivity, $xR_1^c y$ implies $x = d$. So, $P_1^c = R_1^c \subseteq E^c$. To show that $E^c \subseteq \widehat{Q_1^c}$, it is enough to see that $d\widehat{R_2^c}z$ cannot occur. By definition of R_2^c, Exclusivity, and ESQ Exclusivity, $xR_2^c y$ implies $x \neq d$. By definition of $\widehat{R_2^c}$, it follows that $d\widehat{R_2^c}z$ cannot occur. ■

Proof of Remark 2. For the framing model: Since the induced choice function c is TSM-represented by (P_f, P), the only 3-cycles involve the outside option o (and alternatives $a, b \in X$). In particular, $aPoPbP_f a$ and $c(a, b, o) = o$. By way of contradiction, suppose $c(x, y) = x$, $c(w, x, z) = z$, and $c(w, y, z) = w$ for some 3-cycles wxz and wyz. From the simple observation above, it follows that $w = o = z$. But, this contradicts the fact that w and z are distinct.

For the compromise model: Suppose $\mathcal{C}$ is the set of categories and P is the preference of the planner. Then, it is easy to show that the induced choice function c is TSM-represented by the pair $(P_{\mathcal{C}}^{-1}, P)$ where $P_{\mathcal{C}}^{-1}$ is defined by $xP_{\mathcal{C}}^{-1}y$ if yPx and $x, y \in C_i$ for some $C_i \in \mathcal{C}$. As a result, the only 3-cycles involve $a, a' \in C_i$ from one category and $b \in C_j$ from another. In particular, $aPbPa'P_{\mathcal{C}}^{-1}a$ and $c(a, a', b) = b$. So, direct reversals on small menus involve alternatives in the same category while weak reversals involve alternatives in different categories. (For the alternatives specified: there is a direct reversal $\langle a', a\rangle$; and, two weak reversals $\langle a, b\rangle$ and $\langle b, a'\rangle$.)

By way of contradiction, suppose $c(x, y) = x$, $c(w, x, z) = z$, and $c(w, y, z) = w$ for some 3-cycles wxz and wyz (see Definition 1). From the observation in the last paragraph, it follows that $x \approx^c w \not\approx^c z \approx^c y$. Since this means that x and y are in different categories, they are not compared according to the first rationale of $(P_{\mathcal{C}}^{-1}, P)$. Since $P_1^c \subseteq P_{\mathcal{C}}^{-1}$ by Theorem 3 however, this contradicts the inference that xR_1^cy. ■

Proof of Remark 3. Suppose M^* is profit-maximal. First, observe the following: if yP_1x for $x, y \in M^* \cup \{o\}$, then $x = o$. To see this, suppose to the contrary that $x \in M^*$. Since $P_1 \subseteq \widehat{Q_1^c}$ (by Theorem 5), $c_i(M^* \cup \{o\}) = c_i([M^* \cup \{o\}] \setminus \{x\})$ for all $i \in I$ (by Lemma 7). Since x can be removed without affecting choice, M^* is not profit-maximal.

Next, suppose $m \in M^*$ is not purchased by any consumer types. Since M^* maximizes profits, there must be some consumer i such that $c_i(M^* \cup \{o\}) \equiv y \neq x \equiv c_i([M^* \cup \{o\}] \setminus \{m\})$. By Reduction, xym is a 3-cycle for i such that $c_i(x, y, m) = y$. So, mR_1^cx. Since $P_1^c \subseteq P_1$ (by Theorem 5), the observation in the last paragraph implies $x = o$. So, mP_1o and m serves as a decoy for consumer i. To complete the proof, note that m must be the only unchosen alternative in M^*. If there was another unchosen $m' \in M^*$, it could not be optimal to offer both: either m or m' could be removed without affecting choice. This follows from the fact that both mP_1o and $m'P_1o$. ■

Example 6 *Let $X = \{a, b\}$. Suppose it is unprofitable for the monopolist to sell product b; but that this product alone discourages biased consumers from choosing the outside option o. In other words, suppose $\pi(a) > \pi(o) = 0 > \pi(b)$ and $P_1 = \{(b, o)\}$. Then, the six possible types of biased consumer (P_1, P_2^i) would choose as follows:*

