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Dynamic Facility Location with Modular Capacities: Models, Algorithms and Applications in Forestry

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RÉSUMÉ

Les décisions de localisation sont souvent soumises à des aspects dynamiques comme des changements dans la demande des clients. Pour y répondre, la solution consiste à considérer une flexibilité accrue concernant l'emplacement et la capacité des installations. Même lorsque la demande est prévisible, trouver le planning optimal pour le déploiement et l'ajustement dynamique des capacités reste un défi. Dans cette thèse, nous nous concentrons sur des problèmes de localisation avec périodes multiples, et permettant l'ajustement dynamique des capacités, en particulier ceux avec des structures de coûts complexes. Nous étudions ces problèmes sous différents points de vue de recherche opérationnelle, en présentant et en comparant plusieurs modèles de programmation linéaire en nombres entiers (PLNE), l'évaluation de leur utilisation dans la pratique et en développant des algorithmes de résolution efficaces.

Cette thèse est divisée en quatre parties. Tout d'abord, nous présentons le contexte industriel à l'origine de nos travaux : une compagnie forestière qui a besoin de localiser des campements pour accueillir les travailleurs forestiers. Nous présentons un modèle PLNE permettant la construction de nouveaux campements, l'extension, le déplacement et la fermeture temporaire partielle des campements existants. Ce modèle utilise des contraintes de capacité particulières, ainsi qu'une structure de coût à économie d'échelle sur plusieurs niveaux. L'utilité du modèle est évaluée par deux études de cas. La deuxième partie introduit le problème dynamique de localisation avec des capacités modulaires généralisées. Le modèle généralise plusieurs problèmes dynamiques de localisation et fournit de meilleures bornes de la relaxation linéaire que leurs formulations spécialisées. Le modèle peut résoudre des problèmes de localisation où les coûts pour les changements de capacité sont définis pour toutes les paires de niveaux de capacité, comme c'est le cas dans le problème industriel mentionnée ci-dessus. Il est appliqué à trois cas particuliers : l'expansion et la réduction des capacités, la fermeture temporaire des installations, et la combinaison des deux. Nous démontrons des relations de dominance entre notre formulation et les modèles existants pour les cas particuliers. Des expériences de calcul sur un grand nombre d'instances générées aléatoirement jusqu'à

100 installations et 1000 clients, montrent que notre modèle peut obtenir des solutions optimales plus rapidement que les formulations spécialisées existantes. Compte tenu de la complexité des modèles précédents pour les grandes instances, la troisième partie de la thèse propose des heuristiques lagrangiennes. Basées sur les méthodes du sous-gradient et des faisceaux, elles trouvent des solutions de bonne qualité même pour les instances de grande taille comportant jusqu'à 250 installations et 1000 clients. Nous améliorons ensuite la qualité de la solution obtenue en résolvent un modèle PLNE restreint qui tire parti des informations recueillies lors de la résolution du dual lagrangien. Les résultats des calculs montrent que les heuristiques donnent rapidement des solutions de bonne qualité, même pour les instances où les solveurs génériques ne trouvent pas de solutions réalisables. Finalement, nous adaptons les heuristiques précédentes pour résoudre le problème industriel. Deux relaxations différentes sont proposées et comparées. Des extensions des concepts précédents sont présentées afin d'assurer une résolution fiable en un temps raisonnable.

Mots clés: localisation dynamique d'installations, niveaux de capacités modulaires, programation linéaire en nombres entiers, relaxation lagrangienne, heuristiques.

ABSTRACT

Location decisions are frequently subject to dynamic aspects such as changes in customer demand. Often, flexibility regarding the geographic location of facilities, as well as their capacities, is the only solution to such issues. Even when demand can be forecast, finding the optimal schedule for the deployment and dynamic adjustment of capacities remains a challenge. In this thesis, we focus on multi-period facility location problems that allow for dynamic capacity adjustment, in particular those with complex cost structures. We investigate such problems from different Operations Research perspectives, presenting and comparing several mixed-integer programming (MIP) models, assessing their use in practice and developing efficient solution algorithms.

The thesis is divided into four parts. We first motivate our research by an industrial application, in which a logging company needs to locate camps to host the workers involved in forestry operations. We present a MIP model that allows for the construction of additional camps, the expansion and relocation of existing ones, as well as partial closing and reopening of facilities. The model uses particular capacity constraints that involve integer rounding on the left hand side. Economies of scale are considered on several levels of the cost structure. The usefulness of the model is assessed by two case studies. The second part introduces the Dynamic Facility Location Problem with Generalized Modular Capacities (DFLPG). The model generalizes existing formulations for several dynamic facility location problems and provides stronger linear programming relaxations than the specialized formulations. The model can address facility location problems where the costs for capacity changes are defined for all pairs of capacity levels, as it is the case in the previously introduced industrial problem. It is applied to three special cases: capacity expansion and reduction, temporary facility closing and reopening, and the combination of both. We prove dominance relationships between our formulation and existing models for the special cases. Computational experiments on a large set of randomly generated instances with up to 100 facility locations and 1000 customers show that our model can obtain optimal solutions in shorter computing times than the existing specialized formulations. Given the complexity of such models for large instances, the third part of the thesis proposes efficient Lagrangian heuristics. Based on subgradient and bundle methods, good quality solutions are found even for large-scale instances with up to 250 facility locations and 1000 customers. To improve the final solution quality, a restricted model is solved based on the information collected through the solution of the Lagrangian dual. Computational results show that the Lagrangian based heuristics provide highly reliable results, producing good quality solutions in short computing times even for instances where generic solvers do not find feasible solutions. Finally, we adapt the Lagrangian heuristics to solve the industrial application. Two different relaxations are proposed and compared. Extensions of the previous concepts are presented to ensure a reliable solution of the problem, providing high quality solutions in reasonable computing times.

Keywords: dynamic facility location, modular capacities, mixed-integer programming, lagrangian relaxation, heuristics.

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LIST OF ABBREVIATIONS

Combinatorial Optimization Problems

1I	One index (normally refers to a formulation)
2I	Two indices (normally refers to a formulation)
4I	Four indices (normally refers to a formulation)
CFLP	Capacitated Facility Location Problem (synonym: Capacitated Plant Location Problem)
CSLP	Camp Size and Location Problem
DFLP	Dynamic Facility Location Problem
DFLPG	Dynamic Facility Location Problem with Generalized Modular Capacities
DFLP_PC	Dynamic Facility Location Problem with Partial Facility Closing
DFLP_RPC	Dynamic Facility Location Problem with Relocation and Partial Facility Closing
FLP	Facility Location Problem
GMC	Generalized Modular Capacities
MCFLP	Modular Capacitated Facility Location Problem
MCKP	Multiple-Choice Knapsack Problem
UFLP	Uncapacitated Facility Location Problem (synonym: Simple Plant Location Problem)

Mathematical Programming

B&B	Branch-and-Bound
B&C	Branch-and-Cut
LP	Linear Programming / Linear Program
IP	Integer Program
MIP	Mixed-Integer Program
ADC	Aggregated Demand Constraints
RUC	Round-Up Capacity Constraints
SAD	Strengthened Aggregated Demand Constraints
SI	Strong Inequalties (synonym: Strong Linking Constraints)

Miscellaneous

LB	Lower Bound
UB	Upper Bound

To my family.

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CHAPTER 1

INTRODUCTION

The location of facilities is considered one of the most important decisions in logistics. Both the private and public sectors have shown a particular interest in the study of facility location, as they require to strategically locate warehouses, factories, fire stations, schools, telecommunications hubs, and many others. Choosing the ideal location for a facility greatly depends on the application context and may take into consideration aspects that are as diverse as the distance to intermediate storage locations or final customers, the accessibility of the facility terrain, the location's susceptibility to natural disasters, the accessibility and prices of necessary raw materials, the availability of qualified employees, or tax considerations and governmental initiatives. In most of the cases, the locations of facilities are strategic decisions that have a long lasting impact on the operational costs.

Given its economic importance, facility location has been of high interest for Operations Research (OR). In classical facility location, decisions aim to strike a balance between the fixed costs to supply capacity and the allocation costs to serve the demand. The latter often correspond to transportation costs to deliver products or provide services to customers. The vast literature on facility location problems can be traced back to as early as the beginning of the 20th century (Weber, 1929), when a single facility had to be placed to best serve the demand of customers. It has since been extended to a large variety of application contexts, with different objectives and different constraints. Hospitals have to be placed such that the maximum distance to the population is minimal, obnoxious facilities have to be placed as far as possible from the population, and budget constraints may limit the total investment. Among the many extensions that have been proposed, the most common ones include different commodity types for the customer demands, multiple periods in which customers may have different demands, as well as the choice of the facility size.

Most facility location problems are *NP-hard*, i.e., no algorithms are available that may solve these problems in polynomial time. However, many powerful solution methods have been developed to solve these problems. With constant advances in information technology, as well as an increasing understanding of solution algorithms and the structure of combinatorial optimization problems, the OR community has successfully solved increasingly complex and realistic problems.

Today's challenges to advance facility location research may be divided into at least three categories, each of them holding opportunities to achieve significant impact in practice. First, dynamic facility location takes into account the change in planning parameters over time. Uncertainty, in particular concerning the customer demands, may require a robust choice of the facility location. Population shifts, evolving market trends and changes of other environmental factors often require adjustments reaching from facilities to the entire supply chain. These adjustments often concern the decisions of where and when to provide capacity to best satisfy the customer demands. Another important challenge is to represent a problem in a more realistic manner. Typically, this can be done by ensuring that the individual problem constraints are modeled realistically and by representing the cost structure of the problem on a sufficiently detailed level. Economies of scale have often been considered on levels such as the facility's construction, maintenance and production costs. They may also be found on other levels such as the costs to deliver the products to the customers. Finally, integrated planning problems aim at acknowledging the interaction between several planning problems and try to solve them simultaneously, such as in location-routing problems and in integrated facility location with network design. Taking these aspects into consideration when designing facility location problems holds a valuable opportunity to represent the problems in a more realistic manner and therefore provide more tools for supporting decision making processes. However, even today, modeling and solving such problems remains a challenge.

The objective of this thesis is to contribute with models and algorithms to solve dynamic facility location problems, in particular responding to the first two of the above mentioned challenges. This thesis focuses on multi-period facility location problems in which the planning parameters may be subject to significant changes over time. Facilities adapt to the new environment by adjusting the available capacity at each of the locations. The proposed models also take into consideration a very detailed level of the cost structure, enabling a more realistic representation of the problems. The work on these problems has been inspired by an industrial collaboration with *FPInnovations*, one of the world's largest private, non-profit research centers working in forest research. The project aims at providing a decision support tool for a Canadian logging company that has to locate logging camps to host the workers involved in forestry operations. Although it is an extension of classical multi-period facility location problems, this problem does not only possess very specific constraints, but also a very detailed cost structure. As solving these problems exactly by the use of generic mixed-integer programming (MIP) solvers is only successful for small problem instances, we develop heuristics based on Lagrangian relaxation to provide high quality solutions in short computation times, even for large-scale instances.

The remainder of this thesis is organized as follows. In Chapter 2, we review the literature for facility location problems. Given the vast amount of literature that has been produced in this domain in the last decades, we focus on the classical *Capacitated Facility Location Problem (CFLP)* and its variants. In the first part, the CFLP is discussed within the context of location analysis. A classification scheme is provided to guide the discussion on variants and extensions of the classical problem. In particular, we discuss the choice of the facility size, multiple commodities and dynamic adjustment of capacities. The second part of the chapter concerns solution methods that have been proposed for these problems. We then draw conclusions concerning the existing literature.

Chapter 3 introduces an industrial application that can be found in the Canadian forestry sector. The problem is referred to as the *Camp Size and Location Problem (CSLP)* and investigates where to locate and relocate logging camps to host workers involved in the forest operations. This problem can be abstracted to a multi-period facility location problem with multiple commodities that allows for the capacity expansion at facilities, as well as the relocation of facilities from one location to another. This problem contributes to the literature by extending existing problems in several ways. First, facilities may be partially closed during certain time periods. Second, the problem possesses particular capacity constraints in which the total demand allocated to each facility is rounded to the next highest integer value. Finally, the problem has a detailed cost structure for capacity changes, i.e., capacity expansion, as well as for the closing and reopening of capacities. A MIP model based on capacity flows, as well as new valid inequalities for the particular capacity constraints, are presented. All individual problem characteristics are taken into account, while the detailed cost structure for capacity changes is approximated. It is shown how the problem can be tackled more efficiently by solving a simplified version of the problem and using its solutions to warm start the MIP solver. Two case studies exemplify the usefulness of the model in practice.

In Chapter 4, we then introduce a very general dynamic facility location problem, referred to as the *Dynamic Facility Location Problem with Generalized Modular Capacities (DFLPG)*. The problem is characterized by modular capacity changes subject to a detailed cost structure. Due to its generality, the proposed MIP model unifies several existing problems found in the literature. This is illustrated by means of three special cases: the problem with facility closing and reopening, the problem with capacity expansion and reduction, and the combination of the two. The cost structure used in the DFLPG is based on a matrix describing the costs for capacity changes between all pairs of capacity levels, capable to represent complicated cost structures such as the one found in the CSLP. We are not aware of any other work dealing with facility location with a similar level of detail in the cost structure. We analyze the linear programming (LP) relaxation bound obtained by our model, showing that it is at least as strong as the LP relaxation bound of existing specialized formulations. Furthermore, we perform computational studies on a large set of randomly generated instances with up to 100 candidate facility locations (each with up to 10 capacity levels), 1000 customers and 14 time periods. The results show that our model, when solved with a state-of-the-art MIP solver, can obtain optimal solutions in significantly shorter computation times than the specialized formulations for the three special cases.

Chapter 5 is devoted to the solution of a DFLPG extension in which customers have demands for different commodities. We propose Lagrangian based heuristics that find good quality solutions in reasonable computing times. Two methods are used to solve the Lagrangian dual: a subgradient method and a bundle method. After this process, a second optimization step is used to improve the solution quality. This step consists of solving a restricted MIP model, taking into consideration only decisions that have been part of a significant number of the previous Lagrangian solutions. Computational results are given for large instances with up to 250 candidate facility locations and 1000 customers. The results are stable even for large instances, for which general-purpose MIP solvers either consume too much memory or do not solve the problem in reasonable time. To the best of our knowledge, this work is the first to present a Lagrangian relaxation approach to solve large-scale instances of a multi-period facility location problem of this nature, i.e., including modular capacity adjustments and multiple commodity types.

We then close the loop in Chapter 6 by demonstrating how to extend the Lagrangian heuristics to the case of the CSLP or similar problem variants. An alternative formulation, based on the same modeling technique used to model the DFLPG, is presented. Even though the size of this formulation is too large to be handled by generic MIP solvers, the subproblems are of quite reasonable size when decomposed by Lagrangian relaxation. The algorithm from the previous chapter is modified to additionally account for the partial closing and reopening of facilities, the relocation of facilities, and the particular capacity constraints defined in the CSLP. Two different relaxations are presented, each relaxing different sets of constraints. Computational results show the benefit of the Lagrangian heuristics when compared to the use of generic MIP solvers.

Finally, Chapter 7 summarizes the contributions of this thesis and discusses potential future research directions.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we review the literature relevant to the facility location problems we study in this thesis. Section 2.1 introduces classical facility location problems, their variants and extensions, as well as their applications. Section 2.2 focuses on solution methods for the previously introduced problems. Finally, in Section 2.3, we discuss the importance of the existing literature for the work presented in this thesis.

2.1 An Overview of Facility Location Problems and Applications

This section introduces facility location in the broader context of location analysis and reviews classical facility location problems and their variants and extensions. Then, the most common applications are discussed.

2.1.1 Facility Location in the Context of Location Analysis

Location Analysis is concerned with identifying the optimal locations subject to context related constraints. Often, the former are referred to as the locations of facilities, placed to efficiently serve the demand of customers. Literature on location problems can be traced back to as early as 1909 in a book by Alfred Weber, first published in German and later translated into English (Weber, 1929). This work considered the location of a production facility to minimize the total sum of distances to a set of customers. The field of location analysis mostly grew during the 1960's (Smith et al., 2009) and discrete location problems are nowadays a large branch of combinatorial optimization. Many schemes have been proposed to classify location models (Hamacher and Nickel, 1998). One of the basic criteria categorizes the problems into analytic, continuous, network, and discrete location models (Daskin, 2008; Revelle et al., 2008). *Analytic models* are the simplest form of location problems and are based on simplifying assumptions regarding their constraints and their objective function, such as the cost structure. They are typically solved analytically, using calculus or other techniques. *Continuous location models* typically assume a discrete set of demand points, while the locations for the facilities are chosen in the continuous space. A well known example for this class of location models is the above mentioned Weber problem. *Network models* assume that facilities can be placed on the nodes or links of a specified network, while demands are typically placed on the nodes. *Discrete location models* form a special case of network models where demands are given by a discrete set of nodes and facilities may be located on a discrete set of candidate locations.

The category of discrete location models has constantly evolved in the last decades and offers a rich literature on different problems and solution methods to solve them. It can be further classified (Daskin, 2008; Revelle et al., 2008) into median and plant location problems, as well as center and covering problems. A similar classification has also been proposed by Revelle and Eiselt (2005). The development of location analysis from its early beginning and today's most important applications are also reviewed by Smith et al. (2009). Covering problems investigate the minimum number of facilities necessary to guarantee a certain maximum distance between the customers and their assigned facilities. Center problems aim at minimizing the maximum distance between the customers and facilities to which they are allocated.

The *p-median* problem is the simplest problem of the first category of discrete location models. It is known to be NP-hard (Kariv and Hakimi, 1979a,b) and aims at finding the optimal locations for p facilities such that the total average weighted distance between the customers and their assigned facilities is minimal. The essence of this problem, i.e., serving customer demands at minimum cost, is preserved in most of the problem extensions. Typically known as plant or facility location problems, these variants generalize the p-median problem by introducing heterogeneous construction costs and a flexible choice of the number of facilities p.

The authors cited above also comment on other classes of discrete location models. Competitive location problems (Eiselt et al., 1993) deal with facility location in the presence of competitors. Locations have to be chosen such that the market share is maximized. Hub location problems (O'Kelly, 1986; Contreras et al., 2011b; Campbell and O'Kelly, 2012) locate transportation hubs according to a given flow from origins to destinations. While facilities are usually located as close as possible to their customers, some problems may aim at the opposite. For example, undesirable (or obnoxious) facilities (Berman and Wang, 2008) may impose health risk to the population and therefore have to be located in a sufficient distance. The combination of different planning problems has also been considered, such as the Location-Routing problem (Contardo et al., 2013; Prodhon and Prins, 2014) and integrated logistics network design (Cordeau et al., 2006).

In the following, we will focus on facility location problems and their main variants proposed in the literature.

2.1.2 Capacitated Facility Location

In the following, we review classical capacitated facility location problems. We then discuss literature surveys and classical problem extensions. A classification scheme is provided that guides the discussion on different problem characteristics.

2.1.2.1 The Capacitated Facility Location Problem

As has been mentioned in the previous section, facility location problems are important extensions of the classical p-median problem. The *Uncapacitated Facility Location Problem (UFLP)*, also known as the *Simple Plant Location Problem*, aims at selecting a number of facility locations from a discrete set of candidate locations $j \in J$, considering the construction costs f_j for each of the facilities. Customer demands d_i are given for each customer i defined by a discrete set I. These demands have to be satisfied at minimum cost, also taking into consideration the aggregated production and transportation costs c_{ij} to serve one demand unit of customer i by facility j. Clearly, the capacitated case, referred to as the *Capacitated Facility Location Problem (CFLP)*, is a more realistic model, as production capacities are usually limited. It is known to be strongly NP-hard (Cornuéjols and Sridharan, 1991). Here, the binary variables y_j take value 1 if the facility at location j is selected. Its capacity is given by u_j. The continuous variables x_{ij} represent the fraction of the demand from customer i that is served by facility j. Using this notation, the CFLP can be formulated as follows (Sridharan, 1995):

$$(CFLP) \quad \min \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} c_{ij} d_i x_{ij} \tag{2.1}$$

$$s.t. \sum_{j \in J} x_{ij} = 1 \quad \forall i \in I \tag{2.2}$$

$$\sum_{i \in I} d_i x_{ij} \leq u_j y_j \quad \forall j \in J \tag{2.3}$$

$$0 \leq x_{ij} \leq 1 \quad \forall i \in I, \forall j \in J \tag{2.4}$$

$$y_j \in \{0,1\} \quad \forall j \in J. \tag{2.5}$$

The problem minimizes the total cost composed by facility construction and demand allocation. Equalities (2.2) ensure that all customer demands are met. Constraints (2.3) are the capacity constraints at the facilities.

Note that the presented model allows the demand of a customer to be met by different facilities. Certain variants require that each customer is allocated to a single facility, defining x_{ij} as binary, also referred to as *single-sourcing*.

2.1.2.2 Literature Surveys and Problem Classification

The facility location community benefits from a rich and diverse literature dating back to the early 19th century (Krzyzanowski, 1927). The diversity, importance and maturity of the field has been confirmed by many recent literature surveys (Hamacher and Nickel, 1998; Klose and Drexl, 2005; Melo et al., 2009a; Revelle and Eiselt, 2005; Revelle et al., 2008; Smith et al., 2009; Zanjirani Farahani and Hekmatfar, 2009). Several classification schemes have been proposed to point out similarities and differences between the existing models. Melo et al. (2009a) focuses on criteria in the context of Supply Chain Management, whereas Klose and Drexl (2005) specify a classification for facility location problems. One may slightly extend their classification scheme and characterize facility location problems by the following properties:

– **Metric of the underlying network**. Based on the shape or topology of the transportation network, the distances and costs may be based on Euclidean distances or other more complex structures.
– **Type of the objective function**. The problem may minimize the total sum of distances or the maximum distance between the customers and the facilities they are assigned to.
– **Facility capacities**. If facilities possess capacities, they may have fixed or flexible capacities. Capacity modifications may be continuous or chosen according to predefined capacity levels.
– **Single-facility vs. multi-facility**. Each location may either possess a single facility or several facilities, independent or interacting.
– **Single-echelon vs. multi-echelon**. In multi-echelon models, the commodity flow may pass trough several echelons, such as facilities, warehouses, depots, and finally the customer. Reverse flows may be allowed. A direct flow from facilities to customers corresponds to a single-echelon model.
– **Single-commodity vs. multi-commodity**. Customers may have demands for different commodities. Facilities may produce only a certain subset and a certain quantity of commodities.
– **Single-period vs. multi-period**. Models with a single time period rarely correspond to realistic applications. Multiple time periods may involve independent demands and costs for each of the time periods, as well as the opportunity to adjust the locations and capacities of facilities along the planning horizon.
– **Deterministic vs. uncertain**. In practice, certain input data may be subject to uncertainty. Even when data, such as customer demands, can be well predicted, the real values will most probably differ from the predicted value.
– **Single-source vs. multi-source**. Customer demands are either met by a single facility or by different facilities.

Some of these characteristics are similar for most of the works found in facility location literature. For example, the majority of the proposed models aims at minimizing the total costs to serve the customers. Only few assume a fixed budget (e.g., Wang et al., 2003; Sonmez and Lim, 2012). In a similar manner, most of the works assume that customer demands may be served by different facilities (multi-source), while only a few constrain the customer demands to single-sourcing (e.g., Agar and Salhi, 1998; Holmberg et al., 1999; Albareda-Sambola et al., 2009). Some other characteristics have evolved to individual classes of facility location problems, each of them offering a tailored literature on how to model and solve them. In the following, we use the above classification criteria to guide the discussion on the existing facility location literature. We emphasize problem characteristics that are found in the CSLP (see Section 3), the industrial application that has inspired large parts of the research presented in this thesis. Multi-period problem variants are discussed in Section 2.1.3.

2.1.2.3 Choice of the Facility Size and Cost Structure

Given that in most of the application contexts resources are finite, the majority of the proposed facility location models impose capacities on the facilities. In an effort to represent cost structures realistically, many researchers acknowledged the importance of economies of scale (Holmberg and Ling, 1997; Agar and Salhi, 1998; Correia and Captivo, 2003; Correia et al., 2010), i.e., the larger the facility, the cheaper the unit price in terms of facility construction and commodity production. Similarly, some applications involve cost structures that imply inverse economies of scale (Harkness, 2003), where the unit price increases as the facility gets larger. To enable the representation of such cost structures, many models decide not only for the location of facilities, but also for their total capacity instead of assuming a fixed capacity. Problems that involve the choice of the facility size are known under different names, such as the *Dynamic Capacitated Plant Location Problem* (Shulman, 1991) in the multi-period context, and the *Multi-capacitated Plant Location Problem* (Agar and Salhi, 1998) or *Modular Capacitated Location Problem* (Correia and Captivo, 2003) in the single-period context. Early works with a choice of capacities include those of Lee (1991, 1993a,b), Shulman (1991) and Sridharan (1991). In these works, the choice of the capacity level is modeled using an additional variable index, resulting in a facility variable of the form $y_{j\ell}, \ell \in L$, where L is a discrete set of available capacity levels or facility types assigned to a fixed capacity. This intuitive modeling technique has been adapted by several other researchers (Sankaran and Raghavan, 1997).

An alternative modeling technique has been presented in Holmberg (1994) and Holmberg and Ling (1997). The authors use an incremental approach to model staircase functions, where all variables up to the chosen capacity level are active. Similar approaches have since been adapted for more complex location problems (Correia and Captivo, 2003; Gouveia and Saldanha da Gama, 2006). While most of these works consider economies of scales in the construction costs, Correia and Captivo (2003) also represent economies of scale in the total amount of produced commodities. The authors separate the decision of the production level from the demand allocation variables x by using additional binary variables of the form $w_{j\ell}$ to indicate the total amount produced at facility j working at capacity level ℓ.

Most of the works discussed above propose models where a single facility can be located at each location. However, the total capacity available at a site may also be configured by choosing more than one facility at the same location (Wu et al., 2006). Some of these works involving multiple time periods are discussed further below in the context of capacity expansion and reduction over time.

2.1.2.4 Multiple Commodities

Customers may have demands for several distinct commodities and facilities may produce different commodity types. Multiple commodities have become a common extension to classical facility location problems, in particular since they do not further complicate the structure of the model. Models can be distinguished between those that allocate a separate production capacity for each commodity (Canel et al., 2001; Geoffrion and Graves, 1974; Lee, 1991; Warszawski, 1973; Pirkul and Jayaraman, 1998) and those that assume that the production capacity of a facility covers all commodity types at once (Melo et al., 2006). The constraint type depends on the application context. The first type is often used to indicate different technologies or facility types that enable the production of a certain commodity type (Lee, 1991; Pirkul and Jayaraman, 1998). It is modeled by using separate capacity constraints for each commodity. In the second type, the same capacity constraint is used for all commodities and sums up the entire demand allocation.

New research directions also explore the interaction between facilities that produce different commodities, where one facility may benefit from the by-product of another nearby facility (Xie and Ouyang, 2013).

2.1.2.5 Other Generalizations and Variants

Another important class of facility location problems takes into account stochastic and probabilistic elements (Snyder, 2006). In these problems, the input data is not deterministic, but subject to uncertainty. Uncertainty has mostly been assumed to concern the customer demands (Schütz et al., 2008). However, it may also concern other input data such as the traveling times on the transportation network (Berman and LeBlanc, 1984).

Multi-echelon facility location (Zanjirani Farahani et al., 2014), also referred to as multi-level, multi-layer or multi-stage facility location, assumes that the product passes several layers before it reaches its final destination. These kinds of models are very common to model supply chains (Thomas and Griffin, 1996), also known as production-distribution systems (Thanh et al., 2008), where the commodities may be produced in facilities, stored in warehouses and sent to stores or customers. In many applications, there is a natural hierarchy given that the product moves downstream in the supply chain (Gendron and Semet, 2009; Gendron et al., 2011, 2013). In contrast, problems in which the network of commodity flow may contain cycles are said to involve reverse flows and fall in the area of reverse logistics (Alumur et al., 2012).

The majority of works discussed in this thesis have one simple and common objective: the minimization of costs. However, many other objectives are possible. Multi-objective problems aim at combining several, often conflicting, objectives. A very typical example is the combination of traditional economic objectives and the reduction of environmental impact. Given that the public becomes more aware of environmental issues, both the governmental and private sectors will most likely analyze how *green* their supply chain and production process are (Dekker et al., 2012), aiming at the reduction of their carbon footprint and the opportunities to recycle. These are opportunities to explore reverse logistics supply chains as discussed above, but also to combine the traditional economic objective with environmental goals and their impacts (e.g., Hugo and Pistikopoulos, 2005; Harris et al., 2014). Other facility location problems involving multiple objectives include the work of Melachrinoudis (2000), which additionally aims at minimizing the time to access the product.

2.1.3 Dynamic Facility Location

The majority of facility location models are applied to strategic long-term planning. However, customer demands, as well as the prices for production, transportation and commodities tend to change over time. Multi-period models aim at coping with these challenges by defining independent demands and costs for each time period. Early works in the domain of dynamic facility location were initiated by authors such as Ballou (1968), Wesolowsky (1973), Wesolowsky and Truscott (1975) and Sweeney and Tatham (1976). A few authors used the term *dynamic* in a broader context (Arabani and Zanjirani Farahani, 2011; Zanjirani Farahani and Hekmatfar, 2009), also including stochastic aspects. However, the majority of the literature limited the use of the term to the context of multi-period problems. From the modeling viewpoint, the temporal aspect is usually captured by an additional variable index t. While a few models represent the available capacity by an additional flow variable $z_{jt} \in \mathbb{R}^+$, most of the works incorporate modular capacities, using binary variables of type $y_{j\ell t}$, where ℓ is either a capacity level or a facility type linked to a fixed amount of capacity. While the optimal timing of a facility construction, as well as its initial capacity are important decisions (e.g., Shulman, 1991), it has often been found beneficial to adjust capacities at later time periods to better respond to changing demand and market conditions (Owen and Daskin, 1998). Mathematical models that include such features have been applied in both the private and the public sectors to determine locations and capacities for production facilities, entire supply chains (Melo et al., 2006), telecommunications networks (Chardaire et al., 1996), schools (Antunes and Peeters, 2001), ambulances (Brotcorne et al., 2003), emergency services (Hochbaum, 1998) and many more, responding to population shifts and other environmental factors. Several surveys (Owen and Daskin, 1998; Arabani and

Zanjirani Farahani, 2011; Zanjirani Farahani and Hekmatfar, 2009) reviewed the growing literature on dynamic facility location problems, which suggested different ways to adjust capacities throughout a given planning horizon:

- The construction of a facility at a certain time period.
- The expansion or reduction of capacity at an existing facility.
- The temporary closing of a facility and reopening at a later time period.
- The relocation of capacity from one location to another.

The timing of facility construction is part of most of the multi-period facility location problems. We now review the existing literature for the other three features.

2.1.3.1 Capacity Expansion and Reduction

When customer demands of certain regions permanently change and are not likely to return to their previous levels, it may be beneficial to add or reduce (or even permanently shut down) production capacities at an existing facility to permanently adjust to the new conditions.

Luss (1982) discusses modeling techniques for capacity expansion. He points out that the total capacity available at a location may either be provided by a single facility or be composed by several coexisting facilities. The first category includes models that allow one facility at a location that increases or decreases the available capacity over time (Jacobsen, 1990; Canel et al., 2001; Antunes and Peeters, 2001; Melo et al., 2006; Behmardi and Lee, 2008). These models typically use flow variables of type $z_{jt} \in \mathbb{R}^+$ and manage the expansion ($s_{jt} \in \mathbb{R}^+$ variables) or reduction ($r_{jt} \in \mathbb{R}^+$ variables) of capacity by using flow conservation constraints similar to the following:

$$z_{jt} = z_{j(t-1)} + s_{jt} - r_{jt} \quad \forall i \in I \ \ , \forall t \in T \tag{2.6}$$

Models in the second category commonly use integer variables to indicate the number of existing facilities at a location. When the problem allows for capacity expansion, but not reduction, the total capacity can also be composed of several binary variables, one for each constructed facility or expanded capacity (Shulman, 1991; Troncoso and

Garrido, 2005). This modeling technique can also be found in other classes of location problems, such as variants of the *Capacitated Concentrator Problem* (Gouveia and Saldanha da Gama, 2006; Gourdin and Klopfenstein, 2008; Correia et al., 2010).

When the problem involves both capacity expansion and reduction, an alternative modeling technique (Dias et al., 2007) can be used involving binary variables of type $y_{j\ell t_1 t_2}$ to indicate that a capacity of size ℓ is added for a period defined by the interval $[t_1, t_2]$. The total capacity available at a location and time period is then computed by the sum of all facilities (capacity blocks) available at that time period, enabling a flexible expansion and reduction of capacity along time. The two different categories, using flow variables and capacity block variables, are illustrated in Figure 4.1. We refer to Section 4.3 for a detailed discussion of these modeling techniques.

Next to classical capacity expansion and reduction, several special cases with individual restrictions have been presented. In the work of Antunes and Peeters (2001), facilities may either expand or decrease their capacities throughout the planning horizon, but not both. We refer to the book chapter of Jacobsen (1990) for more references to works that consider capacity expansion.

2.1.3.2 Temporary Facility Closing and Reopening

In some situations, it may be beneficial to temporarily close a facility, for example to avoid high maintenance costs. This may be appropriate when demand temporarily decreases, but is likely to return to its previous level afterwards. While, in practice, it may be possible to close only parts of a facility, previous studies focused on the temporary closing of entire facilities. Among the suggested models, certain are limited to a single closing and reopening of each facility, whereas others allow repeated closing and reopening throughout the planning horizon. The uncapacitated facility location problem presented by Van Roy and Erlenkotter (1982), as well as the supply chain model of Hinojosa et al. (2008), allow one-time opening or closing of facilities: new facilities can be opened once and existing facilities can be closed once. Chardaire et al. (1996) and Canel et al. (2001) propose formulations for opening and closing facilities more than once. The former installs and removes terminals in telecommunications networks to adapt to changes in data traffic and costs along time. The authors of both works use binary variables of type y_{jt} to indicate whether a facility is open or closed during a certain period. A closing or reopening is then indicated by a quadratic term $y_{jt}(1-y_{jt})$ in the objective function. A linear formulation for a simplified variant of this problem with fixed capacity levels has been proposed by Dias et al. (2006).

The works cited above interpret facility closing either as temporary (i.e., the facility still exists, but its capacities are temporarily unavailable) or permanent (a facility is shut down). In most cases, maintenance costs for temporarily closed facilities are low and can therefore be ignored in the model. Most of the existing formulations therefore do not explicitly distinguish temporary and permanent facility closing. Furthermore, permanent facility closing may also be seen as a special case of capacity reduction.

2.1.3.3 Facility Relocation

In certain contexts, the relocation of existing capacity from one location to another may be a possibility to shift capacity closer to the demand points. Wesolowsky and Truscott (1975) have been one of the first to consider simple relocation of facilities. The authors use flow conservation constraints similar to (2.6), but with binary variables (instead of flow variables) to indicate whether a facility is available or not. That is, instead of variables representing capacity expansion and reduction, the model contains binary variables to represent the relocation from or to the location.

The relocation of facilities has since been considered by several researchers. Min and Melachrinoudis (1999) document a case study for a company that relocates warehouses and Melachrinoudis (2000) provides an appropriate model. Brotcorne et al. (2003) review location-relocation models for ambulances for deterministic and probabilistic scenarios. Melo et al. (2006) and Melo et al. (2009b) provide an extensive modeling framework for modeling generic multi-level supply chain network structures. Their model is based on flow conservation constraints and focuses on gradual relocation of existing capacity. The authors also show how to link binary variables to indicate the facility type, as well as the origin and destination for the relocated facility. However, it can be noted that most of the other works ignore the distance the facilities are relocated and therefore allocate equal costs to all facility relocations.

Often, the closing of a facility at one location and opening at another location has also been interpreted as a facility relocation, which has been considered under the constraints of a global budget (Wang et al., 2003) and under demand uncertainty (Lim and Sonmez, 2013).

Relocation models have also been proposed under more restricted conditions. Amiri-Aref et al. (2011) present a non-linear mixed-integer formulation to relocate emergency maintenance rooms given that the transit availability for certain regions are subject to uncertainty. Zanjirani Farahani et al. (2008) locate and relocate a single facility under the condition that costs vary according to a continuous weight function. Albareda-Sambola et al. (2009) introduce a problem in which facilities must select a certain number of customers. Once served, customers have to be served in all subsequent periods.

2.1.4 Applications

Facility location problems have been applied in many different contexts. In the private sector, facility location models most often concern the locations of manufacturing and distribution systems (Min and Melachrinoudis, 1999; Broek et al., 2006) and telecommunications networks (Chardaire et al., 1996). Location models have also been often used in the public sector to locate schools (Antunes and Peeters, 2001), hospitals (Vahidnia et al., 2009) or for military logistics (Gue, 2003; Ghanmi, 2010). Many more references can be found in surveys such as those by Arabani and Zanjirani Farahani (2011) and Melo et al. (2009a)

The forestry sector has also been an active user of facility location and supply chain optimization models. Transportation in the forestry domain accounts for a large part of the total operational costs (Audy et al., 2012), reported to be 25-35% in Southern USA, more than 35% in Canada and more than 45% in Chile. Naturally, studies have been strongly contributed from countries with significant log export, such as it is the case in Canada (Haartveit et al., 2004; Vila et al., 2006), Chile (Epstein et al., 1999; Troncoso and Garrido, 2005) and the Scandinavian countries (Rönnqvist, 2003; Bredström et al., 2004; Carlsson and Rönnqvist, 2005). Chan et al. (2009) provide a facility location model to place satellite yards. A comprehensive introduction is provided by D'Amours et al. (2008), also providing further references. We also refer to the work of Vahid and Maness (2010) who have recently reviewed and classified supply chain literature for the forestry sector.

2.2 Solution Methods

We now review solution methods that have commonly been used to solve facility location problems. Solution methods for optimization problems can be distinguished into two broad classes: *exact* and *heuristic* methods. Exact methods solve the problem to optimality, given that sufficient time and memory is available. If, for some reason, the problem is not solved to optimality, exact methods can provide bounds on the optimal solution value. As has been seen in the previous section, it has become common practice to model location problems as MIP models. It is equally common that generic MIP solvers, usually based on elaborate exact methods, are used to solve these models. However, even though generic solvers and information technology constantly advance, OR practitioners try to model real world applications more and more realistically and solve instances as large as possible. Given the complexity of the resulting models, it is often not possible to solve them exactly. Heuristics aim at providing high quality solutions in short computing times even for large problems. Several surveys (Sridharan, 1995; Arabani and Zanjirani Farahani, 2011; Melo et al., 2009a) review some of the many exact and heuristic solution methods that have been proposed. Some of the algorithmic advances have been implemented to improve the performance of generic solvers, while other approaches require to be customized to each problem and therefore form a methodological category by themselves. In the following, we will review the literature for the methods that have been most successful and popular to solve facility location problems.

2.2.1 Exact Methods: Polyhedral Approaches

Most of the facility location problems have been modeled as linear MIP models. Many of them (Sankaran and Raghavan, 1997; Melachrinoudis, 2000; Melo et al., 2006;

Wilhelm et al., 2013) have been solved to optimality within reasonable time by general-purpose solvers. These solvers, such as IBM ILOG CPLEX (IBM, 2010), the Gurobi Optimizer (Gurobi Optimization, Inc., 2014) and COIN-OR (2014), aim at providing an efficient framework to solve generic, in particular linear MIP models, to optimality. To prove optimality, generic MIP solvers are typically based on some sort of *Branch-and-Bound (B&B)* algorithm. Certain binary variables are fixed to one of their feasible values, forming a branch in the B&B tree. Typically, the LP relaxation then provides a lower bound (in the case of a minimization problem) on the optimal integer solution value that may be found in the corresponding branch. Integer feasible solutions are typically used to provide an upper bound on the optimal integer solution. As B&B algorithms rely on the concept of complete enumeration of the decision tree, their performance crucially depends on their ability to prune branches that are not promising to lead to an improved feasible solution, i.e., when the lower bound of a branch is not smaller than the best upper bound available. Therefore, pruning tends to be more successful when the integrality gap of the model (i.e., the relative gap between the optimal integer solution value and the LP relaxation solution value) is small.

A *valid inequality* is an inequality that is satisfied by all feasible integer solutions of the MIP model. Cuts are valid inequalities that are not part of the current problem formulation and are commonly used to approximate the convex hull of the set of integer feasible solutions. The *tighter* the formulation is, the smaller the integrality gap tends to be. *Branch-and-Cut (B&C)* algorithms are B&B algorithms that may add cuts at each node in the tree. B&C algorithms provide the foundation for many generic solvers. To derive effective cuts, the CFLP has been extensively studied in terms of its polyhedral structure. Reference works include those by Leung and Magnanti (1989) and Aardal et al. (1995). Aardal (1998b) strengthens the formulation by introducing redundant variables to derive valid inequalities afterwards. Cutting Plane algorithms for facility location problems are presented by Aardal (1998a) and Avella and Boccia (2007). A few authors also presented customized B&B algorithms (Görtz and Klose, 2012).

As many of the developed cuts are based on sets that are extremely large, one cannot add all cuts to the model. Separation algorithms (Aardal, 1998a) are necessary to identify cuts that actually improve the LP relaxation bound. However, a few valid inequalities have been developed whose number is polynomial in the size of the input data and sufficiently small to be added *a priori* to the model. Some of these cuts are quite effective when added to the model and can be used in combination with other solution approaches to facilitate the solution of the problem. The *Strong Inequalities* (Van Roy, 1986), also referred to as the *Strong Linking Constraints* (Gendron and Crainic, 1994), have been shown to be very effective to increase the value of the facility opening decisions in the LP relaxation. For the CFLP as given by (2.1 - 2.5), they are defined as follows:

$$x_{ij} \leq y_j \ \ \forall i \in I, \ \ \forall j \in J. \tag{2.7}$$

Another important class of valid inequalities are the *Aggregated Demand Constraints (ADC)* (Cornuéjols and Sridharan, 1991):

$$\sum_{j \in J} y_j \leq \sum_{i \in I} d_i. \tag{2.8}$$

Even though they are redundant to the LP relaxation, they often enable generic MIP solvers to derive further cuts.

2.2.2 Mathematical Decomposition

Methods based on mathematical decomposition exploit the structure of linear programs to decompose them into smaller and easier subproblems. Even though, for most of these methods, convergence is formally provided, in practice it is often slow or not achievable. Some of these techniques, in particular those based on Lagrangian relaxation, are therefore often used to solve the problem heuristically. In the following, Benders decomposition, Lagrangian relaxation and cross decomposition will be discussed in the scope of facility location problems.

2.2.2.1 Benders Decomposition

Benders decomposition (Benders, 1962) has proved to be efficient to solve problems under the condition of a special structure. The method relies on the idea that any MIP can be decomposed into two different kinds of problems. If the integer variables are fixed, the resulting subproblem only contains continuous variables. If the continuous variables are fixed, the resulting problem, also referred to as the master problem, only contains integer variables. In algorithms based on Benders decomposition, the dual of the subproblem is solved. Its solution generates a Benders cut in the master problem, stating that the optimal solution has to be at least as costly as the solution found in the subproblem. If the optimal solutions of all possible subproblems were added as a cut to the master problem, the latter would yield the optimal solution. However, in practice, the number of Benders cuts is often too large. Algorithms based on Benders decomposition are therefore only efficient if they provide strong convergence.

Several researchers approached large facility location problems by Benders decomposition. Lee (1991) solves an extended CFLP with multiple commodities and multiple capacity levels. Wentges (1996) shows how to accelerate Benders decomposition for the CFLP by strengthening the Benders cuts. Next to facility location problems, Benders decomposition has also been successfully applied to related problems such as supply chain network design (Cordeau et al., 2006; Easwaran and Üster, 2009) and several hub location problems (e.g., Contreras et al., 2011a).

2.2.2.2 Lagrangian Relaxation

Lagrangian Relaxation (Fisher, 1981, 1985; Guignard, 2003; Frangioni, 2005) decomposes LPs which possess block structures in their coefficient matrix. The constraints that link the different blocks in the original problem are relaxed and penalized in the objective function, referred to as the Lagrange multipliers, or Lagrangian dual variables. Ideally, the relaxed problem, denoted as the Lagrangian subproblem, can be solved efficiently. The optimal Lagrange multipliers can be found by solving the Lagrangian dual problem. Its optimal solution provides a bound on the optimal integer solution and is known to be at least as strong as the LP relaxation bound.

Most of the existing literature that applies Lagrangian relaxation to facility location problems solves the Lagrangian dual by the subgradient method (Held et al., 1974; Guignard, 2003). Several refinements have been proposed to improve the convergence of such methods, such as the deflected subgradient method (e.g., Contreras et al., 2011b) and bundle methods (e.g., Frangioni, 2002, 2005).

Given that they are iterative methods, they typically require hundreds of iterations before converging to the optimal multipliers. At each iteration, the Lagrangian subproblem is solved. Due to these properties, Lagrangian relaxation based methods are typically either embedded in exact methods (e.g., in B&B algorithms) to provide bounds, or as a foundation to heuristically generate feasible solutions, often based on the solutions obtained by solving the Lagrangian subproblems. For a more detailed explanation on Lagrangian relaxation, we refer to Section 5.4.

Lagrangian relaxation has been applied to a large variety of location problems, such as capacitated network design problems (Holmberg and Yuan, 2000) and dynamic hub location problems (Elhedhli and Wu, 2010; Contreras et al., 2011b). When applying Lagrangian relaxation to the CFLP or one of its variants, one has several choices of which constraints to relax. Cornuéjols and Sridharan (1991) present the relaxations of relevant constraint combinations for the CFLP, evaluating and comparing their complexity and strength of bounds. The authors conclude that relaxing the demand constraints or the capacity constraints yields the tightest bounds. Among these two, the relaxation of the capacity constraints may be substantially stronger, but one has to solve a NP-hard subproblem. Correia and Captivo (2003) compare different relaxations for the single period CFLP with modular capacities. They find that the relaxation of the demand constraints performs, on average, better than relaxations of other constraints linked to the capacity limits.

We now explore the two most common relaxations applied to the CFLP as defined by (2.1) - (2.5), the relaxation of the demand constraints (2.2) and the relaxation of the capacity constraints (2.3). When relaxing the demand constraints, one obtains the following Lagrangian subproblem, where α is the vector of Lagrange multipliers:

$$\begin{aligned} L(\alpha) = \quad & \min \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} (c_{ij} d_i - \alpha_i) x_{ij} + \sum_{i \in I} \alpha_i \\ & s.t. \ (2.3) - (2.5). \end{aligned}$$

The Lagrangian subproblem $L(\alpha)$ then decomposes into $|I|$ independent fractional knapsacks. Balas and Zemel (1980) showed how to solve the fractional knapsack in linear time, even though in practice an alternative $\mathscr{O}(|J|\log(|J|))$ time algorithm may be more efficient. Once the knapsacks are solved, the y_j variables are selected by inspection. Given that the Lagrangian subproblem can be solved very efficiently, the relaxation of the demand constraints has been used to solve a large variety of different single period facility location problems (Shetty, 1990; Sridharan, 1991; Pirkul and Jayaraman, 1998; Correia and Captivo, 2003; Wu et al., 2006; Görtz and Klose, 2012; Diabat et al., 2013). As the Lagrangian subproblem does not have the integrality property, the Lagrangian dual may yield bounds which are stronger than the LP relaxation of the original problem (Geoffrion, 1974), which may result in fractional y_j solution values. However, the integrality property can be restored by adding the strong inequalities (2.7) to the problem.

Multi-period capacitated facility location has been addressed in Shulman (1991). The author selects the y_{jt} variables according to the optimal opening schedule for each location j, as can be computed by the use of Dynamic Programming (Bellman, 1957, 1966). The same relaxation has then been adapted to the Uncapacitated Facility Location Problem with facility closing and reopening (Chardaire et al., 1996). Given that the formulation contains the strong inequalities (2.7) (adapted to the multi-period case), relaxing the demand constraints results in a similar Lagrangian subproblem.

Next to the relaxation of the demand constraints, several works also investigated the relaxation of the capacity constraints (Barcelo et al., 1990; Holmberg and Ling, 1997; Gendron et al., 2013; Xie and Ouyang, 2013). When relaxing the capacity constraints, the formulation loses the link between the facility opening decisions and the demand allocation variables. To restore this link and obtain stronger bounds from the Lagrangian dual, it has been common to add the strong inequalities (2.7) to the formulation, referred to as the strong relaxation (in contrast to the weak relaxation, which is defined by the LP relaxation given by (2.1) - (2.5) (Van Roy, 1986). The resulting Lagrangian subproblem is as follows:

$$
\begin{aligned}
L(\alpha) = \quad & \min \sum_{j \in J} (f_j - \alpha_j u_j) y_j + \sum_{i \in I} \sum_{j \in J} (c_{ij} + \alpha_j) d_i x_{ij} \\
& s.t. \ (2.2), (2.4), (2.5), (2.7).
\end{aligned}
$$

Here, the Lagrange multipliers α_j are non-negative. The use of the SIs significantly strengthens the bound provided by the solution of the Lagrangian dual. However, the Lagrangian subproblem $L(\alpha)$ is now a UFLP, which is still NP-hard. When relaxing the capacity constraints, adding the SIs has the opposite effect concerning the integrality gap as when relaxing the demand constraints: with the SIs, the Lagrangian subproblem does not have the integrality property, whereas without the SIs, it has the integrality property, but is very weak. A few works also relaxed both the demand and capacity constraints, in particular for extensions of the classical CFLP (Beasley, 1993; Agar and Salhi, 1998).

Further relaxations may be obtained by performing *Variable Splitting* (Guignard and Kim, 1987). In this technique, certain variables are duplicated and an additional set of constraints is added, ensuring the equality for each pair of the duplicated variables. Ideally, the set of variables that are duplicated is chosen such that the subproblem can be divided into two independent problems, each of which can be efficiently solved. Cornuéjols and Sridharan (1991) report on different combinations for variable splitting for the CFLP. However, the authors do not observe that the relaxations based on variable splitting yield better bounds than the well known relaxation of the capacity constraints.

It has also been found beneficial to perform some sort of local search after the solution of the Lagrangian dual to further improve the quality of the feasible solution. Correia and Captivo (2006) relax capacity constraints for a single-source facility location problem with modular capacities and perform a subsequent tabu search. Li et al. (2009) use the same combination of solution methods for capacitated facility location with multi-commodity flows. Kim and Kim (2013) solve a health-care facility location problem by using a Lagrangian relaxation heuristic. They relax constraints that limit the allocation of clients to at most one facility. Finally, Lagrangian relaxation has not only been used in heuristic frameworks. Their efficient computation of bounds has also been found useful within exact methods (Görtz and Klose, 2012).

2.2.2.3 Cross Decomposition

Some researchers also combined Lagrangian relaxation with Benders decomposition, denoted to as cross decomposition (Van Roy, 1983). The algorithm of Van Roy (1986) is based on the Benders primal subproblem and the Lagrangian dual subproblem. In each iteration, the integer variables are fixed to obtain the primal Benders subproblem. Based on its solution, a solution for its dual is derived. The dual values are then fixed in the Lagrangian subproblem, which is solved to obtain the integer facility opening decisions. These steps produce upper and lower bounds at each iteration and may provide the optimal solution at a certain point. Lee (1993a) and Lee (1993b) apply this technique to their previously presented CFLP with multiple commodities and multiple capacity levels.

2.2.3 Heuristic Methods

Heuristics are algorithms that provide feasible solutions without the guarantee of optimality. However, they are usually much faster than exact algorithms and are therefore useful when an optimization problem is too large to be solved by an exact method or when the application context does not require the optimal solution. Some heuristics, such as those based on mathematical decomposition (Boschetti and Maniezzo, 2007), provide bounds on the solution quality, while others may provide bounds on the total running time.

Two important members in the class of heuristics are *Metaheuristics* and *Approximation algorithms*. A metaheuristic is an algorithmic framework to find solutions to a general class of optimization problems. Metaheuristics such as tabu search, simulated annealing and genetic algorithms have been frequently applied to several families of location problems, from classical facility location problems (Arostegui Jr. et al., 2006) to logistics network design problems that model entire supply chains (Lee and Dong, 2008; Melo et al., 2011a). Approximation algorithms are designed to provide an upper bound for the deviation of the value of the constructed solution from the optimal solution value. Shmoys et al. (1997) present approximation algorithms for several facility location problems. Recent results for *k*-level capacitated facility location problems are given by Du et al. (2009).

Next to the above mentioned classes, heuristics have been developed exploiting different properties of the problems. LP rounding heuristics are methods that construct feasible solutions based on the LP relaxation solution. Melo et al. (2011b) apply this type of heuristic with a subsequent local search to solve a supply chain redesign model. Primal-dual heuristics (Dias et al., 2006) oscillate between the solution of the primal and its dual formulation and aim at finding the optimal solution by satisfying the complementary slackness conditions.

2.3 Discussion and Future Work

The previous review of the related literature is now summarized. Furthermore, we discuss when exact methods are necessary and in which cases the optimal solution may be of less importance.

2.3.1 Summary of Existing Literature

As has been seen throughout this chapter, the existing literature for facility location problems is extensive. A large number of different problem variants has been proposed to take into consideration the different characteristics and extensions as they are relevant in practice. For multi-period capacitated facility location, the most common features to adjust capacities over time have been found to be capacity expansion and reduction, temporary facility closing and reopening and the relocation of facilities. Economies of scale have been considered on all of these levels. However, most of the proposed models consider these features separately.

The majority of the presented facility location problems has been modeled as linear MIP models, many of them solved by generic solvers. Even though generic solvers constantly improve, they are often not capable to solve large models due to limited time and memory resources. Many specialized algorithms have therefore been developed to tackle large-scale problems. In particular, one finds many problems approached by metaheuristics such as Tabu search or algorithms based on mathematical decomposition. In many cases, researchers chose to use Lagrangian relaxation techniques. Interestingly, most of the works are based on the subgradient method, while only few (e.g., Gendron et al., 2013) attempted to use more sophisticated techniques such as bundle methods.

2.3.2 Heuristics vs. Exact Solution Procedures

Given that facility location problems deal with strategic decisions and tend to have a strong financial impact, it may be reasonable to assume that these problems should be solved to optimality. One has several opportunities to reduce the complexity of the problem, as has been suggested in many previous works: one may use a simplified cost structure, reduce the number of candidate facility locations, cluster customers to central demand points or simplify constraints. However, by performing such simplifications, the original problem loses accuracy. As a consequence, cost efficient decisions that may actually be feasible in practice may be ignored by the simplified model. In more severe cases, the suggested solutions may not even be feasible in practice. We may therefore conclude that it is beneficial to model problems as accurately as possible if this has the potential to represent the cost-reducing decisions more realistically, and if the resources to solve the problem appropriately are available. The use of heuristics is therefore a reasonable choice to tackle problems which are complex, and therefore more realistic, than those that can be solved by exact algorithms.

Furthermore, depending on the application context, the question arises whether it is necessary to find an optimal solution. For some applications, a close-to-optimal solution may be satisfactory, while, in other contexts, the optimal solution may be required. Several authors (Cordeau et al., 2006; Melo et al., 2006) argue that, when input data is estimated, an optimal solution is not more useful in practice than a solution that is close to optimality. This is often observed in multi-period planning problems. Even though the input data may be forecast accurately, it is most likely that the real data will slightly deviate from the data used in the planning model.

If the optimal solution cannot be guaranteed, it is important to provide a bound on the quality of the generated solutions, in particular for strategic planning problems. If the problem cannot or does not have to be solved to optimality, how far is the solution from optimality? Mathematical decomposition methods such as Benders decomposition and Lagrangian heuristics provide bounds on the optimal value. They also hold several other advantages, as they facilitate the solution in the case of re-planning and re-optimization, for example when the decision maker needs to evaluate different scenarios.

The previous considerations have a strong impact on how to model and solve complex facility location problems, and have particularly influenced the research presented in this thesis.

CHAPTER 3

OPTIMAL CAMP LOCATIONS IN FORESTRY

3.1 Chapter Preface

In this chapter, we present a complex facility location problem found in the Canadian forestry sector. In this problem, a logging company needs to locate camps to host the workers involved in the forestry operations, either making use of existing camps or constructing new ones. Camps may also be relocated. The decisions have to be made based on a given five year planning that indicates how many workers will be necessary for logging and road construction, at which place and in which seasons.

The research on this problem has been performed in collaboration with FPInnovations, which held contact with the logging company. FPInnovations provided the description of the problem and has been very supportive to verify that all the industrial constraints have been correctly interpreted and taken into consideration in the model. FPInnovations also verified the produced solutions and successfully carried out the cumbersome task of providing the data for two case studies.

From an OR point of view, the problem discussed here can be cast as an extension of classical multi-period facility location. In addition to typical features such as multiple commodities, used to represent different types of workers, the problem extends classical facility location models on several levels, mostly with respect to the complex cost structure. Camps are composed by independent trailers, which may be temporarily closed when not in use. A part of the camp may therefore be closed. To the best of our knowledge, partial temporary facility closing has not been addressed before in the literature. In this chapter, a MIP model for this problem is introduced, which is able to represent the cost structure on a detailed level. Furthermore, it provides a large variety of capacity adjusting features, such as those mentioned above, to represent the problem in a realistic manner and model decisions that are feasible in practice and hold the potential to reduce operational costs.

3.1.1 Approximation of the Cost Structure

Trailers can be divided into two groups: hosting trailers, which host the workers, and supporting trailers, which provide additional infrastructure. The number of supporting trailers depends on the number of hosting trailers. However, this relation is not linear, as can be observed in Table 3.I. Economies of scale can be easily modeled for the case of a simple camp construction. However, previous literature does not answer the question of how to model capacity expansion for an already existing camp, as the costs for additional capacities do not take into consideration the capacity level of the existing camp.

This non-linear relation between hosting and supporting trailers also impacts on the costs to close and reopen parts of the camp. Feasible assumptions have been elaborated with FPInnovations, including a simplification of this non-linear relation that suggests to slightly over-estimate the costs. The model presented here implements this simplification of the cost structure to ensure that the MIP model can be solved by generic solvers. However, the non-linear relation of different types of trailers has been a motivation for further research on models capable of representing such complex cost structures. Chapter 4 focuses on this issue.

3.1.2 Modeling of Relocation

As previously mentioned, existing camps can be relocated from one location to another. The relocation of camps is commonly modeled by direct arcs between each pair of locations (Melo et al., 2006), involving $\Theta(n^2)$ relocation variables, where n is the number of locations. An example of relocation implemented by the use of direct arcs is illustrated in Figure 3.1 (a).

In some applications, the relocation costs may be equal (or almost equal) for the different pairs of origin and destination. In the CSLP, the costs depending on the distance of a relocation only marginally affects the total relocation costs: in the real-world instances provided by FPInnovations, the variable relocation costs make only 0.14% of the fixed relocation costs for each 100km for a camp with one trailer. The information of origin and destination therefore contributes only marginally to the costs.

# Hosting Trailer	Total Capacity	# Supporting Trailer	# Trailer Total	Cost $ Total	Cost $ per Hosting Trailer
1	12	1	2	404,000	140,000
2	24	1	3	606,000	105,000
3	36	1	4	808,000	86,667
4	48	1	5	975,000	81,250
5	60	2	7	1,330,000	91,000
6	72	2	8	1,520,000	86,667
7	84	2	9	1,710,000	77,143
8	96	2	10	1,850,000	75,000
9	108	3	12	2,154,000	80,000
10	120	3	13	2,333,500	78,000
11	132	3	14	2,415,000	76,364
12	144	4	16	2,728,000	76,000
13	156	4	17	2,898,500	74,538
14	168	4	18	3,069,000	73,286
15	180	5	20	3,260,000	76,000
16	192	5	21	3,423,000	72,188
17	204	6	23	3,749,000	74,412
18	216	6	24	3,912,000	73,333
19	228	6	25	4,018,750	72,368
20	240	6	26	4,179,500	71,500
21	252	7	28	4,431,000	72,667
22	264	7	29	4,589,250	71,841
23	276	7	30	4,747,500	71,087
24	288	7	31	4,890,250	70,396
25	300	7	32	5,043,200	69,120

Table 3.I: Relation between the number of hosting and supporting trailers, as well as the corresponding construction costs.

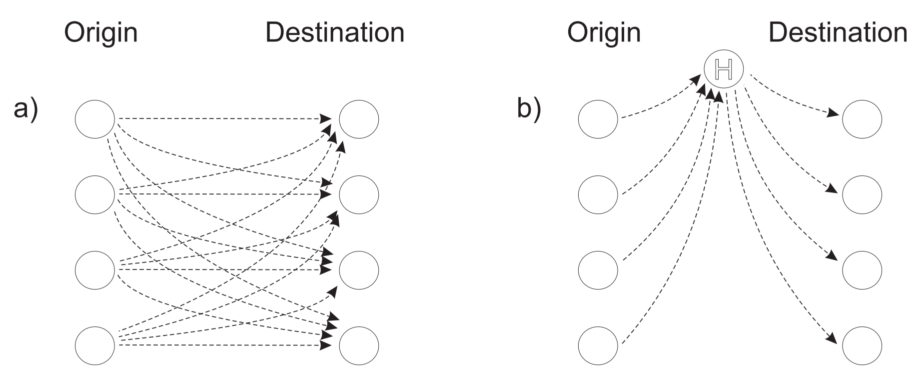

Figure 3.1: Example of facility relocation by the use of (a) direct arcs and (b) hub nodes.

Based on this assumption, we consider an alternative formulation to model relocation, which can be seen as a generalization of the modeling technique suggested by Wesolowsky and Truscott (1975). Instead of sending relocation flow directly from one location to another, all relocation flow is first sent to a central hub node and then redistributed to other locations. This reduces the number of relocation variables to the order of $\Theta(n)$, which should facilitate the solution of the problem. An example of relocation using hub nodes is illustrated in Figure 3.1 (b), in which the hub node is designated with an "H". Contrary to what one may suspect, relocation by direct arcs does not result in a stronger LP relaxation bound than relocation by hub nodes. A proof for the equality of the strength of these two modeling techniques is provided in Appendix A for the case of the CLSP. The appendix also includes a summary of computational results which indicates that, on average, the use of hub nodes significantly facilitates the solution when a generic MIP solver is used.

The camp location problem considered here also considers a new type of capacity constraints, which involves the rounding of the demand allocated to the same facility. Details are found in the remainder of this chapter.

Notes about the chapter

The contents of this chapter correspond to those of the article entitled *Modeling and Solving a Camp Location Problem*, co-authored with Professors Jean-François Cordeau and Bernard Gendron, which has been accepted for publication in Annals of Operations Research (ISSN: 0254-5330), in 2012.

Please note that this paper has been published with a notation slightly different then the one used in the rest of this thesis. In this chapter, as well as in the corresponding Appendix A, candidate facility location are represented by set I and customers by set J, whereas in the rest of the thesis the reverse notation is used.

Modeling and Solving a Camp Location Problem

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Harvesting plans for Canadian logging companies tend to cover wider territories than before. Long transportation distances for the workers involved in logging activities have thus become a significant issue. Often, cities or villages to accommodate the workers are far away. A common practice is thus to construct camps close to the logging regions, containing the complete infrastructure to host the workers. The problem studied in this paper consists in finding the optimal number, location and size of logging camps. We investigate the relevance and advantages of constructing additional camps, as well as expanding and relocating existing ones, since the harvest areas change over time. We model this problem as an extension of the Capacitated Facility Location Problem. Economies of scale are included on several levels of the cost structure. We also consider temporary closing of facility parts and particular capacity constraints that involve integer rounding on the left hand side. Results for real-world data and for a large set of randomly generated instances are presented.

Key words: Logging Camps, Capacitated Facility Location Problem, Mixed Integer Programming.

3.2 Introduction

3.2.1 Context and Scope

Context. Log harvest planning in the forestry sector has changed throughout the last decades. Both silviculture and harvesting in Canada have become more sophisticated and now pose complex planning problems to get the most from the available regions and harvest cycles. Based on a wide variety of considerations, a long-term plan is designed to determine the volume and regions for wood logging. These decisions are commonly divided into smaller time periods, as logging activities and road construction within a single logging region typically take several months.

Due to political and environmental issues, as well as the size of the country, harvesting plans tend to cover wider territories than they used to. Often, sparse logging is necessary to certify the forestry operations. Several questions arise such as the location and capacity for administrative services, sorting yards and central log processing stations. Similarly, the location where the workers involved in forestry activities are accommodated gains in importance. If villages or cities are close, workers can be hosted at their homes or at motels. However, logging regions in Canada are often widely distributed and located far from such hosting options. In that case, accommodating the workers in the closest village or city is rarely an attractive option, as the commuting time and transportation costs are too high. Transportation times would consume a significant portion of the potential productive time. Furthermore, an additional salary is commonly paid when the transportation times exceed a certain threshold.

A common solution to this problem is the construction of logging camps in which the workers are accommodated. Logging camps are typically located close to the logging regions so that the transportation costs for the workers are reasonable. When allocating each work crew to a camp, the accommodation costs are given as a cost per day per worker. In order to host all workers, the construction of new accommodations may be necessary. The larger a camp, the smaller the daily cost per person. Hence, a small number of large camps results in smaller accommodation costs than a large number of small camps. However, the fewer camps are available, the higher the transportation costs tend to be, because their location is less flexible. The construction of a new camp or the relocation of an existing one may pay off in the long term as the traveling costs to the logging regions may be much lower.

Scope. This work investigates the possibility of constructing and relocating camps for the accommodation of workers, considering the harvest planning for the next five years. The problem is motivated by the needs of a Canadian logging company. It consists in finding the number of camps that have to be constructed or relocated, their size and their location such that the total costs for accommodation and transportation are minimized. The interesting question is whether such an investment in camp construction and relocation pays off, considering the operational logging and road construction planning for the next five years. It is important to note that the actual work crew assignment between accommodations and work regions is not relevant in practice. It is only used to determine the minimum capacity level necessary to host all workers. For the operational work crew assignment, other planning tools will be used. It is assumed that all information about work crews, logging regions and distances are known at the beginning of the planning and are not subject to uncertainty.

3.2.2 Contributions and Organization of the Paper

Contributions. Due to the complexity of the problem, manual planning approaches usually do not yield optimal solutions. The main objective of this paper is to propose a formulation for the problem that can be solved by a general-purpose solver for instances of reasonable size. The impact of different instance and model properties on the difficulty of the problem is studied. The presence of economies of scales on several levels of the cost structure as well as partial facility closing are part of the main concerns. Further aspects include particular capacity constraints that involve integer rounding on the left hand side. It is shown how such capacity constraints can be useful in other applications, but increase the integrality gap of the problem. We derive valid inequalities to effectively reduce this integrality gap.

Organization. This paper is organized as follows. Section 3.3 describes the relevant problem details. Since the problem can be modeled as a facility location problem, the literature review in Section 3.4 focuses on relevant extensions in that domain. The mathematical formulation in Section 3.5 gradually extends the Capacitated Facility Location Problem to model the problem being addressed. This includes the particular capacity constraints, valid inequalties and additional features such as the relocation and partial closing of camps. Section 3.6 summarizes the results of the computational experiments performed. Two case studies in Section 3.7 illustrate the benefits of the proposed model when applied in practice. Finally, Section 3.8 concludes the work.

3.3 Problem Description

Based on an existing strategic plan, the logging company provides a harvesting plan for the next five years. Each year is divided into two seasons: winter and summer, each with a certain number of available working days. Depending on the geographical location, some regions will be logged more in winter whereas other regions will be logged more in summer. Each region is defined by its estimated log volume (measured in m^3) that is subject to harvesting (it may be part of the strategic decision that not the entire region will be harvested) within each season and the length of the road (measured in *km*) that has to be constructed in that region in order to access the logging areas and transport the log.

3.3.1 Work Crews, Demands and Hosting Capacities

There are two types of work crews: logging and road construction. Crews of the same type contain the same number of members. The members of a crew always stay together during work and are hosted at the same accommodation. For each logging region and season, a logging and road construction demand is given. Based on given productivity rates for the work crews one can compute the average number of crews necessary to cover the demand at each region for each season.

Example: Logging crews work 100 days within a given season and cut $180m^3$ per day, i.e., 18,000m^3 within the season. A certain region holds a total demand of 27,000m^3 for the season. Throughout 50 days, two logging crews will be working (i.e., $2 \cdot 50 \cdot 180m^3 = 18{,}000m^3$). The other 50 days, a single logging crew will be working (i.e., $1 \cdot 50 \cdot 180m^3 = 9{,}000m^3$). This results in an average allocation of 27,000/18,000 = 1.5 logging crews in that season.

As the operational assignment of logging crews is not our final concern, we can assume that the crews of each working type are flexible with respect to the days they work within each season. That is, if a crew works only a few days in a season, we may assume that the exact days do not matter. In our example, it does not matter in which of the 100 days we use two crews and in which we use only one crew. In practice, a work crew may work a number of days in one region and then in another region in the same season. To determine the minimum capacity necessary to host all work crews allocated to a certain accommodation, consider the following example.

The workers from two regions are hosted at the same camp. One region has an average demand of 1.5 logging crews and 0.7 road construction crew. The other region has an average demand of 1.25 logging crews and 0.5 road construction crew. Figure 3.2 (a) illustrates this scenario for the logging crews. In total, we have a demand of $1.5+1.25 = 2.75$ logging crews and $0.7+0.5 = 1.2$ road construction crews. Hence, for 75% of the time during the season there will be $\lceil 2.75 \rceil = 3$ logging crews and 25% of the time there will be $\lfloor 2.75 \rfloor = 2$ logging crews, which is illustrated in Figure 3.2 (b). In the same way, for 20% of the season there will be $\lceil 1.2 \rceil = 2$ road construction crews and for the other 80% there will be only $\lfloor 1.2 \rfloor = 1$ road construction crew. Assuming that a logging crew has six workers and a road construction crew has three workers, we will need accommodation for $\lceil 2.75 \rceil \cdot 6 + \lceil 1.2 \rceil \cdot 3 = 18+6 = 24$ workers. To determine the minimum capacity of an accommodation, we can add the average numbers of crews allocated to this accommodation and round up the sum to the next highest integer (for each crew type).

3.3.1.0.1 Transportation. Workers are usually transported by pick-ups, using a given road network. Costs are composed of the travel and working time of the workers

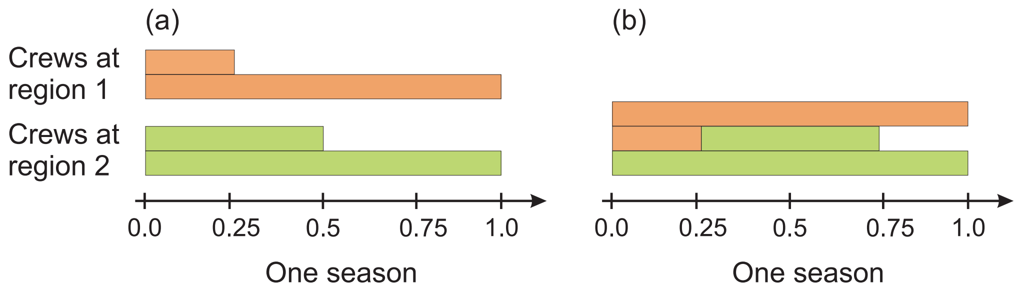

Figure 3.2: Example of logging demands hosted at the same accommodation.

as well as the vehicle costs, i.e., renting and gas. An additional salary has to be paid if a certain transportation time (usually one hour per day) is exceeded. This makes large travel distances very costly. Workers of the same crew are transported in one or more vehicles. Workers of different crews do not share the same vehicle.

3.3.1.0.2 Supervisors. In addition to the work crews, there are fixed numbers of logging and road construction supervisors. Supervisors have to be considered for the accommodation capacities and their individual transportation costs. Although it is not clearly predictable how many days a supervisor will be at which region, one may assume that their presence in a region is proportional to the demand for work crews at that region. Hosting regions for supervisors are often limited to accommodations with administrative units.

3.3.2 Camps and Trailers

Certain accommodations for the workers may already exist. These accommodations can either be hosting options in villages or cities (e.g., apartments, hotels or the employees' own homes) in reasonable distance of the logging regions, or camps that are usually located in the forest close to the logging regions. Accommodations vary in their capacity and their hosting costs. Camps are composed of trailers. A trailer contains the infrastructure to host a certain number of workers. In practice, trailers of different capacities are available. However, for the purpose of this study, we may assume that the trailer with a capacity for twelve persons is the most common one and hence all trailers have the same capacity. In addition to the trailers that host workers, a camp contains a number of additional trailers that provide complementary, but necessary infrastructure, such as a kitchen and leisure facilities. The number of additional trailers directly depends on the total hosting capacity of the camp, i.e., the number of hosting trailers. In the following, we will measure the capacity of a camp by the number of hosting trailers. Hence, the construction costs for a number of hosting trailers already include the costs for the necessary number of additional trailers.

Trailers can be either open or closed. Only open trailers are available for use. Trailers that are not in use have to be closed, involving one-time closing costs. Once a trailer is closed, it cannot be used in subsequent seasons until it is reopened, involving one-time reopening costs. Closing or reopening operations can be performed before each season. Costs for such operations usually involve economies of scale in the number of hosting trailers, since common resources are shared. The use of hosting trailers to accommodate workers involves two types of daily costs: fixed costs for each open trailer (including the cost for the trailer itself, its equipment, the cook, etc.) and variable costs (food, etc.) for each worker. The fixed costs are paid for each open trailer per day. Costs for closed trailers are so small that they do not have to be considered. Variable costs are paid for each worker hosted at the camp. If a trailer is open, its fixed costs have to be paid throughout the entire season, independent of its use. All costs may follow the principle of economies of scale, i.e., the larger the quantity, the lower the price-per-worker/trailer. New camps can only be constructed at certain places from a given set of potential locations. It is very common that several logging regions are served by workers from the same accommodation. Though it is rare, one logging region may also be served by workers from different accommodations.

3.3.3 Capacity Expansion and Camp Relocation

At certain points during the planning it may be interesting to increase the capacity of existing camps. Such capacity expansion is performed by adding new trailers. It is assumed that the cost of adding n trailers is the same as the construction of a new camp with n trailers. Trailers may also be permanently shut down. For the sake of simplicity, it is assumed that this is done by closing these trailers.

Logging regions are not equally harvested every year. That is, a camp may be close to logging regions with demands in certain years, but far away from logging regions that will be harvested afterwards. Instead of constructing a new camp, which involves high costs, camps can be moved from one location to another. The relocation of camps can only be performed once a year, before the summer season. The distance between the origin and destination for a relocation has very little impact on the total relocation costs. We may thus assume that the total cost for relocating a camp depends only on the camp size (i.e., the number of trailers it includes). All trailers have to be closed before relocation. After the relocation, all trailers that are supposed to be in use have to be reopened again. In theory, camps from two distinct locations can also be joined to further reduce the costs per unit. Trailers from the same camp could also be relocated to distinct locations. In practice, these features are observed rather rarely. For the sake of simplicity, it is hence assumed that camps can only be relocated as a whole and that two different camps cannot be merged at the same location.

3.3.4 Objective

Given that all logging and road construction demands must be covered, we must ensure that sufficient accommodations are available to host the workers. We want to minimize the total costs, which are composed of two parts:

– All costs involved in providing the necessary accommodations: camp construction, camp relocation, maintenance for open trailers, closing and reopening of trailers and hosting costs for workers.
– The transportation costs between the accommodations and the logging regions. This includes the costs for using the vehicles and an additional salary for long transportation times.

A solution to the problem consists of the following information, given for each of the seasons in each of the years of the planning horizon:

– For each camp construction: the location and camp size.
– For each camp relocation: the origin, destination and size of the relocated camp.

– For each camp: the number of trailers that will be closed or reopened.

An insight into the suggested assignment of work crew demands to the accommodations may also be interesting for decision-makers. The assignment is necessary to determine the minimum level of camp capacities. However, it is not explicitly part of the problem solution.

Throughout this work, we will refer to this problem as the *Camp Size and Location Problem (CSLP)*.

3.4 Literature Review

The forestry sector has been an extensive user of Operations Research (OR) methods for strategic, tactical and operational planning. Optimization is mainly used for supply chain design (D'Amours et al., 2008), harvesting (Bredström et al., 2010) and transportation planning (Carlsson et al., 2009). Strong interest is shown by both the public and private sector, typically in countries where logs represent a large portion of the net exports, such as Canada, Chile, New Zealand and the Scandinavian countries. Several recent surveys provide broad overviews of optimization in the forestry sector (see, e.g., D'Amours et al., 2008; Rönnqvist, 2003; Weintraub and Romero, 2006).

Rönnqvist (2003) compares different planning levels in terms of planning horizon, allowable solution time and required solution quality. These characteristics strongly vary among the different applications. Board cutting is individually decided for each tree and has to be optimally solved within less than a second. Harvesting plans typically cover an entire year. Such forest management plans have to be evaluated quickly to allow manual comparisons. Thus, for problems of this category, near optimal solutions are desired within a few hours of computation time. However, the planning includes a strategic outlook for more than 100 years. To the best of our knowledge, the problem of locating logging camps has not yet been addressed in the OR literature. Its solution requirements are similar to those of road planning: one aims at near-optimal solutions, planning includes decisions for five years and one can allow computation times of several hours. Mathematical programming appears to be an appropriate tool, since it provides high quality solutions and it allows to model particular industrial constraints.

Several known problems present features similar to those found in the CSLP. Such problems typically belong to the family of Facility Location Problems. The CSLP can be formulated as an extension of the well studied Capacitated Facility Location Problem (CFLP), which aims at finding the optimal locations to construct an unknown number of facilities with capacity constraints. All customer demands have to be covered and the total costs, usually composed by costs for facility construction, production and transportation, are minimized. In the last decades, practical needs led to many extensions of the CFLP such as multiple periods, multiple commodities, multiple capacity levels and multiple stages. Since demands are likely to change over time, many models focused on the dynamic (i.e., multi-period) case of the problem in order to address dynamic aspects such as capacity reduction, expansion and relocation.

The diversity, importance and maturity of facility location problems has been confirmed by many recent literature surveys (Hamacher and Nickel, 1998; Klose and Drexl, 2005; Melo et al., 2009a; Revelle and Eiselt, 2005; Revelle et al., 2008). Melo et al. (2009a) focus on the context of supply chains. Smith et al. (2009) review the development of location analysis from its early beginning and highlights today's most important applications. Many of the extensions proposed for the CFLP can be found in the proposed CSLP. Camps are translated to facilities and hosting demands to customers. The relevant literature regarding these features will now be reviewed.

Dynamic Facility Location Problems. The CSLP contains strong dynamic aspects, since logging regions tend to be harvested within a few seasons. Hence, a customer may have high demands in some time periods and no demand at all in the other periods. Early works in the domain of dynamic facility location were initiated by Ballou (1968) and Wesolowsky (1973). Recent works include Albareda-Sambola et al. (2009), Canel et al. (2001), Dias et al. (2006), Melo et al. (2006), Antunes and Peeters (2001), Shulman (1991) and Troncoso and Garrido (2005). Many more references can be found in the previously cited reviews as well as in the one of Owen and Daskin (1998), which focuses on approaches that are based on either dynamic or stochastic facility location problems.

In addition to the optimal timing and sizes for facility construction, further dynamic features have been found beneficial to adapt to changing demand and market conditions. Capacity expansion has been incorporated by Melo et al. (2006), Antunes and Peeters (2001) and Troncoso and Garrido (2005). Capacity reduction or facility shut-down is addressed by Canel et al. (2001), Dias et al. (2006), Melo et al. (2006) and Antunes and Peeters (2001). In an early work, Wesolowsky and Truscott (1975) considered a simple case of relocation of facilities. Melo et al. (2006) provide an extensive modeling framework for dynamic multi-commodity facility location problems. Their model focuses on the relocation of existing facilities and gradual capacity transfer from existing facilities to new ones while considering generic multi-level supply chain network structures.

Multiple Commodities. In some applications, customers have demands for several distinct commodities. The models must then distinguish between the different commodities to satisfy the demand for each of them as well as to control their capacity at the facilities. In the context of the CSLP, the different work crew types (i.e., logging crews and road construction crews) and supervisors can be modeled as different commodities.

In the multi-commodity facility location literature, models commonly assume that the customers have an individual demand for each commodity. However, on the facility side, the capacity constraints can be formulated in two different ways:

i. Each facility holds an individual capacity for each of the commodities.

ii. Each facility holds a global capacity for the sum of all commodities.

The first option is the more common one in the literature (Canel et al., 2001; Geoffrion and Graves, 1974; Lee, 1991; Warszawski, 1973). In the CSLP, we rather consider the second case. While customers have a demand distinguished between the different commodities, the total capacity at the camps applies to the sum of all workers, whether they are logging or road construction workers. This idea of a common capacity for all commodities is also followed in the modeling framework of Melo et al. (2006).

Multiple Capacity Levels. The presence of production capacities automatically raises the question of the dimension of such capacities. While some applications allow for several facilities at the same place, most consider only one facility per location. Facilities may have fixed capacities or may choose among different capacity levels. Often, facility construction and unit production costs follow the principle of economies of scale, i.e., the larger the facility, the cheaper the price per unit in terms of facility construction and commodity production. One finds this feature in the CSLP, where camps are composed of trailers. The more hosting trailers exist, the larger the capacity and the better common resources (such as supplementary infrastructure) are shared. The choice of different capacity levels allows to represent such economies of scale.

Early works considering different capacity levels are Lee (1991), Shulman (1991) and Sridharan (1991). The choice of the capacity level is modeled as an additional variable index, having only one variable of a certain capacity level active for each facility. The cost part in the objective function thus corresponds to a piecewise linear function. In the literature, this has been the most common way to represent such cost functions (Paquet et al., 2004; Troncoso and Garrido, 2005).

Holmberg (1994) and Holmberg and Ling (1997) introduce an incremental approach to model staircase functions, where all variables up to the chosen capacity level are active. Similar approaches have since been adapted to more complex problems (Correia and Captivo, 2003; Gouveia and Saldanha da Gama, 2006).

Conclusions. Many of the features found in the CSLP have already been addressed in isolation in the facility location literature. However, very few models consider modular capacity levels in a dynamic context (Melo et al., 2006; Antunes and Peeters, 2001; Shulman, 1991; Troncoso and Garrido, 2005). These works do not address dynamic features such as facility closing/reopening or relocation. The closest related works are those of Melo et al. (2006) and Troncoso and Garrido (2005). The latter authors represent economies of scale for facility construction, but not for operational costs. Capacity relocation is also not considered. Melo et al. (2006) focus on capacity relocation, but consider modular capacity decisions only for relocation.

While many models consider closing an entire facility or reducing its capacity, none of the reviewed works present the possibility of partially or entirely deactivating a facility for a certain time period, as it is possible with trailers in logging camps. In addition, the capacity constraints found in the CSLP have not yet been addressed in the context of facility location problems.

3.5 Mathematical Formulation

The CSLP can be modeled as an extension of the CFLP. Some of the additional features have been considered in variations of that classical problem. However, to the best of our knowledge, no extension of the CFLP considered all features at the same time. In particular, two of them have not been mentioned in the related literature:

i. Round-up (integer) capacity constraints for the camps.

ii. Partial closing and reopening of trailers throughout the planning periods.

In the following, we will model the CSLP by extending the CFLP in two steps. In a first step, a formulation for a dynamic modular (i.e., multiple capacity levels) multi-commodity Facility Location Problem is studied. This problem will be referred to as the *Dynamic Modular Multi-Commodity Capacitated Facility Location Problem (DMCFLP)*. Then, the dynamic features are added, namely the relocation of camps and the closing and reopening of trailers. This problem represents the CSLP as described above.

The intermediate problem, namely the DMCFLP, is explored mainly due to two reasons. First, to explore the impact of the additional features on the solution difficulty. Second, all DMCFLP solutions are essentially feasible for the CSLP. As we will see later on, DMCFLP solutions of good quality can be obtained much easier than solutions for the CSLP. Using DMCFLP solutions as starting solutions can be helpful to solve the complete CSLP.

3.5.1 The DMCFLP – An Extension of the CFLP

The classical CFLP, as presented by Sridharan (1995), is extended. To be more precise, the following features are added:

- Multiple periods. We study the problem in a dynamic context, i.e., over multiple time periods with independent demands.

- Multiple commodities. We assume the existence of different commodities, one for each work crew type. Each customer may have independent demands for each of these commodities.
- Multiple capacity levels. We assume that a facility may have different capacities, i.e., different numbers of hosting trailers. These capacities are modular and can represent cost structures involving economies of scale.

Due to its additional characteristics, we refer to this problem as the *Dynamic Modular Multi-Commodity Facility Location Problem (DMCFLP)*.

3.5.1.1 Input Data and Decision Variables

Input Data. Consider the following input data:

- I - set of potential camp locations (facilities).
- J - set of logging/road construction regions (customers).
- K - set of possible camp sizes (with respect to the number of hosting trailers), $K = \{1, 2, .., \overline{K}\}$.
- P - set of existing work crew types (commodities).
- T - set of seasons (time periods), $T = \{1, 2, 3, .., |T|\}$.
- N_p - number of workers in a crew of type p.
- d_{jpt} - demand (in number of crews) for commodity $p \in P$ in region $j \in J$ and period $t \in T$.
- u_{ik} - total capacity (in number of workers) of a camp of size $k \in K$ at location $i \in I$.
- c^C_{ik} - construction cost of a camp of size $k \in K$ at location $i \in I$.
- c^V_{ijkpt} - variable operational costs (including transportation and hosting costs) for the entire time period $t \in T$ for one crew of working type $p \in P$ accommodated at a camp of size $k \in K$ at location $i \in I$ and working at region $j \in J$. The total cost is typically not linear with respect to the Euclidean distance between the work region and the accommodation.

Decision Variables. The decision variables are:

– $x_{ijkpt} \in \mathbb{R}_0^+$ - total demand (in number of crews) of crew type $p \in P$ assigned from a camp of size $k \in K$ at location $i \in I$ to region $j \in J$ at time period $t \in T$.
– $y_{ik} \in \{0,1\}$ - 1, if a camp of size $k \in K$ is constructed at location $i \in I$ at the beginning of the horizon, 0 otherwise.

3.5.1.2 Mathematical Model

The model is given by:

$$\min \sum_{i\in I}\sum_{k\in K} c^C_{ik} y_{ik} + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{p\in P}\sum_{t\in T} c^V_{ijkpt} x_{ijkpt} \tag{3.1}$$

$$s.t. \sum_{i\in I}\sum_{k\in K} x_{ijkpt} = d_{jpt} \quad ;\forall j \in J \ ;\forall p \in P \ ;\forall t \in T \tag{3.2}$$

$$\sum_{p\in P}\sum_{j\in J} N_p x_{ijkpt} \leq u_{ik} y_{ik} \quad ;\forall i \in I \ ;\forall k \in K \ ;\forall t \in T \tag{3.3}$$

$$\sum_{k\in K} y_{ik} \leq 1 \quad ;\forall i \in I \tag{3.4}$$

$$x_{ijkpt} \leq d_{jpt} y_{ik} \quad ;\forall i \in I \ ;\forall j \in J \ ;\forall k \in K \ ;\forall p \in P \ ;\forall t \in T \tag{3.5}$$

$$x_{ijkpt} \in \mathbb{R}_0^+ \quad ;\forall i \in I \ ;\forall j \in J \ ;\forall k \in K \ ;\forall p \in P \ ;\forall t \in T \tag{3.6}$$

$$y_{ik} \in \{0,1\} \quad ;\forall i \in I \ ;\forall k \in K \tag{3.7}$$

The objective function (3.1) minimizes the camp construction cost and the operational costs. Note that the operational costs c^V_{ijkpt} are composed by both transportation and hosting costs. The transportation costs depend on the distance between both locations i and j as well as the type of crew p. The hosting costs depend on the camp size k as well as the crew type p.

The set of constraints (3.2) guarantees that all customer demands are satisfied. Note that demands are likely to be fractional, as illustrated in Figure 3.2. Constraints (3.3) require that the hosting demands assigned to each camp do not exceed the camp capacities. Constraints (3.4) ensure that only one capacity level is selected for each facility. The set of valid inequalities (3.5), also referred to as *Strong Inequalities (SI)* (Gendron and Crainic, 1994), provide a stronger upper bound for the demand assignment variables.

Computational experiments show that CPLEX solves the problem more effectively when adding only the violated SIs (using CPLEX user cuts) than when adding all SIs *a priori* or not adding them at all.

Non-movable accommodations. In addition to logging camps, we may model accommodations such as motels and apartments to host workers. We do so by representing them as a restricted case of a camp, with two types of information: hosting costs and total capacity. Such accommodations possess a single capacity level and cannot be relocated.

3.5.2 Round-Up Capacity Constraints

As explained above, the CSLP involves particular capacity constraints where the sum of all demands assigned to a certain accommodation is rounded up to the next integer value. Adding, for example, demands of 1.5 crews and 1.25 crews, one only needs a total capacity for three crews (if all crews are hosted at the same camp) instead of four (compare Figure 3.2).

We introduce additional integer variables z_{ikpt} for the integer rounding, indicating the total number of crews of type p assigned to a size k camp at location $i \in I$ at period $t \in T$. The existing capacity constraints (3.3) are replaced by two new constraints (3.8) and (3.9), which we will refer to as the *round-up capacity constraints (RUC)*. Instead of using the continuous sum of the facility/customer assignment variables (x variables), the capacity constraints (3.9) take into account the next highest integer value, bounded by the z variables in constraints (3.8):

$$\sum_{j \in J} x_{ijkpt} \leq z_{ikpt} \quad ;\forall i \in I \;\; ;\forall k \in K \;\; ;\forall p \in P \;\; ;\forall t \in T \tag{3.8}$$

$$\sum_{p \in P} N_p z_{ikpt} \leq u_{ik} y_{ik} \quad ;\forall i \in I \;\; ;k \in K \;\; ;\forall t \in T \tag{3.9}$$

$$z_{ikpt} \in \mathbb{Z}_0^+ \quad ;\forall i \in I \;\; ;\forall k \in K \;\; ;\forall p \in P \;\; ;\forall t \in T \tag{3.10}$$

This type of capacity constraints is likely to appear in other applications. In the context of facility location problems, scenarios can be modeled where a facility may not be able to produce any arbitrary amount of a product, but only modular sized packages of products.

3.5.2.1 Strengthening the Formulation

Experiments have shown that the average integrality gap increases significantly (see Section 3.6.2 for details) when using round-up capacity constraints (3.8)–(3.10) instead of the usual constraints (3.3). Consider the following *aggregrated demand* inequalities which are known to be redundant for the linear relaxation of the model:

$$\sum_{i \in I} \sum_{k \in K} u_{ik} y_{ik} \geq \sum_{p \in P} \sum_{j \in J} d_{jpt} N_p \quad ; \forall t \in T$$

We will now strengthen these inequalities, based on the fact that z is integer. Substituting (3.2) in (3.8) shows that one can always round up the sum of all demands from different regions for the same product. We replace the right hand side (RHS) of the previous inequality by D_t, where:

$$D_t = \sum_{p \in P} \left\lceil \sum_{j \in J} d_{jpt} \right\rceil N_p \quad ; \forall t \in T$$

We now express the resulting inequality in terms of the number of trailers instead of the number of crews. Assuming that each trailer hosts exactly M workers, i.e., $u_{ik} = Mk$, we have:

$$\sum_{i \in I} \sum_{k \in K} k y_{ik} \geq \frac{D_t}{M} \quad ; \forall t \in T$$

These inequalities state the minimum number of open trailers necessary to satisfy all customer demands. We know that the RHS, the minimum number of open trailers, is always integer. We can thus replace the RHS by S_t, where:

$$S_t = \left\lceil \frac{D_t}{M} \right\rceil \quad ; \forall t \in T$$

In a final step, we aim at reducing the coefficients of the y variables on the left hand side. Suppose that $\overline{K} > S_t$. It is then sufficient that only one $y_{ik'}$ with $k' \geq S_t$ is active in order to satisfy the entire customer demand in the integer solution. That is, we may set the coefficient of a variable $y_{ik'}$ to S_t whenever $k' \geq S_t$:

$$\sum_{i \in I} \sum_{k \in K} \min\{k, S_t\} \, y_{ik} \geq S_t \quad ; \forall t \in T \tag{3.11}$$

In the following, we will refer to these constraints as the *strengthened aggregated demand (SAD)* inequalities.