Type	P_2^i	p_i	$c_i(\{a, o\})$	$c_i(\{b, o\})$	$c_i(\{a, b, o\})$
1	a, o, b	1/4	a	b	a
2	a, b, o	1/4	a	b	a
3	o, a, b	1/4	o	b	a
4	o, b, a	1/4	o	b	b
5	b, a, o	0	a	b	b
6	b, o, a	0	o	b	b

Given the type distribution p, it is optimal for the monopolist to offer biased consumers $M^* = \{a, b\}$ if and only if $\pi(a) + \pi(b) > 0$. In contrast, it is optimal to offer $M_\emptyset^* = \{a\}$ when consumers are unbiased (since b imposes a menu cost $\epsilon(b)$ but is not chosen by types 1 to 4). In other words, the monopolist offers different menus to biased and unbiased consumers when $\pi(a) + \pi(b) > 0$. Clearly, biased types 3 and 4 are better off with the menu $M_\emptyset^*$ offered to unbiased consumers. In contrast, unbiased types 5 and 6 are better off with the menu M^* offered to biased consumers.

Remark 4 (Nash Implementation) *The choice behavior $c_i(\cdot, \theta)$ of each agent $i \in I$ depends on a state $\theta \in \Theta$ that is unknown to the planner. The social welfare function f defines the set of acceptable outcomes $f(\theta) \subseteq X$ in each state $\theta \in \Theta$. The objective is to design a simultaneous game form G such that $x \in X$ is a Nash equilibrium outcome of the game (G, θ) if and only if it is acceptable at θ. Where $NE(G, \theta)$ denotes the Nash equilibrium outcomes of (G, θ), the social welfare function f is Nash implementable if there exists a game form G such that $NE(G, \theta) = f(\theta)$ for all $\theta \in \Theta$. Korpela [2012] and de Clippel [2014] show that the following is necessary for f to be Nash implementable:*

Weak μ For every $\theta \in \Theta$ and $x \in f(\theta)$, there exists a collection $\mathcal{X}(\theta, x) \equiv \{X_i(\theta, x) \subseteq X : i \in I\}$ such that

(i) $x \in c_i(X_i(\theta, x), \theta)$ *for all* $i \in I$*; and,*

(ii) $x \in c_i(X_i(\theta, x), \phi)$ *for all* $i \in I$ *only if* $x \in f(\phi)$.

B.4 Related Models

B.4.1 The Transitive RSM Model

Remark 5 *Several papers characterize the transitive RSM model by defining a "revealed 2-rationale" and requiring this binary relation to be acyclic. For convenience, I restate the definitions used in these papers:*

$$xR_2^a y \text{ if } c(x, y) = x \text{ and } c(A) \neq c(A \setminus \{y\}) \text{ for some } A \supset \{x, y\} \qquad \text{(Au and Kawai [2011, Theorem 1])}$$

$$xR_2^l y \text{ if } c(B) = x \text{ and } c(A) \neq c(A \setminus \{y\}) \text{ for some } \{x, y\} \subseteq B \subset A \qquad \text{(Lleras et al. [2011, Theorem 4])}$$

$$xR_2^y y \text{ if } \begin{cases} c(x, y) = x \text{ and } c(A) = y \text{ for some } A \supset \{x, y\}\text{; or,} \\ c(B \cup \{y\}) = x \neq c(B). \end{cases} \qquad \text{(Yildiz [2015, Proposition 2])}$$

By Lemma 6, R_2^a *is equivalent to* $\widehat{R_2^c}$ *(given Weak WARP, Expansion, and 3-Acyclicity). Lemma 30 treats* R_2^l *and* R_2^y*.*

Lemma 30 *If c satisfies Weak WARP, Expansion, and 3-Acyclicity, then: (i)* $R_2^l = \widehat{R_2^c}$*; and, (ii)* $R_2^y = R_2^c$*.*