3.5.3 The CSLP - Adding Partial Camp Closing, Relocation and Modular Costs

In this section, the previous model will be extended with the following features that may appear in a dynamic context:

i. Construction of new camps/trailers at any time period.

ii. Closing and reopening of trailers at any time period.

iii. Relocation of camps at any time period.

iv. Modular costs for trailer closing/reopening and camp relocation.

This problem corresponds to the CSLP. A network flow structure, illustrated in Figure 3.3, is added on top of the previously introduced model to manage the first three features. For each time period, two nodes for open trailers and two nodes for closed trailers are used. Arcs between these nodes represent certain operations to modify the number of open and closed trailers at each location and to relocate them to other locations. The flow on these arcs indicates the number of trailers involved in the corresponding operation. New trailers can be constructed at the beginning of any season (s arcs). Open trailers can be closed (v^{OC} arcs) and closed trailers can be reopened (v^{CO} arcs). The arcs v^{OO} represent trailers that were open at the beginning of the season and remain open during the current season. The arcs v^{CC} indicate closed trailers that are not relocated to another region. These trailers can still be reopened for the current season. Finally, l^O and l^C

indicate the number of trailers that are open and closed, respectively, at each location throughout the entire season.

Relocation is allowed only for closed trailers. One could model relocation by the use of direct arcs between all location pairs. However, this would result in very large models. Experiments showed that this significantly increases the model size and therefore also the difficulty of solving the problem. Instead, relocation is modeled by the use of a central node, here referred to as a *hub node (H)*. The flow of relocated trailers is first passed to the hub node (w^O arcs) and then further distributed to another location (w^I arcs).

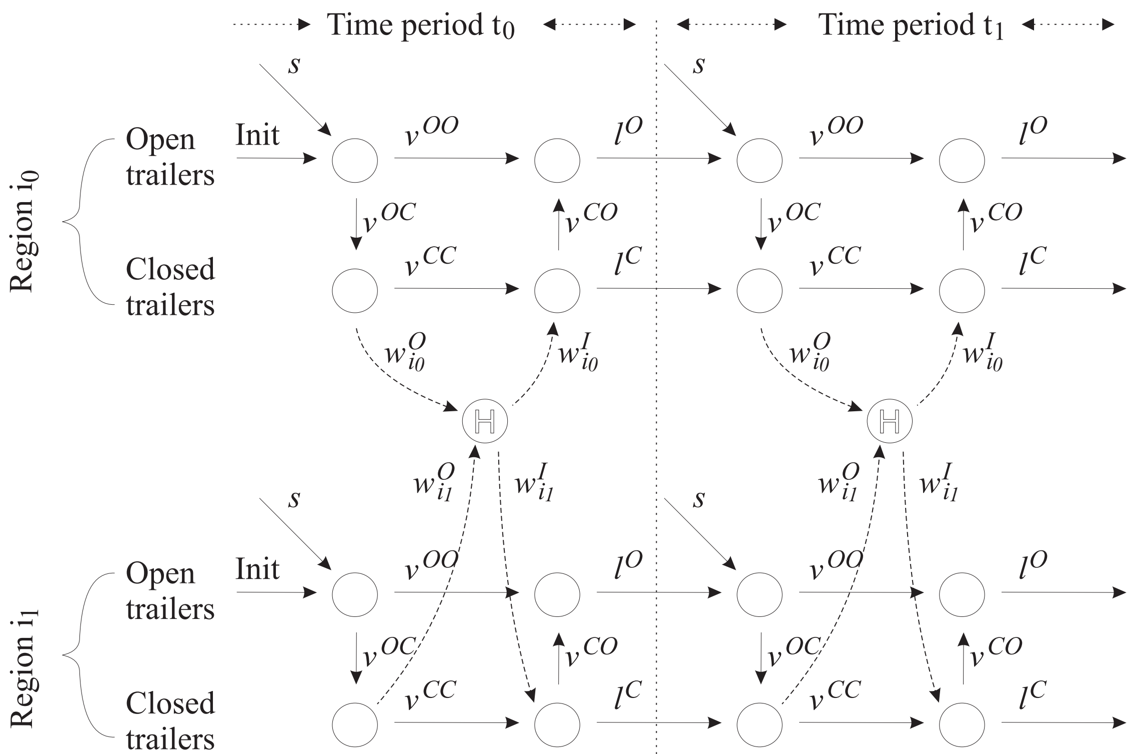

Figure 3.3: Network model to manage open and closed trailers at each location.

3.5.3.1 Input Data and Decision Variables

Additional Input Data. In addition to the previously introduced input data, additional parameters are considered. These data may already consider economies of scale with respect to k, the number of trailers involved in the operation: c_k^{TO} and c_k^{TC} are the costs to reopen and close k trailers of the same camp, respectively. The maintenance costs for a camp with k open trailers during season t is given by c_{kt}^M. Finally, c_k^R represents the costs for relocating a camp with k closed trailers.

Additional Decision Variables. To incorporate the new features, some variables have to be extended and new variables have to be added to the model. Binary variables y_{ikt} now indicate whether the camp located at i has k open hosting trailers during period t. A separate binary variable s_{iqt} indicates the construction of q new trailers at location i before period t. In addition, arc flow variables for the network are added to manage the closing and reopening of trailers: l_{it}^O, l_{it}^C, v_{it}^{OO}, v_{it}^{OC}, v_{it}^{CO}, v_{it}^{CC}, w_{it}^O and w_{it}^I.

Finally, binary variables are needed to incorporate modular costs: v_{ikt}^{BCO} and v_{ikt}^{BOC} indicate whether k trailers are reopened or closed, respectively, at location i before time period t. Variables w_{ikt}^{BO} and w_{ikt}^{BI} indicate whether a size k camp is relocated from or to, respectively, location i before period t. The relocation of a camp of size k' from location i_1 to location i_2 at time period t' is thus performed by selecting the two variables $w_{i_1k't'}^{BO}$ and $w_{i_2k't'}^{BI}$.

3.5.3.2 Mathematical Model

Objective Function. The objective function minimizes all costs: maintenance for open trailers, operational hosting and transportation, trailer construction, camp relocation and trailer reopening and closing. Note that each camp relocation involves two binary variables w_{ikt}^{BO} and w_{ikt}^{BI}, while only one of them has to be considered in the objective function to attribute the relocation costs:

$$\begin{aligned}\min \sum_{i\in I}\sum_{k\in K}\sum_{t\in T} c_{kt}^M y_{ikt} + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\sum_{p\in P}\sum_{t\in T} c_{ijkpt}^V x_{ijkpt} \\ + \sum_{i\in I}\sum_{q\in K}\sum_{t\in T} c_{iq}^C s_{iqt} + \sum_{i\in I}\sum_{k\in K}\sum_{t\in T} c_k^R w_{ikt}^{BO} \\ + \sum_{i\in I}\sum_{k\in K}\sum_{t\in T} c_k^{TO} v_{ikt}^{BCO} + \sum_{i\in I}\sum_{k\in K}\sum_{t\in T} c_k^{TC} v_{ikt}^{BOC}\end{aligned} \tag{3.12}$$

Demand and Capacity Constraints. The constraints representing the part of the facility location problem are identical to the ones in the previously presented model. However, the y variables now represent the number of open trailers at each location and time period:

$$\sum_{i\in I}\sum_{k\in K} x_{ijkpt} = d_{jpt} \;\; ;\forall j \in J \;\; ;\forall p \in P \;\; ;\forall t \in T \tag{3.13}$$

$$\sum_{j\in J} x_{ijkpt} \leq z_{ikpt} \;\; ;\forall i \in I \;\; ;\forall k \in K \;\; ;\forall p \in P \;\; ;\forall t \in T \tag{3.14}$$

$$\sum_{p\in P} N_p z_{ikpt} \leq u_{ik} y_{ikt} \;\; ;\forall i \in I \;\; ;\forall k \in K \;\; ;\forall t \in T \tag{3.15}$$

$$\sum_{k\in K} y_{ikt} \leq 1 \;\; ;\forall i \in I \;\; ;\forall t \in T \tag{3.16}$$

$$x_{ijkpt} \leq d_{jpt} y_{ikt} \;\; ;\forall i \in I \;\; ;\forall j \in J \;\; ;\forall k \in K \;\; ;\forall p \in P \;\; ;\forall t \in T \tag{3.17}$$

Flow Conservation and Consistency Constraints. The network is modeled by the following constraints. Constraints (3.18), (3.19), (3.20) and (3.21) represent the first nodes for open and closed trailers and the second nodes for open and closed trailers, respectively. Note that the variables l^O_{it} and l^C_{it} do not exist for $t = 0$, i.e., in constraints (3.18) and (3.19), we have $l^O_{i(t=0)} = 0$ and $l^C_{i(t=0)} = 0$. If a region $i \in I$ already possesses a camp at the beginning of the planning horizon, then a constant $\Gamma_{it} > 0$ (with $t = 1$) indicates the number of hosting trailers of that camp. Clearly, $\Gamma_{it} = 0$ for all $t > 1$. Constraints (3.22) guarantee that the number of existing trailers at a camp never exceeds the maximum camp size, while (3.23) link the y variables to the number of open trailers:

$$\Gamma_{it} + l^{O}_{i(t-1)} + \sum_{q \in K} q s_{iqt} = v^{OO}_{it} + v^{OC}_{it} \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.18}$$

$$l^{C}_{i(t-1)} + v^{OC}_{it} = v^{CC}_{it} + w^{O}_{it} \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.19}$$

$$v^{OO}_{it} + v^{CO}_{it} = l^{O}_{it} \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.20}$$

$$v^{CC}_{it} + w^{I}_{it} = v^{CO}_{it} + l^{C}_{it} \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.21}$$

$$l^{O}_{it} + l^{C}_{it} \leq \overline{K} \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.22}$$

$$\sum_{k \in K} k y_{ikt} = l^{O}_{it} \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.23}$$

Relocation Consistency Constraints. Equalities (3.26) enforce that if a camp of size k is removed from a location, then a camp of the same size must be placed at another region. They ensure that trailers of different camps will not be mixed if they are relocated at the same time period. Constraints (3.24) ensure that camps are only relocated as a whole, i.e., no trailers remain at the location if a camp is relocated. Constraints (3.25) say that a camp can only be relocated to locations where no other camps exist. Constraints (3.26) ensure that camps from different locations are not merged. Although redundant, constraints $\sum_{k \in K} w^{BO}_{ikt} \leq 1$ and $\sum_{k \in K} w^{BI}_{ikt} \leq 1$ are explicitly added to the model, since they help CPLEX generate further cuts.

$$v^{CC}_{it} + v^{OO}_{it} \leq \overline{K} \left(1 - \sum_{k \in K} w^{BO}_{ikt} \right) \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.24}$$

$$v^{CC}_{it} + v^{OO}_{it} - \sum_{q \in K} q s_{jqt} \leq \overline{K} \left(1 - \sum_{k \in K} w^{BI}_{ikt} \right) \quad ;\forall i \in I \;\; ;\forall t \in T \tag{3.25}$$

$$\sum_{i \in I} w^{BO}_{ikt} = \sum_{i \in I} w^{BI}_{ikt} \quad ;\forall k \in K \;\; ;\forall t \in T \tag{3.26}$$

Linking Constraints for Modular Costs. Linking constraints as suggested by Melo et al. (2006) are used to link the continuous arc flow variables to the binary variables for modular decisions:

$$\sum_{k \in K} k v_{ikt}^{BCO} = v_{it}^{CO} \quad ;\forall i \in I \ ;\forall t \in T \tag{3.27}$$

$$\sum_{k \in K} k v_{ikt}^{BOC} = v_{it}^{OC} \quad ;\forall i \in I \ ;\forall t \in T \tag{3.28}$$

$$\sum_{k \in K} k w_{ikt}^{BO} = w_{it}^{O} \quad ;\forall i \in I \ ;\forall t \in T \tag{3.29}$$

$$\sum_{k \in K} k w_{ikt}^{BI} = w_{it}^{I} \quad ;\forall i \in I \ ;\forall t \in T \tag{3.30}$$

Variable Domains. Once the y variables are fixed, the remaining subproblem defined by the network flow structure can be stated as a *Mininum Cost Network Flow Problem*. All l^O arcs are then fixed according to the y values due to the equality constraints (3.23). Thus, the remaining network matrix has the *unimodularity property*. We could thus state all arc variables as continuous without losing their integrality property in the solution. However, we keep integrality on the arc variables, since experiments showed that it slightly facilitates the solution by CPLEX.

$$x_{ijkpt} \in \mathbb{R}^+ \quad ;\forall i \in I \ ;\forall j \in J \ ;\forall k \in K \ ;\forall p \in P \ ;\forall t \in T \tag{3.31}$$

$$z_{ikpt} \in \mathbb{Z}^+ \quad ;\forall i \in I \ ;\forall k \in K \ ;\forall p \in P \ ;\forall t \in T \tag{3.32}$$

$$y_{ikt} \in \{0,1\} \quad ;\forall i \in I \ ;\forall k \in K \ ;\forall t \in T \tag{3.33}$$

$$s_{iqt} \in \{0,1\} \quad ;\forall i \in I \ ;\forall q \in K \ ;\forall t \in T \tag{3.34}$$

$$l_{it}^{O}, l_{it}^{C}, v_{it}^{CC}, v_{it}^{CO}, v_{it}^{OO}, v_{it}^{OC}, w_{it}^{O}, w_{it}^{I} \in \mathbb{Z}^+ \quad ;\forall i \in I \ ;\forall t \in T \tag{3.35}$$

$$v_{ikt}^{BCO}, v_{ikt}^{BOC}, w_{ikt}^{BO}, w_{ikt}^{BI} \in \{0,1\} \quad ;\forall i \in I \ ;\forall k \in K \ ;\forall t \in T \tag{3.36}$$

Note that, for the CSLP, the SAD inequalities given by (3.11) are modified, replacing each variable y_{ik} by a variable y_{ikt}.

3.6 Computational Experiments

3.6.1 Instance Generation and Experimentation Environment

In order to test the robustness of the model, instances have been generated with different parameters. Certain data have been adapted from a real-world (RW) instance, based on data provided by a Canadian logging company (see Section 3.7.2). Key parameters are found to be the ones that may change the difficulty of the problem, namely:

- **Problem dimension**. Instances have been generated with the following dimensions *(#facility locations/#customers)*: *(10/20)*, *(10/50)*, *(50/50)* and *(50/100)*.
- **Distances and transportation costs**. For each of the problem sizes, three different networks have been randomly generated on squares of the following sizes: $300km \times 300km$, $380km \times 380km$ and $450km \times 450km$. Transportation costs have been computed as explained in Section 3.3.1.0.1.
- **Number of commodities**. Demands are generated either only for logging and road construction (i.e., two commodities) or additionally for the corresponding supervisors (i.e., four commodities).
- **Concavity of the cost curves**. Two extreme cases are considered: construction and operational costs are either linear or concave. In addition, the cost curves given in the RW instance with linear construction costs and concave operational costs are considered.
- **Demand distribution**. The demand for each region within each season is randomly generated so that the total demand in each season throughout all regions is similar. For each region, the demand is either uniformly distributed over all seasons or randomly distributed over up to four seasons.
- **Cost distribution**. Costs are generated to result in different ratios between camp construction/relocation and transportation costs. The transportation costs were set to 20%, 100% and 200% of the original transportation costs indicated in the RW instance.
- **Initial demand coverage**. Instances are generated with different numbers of ini-

tially existing camps. The total capacity of such camps covers either 0%, 50% or 100% of the total demand.

All generated instances contain ten time periods. Camp relocation costs and the costs to close or reopen trailers have been adapted from the RW instance. The maximum camp size $\overline{K}$ has been chosen so that a single camp with $\overline{K}$ trailers is capable to host the entire worker demand. The combination of all different configurations explained above resulted in 1296 instances. Experiments on all instances showed that instances are significantly easier to solve when the cost curves are linear or only two commodities (i.e., no demands for supervisors) are used. On the other hand, instances with 50 or more potential facility locations could virtually not be solved within the imposed time limit of one hour of computation time. The results presented throughout this paper are thus based on a subset of the instances described above. This subset includes 216 instances: all instances of reasonable size, i.e., (10/20) and (10/50), excluding those which are known to be easily solved, i.e., having only two commodities or linear cost curves.

The code has been written in C/C++ using the Callable Library of IBM ILOG CPLEX 12.3 and has been compiled and executed on openSUSE 11.3. Each problem instance has been run on a single AMD Opteron 250 processor (2.4 GHz), limited to 4GB of RAM. If not stated otherwise, CPLEX computation times have been limited to 60 minutes.

3.6.2 Computational Results

The following variants of the problem have been considered to investigate the impact of the different problem features on the difficulty of solving the problem:

– The DMCFLP as described in Section 3.5.1. Both versions without and with RUC constraints (round-up capacity constraints, see Section 3.5.2) and SAD inequalities (strengthened aggregated demand inequalities, see Section 3.5.2.1) are considered.
– The CSLP, as described in Section 3.5.3.

The SI valid inequalities, given by (3.5) and (3.17) for the DMCFLP and the CSLP, respectively, are very effective to strengthen the model. The integrality gap of the DMCFLP with RUC constraints and SAD inequalities was found to be 20.3% (average over the 216 selected instances). Adding the SI inequalities (3.5) to the model decreased the integrality gap to an average of 2.2%. In CPLEX, valid inequalities can be added to the model either all *a priori* or dynamically (called *user cuts*), only those that are violated during the solution of the linear relaxation. In the following experiments, the SIs have been added as user cuts in the case of the DMCFLP. For the CSLP, all SIs have been added to the model *a priori*. Further experiments indicate that CPLEX performs best when the parameter *MIPEmphasis* is set to *feasibility*.

3.6.2.1 Impact of the RUC constraints and SAD inequalities

Computational experiments for the DMCFLP (performed on all 1296 instances described above) showed that the average integrality gap increased from 2.8% to 6.0% when the RUC (round-up capacity) constraints are used within the model. This indicates that the RUC constraints significantly complicate the solution of the problem. However, the additional use of the SAD inequalities reduces the average integrality gap to 1.4%.

Table 3.II summarizes the average optimality gaps after one hour of computation time. We compare three different versions for the DMCFLP. The version *w/o RUC* indicates the DMCFLP, defined by (3.1) – (3.7), with common capacity constraints (i.e., no round-up capacity constraints). The second version, denoted by *w/ RUC w/o SAD*, explores the impact of the round-up capacity constraints. This problem version is thus defined by by (3.1), (3.2) and (3.4) – (3.10). Finally, we investigate the impact of the SAD inequalities. The version, denoted by *w/ RUC w/ SAD*, is thus defined by (3.1), (3.2) and (3.4) – (3.11). As previously mentioned, all SIs, given by inequalities (3.5) are added as CPLEX user cuts.

For each of the three versions we report average and maximum optimality gaps. The column *# ns* indicates the number of instances where either no feasible integer solution has been found or the solver ran out of memory. The results indicate that adding the round-up capacity constraints significantly complicates the solution of the problem. For ten instances, no feasible solution could be found. However, the additional use of the SAD inequalities proved quite effective to improve the optimality gap.

Inst size	# Inst	w/o RUC gap % avg	max	# ns	w/ RUC w/o SAD gap % avg	max	# ns	w/ RUC w/ SAD gap % avg	max	# ns
10/20	108	0.00	0.01	0	7.73	41.05	0	0.39	26.01	0
10/50	108	8.60	38.26	0	25.06	59.88	10	17.82	57.83	10
All	216	4.30	38.26	0	16.02	59.88	10	8.68	57.83	10

Table 3.II: Comparing the solution quality for the DMCFLP without/with RUC constraints as well as without/with SAD inequalities after one hour of computation time.

3.6.2.2 Solving the CSLP and Solution Properties

We now explore how the difficulty of solving the CSLP is affected by the RUC constraints and SAD inequalities. We also investigate the impact of different instance characteristics. We show relations between the optimal solutions of the DMCFLP and the CSLP by comparing the number of constructed and relocated trailers. This leads to the idea of using DMCFLP solutions as starting solutions for the CSLP. The impact of certain properties such as the demand distribution over time, the initial camp capacity and the dimension of transportation costs is evaluated.

Table 3.III compares the results for different solution approaches: two approaches based on conventional CPLEX optimization and a third approach which is explained further below. The first approach, denoted by *CSLP w/o SAD*, involves the solution of the CSLP defined by (3.12) – (3.36) using CPLEX. The second approach, denoted by *CSLP w/ SAD* additionally uses the SAD inequalities (3.11).

The table presents average and maximum optimality gaps when compared with the best known lower bound for each instance. In addition, the number of instances where no feasible integer solution has been found or the solver ran out of memory (*# ns*) is reported. As the results indicate, the SAD inequalities improve the performance of CPLEX. Feasible solutions can be found for 36 further instances and the solution quality improves significantly.

DMCFLP warm start solutions for the CSLP. As we observed in the previous section, DMCFLP solutions of fair quality can easily be obtained. For the CSLP, we may

Inst size	# Inst	CSLP w/o SAD gap % avg	max	# ns	CSLP w/ SAD gap % avg	max	# ns	CSLPHeur w/ SAD gap % avg	max	# ns
10/20	108	28.71	55.32	41	9.13	54.06	22	6.42	20.24	0
10/50	108	-	-	108	3.53	22.57	91	18.24	49.95	31
All	216	28.71	55.32	149	8.21	54.06	113	11.89	49.95	13

Table 3.III: Comparing the solution quality after one hour of computation time using different solution approaches.

have trouble to find any feasible integer solution at all. However, a feasible solution for the DMCFLP is also feasible for the CSLP. To convert an optimal DMCFLP solution into a feasible CSLP solution, the y variable values of the DMCFLP solution are fixed. CPLEX then heuristically finds feasible values for the missing variables (parameter *effortLevel* has been set to 3). Table 3.IV shows the average optimality gaps of the optimal DMCFLP solutions in the CSLP. The average optimality gap of such solutions (except for five instances of size (10/50) where no optimal DMCFLP solution has been found) is around 15%. The results are then separated by instances with certain characteristics, namely the demand distribution along time as well as the initial demand coverage by existing camps. One would assume that DMCFLP solutions perform better for instances where the demand is uniformly distributed over time, since the relocation of camps seems less probable. However, the results do not show any clear evidence of a better performance.

On the other hand, the total capacity of existing camps seems to have more impact on the DMCFLP solution quality in the CSLP. The less camps initially exist, the better the DMCFLP solution quality. This is because in both versions camps have to be constructed. This is summarized in Table 3.V, which reports the average number of constructed and relocated trailers according to the demand distribution and the number of initially existing camps (only solutions with a proven optimality gap smaller than or equal to 10% have been considered). Instances with demand uniformly distributed over all time periods tend to have less constructions and relocations than instances in which demand is irregularly distributed over time. In addition, the less camp capacity is initially available, the smaller the chance that existing camps are relocated instead of constructing new ones. Thus, new optimal placed camps in a DMCFLP solution are more likely to be a good choice for the CSLP as well.

Inst Size	all	Demand distribution		Initial demand coverage		
		Uniform	Clustered	0%	50%	100%
10/20	12.8	11.6	13.9	9.2	12.9	16.1
10/50	17.2	18.8	15.7	12.1	15.8	23.9
Total	14.9	15.0	14.8	10.6	14.4	19.8

Table 3.IV: The average optimality gaps of optimal DMCFLP solutions in the CSLP.

Inst Size	Demand distribution		Initial demand coverage		
	Uniform	Clustered	0%	50%	100%
# Constructions	4.9	6.7	7.8	4.6	2.3
# Relocations	0.8	1.1	0.0	1.2	1.8

Table 3.V: The average number of constructed and relocated trailers within near optimal CSLP solutions.

We may thus use DMCFLP solutions as warm start solutions for the CSLP. The last three columns, denoted by *CSLPHeur w/ SAD*, in Table 3.III indicate the results after one hour of computation time for the CSLP (w/ SAD), when the best DMCFLP (w/ RUC w/ SAD) solution obtained after one hour of computation time is used as a warm start solution. Compared with the conventional execution of the CSLP, CPLEX now finds feasible solutions for most of the instances while maintaining a similar average optimality gap.

The impact of the cost ratio. The ratio between transportation costs and the costs to construct or relocate camps has also been found to have a strong impact on the difficulty of solving the problem. A total of 264 additional instances of the sizes (10/20) and (10/50) have been generated with eleven different transportation costs, set between 1% and 3000% of the original transportation costs given in the RW instance. We refer to this percentage as $TC\%$. All instances contain sufficient camp capacities to cover 50% of the average demand per season.

Figure 3.4 (a) and (b) illustrate the difficulty of solving the generated instances for the CSLP subject to their $TC\%$ ratios (in one hour of computation time). For each of the $TC\%$ cost ratios, the number of instances where no feasible solution has been found (see Figure 3.4 (a)) and the average optimality gap of the final solutions (see Figure 3.4 (b)) are reported. The results indicate that the problem gets more difficult to solve when $TC\% = 100$. With $TC\%$ values greater than 1500, it seems that the solution of the problem gets slightly easier again. Figure 3.4 (c) shows the average number of constructed and relocated trailers within the final solutions (again, only solutions with a proven optimality gap smaller than or equal to 10% have been considered). The results indicate that the number of constructed trailers grows faster than the number of relocations when the transportation costs increase.

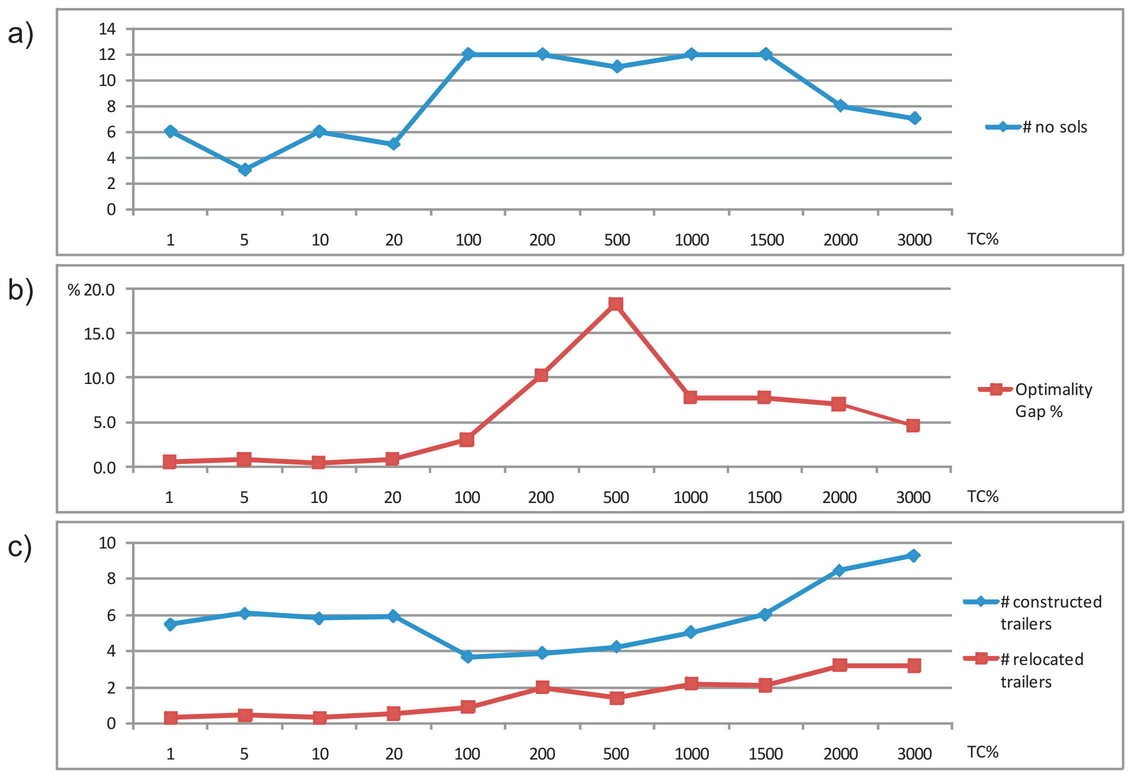

Figure 3.4: The impact of the transportation cost ratio on (a) the number of CSLP instances where no solutions have been found, (b) the average optimality gaps and (c) the average number of constructed and relocated trailers in near optimal solutions.

Yearly camp relocation. All previous experiments have assumed that camp relocation is allowed after each season. In the case of the Canadian logging company that provided the real-world instance, relocation is possible only once a year. We investigate the difficulty of solving this slightly simplified problem, considering all instances (*ISall*). We use the *CSLPheur* approach, i.e., we first solve the DMCFLP with a time limit of one hour and then use the best solution as a starting solution for the CSLP, also limited to one hour of computation time. The results, summarized in Table 3.VI, show that instances of reasonable size (i.e., 10/20 and 10/50) can be fairly well solved. Most of the larger instances exceed either the given memory limit of 4GB or CPLEX capabilities to solve the problem in the given time limit.

Inst Size	# Inst	gap %	# ns	# opt	time (sec)
10/20	324	4.3	0	134	3992
10/50	324	14.6	17	24	5664
50/50	324	24.2	134	12	7173
50/100	324	19.7	295	31	7447
Total Avg	1296	11.5	446	201	4984

Table 3.VI: Results (ISall) with CSLPheur when camp relocation is allowed only once a year.

3.7 Case Study

In this section, we analyze the planning solutions proposed by our model for two planning periods of our industrial partner. Each of the two planning periods spans five years. Each year is divided into a summer and a winter season. For the first planning period, we consider the activities performed by the company throughout the harvest period 2006 to 2010. We aim at simulating the decisions made by the company and compare them with the decisions suggested by the mathematical model. The second planning horizon considers the harvest planning for the next five years, starting in 2011.

3.7.1 Comparative Study for Planning Period 2006 to 2010

In this study, we simulate the activities performed by the company on two different levels: first, construction and relocation of logging camps and, second, the allocation of worker demand to accommodations. The results are then compared to the solution provided by the mathematical model.

3.7.1.1 Data Description

The company performed logging and road construction activities in an area which is divided into approximately 4000 different regions. These regions are geographically clustered to a total of 38 regions. The planning period starts at the summer season in 2006 and ends after the winter season in 2010. The logging and road construction activities were subject to significant variations throughout the seasons. The total average demands (in number of workers per day) for logging and road construction in each season are illustrated in Figure 3.5. Note that the demands at each region are not necessarily clustered within subsequent seasons. Logging crews are formed by six workers, while road construction crews contain three workers. Demands for three logging supervisors and one road construction supervisor are estimated in proportion to the regions' work crew demands. All 38 working regions as well as the locations of the company's camps are available for potential camp construction or relocation. Detailed data for the entire

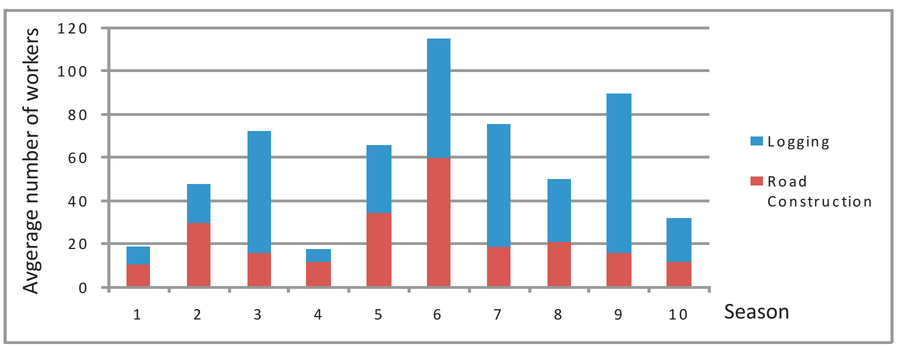

Figure 3.5: Total demand (in average number of workers per day) throughout all seasons.

road network, composed by roads categorized into four different conditions, are available. A simplified version is illustrated in Figure 3.6. Logging and road construction regions are indicated by the green areas. Each road type allows a different vehicle average speed, such that transportation times and costs are computed fairly accurately. Costs take into account gasoline, vehicle renting and additional salary due to long travel times.

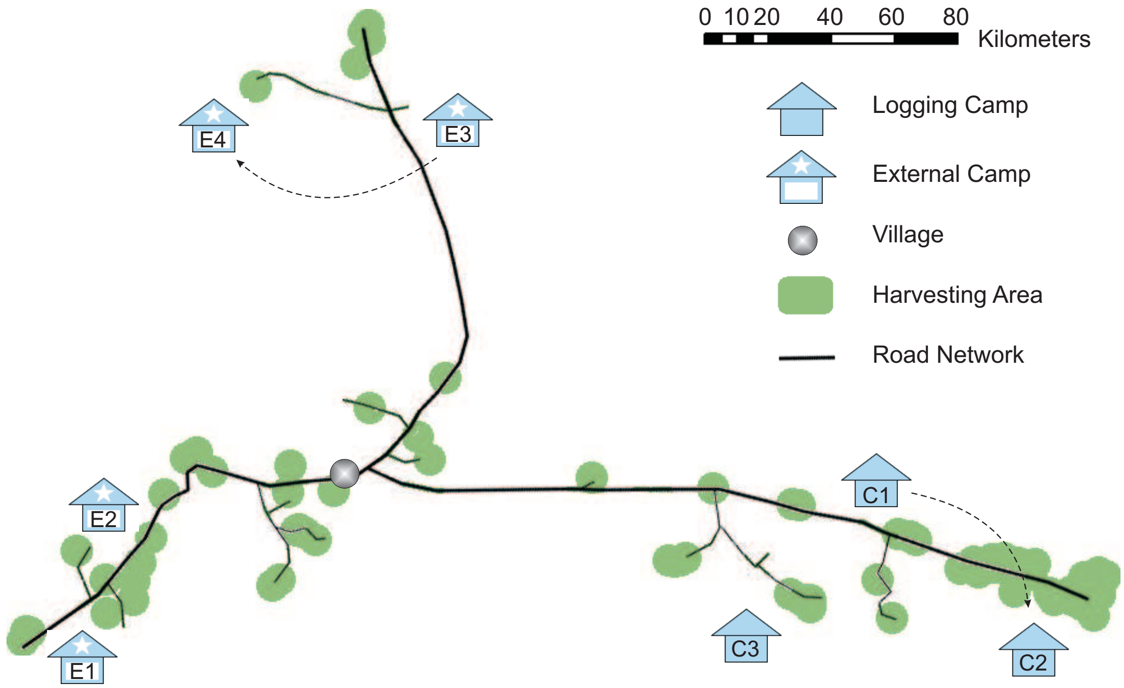

Figure 3.6: Simplified illustration of the logging regions and the road network.

Available accommodations and camp relocations. A village is located in a central location between the regions of forestry activities. According to the company, approximately two out of five crews live in the village and may thus be hosted at zero costs. We thus roughly estimate that 40% of the total worker demand may be hosted at the village, paying only the transportation costs. In addition, a practically unlimited number of hotel accommodations is available at the village for a price of 170$ per person per night, including 54$ for food.

In addition to the village, three logging camps from an external company are available. In the map, these camps are indicated by (capacity in parentheses in number of workers) E1 (65), E2 (40) and E3 (120). The latter has been relocated to location E4 after the winter season in 2008. External camps can be used on demand at an estimated price of 170$ per person per night (food included).

The company itself held three camps in the beginning of 2006, indicated by C1 (60), C2 (96) and C3 (48). We assume that these camps hold trailers each with a capacity for twelve workers. After a few years, the location of camp C1 was two far from the new logging regions. Parts of this camp have thus been relocated to join the camp located at C2 after the winter season 2009, resulting in a larger camp for up to 144 workers. The costs for these camps involve significant economies of scale and thus depend on the size of the camp. The maintenance costs are around 1020$ per day for a camp with a single trailer (capacity for twelve workers) and around 3400$ for a camp with ten trailers (capacity for 120 workers). In addition, we assume a daily cost of 54$ per worker for food. Maintaining large camps may thus be much cheaper than using the external camps and accommodations.

As can be recognized, the available capacities are very large compared to the total number of workers active in logging and road construction. This is due to the fact that other workers involved in forestry activities, such as forest management, tree planting, etc. use the camps. In addition, some capacities are used by the mining industry. However, the priority is always given to logging and road construction workers. We can thus assume that the entire capacity is available.

3.7.1.2 Comparison to the Proposed Planning

We now compare the activities performed by the company with the planning proposed by the mathematical model. As previously mentioned, we compare the planning decisions on two different levels:

i. Availability of capacities. We compare the decisions regarding camp construction and relocation.

ii. Worker demand allocation. We compare the allocation of workers from working regions to accommodations.

Decisions regarding camp construction and relocation. All optimization models are based on the CSLP model, defined by (3.12) – (3.36) and using the SAD inequalities (3.11) to facilitate the solution. We simulate the activities performed by the company by fixing all decisions regarding available capacities exactly as stated in Section 3.7.1.1. To be precise, we fix the relocation of external camp E3, the relocation of camp C1 and the construction of camp C3. As all construction and relocation decisions are fixed, the model is solved to optimality in a few minutes. The results for this scenario are compared to the optimal solution for a scenario where only the initial capacities of the company are fixed. External camp E3 is still relocated (as this is not a decision made by the company). However, the relocation of C1 and the construction of C3 are not fixed. The optimal solution for this scenario has been obtained within 20 hours of computation

Costs ($)	Simulated company activities	Optimized decisions
Hosting	2,683,588	2,770,588
Transportation	1,848,021	1,931,367
Maintenance	1,600,222	1,419,187
Trailer Change	164,328	110,713
Sub-total	6,296,159	6,231,855
Construction	975,000	0
Relocation	302,470	0
Total	7,573,630	6,231,855

Table 3.VII: Cost distribution for the simulated company activities and the optimized solution.

time (nine hours to prove optimality $< 1\%$). In contrast to the decisions made by the company, the optimal solution does not suggest any camp construction or relocation. Instead, the cheapest solution is obtained when using the available capacities at the same locations as found in the beginning. Table 3.VII summarizes the cost distribution for both scenarios. Without costs for construction and relocation, the costs for both scenarios are very similar. This suggests that, for the activities in the given time period, the initial locations of the existing camps, as well as their capacities were just as good as the locations and capacities achieved by the construction and relocation of camps. Adding the costs for construction and relocation to the costs for the company's activities results in a total cost that is much higher. Note that both scenarios assume optimal demand allocation. We will explore this topic further below.

Decisions regarding the demand allocation. The previous analysis simulates the company's activities regarding the decisions of where to locate or relocate camps. For both scenarios, the results assume that the demand allocation is optimal, i.e., the amount of workers from each region hosted at each accommodation, as well as the capacity level maintained at each camp during each season. As many cost factors have to be considered when allocating the workers to the accommodations, a manual allocation planning is likely to be far from optimal. Of course, many other factors may impact the decisions when allocating certain working regions to accommodations, such as the preferences of certain workers.

For the given planning period, the actual allocation of the workers to the available accommodations is not known. We thus use a simple heuristic to simulate the manual allocation planning. The allocation is performed for each season. We give priority to regions with large worker demands. Logging and road construction demands are thus considered in non-increasing order. We then select the accommodation that has the lowest cost for transportation and hosting. To compute the real cost, one should also consider the maintenance costs for open trailers at the company's camps. However, these costs depend on the actual occupation level of the camp and are thus difficult to estimate at the beginning of the heuristic planning. We thus do not include the maintenance costs in the total costs to emphasize the use of the company's own camps. Table 3.VIII compares the

Costs ($)	Optimal	Heuristic
Hosting	2,683,588	1,437,822
Transportation	1,848,021	3,093,429
Maintenance	1,600,222	2,424,474
Trailer Change	164,328	186,881
Sub-total	6,296,159	7,142,606
Construction	975,000	975,000
Relocation	302,470	302,470
Total	7,573,630	8,420,076

Table 3.VIII: Cost distribution for optimal and heuristic demand allocation.

cost distribution for the optimal and the heuristic demand allocation. From an economic point of view, the optimal demand allocation is 11.8% cheaper when comparing the costs involved in hosting, transportation and camp/trailer maintenance. Note, however, that a planning as suggested in the optimal scenario is likely not to be completely feasible in practice. Many other factors may impact on such planning, such as the preferences of workers, changes in demand and other uncertainties. Workers will most likely prefer not to change their accommodation too often throughout a working season. An analysis of the suggested demand allocation in the optimal scenario shows that in most cases this criterion is met. Working crews are allocated to the same accommodation throughout the entire season. Only in a small number of cases, two different accommodations are suggested to host the same crew throughout a season. Although the allocation of workers to accommodations includes many other factors not considered in our analysis, the large cost difference found in our comparison emphasizes the difficulty of a cost efficient manual planning and suggests a potential to perform demand allocation in a more cost efficient manner, while respecting all other requirements.

3.7.2 Analysis of Proposed Planning for Period Starting in 2011

Based on the logging and road construction demands for the harvesting period 2011 to 2015, we now analyze the decisions proposed by the mathematical model.

Data Description. The data contains 29 clusters of logging regions. The road network is similar to the one shown in Figure 3.6. However, logging regions, as well as the locations of available accommodations, are different. The demands in this planning are much more balanced over the seasons than it was the case in the previous planning period. Demands require up to eight logging and four road construction crews. The complete demand is easily covered by five existing accommodations: the village and four camps (with 2, 3, 4 and 4 trailers, respectively). All other assumptions are similar to the ones made for the previous planning period.

Solution analysis for different scenarios. Data about whether or not the company intends to construct or relocate camps were not available. We therefore do not compare to decisions of the company, but rather to two extreme scenarios to show how the proposed model may help in future decisions: one scenario where available capacities are not changed at all and one where capacities may be changed if beneficial. The first scenario thus considers only the existing accommodations at their original locations. Camp construction and relocation are thus not allowed. The second scenario assumes the original locations of the initially existing accommodations, but additionally allows the construction of new camps and the relocation of existing ones (once a year). Both scenarios are based on the CSLP model, defined by (3.12) – (3.36) and the SAD inequalities (3.11). Table 3.IX shows how costs are distributed in the optimal solution of each scenario. Scenario 2 suggests the relocation of a camp with four trailers after the fifth season. The additional camp relocation costs are outweighed by the savings in the transportation costs, which reduced by more than 40%. This results in a very beneficial solution, reducing the total costs by 8.6%.

Costs ($)	Scenario 1	Scenario 2
Hosting	1,879,905	1,476,083
Transportation	2,261,809	1,353,561
Maintenance	2,983,112	3,365,490
Trailer Change	252,521	242,751
Construction	0	0
Relocation	0	302,470
Total	7,377,347	6,740,355

Table 3.IX: Cost distribution in the optimal solutions for both scenarios.

Clearly, the reduction of the transportation costs is directly linked to the traveled time and distance. As can be seen in Table 3.X, the average distance traveled by the crews is reduced significantly (23% and 16%, respectively, in Scenario 2) when the camp is relocated. Finally, Table 3.X also reports the proportion of time during which existing trailers are open. This percentage considers all existing trailers throughout all time periods. One can observe that slightly more trailers are opened in Scenario 2, i.e., the existing camps are better used than in the previous two scenarios. Maintenance costs increase, but lower transportation costs may be involved as such trailers are closer to certain logging regions.

	Scenario 1	Scenario 2
Trailers open	49.4%	54.8%
Average travel distance (km):		
Logging crews	114	88
Road construction crews	129	109

Table 3.X: Usage of existing trailers and travel distances for the both scenarios.

3.8 Conclusions and Future Research

A mixed-integer programming model for the location of logging camps has been presented. This model extends the classical Capacitated Facility Location Problem by several features. Next to the well known features of multiple periods, multiple commodities and multiple capacity levels, further extensions include the partial and temporary closing of facilities, particular capacity constraints that include integer rounding and the integration of economies of scale on several levels of the cost structure. In addition, the model allows the extensions and relocation of existing facilities. Such integer rounding capacity constraints can be useful in other applications. As they increase the integrality gap and therefore the difficulty to solve the problem, new valid inequalities are derived to effectively reduce this integrality gap.

Instances based on a large variety of different properties have been generated. Experiments on these instances illustrated the impact of the different problem features on the difficulty to solve the problem. It is shown that general purpose solvers such as CPLEX are capable of solving most of the instances up to a realistic size in reasonable time, when using optimal solutions of a simplified problem as warm start solution for the entire problem. Case studies based on data from a Canadian logging company for two planning periods have been presented. The first study indicates a strong potential for economic savings on two different decision levels: where to locate the logging camps, as well as how to allocate worker demand from the working regions to the accommodations. The second study proposes a planning for the upcoming planning period of the company. It proposes the relocation of an existing camp, resulting in potential savings of more than 8% of the total costs when compared to the scenario where camps stay at their current location.

Though most of the smaller and medium sized instances can be solved in reasonable time, some of the instances remain unsolved. The models for larger instances typically exceed the memory limitations of current standard computers, such as the ones used in the experiments. In order to solve these instances, more sophisticated solution techniques are necessary, such as mathematical decomposition. Interesting extensions of the model for future research include the possibility of partial relocation of camps, as well as the use of trailers of different sizes.

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CHAPTER 4

DYNAMIC FACILITY LOCATION WITH GENERALIZED MODULAR CAPACITIES

4.1 Chapter Preface

In Chapter 3, an industrial application has been introduced that possesses a very detailed cost structure for capacity changes. In that application, the costs do not only depend on the total capacity involved in the operation, such as capacity expansion or the temporary closing of a part of a facility, but also on the current capacity level. As can be observed in Table 3.I, the relation between the number of hosting and supporting trailers cannot be represented by a polynomial function. In fact, to represent the cost structure of the problem on a detailed level, one may construct a matrix indicating the exact costs to expand capacities from capacity level ℓ_1 to capacity level ℓ_2. One may do the same for the temporary closing of capacity levels.

In this chapter, we consider a simplified variant of the CSLP, a multi-period facility location problem with modular capacities where the costs for capacity changes are based on a cost matrix. As we will see, the resulting MIP model is simple, yet powerful, generalizing several facility location problems found in the existing literature and providing strong LP relaxation bounds. This is illustrated by comparing the new formulation to specialized formulations for three facility location problems. Using generic MIP solvers, the proposed formulation finds optimal solutions in significantly shorter computing times than the specialized formulations. The dominance of the formulation in terms of the LP relaxation bound is proved in Appendix B.1. The results of this chapter provide an ideal foundation to model problems with complex cost structures, such as the one found in the CSLP, in full detail.

Notes about the chapter

The contents of this chapter correspond to those of the article entitled *Dynamic Facility Location with Generalized Modular Capacities*, co-authored with Professors Jean-François Cordeau and Bernard Gendron, which has been submitted for publication to Transportation Science (ISSN: 0041-1655), in March 2013 (revised version submitted in March 2014).

While the above mentioned paper only contains formal dominance proofs for the two special cases DMCFLP_CR and DMCFLP_ER, a dominance proof for the DMCFLP_CR_ER, as well as its explicit flow model can be found in Appendix B.1.3.

Dynamic Facility Location with Generalized Modular Capacities

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Location decisions are frequently subject to dynamic aspects such as changes in customer demand. Often, flexibility regarding the geographic location of facilities, as well as their capacities, is the only solution to such issues. Even when demand can be forecast, finding the optimal schedule for the deployment and dynamic adjustment of capacities remains a challenge, especially when the cost structure for these adjustments is complex. In this paper, we introduce a unifying model that generalizes existing formulations for several dynamic facility location problems and provides stronger linear programming relaxations than the specialized formulations. In addition, the model can address facility location problems where the costs for capacity changes are defined for all pairs of capacity levels. To the best of our knowledge, this problem has not been addressed in the literature. We apply our model to special cases of the problem with capacity expansion and reduction or temporary facility closing and reopening. We prove dominance relationships between our formulation and existing models for the special cases. Computational experiments on a large set of randomly generated instances with up to 100 facility locations and 1000 customers show that our model can obtain optimal solutions in shorter computing times than the existing specialized formulations.

Key words: Mixed-Integer Programming, Facility Location, Modular Capacities.

4.2 Introduction

Dynamic facility location consists in deciding *where* and *when* to provide capacity to satisfy customer demand at the lowest cost. This demand is rarely stable, but rather increases, decreases or oscillates over time. Therefore, facility capacities often have to be adjusted dynamically. Many variants of dynamic facility location problems have been studied, suggesting different ways to adjust capacities throughout a given planning horizon. The most common features include capacity expansion and reduction (Luss, 1982; Jacobsen, 1990; Antunes and Peeters, 2001; Troncoso and Garrido, 2005; Dias et al., 2007), temporary facility closing (Chardaire et al., 1996; Canel et al., 2001; Dias et al., 2006), as well as the relocation of capacities (Melo et al., 2006). Mathematical models that include such features have been applied in both the private and the public sectors to determine locations and capacities for production facilities, schools, hospitals, libraries and many more.

Facility location decisions aim to strike a balance between the fixed costs to supply capacity and the allocation costs to serve the demand. The latter often correspond to transportation costs to deliver products or provide services to customers. The ratio between these two types of costs has a strong impact on the solution and the difficulty of solving the problem (see, e.g., Shulman, 1991; Melkote and Daskin, 2001). In dynamic facility location problems, a detailed representation of the transportation costs not only affects the facility locations, but also their capacity throughout the planning horizon as capacity tends to follow the demand along time.

Regarding the fixed costs to provide the capacity, many studies acknowledge the existence of economies of scale (Correia and Captivo, 2003; Correia et al., 2010). While previous works considered economies of scale mainly for the construction and production costs, the costs for adjusting the capacities of the facilities have commonly been modeled in less detail. However, the latter is necessary to ensure a fair representation of the cost structure found in practice. The costs to adjust capacities often do not only depend on the size of the adjustment, but also on the current capacity level. This is true in a large class of applications, especially in transportation, logistics and telecommunications, where additional capacity gets cheaper (or more expensive) when approaching the maximum capacity limit.

In this work, we introduce a very general dynamic facility location problem, referred to as the *Dynamic Facility Location Problem with Generalized Modular Capacities (DFLPG)*. The problem allows modular capacity changes subject to a detailed cost structure and is modeled as a mixed-integer programming (MIP) formulation. Due to its generality, this model unifies several existing problems found in the literature. The cost structure used in the model is based on a matrix describing the costs for capacity changes between all pairs of capacity levels. We are not aware of any other work dealing with facility location with a similar level of detail in the cost structure.

Our study is motivated by an industrial project with a Canadian logging company that must locate camps to host workers involved in wood harvest activities while optimizing the overall logistics and transportation costs (Jena et al., 2012). In this problem, the total capacity of a camp is represented by its number of hosting units, while additional units provide supporting infrastructure. As the relation between the number of different units is non-linear, the costs for capacity changes are described in a transition matrix.

The contribution of this work is threefold. First, we introduce a general dynamic facility location model that comprises a large set of existing formulations. Second, we analyze the linear programming (LP) relaxation bound obtained by our model, showing that it is at least as strong as the LP relaxation bound of existing specialized formulations. Third, we perform a computational study on a large set of randomly generated instances, showing that our model, when solved with a state-of-the-art MIP solver, can obtain optimal solutions in shorter computation times than the specialized formulations.

The paper is organized as follows. In Section 4.3, we present a survey of the relevant literature. Section 4.4 introduces a linear MIP formulation for the DFLPG and shows how this model can be used to represent two important special cases. To compare the resulting models with alternative formulations, Section 4.5 derives specialized formulations for the two special cases, based on existing models from the literature. We identify a weak point in one of the existing formulations and suggest a set of valid inequalities to make it as strong as our model. Dominance relations are proved between all formulations, showing that our model is at least as strong as each of the specialized formulations. The presented models are then compared by means of computational experiments in Section 4.6. Finally, conclusions follow in Section 4.7.

4.3 Literature Review

Most dynamic facility location problems can be seen as multi-periodic extensions of classical location problems, such as the Capacitated Facility Location Problem (CFLP). However, dynamic facility location problems commonly involve further extensions. As pointed out by Arabani and Zanjirani Farahani (2011), the notion of what dynamic means may differ when dealing with different areas of facility location. Its definition thus strongly depends on the application context. For example, school capacities may be increased or decreased to meet demographic trends (e.g., Antunes and Peeters, 2001), terminals in telecommunications networks may be installed and removed along time to adapt to changes in data traffic and costs (e.g., Chardaire et al., 1996) and hospitals may relocate ambulances to cope with unpredictable demand (e.g., Brotcorne et al., 2003). Owen and Daskin (1998) review works that treat either dynamic or stochastic facility location problems. A chapter in the textbook of Zanjirani Farahani and Hekmatfar (2009) deals with dynamic aspects of facility location problems. Several classification criteria are proposed. A book chapter by Jacobsen (1990) dedicated to multi-period capacitated location models thoroughly discusses models that allow capacity expansion. Luss (1982) focuses on capacity expansion and reviews the literature and applications in the context of problems with a single facility, two facilities and multiple facilities. Although not explicitly focusing on dynamic aspects, many other works introduced classifications for location problems which often also apply to features that can be found in dynamic location problems. These include, among many others, the works of Hamacher and Nickel (1998), Owen and Daskin (1998), Klose and Drexl (2005), Daskin (2008) and Melo et al. (2009a).

The choice of the facility type or size has also been considered in several works. In particular, Shulman (1991), Correia and Captivo (2003) and Troncoso and Garrido

(2005) consider such choice, which implies different capacities and costs for each facility type. The last authors apply the model to the forestry sector, where facilities of different sizes may also be expanded. Dias et al. (2007) focus on modular capacity expansion and reduction. Wu et al. (2006) present a facility location problem where the facility setup costs depend on the number of facilities placed at a site. To represent economies of scale, all of the cited works use binary variables to distinguish different facility sizes. Capacity level changes consider only the amount of capacity added or removed. However, the previous capacity level is not taken into consideration. Some authors such as Harkness (2003) also recognize the importance of inverse economies of scale, where the unit price increases as the facility gets larger.

To dynamically adjust capacity to demand changes, the best choice depends on the demand forecast and the costs involved in capacity changes. For example, if capacity is leased, it may be possible to terminate a leasing contract at any time. In other situations, it may be beneficial to temporarily close a facility to avoid high maintenance costs. This may be appropriate when demand temporarily decreases, but is likely to return to its previous level afterwards. The closing and reopening of facilities may be partial or complete. Previous studies focused mostly on temporarily closing entire facilities. Among the suggested models, certain are limited to a single closing and reopening of each facility, whereas others allow repeated closing and reopening throughout the planning horizon. The uncapacitated facility location problem presented by Van Roy and Erlenkotter (1982), as well as the supply chain model of Hinojosa et al. (2008), allow one-time opening or closing of facilities: new facilities can be opened once and existing facilities can be closed once. Chardaire et al. (1996) and Canel et al. (2001) propose formulations for opening and closing facilities more than once. Both works use binary variables to represent the state of the facility. The objective function contains a bilinear term to represent a state change from open to closed or vice-versa. A linear formulation for a simplified version of this problem, treating only a single capacity level, has been proposed by Dias et al. (2006). Binary variables with two time indices indicate the period throughout which a facility is open. The cited works interpret facility closing either as temporary (i.e., the facility still exists, but its capacities are temporarily unavailable) or permanent. In most cases, maintenance costs for temporarily closed facilities are low and can therefore be ignored in the model. Most of the existing formulations therefore do not explicitly distinguish temporary and permanent facility closing.

When the customer demand permanently changes in a certain region and is not likely to return to its previous level, one may want to expand or reduce the facility capacities to permanently adjust to these new conditions. Luss (1982) observes that models for capacity expansion can be classified into two categories: capacity expansion at a single facility and capacity expansion via a finite set of projects, each holding a certain capacity. The first category includes models that allow one facility at a location and increases or decreases of the available capacity along time. The second category consists of models where multiple facilities are allowed in the same location, each specified by a time interval (a capacity block) of production availability. Figure 4.1 illustrates both classes. The first class is shown in (a), where capacities at the same facility are either increased or decreased. The second class may be illustrated by (b) and (c), representing two extreme configurations of the capacity blocks. Any configuration between these two is also feasible for the second class.

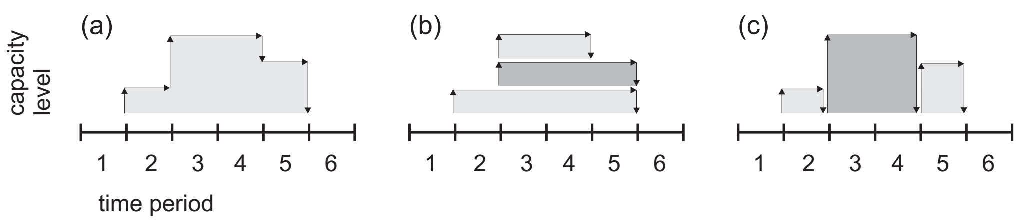

Figure 4.1: Capacity expansion/reduction by use of a single facility (a), horizontal capacity blocks (b) and vertical capacity blocks (c).

Models in the first category include those of Melo et al. (2006) and Behmardi and Lee (2008). Both works model capacity expansion and reduction by relocating capacity from or to a dummy location. The authors of the former work model capacities as a continuous flow, but demonstrate how to link the flow to binary variables to restrict capacity changes to modular sizes. Models in the second category do not allow the capacity modification of a facility once it is constructed. However, they allow multiple facilities of different sizes (capacity blocks) at the same location, which is equivalent to the adjustment of the total capacity sum along time. Examples for this class include the works of Shulman (1991), Troncoso and Garrido (2005) and Dias et al. (2007). More restricted types of capacity expansion or reduction have also been presented. In the work of Antunes and Peeters (2001), either a facility expands or decreases its capacity throughout the entire planning horizon. Capacity expansion and reduction at the same location is thus not allowed.

4.4 Mathematical Formulation

In this section, we give a more formal description of the DFLPG and introduce a MIP model for the problem. We also explain how the different cases described in Section 4.3 can be modeled as a DFLPG.

4.4.1 DFLPG Formulation

We denote by J the set of potential facility locations and by $L = \{0, 1, 2, \ldots, q\}$ the set of possible capacity levels for each facility. We also denote by I the set of customer demand points and by $T = \{1, 2, \ldots, |T|\}$ the set of time periods in the planning horizon. We assume throughout that the beginning of period $t+1$ corresponds to the end of period t.

The demand of customer i in period t is denoted by d_{it}. The cost to serve one unit from facility j operating at capacity level ℓ to customer i during period t is denoted by $g_{ij\ell t}$. This term is typically a cost function for handling and transportation costs, based on the distance between customer i and facility j. The capacity of a facility of size ℓ at location j is given by $u_{j\ell}$ (with $u_{j0} = 0$). The cost matrix $f_{j\ell_1\ell_2 t}$ describes the combined cost to change the capacity level of a facility at location j from ℓ_1 to ℓ_2 at the beginning of period t and to operate the facility at capacity level ℓ_2 throughout time period t. Furthermore, we let ℓ^j be the capacity level of an existing facility at location j. The constant ℓ^j is 0 if location j does not possess an existing facility at the beginning of the planning horizon.

To formulate the problem, we use binary variables $y_{j\ell_1\ell_2 t}$ equal to 1 if and only if the facility at location j changes its capacity level from ℓ_1 to ℓ_2 at the beginning of period t and maintains capacity level ℓ_2 throughout period t. The allocation variables $x_{ij\ell t}$ denote the fraction of the demand of customer i in period t that is served from a facility of size ℓ located at j. Based on these definitions, we define the following MIP model, referred to as the *Generalized Modular Capacities (GMC)* formulation:

$$\text{(GMC)} \quad \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2 t} y_{j\ell_1\ell_2 t} + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} g_{ij\ell t} d_{it} x_{ij\ell t} \tag{4.1}$$

$$s.t. \ \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell t} = 1 \ \ \forall i \in I, \ \forall t \in T \tag{4.2}$$

$$\sum_{i\in I} d_{it} x_{ij\ell t} \leq \sum_{\ell_1\in L} u_{j\ell} y_{j\ell_1\ell t} \ \ \forall j \in J, \ \forall \ell \in L, \ \forall t \in T \tag{4.3}$$

$$\sum_{\ell_1\in L} y_{j\ell_1\ell(t-1)} = \sum_{\ell_2\in L} y_{j\ell\ell_2 t} \ \ \forall j \in J, \ \forall \ell \in L, \ \forall t \in T\setminus\{1\} \tag{4.4}$$

$$\sum_{\ell_2\in L} y_{j\ell^j\ell_2 1} = 1 \ \ \forall j \in J \tag{4.5}$$

$$x_{ij\ell t} \geq 0 \ \ \forall i \in I, \ \forall j \in J, \ \forall \ell \in L, \ \forall t \in T \tag{4.6}$$

$$y_{j\ell_1\ell_2 t} \in \{0,1\} \ \ \forall j \in J, \ \forall \ell_1 \in L, \ \forall \ell_2 \in L, \ \forall t \in T. \tag{4.7}$$

The objective function (4.1) minimizes the total cost for changing the capacity levels and allocating the demand. Constraints (4.2) are the demand constraints for the customers. Constraints (4.3) are the capacity constraints at the facilities. Constraints (4.4) link the capacity change variables in consecutive time periods. Finally, constraints (4.5) specify that exactly one capacity level must be chosen at the beginning of the planning horizon. The flow constraints (4.4) further guarantee that, at each time period, exactly one capacity change variable is selected. Note that, taking into consideration the initial locations of facilities, we necessarily have: $y_{j\ell_1\ell_2 1} = 0, \forall j \in J, \forall \ell_1 \in L\setminus\{\ell^j\}, \forall \ell_2 \in L$.

Valid Inequalities. To facilitate the solution of the GMC, we may additionally use two types of valid inequalities. The *Strong Inequalities (SI)* used in facility location and network design problems (see, for instance, Gendron and Crainic, 1994) are known to provide a tight upper bound for the demand assignment variables. These inequalities can be adapted to our model as follows:

$$x_{ij\ell t} \leq \sum_{\ell_1 \in L} y_{j\ell_1 \ell t} \quad \forall i \in I, \ \forall j \in J, \ \forall \ell \in L, \ \forall t \in T. \tag{4.8}$$

The SIs may be added to the model either *a priori* or in a branch-and-cut manner only when they are violated in the solution of the LP relaxation. The second set of valid inequalities is referred to as the *Aggregated Demand Constraints (ADC)*. Although they are redundant for the LP relaxation, adding them to the model enables MIP solvers to generate cover cuts that further strengthen the formulation:

$$\sum_{j \in J} \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} u_{j\ell_2} y_{j\ell_1 \ell_2 t} \geq \sum_{i \in I} d_{it} \quad \forall t \in T. \tag{4.9}$$

4.4.2 DFLPG Based Models for the Special Cases

We now explain how two important special cases can be modeled with the GMC formulation: first, *Facility closing and reopening* and, second, *Capacity expansion and reduction*.

The first problem considered here allows the construction of at most one facility per location. The size of the facility is chosen from a discrete set of capacity levels. Existing facilities may be closed and reopened multiple times. Note that, in this problem, facility closing does not refer to permanent closing, but only to the temporary closing of a facility. We therefore distinguish costs for the construction of a facility, for temporarily closing an open facility, for reopening a closed facility and for maintenance of open facilities. As most of the previous literature, we do not consider maintenance costs for temporarily closed facilities. We denote this problem as the *Dynamic Modular Capacitated Facility Location Problem with Closing and Reopening (DMCFLP_CR)*.

In the second problem considered, capacities can be adjusted by the use of a single facility at each location. At each facility, the capacity can be expanded or reduced from one capacity level to another. We assume that an expansion of ℓ capacity levels has always the same costs, regardless of the previous capacity level. We assume the same for the reduction of capacities. We denote this problem as the *Dynamic Modular Capacitated*

Facility Location Problem with Capacity Expansion and Reduction (DMCFLP_ER).

In addition to the input data already defined for the DFLPG, we define the following fixed costs to characterize these two special cases:

- $c^c_{j\ell}$ and $c^o_{j\ell}$ are the costs to temporarily close and reopen a facility of size ℓ at location j, respectively;
- $f^c_{j\ell}$ and $f^o_{j\ell}$ are the costs to reduce and to expand the capacity of a facility at location j by ℓ capacity levels, respectively;
- $F^o_{j\ell}$ is the cost to maintain an open facility of size ℓ at location j throughout one time period.

For the sake of simplicity and without loss of generality, we assume that all these costs do not change during the planning horizon.

In the GMC, capacity level changes are represented by the $y_{j\ell_1\ell_2 t}$ variables. These transitions from one capacity level to another can be represented in a graph, where each node represents a capacity level and each arc a capacity level transition. To model the special cases, we choose a certain subset of arcs, as well as their corresponding objective function coefficients $f_{j\ell_1\ell_2 t}$. Note that, while the costs for the GMC can be based on a cost matrix, the costs for the special cases are based on a cost vector. The cost coefficients $f_{j\ell_1\ell_2 t}$ correspond to combinations of different operations, for example the cost to expand capacity plus the maintenance costs for the new capacity level.

For the problem variant involving facility closing and reopening, we create an artificial capacity level $\overline{\ell}$ for each capacity level $\ell \in L\backslash\{0\}$. Capacity level $\overline{\ell}$ represents the state in which a facility of size ℓ is temporarily closed. At each time period $t \in T$ and location $j \in J$, we may find different arc types $y_{j\ell_1\ell_2 t}$ to model capacity level changes (note that the cost for an arc is usually composed by the cost to perform the capacity transition, as well as the maintenance costs for the new capacity level):

i. Facility construction and capacity expansion. The expansion of the capacity is represented by an arc from capacity level ℓ_1 to any other capacity level $\ell_2 > \ell_1$. If the arc represents a facility construction, then ℓ_1 is 0. The capacity is thus expanded by $\ell_2 - \ell_1$ capacity levels. The cost for this arc is set to $f_{j\ell_1\ell_2 t} = f^o_{j(\ell_2-\ell_1)} + F^o_{j\ell_2}$.

ii. Capacity reduction. The reduction of the capacity is represented by an arc from capacity level ℓ_1 to any other capacity level $\ell_2 < \ell_1$. The capacity is thus reduced by $\ell_1 - \ell_2$ capacity levels. The cost for this arc is set to $f_{j\ell_1\ell_2t} = f^c_{j(\ell_1-\ell_2)} + F^o_{j\ell_2}$.

iii. Maintaining the current capacity level. A facility may neither expand nor reduce the current capacity level. The cost of this arc is thus only composed of the maintenance cost, i.e., $f_{j\ell_1\ell_1t} = F^o_{j\ell_1}$ if the capacity level represents an open facility, $f_{j\bar{\ell}_1\bar{\ell}_1t} = 0$ if the capacity level represents a temporarily closed facility and $f_{j00t} = 0$ if no facility exists.

iv. Temporary closing. An open facility of size ℓ_1 can be temporarily closed, i.e., it changes to capacity level $\bar{\ell}_1$. The total cost is $f_{j\ell_1\bar{\ell}_1t} = c^c_{j\ell_1}$.

v. Reopening a closed facility. A temporarily closed facility of size ℓ_1 can be reopened, i.e., it changes its capacity level from $\bar{\ell}_1$ to ℓ_1. The total cost for this arc is $f_{j\bar{\ell}_1\ell_1t} = c^o_{j\ell_1} + F^o_{j\ell_1}$.

The DMCFLP_CR is represented by arcs of type 1 (for construction only), 3, 4 and 5. We denote the resulting model as the *CR-GMC* formulation. The DMCFLP_ER is represented by arcs of type 1, 2 and 3. The resulting model is denoted as the *ER-GMC* formulation.

4.5 Comparisons with Specialized Formulations

We now present alternative formulations for the two special cases discussed in Section 4.4.2. These formulations are adaptations of existing models proposed in the literature. For each problem, we present formulations based on two different modeling approaches as presented in Section 4.3: location variables with one time index and location variables with two time indices.

4.5.1 Facility Closing and Reopening

We consider models for the problem with facility closing and reopening, the DMCFLP_CR.

4.5.1.1 Single Time Index Flow Formulation

This model can be seen as an extension of existing dynamic facility location problems (Shulman, 1991). Flow conservation constraints such as those used in the relocation model of Wesolowsky and Truscott (1975) are adapted to model facility closing and reopening. The model is based on the following variables. The demand allocation from facilities to customers is given by $x_{ij\ell t}$. Binary variable $s_{j\ell t}$ is 1 if a facility of size ℓ is constructed at the beginning of period t at location j, while binary flow variable $y_{j\ell t}$ indicates whether a facility of size ℓ is available at location j during time period t. Finally, binary variables $v^o_{j\ell t}$ and $v^c_{j\ell t}$ are equal to 1 if a temporarily closed facility at location j of size ℓ is reopened at the beginning of period t and if an open facility at location j of size ℓ is temporarily closed at the beginning of period t, respectively. The input data is as defined in Section 4.4.2. Note that certain equations may include terms which are not defined for a certain variable index, e.g., index $(t-1)$ is not defined for $t=1$. Undefined terms are assumed to take the value 0. The *single time index flow formulation (CR-1I)* is given by:

$$\text{(CR-1I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t}+F^o_{j\ell}y_{j\ell t}+c^o_{j\ell}v^o_{j\ell t}+c^c_{j\ell}v^c_{j\ell t}\right)+\sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}g_{ij\ell t}d_{it}x_{ij\ell t} \tag{4.10}$$

$$s.t. \sum_{j\in J}\sum_{\ell\in L}x_{ij\ell t}=1 \;\; \forall i\in I,\;\; \forall t\in T \tag{4.11}$$

$$\sum_{i\in I}d_{it}x_{ij\ell t}\le u_{j\ell}y_{j\ell t} \;\; \forall j\in J,\;\; \forall \ell\in L,\;\; \forall t\in T \tag{4.12}$$

$$y_{j\ell t}=y_{j\ell(t-1)}+s_{j\ell t}+v^o_{j\ell t}-v^c_{j\ell t} \;\; \forall j\in J,\;\; \forall \ell\in L,\;\; \forall t\in T \tag{4.13}$$

$$\sum_{t'=1}^{t}v^o_{j\ell t'}\le\sum_{t'=1}^{t}v^c_{j\ell t'} \;\; \forall j\in J,\;\; \forall \ell\in L,\;\; \forall t\in T \tag{4.14}$$

$$\sum_{\ell\in L}\sum_{t\in T}s_{j\ell t}\le 1 \;\; \forall j\in J \tag{4.15}$$

$$x_{ij\ell t}\ge 0 \;\; \forall i\in I,\;\; \forall j\in J,\;\; \forall \ell\in L,\;\; \forall t\in T \tag{4.16}$$

$$s_{j\ell t},v^o_{j\ell t},v^c_{j\ell t},y_{j\ell t}\in\{0,1\} \;\; \forall j\in J,\;\; \forall \ell\in L,\;\; \forall t\in T. \tag{4.17}$$

The objective function (4.10) minimizes the total costs composed by facility construction, maintenance of open facilities and facility reopening and closing, as well as the costs to satisfy the customer demand. Constraints (4.11) are the demand constraints. Constraints (4.12) are the capacity constraints. The flow constraints (4.13) manage the state of a facility of a certain size, either open or closed. Constraints (4.14) ensure that a facility has to be temporarily closed before it can be reopened. Finally, constraints (4.15) state that at most one facility can be constructed at each location.

The Strong Inequalities (4.8) can be adapted by replacing the right-hand side by $y_{j\ell t}$, while the Aggregated Demand Constraints (4.9) can be used by replacing the left-hand side by $\sum_{j\in J}\sum_{\ell\in L} u_{j\ell} y_{j\ell t}$.

4.5.1.2 Double Time Index Block Formulations

Dias et al. (2006) presented a linear MIP model that allows the repeated closing and reopening of facilities. The model uses binary decision variables with two time indices, one for the opening and one for the closing of a facility. We extend this model by adding the choice of different facility capacity levels (note that we remove the constraints that require a minimum availability of open facilities). We also use a different notation to be consistent with our previously introduced notations. Binary variable $s_{j\ell t_1 t_2}$ is 1 if a facility of size ℓ is constructed at location j at the beginning of time period t_1 and stays open until the end of period t_2. Binary variable $y_{j\ell t_1 t_2}$ is 1 if an existing facility of size ℓ, located at j, is reopened at the beginning of time period t_1 and stays open until the end of period t_2. We let $\hat{f}^C_{j\ell t_1 t_2}$ denote the aggregated cost to construct a facility of size ℓ at location j at time period t_1, its maintenance costs from the beginning of period t_1 to the end of period t_2, and the costs to temporarily close it at the end of period t_2. We also let $\hat{f}^R_{j\ell t_1 t_2}$ denote the same type of cost for reopening an existing facility of size ℓ instead of its construction. These constants are computed as follows:

$$\hat{f}^C_{j\ell t_1 t_2} = f^o_{j\ell} + c^c_{j\ell} + (t_2 - t_1 + 1)F^o_{j\ell} \quad \text{and} \quad \hat{f}^R_{j\ell t_1 t_2} = c^o_{j\ell} + c^c_{j\ell} + (t_2 - t_1 + 1)F^o_{j\ell}.$$

Since the binary variables with two time indices describe capacity blocks through time, we refer to this formulation as the *double time index block formulation (CR-2I)*:

$$\text{(CR-2I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t_1\in T}\sum_{t_2=t_1}^{|T|}\left(\hat{f}^C_{j\ell t_1 t_2} s_{j\ell t_1 t_2} + \hat{f}^R_{j\ell t_1 t_2} y_{j\ell t_1 t_2}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} g_{ij\ell t} d_{it} x_{ij\ell t} \tag{4.18}$$

$$\text{s.t.} \quad \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell t} = 1 \quad \forall i\in I,\ \forall t\in T \tag{4.19}$$

$$\sum_{t_2=t}^{|T|} y_{j\ell t t_2} \le \sum_{t_1=1}^{t-1}\sum_{t_2=t_1}^{t-1} s_{j\ell t_1 t_2} \quad \forall j\in J,\ \forall \ell\in L,\ \forall t\in T \tag{4.20}$$

$$\sum_{\ell\in L}\sum_{t_1\in T}\sum_{t_2=t_1}^{|T|} s_{j\ell t_1 t_2} \le 1 \quad \forall j\in J \tag{4.21}$$

$$\sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} (s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}) \le 1 \quad \forall j\in J,\ \forall t\in T \tag{4.22}$$

$$\sum_{i\in I} d_{it} x_{ij\ell t} \le \sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} u_{j\ell}(s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}) \quad \forall j\in J,\ \forall \ell\in L,\ \forall t\in T \tag{4.23}$$

$$x_{ij\ell t} \ge 0 \quad \forall i\in I,\ \forall j\in J,\ \forall \ell\in L,\ \forall t\in T \tag{4.24}$$

$$s_{j\ell t_1 t_2}, y_{j\ell t_1 t_2} \in \{0,1\} \quad \forall j\in J,\ \forall \ell\in L,\ \forall t_1\in T,\ \forall t_2\in T. \tag{4.25}$$

Constraints (4.19) are the demand constraints. Constraints (4.20) guarantee that a facility can only be reopened if it has been constructed and temporarily closed in an earlier period. Inequalities (4.21) impose that a facility can be constructed only once throughout the entire planning horizon. Constraints (4.22) guarantee that the intervals of open facilities (i.e., the capacity blocks) at the same location do not intersect. In other words, a facility can only be reopened if it is currently closed. In addition, these constraints also require that only one facility size ℓ is selected at each location. Constraints (4.23) are the facility capacity constraints.

The Strong Inequalities (4.8) can be adapted by replacing the right-hand side by $\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|}\left(s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}\right)$. The Aggregated Demand Constraints (4.9) can be used by replacing the left-hand side by $\sum_{j\in J}\sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} u_{j\ell}(s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2})$.

Strengthening the CR-2I formulation. Constraints (4.20) specify that, at each time period t, the capacity that is reopened at this period cannot be greater than the capacity that has been previously constructed. Consider the following LP relaxation solution scenario, where demands exist for three time periods t_1, t_2 and t_3. A facility construction variable is selected with solution value 0.5, opening at the beginning of t_1 and closing at the end of t_1 (i.e., $s_{j\ell t_1 t_1} = 0.5$). Facility reopening variables are then selected twice, each time with the same solution value 0.5. The first reopening spans the time interval from the beginning of t_2 until the end of t_3 (i.e., $y_{j\ell t_2 t_3} = 0.5$), whereas the second reopening spans the time interval from the beginning of t_3 until the end of t_3 (i.e., $y_{j\ell t_3 t_3} = 0.5$). Separately, each of the last two reopenings is feasible in constraints (4.20). However, in total the solution reopens more capacity than has been made available through construction. To avoid such behaviour in the LP relaxation solution, we may replace constraints (4.20) with the tighter set of constraints:

$$\sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} y_{j\ell t_1 t_2} \leq \sum_{t_1=1}^{t} \sum_{t_2=t_1}^{t} s_{j\ell t_1 t_2} \quad \forall j \in J, \ \forall \ell \in L, \ \forall t \in T. \tag{4.26}$$

We denote the formulation given by (4.18), (4.19) and (4.21) – (4.26) as the CR-2I+ formulation.

4.5.1.3 Dominance Relationships

For any integer linear programming model P, let $\overline{P}$ be the corresponding LP relaxation. For any model P, we denote by $v(P)$ its optimal value. For the three models presented for the DMCFLP_CR, the following relationships hold:

Theorem 4.5.1. $v(\overline{\textit{CR-GMC}}) = v(\overline{\textit{CR-1I}}) \geq v(\overline{\textit{CR-2I}})$.

Proof. See Appendices B.1.1.1 (Theorem B.1.1) and B.1.1.2 (Theorem B.1.2).

If constraints (4.20) in the CR-2I formulation are replaced by the strengthening constraints (4.26), all three formulations are equally strong:

Theorem 4.5.2. $v(\overline{\textit{CR-GMC}}) = v(\overline{\textit{CR-1I}}) = v(\overline{\textit{CR-2I+}})$.

Proof. See Appendix B.1.1.3 (Theorems B.1.1 and B.1.4).

4.5.2 Capacity Expansion and Reduction

We consider models for the facility location problem with capacity expansion and reduction, the DMCFLP_ER.

4.5.2.1 Single Time Index Flow Formulation

We modify the CR-1I as follows. Binary variables $s_{j\ell t}$ now represent the total capacity expansion. A variable $s_{j\ell t}$ is 1 if the capacity of the facility located at j is expanded by ℓ capacity levels at the beginning of period t. Binary variable $w_{j\ell t}$ is 1 if the capacity of a facility located at j is reduced by ℓ capacity levels at the beginning of period t. We refer to this formulation as the *single time index flow formulation (ER-1I)*:

$$\text{(ER-1I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t} + f^c_{j\ell}w_{j\ell t} + F^o_{j\ell}y_{j\ell t}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} g_{ij\ell t}d_{it}x_{ij\ell t} \quad (4.27)$$

$$s.t. \ (4.11), (4.12)$$

$$\sum_{\ell\in L}\ell y_{j\ell t} = \sum_{\ell\in L}\left(\ell y_{j\ell(t-1)} + \ell s_{j\ell t} - \ell w_{j\ell t}\right) \ \ \forall j\in J, \ \forall t\in T \quad (4.28)$$

$$\sum_{\ell\in L} y_{j\ell t} \le 1 \ \ \forall j\in J, \ \forall t\in T \quad (4.29)$$

$$\sum_{\ell\in L} s_{j\ell t} \le 1 \ \ \forall j\in J, \ \forall t\in T \quad (4.30)$$

$$\sum_{\ell\in L} w_{j\ell t} \le 1 \ \ \forall j\in J, \ \forall t\in T \quad (4.31)$$

$$x_{ij\ell t} \ge 0 \ \ \forall i\in I, \ \forall j\in J, \ \forall \ell\in L, \ \forall t\in T \quad (4.32)$$

$$s_{j\ell t}, w_{j\ell t}, y_{j\ell t} \in \{0,1\} \ \ \forall j\in J, \ \forall \ell\in L, \ \forall t\in T. \quad (4.33)$$

Now, the flow conservation constraints (4.28) manage the size of the facilities throughout the planning periods. Constraints (4.29) – (4.31), referred to as the *limiting constraints*, guarantee that the solution selects at most one capacity level for each type of variable y, s and w, respectively. If the costs for facility maintenance, capacity expansion and capacity reduction include economies of scale, these constraints are redundant, because the optimal solution will always choose a single capacity level: the one with the lowest cost in relation to its capacity.

The model may be seen as an adaptation of the relocation model of Wesolowsky and Truscott (1975), where capacity is expanded or reduced instead of relocated. It is also similar to the model presented by Jacobsen (1990) and to simplifications of the models presented by Melo et al. (2006) and Behmardi and Lee (2008).

4.5.2.2 Double Time Index Block Formulations

Dias et al. (2007) allow multiple capacity blocks of different sizes at the same location. For each block, binary variables define the exact time interval during which the block is active. This accumulation of capacity blocks allows flexible capacity expansion and reduction as previously discussed and exemplified in Figure 4.1 (b) and (c). We extend this formulation to model the DMCFLP_ER.

Binary variables $y'_{j\ell t_1 t_2}$ indicate whether a capacity block of size ℓ is available at location j from the beginning of time period t_1 until the end of time period t_2. Each capacity block may thus represent economies of scale in function of its own size. However, in contrast to the ER-1I, the total capacity available at a location can now be composed by several capacity blocks. To consider economies of scale on the entire capacity involved at each location, we introduce additional binary variables $y_{j\ell t}$, which are 1 if the total capacity summed over all capacity blocks at location j available at time period t equals ℓ. In the same manner, we introduce variables $s_{j\ell t}$ and $w_{j\ell t}$ to represent the total capacity that is added at a location (i.e., the construction of capacity blocks) or removed at a location (i.e., the closing of capacity blocks), respectively. Finally, as in the previous models, $x_{ij\ell t}$ is the fraction of customer i's demand that is served by a facility of size ℓ at location j. The *double time index block formulation (ER-2I)* is given by:

$$\text{(ER-2I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t}+f^c_{j\ell}w_{j\ell t}+F^o_{j\ell}y_{j\ell t}\right)+\sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}g_{ij\ell t}d_{it}x_{ij\ell t} \quad (4.34)$$

$$s.t.\ (4.11),(4.12),(4.29),(4.30),(4.31)$$

$$\sum_{\ell\in L}\ell s_{j\ell t}=\sum_{\ell\in L}\sum_{t_2=t}^{|T|}\ell y'_{j\ell tt_2} \quad \forall j\in J,\ \forall t\in T \quad (4.35)$$

$$\sum_{\ell\in L}\ell w_{j\ell t}=\sum_{\ell\in L}\sum_{t_1=1}^{t-1}\ell y'_{j\ell t_1(t-1)} \quad \forall j\in J,\ \forall t\in T \quad (4.36)$$

$$\sum_{\ell\in L}\ell y_{j\ell t}=\sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|}\ell y'_{j\ell t_1t_2} \quad \forall j\in J,\ \forall t\in T \quad (4.37)$$

$$x_{ij\ell t}\geq 0 \quad \forall i\in I,\ \forall j\in J,\ \forall \ell\in L,\ \forall t\in T \quad (4.38)$$

$$y'_{j\ell t_1t_2},s_{j\ell t},w_{j\ell t},y_{j\ell t}\in\{0,1\} \quad \forall j\in J,\ \forall \ell\in L,\ \forall t_1\in T,\ \forall t_2\in T. \quad (4.39)$$

We adapt the demand and capacity constraints (4.11) and (4.12), respectively, from the previous models. Constraints (4.35), (4.36) and (4.37) are the linking constraints that set the binary variables to benefit from economies of scale in function of the total capacity involved in each operation and location. As for the ER-1I formulation, we also add the limiting constraints (4.29) – (4.31) as introduced in Section 4.5.2.1. The limiting constraints are necessary to ensure that feasible solutions use only one active variable of each type y, s and w for each location and time period. These constraints have also proved to facilitate the solution process. We may also add the Strong Inequalities and the Aggregated Demand Constraints.

4.5.2.3 Dominance Relationships

For the DMCFLP_ER, the ER-GMC formulation is stronger (strictly stronger for some instances) than the other two formulations:

Theorem 4.5.3. $v(\overline{ER\text{-}GMC})\geq v(\overline{ER\text{-}1I})=v(\overline{ER\text{-}2I})$.

Proof. See Appendices B.1.2.2 (Theorem B.1.6) and B.1.2.1 (Theorem B.1.7).

4.6 Computational Experiments

In this section, computational results are reported to illustrate the strength of the different formulations and their performance when using a state-of-the-art MIP solver to find optimal integer solutions. Computational experiments were performed for the two problem variants, DMCFLP_CR and DMCFLP_ER.

A large set of instances has been generated to evaluate the performance of the proposed formulations, varying a set of key parameters that were found to affect the difficulty of the problem. Instances have been generated with the following dimensions *($|J|/|I|$)*: *(10/20)*, *(10/50)*, *(50/50)*, *(50/100)*, *(50/250)*, *(100/250)*, *(100/500)* and *(100/1000)*. The highest capacity level at any facility, denoted by q, has been selected such that $q \in \{3,5,10\}$. Three different networks have been randomly generated on squares of the following sizes: 300*km*, 380*km* and 450*km*. We consider two different demand scenarios. In both scenarios, the demand for each of the customers is randomly generated and randomly distributed over time. The two scenarios differ in their total demand summed over all customers in each time period. In the first scenario (*regular*), the total demand is similar in each time period. The second scenario (*irregular*) assumes that the total demand follows strong variations along time and therefore varies at each time period. Construction and operational costs follow concave cost functions, i.e., they involve economies of scale. All instances have also been generated with a second cost scenario in which the transportation costs are five times higher. Instances have been generated with different numbers of time periods $|T| \in \{6,8,10,12,14\}$. However, if not otherwise stated, the following computational experiments are based on instances with $|T| = 12$, which may be interpreted as a planning horizon of one year divided into 12 months. This instance set contains a total of 288 instances. Note that we assume that the problem instances do not contain initially existing facilities. We refer to Appendix B.2 for a detailed description of the parameters used to generate the instances. Furthermore, we refer to Appendix B.3 for details on the model sizes.

All mathematical models have been implemented in C/C++ using the IBM CPLEX 12.6.0 Callable Library. The code has been compiled and executed on openSUSE 11.3.

Each problem instance has been run on a single Intel Xeon X5650 processor (2.67GHz), limited to 24GB of RAM.

4.6.1 Linear Relaxation Solution and Integrality Gaps

The different formulations for the two problem variants are now compared by means of their LP relaxation bounds as well as the time necessary to solve the LP relaxations. All SIs have been added *a priori*. The Aggregated Demand Constraints have not been added to these models, since they do not have any impact on the strength of the LP relaxation. For all instances, the LP relaxation has been solved to optimality. Table 4.I shows the average times to solve the LP relaxation as well as the average integrality gaps, for each problem dimension and each number of maximum capacity levels q. The optimal integer solutions used to compute the integrality gaps have been obtained by running CPLEX for up to 24hs.

As previously shown, the CR-1I, the CR-2I+ and the CR-GMC formulations provide the same LP relaxation bound and thus the same integrality gap.

However, the CR-GMC formulation solves the relaxation in slightly shorter computing times than the CR-1I and CR-2I+ formulations.

For the DMCFLP_ER, the ER-1I and ER-2I formulations provide the same integrality gaps.

Even though the computing times for the ER-GMC formulation are higher than for the previous two formulations, the ER-GMC formulation provides a significantly smaller integrality gap.

4.6.2 CPLEX Optimization

Generic MIP solvers such as CPLEX incorporate several heuristics to find good quality solutions early in the search tree and to improve the final solution quality. However, the use of such heuristics often leads to an unforeseeable behavior and does not allow for a proper comparison of different formulations for the same problem. We therefore compare the performance of the different formulations by considering two different opti-

		DMCFLP_CR				DMCFLP_ER					
						ER-1I		ER-2I		ER-GMC	
q	Instance	Time (sec)			Integr.	Time	Integr.	Time	Integr.	Time	Integr.
	size	1I	2I+	GMC	Gap %	(sec)	Gap %	(sec)	Gap %	(sec)	Gap %
3	10/20	0.0	0.2	0.0	1.36	0.1	2.54	0.0	2.54	0.2	0.97
	10/50	0.3	0.3	0.1	0.33	0.2	0.96	0.0	0.96	0.3	0.34
	50/50	1.1	1.7	0.4	0.28	0.7	2.97	0.4	2.97	0.8	0.31
	50/100	1.7	2.6	0.8	0.01	0.7	1.34	0.7	1.34	0.6	0.03
	50/250	2.5	3.8	1.4	0.00	1.0	0.61	1.2	0.61	1.6	0.01
	100/250	8.3	10.3	4.4	0.02	3.8	0.99	4.4	0.99	5.3	0.02
	100/500	17.5	21.6	11.3	0.01	8.3	0.58	8.8	0.58	12.3	0.01
	100/1000	34.3	51.4	28.4	0.01	19.7	0.37	23.3	0.37	28.4	0.00
	Avg All	**8.2**	**11.5**	**5.9**	**0.25**	**4.3**	**1.29**	**4.8**	**1.29**	**6.2**	**0.21**
5	10/20	0.3	0.5	0.1	2.33	0.3	5.19	0.0	5.19	0.3	1.86
	10/50	0.6	1.0	0.4	0.80	0.2	2.08	0.1	2.08	0.7	0.68
	50/50	3.8	5.8	2.8	0.93	1.3	6.60	1.5	6.60	3.0	1.15
	50/100	6.5	8.4	2.8	0.18	1.9	2.79	2.4	2.79	3.5	0.19
	50/250	7.1	8.8	3.2	0.01	2.6	1.17	2.8	1.17	3.8	0.01
	100/250	26.6	24.3	10.8	0.03	8.3	1.93	9.4	1.93	15.1	0.03
	100/500	39.8	47.6	18.8	0.01	16.5	1.12	15.3	1.12	23.8	0.01
	100/1000	74.6	85.5	43.6	0.01	33.4	0.70	46.9	0.70	49.8	0.00
	Avg All	**19.9**	**22.7**	**10.3**	**0.54**	**8.1**	**2.70**	**9.8**	**2.70**	**12.5**	**0.49**
10	10/20	2.1	3.7	1.8	2.30	0.3	7.27	0.7	7.27	1.8	1.15
	10/50	3.6	7.1	4.1	1.38	0.8	4.81	1.1	4.81	3.4	0.70
	50/50	73.9	128.9	66.8	3.78	29.7	14.10	31.2	14.10	103.1	2.44
	50/100	125.3	207.7	96.2	1.25	38.9	7.15	45.0	7.15	101.1	1.07
	50/250	212.3	163.3	140.3	0.44	47.9	3.22	48.2	3.22	101.6	0.42
	100/250	1126.1	1011.4	940.0	0.53	274.2	4.73	285.7	4.73	829.2	0.47
	100/500	647.7	451.3	236.9	0.06	122.5	2.46	152.5	2.46	303.7	0.09
	100/1000	325.6	421.3	138.9	0.01	116.6	1.47	140.0	1.47	158.2	0.01
	Avg All	**190.8**	**184.0**	**120.8**	**1.10**	**50.3**	**5.85**	**54.8**	**5.85**	**125.7**	**0.78**

Table 4.I: Average LP relaxation solution time and average integrality gaps for all formulations.

mization environments. The first one is a traditional branch-and-cut environment, which aims at testing the formulations' ability to prove optimality. We used the MIP branch-and-cut algorithm of CPLEX 12.6.0 and turned off all heuristics (i.e., MIP heuristics, Feasibility Pump, Local Branching and RINS). Instead, we used the solution value of the optimal integer solution as an artificial upper bound. This value is passed as a cut-off value in the branch-and-cut tree. In the second optimization scenario, we used CPLEX default settings, which reflects a typical use in practice.

For all experiments, computation times have been limited to six hours. Furthermore, all Strong Inequalities have been added *a priori* to the models. Even though the number of SIs may increase significantly, adding them *a priori* (instead of as *CPLEX user cuts* or even not at all) significantly facilitates the solution of the problems. Experiments showed that, for most of the problem instances, a large number of SIs are violated. CPLEX thus spends much time identifying and adding violated SIs when treated as *CPLEX user cuts*. Although redundant to the LP relaxation of the presented formulations, the Aggregated Demand Constraints tend to slightly facilitate the solution of the problems. Therefore, they also have been added to the formulations. For some models, the limiting constraints as shown in Section 4.5.2 may not change the set of feasible integer solutions, but still facilitate the solution of the problem. For example, for the ER-1I formulation, the average solution time for our test instances decreased by around 35%. The constraints are thus added to the models even if they are redundant.