Proof. (i) Since $\widehat{R_2^c} = R_2^a$ by Lemma 6, the proof can be carried out with R_2^a in place of $\widehat{R_2^c}$. ($\Leftarrow$) Suppose $xR_2^a y$. Then, by definition of R_2^a, the implication holds with $B = \{x, y\}$. ($\Rightarrow$) Suppose $c(B) = x$ and $c(A) \neq c(A \setminus \{y\})$ for some $\{x, y\} \subseteq B \subset A$. It suffices to show that $c(x, y) = x$ when $B \neq \{x, y\}$. By Lemma 5, c satisfies Reduction. If $c(A) = x$, Reduction implies that xya' is a 3-cycle where $a' \equiv c(A \setminus \{y\})$. So, $c(x, y) = x$ as required. If $c(A) = y$, then $c(x, y) = x$ as required. Otherwise, $c(x, y) = y = c(A)$ and $c(B) = x$ contradict Weak WARP. Finally, suppose $c(A) = a \notin \{x, y\}$. If $c(A \setminus \{y\}) = x$, $c(A) = c(A \setminus \{y\} \cup B) = x$ by Expansion, which contradicts $c(A) \neq x$. So, $c(A \setminus \{y\}) = a' \neq x$. By way of contradiction, suppose $c(x, y) = y$. Since $c(A) = a$ and $c(B) = x$, the same kind of argument given in the previous paragraph establishes $c(a, x) = x$ by Weak WARP. Since $c(A) = a \neq a' = c(A \setminus \{y\})$, Reduction implies that $a'ay$ is a 3-cycle s.t. $c(a, a', y) = a$. Since $c(a, y) = a$, xay is also a 3-cycle. By 3-Acyclicity, $c(a, a', y) = a$ implies $c(a, x, y) \neq x$. If $c(a, x, y) = y$, then $c(a, y) = a = c(A)$ and $c(a, x, y) = y$ contradict Weak WARP. So, $c(a, x, y) = a$. Since $c(x, a) = x = c(B)$, Expansion implies $c(B \cup \{a\}) = x$. Since $c(a, x, y) = a = c(A)$ however, this contradicts Weak WARP. So, $c(x, y) = x$ as required.

(ii) ($\Leftarrow$) Suppose $xR_2^c y$. Then, there exists a 3-cycle xyz s.t. $c(x, y, z) \neq z$. If $c(x, y, z) = y$, the first branch is satisfied with $A = \{x, y, z\}$. If $c(x, y, z) = x$, the second branch is satisfied with $B = \{x, z\}$. ($\Rightarrow$) First, suppose $c(x, y) = x$ and $c(A) = y$ for some $A \supset \{x, y\}$. By Reduction, there exists a 3-cycle xyz s.t. $c(x, y, z) = y$. By definition, $xR_2^c y$. Next, suppose $c(B \cup \{y\}) = x \neq b = c(B)$. By Reduction, bxy is a 3-cycle s.t. $c(b, x, y) = x$. By definition, $xR_2^c y$. ■

Remark 6 *Au and Kawai show that any transitive RSM can be represented by* $(\widehat{Q_1^a}, \succ_2^a)$ *where:* $x\widehat{Q_1^a}y$ *if* $c(A) = c(A \setminus y)$ *for all* $A \supseteq \{x, y\}$*; and,* $\succ_2^a$ *is any linear order that completes* R_2^a*. Lemma 7 shows that* $\widehat{Q_1^c}$ *and* $\widehat{Q_1^a}$ *are equivalent.*

Lemma 31 *If c can be represented in terms of shortlisting, then* $xP_1^c y$ *iff* $c(x, y) = x$ *and* $yP_2^c x$*.*

Proof. ($\Rightarrow$) Suppose $xP_1^c y$. By Lemma 19, $yP_2^c x$. By Lemmas 16-17, $P_1^c \subseteq \succ^c$. So, $c(x, y) = x$. ($\Leftarrow$) Suppose $c(x, y) = x$ and $yP_2^c x$. Since $P_2^c \subseteq Q_2^c$ by Lemma 20, $yQ_2^c x$. By way of contradiction, suppose $\neg[xP_1^c y]$. Then, $x(\succ^c \setminus P_1^c)y$ so that $xQ_2^c y$. Since $yQ_2^c x$, this contradicts Lemma 22. ■

Remark 7 *Au and Kawai define a revealed preference* P_1^a *for the first rationale by* $xP_1^a y$ *if* $x\widehat{Q_1^a}y$ *and* $yR_2^a...R_2^a x$. *Lemma 31 above effectively shows the equivalence between* P_1^c *and* P_1^a. *By Lemma 6,* $yP_2^c x$ *is equivalent to* $yR_2^a...R_2^a x$. *To see that* $x\widehat{Q_1^a}y$ *can be replaced by* $c(x,y)=x$ *in their definition:*