4.6.2.1 Optimization in Branch-and-Cut Environment

We now present computational results for the branch-and-cut environment. CPLEX offers three different search strategies (parameter `MIPsearch`): traditional branch-and-cut, dynamic search and an automatic choice based on internal rules. Our experiments showed that the traditional branch-and-cut performed slightly better than the other two options. All of the following results are therefore based on the traditional branch-and-cut scheme. Furthermore, all heuristics are turned off and the optimal integer solution value is passed to the solver as an upper bound cut-off value.

For each problem, the results have been separated into two groups: instances that

q	Instance size	DMCFLP_CR # Inst	CR-1I	CR-2I+	CR-GMC	DMCFLP_ER # Inst	ER-1I	ER-2I	ER-GMC
3	10/20	12	0.9	11.3	1.2	12	0.3	1.4	0.3
	10/50	12	0.5	5.1	0.4	12	0.5	1.7	0.5
	50/50	11	6.3	7.8	2.3	12	421.8	1,015.9	161.4
	50/100	12	3.6	9.9	2.3	12	3.1	5.9	2.5
	50/250	12	3.4	23.2	3.8	12	6.0	9.6	5.2
	100/250	12	14.8	56.7	14.6	12	17.4	29.3	14.3
	100/500	12	28.2	127.0	31.3	12	34.7	65.7	29.4
	100/1000	12	66.9	370.3	75.8	12	82.2	179.1	54.8
	All	**95**	**15.7**	**77.1**	**16.6**	**96**	**70.7**	**163.6**	**33.6**
5	10/20	12	13.7	470.4	11.8	12	7.8	199.8	4.3
	10/50	12	12.3	1,141.8	6.8	12	3.4	16.7	2.6
	50/50	9	13.9	18.9	4.3	8	24.4	139.9	7.8
	50/100	11	72.6	824.5	17.3	12	32.3	142.8	23.6
	50/250	12	8.8	46.8	9.8	12	11.4	21.0	10.1
	100/250	12	30.2	107.7	31.3	12	50.8	71.7	32.8
	100/500	12	45.9	230.2	47.8	12	83.6	131.4	56.4
	100/1000	12	109.3	652.8	115.2	12	198.5	301.4	97.6
	All	**92**	**38.8**	**446.0**	**31.5**	**92**	**52.7**	**127.6**	**30.3**
10	10/20	8	529.1	899.4	159.5	3	26.0	8,762.3	5.3
	10/50	7	115.1	3,584.3	81.7	2	3.0	1,104.0	3.5
	50/50	4	85.5	37.3	16.0	1	53.0	2,387.0	19.0
	50/100	6	102.7	2,168.3	18.8	6	133.5	4,194.0	23.3
	50/250	8	243.9	1,830.3	101.0	6	47.8	203.0	36.0
	100/250	7	112.0	306.6	88.3	7	165.9	659.6	111.4
	100/500	11	198.8	931.0	165.5	11	531.6	2,131.2	284.7
	100/1000	5	136.6	1,226.6	155.4	12	887.2	2,291.4	313.0
	All	**56**	**207.2**	**1,403.7**	**108.0**	**48**	**393.3**	**2,350.4**	**168.0**

Table 4.II: CPLEX branch-and-cut computation times (in seconds) for instances solved to optimality by all formulations for each problem.

			CR-1I			CR-2I+			CR-GMC		
q	Instance	#	Gap %		#	Gap %		#	Gap %		#
	size	Inst	Avg	Max	ns	Avg	Max	ns	Avg	Max	ns
3	50/50	1	0.01	0.01	0	0.06	0.06	0	0.01	0.01	0
5	50/50	3	0.12	0.12	2	-	-	3	0.01	0.01	2
	50/100	1	0.01	0.01	0	0.02	0.02	0	0.01	0.01	0
10	10/20	4	0.05	0.13	1	0.63	0.63	3	0.12	0.31	0
	10/50	5	0.01	0.01	2	0.45	0.45	4	0.01	0.01	2
	50/50	8	0.10	0.10	7	0.01	0.01	7	0.01	0.01	6
	50/100	6	0.01	0.01	5	-	-	6	0.01	0.01	5
	50/250	4	0.01	0.01	3	-	-	4	0.00	0.01	2
	100/250	5	0.04	0.04	4	-	-	5	0.01	0.01	3
	100/500	1	0.01	0.01	0	-	-	1	0.01	0.01	0
	100/1000	7	0.00	0.00	0	-	-	7	0.00	0.01	0
	All	**40**	**0.02**	**0.13**	**22**	**0.37**	**0.63**	**37**	**0.03**	**0.31**	**18**

Table 4.III: CPLEX branch-and-cut optimality gaps for instances of the DMCFLP_CR not solved within 6hs.

			ER-1I			ER-2I			ER-GMC		
q	Instance	#	Gap %		#	Gap %		#	Gap %		#
	size	Inst	Avg	Max	ns	Avg	Max	ns	Avg	Max	ns
5	50/50	4	0.01	0.01	3	-	-	4	0.05	0.12	1
10	10/20	9	0.01	0.01	0	0.37	0.37	8	0.01	0.01	0
	10/50	10	0.01	0.01	1	-	-	10	0.01	0.01	0
	50/50	11	0.01	0.01	8	-	-	11	0.00	0.01	7
	50/100	6	0.01	0.01	5	-	-	6	0.00	0.01	4
	50/250	6	0.01	0.01	2	-	-	6	0.03	0.12	1
	100/250	5	0.01	0.01	4	-	-	5	0.01	0.01	3
	100/500	1	-	-	1	-	-	1	0.01	0.01	0
	All	**48**	**0.01**	**0.01**	**21**	**0.37**	**0.37**	**47**	**0.01**	**0.12**	**15**

Table 4.IV: CPLEX branch-and-cut optimality gaps for instances of the DMCFLP_ER not solved within 6hs.

have been solved to optimality by all formulations and instances where at least one formulation could not prove optimality within the given time limit. Table 4.II summarizes the results for the instances that have been solved by all formulations for each problem. The table reports the number of instances that have been solved to optimality, as well as the average computation times to solve the instances for each of the formulations. For both problem variants, we observe that the 2I formulation performs worst. Among the 1I and the GMC based formulations, the GMC based models provide substantially better results.

Tables 4.III and 4.IV summarize the results for instances where at least one of the formulations did not solve the instances in the given time limit. The tables show average and maximum optimality gaps as reported by CPLEX, as well as the number of instances where the optimal solution has not been found within the given time limit (*#ns*). Note that a positive optimality gap indicates that an optimal solution (i.e., the one with the cut-off value) has been found, but optimality has not been proven. For $q = 3$ and $q = 5$, a few instances with 50 facility locations have been found to be difficult to solve. All other instances are for $q = 10$. Again, the 2I formulations perform worst, having the highest number of instances where the optimal solution has not been found. For both problem variants, the GMC finds more solutions than the 1I and 2I formulations. If the optimal solutions are found, the optimality gaps are low for all three formulations.

4.6.2.2 Optimization with CPLEX Default Settings

As shown in the previous section, the GMC based formulation outperforms the 1I and 2I formulations for both problem variants in a traditional branch-and-cut environment, allowing for a clear comparison of the formulations without the interference of heuristics. In practice, however, the objective is most often to find high quality solutions in short computing times. Generic MIP solvers such as CPLEX incorporate several heuristics to find good quality solutions early in the search tree and to improve the final solution quality. We now compare the different formulations using CPLEX with default settings, making full use of the heuristic capabilities of the MIP solver.

Computational experiments on the same set of test instances indicate trends similar

q	Instance size	DMCFLP_CR # Inst	CR-1I	CR-2I+	CR-GMC	DMCFLP_ER # Inst	ER-1I	ER-2I	ER-GMC
3	10/20	12	1.1	5.7	1.5	12	0.3	1.4	0.3
	10/50	12	0.8	3.8	1.2	12	0.5	1.6	1.1
	50/50	12	121.8	158.4	18.3	12	302.4	1,402.6	116.2
	50/100	12	4.2	13.4	3.3	12	4.8	7.3	3.7
	50/250	12	4.3	25.3	5.6	12	7.5	12.3	6.8
	100/250	12	13.9	70.0	20.4	12	22.7	36.6	19.1
	100/500	12	36.5	155.0	36.3	12	45.9	75.5	36.8
	100/1000	12	76.3	440.4	89.3	12	92.7	156.0	64.4
	All	**96**	**32.4**	**109.0**	**22.0**	**96**	**59.6**	**211.7**	**31.0**
5	10/20	12	10.2	43.0	10.4	12	7.3	42.3	5.8
	10/50	12	10.8	121.2	12.9	12	5.0	25.1	5.0
	50/50	10	194.6	176.1	62.0	9	663.0	2,126.3	84.2
	50/100	12	447.9	518.8	143.3	12	84.6	161.8	35.3
	50/250	12	10.2	51.8	11.7	12	14.8	29.2	13.8
	100/250	12	40.3	136.5	41.1	12	61.1	104.4	46.0
	100/500	12	65.3	270.9	56.1	12	119.5	160.3	69.5
	100/1000	12	128.1	741.3	143.4	12	192.8	331.8	126.8
	All	**94**	**111.7**	**259.2**	**60.1**	**93**	**126.8**	**316.1**	**47.1**
10	10/20	8	59.8	903.8	52.9	8	55.0	2,808.0	10.9
	10/50	7	119.3	1,033.6	108.3	8	180.9	4,310.8	28.6
	50/50	5	184.4	61.6	44.4	5	392.0	2,946.4	67.0
	50/100	7	744.1	1,595.0	97.4	7	577.0	3,186.6	162.1
	50/250	10	1,824.1	2,018.3	289.4	9	1,747.9	4,865.4	257.2
	100/250	8	2,009.1	1,049.3	503.8	7	258.3	963.0	125.6
	100/500	11	208.0	701.5	215.3	11	806.2	3,565.6	416.9
	100/1000	8	420.3	1,760.5	355.8	12	957.1	2,809.6	389.8
	All	**64**	**740.8**	**1,192.4**	**222.2**	**67**	**683.3**	**3,245.6**	**212.6**

Table 4.V: Computation times (in seconds) using CPLEX with default settings for instances solved to optimality by all formulations for each problem.

			CR-1I			CR-2I+			CR-GMC		
q	Instance	#	Gap %		#	Gap %		#	Gap %		#
	size	Inst	Avg	Max	ns	Avg	Max	ns	Avg	Max	ns
5	50/50	2	0.99	1.18	0	1.17	1.32	0	0.18	0.35	0
10	10/20	4	0.01	0.01	0	0.72	0.96	0	0.01	0.01	0
	10/50	5	0.12	0.56	0	0.56	1.36	0	0.26	0.87	0
	50/50	7	1.85	3.73	0	1.46	4.21	0	1.36	3.42	0
	50/100	5	1.14	2.54	0	0.87	1.84	0	0.58	1.43	0
	50/250	2	0.59	0.85	0	0.59	0.89	0	0.42	0.75	0
	100/250	4	1.10	2.76	0	0.67	1.61	0	0.69	1.69	0
	100/500	1	0.01	0.01	0	0.04	0.04	0	0.01	0.01	0
	100/1000	4	0.00	0.00	0	-	-	4	0.00	0.01	0
	All	**32**	**0.78**	**3.73**	**0**	**0.86**	**4.21**	**4**	**0.54**	**3.42**	**0**

Table 4.VI: Optimality gaps using CPLEX with default settings for instances of the DM-CFLP_CR not solved within 6hs.

			ER-1I			ER-2I			ER-GMC		
q	Instance	#	Gap %		#	Gap %		#	Gap %		#
	size	Inst	Avg	Max	ns	Avg	Max	ns	Avg	Max	ns
5	50/50	3	0.50	1.00	0	1.01	1.33	0	0.00	0.00	0
10	10/20	4	0.01	0.01	0	1.60	2.76	0	0.01	0.01	0
	10/50	4	0.01	0.01	0	1.22	1.65	0	0.01	0.01	0
	50/50	7	1.43	3.23	0	3.12	5.08	1	0.55	1.30	0
	50/100	5	0.83	1.47	0	1.47	2.45	0	0.45	1.09	0
	50/250	3	0.32	0.74	0	0.60	1.06	0	0.12	0.35	0
	100/250	5	0.52	1.22	0	2.05	6.85	0	0.34	1.11	0
	100/500	1	0.12	0.12	0	0.55	0.55	0	0.01	0.01	0
	All	**29**	**0.62**	**3.23**	**0**	**1.78**	**6.85**	**1**	**0.29**	**1.30**	**0**

Table 4.VII: Optimality gaps using CPLEX with default settings for instances of the DMCFLP_ER not solved within 6hs.

to those observed in the experiments of Section 4.6.2.1. The results for the instances that have been solved by all formulations for each problem are summarized in Table 4.V. The table reports the number of instances that have been solved to optimality, as well as the average computing times to solve the instances for each of the formulations. As in the previous experiments, the 2I formulation performs worst. Among the 1I and the GMC based formulations, the GMC based models are solved in substantially shorter computing times.

Tables 4.VI and 4.VII summarize the results for instances where at least one of the formulations did not solve the instances in the given time limit. The tables report average and maximum optimality gaps as reported by CPLEX, as well as the number of instances where no feasible solution has been found (*#ns*). For $q = 5$, the few instances that have been found to be difficult to solve are those with 50 facility locations. All other instances are for $q = 10$. Again, the 2I formulations perform worst. For some of the instances, the formulation did not find any feasible solution. The GMC formulation performs similar to the 1I formulation for the DMCFLP_CR and presents slightly better results than the 1I formulation for the DMCFLP_ER.

4.6.3 Closing and Reopening with Capacity Expansion and Reduction.

The two problem variants treated above consider either facility closing/reopening or capacity expansion/reduction. Experiments have also been performed for a third problem variant combining both features, referred to as the DMCFLP_CRER. The problem is modeled by the use of the DFLPG by using the transition arcs for both problems as shown in Section 4.4.2. Additionally, arcs are added representing combined decisions such as facility reopening with subsequent capacity expansion (in the same time period), as well as capacity reduction with subsequent facility closing. Alternatively, a specialized flow formulation can be used with two types of flow constraints: one to manage the capacity of open facilities and one to manage the capacity of closed facilities. The observations made above regarding the CPLEX root node solution were also confirmed for this more complex problem variant. In addition, the advantage of the GMC model for this variant is even more obvious than what was observed for the DMCFLP_ER. We proved that the GMC based model provides a stronger LP relaxation than the specialized flow formulation. Computationally, the average integrality gap (for all instances with $q = 10$) improved from 6.00% to 1.06% when using the GMC based model instead of the specialized formulation. In the traditional branch-and-cut environment, using CPLEX without heuristics and providing it with the optimal integer solution value as cut-off, the flow formulation takes on average 1,820 seconds to solve the instances of size $q = 10$, while the GMC based formulation solves the same instances in an average time of only 206 seconds, about nine times faster. Using CPLEX default settings, the dominance of the GMC based formulation is mainly preserved. The average computation time improves from 1,924 to 313 seconds.

4.6.4 Solution Structure and Instance Properties

We now analyze the structure of the optimal or near-optimal solutions. Figure 4.2 illustrates for each problem variant and problem size (10, 50 and 100 candidate facility locations) the minimum (*Min*), maximum (*Max*) and average number (*Avg*) of selected facility locations. Since a facility may not be available in all of the subsequent time periods after its construction, a second average value (*Avg open*) indicates the average number of facilities that are available (i.e., having $\ell \geq 1$) at each time period. The results are surprisingly similar for the three problem variants CR, ER and CRER. On average, about half of the candidate locations have been selected. These facilities are active only in about two thirds of the planning horizon. For the CR, this is done by closing a facility. For the ER, the capacity is reduced to level 0.

Figure 4.3 shows different indicators of the solutions structure: the average number of facility closings and reopenings, as well as the average number of capacity expansions and reductions. It can be observed that the average values for certain indicators such as capacity expansion and reduction are similar for the three problem variants. Based on these results, one may conclude that the main driver to adjust capacities are high maintenance costs and therefore high quality solutions tend to provide a total capacity that only slightly exceeds the total demand. However, an analysis of the solutions for smaller instances reveals that the selected opening schedules are very different for the three

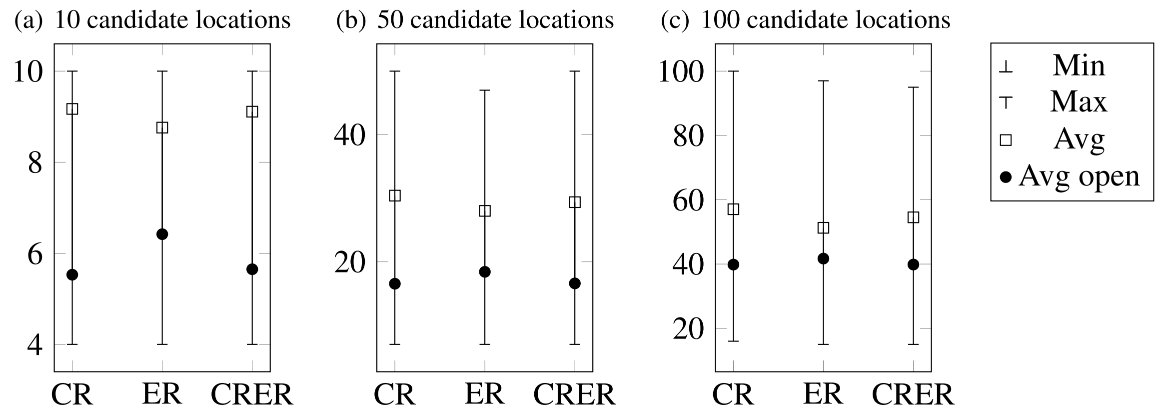

Figure 4.2: Structure of optimal solutions: minimum, average and maximum number of selected facility locations, as well as the average number of open facilities per time period throughout the entire planning horizon.

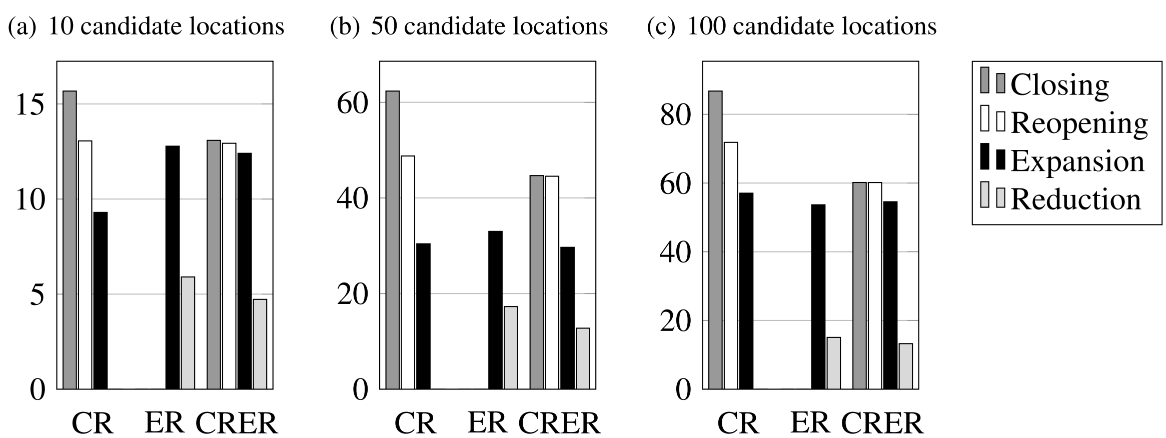

Figure 4.3: Structure of optimal solutions: average number of facility closings and re-openings, as well as capacity reductions and expansions.

problem variants when the original transportation costs are used. In contrast, the opening schedules are very similar when the transportation costs are set five times higher. Table 4.VIII presents the impact of these instance properties on the solution structure. The table shows, for each of the indicators, the average number of occurrences in instances with the original transportation costs and in instances where the transportation costs are set five times higher. In the same way, it indicates the number of occurrences in instances with regular demand distribution and with irregular demand distribution. The impact of these instance properties has been found to be very similar for all three problem variants and is here exemplified for the DMCFLP_CRER, showing the average values over all instances. We can identify a clear trend. Solutions for instances with original transportation costs involve only a few operations that adjust the capacities throughout the planning horizon and therefore tend to serve the demand from a similar set of facility locations. Solutions for instances with high transportation costs provide capacities that tend to geographically follow the demand along time, constructing on average more than twice the number of facilities and performing two to three times the operations that adjust capacities along time. As in both cases the maintenance cost are the same, the motivating factor to geographically shift capacity is rather given by high transportation costs and the effort to bring capacities closer to the demand. Regarding the demand distribution, an irregular demand distribution results in only slightly more capacity adjustments than a regular demand distribution.

	Transportation costs		Demand distribution	
#	original	5× higher	regular	irregular
Constructions	21.7	44.9	33.3	33.3
Closings	21.0	63.3	40.3	43.9
Reopenings	21.0	63.2	40.3	43.8
Capacity expansions	22.3	46.3	33.8	34.7
Capacity reductions	5.3	16.4	10.6	11.1
Avg. open facilities	16.2	28.4	23.2	21.3

Table 4.VIII: Impact of instance characteristics (transportation costs and demand distribution) on the solution structure for the DMCFLP_CRER.

Impact on problem difficulty. The instance characteristics not only impact the solution structure, but also the difficulty of solving the problem. The computing time for instances with irregular total customer demand is, on average, 30% lower than for instances where the total customer demand is regular at each time period. In contrast, the ratio between transportation and facility construction costs has a much larger impact. Instances where the transportation costs are five times higher than the original transportation costs are, on average, solved around 60 times faster.

	$\|T\| = 6$		$\|T\| = 8$		$\|T\| = 10$		$\|T\| = 12$		$\|T\| = 14$	
	Gap %	Time (sec)	Gap %	Time (sec)	Gap %	Time (sec)	Gap %	Time (sec)	Gap %	Time (sec)
10/20	0.00	112.1	0.01	180.8	0.01	140.0	0.01	940.7	0.01	2,569.4
10/50	0.00	73.0	0.01	129.1	0.01	302.4	0.01	1,822.0	0.06	6,203.6
50/50	0.17	7,842.3	0.25	8,818.4	0.45	10,820.0	1.23	10,913.3	0.96	12,800.9
50/100	0.02	2,126.8	0.11	4,107.6	0.17	4,201.6	0.56	7,582.8	0.40	9,225.2
50/250	0.01	446.0	0.02	2,002.3	0.01	1,945.0	0.14	7,304.1	0.12	5,655.8
100/250	0.05	2,883.7	0.08	5,516.9	0.14	6,940.4	0.57	8,899.3	0.31	9,481.0
100/500	0.00	339.3	0.00	988.9	0.00	940.8	0.01	2,687.5	0.01	2,571.3
100/1000	0.00	414.5	0.00	463.9	0.00	538.4	0.00	690.2	0.00	549.1
All	**0.03**	**1,779.7**	**0.06**	**2,776.0**	**0.10**	**3,228.6**	**0.19**	**4,201.8**	**0.23**	**6,132.0**

Table 4.IX: Impact of number of time periods in problem instances ($q = 10$) for the CRER-GMC formulation when using CPLEX with default settings.

Finally, we also analyzed the impact of the length of the planning horizon in the problem instances, using CPLEX with its default settings. Table 4.IX summarizes the average computation times and average optimality gaps for the CRER-GMC formulation. The computational results are presented for five different numbers of time periods $|T|$: 6, 8, 10, 12 and 14. The results are very consistent, showing that the difficulty of the problems increases proportionally to the number of time periods. For the CRER-1I formulation, a similar trend was observed. However, the CRER-1I was clearly outperformed by the CRER-GMC for all tested lengths of the planning horizon.

4.7 Conclusions and Future Research

We have introduced a new general facility location problem that unifies several existing multi-period facility location problems. We showed the flexibility of this generalization by focusing on two problem variants: facility closing and reopening and capacity expansion and reduction. In addition, we also reported results on a variant that combines both of these features. For the two first cases, we derived specialized models based on two well-known formulation approaches. We formally proved that, even though our model is more general, it provides LP relaxation bounds as strong as the other formulations for the case of facility closing/reopening and stronger LP relaxation bounds than the formulations for the other two cases. Computational experiments showed that, for the two variants involving capacity expansion and reduction, the integrality gap of our model is up to seven times smaller than the integrality gaps of the specialized formulations. When assessing the performance of the models in a traditional branch-and-cut environment, the GMC based models solved the instances, on average, up to nine times faster than the specialized formulations. Using CPLEX default settings to solve the problem, the GMC based models are, on average, up to six times faster.

The general model may also be used to model other problem variants not addressed in this work, e.g., the closing and reopening model of Chardaire et al. (1996) or the dynamic location problem of Sridharan (1995). In addition, problem variants that involve capacity changes may benefit from the proposed modeling technique to strengthen the existing models. Problems such as those presented by Shulman (1991) and Correia and Captivo (2003) can be modeled by the DFLPG when adding individual constraints such as minimum production bounds for the facilities. Finally, as the general model is already very strong, it may also be an ideal candidate for decomposition techniques such as Lagrangian relaxation to find good quality solutions in short computation times.

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CHAPTER 5

LAGRANGIAN RELAXATION FOR DYNAMIC FACILITY LOCATION

5.1 Chapter Preface

The DFLPG, presented in Chapter 4, has shown to be a fairly general facility location problem that provides strong LP relaxation bounds, in particular when the problem involves different capacity levels. When applied to other facility location problems, it has been shown that generic MIP solvers find optimal solutions in significantly shorter computing times when using the proposed GMC formulation instead of existing specialized formulations. However, no matter which formulation is used, models grow quickly when considering large problem instances. Generic solvers may therefore not solve the problem in reasonable computing times or, even worse, not find feasible solutions at all due to the complexity of the model and the given time and memory restrictions.

In this chapter, we explore solution methods that are capable to solve large-scale instances for the DFLPG. The widely recognized success of Lagrangian relaxation based methods for facility location problems suggests applying this technique to the GMC formulation. The Lagrangian heuristic proposed in this chapter relaxes the demand constraints of the problem. If, instead, the capacity constraints are relaxed, the difficulty of the resulting problem strongly depends on whether the Strong Inequalities are used or not. Without the SIs, the Lagrangian subproblem decomposes into two independent problems, which can be solved efficiently: a fractional knapsack to fill up the demand and a shortest path problem with negative costs. However, as pointed out in Section 2.2.2.2, relaxing the capacity constraints without the presence of the SIs decouples the demand allocation variables from the facility opening decisions. The resulting bound is therefore expected to be rather weak. For many facility location problems, using the SIs when relaxing the capacity constraints results in a bound which is stronger than the one obtained by relaxing the demand constraints. In the case of the GMC model, the resulting Lagrangian subproblem involves solving the Dynamic Uncapacitated Facility Location

Problem. This relaxation has two major drawbacks. First, the subproblem is known to be NP-hard and it may be difficult to find an efficient algorithm to solve it. Second, the strength and the essence of the DFLPG lay in its ability to represent capacity changes on a detailed level. Relaxing the capacity constraints is therefore not promising. The same reason argues against the relaxation of the flow conservation, since the strength of the GMC's LP relaxation stems from these constraints.

As will be shown in this chapter, a Lagrangian heuristic based on the relaxation of the demand constraints may solve a large part of the instances. However, for more difficult instances, the feasible solutions generated through the solution of the Lagrangian dual have been found to have a large optimality gap. Several local search approaches have therefore been tested to improve the feasible solutions in a second optimization phase. Satisfactory results have been obtained with a local search including relatively simple neighborhoods. However, given the generality of the DFLPG, it has been found beneficial to use a more generic approach. We therefore use a restricted MIP model, limited to decisions which have been judged important in the Lagrangian solution throughout the solution process. This approach solves fairly well even large-scale instances for all three special cases tested in this work, as well as the DFLPG itself. This type of matheuristic may also be applied to problems other than facility location problems. Using information from the Lagrangian solutions to construct such a restricted MIP model may be a promising avenue for a general heuristic framework. The results of this chapter confirm the potential of Lagrangian relaxation techniques to solve complex and large facility location problems and are even encouraging to approach the CSLP with similar techniques.

Notes about the chapter

The contents of this chapter correspond to those of the article entitled *Lagrangian Heuristics for Large-Scale Dynamic Facility Location with Generalized Modular Capacities*, co-authored with Professors Jean-François Cordeau and Bernard Gendron, which has been submitted for publication to the INFORMS Journal on Computing (ISSN: 1091-9856), in April 2014.

Lagrangian Heuristics for Large-Scale Dynamic Facility Location with Generalized Modular Capacities

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We consider the Dynamic Facility Location Problem with Generalized Modular Capacities, a multi-period facility location problem in which the costs for capacity changes are defined for all pairs of capacity levels. The problem embeds a complex cost structure and generalizes several existing facility location problems, such as those that allow temporary facility closing or capacity expansion and reduction. As the model may grow very large, general-purpose mixed-integer programming solvers are limited to solving instances of small to medium size. In this paper, we extend the generalized model to the case of multiple commodities. We propose Lagrangian heuristics, based on subgradient and bundle methods, to find good quality solutions for large-scale instances with up to 250 facility locations and 1000 customers. To improve the final solution quality, a restricted model is solved based on the information collected through the solution of the Lagrangian dual. Computational results show that the Lagrangian based heuristics provide highly reliable results for all problem variants considered. They produce good quality solutions in short computing times even for instances where state-of-the-art mixed-integer programming solvers do not find feasible solutions. The strength of the formulation also allows the method to provide tight bounds on the solution quality.

Key words: Dynamic Facility Location, Modular Capacities, Lagrangian Relaxation.

5.2 Introduction

Dynamic facility location problems aim at providing capacity planning over a multiple-period planning horizon. Given that customer demands may vary significantly over time, facilities often adjust their capacities. These problems find applications in both the public and private sectors, for the location of production facilities (Fleischmann et al., 2006), schools (Antunes and Peeters, 2001), health care facilities (Correia and Captivo, 2003; Kim and Kim, 2013) and many more, as documented in several recent literature surveys (Thomas and Griffin, 1996; Brotcorne et al., 2003; Revelle et al., 2008; Melo et al., 2009a; Smith et al., 2009). To represent the adjustment of capacities in such problems, common actions include capacity expansion and reduction (Luss, 1982; Jacobsen, 1990; Antunes and Peeters, 2001; Troncoso and Garrido, 2005; Dias et al., 2007), temporary closing of facilities (Chardaire et al., 1996; Canel et al., 2001; Dias et al., 2006) and the relocation of facilities (Melo et al., 2006). Although mathematical models often take into account complex environments such as complete supply chains, the cost structure to adjust capacities along time is commonly modeled in less detail. Economies of scale are often represented on the level of the total capacity involved in each operation, but do not take into consideration the capacity level before the change. A more detailed representation of the cost structure is necessary in a number of applications, especially in the domains of transportation, logistics and telecommunications, where additional capacity gets cheaper (or more expensive) when approaching the maximum capacity limit. For instance, in the problem introduced by Jena et al. (2012), logging camps are located to host workers in the forest industry. In this problem, the total capacity of a camp is represented by its number of different hosting units, while additional units provide supporting infrastructure. As the relation between the number of different units cannot be captured by a simple function, the costs for capacity changes need to be described in a transition matrix.

Jena et al. (2013) recently introduced the Dynamic Facility Location Problem with Generalized Modular Capacities (DFLPG), in which the costs for capacity changes are based on a cost matrix. The mixed-integer programming (MIP) model presented by the authors therefore allows taking into account not only the total capacity involved in the capacity change, but also the current capacity level. This model generalizes several multi-period facility location problems: the problem with facility closing and reopening, the problem with capacity expansion and reduction, and the combination of the two. In addition, the DLFPG provides a strong linear programming (LP) relaxation bound. Compared to alternative MIP formulations, the DFLPG based models can often be solved twice as fast using a general-purpose MIP solver. Although it is possible to solve the models for small and medium size instances, they usually grow too large when considering more complex problem variants or larger instances. In this case, heuristics are an interesting alternative. They also provide an advantage when performing "what-if" analysis, which requires repeatedly solving the problem with different scenarios. Heuristics are usually capable of using solutions for a certain scenario to quickly find solutions for a different one.

Metaheuristics such as tabu search, simulated annealing and genetic algorithms have been frequently applied to several families of location problems, from classical facility location problems (Arostegui Jr. et al., 2006) to logistics network design that model entire supply chains (Lee and Dong, 2008; Melo et al., 2011a). Lagrangian relaxation based heuristics have been developed for several variants of single-period facility location problems (Barcelo et al., 1990; Sridharan, 1991; Beasley, 1993; Sridharan, 1995; Holmberg and Ling, 1997; Agar and Salhi, 1998; Holmberg and Yuan, 2000; Correia and Captivo, 2003; Wu et al., 2006), some of which combined Langrangian relaxation and local search (Correia and Captivo, 2006; Li et al., 2009). Lagrangian bounds have also been used within exact methods (Görtz and Klose, 2012). For multi-period facility location, approaches based on Lagrangian relaxation have been proposed for problems without capacities (Chardaire et al., 1996), with fixed capacities (Shulman, 1991), and for multi-echelon problems in the context of supply chain design (Diabat et al., 2013). Furthermore, Lagrangian based methods have been successfully applied to other location problems such as dynamic hub location (Elhedhli and Wu, 2010; Contreras et al., 2011b).

In this paper, we present an extension of the DFLPG in which customers have demands for different commodities. We propose Lagrangian based heuristics that find good quality solutions in reasonable computing times. Two methods are introduced to solve the Lagrangian dual: a subgradient method and a bundle method. After this process, a second optimization step is used to improve the solution quality. This step consists of solving a restricted MIP model, taking into consideration only decisions that have been part of a significant number of the previous Lagrangian solutions. To the best of our knowledge, this work is the first to present a Lagrangian relaxation approach to solve large-scale instances of a multi-period facility location problem of this nature, i.e., including modular capacity levels and multiple commodities. Given the strength of the formulation used to model the DFLPG, the Lagrangian heuristics are capable of providing relatively tight bounds on the optimal value. The results are stable even for large instances, where general-purpose MIP solvers either consume too much memory or do not solve the problem in reasonable time. Given the generality of the DFLPG, the proposed heuristic can handle an entire class of problems, consisting of all those that can be modeled by the DFLPG.

The remainder of the paper is organized as follows. Section 5.3 reviews and extends the MIP formulation for the DFLPG and shows how it can be used to model three different special cases. Section 5.4 explains how the problem is decomposed via Lagrangian relaxation and outlines the resulting heuristics. Section 5.5 then discusses how the final solution quality can be improved in a second optimization phase, using information from the solution of the Lagrangian dual to generate a restricted MIP model. The Lagrangian heuristics are then compared by means of computational experiments in Section 5.6. First, general results are presented for each of the different problem variants. Then, the advantages of the Lagrangian heuristics are illustrated with more detailed results comparing their performance to a state-of-the-art MIP solver. Finally, conclusions are drawn and future research directions are discussed in Section 5.7.

5.3 Mixed Integer Programming Formulation

This section first introduces a general formulation for the DFLPG and then explains how it can be used to model three special cases.

5.3.1 General Model

We consider the mixed-integer programming formulation introduced by Jena et al. (2013) and extend it to include multiple commodities. We denote by J the set of candidate facility locations and by $L = \{0,1,2,\ldots,q\}$ the set of possible capacity levels for each facility. We also denote by I the set of customer demand points and by $T = \{1,2,\ldots,|T|\}$ the set of time periods in the planning horizon. We assume throughout that the beginning of period $t+1$ corresponds to the end of period t. Additionally, we denote by $P = \{1,2,\ldots,|P|\}$ the set of different commodities. The demand of customer i for commodity p in period t is denoted by d_{ipt}, while the cost to serve one unit of commodity p from facility j operating at capacity level ℓ to customer i during period t is denoted by $g_{ij\ell pt}$. The capacity of a facility of size ℓ at location j is given by $u_{j\ell}$ (with $u_{j0} = 0$). For each j and t, the cost matrix $f_{j\ell_1\ell_2t}$ describes the combined cost to change the capacity level of a facility at location j from ℓ_1 to ℓ_2 at the beginning of period t and to operate the facility at capacity level ℓ_2 throughout time period t. Furthermore, we let ℓ^j be the initial capacity level of an existing facility at location j.

To formulate the problem, we use binary variables $y_{j\ell_1\ell_2t}$ equal to 1 if and only if the facility at location j changes its capacity level from ℓ_1 to ℓ_2 at the beginning of period t. The allocation variables $x_{ij\ell pt}$ denote the fraction of the demand of customer i for commodity p in period t that is served from a facility of size ℓ located at j. Using this notation, we define the following MIP model, referred to as the *Generalized Modular*

Capacities (GMC) formulation:

$$\text{(GMC)} \quad \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2t}y_{j\ell_1\ell_2t} + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T} g_{ij\ell pt}d_{ipt}x_{ij\ell pt} \tag{5.1}$$

$$s.t. \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell pt} = 1 \;\; \forall i\in I,\; \forall p\in P,\; \forall t\in T \tag{5.2}$$

$$\sum_{i\in I}\sum_{p\in P} d_{ipt}x_{ij\ell pt} \le \sum_{\ell_1\in L} u_{j\ell}y_{j\ell_1\ell t} \;\; \forall j\in J,\; \forall \ell\in L,\; \forall t\in T \tag{5.3}$$

$$\sum_{\ell_1\in L} y_{j\ell_1\ell(t-1)} = \sum_{\ell_2\in L} y_{j\ell\ell_2t} \;\; \forall j\in J,\; \forall \ell\in L,\; \forall t\in T\setminus\{1\} \tag{5.4}$$

$$\sum_{\ell_2\in L} y_{j\ell^j\ell_2 1} = 1 \;\; \forall j\in J \tag{5.5}$$

$$x_{ij\ell pt} \ge 0 \;\; \forall i\in I,\; \forall j\in J,\; \forall \ell\in L,\; \forall p\in P,\; \forall t\in T \tag{5.6}$$

$$y_{j\ell_1\ell_2t} \in \{0,1\} \;\; \forall j\in J,\; \forall \ell_1\in L,\; \forall \ell_2\in L,\; \forall t\in T. \tag{5.7}$$

The objective function (5.1) minimizes the total cost for changing the capacity levels and allocating the demand. Constraints (5.2) are the demand constraints for the customers. Constraints (5.3) are the capacity constraints at the facilities. Constraints (5.4) link the capacity change variables in consecutive time periods. Finally, constraints (5.5) specify that exactly one capacity level must be chosen at the beginning of the planning horizon. Note that the flow constraints (5.4) and (5.5) further guarantee that, in each time period, exactly one capacity change variable is selected.

We may also adapt two types of valid inequalities to be used in the GMC formulation:

$$x_{ij\ell pt} \le \sum_{\ell_1\in L} y_{j\ell_1\ell t} \;\; \forall i\in I,\; \forall j\in J,\; \forall \ell\in L,\; \forall p\in P,\; \forall t\in T. \tag{5.8}$$

$$\sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L} u_{j\ell_2}y_{j\ell_1\ell_2t} \ge \sum_{i\in I}\sum_{p\in P} d_{ipt} \;\; \forall t\in T. \tag{5.9}$$

The *Strong Inequalities (SI)* (5.8), typically used in facility location and network design problems (see, for instance, Gendron and Crainic, 1994), are known to provide a tight upper bound for the demand assignment variables. The SIs may be added to the model either *a priori* or in a branch-and-cut manner only when they are violated in the solution of the LP relaxation. The set of valid inequalities (5.9) is referred to as the *Aggregated Demand Constraints (ADC)*. Although they are redundant for the LP relaxation, adding them to the model enables MIP solvers to generate cover cuts that further strengthen the formulation.

5.3.2 Special Cases

We now illustrate how special cases can be modeled by using the GMC formulation. As will be explained in Section 5.5, our solution approach can be tailored to take advantage of the special structure of each problem variant. Jena et al. (2013) explicitly show how to model two problem variants, using the GMC formulation: facility location with closing and reopening of facilities and facility location with capacity expansion and reduction. In the first problem, the size of the facility is chosen from a discrete set of capacity levels. Existing facilities may then be closed and reopened multiple times. In the second problem considered, capacities can be adjusted by the use of a single facility at each location. At each facility, the capacity can be expanded or reduced from one capacity level to another. It is assumed that an expansion of ℓ capacity levels has always the same cost, regardless of the previous capacity level. These two problems are denoted as the *Dynamic Modular Capacitated Facility Location Problem with Closing and Reopening (DMCFLP_CR)* and the *Dynamic Modular Capacitated Facility Location Problem with Capacity Expansion and Reduction (DMCFLP_ER)*, respectively.

A subset of capacity change variables $y_{j\ell_1\ell_2t}$ is chosen to model these special cases. The cost coefficients $f_{j\ell_1\ell_2t}$ for these variables are based on the following fixed costs, defined to characterize the special cases:

- $c^c_{j\ell}$ and $c^o_{j\ell}$ are the costs to temporarily close and reopen a facility of size ℓ at location j, respectively;
- $f^c_{j\ell}$ and $f^o_{j\ell}$ are the costs to reduce and to expand the capacity of a facility at location j by ℓ capacity levels, respectively;
- $F^o_{j\ell}$ is the cost to maintain an open facility of size ℓ at location j throughout one time period.

For the problem variant involving facility closing and reopening, we create an artificial capacity level $\overline{\ell}$ for each capacity level $\ell \in L\backslash\{0\}$. Capacity level $\overline{\ell}$ represents the state in which a facility of size ℓ is temporarily closed. At each time period $t \in T$ and location $j \in J$, we may find capacity transition decisions $y_{j\ell_1\ell_2t}$ that represent different types of operations (note that the costs for these decisions are usually composed by the cost to perform the capacity transition, as well as the maintenance cost for the new capacity level):

i. Facility construction and capacity expansion. The expansion of the capacity is represented by a capacity transition from capacity level ℓ_1 to any other capacity level $\ell_2 > \ell_1$. If the decision represents a facility construction, then ℓ_1 is 0. The capacity is thus expanded by $\ell_2 - \ell_1$ capacity levels. The cost for this decision is set to $f_{j\ell_1\ell_2t} = f^o_{j(\ell_2-\ell_1)} + F^o_{j\ell_2}$.

ii. Capacity reduction. The reduction of the capacity is represented by a transition from capacity level ℓ_1 to any other capacity level $\ell_2 < \ell_1$. The capacity is thus reduced by $\ell_1 - \ell_2$ capacity levels. The cost for this decision is set to $f_{j\ell_1\ell_2t} = f^c_{j(\ell_1-\ell_2)} + F^o_{j\ell_2}$.

iii. Maintaining the current capacity level. A facility may neither expand nor reduce the current capacity level. The cost of this transition is thus only composed of the maintenance cost, i.e., $f_{j\ell_1\ell_1t} = F^o_{j\ell_1}$ if the capacity level represents an open facility, $f_{j\overline{\ell}_1\overline{\ell}_1t} = 0$ if the capacity level represents a temporarily closed facility and $f_{j00t} = 0$ if no facility exists.

iv. Temporary closing. An open facility of size ℓ_1 can be temporarily closed, i.e., it changes to capacity level $\overline{\ell}_1$. The total cost is $f_{j\ell_1\overline{\ell}_1t} = c^c_{j\ell_1}$.

v. Reopening a closed facility. A temporarily closed facility of size ℓ_1 can be reopened, i.e., it changes its capacity level from $\overline{\ell}_1$ to ℓ_1. The total cost for this decision is $f_{j\overline{\ell}_1\ell_1t} = c^o_{j\ell_1} + F^o_{j\ell_1}$.

The DMCFLP_CR is represented by transition decisions of type 1 (for construction only), 3, 4 and 5. We denote the resulting model as the *CR-GMC* formulation. The DMCFLP_ER is represented by transition decisions of type 1, 2 and 3. The resulting model is denoted as the *ER-GMC* formulation.

Jena et al. (2013) also refer to a third problem variant, which combines both features of the two special cases. It is denoted as the *Dynamic Modular Capacitated Facility Location Problem with Closing/Reopening and Capacity Expansion/Reduction (DM-CFLP_CR_ER)*. The problem variant is modeled by using the transition decisions of type 1 – 5 presented above. However, these decisions allow only one single operation, for example either capacity reduction or facility closing, at each time period. In practice, it is very likely that one may want to reduce or expand the capacity before closing or after reopening a facility at the same time period. We may therefore consider four additional decision types that represent combinations of such operations:

(a) A facility is reopened at level ℓ_1 and its capacity is expanded to level $\ell_2 > \ell_1$ at the same time period.

(b) A facility is reopened at level ℓ_1 and its capacity is reduced to level $\ell_2 < \ell_1$ at the same time period.

(c) The capacity of a facility at level ℓ_1 is expanded to level $\ell_2 > \ell_1$ and the facility is closed right after.

(d) The capacity of a facility at level ℓ_1 is reduced to level $\ell_2 < \ell_1$ and the facility is closed right after.

By making the realistic assumption that the costs for closing and reopening a facility are non-decreasing as the size of the facility increases, we may discard two of the four possibilities.

Proposition 5.3.1. *Let $c^o_{j\ell} \leq c^o_{j(\ell+1)}$ and $c^c_{j\ell} \leq c^c_{j(\ell+1)}$ for $\ell = 0,1,2,\ldots,(q-1)$, then there is at least one optimal solution that does neither use decisions of type (b) nor of type (c).*

PROOF. Note that case (c) may only occur in two situations: either the facility stays closed until the end of the planning horizon or the facility is reopened at a later moment. If the facility stays closed, then closing it at level ℓ_1 is at most as expensive as combined capacity expansion and closing as suggested in case (c): $c^c_{j\ell_1} \leq f^o_{j(\ell_2-\ell_1)} + c^c_{j\ell_2}$. If the facility is closed at the beginning of time period t_1, but it will be reopened at the beginning of period $t_2 > t_1$, then the corresponding costs using case (c) are given by: $C^c =$

$c^c_{j\ell_1} + f^o_{j(\ell_2-\ell_1)} + c^o_{j\ell_2} + F^o_{j\ell_2}$. However, the same solution may be reproduced by closing the facility at level ℓ_1 and expanding its capacity only after it has been reopened using case (a), which corresponds to the following costs: $C^a = c^c_{j\ell_1} + c^o_{j\ell_1} + f^o_{j(\ell_2-\ell_1)} + F^o_{j\ell_2}$. Now, because $c^o_{j\ell_1} \leq c^o_{j\ell_2}$, we have: $C^a \leq C^c$. Therefore, a solution using case (a) is at most as expensive as a solution using case (c).

The same can be shown for the relation between cases (d) and (b), where reducing the capacity before temporary closing is as most as costly as reducing the capacity after temporary closing. □

We thus add only the transition decisions given by the cases (a) and (d) to the model:

vi. Reopening and capacity expansion. A closed facility of capacity level ℓ_1 is reopened and its capacity is expanded to level ℓ_2 (with $\ell_1 < \ell_2$). The cost for this decision, including the maintenance costs at capacity level ℓ_2 is thus set to $f_{j\overline{\ell_1}\ell_2 t} = c^o_{j\ell_1} + f^o_{j(\ell_2-\ell_1)} + F^o_{\ell_2}$.

vii. Capacity reduction and facility closing. An open facility reduces its capacity from level ℓ_1 to level ℓ_2 (with $\ell_1 > \ell_2$) and is temporarily closed afterwards. The cost for this decision is thus set to $f_{j\ell_1\overline{\ell_2}t} = f^c_{j(\ell_1-\ell_2)} + c^c_{j\ell_2}$.

5.4 Lagrangian Relaxation

When applying Lagrangian relaxation to capacitated facility location problems, it is common to relax either the capacity constraints or the demand constraints. Relaxing the capacity constraints results in a subproblem that is NP-hard (Van Roy and Erlenkotter, 1982; Barcelo et al., 1990). Furthermore, given that the focus of the GMC model is the detailed representation of capacity changes, it is intuitive to keep the capacity constraints. A more promising and popular choice in the literature (e.g., Shulman, 1991; Beasley, 1993; Wu et al., 2006) is to relax the demand constraints (5.2), which yields a Lagrangian subproblem that can be solved efficiently. Let α be the vector of Lagrange multipliers. After relaxing the demand constraints (5.2) and rearranging the terms in the objective function, we obtain the following Lagrangian subproblem:

$$
\begin{aligned}
L(\alpha) = \quad & \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2 t}y_{j\ell_1\ell_2 t} \\
& + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T}(g_{ij\ell pt}d_{ipt} - \alpha_{ipt})x_{ij\ell pt} + \sum_{i\in I}\sum_{p\in P}\sum_{t\in T}\alpha_{ipt} \\
& s.t.\ (5.3) - -(5.8).
\end{aligned}
$$

Note that the Strong Inequalities (5.8) are included in the Lagrangian subproblem, since they are easy to handle, as shown next.

5.4.1 Solution of the Lagrangian Subproblem

Let $\tilde{c}_{ij\ell pt} = g_{ij\ell pt}d_{ipt} - \alpha_{ipt}$ denote the modified costs for the $x_{ij\ell pt}$ variables. We separate the Lagrangian subproblem into $|J|$ independent subproblems, one for each candidate facility location for a fixed set of Lagrangian multipliers α. The Lagrangian subproblem is solved as $L(\alpha) = \sum_{j\in J} L_j(\alpha) + \sum_{i\in I}\sum_{p\in P}\sum_{t\in T}\alpha_{ipt}$, where $L_j(\alpha)$ is defined as follows:

$$
\begin{aligned}
L_j(\alpha) = \quad & \min \sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2 t}y_{j\ell_1\ell_2 t} + \sum_{i\in I}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T}\tilde{c}_{ij\ell pt}x_{ij\ell pt} \\
& s.t.\ \sum_{i\in I}\sum_{p\in P} d_{ipt}x_{ij\ell pt} \le \sum_{\ell_1\in L} u_{j\ell}y_{j\ell_1\ell t} \quad \forall \ell\in L,\ \forall t\in T \\
& \sum_{\ell_1\in L} y_{j\ell_1\ell(t-1)} = \sum_{\ell_2\in L} y_{j\ell\ell_2 t} \quad \forall \ell\in L,\ \forall t\in T\setminus\{1\} \\
& \sum_{\ell_2\in L} y_{j\ell j\ell_2 1} = 1 \\
& x_{ij\ell pt} \le \sum_{\ell_1\in L} y_{j\ell_1\ell t} \quad \forall i\in I,\ \forall \ell\in L,\ \forall p\in P,\ \forall t\in T \\
& x_{ij\ell pt} \ge 0 \quad \forall i\in I,\ \forall \ell\in L,\ \forall p\in P,\ \forall t\in T \\
& y_{j\ell_1\ell_2 t} \in \{0,1\} \quad \forall \ell_1\in L,\ \forall \ell_2\in L,\ \forall t\in T.
\end{aligned}
$$

Each of these subproblems (one for each location $j \in J$) is concerned with finding the optimal capacity planning over time, i.e., an optimal schedule to open facilities of a certain size such that the total cost composed by demand allocation costs (considering the modified costs $\tilde{c}_{ij\ell pt}$) and the costs to change capacity levels is minimal. We can solve this problem using dynamic programming by adapting the approach presented by Shulman (1991). Let $L_j^{\alpha}(\ell,t)$ denote the cost for an optimal demand allocation at period t assuming that a facility of size ℓ is available. For a given set of multipliers α, let $O_j^{\alpha}(\ell,t)$ denote the optimal cost to serve all demands by facility j throughout the time periods $0,\ldots,t$, with a facility of size ℓ at the end of period t. For $\ell > 0, t > 0$, the optimal value of $O_j^{\alpha}(\ell,t)$ is composed of the costs for demand allocation in period t, the capacity transition to level ℓ, the facility maintenance at level ℓ, and the optimal cost to serve all demands in previous time periods at the capacity level that minimizes the total cost. They can be computed by the following recurrence formula:

$$O_j^{\alpha}(\ell,t) = L_j^{\alpha}(\ell,t) + \min_{0\leq\ell_1\leq q}\{f_{j\ell_1\ell t} + O_j^{\alpha}(\ell_1,t-1)\}.$$

Note that $L_j^{\alpha}(0,t) = 0$ since demand cannot be allocated to a facility with capacity level 0. Furthermore, for $t = 0$ the size of the facility that exists at the beginning of the planning horizon is ℓ^j. We therefore have: $O_j^{\alpha}(\ell,0) = f_{i\ell^j\ell t} + L_j^{\alpha}(\ell,0)$.

The subproblem is then solved by selecting the facility size at the last time period that has the lowest total cost:

$$L_j(\alpha) = \min_{0\leq\ell\leq q}\left\{O_j^{\alpha}(\ell,|T|)\right\}.$$

Note that, without the use of the SIs, the Lagrangian subproblem does not possess the integrality property (Geoffrion, 1974), since facility capacities will only be opened as much as forced by the capacity constraints, i.e., $\sum_{\ell_1\in L} y_{j\ell_1\ell t} = \sum_{i\in I}\sum_{p\in P}(d_{ipt}x_{ij\ell pt})/u_{j\ell}$, which may be fractional. Adding the SIs to the problem strengthens the dependence between the opening decisions and the demand allocation: $\sum_{\ell_1\in L} y_{j\ell_1\ell t} = \max_{i\in I,p\in P}\{x_{ij\ell pt}\}$. The variables $x_{ij\ell pt}$ (and therefore also one of the corresponding $y_{j\ell_1\ell_2 t}$ variables) will take value 1 if their modified costs $\tilde{c}_{ij\ell pt}$ compensate the costs for the open facility. As a consequence, using the SIs, the Lagrangian subproblem also has the integrality property.

The lower bound provided by the Lagrangian dual will therefore never be better than the bound provided by the LP relaxation of the original problem using the SIs.

Computation of the Optimal Demand Allocation. The optimal demand allocation $L_j^{\alpha}(\ell,t)$ at location j assumes that a facility of size ℓ is available and can be computed by solving a fractional knapsack problem (subject to the capacity constraints and the SIs):

$$\begin{aligned} L_j^{\alpha}(\ell,t) = \quad & \min \sum_{i\in I}\sum_{p\in P} \tilde{c}_{ij\ell pt} x_{ij\ell pt} \\ & s.t. \sum_{i\in I}\sum_{p\in P} d_{ipt} x_{ij\ell pt} \le u_{j\ell} \\ & \quad 0 \le x_{ij\ell pt} \le 1 \ \ \forall i \in I, \forall p \in P. \end{aligned}$$

This problem can be solved by sorting all x variables in increasing order of their ratio $\tilde{c}_{ij\ell pt}/d_{ipt}$, selecting those with the most negative ratio until the capacity is completely filled or all variables with negative ratios have been selected. To be precise, we repeatedly select the variables with the most negative ratio for $< i, p >$ and increase the variable value to the maximum value possible, updating the remaining knapsack capacity $u'_{j\ell}$ after each variable selection:

$$< i^*, p^* > = \underset{i\in I, p\in P}{\operatorname{argmin}} \left\{ \frac{\tilde{c}_{ij\ell pt}}{d_{ipt}} \right\}, \quad x_{i^* j\ell p^* t} = \min \left\{ 1, \frac{u'_{j\ell}}{d_{ipt}} \right\}.$$

Clearly, all other x variables are set to 0.

5.4.2 Solution of the Lagrangian Dual

The solution of the Lagrangian subproblem, for any choice of the Lagrange multipliers α, provides a lower bound to the DFLPG. To obtain the best possible lower bound, one must solve the Lagrangian dual:

$$z^* = \max_{\alpha} L(\alpha).$$

The Lagrangian function $L(\alpha)$ is non-differentiable. However, a subgradient direction can be easily computed. We consider two different methods to solve the Lagrangian dual: a subgradient method and a bundle method.

Subgradient Method. The subgradient direction γ_{ipt} at the $k-th$ iteration is computed as the violation of the relaxed constraints when x is fixed to the values found by solving the Lagrangian subproblem:

$$\gamma_{ipt}^{k} = 1 - \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell pt} \quad \forall i \in I,\ \ \forall p \in P,\ \ \forall t \in T.$$

We choose the step size λ^k at iteration k as suggested by Held et al. (1974) and often used in other works (Shulman, 1991; Sridharan, 1991; Correia and Captivo, 2003):

$$\lambda^k = \delta^k \frac{\widehat{Z} - L^k(\alpha)}{\sum_{i\in I}\sum_{p\in P}\sum_{t\in T}(\gamma_{ipt}^{k})^2},$$

where δ^k is a scalar, $L^k(\alpha)$ equals the value of $L(\alpha)$ at iteration k and $\widehat{Z}$ is the cost of the best feasible solution found so far. The Lagrange multipliers for the $(k+1)-st$ iteration are then updated by:

$$\alpha_{ipt}^{(k+1)} = \alpha_{ipt}^{k} + \lambda^k \gamma_{ipt}^{k} \quad \forall i \in I,\ \ \forall p \in P,\ \ \forall t \in T.$$

Bundle Method. The second method used to solve the Langrangian dual is an implementation of the bundle method (Frangioni, 2005). The method uses a subset of the tuples $< L(\alpha^s), \gamma^s >$ with $s \in B$ and B is referred to as the bundle of subgradients γ^s. From the primal view point, the following quadratic problem has to be solved at each iteration (Frangioni and Gallo, 1999):

$$\min_{\theta^s} \left\{ \tfrac{1}{2} \parallel \sum_{s\in B} \gamma^s \theta^s \parallel^2 + \tfrac{1}{R} E_B \theta;\ s.t.\ \sum_{s\in B} \theta^s = 1,\ \theta \geq 0 \right\},$$

where R is the so called trust region for the tentative ascent direction, and $E_s = L(\alpha) + \gamma(\hat{\alpha} - \alpha) - L(\hat{\alpha})$ is the linearization error from the current point $\hat{\alpha}$. The solution values for θ^s, given for each bundle member, hold valuable information and can be used to construct feasible integer solutions (see Section 5.5.2). The tentative ascent direction is then computed by the convex combination of the subgradients, using the convex multipliers θ. Alternatively, the dual problem can be solved to compute the ascent direction, or directly the new point. Frangioni and Gallo (1999) elaborate on this relationship in detail.

Bundle methods usually possess stronger convergence properties than the subgradient method. However, they also tend to require more time to compute the Lagrange multipliers. They are therefore beneficial when a small number of iterations is performed to reach the desired accuracy.

5.4.3 Upper Bound Generation

At each iteration, a feasible solution is generated based on the Lagrangian solution obtained by solving the Lagrangian subproblem. This solution provides an upper bound for the optimal integer solution of the problem that directly impacts the convergence of the subgradient and bundle methods. Even though high quality upper bounds are desirable, it is important that they are generated in an efficient manner, as the solution of the Lagrangian dual typically involves hundreds of iterations.

The solution of the Lagrangian subproblem provides a facility opening schedule for the entire planning horizon. This schedule is defined by capacity levels ℓ'_{jt} indicating the facility size at locaton j at time period t. In addition to the schedule, the Lagrangian solution provides a demand allocation. As the demand constraints (5.2) have been relaxed, the customer demands d_{ipt} are either exactly met, under-served or over-served.

The set of all customer demands can therefore be separated into three subsets, where Σ_1, Σ_2 and Σ_3 denote the demands defined by triplets $< i, p, t >$, which are exactly met, over-served and under served, respectively:

$$\Sigma_1 = \left\{ < i,p,t >: \sum_{j \in J} x_{ij(\ell'_{jt})pt} = 1 \right\}, \Sigma_2 = \left\{ < i,p,t >: \sum_{j \in J} x_{ij(\ell'_{jt})pt} > 1 \right\}$$

$$\text{and } \Sigma_3 = \left\{ < i,p,t >: \sum_{j \in J} x_{ij(\ell'_{jt})pt} < 1 \right\}.$$

To obtain an integer feasible solution, we heuristically reduce redundant demand allocation for the pairs in Σ_2 and increase missing demand allocation for the pairs in Σ_3. Note that the heuristic to increase available capacity is very simple. The difficulty here is to find general rules that perform well on the different problem variants that may be modeled by the use of the GMC formulation. The heuristic procedure used to obtain a feasible solution is composed of the following steps:

i. **Reduce demand allocation**: For each $< i,p,t > \in \Sigma_2$, all facility/size pairs $(j,(\ell'_{jt}))$ are sorted in decreasing order of their allocation costs $g_{ij\ell pt}$. The allocated flow is removed until the total allocated demand for $< i,p,t >$ equals 1.

ii. **Increase capacities**: If the total remaining capacity is smaller than the total remaining demand, we increase the capacity sequentially for each time period according to the following steps until the total demand can be met. Facilities are considered without a specific order. We consider two simple possibilities to increase capacity: if a facility is already open at any moment in the planning horizon, we increase the capacity for the current time period to its maximum capacity level throughout the planning; if no facility exists, we increase the capacity level until the missing capacity is covered or the maximum capacity level for this facility is reached.

iii. **Increase the demand allocation**: For each $< i,p,t > \in \Sigma_3$, all facility/size pairs $(j,(\ell'_{jt}))$ with remaining capacity are sorted in increasing order of their allocation costs $g_{ij\ell pt}$. Demand is allocated to these pairs until the total allocated demand for $< i,p,t >$ equals 1.

iv. **Reduce unused capacities of open facilities**: For each facility, we use a dynamic programming algorithm, similar to the one used to solve Lagrangian subproblem, to compute the optimal opening schedule (i.e., the one with the lowest costs) that guarantees sufficient capacity to satisfy the demand allocated to that facility.

Even though the resulting solution is integer feasible, its demand allocation may still be improved. Therefore, a final step consists in computing the optimal demand allocation for the current opening schedule using the CPLEX network algorithm.

5.5 Upper Bound Improvement: Restricted MIP Model

The previous section outlined the heuristic procedure to generate integer feasible solutions. This heuristic focuses on efficiency rather than on the quality of the upper bound. However, the objective of the Lagrangian heuristic is to provide high quality solutions. It is therefore beneficial to add an optimization phase that aims at finding solutions of higher quality than those already found during the solution of the Lagrangian dual. Either one tries to improve promising solutions that have been found during the Lagrangian dual method, or one constructs new solutions based on information gathered during the process.

Local improvement heuristics, based on already available solutions, have been successfully applied in a second optimization phase after performing a Lagrangian relaxation method (e.g., Correia and Captivo, 2006; Li et al., 2009). However, they require a detailed knowledge of the problem structure. As seen in Section 5.3.2, the GMC is a fairly general model, capable of representing different facility location problems. In some cases, certain capacity levels represent open facilities, whereas other capacity levels represent closed facilities. Given the flexibility regarding the usage of capacity levels, it is beneficial to use a more general mechanism to find high quality solutions.

5.5.1 MIP Model Based on Lagrangian Solutions

The Lagrangian heuristic proposed in this work involves a second optimization phase using information collected during the solution of the Lagrangian dual. We solve a restricted MIP, taking into consideration the decisions made by the Lagrangian solutions.

One would expect that the larger the decision space is, the better the quality of the final solution will be. However, this is only true without memory and computing time limitations. Given those limitations, a large MIP may result in a low overall performance, as the model is too large to be solved with the available time and memory resources. We therefore filter the decisions considered in the restricted MIP to sufficiently reduce the size of the model.

Let n^{Iter} denote the number of iterations performed by the subgradient or by the bundle method. Let $n^{C}_{j\ell t}$ be the number of Lagrangian solutions where capacity level ℓ has been selected for location j at time period t (note that we have $\sum_{\ell \in L} n^{C}_{j\ell t} = n^{Iter}$ for each j and t). Furthermore, let L^{R}_{jt} be the set of capacity levels for location j and period t available in the restricted MIP. The restricted MIP is then defined as follows:

- **Decision fixing.** For each j and t, a decision is fixed to capacity level ℓ if it appears in at least $100 \times pFix$ (with $pFix \in]0.5, 1]$) percent of all iterations, i.e, $L^{R}_{jt} = \{\ell\}$, if $n^{C}_{j\ell t}/n^{Iter} \geq pFix$.
- **Selection of available capacity levels.** If the capacity level for location j and time period t is not fixed, L^{R}_{jt} is composed by the n^{S} capacity levels that appear the most often in the Lagrangian solutions (i.e., have the highest $n^{C}_{j\ell t}$) and appear in at least one Lagrangian solution (i.e., $n^{C}_{j\ell t} \geq 1$).
- **Defining the set of capacity transitions.** Decisions $y_{j\ell_1\ell_2 t}$ are defined for all combinations between ℓ_1 and ℓ_2, with $\ell_1 \in L^{R}_{jt}$ and $\ell_2 \in L^{R}_{j(t+1)}$, if available in the original GMC formulation.

Using appropriate values for the parameters $pFix$ and n^{S}, the original GMC model can be reduced to a restricted version with reasonable memory and computing time requirements, taking into consideration only decisions that have been found to be significant by the Lagrangian solutions.

5.5.2 MIP Model Based on Convexified Bundle Solutions

When using the bundle method to solve the Lagrangian dual, we may take advantage of the information the method holds concerning the set of solutions that are linked to the subgradients in the bundle, as demonstrated by Borghetti et al. (2003).

As explained in Section 5.4.2, the bundle method provides a multiplier θ^s for each Lagrangian solution s such that $\sum_s \theta^s = 1$. The value θ^s can be seen as a probability that solution s provides a good opening schedule. We may therefore derive probabilities for each of the opening decisions $\tilde{y}_{j\ell t} = \sum_s \theta^s y^s_{j\ell t}$, where $y^s_{j\ell t}$ is 1 if solution s selects capacity level ℓ for location j at period t.

We may now construct a restricted MIP, as previously shown based on the Lagrangian solutions. Instead of using the number of occurrences $n^C_{j\ell t}$ in Lagrangian solutions, we use the value of $\tilde{y}_{j\ell t} \in [0,1]$, defining its importance according to the multipliers θ^s provided by the bundle method. In this case, a capacity level ℓ is fixed at location j and period t if $\tilde{y}_{j\ell t} \geq pFix$, where $pFix \in]0.5,1]$. Otherwise, L^R_{jt} is composed by the n^S capacity levels with the highest $\tilde{y}_{j\ell t}$ values, with $\tilde{y}_{j\ell t} \geq 0.001$. Note that the Lagrangian solutions linked to the subgradients that are stored in the bundle are only a subset of those generated in all iterations. The set of decisions considered in the restricted MIP based on the convexified bundle solution is therefore very likely to be much smaller than the restricted MIP based on all Lagrangian solutions.

5.6 Computational Results

In this section, the performance of different configurations for the Lagrangian heuristics and that of the MIP solver CPLEX will be evaluated and compared by means of computational experiments. First, we discuss how test instances were generated. Then, we elaborate on the integrality gap of the different problems. Finally, computational results are presented to explore the impact of parameter choices for the Lagrangian heuristics and to compare different configurations with each other and with CPLEX.

Test instances have been generated by following a scheme similar to that described in Jena et al. (2013). However, the instances used in this previous work included only one commodity, up to 100 candidate facility locations and up to 1000 customer locations. In this work, we use instances that are significantly larger with respect to the number of candidate facility locations and the number of commodities. Instances have been generated with different numbers of candidate facility locations $|J|$ and customers $|I|$, combining all pairs of $|J| \in \{50,100,150,200,250\}$ and $|I| \in \{|J|, 4 \cdot |J|\}$. The highest capacity level at any facility, denoted by q, has been selected such that $q \in \{3,5,10\}$. Three different networks have been randomly generated on squares of the following sizes: 300*km*, 380*km* and 450*km*. We consider two different demand scenarios. In both scenarios, the demand for each of the customers is randomly generated and randomly distributed over time. The two scenarios differ in their total demand summed over all customers in each time period. In the first scenario (*regular*), the total demand is similar in each time period. The second scenario (*irregular*) assumes that the total demand follows strong variations along time and therefore varies at each time period. The number of commodities $|P|$ has been selected such that $|P| \in \{1,3,5\}$. The demands for the second to fifth commodities are computed based on the demand for the first commodity. To be precise, the demand d_{jpt} for $p \geq 2$ is computed as $d_{jpt} = d_{j1t} \cdot rand(1.0, 0.2) \cdot avgDem_p / avgDem_1$, where $avgDem_1 = 10$, $avgDem_2 = 6$, $avgDem_3 = 9$, $avgDem_4 = 5$, $avgDem_5 = 8$, and $rand(1.0, 0.2)$ is a random variable with normal distribution, mean value of 1.0 and standard deviation of 0.2. Construction and operational costs follow concave cost functions, i.e., they involve economies of scale. Jena et al. (2013) also tested a second cost scenario in which the transportation costs are five times higher. The authors found that these instances are significantly easier to solve. In this work, we only consider the instances that are more difficult to solve, i.e., the ones with their original level of transportation costs. The combination of the different properties listed above results in a total of ($5 \times 2 \times 3 \times 3 \times 2 \times 3 =$) 540 instances. All instances contain ten time periods, which is found to be sufficient to demonstrate capacity changes along time and small enough to not increase the size of the models too much. Note that we assume that the problem instances do not contain initially existing facilities. We refer to Appendix C.1 for a detailed description of the parameters used to generate the instances.

All mathematical models and the Lagrangian based heuristics have been implemented in C/C++ using the IBM CPLEX 12.6.0 Callable Library. The code has been compiled and executed on openSUSE 11.3. Each problem instance has been run on a single Intel Xeon X5650 processor (2.67GHz), limited to 24GB of RAM.

5.6.1 Integrality Gaps of the Test Instances

The integrality gap is defined as the difference between the optimal LP relaxation solution value and the cost of an optimal integer solution, divided by the latter. For many instances, the GMC models are very large and exceed the available memory of 24GB. It was therefore not possible to find all of these optimal values. The integrality gap has been exactly determined only for a subset of the 540 instances. Considering the best lower and upper bounds obtained throughout all computational experiments, optimality has been proved for 302, 388, 384 and 382 instances for the DFLPG, the DMCLFP_CR, the DMCFLP_ER and the DMCFLP_CR_ER, respectively.

As observed in Jena et al. (2013), the integrality gap for the GMC based formulations tend to be very small. This turns out to be useful for two reasons. First, when using Lagrangian relaxation, the provided bounds are more meaningful. Low integrality gaps may help to prove optimality within a certain tolerance. Second, the input data for multi-period facility location problems usually comes from forecasts, and it is very likely that the real data will slightly deviate from the forecast, especially for the last time periods. An optimal solution may therefore not be more relevant in practice than a solution that guarantees optimality within a certain tolerance. Melo et al. (2011a) therefore aim at a finding solutions within 1% from the optimal solution. On the instances used in this work, the integrality gap has been found to be smaller than or equal to 1% for a fairly large part of the instances. To be precise, the integrality gap is smaller than or equal to 1% for at least 413, 397, 410 and 397 instances for each the four problems, respectively. The Lagrangian relaxation may therefore prove optimality within a deviation of 1% for a large part of the instances if its lower bounds are close to the LP relaxation bounds and its generated upper bounds (i.e., the feasible solutions generated throughout the Lagrangian relaxation) are close to optimal.

5.6.2 Comparison of Different Configurations for the Lagrangian Heuristics

We now compare the performance of different configurations for the Lagrangian relaxation based heuristics. Section 5.4 discussed two different methods to solve the

Lagrangian dual, the subgradient method and the bundle method. These methods can be used to generate feasible solutions at each iteration. Furthermore, it has been shown in Section 5.5 how information from the Lagrangian solutions and the convexified bundle solutions can be collected throughout the solution of the Lagrangian dual, and then be used to generate a restricted MIP to find solutions of even better quality.

Parameter Settings. The subgradient method is used with an initial scalar $\delta^k = 2.0$. This scale factor halves every 25 consecutive iterations without improvement in the lower bound. The algorithm terminates if δ^k falls below 0.005. For the bundle method, an implementation similar to the one described by Frangioni (2005) has been used as a black box. The bundle implementation has four principal internal performance and termination criteria, which are set as follows. Parameters *tStar*, *EpsLin* have been set to 10^4 and 10^{-6}, respectively. The long-term *t-strategy* has been set to *"soft"* with a parameter value of *0.1*. In addition to the stopping criteria mentioned above, a 1% optimality stopping criterion has been used, i.e., the algorithms stop as soon as the best lower and upper bounds found are within 1%. All experiments have been limited to a maximum of 2 hours of computing time.

5.6.2.1 Combining the Lagrangian Dual Solution Methods with a Restricted MIP

After performing the subgradient method, a restricted MIP can be solved based on the Lagrangian solutions (see Section 5.5.1). When using the bundle method, the restricted MIP can be generated based on either the Lagrangian solutions or on the convexified bundle solution (see Section 5.5.2). We now compare the performance of different combinations for the heuristic, i.e., the use of the subgradient method and the bundle method to solve the Lagrangian dual, and the use of the restricted MIP based on Lagrangian solutions and the convexified solutions to further improve the solution quality. The bundle method has shown significantly faster convergence than the subgradient method. We therefore stop the method when a maximum of 500 iterations has been performed. For the subgradient method, due to its slower convergence, we also tested configurations with a maximum of 1000 iterations.

Table 5.I summarizes the results for seven different solution strategies: the subgradient method without (*"only"*) and with a restricted MIP based on the Lagrangian Solutions (*"w/ LS R-MIP"*), as well as the bundle method without (*"only"*) and with a restricted MIP, based either on the Lagrangian solutions (*"w/ LS R-MIP"*) or on the convexified bundle solution (*"w/ CS R-MIP"*). As mentioned above, the subgradient method has been tested in two variants, stopping either after a maximum of 500 iterations or after a maximum of 1000 iterations. When using the restricted MIP based on the Lagrangian solutions, we use parameter values that have led to good performance (see Section 5.6.2.2): $pFix = 70\%$ and $n^S = 3$. For the bundle method with the restricted MIP based on the convexified solutions, we used $pFix = 0.85$ and $n^S = 4$, which led to smaller average and maximum optimality gaps than setting $pFix$ to 0.7, 0.8 or 0.9. Note that for the restricted MIP based on the convexified solutions, we only tested n^S values of 2, 3 and 4.

The results take into account all 540 instances and are reported for each of the four problem variants. We indicate the average and maximum gap (when compared to the best lower bounds known for the instances), the average computing time and the number of instances for which a 1% optimality has been proved (*"# prov. 1% gap"*).

The results are consistent for the four different problem variants. Solving only the Lagrangian dual, the bundle method clearly stays ahead of the subgradient method. Given its stronger convergence properties, it finishes, on average, in significantly shorter computing times. For the subgradient method, allowing 1000 instead of 500 iterations strongly improves the solution quality. After this first phase, a 1% optimality has been proved for more than half of the instances.

Adding the Lagrangian solution based restricted MIP to the subgradient method significantly improved the optimality gap when up to 1000 iterations are performed. With only 500 iterations, the improvement is less significant. This illustrates the importance of reasonably solving the Lagrangian dual before constructing a restricted MIP, because "high-quality" decisions tend to appear in the later stage of the subgradient method.

For the bundle method, a larger improvement of the maximum optimality gap can be observed. Both versions of the restricted MIP result in very competitive results. The

	Subgradient method				Bundle method		
	500 max iter		1000 max iter		500 max iter		
	only	w/ LS R-MIP	only	w/ LS R-MIP	only	w/ LS R-MIP	w/ CS R-MIP
DFLPG							
Avg Gap %	3.01	0.81	1.67	0.72	1.64	0.76	0.72
Max Gap %	17.31	17.31	13.06	7.28	8.83	7.85	2.88
Avg Time (sec)	374.8	1,140.0	486.5	714.5	258.9	844.6	359.3
# prov. 1% gap	211	386	329	407	331	405	407
DMCFLP_CR							
Avg Gap %	3.85	0.82	2.28	0.88	2.10	0.86	0.81
Max Gap %	22.26	12.96	14.38	9.73	10.39	10.39	3.78
Avg Time (sec)	846.9	1,532.2	1,130.1	1,447.2	617.8	1,370.0	884.6
# prov. 1% gap	160	378	287	385	292	392	397
DMCFLP_ER							
Avg Gap %	3.18	0.78	1.72	0.70	1.64	0.74	0.69
Max Gap %	17.58	14.68	12.45	6.63	10.47	8.56	2.57
Avg Time (sec)	379.5	1,137.0	484.4	712.9	247.7	833.9	348.8
# prov. 1% gap	205	391	317	405	325	410	411
DMCFLP_CR_ER							
Avg Gap %	3.77	0.87	2.11	0.82	1.96	0.83	0.77
Max Gap %	21.00	17.76	15.92	8.37	10.14	9.15	3.44
Avg Time (sec)	840.3	1,703.7	1,091.7	1,493.7	569.9	1,335.4	857.6
# prov. 1% gap	174	373	295	389	310	399	395

Table 5.I: Comparison of different configurations for the Lagrangian based heuristics for the four problems.

maximum optimality gap is always kept below 4.25%, while the average computing time is very reasonable. Using the restricted MIP to improve the solution quality, the number of instances where a 1% optimality gap could be proved increased to over 75% of all instances. While both approaches show similar maximum optimality gaps, the convexified solution presents better average gaps and is capable of proving a 1% gap for more instances.

The results based on the bundle method are clearly better than those based on the subgradient method, as the subgradient method itself already takes a significant portion of the available computing time. Therefore, there is often not enough time left to solve the restricted MIP. However, a heuristic based on the latter could still be effective. Tuning the maximum number of subgradient iterations and the parameters used to define the restricted MIP will hereby make the crucial difference. Such tuning is exemplified in the next section.

5.6.2.2 Restricted MIP Parameter Tuning

The restricted MIP, performed after the solution of the Lagrangian dual, has to be sufficiently restricted in a way in that it can be reasonably solved within the remaining time. This is done by appropriately setting the two parameters n^S and $pFix$, indicating the maximum number of decisions considered for each location and time period, and the percentage necessary to fix a decision, respectively.

Table 5.II summarizes the results for different parameter values, using the bundle method with a restricted MIP based on the Lagrangian solutions applied to the DFLPG. The results are given for all combinations between different $pFix$ and n^S values, reporting the average and maximum optimality gap, as well as the average computation time. The average computation times increase due to two factors: more capacity level decisions in the MIP (i.e., higher values of n^S), and less variable fixing (i.e., higher values of $pFix$). For the given time limit of 2 hours, well performing values can be found by balancing these two parameters. Setting n^S to 3, 4 or even 10, and $pFix$ between 80% and 90% results in a maximum optimality gap of around 3.36%, while other parameter values may result in gaps of up to 8.83%. Clearly, if more computing time is available, one

pFix		$n^S = 2$	$n^S = 3$	$n^S = 4$	$n^S = 10$
51%	Avg / Max gap %	1.48 / 8.83	1.48 / 8.83	1.48 / 8.83	1.48 / 8.83
	Avg time	233.3 sec	231.5 sec	231.2 sec	228.4 sec
70%	Avg / Max gap %	0.82 / 5.37	0.80 / 5.37	0.80 / 5.37	0.80 / 5.37
	Avg time	235.6 sec	239.3 sec	243.6 sec	242.3 sec
80%	Avg / Max gap %	0.75 / 3.47	0.74 / 3.36	0.73 / 3.15	0.73 / 3.20
	Avg time	286.6 sec	296.2 sec	301.1 sec	302.3 sec
90%	Avg / Max gap %	0.74 / 7.85	0.72 / 3.35	0.71 / 3.35	0.71 / 3.35
	Avg time	385.7 sec	415.0 sec	422.7 sec	419.6 sec
100%	Avg / Max gap %	0.74 / 7.85	0.72 / 7.85	0.72 / 7.85	0.72 / 7.85
	Avg time	597.7 sec	620.0 sec	623.3 sec	614.1 sec
No	Avg / Max gap %	0.74 / 7.85	0.72 / 7.85	0.72 / 7.85	0.72 / 7.85
Fixing	Avg time	582.2 sec	595.9 sec	613.6 sec	601.6 sec

Table 5.II: Comparison of results for different parameters for the bundle method with MIP based on Lagrangian solutions, applied to the DFLPG.

may allow higher values for these parameters, which may further improve the solution quality.

Similar experiments were performed for different parameter values for the restricted MIP based on the convexified bundle solutions. Not restricting the MIP at all resulted in significantly better results than for the non-restricted MIP based on the Lagrangian solutions. Furthermore, it was found that the restricted MIP based on the convexified bundle solution is less sensitive to changes in the parameter value $pFix$ than the one based on the Lagrangian solutions. These results suggest that the decisions that are part of solutions selected by the bundle are those which are also present in high quality solutions.

5.6.3 Comparisons with CPLEX

The performance of one of the Lagrangian based heuristics is now compared to CPLEX. We chose the configuration that provided the lowest average and maximum optimality gaps: the bundle method with restricted MIP based on its convexified solution, with $n^S = 4$ and $pFix = 0.85$. CPLEX has been used with standard parameters. As in the previous experiments, a 1% optimality stopping criterion and a time limit of 2 hours have been applied.

Computational Results. Tables 5.III, 5.IV, 5.V and 5.VI summarize the results for CPLEX, as well as for the Lagrangian based heuristic outlined above for the four different problems DFLPG, DMCLFP_CR, DMCFLP_ER and DMCFLP_CR_ER, respectively. All results are grouped by the number of capacity levels q and the problem dimension defined by the number of candidate facility locations and the number of customers. Each group given by such a combination includes 18 instances. The tables report the average and maximum gaps of the best feasible integer solutions found by the algorithm when compared to the best lower bound known for the corresponding problem instance, as well as the average computing times. Note that the results shown in the Tables 5.III – 5.VI only take into account the instances where CPLEX found a feasible integer solution within the time limit of 2 hours. The number of instances for which CPLEX did not find any feasible solution is indicated by column *"#ns"*. Furthermore, column *"# prov. 1% gap"* gives the number of instances (out of those for which CPLEX found a feasible solution) where a 1% optimality gap has been proven by the algorithm. For the Lagrangian heuristic, the number in brackets to the right represents the same count, but for all instances.

The observations made for the results of CPLEX and the Lagrangian based heuristics are similar for all four problems. The number of instances where CPLEX did not find feasible solutions is fairly high, at least 25% of the instances for each of the four problems. In most of the cases, this happens due to memory limitations when the number of capacity levels or the number of candidate facility locations is high. Even though the average quality of solutions found by CPLEX is quite good, the solver provides large optimality gaps on many instances. This is mostly the case when a large number of capacity levels ($q = 10$) is available. As the solver constantly improves its bounds, the optimality gaps proven by the algorithm (shown in brackets) are very close to the gaps when compared to the best known lower bound for the instances. CPLEX is capable of proving a 1% optimality gap for at least 342 out of the 540 instances for each of the four problems.

The Lagrangian based heuristic provides stable results for each of the four problems. When compared to the same instances, it provides an average gap lower than that of CPLEX in computing times that are, on average, significantly lower. For the DFLPG and the DMCFLP_ER, the Lagrangian heuristic is, on average, twelve times faster than CPLEX. For the DMCFLP_CR and the DMCFLP_CR_ER, the heuristic is, on average, five times faster. Most importantly, the maximum optimality gap is at most 3.78%. Due to the strength of the GMC formulation, the maximum optimality gap proven by the Lagrangian heuristic is 4.87%. Furthermore, considering the same set of instances, the heuristic proves a 1% gap for almost the same number of instances as CPLEX. When considering all 540 instances (even those for which CPLEX does not find feasible solutions), the Lagrangian heuristic proves a 1% gap for 395 or more of the 540 instances for each of the four problems.

Interestingly, the difficulty of a problem is not always linked to its dimension. Instances where the number of customers is close to the number of candidate facility locations are significantly harder to solve than those where the number of customers is higher. In particular, this can be observed for instances of dimension (50/50). An analysis showed that these instances tend to possess larger integrality gaps, which may be linked to the fact that the more customers are available, the easier it is to make efficient use of a facility (in terms of allocation costs and capacity usage) in an integer solution.

A Note on the Model Size. As the previous results show, general-purpose MIP solvers such as CPLEX may perform very well on small instances, i.e., when the number of capacity levels is low ($q \in \{3,5\}$) and the number of candidate facility locations is small ($|J| \leq 100$). Clearly, adding the SIs (5.8) a priori to the model significantly increases the number of constraints and, therefore, the memory requirements of the model. As noted by Jena et al. (2013), the addition of the SIs to the GMC based models significantly facilitates the solution of the problems. In fact, for the instances used in this work, without the use of the SIs, CPLEX provides very low solution quality even for small instances. Other studies, such as the one by Gendron and Larose (2014) applied to a network design problem, confirm that it may be beneficial to add these inequalities in a branch-and-cut scheme. However, this only yields good performance if only a small number of SIs are violated and therefore added to the model. In the case of the DFLPG, a significant number of SIs are violated in its LP relaxation. Adding the inequalities as CPLEX user cuts to reduce the size of the model showed less competitive results. For more than 40% of the instances, the solver could not find feasible solutions. When feasible solutions were found, the average optimality gap was consistently high, on average, more than 10%.

We also note that, even though we use information from the Lagrangian solutions, other mechanism could be used to rate the importance of opening decisions to generate a MIP that is significantly restricted in its size. Theoretically, using the LP relaxation solution would be one alternative. However, as the LP relaxation cannot be efficiently solved (or not at all) for large instances, such a solution strategy would be applicable only to small and medium sized instances, or in computing environments with significantly larger memory and time resources.

		CPLEX (with SIs a priori)					Lagrangian Heuristic			
q	Instance size	Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap	# ns	Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap
3	50/50	0.26	0.99	531.7	18	0	0.44	1.32	6.7	7 [7]
	50/200	0.02	0.12	15.1	18	0	0.43	0.92	7.6	18 [18]
	100/100	0.11	0.51	54.1	18	0	0.43	0.95	13.0	17 [17]
	100/400	0.04	0.37	63.9	18	0	0.58	0.95	39.7	18 [18]
	150/150	0.13	0.76	82.4	18	0	0.38	0.87	33.3	18 [18]
	150/600	0.06	0.64	179.7	18	0	0.66	0.96	104.6	18 [18]
	200/200	0.17	0.86	116.3	18	0	0.51	0.98	49.2	18 [18]
	200/800	0.09	0.52	370.1	12	6	0.67	0.90	184.1	12 [18]
	250/250	0.04	0.37	179.7	18	0	0.44	0.92	88.8	18 [18]
	250/1000	0.15	0.86	373.5	6	12	0.52	0.94	262.3	6 [18]
	All	0.10	0.99	177.1	162	18	0.50	1.32	61.5	150 [168]
		[0.18]	[1.00]				[0.67]	[2.32]		
5	50/50	0.71	2.11	3,122.8	13	0	0.88	2.06	28.4	3 [3]
	50/200	0.17	0.79	90.7	18	0	0.48	0.89	17.5	16 [16]
	100/100	0.46	1.30	1,268.3	16	0	0.64	1.26	36.8	9 [9]
	100/400	0.04	0.16	145.8	18	0	0.55	0.87	58.3	18 [18]
	150/150	0.39	1.13	1,106.5	15	1	0.61	1.24	98.5	12 [13]
	150/600	0.08	0.67	255.0	12	6	0.63	0.96	116.2	12 [18]
	200/200	0.22	0.84	762.6	16	2	0.52	0.92	89.3	15 [16]
	200/800	0.06	0.20	552.0	6	12	0.53	0.89	243.8	6 [18]
	250/250	0.15	0.52	885.7	17	1	0.46	0.95	151.2	17 [17]
	250/1000	0.13	0.75	683.3	6	12	0.49	0.94	348.5	6 [18]
	All	0.27	2.11	957.8	137	34	0.59	2.06	90.2	114 [146]
		[0.39]	[2.11]				[0.86]	[4.59]		
10	50/50	23.11	92.72	6,472.0	2	0	1.90	2.88	282.7	0 [0]
	50/200	0.86	2.19	2,823.1	12	3	0.73	1.28	108.0	8 [9]
	100/100	3.03	14.82	5,312.7	4	7	1.26	2.44	131.1	2 [2]
	100/400	0.30	1.44	991.3	10	7	0.55	0.91	123.4	11 [18]
	150/150	2.59	11.93	5,014.6	3	11	0.85	1.31	105.0	2 [2]
	150/600	0.07	0.17	541.2	6	12	0.43	0.67	125.2	6 [17]
	200/200	0.88	1.66	3,400.7	4	12	0.80	1.62	193.3	3 [4]
	200/800	0.12	0.12	1,743.0	1	17	0.15	0.15	681.0	1 [18]
	250/250	0.20	0.36	1,052.7	3	15	0.27	0.40	171.3	3 [5]
	250/1000	-	-	-	0	18	-	-	-	- [18]
	All	6.28	92.72	3,741.6	45	102	1.02	2.88	171.1	36 [93]
		[6.36]	[92.72]				[1.18]	[3.63]		
All	All	1.42	92.72	1,192.7	344	154	0.64	2.88	94.5	300 [407]
		[1.51]	[92.72]				[0.90]	[4.59]		

Table 5.III: Comparison of CPLEX and Lagrangian based heuristics for the DFLPG: average and maximum optimality gap when compared to the best known lower bound.

q	Instance size	CPLEX (with SIs a priori) Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap	# ns	Lagrangian Heuristic Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap
3	50/50	0.29	1.14	650.6	17	0	0.51	1.54	13.9	7 [7]
	50/200	0.08	0.67	18.5	18	0	0.61	0.99	15.6	18 [18]
	100/100	0.11	0.55	41.2	18	0	0.44	0.84	27.3	17 [17]
	100/400	0.08	0.66	90.6	18	0	0.62	0.94	86.8	18 [18]
	150/150	0.04	0.23	102.4	18	0	0.52	0.99	62.4	18 [18]
	150/600	0.10	0.85	275.5	18	0	0.72	0.98	243.4	18 [18]
	200/200	0.13	0.92	198.0	18	0	0.56	0.95	137.8	16 [16]
	200/800	0.19	0.93	589.3	12	6	0.57	0.95	530.9	12 [18]
	250/250	0.06	0.34	496.3	18	0	0.56	0.95	219.1	18 [18]
	250/1000	0.15	0.73	791.0	6	12	0.67	0.96	596.7	6 [18]
	All	0.12	1.14	281.1	161	18	0.57	1.54	151.0	148 [166]
		[0.19]	[1.27]				[0.74]	[2.75]		
5	50/50	0.67	2.44	1,973.7	14	0	0.91	2.24	32.2	3 [3]
	50/200	0.26	0.69	86.4	18	0	0.54	1.04	46.9	16 [16]
	100/100	0.37	0.93	1,144.2	17	0	0.45	1.05	91.8	13 [13]
	100/400	0.11	0.89	203.6	18	0	0.61	0.97	136.9	18 [18]
	150/150	0.39	1.00	1,104.5	17	0	0.54	1.19	191.7	15 [15]
	150/600	0.09	0.85	413.8	12	6	0.69	0.95	280.6	12 [18]
	200/200	0.23	0.88	992.8	18	0	0.52	0.96	322.6	18 [18]
	200/800	0.16	0.60	868.0	6	12	0.36	0.94	597.0	6 [18]
	250/250	0.48	3.88	1,473.8	16	1	0.54	1.12	457.3	16 [17]
	250/1000	0.16	0.81	1,214.5	6	12	0.64	0.98	734.3	6 [18]
	All	0.32	3.88	950.4	142	31	0.59	2.24	227.7	123 [154]
		[0.47]	[3.95]				[0.89]	[3.70]		
10	50/50	4.27	19.79	5,779.7	4.00	1	2.32	3.78	977.8	0 [0]
	50/200	0.74	1.51	3,308.0	9	5	0.82	1.53	202.8	4 [4]
	100/100	7.84	66.50	5,449.2	4	5	1.67	3.19	376.4	0 [0]
	100/400	0.92	8.29	1,151.1	11	6	0.66	0.99	370.7	11 [15]
	150/150	17.03	88.13	5,172.8	4	7	1.26	2.40	331.3	1 [1]
	150/600	0.26	0.89	1,010.3	6	12	0.55	0.88	440.5	6 [15]
	200/200	23.45	89.67	4,217.1	4	10	1.09	1.73	506.6	3 [3]
	200/800	0.22	0.63	3,467.8	6	12	0.44	0.95	1,395.0	6 [17]
	250/250	1.02	2.91	3,064.0	3	14	0.52	0.80	664.8	4 [6]
	250/1000	-	-	-	0	18	-	-	-	- [16]
	All	6.41	89.67	3,951.9	51	90	1.23	3.78	555.2	35 [77]
		[7.08]	[100.00]				[1.53]	[4.50]		
All	All	1.61	89.67	1,353.7	354	139	0.72	3.78	270.2	306 [397]
		[1.84]	[100.00]				[1.05]	[4.50]		

Table 5.IV: Comparison of CPLEX and Lagrangian based heuristics for the DMCFLP_CR: average and maximum optimality gap when compared to the best known lower bound.