Proof. **(⇒)** Suppose $x\widehat{Q_1^a}y$. By definition of $\widehat{Q_1^a}$, $c(x,y)=x$. **(⇐)** Suppose $c(x,y)=x$ and $yR_2^a...R_2^a x$. To see that $x\widehat{Q_1^a}y$, suppose otherwise. Then, $c(x,y)=y$ (which contradicts $c(x,y)=x$) or $xR_2^a y$ (which contradicts the acyclicity of R_2^a). Thus, $x\widehat{Q_1^a}y$ as required. ■

To conclude, I correct an oversight in the proof of Au and Kawai's Theorem 1:

Remark 8 *While Claim 5 of their paper is correct, the proof contains an error. Along the lines of Lemma 5, they establish that* $c(B')=x$ *by Expansion. Using Weak WARP, Expansion, and* R_2^a*-Acyclicity, they also show that* $c(A'')=x$ *where* $A''\equiv A\setminus\{z\in A\setminus B' : c(y,z)=z \text{ or } xyz \text{ is a 3-cycle s.t. } c(x,y,z)=x\}$. *Then, they claim that Weak WARP and Expansion imply* $c(B'\cup\{w\})=x$ *for some* $w\in A''\setminus B'$ *such that* $c(w,y)=w$. *The choice function c in Example 2 provides a counter-example to this claim. It satisfies Expansion and Weak WARP but violates the property claimed by Au and Kawai (as well as Selective IIA) since* $c(w,y)=w$ *but* $c(w,x,y,z)=y\neq x=c(x,y,z)$.

By Lemma 32 below however, Claim 5 is nonetheless correct by the argument given in Lemma 5.

Lemma 32 *If c satisfies Expansion, Weak WARP, and* R_2^a*-Acyclicity, then it satisfies Selective IIA.*

Proof. By Lemma 4, it suffices to show that R_2^a-Acyclicity implies 3-Acyclicity. By way of contradiction, suppose there exist 3-cycles wxz, wyz s.t. $c(w,x,z)=x$ and $c(w,y,z)\neq y$. Then, $wR_2^a xR_2^a z$ and $zR_2^a w$. But, this contradicts R^a-Acyclicity, which establishes that c satisfies 3-Acyclicity. ■

B.4.2 The T_1SM Model

Remark 9 *Matsuki and Tadenuma [2013] axiomatize the* T_1SM *model with Expansion, Weak WARP, and the following:*

Elimination *If* $c(A)\neq y$ *for all* $A\supseteq\{x,y\}$, *then:*

(i) $c(B\cup\{x\})=x$ *for all B s.t.* $c(B)=y$; *or,* ***(ii)*** $c(B)=c(B\setminus\{y\})$ *for all B s.t.* $B\supseteq\{x,y\}$.

Formally, Elimination limits Exclusivity to pairs $\langle x,y\rangle$ *where* $c(A)\neq y$ *for all* $A\supseteq\{x,y\}$.

Remark 10 *Matsuki and Tadenuma also show that behavior consistent with the model can be* T_1SM*-represented by* $(\widehat{P_1^m},\widehat{Q_2^m})$ *where:* $\widehat{P_1^m}\equiv tc(\widehat{R_1^m})$; $x\widehat{R_1^m}y$ *if there exists a 3-cycle xyz s.t.* $c(x,y,z)=z$; *and,* $\widehat{Q_2^m}\equiv\succ^c\setminus\widehat{P_1^m}$. *Lemma 1 establishes that* $(\widehat{Q_1^c},\widehat{R_2^c})$ *defines another* T_1SM*-representation. The proof below shows* $\widehat{P_1^m}\subseteq P_1$ *and* $\widehat{R_2^c}\subseteq P_2$. *Together, these facts establish that* $\widehat{P_1^m}$ *and* $\widehat{R_2^c}$ *define revealed preference in the* T_1SM *model:*

- $x\widehat{P_1^m}y$ *iff* xP_1y *for every* T_1SM*-representation* (P_1,P_2) *of c; and,*
- $x\widehat{R_2^c}y$ *iff* xP_2y *for every* T_1SM*-representation* (P_1,P_2) *of c.*

Proof. Given a T_1SM-representation (P_1,P_2), the inclusions $\widehat{R_1^m}\subseteq P_1$ and $R_2^c\subseteq P_2$ are straightforward. I show: (a) $\widehat{P_1^m}\subseteq P_1$; and, (b) $\widehat{R_2^c}\subseteq P_2$. **(a)** Since $\widehat{R_1^m}\subseteq P_1$, $\widehat{P_1^m}\equiv tc(\widehat{R_1^m})\subseteq tc(P_1)=P_1$. **(b)** Since $R_2^c\subseteq P_2$, I show that $x\widehat{R_2^c}y$ implies xP_2y under branch (ii) of $\widehat{R_2^c}$. In that case: $c(x,y)=x$; and, $c(w,x,z)=z$, $c(w,y,z)=w$ for some 3-cycles wxz and wyz. By definition, $w\widehat{R_1^c}x$ and $y\widehat{R_1^c}z$. By (a), wP_1x and yP_1z. By way of contradiction, suppose $\neg(xP_2y)$. Since $c(x,y)=x$, xP_1y. Since $wP_1xP_1yP_1z$, wP_1z by transitivity. But, this contradicts $c(w,z)=z$. ■

B.4.3 The RSM Model

Remark 11 *Dutta and Horan's definitions of the revealed rationales* P_1^{RSM} *and* P_2^{RSM} *may be restated as follows:*

(1) $xP_1^{RSM}y$ *if c displays a direct* $\langle x,y\rangle$ *reversal; and,*

(2) $xP_2^{RSM}y$ *if c displays a weak* $\langle x,y\rangle$ *reversal for some menu B such that* $c(B)=y$.

The restatement of (1) is only a matter of definitions. However, the restatement of (2) depends on the additional observation that $c(A)=x$ *and* $c(B)=y$ *for* $B\supset A$ *imply* $c(x,y)=x$ *by Weak WARP.*

Example 7 *Let* (P_1,P_2) *denote the* T_2SM *on* $X\equiv\{x_i\}_{i=1}^n$ *defined by:*

– $x_iP_1x_j$ *if and only if* $j=i+1$*; and*

– P_2 *defined as a transitive rationale such that* $x_nP_2...P_2x_1$.

For $n\geq 4$, $c_{(P_1,P_2)}$ *is not TSM-representable. This follows from the fact that it violates 3-Acyclicity. To see this, note that* $x_{i-1}x_ix_{i+1}$ *is a 3-cycle s.t.* $c_{(P_1,P_2)}(x_{i-1},x_i,x_{i+1})=x_{i-1}$ *for* $1<i<n$. *Moreover, one cannot infer* $x_nP_2^{RSM}x_1$ *without* $c_{(P_1,P_2)}(X)=x_1$ *– since* $c_{(P_1,P_2)}(A)\neq x_1$ *for all* $\{x_1,x_n\}\subset A\subset X$.

Example 8 *Let* (P_1,P_2) *denote the* T_2SM *on* $X\equiv\{x_i\}_{i=1}^n\cup\{w,y,z\}$ *defined by:*

– $x_iP_1x_j$ *if and only if* $j=i+1$ *and* $x_nP_1yP_1zP_1w$*; and*

– P_2 *defined as a transitive rationale such that* $zP_2yP_2x_2P_2...P_2x_nP_2wP_2x_1$.

For $n\geq 1$, $c_{(P_1,P_2)}$ *is not TSM-representable. This follows from the fact that it violates 3-Acyclicity. To see this, note that* x_1x_2w *and* $x_1x_2x_3$ *are 3-cycles s.t.* $c_{(P_1,P_2)}(x_1,x_2,w)=w$ *and* $c_{(P_1,P_2)}(x_1,x_2,x_2)=x_1$. *Moreover, one cannot infer* $zP_1^{RSM}w$ *without* $c_{(P_1,P_2)}(X)=x_1$ *– since* $c_{(P_1,P_2)}(A\cup\{z\})=z$ *for all* $A\subset X\setminus\{z\}$ *such that* $c_{(P_1,P_2)}(A)=w$.