		CPLEX (with SIs a priori)					Lagrangian Heuristic			
q	Instance size	Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap	# ns	Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap
3	50/50	0.25	0.94	779.8	17	0	0.25	0.86	6.7	11 [11]
	50/200	0.01	0.10	20.5	18	0	0.59	0.97	6.9	18 [18]
	100/100	0.17	0.67	40.7	18	0	0.37	0.94	13.8	17 [17]
	100/400	0.04	0.37	89.0	18	0	0.65	0.99	40.7	18 [18]
	150/150	0.20	0.74	120.3	18	0	0.46	0.96	26.4	18 [18]
	150/600	0.01	0.17	263.9	18	0	0.67	0.95	105.0	18 [18]
	200/200	0.06	0.43	189.7	18	0	0.31	0.86	46.3	18 [18]
	200/800	0.11	0.76	466.3	12	6	0.75	0.96	190.3	12 [18]
	250/250	0.04	0.40	393.0	18	0	0.50	0.91	97.1	18 [18]
	250/1000	0.31	0.91	538.8	6	12	0.50	0.91	247.3	6 [18]
	All	0.11	0.94	265.3	161	18	0.50	0.99	61.4	154 [172]
		[0.20]	[1.39]				[0.68]	[2.11]		
5	50/50	0.56	1.60	2,521.3	15	0	0.85	1.79	34.7	3 [3]
	50/200	0.19	0.65	100.7	18	0	0.48	0.86	18.3	15 [15]
	100/100	0.44	1.83	1,376.8	15	0	0.58	1.35	35.1	10 [10]
	100/400	0.10	0.45	240.1	18	0	0.57	0.83	58.2	18 [18]
	150/150	0.43	1.44	1,286.9	16	0	0.50	1.21	91.9	13 [13]
	150/600	0.01	0.11	393.4	12	6	0.67	0.97	118.4	12 [18]
	200/200	0.38	2.58	1,061.8	16	1	0.53	0.98	113.6	16 [16]
	200/800	0.05	0.11	796.5	6	12	0.51	0.94	239.0	6 [18]
	250/250	0.17	0.63	1,211.4	16	2	0.51	0.91	174.9	16 [17]
	250/1000	0.32	0.82	1,076.8	6	12	0.54	0.94	328.2	6 [18]
	All	0.29	2.58	1,039.8	138	33	0.58	1.79	94.2	115 [146]
		[0.43]	[2.60]				[0.87]	[3.84]		
10	50/50	6.31	87.99	5,595.0	6.00	0	1.44	2.57	149.9	1 [1]
	50/200	0.83	5.76	2,029.6	15	1	0.64	1.19	125.5	9 [10]
	100/100	8.18	91.36	4,867.1	6	3	1.22	2.43	157.3	2 [2]
	100/400	0.23	0.63	1,012.0	11	7	0.51	0.86	125.1	11 [17]
	150/150	11.78	95.92	5,188.2	4	9	1.06	1.92	570.4	2 [2]
	150/600	0.09	0.31	1,139.2	6	12	0.44	0.67	154.3	6 [16]
	200/200	0.61	1.91	2,713.2	4	13	0.80	1.48	166.0	2 [3]
	200/800	3.20	6.33	3,544.0	1	16	0.02	0.04	507.0	2 [18]
	250/250	0.10	0.19	1,570.0	3	15	0.40	0.84	165.3	3 [6]
	250/1000	-	-	-	0	18	-	-	-	- [18]
	All	4.30	95.92	3,468.0	56	94	0.91	2.57	197.3	38 [93]
		[4.62]	[100.00]				[1.18]	[3.83]		
All	All	1.09	95.92	1,250.8	355	145	0.62	2.57	103.2	307 [411]
		[1.25]	[100.00]				[0.91]	[3.84]		

Table 5.V: Comparison of CPLEX and Lagrangian based heuristics for the DMCFLP_ER: average and maximum optimality gap when compared to the best known lower bound.

q	Instance size	CPLEX (with SIs a priori) Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap	# ns	Lagrangian Heuristic Avg Gap %	Max Gap %	Avg Time	# prov. 1% gap
3	50/50	0.23	1.20	738.9	17	0	0.48	1.40	13.9	7 [7]
	50/200	0.05	0.44	20.6	18	0	0.46	0.94	16.0	18 [18]
	100/100	0.15	0.73	70.4	18	0	0.38	0.93	32.5	17 [17]
	100/400	0.13	0.87	99.0	18	0	0.53	0.90	84.5	18 [18]
	150/150	0.09	0.39	127.3	18	0	0.49	0.94	58.3	18 [18]
	150/600	0.03	0.28	302.9	18	0	0.74	0.97	217.0	18 [18]
	200/200	0.13	0.79	248.5	18	0	0.52	0.96	126.4	18 [18]
	200/800	0.15	0.93	666.7	12	6	0.64	0.98	422.9	12 [18]
	250/250	0.11	0.83	453.1	18	0	0.41	0.87	201.0	18 [18]
	250/1000	0.04	0.12	812.3	6	12	0.69	0.95	428.2	6 [18]
	All	0.12	1.20	308.4	161	18	0.52	1.40	130.5	150 [168]
		[0.19]	[1.23]				[0.68]	[2.39]		
5	50/50	0.71	2.96	2,951.6	13	0	0.91	2.42	48.4	3 [3]
	50/200	0.23	0.81	187.3	18	0	0.45	0.82	41.4	14 [14]
	100/100	0.47	1.85	1,286.9	16	0	0.54	1.42	88.8	10 [10]
	100/400	0.12	0.63	273.7	18	0	0.55	0.95	129.6	18 [18]
	150/150	0.41	1.13	1,242.1	16	1	0.52	0.87	164.8	15 [15]
	150/600	0.10	0.89	475.3	12	6	0.67	0.96	242.3	12 [18]
	200/200	0.26	0.84	1,193.9	18	0	0.40	0.86	319.4	17 [17]
	200/800	0.17	0.89	1,202.3	6	12	0.68	0.97	599.5	6 [18]
	250/250	0.47	4.61	1,230.1	15	2	0.52	0.98	358.8	16 [17]
	250/1000	0.03	0.15	1,347.3	6	12	0.66	0.86	640.3	6 [18]
	All	0.33	4.61	1,142.0	138	33	0.57	2.42	205.3	117 [148]
		[0.47]	[4.61]				[0.87]	[4.00]		
10	50/50	8.06	87.78	6,168.2	3.00	0	2.14	3.44	1,632.8	0 [0]
	50/200	0.63	1.55	2,561.8	11	5	0.70	1.23	172.5	6 [7]
	100/100	15.65	94.16	5,781.8	3	4	1.72	2.59	776.1	0 [0]
	100/400	0.28	0.68	705.1	10	8	0.66	0.96	194.3	10 [14]
	150/150	0.98	1.82	4,655.6	4	11	1.02	1.66	289.4	1 [1]
	150/600	0.04	0.19	1,394.0	6	12	0.53	0.99	492.5	6 [15]
	200/200	19.75	96.06	4,567.4	3	13	0.97	1.69	384.0	2 [2]
	200/800	-	-	-	0	18	-	-	-	- [17]
	250/250	0.23	0.50	3,733.7	3	15	0.40	0.57	453.3	3 [6]
	250/1000	-	-	-	0	18	-	-	-	- [17]
	All	6.34	96.06	4,043.6	43	104	1.24	3.44	693.4	28 [79]
		[6.74]	[100.00]				[1.44]	[4.87]		
All	All	1.43	96.06	1,364.1	342	155	0.68	3.44	270.2	295 [395]
		[1.59]	[100.00]				[1.00]	[4.87]		

Table 5.VI: Comparison of CPLEX and Lagrangian based heuristics for the DMCFLP_CR_ER: average and maximum optimality gap when compared to the best known lower bound.

We finally would like to remark that, even though the computational results are reported using CPLEX v12.6, all experiments had previously been performed with v12.4. We observed a significant improvement of the computing times: solving the models with CPLEX was, on average, about 20% to 50% faster for each of the four problem variants. The improvement for the Lagrangian heuristics has been found to be, on average, between 35% and 80%. This may be due to the fact that the solver improved particularly for small problems, as it is the case for the restricted MIP used in the Lagrangian heuristics.

5.7 Conclusions and Future Research

In this work, we have extended the Dynamic Facility Location Problem with Generalized Modular Capacities by considering demands for multiple commodities. We addressed the solution of large-scale instances and proposed a heuristic based on two optimization phases. First, the Lagrangian dual is solved, involving the iterated solution of the Lagrangian subproblem. In this phase, feasible solutions of reasonable quality are found in very short computing times. Then, a restricted MIP is generated taking into consideration only decisions that have been found important during the solution of the Lagrangian dual. Using this approach, the final solution quality is consistently within 3.78% from the best known lower bound, even for instances for which CPLEX does not find feasible solutions due to the large memory and solution time required by the model.

The general cost structure of this problem allows for representing several existing facility location problems. In addition to the DFLPG, in which the capacity change costs are based on a cost matrix, this has been exemplified on three special cases. Given the strength of the GMC formulation, the Lagrangian heuristic was able to prove optimality within 1% for most of the small and medium sized instances. The proposed model and solution method may be applied to other problems, especially to those where the model size passes the limits of state-of-the-art MIP solvers. It may also be applied to larger instances than those addressed in this work, as the method consumes very little memory.

The Lagrangian dual has been solved by the classical subgradient method and a bundle implementation. Although the bundle method requires more time to compute the Lagrangian multipliers, it consistently outperformed the subgradient approach due to its strong convergence properties. On average, it required half of the time and resulted in a higher solution quality.

While local improvement heuristics such as tabu search have been common as a second phase optimization, the use of a restricted MIP is an interesting alternative, as general-purpose MIP solvers constantly improve. The implementation of a restricted MIP is very simple. Furthermore, one can handle any kind of problem structure that can be defined as a MIP. Even though one does not have to worry about finding the right trade-off between size and inspection time of a neighborhood, the question of how to significantly restrict the size of the original MIP is crucial. The bundle method with restricted MIP resulted in very competitive results, especially since the use of the convexified solutions already limits the decisions to those stored in the bundle. For the subgradient method, a well performing filtering approach based on the Lagrangian solutions may be designed, for example, by better tuning the maximum number of subgradient iterations and the parameter values for the restricted MIP.

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CHAPTER 6

LAGRANGIAN RELAXATION FOR DYNAMIC FACILITY LOCATION WITH RELOCATION AND PARTIAL FACILITY CLOSING

6.1 Introduction

This chapter elaborates on Lagrangian heuristics for the Camp Size and Location Problem. Chapter 3 introduced this problem and presented a MIP formulation. It has been shown that generic MIP solvers can solve problems of reasonable size. However, they fail to solve models for large instances. Furthermore, the model uses a simplified representation of the cost structure, not taking into consideration the current capacity level, but only the total capacity involved in the operation. The GMC model, presented in Chapter 4, addresses this issue and allows to represent problems that involve such cost structures. In Chapter 5, it has then been shown how the GMC model can be solved by heuristics based on Lagrangian relaxation.

In this chapter, we develop an alternative formulation for the CSLP that is based on the GMC modeling technique in order to obtain strong LP relaxation bounds and enable the representation of a more realistic cost structure. In particular, we extend the DFLPG to allow for partial facility closing and the relocation of facilities.

In the following, we discuss questions that arise when relocation is modeled in the context of facility location, in particular in the case of the CSLP. Then, a mathematical model for the CSLP with common capacity constraints is presented in Section 6.2. This model is based on the GMC formulation and allows for the partial closing and reopening of capacities, as well as for the relocation of facilities. Section 6.3 then illustrates how this problem can be decomposed via Lagrangian relaxation. It elaborates on two different relaxations, as well as on the combinatorial nature of the problem when round-up capacity constraints are used. Computational results are presented in Section 6.4, including an analysis of the integrality gaps of the problems, a computational comparison between the performance of the flow formulation presented in Chapter 3 and the GMC based formulation. The results for the different variants of Lagrangian heuristics indicate that these heuristics are a promising avenue to provide robust results, even for large scale instances. Finally, Section 6.5 concludes this chapter and discusses future research directions.

Preliminary Discussions Concerning Facility Relocation

When modeling relocation in facility location problems, several issues have to be discussed in the context of the actual application context. In the Camp Size and Location Problem, facilities have a particularly complex structure, which raises several questions. To ensure an unambiguous modeling of facility relocation, we establish the following assumptions, which are identical to those assumed for the CSLP in Chapter 3:

– Facilities can only be relocated as a whole, not partially.
– Before relocating a facility, its entire capacity has to be closed.
– After a facility has been relocated to another location, its capacity has to be re-opened before it is available.
– Each location can only hold a single facility. A facility cannot be relocated to a location with an already existing facility.
– In the same way, facilities cannot be merged.
– One may relocate a facility from a location a to another location and relocate a facility to location a at the same time period.
– One may relocate a facility from a location a to another location and construct a new facility at location a at the same time period.

6.2 GMC Based Mathematical Formulation

In this section, we develop a GMC based formulation for the CSLP, using common capacity constraints instead of round-up capacity constraints. This problem extends the DFLPG by including partial facility closing and relocation of facilities. It is denoted as the *Dynamic Facility Location Problem with Relocation and Partial Facility Closing (DFLP_RPC)*. The problem variant without relocation, a straight-forward simplification of the DFLP_RPC, will be denoted as the *Dynamic Facility Location Problem with Partial Facility Closing (DFLP_PC)*.

Input Data. We consider the same input data as used for the CSLP. We denote by J the set of potential facility locations and by $L = \{0, 1, 2, \ldots, q\}$ the set of possible capacity levels for each facility. We also denote by I the set of customer demand points and by $T = \{1, 2, \ldots, |T|\}$ the set of time periods in the planning horizon. We assume throughout that the beginning of period $t+1$ corresponds to the end of period t. The set of different commodities is denoted by P. The demand of customer i for commodity $p \in P$ in period t is denoted by d_{ipt}, while $g_{ij\ell pt}$ denotes the cost to produce one unit of commodity $p \in P$ at a facility of size $\ell \in L$ at location $j \in J$ and deliver it to customer $i \in I$ at time period $t \in T$. The capacity of a facility of size ℓ at location j is given by $u_{j\ell}$ (with $u_{j0} = 0$). Furthermore, we let J^0 be the set of locations that already possess facilities at the beginning of the planning horizon and ℓ^j be the capacity level of an existing facility at location j. The construction cost of a facility of size $\ell \in L$ at location $j \in J$ is denoted by $c^C_{j\ell}$. The costs c^{TO}_{ℓ} and c^{TC}_{ℓ} are the costs to reopen and close ℓ capacity levels of the same facility, respectively. The maintenance costs for a facility with ℓ open trailers during period t is given by $c^M_{\ell t}$.

Decision Variables and Aggregated Coefficients. The GMC formulation uses binary variables of type $y_{j\ell_1\ell_2 t}$ to represent a capacity change from level ℓ_1 to ℓ_2. In the DFLP_RPC, one needs to simultaneously manage capacity on two levels: the existing capacity and the open capacity. We therefore extend this modeling technique and use binary variables $y_{j\ell_1\ell_2 n_1 n_2 t}$ that are 1 if a facility at location j changes its level of existing capacity from n_1 to n_2 and its level of open capacity from level ℓ_1 to ℓ_2 at the beginning of time period t. Clearly, variables are defined only for $\ell_1 \leq n_1$ and $\ell_2 \leq n_2$. Binary variables $w^{BO}_{j\ell nt}$ indicate whether a facility of size n, open at capacity level ℓ, is closed and relocated from location j to another location before period t. Binary variables w^{BI}_{jnt}

indicate whether a facility of size n is relocated to location j before period t. The continuous variables $x_{ij\ell pt} \in [0,1]$ denote the fraction of the demand d_{ipt} satisfied by a facility at location j open at facility level ℓ.

A cost matrix $f_{j\ell_1\ell_2n_1n_2t}$ describes the aggregated costs to change the open capacity of a facility at location j from level ℓ_1 to ℓ_2 and the existing capacity from level n_1 to n_2 at the beginning of period t, as well as the costs to operate the facility at levels ℓ_2 and n_2 throughout time period t. The number of capacity levels constructed (nFC), the number of capacity levels reopened (nRE) and the number of capacity levels closed (nCL) that are represented by a decision variable $y_{j\ell_1\ell_2n_1n_2t}$ can be computed as:

$$
\begin{aligned}
nFC &= \max\{0,(n_2-n_1)\} \\
nRE &= \max\{0,(\ell_2-\ell_1)-nFC\} \\
nCL &= \max\{0,(\ell_1-\ell_2)+nFC\}
\end{aligned}
$$

The cost coefficients are then defined as:

$$
f_{j\ell_1\ell_2n_1n_2t} = c^M_{\ell_2t} + c^C_{(nFC)t} + c^{TO}_{(nRE)t} + c^{TC}_{(nCL)t}
$$

The costs to relocate a facility of size n are given by c^R_n. Based on these definitions, we define the following MIP model, referred to as the *Generalized Modular Capacities* formulation for the DFLP_RPC (RPC-GMC):

(RPC-GMC)

$$\min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{n_1\in L}\sum_{n_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2n_1n_2t}\, y_{j\ell_1\ell_2n_1n_2t} + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T} g_{ij\ell pt} d_{ipt} x_{ij\ell pt}$$

$$+ \sum_{j\in J}\sum_{\ell\in L}\sum_{n\in L}\sum_{t\in T}(c^{TC}_{\ell} + \frac{c^R_n}{2}) w^{BO}_{j\ell nt} + \sum_{j\in J}\sum_{n\in L}\sum_{t\in T}\frac{c^R_n}{2} w^{BI}_{jnt} \tag{6.1}$$

$$s.t.\ \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell pt} = 1 \ \ \forall i\in I,\ \forall p\in P,\ \forall t\in T \tag{6.2}$$

$$\sum_{i\in I}\sum_{p\in P} d_{ipt} x_{ij\ell pt} \le \sum_{\ell_1\in L}\sum_{n_1\in L}\sum_{n_2\in L} u_{j\ell} y_{j\ell_1\ell n_1n_2t} \ \ \forall j\in J,\ \forall \ell\in L,\ \forall t\in T \tag{6.3}$$

$$\sum_{\ell_1\in L}\sum_{n_1\in L} y_{j\ell_1\ell n_1n(t-1)} = \sum_{\ell_2\in L}\sum_{n_2\in L} y_{j\ell\ell_2nn_2t} + w^{BO}_{j\ell nt}$$

$$\forall j\in J,\ \forall \ell\in L\backslash\{0\},\ \forall n\in L\backslash\{0\},\ \forall t\in T\backslash\{1\} \tag{6.4}$$

$$\sum_{\ell_1\in L}\sum_{n_1\in L} y_{j\ell_10n_1n(t-1)} + w^{BI}_{jnt} = \sum_{\ell_2\in L}\sum_{n_2\in L} y_{j0\ell_2nn_2t} + w^{BO}_{j0nt} \ \ \forall j\in J,\ \forall n\in L\backslash\{0\},\ \forall t\in T\backslash\{1\} \tag{6.5}$$

$$\sum_{\ell_2\in L}\sum_{n_2\in L} y_{j\ell^j\ell_2(n_1=\ell^j)n_21} = 1 \ \ \forall j\in J^0 \tag{6.6}$$

$$\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{n_1\in L}\sum_{n_2\in L} y_{j\ell_1\ell_2n_1n_2t} \le 1 \ \ \forall j\in J,\ \forall t\in T \tag{6.7}$$

$$\sum_{j\in J}\sum_{\ell\in L} w^{BO}_{j\ell nt} = \sum_{j\in J} w^{BI}_{jnt} \ \ \forall n\in L,\ \forall t\in T \tag{6.8}$$

$$x_{ij\ell pt} \ge 0 \ \ \forall i\in I,\ \forall j\in J,\ \forall \ell\in L,\ \forall p\in P,\ \forall t\in T \tag{6.9}$$

$$y_{j\ell_1\ell_2n_1n_2t} \in \{0,1\} \ \ \forall i\in I,\ \forall j\in J,\ \forall \ell_1\in L,\ \forall \ell_2\in L$$

$$,\ \forall n_1\in L^{\ge \ell_1},\ \forall n_2\in L^{\ge \ell_2},\ \forall t\in T \tag{6.10}$$

$$w^{BO}_{j\ell nt} \in \{0,1\},\ \forall j\in J,\ \forall \ell\in L,\ \forall n\in L,\ \forall t\in T \tag{6.11}$$

$$w^{BI}_{jnt} \in \{0,1\},\ \forall j\in J,\ \forall n\in L,\ \forall t\in T. \tag{6.12}$$

The objective function (6.1) minimizes the total costs for changing the capacity levels and allocating the demand. Note that the relocation costs c^R_n are equally split on both variables w^{BO} and w^{BI}. This way, we intend to better use both variables within the Lagrangian relaxation in which the relocation linking constraints are relaxed. Constraints (6.2) are the demand constraints for the customers. Constraints (6.3) are the capacity constraints at the facilities. Constraints (6.4) and (6.5) are the flow conservation constraints that link the capacity change variables in consecutive time periods. Constraints (6.6) are the flow initialization constraints that specify that exactly one capacity level is chosen at the beginning of the planning horizon. Constraints (6.7) guarantee that exactly one capacity change variable is selected at each time period. Finally, constraints (6.8) are the relocation linking constraints, ensuring that all facilities of size ℓ that have been removed from a location are located at another location at the same time period.

The DFLP_PC is modeled in the same manner, without the relocation linking constraints (6.8) and relocation variables as defined by (6.11) and (6.12). Furthermore, equalities (6.6) have to be defined for all $j \in J$, and contain all possible capacity levels on the left-hand side of the equation. Given that these constraints in combination with the flow conservation constraints guarantee that a at most one y variable is selected at each location and time period, constraints (6.7) become redundant. This formulation is referred to as the *PC-GMC* formulation. The network structure used in both problems, represented by the flow conservation constraints, is illustrated in Figure 6.1 for a small example with four time periods and two capacity levels. Each node represents the number of open and existing capacity (*open capacity level / existing capacity level*). The binary capacity change variables are represented by arcs, which allow for the construction of new capacity, as well as the closing and reopening of open capacity.

Valid Inequalities. We also adapt the Strong Inequalities and the Aggregated Demand Constraints to the PC-GMC and RPC-GMC formulations:

$$x_{ij\ell pt} \leq \sum_{\ell_1 \in L} \sum_{n_1 \in L} \sum_{n_2 \in L} y_{j\ell_1 \ell n_1 n_2 t} \quad \forall i \in I,\ \forall j \in J,\ \forall \ell \in L,\ \forall p \in P,\ \forall t \in T \tag{6.13}$$

$$\sum_{i \in I} \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \sum_{n_1 \in L} \sum_{n_2 \in L} u_{j\ell_2} y_{j\ell_1 \ell_2 n_1 n_2 t} \geq \sum_{j \in J} \sum_{p \in P} d_{ipt} \quad \forall t \in T. \tag{6.14}$$

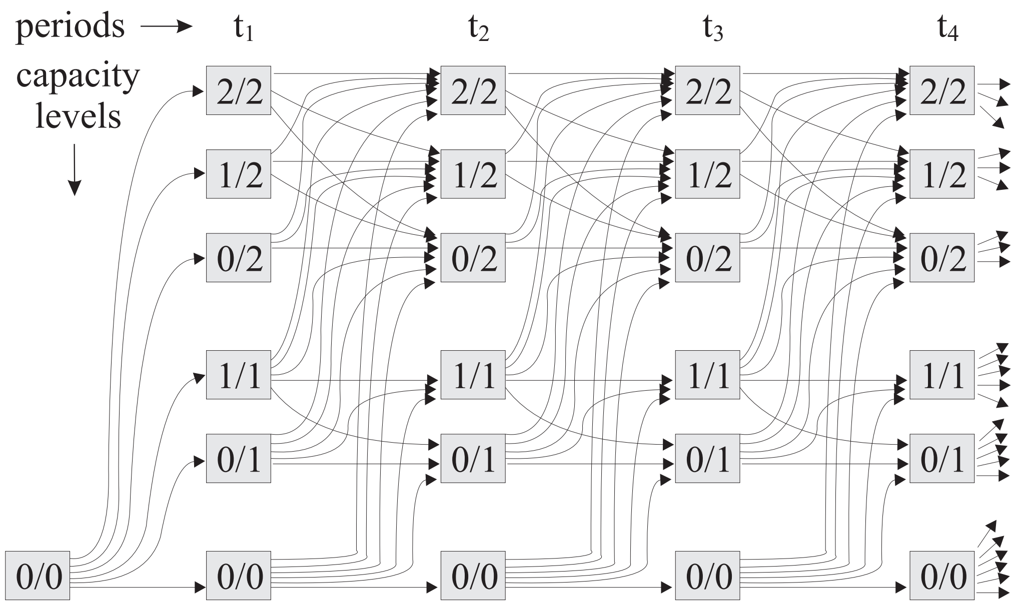

Figure 6.1: Network model to manage partial facility closing and reopening used in the PC-GMC and RPC-GMC models. Each node indicates the level of open and existing capacity.

6.3 Lagrangian Heuristics

We now discuss how to apply Lagrangian relaxation to the GMC based formulation with and without relocation. It has been shown in Chapter 5 that, when relaxing the demand constraints in the DFLPG, the Lagrangian subproblem can be decomposed into independent subproblems, one for each candidate facility location. These independent subproblems can then be efficiently solved by dynamic programming.

When applying Lagrangian relaxation to the DFLPG_PC, relaxing the demand constraints also results in independent subproblems, which can be solved with a similar algorithm. When the relocation of facilities is allowed, the relocation linking constraints (6.8) are an additional link between the candidate facility locations. Therefore, relaxing the demand constraints in the DFLPG_RPC is not sufficient to decompose the Lagrangian subproblem into independent problems. We have two possibilities to apply Lagrangian relaxation to the problem. We can relax both the demand constraints (6.2) and the relocation linking constraints (6.8) in order to obtain a subproblem that can be decomposed by location. This approach is explained in Section 6.3.1. Alternatively, we relax only the demand constraints (6.2). The remaining Lagrangian subproblem then still includes the relocation linking constraints and, therefore, cannot be decomposed by location. Instead, the subproblem may be transformed into an IP and be solved by a generic MIP solver. This approach is discussed in Section 6.3.2.

6.3.1 Relaxation of Demand and Relocation Linking Constraints

We relax both the demand constraints (6.2) and the relocation linking constraints (6.8). Let α be the vector of Lagrange multipliers associated to the relaxed demand constraints and β the one associated to the relaxed relocation linking constraints. This results in the Lagrangian subproblem, which, after rearranging the terms in the objective function, can be stated as follows:

$$
\begin{aligned}
L(\alpha,\beta) = \quad & \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{n_1\in L}\sum_{n_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2n_1n_2t}\, y_{j\ell_1\ell_2n_1n_2t} \\
& + \sum_{j\in J}\sum_{\ell\in L}\sum_{n\in L}\sum_{t\in T}\Big(c_\ell^{TC} + \frac{c_n^R}{2} - \beta_{nt}\Big) w_{j\ell nt}^{BO} + \sum_{j\in J}\sum_{n\in L}\sum_{t\in T}\Big(\frac{c_n^R}{2} + \beta_{nt}\Big) w_{jnt}^{BI} \\
& + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T}\big(g_{ij\ell pt} d_{ipt} - \alpha_{ipt}\big) x_{ij\ell pt} \\
& + \sum_{i\in I}\sum_{p\in P}\sum_{t\in T}\alpha_{ipt} \\
& s.t.\ (6.3)-(6.7), (6.9)-(6.12).
\end{aligned}
$$

The Lagrangian dual problem is then solved to obtain the optimal Lagrange multipliers. If solved to optimality, the lower bound provided is equivalent to the LP relaxation bound:

$$\max_{\alpha,\beta} L(\alpha,\beta)$$

The subgradient direction is composed of the two vectors γ_{ipt} and μ_{nt}, which represent the subgradients for the relaxed demand and relocation linking constraints, respectively. At the $k-th$ iteration, they are computed as the derivative of the relaxed constraints in α and β, respectively, with variables x, w^{BO} and w^{BI} fixed to the values found in the Lagrangian subproblem:

$$\gamma_{ipt}^{k} = 1 - \sum_{j \in J} \sum_{\ell \in L} x_{ij\ell pt} \quad \forall i \in I,\ \forall p \in P,\ \forall t \in T$$
$$\mu_{nt}^{k} = \sum_{j \in J} w_{jnt}^{BI} - \sum_{j \in J} \sum_{\ell \in L} w_{j\ell nt}^{BO} \quad \forall n \in L \setminus \{0\},\ \forall t \in T.$$

We refer to Section 5.4 for more explanations on the subgradient and bundle methods.

6.3.1.1 Solution of the Lagrangian Subproblem

Let $\tilde{c}_{ij\ell pt} = g_{ij\ell pt} d_{ipt} - \alpha_{ipt}$ denote the modified variable coefficients for the x variables. We separate the Lagrangian subproblem into $|J|$ independent subproblems, one for each potential facility location for a fixed set of Lagrange multipliers α. The Lagrangian subproblem is then defined as $L(\alpha,\beta) = \sum_{j \in J} L_j(\alpha,\beta) + \sum_{i \in I} \sum_{p \in P} \sum_{t \in T} \alpha_{ipt}$, where $L_j(\alpha,\beta)$ is defined as follows:

$$L_j(\alpha,\beta) =$$

$$\min \sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{n_1\in L}\sum_{n_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2n_1n_2t}y_{j\ell_1\ell_2n_1n_2t}$$

$$+\sum_{\ell\in L}\sum_{n\in L}\sum_{t\in T}(c_\ell^{TC}+\frac{c_n^R}{2}-\beta_{nt})w_{j\ell nt}^{BO}+\sum_{n\in L}\sum_{t\in T}(\frac{c_n^R}{2}+\beta_{nt})w_{jnt}^{BI}$$

$$+\sum_{i\in I}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T}\tilde{c}_{ij\ell pt}x_{ij\ell pt}+\sum_{i\in I}\sum_{p\in P}\sum_{t\in T}\alpha_{ipt}$$

$$s.t. \sum_{i\in I}\sum_{p\in P} d_{ipt}x_{ij\ell pt} \le \sum_{\ell_1\in L}\sum_{n_1\in L}\sum_{n_2\in L} u_{j\ell}y_{j\ell_1\ell n_1n_2t} \quad \forall \ell\in L,\ \forall t\in T \tag{6.15}$$

$$\sum_{\ell_1\in L}\sum_{n_1\in L} y_{j\ell_1\ell n_1n(t-1)} = \sum_{\ell_2\in L}\sum_{n_2\in L} y_{j\ell\ell_2nn_2t}+w_{j\ell nt}^{BO} \quad \forall \ell\in L\backslash\{0\},\ \forall n\in L,\ \forall t\in T\backslash\{1\} \tag{6.16}$$

$$\sum_{\ell_1\in L}\sum_{n_1\in L} y_{j\ell_1 0n_1n(t-1)}+w_{jnt}^{BI} = \sum_{\ell_2\in L}\sum_{n_2\in L} y_{j0\ell_2nn_2t}+w_{j0nt}^{BO} \quad \forall n\in L,\ \forall t\in T\backslash\{1\} \tag{6.17}$$

$$\sum_{\ell_2\in L}\sum_{n_2\in L} y_{j\ell^j\ell_2(n_1=\ell^j)n_21} = 1 \quad ,\forall j\in J^0 \tag{6.18}$$

$$\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{n_1\in L}\sum_{n_2\in L} y_{j\ell_1\ell_2n_1n_2t} \le 1 \quad \forall t\in T \tag{6.19}$$

$$x_{ij\ell pt} \ge 0 \quad \forall i\in I,\ \forall \ell\in L,\ \forall p\in P,\ \forall t\in T \tag{6.20}$$

$$y_{j\ell_1\ell_2n_1n_2t}\in\{0,1\} \quad \forall i\in I,\ \forall \ell_1\in L,\ \forall \ell_2\in L,\ \forall n_1\in L,\ \forall n_2\in L,\ \forall t\in T \tag{6.21}$$

$$w_{j\ell nt}^{BO}\in\{0,1\} \quad \forall \ell\in L,\ \forall n\in L,\ \forall t\in T \tag{6.22}$$

$$w_{jnt}^{BI}\in\{0,1\} \quad \forall n\in L,\ \forall t\in T. \tag{6.23}$$

The dynamic programming algorithm to solve this subproblem extends the one presented in Chapter 5. Let $O_j^{\alpha,\beta}(\ell,n,t)$ denote the value of the optimal opening schedule from time period 1 to t, including the costs to satisfy the customer demand during these time periods and assuming that a facility of size n, open at capacity level ℓ is available at the end of time period t. To compute these values, one needs to evaluate the following four cases, where an "IN relocation" represents the case in which a facility is relocated to the location and an "OUT relocation" represents the case in which a facility is relocated from the location:

i. No IN relocation, no OUT relocation. In this standard case, the cheapest capacity level is chosen, including the costs to satisfy demand until period $t-1$ and the costs for the capacity transition:

$$\hat{C}_1(\ell,n,t) = \min_{0\leq n_1\leq n, 0\leq \ell_1\leq n_1} \{f_{j\ell_1\ell n_1 nt} + O_j^{\alpha,\beta}(\ell_1,n_1,t-1)\}$$

ii. IN relocation, no OUT relocation. The location did not have a facility at period $t-1$ (since no facility has been relocated from the location to another, and merging of facilities is not allowed). A facility of size n_1 has been relocated to location j and possibly expanded by additional capacity, resulting in a final level of existing capacity n. Furthermore, some unused capacity may have been closed, resulting in a final level of open capacity ℓ:

$$\hat{C}_{RelocIN}(\ell,n,t) = \min_{1\leq n_1\leq n} \{\frac{c^R_{n_1}}{2} + c^{TO}_{(nRE)t} + c^{TC}_{(nCL)t} + \beta_{n_1 t} + c^c_{j(n-n_1)} | \\ nRE = \max\{0,\ell-n+n_1\}, nCL = \max\{0,n-n_1-\ell\}\}$$

$$\hat{C}_2(\ell,n,t) = \hat{C}_{RelocIN}(\ell,n,t) + c^M_{\ell t} + O_j^{\alpha,\beta}(0,0,t-1)$$

iii. No IN relocation, OUT relocation. A facility of unknown size has been relocated to another location. Since we know that currently a $<\ell,n>$ facility is available, it must have been constructed after the relocation of the outgoing facility:

$$\hat{C}_{RelocOUT} = \min_{1\leq n_1\leq q, 0\leq \ell_1\leq n_1} \{c^{TC}_{\ell_1 t} + \frac{c^R_{n_1}}{2} - \beta_{n_1 t} + O_j^{\alpha,\beta}(\ell_1,n_1,t-1)\}$$

$$\hat{C}_3(\ell,n,t) = \hat{C}_{RelocOUT} + f_{j0\ell 0nt}$$

iv. IN relocation, OUT relocation. A facility of unknown size has been relocated to another location. Furthermore, a facility has been located to the current location and/or capacity has been constructed such that now a facility of size n is available:

$$\hat{C}_4(\ell,n,t) = \hat{C}_{RelocIN}(\ell,n,t) + \hat{C}_{RelocOUT} + c^M_{\ell t}$$

The optimal value for $O^{\alpha}_j(\ell,n,t)$ is then computed as follows:

$$O^{\alpha,\beta}_j(\ell,n,t) = L^{\alpha}_j(\ell,n,t) + \min\{\hat{C}_1(\ell,n,t), \hat{C}_2(\ell,n,t), \hat{C}_3(\ell,n,t), \hat{C}_4(\ell,n,t)\}$$

If we assume that the location does not have existing facilities at the beginning of the planning horizon, facility relocation is not possible at period 1 and the initial capacity level is 0:

$$O^{\alpha}_j(\ell,n,0) = L^{\alpha}_j(\ell,n,0) + f_{j(\ell_1=0)\ell(n_1=0)nt}$$

We solve the subproblem for location j by selecting the minimum among all possible facility sizes:

$$L_j(\alpha,\beta) = \min_{0\leq\ell\leq q, \ell\leq n\leq q} \left\{O^{\alpha}_j(\ell,n,|T|)\right\}$$

For each cell $< j,\ell,n,t >$, its predecessors $< j,\ell',n',t-1 >$ of the optimal opening schedule are stored. In addition, the following information are stored, indicating whether relocations are part of the optimal schedule:

– A flag (yes/no), indicating whether a facility of size n has been relocated to this location.
– The size and number of open capacity levels of a facility that has been relocated from the current to another location.

The solution provided by this algorithm contains, for each time period, one decision for facility closing, facility opening and facility maintenance (each indicating the size ℓ of the corresponding facility).

6.3.1.2 Generation of Upper Bounds

Feasible solutions for the DFLP_PC and the DFLP_RPC are generated in a similar fashion as for the DFLPG (see Section 5.4.3). First, redundant demand is reduced as it is the case for the DFLPG. If, at a certain time period, capacity is missing, we increase the open capacity level ℓ of the facilities until the total demand can be met or the existing capacity level n is reached. Note that facilities are considered without a specific order. If the total demand can still not be covered by the available (i.e., open) capacity, we consider locations without facilities and increase both ℓ and n until the missing capacity is covered or the maximum capacity level for this facility is reached. Missing demand allocation is then increased as it has been shown for the DFLPG. Afterwards, unused capacities are closed using a dynamic programming algorithm similar to the one shown for the solution of the Lagrangian subproblem. In a final step, the optimal demand allocation for the current opening schedule is computed using the CPLEX network algorithm.

For the DFLP_RPC, the relocation decisions provided by the Lagrangian solution can be used to generate a feasible solution. This requires to match the $w^{BO}_{j\ell nt}$ and w^{BI}_{jnt} variables. However, experiments showed that these pairs rarely match in the Lagrangian solutions, i.e., the number of selected $w^{BO}_{j\ell nt}$ variables does not necessarily equal the number of selected w^{BI}_{jnt} variables for each facility size n and time period t. We therefore apply the following procedure to select feasible pairs of outgoing and incoming relocations. For each pair of facility size n and time period t, we collect the maximum number of facility matches j' and j'' (with $j' \neq j''$) such that $w^{BO}_{j'\ell nt}$ and $w^{BI}_{j''nt}$ are part of the Lagrangian solution. To find those pairs, facilities are considered without a specific order. The procedure excludes configurations that may lead to an infeasible opening schedule at each of the facility locations. To be precise, we ensure that no outgoing relocation of a facility is smaller than a previous incoming relocation and that multiple incoming relocations at the same location are separated by outgoing relocation.

For locations for which no relocation pairs have been selected, we use the Lagrangian solution for the problem variant without relocation.

6.3.2 Relaxation of the Demand Constraints

When relaxing only the demand constraints, the relocation linking constraints still provide a link between the different facility locations. To the best of our knowledge, this problem cannot be decomposed by location. A feasible solution can be obtained by solving a MIP model in which all demand allocation variables are eliminated. This is explained next.

The Lagrangian subproblem is transformed into an integer problem as follows. The demand allocation variables x can be eliminated by observing that the optimal solution values for x variables linked to a facility opening decision (j, ℓ, t) can be easily obtained by solving a fractional knapsack, taking into consideration the modified cost vector $\tilde{c}$ as OF coefficients for the x variables and assuming that facility j has an open capacity of level ℓ at time period t. The procedure is equivalent to the one outlined in Section 5.4.1. The solution value of the knapsack is then added to the OF coefficient of the corresponding facility opening decisions, i.e., all $y_{j\ell_1\ell_2n_1n_2t}$ variables with $\ell_2 = \ell$ and $n_2 = n$. The resulting problem is a pure IP, consisting only of the $y_{j\ell_1\ell_2n_1n_2t}$ binary variables. To ensure feasibility in the original problem, we also add the Aggregated Demand Constraints (ADC). The problem is then solved by a MIP solver.

6.3.2.1 Generation of Upper Bounds

If the ADCs are added to the problem formulation, the solution obtained by the Lagrangian subproblem is feasible for the DFLPG_RPC. The Lagrangian solution provides an opening schedule for the facilities, as well as their relocation decisions. Given that the demand allocation in this solution has been computed based on demand allocation costs that have been modified by the Lagrange multipliers α, we first remove the current demand allocation suggested by the Lagrangian solution and then compute the optimal demand allocation for the given opening schedule based on the original allocation costs.

6.3.3 Using Round-up Capacity Constraints

It has been shown how Lagrangian relaxation can be applied to dynamic facility location problems that involve partial facility closing and reopening, as well as facility relocation. We now address the last particularity of the CSLP, namely the round-up capacity constraints (see Section 3.5.2). These constraints assume that demand can be satisfied only in batches, i.e., facilities produce entire batches of products, even when the requested quantity is less (and fractional). We denote by nS_p the number of units within a produced batch of commodity type p. We introduce new integer variables $z_{j\ell pt} \in \mathbb{Z}^+$, which represent the total number of batches of product type p assigned to a size ℓ facility at location $j \in J$. The DFLP_RPC with RUC constraints is then given by (6.1), (6.2), (6.4) - (6.12), with the following RUC constraints that substitute the former capacity constraints (6.3):

$$\sum_{i \in I} d_{ipt} x_{ij\ell pt} \leq z_{j\ell pt} \quad \forall j \in J,\ \forall \ell \in L,\ \forall p \in P,\ \forall t \in T \tag{6.24}$$

$$\sum_{p \in P} nS_p z_{j\ell pt} \leq \sum_{\ell_1 \in L} \sum_{n_1 \in L} \sum_{n_2 \in L} u_{j\ell} y_{j\ell_1 \ell n_1 n_2 t} \quad \forall j \in J,\ \forall \ell \in L,\ \forall t \in T \tag{6.25}$$

$$z_{j\ell pt} \in \mathbb{Z}_0^+ \quad \forall j \in J,\ \forall \ell \in L,\ \forall p \in P,\ \forall t \in T. \tag{6.26}$$

The Lagrangian Subproblem

Relaxing the demand and relocation linking constraints, the Lagrangian subroblem is as follows:

$$\begin{aligned} L(\alpha,\beta) = \quad & \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{n_1\in L}\sum_{n_1\in L}\sum_{t\in T} f_{j\ell_1\ell_2 n_1 n_2 t}\, y_{j\ell_1\ell_2 n_1 n_2 t} \\ & + \sum_{j\in J}\sum_{\ell\in L}\sum_{n\in L}\sum_{t\in T}\left(c_\ell^{TC} + \frac{c_n^R}{2} - \beta_{nt}\right) w_{i\ell nt}^{BO} + \sum_{j\in J}\sum_{n\in L}\sum_{t\in T}\left(\frac{c_n^R}{2} + \beta_{nt}\right) w_{int}^{BI} \\ & + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{p\in P}\sum_{t\in T}\left(g_{ij\ell pt} d_{ipt} - \alpha_{ipt}\right) x_{ij\ell pt} \\ & + \sum_{i\in I}\sum_{p\in P}\sum_{t\in T} \alpha_{ipt} \\ & s.t.\ (6.4)-(6.12), (6.24)-(6.26). \end{aligned}$$

Note that, using the RUC constraints, the Lagrangian subproblem does not have the integrality property. The lower bound provided by the Lagrangian subproblem may thus be better than the bound provided by the LP relaxation of the original problem.

Solution of the Lagrangian Subproblem

As has been shown for the case with common capacity constraints, the Lagrangian subproblem can be decomposed into $|J|$ independent problems, one for each candidate facility location. These problems are equivalent to finding the optimal opening schedule for each facility location. With common capacity constraints, the optimal demand allocation for each location, time period and capacity level can be found by solving a fractional knapsack problem. When using RUC constraints, the optimal demand allocation is obtained by solving problems of the following type, with j, ℓ and t fixed:

$$\begin{aligned}
\min & \sum_{i\in I}\sum_{p\in P}\tilde{c}_{ij\ell pt}x_{ij\ell pt} \\
s.t. & \sum_{i\in I} d_{ipt}x_{ij\ell pt} \leq z_{j\ell pt} \quad \forall p\in P \\
& \sum_{p\in P} nS_p z_{j\ell pt} \leq u_{j\ell} \\
& x_{ij\ell pt} \leq 1 \quad \forall i\in I,\ \forall p\in P \\
& x_{ij\ell pt} \in \mathbb{R}^+ \quad \forall i\in I,\ \forall p\in P \\
& z_{j\ell pt} \in \mathbb{Z}_0^+ \quad \forall p\in P.
\end{aligned}$$

The problem contains two encapsulated knapsack problems: for each p, an integer value for $z_{j\ell pt}$ has to be selected such that the total capacity is respected and the costs are minimal. However, the cost for choosing a certain integer value is not linear, but depends on the choice of x. For each integer value of $z_{j\ell pt}$, a continuous knapsack has to be solved to choose a number of x variables such that the total costs are minimal.

The steps to solve this problem are as follows:

i. In a first step, we identify all feasible integer values for $z_{j\ell pt}$. These values are bounded by $\frac{u_{j\ell}}{nS_p}$. For each p, we then have a number of integer values that $z_{j\ell pt}$ may take, represented by set $\Omega_p = \left\{0,1,2,...,\left\lfloor\frac{u_{j\ell}}{nS_p}\right\rfloor\right\}$.

ii. The costs for each of the integer values of $z_{j\ell pt}$ are computed by solving a continuous knapsack problem, just as performed in the Lagrangian Relaxation for the version with common capacity constraints. Thus, for each $p\in P$, we solve such a continuous knapsack. The capacity of the knapsack is given by the integer value of $z_{j\ell pt}$, which we denote by $cap_{j\ell pt}$. Note that the weight of each object is given by d_{ipt}. Thus, for each $p\in P$, we solve a continuous knapsack for each $cap_{j\ell pt}\in\Omega_p$ as follows:

We choose the x variables with the lowest ratio between its coefficient within the objective function and the coefficient within the constraint, i.e., $\frac{\tilde{c}_{ij\ell pt}}{d_{ipt}}$. To be precise, we choose the $x_{i^*j\ell p^*t}$ variable that has the most negative ratio $\frac{\tilde{c}_{ij\ell pt}}{d_{ipt}}$ and has not yet been chosen:

$$i^* = \min_{i \in I} \left\{ i : \frac{\tilde{c}_{ij\ell pt}}{d_{ipt}} \right\}$$

Let $cap'_{j\ell pt}$ be the remaining capacity. We then increase the value of the x variable to the maximum value possible, i.e., either until the entire demand is met or the capacity limit is reached:

$$x_{i^* j\ell pt} = \min \left\{ 1.0, \frac{cap'_{j\ell pt}}{d_{i^* pt}} \right\}$$

All other x variables are set to 0. Clearly, only x variables with negative ratios are considered.

iii. The cost coefficients for each $z_{j\ell pt}$ are given by the solution of the previously solved continuous knapsack for $z_{j\ell pt}$. In the following knapsack problem, each integer value for a $z_{j\ell pt}$ variable represents an object. The weight of object $z_{j\ell pt}$ is given by $nS_p z_{j\ell pt}$. The total capacity is given by $u_{j\ell}$. In this type of integer knapsack, we need to choose exactly one object for each $p \in P$. This knapsack variant is also known as the *Multiple-Choice Knapsack Problem (MCKP)*, presented by Martello and Toth (1990), and is solved by a Dynamic Programming algorithm.

Note that, in step (ii), solving the series of continuous knapsacks, one for each value of $z_{j\ell pt}$, for a given p can be performed efficiently by noting that the optimal solution for the continuous knapsack of a capacity z' is necessarily part of an optimal solution for any capacity higher than z'.

Upper Bound Generation

Generating upper bounds for the problem variant with RUC constraints is similar to the one without RUC constraints. However, the integer-rounding needs to be taken into account in several situations. When increasing capacities, the capacity for each product is given by the value of the current $z_{j\ell pt}$ value. Furthermore, the integer-rounding need to be considered when closing unused capacities.

6.3.4 Restricted MIP model

We may use a restricted MIP similar to the one introduced in Section 5.5 to improve the final solution quality. For the DFLPG, we consider a restricted MIP based on the convexified solutions provided by the bundle method (see Section 5.5.2). In contrast to the restricted MIP model for the DFLPG, the restricted MIP models for the DFLP_PC and DFLP_RPC need to decide for the level of open and existing capacity levels for each location and time period. We therefore define the restricted MIP in terms of capacity level pairs (ℓ, n) instead of a capacity level ℓ only.

As explained in Section 5.4.2, the bundle method provides a multiplier θ^s for each Lagrangian solution s such that $\sum_s \theta^s = 1$. The value θ^s can be seen as a probability that solution s provides a good opening schedule. We may therefore derive probabilities for each of the opening decisions $\tilde{y}_{j\ell nt} = \sum_s \theta^s y^s_{j\ell nt}$, where $y^s_{j\ell nt}$ is 1 if solution s selects capacity level pair (ℓ, n) for location j at period t. Furthermore, let L^R_{jt} be the set of (ℓ, n) pairs for location j and period t available in the restricted MIP. The restricted MIP is then defined as follows:

- **Decision fixing.** For each j and t, a decision is fixed to capacity level pair (ℓ, n) if $\tilde{y}_{j\ell nt} \geq pFix$, where $pFix \in]0.5, 1]$.
- **Selection of available capacity levels.** If the decision for location j and period t is not fixed, L^R_{jt} is composed by the n^S capacity level pairs (ℓ, n) that have the highest $\tilde{y}_{j\ell nt}$ values, with $\tilde{y}_{j\ell nt} > 0.001$.
- **Defining the set of capacity transitions.** Decisions $y_{j\ell_1\ell_2 n_1 n_2 t}$ are defined for all combinations between (ℓ_1, n_1) and (ℓ_2, n_2), with $(\ell_1, n_1) \in L^R_{jt}$ and $(\ell_2, n_2) \in L^R_{j(t+1)}$, if available in the original PC-GMC or RPC-GMC formulation.

For the DFLP_RPC, we additionally consider relocation decisions in the restricted MIP. Probabilities for outgoing and incoming relocations $\tilde{w}^{BO}_{j\ell nt}$ and $\tilde{w}^{BI}_{jnt}$, respectively, can be computed as follows. We set $\tilde{w}^{BO}_{j\ell nt} = \sum_s \theta^s w^{BOs}_{j\ell nt}$, where $w^{BOs}_{j\ell nt}$ is 1 if solution s relocates a facility of size n open at level ℓ from location j to another location at period t. In the same way, we set $\tilde{w}^{BI}_{jnt} = \sum_s \theta^s w^{BIs}_{jnt}$, where w^{BIs}_{jnt} is 1 if solution s relocates a facility of size n open at level ℓ from location j to another location at period t. All relocation variables with their corresponding $\tilde{w}^{BOs}_{j\ell nt}$ and $\tilde{w}^{BI}_{jnt}$ greater or equal to 0.001 are added to the restricted model. To ensure their feasibility with respect to the flow conservation constraints, certain capacity levels are added to the sets of available capacity levels L^R. To be precise, when adding a relocation decision w^{BI}_{jnt} to the restricted MIP, capacity level pair $(0,n)$ is added to $L^R_{j(t-1)}$ and capacity level pairs $(0,n)$ and (n,n) are added to L^R_{jt} to ensure that the flow conservation constraints contain the capacity transition variables $y_{j\ell_1\ell_2 n_1 n_2 t}$ that either maintain the facility closed or reopen it at its maximum capacity level n.

6.4 Computational Results

This section focuses on the computational results concerning the models and Lagrangian based heuristics for the DFLP_PC and DFLP_RPC. Section 6.2 introduced the GMC based models for both problem variants, referred to as the PC-GMC and RPC-GMC formulations, respectively. Note that these problem variants do not involve the round-up capacity constraints.

A flow formulation for the CSLP has been introduced in Section 3.5. As the CSLP corresponds to the DFLP_RPC, but with round-up capacity constraints (see Section 3.5.2), one obtains similar flow formulations for the DFLP_PC and the DFLP_RPC. We denote the corresponding formulation for the DFLP_RPC as the RPC-2i formulation, which is defined by (3.12) – (3.13), (3.16) – (3.36) and the common capacity constraints, which replace the round-up capacity constraints (3.14) – (3.15) and are defined as follows (note the inverse notation of sets I and J, as well as the use of K instead of L for the set of capacity levels):

$$\sum_{j\in J}\sum_{p\in P} x_{ijkpt} \leq u_{ik} y_{ikt} \quad \forall i \in I,\ \forall k \in K,\ \forall t \in T. \tag{6.27}$$

The corresponding flow formulation for the DFLP_PC, referred to as the PC-2i formulation, is defined in a similar manner, but without the relocation linking constraints (3.26) and without the relocation variables w^{BO}_{jkt} and w^{BI}_{jkt}.

In the following, computational results are presented to evaluate both formulations for the DFLP_PC and the DFLP_RPC. For each problem variant, the integrality gaps and the performance of CPLEX to find optimal integer solutions are investigated, taking into consideration the two different formulations. Then, results are presented that indicate the potential of the Lagrangian heuristics.

Note that all computational experiments have been performed on the instance set used in Chapter 5. We refer to Appendix C.1 for a detailed description of the parameters used to generate the instances. All mathematical models and the Lagrangian based heuristics have been implemented in C/C++ using the IBM CPLEX 12.6.0 Callable Library. The code has been compiled and executed on openSUSE 11.3. Each problem instance has been run on a single Intel Xeon X5650 processor (2.67GHz), limited to 24GB of RAM.

6.4.1 Computational Results for the DFLP_PC

We now present computational results for the DFLP_PC, including an analysis of the integrality gap for the PC-2i and PC-GMC formulations, the performance of CPLEX to find optimal integer solutions and results for the Lagrangian heuristics.

Integrality gaps

We now elaborate on the average integrality gaps of the proposed formulations. Table 6.I summarizes the deviation of the LP relaxation values from the best known upper bounds for two sets of instances: in the first set, instances are considered if their best known upper bounds are within 0.1% of the optimal solution. In the second set, instances are considered regardless of the quality of the available upper bound. Given that the models become very large, CPLEX could not solve all LP relaxations within the available memory of 24 gigabyte and time limit of 12 hours computing time. The table therefore considers only those instances for which the LP relaxation of both formulations has been solved by CPLEX. CPLEX was not able to solve the LP relaxation for the majority of the large instances with $q = 10$, mostly due to memory limitations.

q	Instance size	Integr. Gap % compared to best known UB < 0.1% from opt # inst.	Integr. Gap* % PC-2i	PC-GMC	Integr. Gap % compared to best known UB # inst.	Integr. Gap* % PC-2i	PC-GMC
3	50/50	15	3.64	0.70	18	3.87	0.89
	50/200	18	1.06	0.04	18	1.06	0.04
	100/100	18	2.54	0.13	18	2.54	0.13
	100/400	18	0.98	0.01	18	0.98	0.01
	150/150	18	2.09	0.08	18	2.09	0.08
	150/600	18	0.94	0.01	18	0.94	0.01
	200/200	18	1.91	0.05	18	1.91	0.05
	200/800	12	0.85	0.01	18	1.06	0.18
	250/250	18	1.72	0.02	18	1.72	0.02
	250/1000	6	0.74	0.02	16	1.29	0.42
	Avg All	159	1.71	0.11	178	1.75	0.18
5	50/50	9	5.91	1.38	18	7.13	1.99
	50/200	16	1.90	0.17	18	1.98	0.22
	100/100	11	3.83	0.30	18	4.58	0.59
	100/400	18	1.65	0.03	18	1.65	0.03
	150/150	13	3.49	0.21	18	3.70	0.33
	150/600	12	1.50	0.01	18	1.76	0.22
	200/200	14	3.11	0.13	18	3.26	0.21
	200/800	6	1.29	0.02	12	1.70	0.29
	250/250	10	2.56	0.02	15	2.92	0.09
	250/1000	5	1.24	0.02	6	1.30	0.06
	Avg All	114	2.65	0.21	159	3.18	0.44
10	50/50	2	11.97	2.33	16	12.48	2.86
	50/200	4	4.39	0.46	13	4.93	0.82
	100/100	0	-	-	11	10.03	2.05
	100/400	5	2.98	0.04	5	2.98	0.04
	150/150	0	-	-	5	7.08	1.02
	150/600	1	3.57	0.05	1	3.57	0.05
	200/200	1	4.78	0.21	2	5.64	0.53
	200/800	0	-	-	0	-	-
	250/250	0	-	-	0	-	-
	250/1000	0	-	-	0	-	-
	Avg All	13	4.98	0.53	53	8.29	1.61
All	**Avg All**	286	2.23	0.17	390	3.22	0.48

Table 6.I: Deviations of LP relaxation values from best known upper bounds for the two formulations of the DFLP_PC.

When the LP relaxation has been solved and the (near) optimal solutions are known, the PC-GMC clearly dominates the PC-2i flow formulation, presenting an average integrality gap more than 10 times lower than the one given by the flow formulation.

CPLEX Optimization for the DFLP_PC

Table 6.II summarizes the results given by CPLEX, using the PC-2i and the PC-GMC formulations. For each of the two formulations, the table summarizes the number of instances where feasible solutions have been found (*"# inst."*), as well as the average and maximum deviation from the best known lower bounds. Given that the models generated by the PC-2i formulation are smaller than those generated by the PC-GMC formulation, this formulation finds feasible solutions for slightly more (323 vs. 291) instances. However, the PC-GMC formulation provides a significantly better solution quality.

Table 6.III reports a direct comparison between both formulations for the instances where both formulations found feasible solutions. Here, the PC-GMC formulation confirms its advantage when compared to the 2i formulation: computing times and solution quality are significantly better.

Performance of the Lagrangian Heuristic for the DFLP_PC

The Lagrangian heuristic for the DFLP_PC relaxes the demand constraints. The resulting Lagrangian subproblem is identical to the one presented in Section 6.3.1. However, it does neither contain the relocation variables nor the relocation linking constraints. Feasible solutions are generated based on the Lagrangian solutions by adapting the steps indicated in Section 5.4.3. A 0.01% optimality stopping criteria has been used. Furthermore, the Lagrangian dual is solved by the bundle method, limited to a maximum of 500 iterations.

Table 6.IV compares the performance of two different configurations of the Lagrangian heuristic. The first heuristic only uses the bundle method, whereas the second heuristic adds the solution of the restricted MIP model based on the convexified solu-

q	Instance size	PC-2i # inst.	PC-2i Avg % Gap*	PC-2i Max % Gap*	PC-GMC # inst.	PC-GMC Avg % Gap*	PC-GMC Max % Gap*
3	50/50	18	0.13	1.15	18	0.11	1.09
	50/200	18	0.00	0.01	18	0.00	0.01
	100/100	18	0.03	0.41	18	0.01	0.09
	100/400	18	0.00	0.01	18	0.00	0.01
	150/150	18	0.15	2.73	18	0.00	0.01
	150/600	18	0.00	0.01	18	0.00	0.01
	200/200	18	0.30	5.14	18	0.00	0.01
	200/800	11	0.43	3.39	9	0.00	0.01
	250/250	16	0.01	0.16	18	0.01	0.06
	250/1000	6	7.40	35.80	6	0.00	0.01
	Avg All	159	0.38	35.80	159	0.02	1.09
5	50/50	18	0.96	6.15	18	0.56	2.38
	50/200	18	0.05	0.51	18	0.05	0.45
	100/100	18	0.27	1.53	18	0.22	1.14
	100/400	18	0.00	0.01	12	0.00	0.01
	150/150	14	0.13	0.75	18	0.22	1.38
	150/600	12	0.01	0.04	6	0.00	0.01
	200/200	14	0.14	0.66	11	0.04	0.39
	200/800	2	3.24	5.21	0	-	-
	250/250	12	0.12	0.57	6	0.00	0.01
	250/1000	0	-	-	0	-	-
	Avg All	126	0.28	6.15	107	0.18	2.38
10	50/50	11	19.36	90.83	15	2.26	3.78
	50/200	10	10.39	80.50	6	0.15	0.60
	100/100	7	2.62	10.06	4	0.58	1.50
	100/400	7	0.16	1.09	0	-	-
	150/150	2	0.73	1.28	0	-	-
	150/600	1	0.01	0.01	0	-	-
	200/200	0	-	-	0	-	-
	200/800	0	-	-	0	-	-
	250/250	0	-	-	0	-	-
	250/1000	0	-	-	0	-	-
	Avg All	38	8.89	90.83	25	1.48	3.78
All	**Avg All**	323	1.34	90.83	291	0.20	3.78

Table 6.II: CPLEX optimization, using the PC-2i and PC-GMC formulations for the DFLP_PC.

q	Instance size	# inst.	PC-2i Avg % Gap*	PC-2i Max % Gap*	PC-2i Time (sec)	PC-GMC Avg % Gap*	PC-GMC Max % Gap*	PC-GMC Time (sec)
3	50/50	18	0.13	1.15	2,171.1	0.11	1.09	1,436.6
	50/200	18	0.00	0.01	233.6	0.00	0.01	101.2
	100/100	18	0.03	0.41	868.9	0.01	0.09	550.2
	100/400	18	0.00	0.01	268.4	0.00	0.01	173.0
	150/150	18	0.15	2.73	1,634.1	0.00	0.01	404.7
	150/600	18	0.00	0.01	1,116.3	0.00	0.01	498.9
	200/200	18	0.30	5.14	2,578.4	0.00	0.01	707.9
	200/800	9	0.52	3.39	4,984.6	0.00	0.01	1,119.4
	250/250	16	0.01	0.16	2,104.6	0.00	0.01	681.6
	250/1000	6	7.40	35.80	6,881.3	0.00	0.01	1,880.0
	Avg All	157	0.38	35.80	1,780.2	0.02	1.09	649.5
5	50/50	18	0.96	6.15	5,298.3	0.56	2.38	4,042.3
	50/200	18	0.05	0.51	2,499.5	0.05	0.45	1,476.7
	100/100	18	0.27	1.53	4,207.6	0.22	1.14	3,455.7
	100/400	12	0.00	0.01	653.8	0.00	0.01	442.8
	150/150	14	0.13	0.75	3,362.4	0.06	0.74	2,343.1
	150/600	6	0.00	0.01	2,694.3	0.00	0.01	641.5
	200/200	10	0.12	0.66	3,390.1	0.01	0.01	1,317.8
	200/800	0	-	-	-	-	-	-
	250/250	6	0.08	0.26	5,006.8	0.00	0.01	1,159.8
	250/1000	0	-	-	-	-	-	-
	Avg All	102	0.26	6.15	3,442.4	0.15	2.38	2,192.6
10	50/50	11	19.36	90.83	7,200.1	2.07	3.78	6,911.6
	50/200	6	0.18	0.58	6,072.5	0.15	0.60	3,541.7
	100/100	4	0.72	1.93	7,200.0	0.58	1.50	7,200.3
	100/400	0	-	-	-	-	-	-
	150/150	0	-	-	-	-	-	-
	150/600	0	-	-	-	-	-	-
	200/200	0	-	-	-	-	-	-
	200/800	0	-	-	-	-	-	-
	250/250	0	-	-	-	-	-	-
	250/1000	0	-	-	-	-	-	-
	Avg All	21	10.33	90.83	6,877.9	1.23	3.78	6,003.8
All	**Avg All**	280	1.09	90.83	2,768.1	0.16	3.78	1,613.2

Table 6.III: CPLEX results comparing the the two DFLP_PC formulations, considering instances where both formulations found feasible solutions.

tions. The restricted MIP model has been used with parameter $n^S = 3$ and does not apply decision fixing. The table reports, for each of the two configurations, the average and maximum deviations of the obtained upper bounds from the best known lower bounds, as generated by the Lagrangian heuristic. Note that these results contain all 540 instances. We observed that the lower bound provided by the bundle method is very close to the LP relaxation bound. For the upper bounds, the use of the restricted MIP model significantly improves the final solution quality, generating solutions of reasonable quality even for larger instances. The maximum deviation from the best known lower bound is 3.72%, while the average deviation is 0.77%.

Table 6.V compares the performance of CPLEX and the Lagrangian heuristic (bundle with restricted MIP model) for the instances for which CPLEX found feasible solutions. The table reports the number of instances where CPLEX does not find feasible solutions (*"# ns"*), as well as the average and maximum gaps from the best known lower bounds. The results indicate that the Lagrangian heuristic provides competitive results with a maximum optimality gap from the best known lower bounds lower than CPLEX.

6.4.2 Computational Results for the DFLP_RPC

We now present computational results for the DFLP_RPC, including an analysis of the integrality gaps for the RPC-2i and RPC-GMC formulations, the performance of CPLEX to find optimal integer solutions and results for the Lagrangian heuristics.

Integrality gaps

Table 6.VI provides information on the integrality gap for the two formulations. As for the DFLP_PC, the results only include instances where the LP relaxation has been solved for both formulations. The table summarizes the deviation of the LP relaxation values from the best known upper bounds for two sets of instances: in the first set, instances are considered if their best known upper bounds are within 0.1% of the optimal solution. In the second set, instances are considered regardless of the quality of the available upper bound. The results show similar trends as observed for the DFLP_PC.

q	Instance size	Bundle only Avg Gap %	Max Gap %	Avg Time	Bundle + R-MIP UB* Gap %	Max Gap %	Avg Time
3	50/50	1.47	3.86	7.2	0.39	1.16	22.3
	50/200	0.53	0.99	9.0	0.57	0.95	11.4
	100/100	0.74	2.55	14.6	0.41	0.97	15.7
	100/400	0.59	0.94	48.7	0.62	0.86	48.1
	150/150	1.24	9.00	32.4	0.57	0.99	32.2
	150/600	0.73	0.97	129.5	0.71	0.98	129.9
	200/200	1.46	6.82	76.2	0.46	0.90	81.9
	200/800	0.93	2.39	307.8	0.53	0.95	306.6
	250/250	1.03	3.85	118.1	0.52	0.95	119.3
	250/1000	1.11	2.63	684.4	0.67	0.98	740.1
	Avg All	0.98	9.00	143.9	0.54	1.16	150.8
5	50/50	3.66	6.64	13.3	0.72	2.43	883.4
	50/200	0.49	1.25	25.0	0.39	0.90	26.1
	100/100	2.20	5.25	41.9	0.40	1.00	183.3
	100/400	0.66	0.98	80.0	0.65	0.95	69.8
	150/150	2.38	6.01	91.2	0.48	0.93	117.9
	150/600	0.72	0.95	193.7	0.73	0.95	185.3
	200/200	2.07	7.70	169.6	0.49	0.88	163.2
	200/800	0.89	2.10	420.3	0.65	1.00	418.7
	250/250	1.93	5.28	239.9	0.54	0.99	268.2
	250/1000	1.10	2.66	953.6	0.69	0.96	925.8
	Avg All	1.61	7.70	223.2	0.57	2.43	324.2
10	50/50	5.35	8.00	39.1	2.15	3.07	3,169.7
	50/200	2.13	3.41	111.0	0.77	1.48	2,185.6
	100/100	7.45	9.18	111.4	2.08	3.72	4,971.2
	100/400	1.45	3.90	289.5	0.76	0.99	469.1
	150/150	6.86	8.29	245.3	1.66	2.50	5,350.0
	150/600	1.56	4.01	716.4	0.71	1.00	1,036.3
	200/200	7.47	12.30	421.2	1.45	2.10	5,293.8
	200/800	1.71	3.41	1,238.3	0.58	0.99	1,653.0
	250/250	7.15	10.47	576.4	1.19	2.33	4,665.2
	250/1000	1.23	2.96	1,614.1	0.62	0.99	1,779.1
	Avg All	4.23	12.30	536.3	1.20	3.72	3,057.3
All	**Avg All**	2.28	12.30	301.9	0.77	3.72	1,177.4

Table 6.IV: Results of the Lagrangian Heuristic for all 540 instances of the DFLP_PC.

		CPLEX			Lagrangian Heuristic	
q	Instance size	# ns	Avg Gap %	Max Gap %	Avg Gap %	Max Gap %
3	50/50	0	0.11	1.09	0.39	1.16
	50/200	0	0.00	0.01	0.57	0.95
	100/100	0	0.01	0.09	0.41	0.97
	100/400	0	0.00	0.01	0.62	0.86
	150/150	0	0.00	0.01	0.57	0.99
	150/600	0	0.00	0.01	0.71	0.98
	200/200	0	0.00	0.01	0.46	0.90
	200/800	9	0.00	0.01	0.44	0.95
	250/250	0	0.01	0.06	0.52	0.95
	250/1000	12	0.00	0.01	0.45	0.96
	Avg All	21	0.02	1.09	0.52	1.16
5	50/50	0	0.56	2.38	0.72	2.43
	50/200	0	0.05	0.45	0.39	0.90
	100/100	0	0.22	1.14	0.40	1.00
	100/400	6	0.00	0.01	0.68	0.95
	150/150	0	0.22	1.38	0.48	0.93
	150/600	12	0.00	0.01	0.62	0.82
	200/200	7	0.04	0.39	0.38	0.78
	200/800	18	-	-	-	0.00
	250/250	12	0.00	0.01	0.47	0.94
	250/1000	18	-	-	-	0.00
	Avg All	73	0.18	2.38	0.51	2.43
10	50/50	3	2.26	3.78	2.10	3.04
	50/200	12	0.15	0.60	0.35	0.61
	100/100	14	0.58	1.50	0.65	1.54
	100/400	18	-	-	-	0.00
	150/150	18	-	-	-	0.00
	150/600	18	-	-	-	0.00
	200/200	18	-	-	-	0.00
	200/800	18	-	-	-	0.00
	250/250	18	-	-	-	0.00
	250/1000	18	-	-	-	0.00
	Avg All	155	1.48	3.78	1.45	3.04
All	**Avg All**	249	0.20	3.78	0.60	3.04

Table 6.V: Comparison of solution quality for CPLEX and the Lagrangian heuristic for the DFLP_PC, considering instances where CPLEX found feasible solutions.

However, the problem seems to be more difficult to solve. CPLEX was not able to solve the LP relaxation for most of the large instances with $q = 5$ and $q = 10$. When the LP relaxation has been solved and the (near) optimal solutions are known, the RPC-GMC formulation clearly dominates the flow formulation, presenting an average integrality gap more than 20 times lower than the one given by the 2i formulation.

CPLEX Optimization for the DFLP_RPC

Tables 6.VII and 6.VIII summarize the results given by CPLEX. Tables 6.VII summarizes, for each of the two formulations, the number of instances where feasible solutions have been found, as well as the average and maximum deviation from the best known lower bounds. Interestingly, the observations regarding the number of instances where feasible solutions are found are now opposed to those made for the DFLP_PC. Here, the RPC-GMC formulation finds feasible solutions for siginificantly more instances (285 vs. 179).

Table 6.VIII reports a direct comparison between both formulations for the instances where both formulations found feasible solutions. The RPC-GMC formulation confirms its advantage: computing times and solution quality are significantly better.

As the DFLP_PC seems to be significantly easier to solve, one may expect that the quality of solutions obtained from the DFLP_PC may be better than the quality of solutions obtained by the DFLP_RPC. However, a comparison of the solution values from the two problem variants did not confirm this hypothesis. Using the 2i formulations, the DFLP_PC resulted in an average deviation of 2.85% from the best known lower bound of the DFLP_RPC, whereas the DFLP_RPC solutions averaged a 0.44% deviation. For the GMC based formulations, the DFLP_PC resulted in an average deviation of 3.14% from the best known lower bound of the DFLP_RPC, whereas the DFLP_RPC solutions averaged a 0.52% deviation. Note that this comparison only considers instances where feasible solutions have been found for both formulations.

q	Instance size	Integr. Gap % compared to best known UB < 0.1% from opt # inst.	Integr. Gap % RPC-2i	RPC-GMC	Integr. Gap % compared to best known UB # inst.	Integr. Gap* % RPC-2i	RPC-GMC
3	50/50	14	5.06	0.66	18	5.38	0.93
	50/200	18	1.52	0.05	18	1.52	0.05
	100/100	18	3.97	0.11	18	3.97	0.11
	100/400	18	1.45	0.01	18	1.45	0.01
	150/150	18	3.48	0.06	18	3.48	0.06
	150/600	18	1.45	0.01	18	1.45	0.01
	200/200	18	3.19	0.03	18	3.19	0.03
	200/800	14	1.42	0.01	18	1.49	0.09
	250/250	18	3.02	0.02	18	3.02	0.02
	250/1000	9	1.32	0.03	15	1.57	0.21
	Avg All	163	2.62	0.09	177	2.67	0.15
5	50/50	5	6.77	1.00	18	8.35	2.15
	50/200	14	2.34	0.16	18	2.46	0.25
	100/100	9	4.86	0.22	18	5.83	0.65
	100/400	13	2.08	0.02	18	2.12	0.08
	150/150	10	4.52	0.15	18	4.97	0.35
	150/600	9	2.03	0.03	18	2.25	0.23
	200/200	9	3.89	0.04	18	4.48	0.26
	200/800	6	2.07	0.05	10	2.18	0.14
	250/250	7	3.27	0.05	18	4.19	0.21
	250/1000	2	2.16	0.03	4	1.94	0.08
	Avg All	84	3.28	0.15	158	4.14	0.49
10	50/50	0	-	-	18	12.74	3.30
	50/200	1	4.02	0.31	16	5.42	1.13
	100/100	0	-	-	16	11.05	2.57
	100/400	2	3.19	0.03	6	3.55	0.20
	150/150	0	-	-	11	8.68	1.83
	150/600	1	3.94	0.09	1	3.94	0.09
	200/200	0	-	-	5	6.74	8.88
	200/800	0	-	-	0	-	-
	250/250	0	-	-	5	5.78	0.95
	250/1000	0	-	-	0	-	-
	Avg All	4	3.58	0.11	78	8.67	2.43
All	**Avg All**	251	2.86	0.11	413	4.36	0.71

Table 6.VI: Deviations of LP relaxation values from best known upper bounds of the RPC-2i and RPC-GMC formulations for the DFLP_RPC.

q	Instance size	RPC-2i # inst.	RPC-2i Avg % Gap*	RPC-2i Max % Gap*	RPC-GMC # inst.	RPC-GMC Avg % Gap*	RPC-GMC Max % Gap*
3	50/50	17	0.33	0.92	18	0.33	0.99
	50/200	18	0.17	0.87	18	0.11	0.98
	100/100	18	0.31	0.68	18	0.16	0.71
	100/400	18	0.33	0.92	18	0.17	0.78
	150/150	17	0.25	0.49	18	0.16	0.65
	150/600	16	0.75	4.83	18	0.34	0.84
	200/200	14	0.25	0.61	18	0.16	0.43
	200/800	2	0.53	0.92	9	1.14	5.24
	250/250	13	0.78	2.60	18	0.22	0.83
	250/1000	0	-	-	2	0.03	0.04
	Avg All	133	0.39	4.83	155	0.26	5.24
5	50/50	5	0.75	2.12	18	1.00	3.50
	50/200	10	0.43	1.42	18	0.36	0.81
	100/100	10	0.86	3.61	18	0.45	1.41
	100/400	7	0.53	0.86	12	0.18	0.98
	150/150	8	0.54	1.65	18	0.42	0.91
	150/600	1	0.45	0.45	6	1.20	5.75
	200/200	3	0.53	0.61	12	0.35	0.87
	200/800	0	-	-	1	0.40	0.40
	250/250	1	2.10	2.10	6	2.51	13.53
	250/1000	0	-	-	-	-	-
	Avg All	45	0.64	3.61	109	0.64	13.53
10	50/50	0	-	-	11	2.65	4.12
	50/200	1	0.53	0.53	6	0.69	1.05
	100/100	0	-	-	4	1.38	1.92
	100/400	0	-	-	-	-	-
	150/150	0	-	-	-	-	-
	150/600	0	-	-	-	-	-
	200/200	0	-	-	-	-	-
	200/800	0	-	-	-	-	-
	250/250	0	-	-	-	-	-
	250/1000	0	-	-	-	-	-
	Avg All	1	0.53	0.53	21	1.85	4.12
All	**Avg All**	179	0.45	4.83	285	0.52	13.53

Table 6.VII: CPLEX optimization, using the RPC-2i and RPC-GMC formulations for the DFLP_RPC.

q	Instance size	# inst.	RPC-2i Avg % Gap*	RPC-2i Max % Gap*	RPC-2i Time (sec)	RPC-GMC Avg % Gap*	RPC-GMC Max % Gap*	RPC-GMC Time (sec)
3	50/50	17	0.33	0.92	821.9	0.67	0.99	144.9
	50/200	18	0.17	0.87	215.8	0.16	0.99	64.1
	100/100	18	0.31	0.68	363.9	0.26	0.86	77.6
	100/400	18	0.33	0.92	1,292.5	0.17	0.79	407.6
	150/150	17	0.25	0.49	994.6	0.22	0.69	217.9
	150/600	16	0.75	4.83	3,452.4	0.30	0.85	1,610.5
	200/200	14	0.25	0.61	2,499.4	0.15	0.48	570.6
	200/800	2	0.53	0.92	4,058.0	0.00	0.00	1,839.0
	250/250	13	0.78	2.60	4,173.5	0.21	0.84	1,067.0
	250/1000	0	-	-	-	-	-	-
	Avg All	133	0.39	4.83	1,633.0	0.27	0.99	496.4
5	50/50	5	0.75	2.12	2,997.2	0.93	1.39	1,934.2
	50/200	10	0.43	1.42	1,382.1	0.37	0.97	228.6
	100/100	10	0.86	3.61	4,107.0	0.52	0.97	570.7
	100/400	7	0.53	0.86	3,185.4	0.18	0.36	950.0
	150/150	8	0.54	1.65	3,177.3	0.29	0.90	1,062.1
	150/600	1	0.45	0.45	7,058.0	0.00	0.00	2,406.0
	200/200	3	0.53	0.61	3,494.3	0.24	0.64	1,072.3
	200/800	0	-	-	-	-	-	-
	250/250	1	2.10	2.10	7,201.0	-	-	-
	250/1000	0	-	-	-	-	-	-
	Avg All	45	0.64	3.61	3,163.0	0.41	1.39	871.7
10	50/50	0	-	-	-	-	-	-
	50/200	1	0.53	0.53	865.0	0.46	0.46	1,241.0
	100/100	0	-	-	-	-	-	-
	100/400	0	-	-	-	-	-	-
	150/150	0	-	-	-	-	-	-
	150/600	0	-	-	-	-	-	-
	200/200	0	-	-	-	-	-	-
	200/800	0	-	-	-	-	-	-
	250/250	0	-	-	-	-	-	-
	250/1000	0	-	-	-	-	-	-
	Avg All	1	0.53	0.53	865.0	0.46	0.46	1,241.0
All	**Avg All**	179	0.45	4.83	2,013.3	0.30	1.39	592.3

Table 6.VIII: CPLEX results comparing the two DFLP_RPC formulations, considering instances where both formulations found feasible solutions.

Performance of the Lagrangian Heuristic for the DFLP_RPC Relaxing Demand Constraints

Solution Strategy. Solving the Lagrangian problem with a MIP solver such as CPLEX can consume significant computing times. We observed that the solution of the Lagrangian problem is significantly easier when the Lagrange multipliers, which directly impact the OF coefficients of the problem, are well chosen. This may take a considerable number of iterations, often several hundreds. We therefore initialize the Lagrange multipliers by using the bundle method for a certain number of iterations on a problem variant that can be decomposed, which is relatively quick. Clearly, that may be the DFLP_RPC with relaxed demand and relocation constraints. Furthermore, the DFLP_PC can be seen as a special case of the DFLP_RPC and it is likely that good Lagrange multipliers for one problem perform well for the other. Therefore, we may also use the DFLP_PC to initialize the Lagrange multipliers. After that initialization phase, the best multipliers found are used to initialize the bundle method on the DFLP_RPC where we solve the subproblem with CPLEX. In case we have found good multipliers, the resulting IP is not that hard to solve.

Note that, even though we are not decomposing the original problem, the problem solved in each iteration is significantly smaller than the original problem, because we do not explicitly use the demand allocation variables x.

Computational Results. In the computational experiments, we allocate up to 40% of the maximum number of iterations to the initialization phase. Interestingly, using the initialization phase based on the DFLP_PC resulted in slightly better results than using the initialization based on the DFLP_RPC. As in previous experiments, a 1% optimality stopping criterion has been used. Furthermore, the Lagrangian dual is solved by the bundle method, limited to a maximum of 1000 iterations. CPLEX is used with standard parameters.

Table 6.IX compares the performance of CPLEX and the Lagrangian heuristic for the instances for which CPLEX found feasible solutions. The table reports the number of instances where CPLEX did not find feasible solutions (*"# ns"*), as well as the average

and maximum gaps from the best known lower bounds. For the instances for which CPLEX found feasible solutions, the Lagrangian heuristic provides similar results. Even though the average optimality gap is higher, the maximum optimality gap is slightly lower than for CPLEX.

Table 6.X reports the average and maximum deviations of the obtained upper bounds from the best known lower bounds, as generated by the Lagrangian heuristic. Note that these results contain all 540 instances. The heuristic generates solutions of reasonable quality even for larger instances. The maximum deviation from the best known lower bound is 15.42%, while the average deviation is 3.08%.

Performance of the Lagrangian Heuristic Relaxing Demand and Relocation Linking Constraints

When demand and relocation linking constraints are relaxed, the Lagrangian subproblem can be efficiently solved. Feasible solutions are constructed taking into consideration the relocation decisions suggested by the Lagrangian solutions. Table 6.XI shows the results for the bundle method, as well as for the bundle method with the subsequent solution of a restricted MIP model. The latter has been used with parameter $n^S = 4$ and without decision fixing. For each configuration of the Lagrangian heuristic, the table reports the average and maximum deviations of the obtained upper bounds from the best known lower bounds, as generated by the Lagrangian heuristic. Note that these results contain all 540 instances. The results are similar to those observed for the DFLP_PC. Whereas the bundle method itself may produce solutions with high optimality gaps, the Lagrangian heuristic using the restricted MIP model generates solutions of reasonable quality even for larger instances. The maximum deviation from the best known lower bound is 5.43%, while the average deviation is 0.96%.

Table 6.XI compares the performance of CPLEX and the Lagrangian heuristic (bundle with restricted MIP model) for the instances for which CPLEX found feasible solutions. The table reports the number of instances where CPLEX does not find feasible solutions (*"# ns"*), as well as the average and maximum gaps from the best known lower bounds. The results indicate that the Lagrangian heuristic provides competitive results

			CPLEX		Lagrangian Heuristic	
q	Instance size	# ns	Avg Gap %	Max Gap %	Avg Gap %	Max Gap %
3	50/50	0	0.33	0.99	0.69	2.30
	50/200	0	0.11	0.98	0.60	0.89
	100/100	0	0.16	0.71	0.57	1.00
	100/400	0	0.17	0.78	0.83	1.98
	150/150	0	0.16	0.65	0.78	2.19
	150/600	0	0.34	0.84	0.85	1.87
	200/200	0	0.16	0.43	1.04	4.42
	200/800	9	1.14	5.24	1.14	2.91
	250/250	0	0.22	0.83	0.79	1.98
	250/1000	16	0.03	0.04	0.96	1.22
	Avg All	25	0.26	5.24	0.79	4.42
5	50/50	0	1.00	3.50	2.73	5.91
	50/200	0	0.36	0.81	0.69	1.72
	100/100	0	0.45	1.41	1.61	4.16
	100/400	6	0.18	0.98	0.78	2.02
	150/150	0	0.42	0.91	1.55	4.97
	150/600	12	1.20	5.75	0.97	2.03
	200/200	6	0.35	0.87	1.51	4.99
	200/800	17	0.40	0.40	1.44	1.44
	250/250	12	2.51	13.53	1.90	3.21
	250/1000	18	-	-	-	0.00
	Avg All	71	0.64	13.53	1.51	5.91
10	50/50	7	2.65	4.12	8.31	10.22
	50/200	12	0.69	1.05	2.37	3.89
	100/100	14	1.38	1.92	9.50	11.18
	100/400	18	-	-	-	0.00
	150/150	18	-	-	-	0.00
	150/600	18	-	-	-	0.00
	200/200	18	-	-	-	0.00
	200/800	18	-	-	-	0.00
	250/250	18	-	-	-	0.00
	250/1000	18	-	-	-	0.00
	Avg All	159	1.85	4.12	6.84	11.18
All	**Avg All**	255	0.52	13.53	1.51	11.18

Table 6.IX: Comparison of solution quality for CPLEX and the Lagrangian heuristic based on the relaxation of the demand constraints, considering instances for the DFLP_RPC where CPLEX found feasible solutions.

		Lagrangian Heuristic		
q	Instance size	Avg Gap %	Max Gap %	Avg Time
3	50/50	0.69	2.30	590.5
	50/200	0.60	0.89	86.3
	100/100	0.57	1.00	620.2
	100/400	0.83	1.98	480.0
	150/150	0.78	2.19	674.4
	150/600	0.85	1.87	1,313.9
	200/200	1.04	4.42	1,735.2
	200/800	1.00	2.91	2,890.6
	250/250	0.79	1.98	2,452.0
	250/1000	1.20	3.29	4,990.3
	Avg All	0.83	4.42	1,583.4
5	50/50	2.73	5.91	4,945.4
	50/200	0.69	1.72	1,221.7
	100/100	1.61	4.16	5,116.3
	100/400	0.74	2.02	1,794.0
	150/150	1.55	4.97	5,879.0
	150/600	0.87	2.03	3,558.2
	200/200	1.59	4.99	5,334.7
	200/800	1.10	2.67	5,603.8
	250/250	1.62	3.21	6,051.3
	250/1000	1.67	3.31	6,700.8
	Avg All	1.42	5.91	4,620.5
10	50/50	9.16	14.21	7,495.4
	50/200	3.43	5.39	7,283.1
	100/100	12.12	14.89	7,377.6
	100/400	2.82	6.09	7,386.3
	150/150	11.11	15.42	7,394.3
	150/600	2.86	5.16	7,404.1
	200/200	11.84	15.36	7,626.7
	200/800	2.96	4.41	7,542.3
	250/250	11.27	15.28	7,623.0
	250/1000	2.31	3.74	7,526.0
	Avg All	6.99	15.42	7,465.9
All	**Avg All**	3.08	15.42	4,556.6

Table 6.X: Results of the Lagrangian Heuristic based on the relaxation of the demand constraints for all 540 instances for the DFLP_RPC.

q	Instance size	Bundle only Avg Gap %	Bundle only Max Gap %	Bundle only Avg Time	Bundle + R-MIP Avg Gap %	Bundle + R-MIP Max Gap %	Bundle + R-MIP Avg Time
3	50/50	3.39	10.45	10.4	0.25	0.96	22.3
	50/200	0.79	2.04	21.9	0.56	0.93	11.4
	100/100	1.76	5.99	30.2	0.33	0.85	15.7
	100/400	0.85	2.67	95.7	0.65	0.92	48.1
	150/150	2.39	6.43	67.8	0.31	0.98	32.2
	150/600	1.07	3.21	217.5	0.58	0.97	129.9
	200/200	2.07	6.84	121.6	0.49	0.97	81.9
	200/800	1.55	4.82	418.3	0.65	0.99	306.6
	250/250	1.79	6.42	178.2	0.50	0.90	119.3
	250/1000	1.84	4.67	917.3	0.50	0.98	740.1
	Avg All	1.75	10.45	207.9	0.48	0.99	150.8
5	50/50	8.11	18.95	17.7	1.16	2.53	883.4
	50/200	1.33	4.01	58.3	0.45	0.91	26.1
	100/100	5.78	16.18	59.2	0.58	1.80	183.3
	100/400	1.25	3.52	170.1	0.65	0.98	69.8
	150/150	5.16	10.60	131.5	0.51	1.70	117.9
	150/600	1.15	3.16	373.6	0.67	0.94	185.3
	200/200	4.23	7.66	237.7	0.42	0.99	163.2
	200/800	1.63	5.16	693.9	0.58	0.95	418.7
	250/250	3.47	8.01	340.9	0.33	0.92	268.2
	250/1000	1.88	5.76	1,256.9	0.42	0.95	925.8
	Avg All	3.40	18.95	334.0	0.58	2.53	324.2
10	50/50	14.45	20.19	62.4	3.07	5.32	3,169.7
	50/200	6.34	12.55	144.7	1.65	4.52	2,185.6
	100/100	14.26	24.62	168.7	2.94	4.65	4,971.2
	100/400	3.99	8.61	498.1	0.78	1.64	469.1
	150/150	14.59	23.05	329.6	2.77	5.37	5,350.0
	150/600	3.28	6.19	1,023.8	1.03	5.41	1,036.3
	200/200	14.55	21.81	508.7	2.36	5.02	5,293.8
	200/800	3.39	6.58	1,638.2	0.89	5.43	1,653.0
	250/250	11.89	17.91	719.8	1.97	4.36	4,665.2
	250/1000	2.63	5.38	2,216.3	0.62	3.44	1,779.1
	Avg All	8.94	24.62	731.0	1.81	5.43	3,057.3
All	**Avg All**	4.70	24.62	424.3	0.96	5.43	1,177.4

Table 6.XI: Results of the Lagrangian Heuristic, relaxing demand and relocation linking constraints, for all 540 instances for the DFLP_RPC.

with a maximum optimality gap from the best known lower bounds significantly lower than CPLEX.

6.4.3 Computational Results for the DFLP_RPC with RUC Constraints

When adding the RUC constraints to the DFLP_RPC, the problem becomes significantly more difficult to solve. In the following, we will perform a similar analysis as has been done for the DFLP_PC and DFLP_RPC. We compare the integrality gaps for the two different formulations, the performance of CPLEX to find optimal integer solutions and we present results for the Lagrangian heuristics.

Integrality gaps

Table 6.XIII provides information on the integrality gap for the two formulations. As before, the results only include instances where the LP relaxation has been solved for both formulations. The table summarizes the deviation of the LP relaxation values from the best known upper bounds for two sets of instances: in the first set, instances are considered if their best known upper bounds are within 0.1% of the optimal solution. In the second set, instances are considered regardless of the quality of the available upper bound. The results show similar trends as observed for the DFLP_RPC without RUC constraints. When the LP relaxation has been solved and the (near) optimal solutions are known, the RPC-GMC formulation clearly dominates the flow formulation, presenting an average integrality gap more than 10 times lower than the one given by the 2i formulation.

CPLEX Optimization

Tables 6.XIV and 6.XV summarize the results given by CPLEX. Tables 6.XIV summarizes, for each of the two formulations, the number of instances where feasible solutions have been found, as well as the average and maximum deviation from the best known lower bounds. Interestingly, the observations regarding the number of instances where feasible solutions are found are now opposed to those made for the DFLP_PC.

			CPLEX		Lagrangian Heuristic	
q	Instance size	# ns	Avg Gap %	Max Gap %	Avg Gap %	Max Gap %
3	50/50	0	0.33	0.99	0.25	0.96
	50/200	0	0.11	0.98	0.56	0.93
	100/100	0	0.16	0.71	0.33	0.85
	100/400	0	0.17	0.78	0.65	0.92
	150/150	0	0.16	0.65	0.31	0.98
	150/600	0	0.34	0.84	0.58	0.97
	200/200	0	0.16	0.43	0.49	0.97
	200/800	9	1.14	5.24	0.42	0.82
	250/250	0	0.22	0.83	0.50	0.90
	250/1000	16	0.03	0.04	0.08	0.11
	Avg All	25	0.26	5.24	0.45	0.98
5	50/50	0	1.00	3.50	1.16	2.53
	50/200	0	0.36	0.81	0.45	0.91
	100/100	0	0.45	1.41	0.58	1.80
	100/400	6	0.18	0.98	0.60	0.98
	150/150	0	0.42	0.91	0.51	1.70
	150/600	12	1.20	5.75	0.48	0.89
	200/200	6	0.35	0.87	0.33	0.99
	200/800	17	0.40	0.40	0.26	0.26
	250/250	12	2.51	13.53	0.09	0.34
	250/1000	18	-	-	-	-
	Avg All	71	0.64	13.53	0.58	2.53
10	50/50	7	2.65	4.12	2.59	4.04
	50/200	12	0.69	1.05	0.63	0.82
	100/100	14	1.38	1.92	1.54	2.31
	100/400	18	-	-	-	-
	150/150	18	-	-	-	-
	150/600	18	-	-	-	-
	200/200	18	-	-	-	-
	200/800	18	-	-	-	-
	250/250	18	-	-	-	-
	250/1000	18	-	-	-	-
	Avg All	159	1.85	4.12	1.83	4.04
All	**Avg All**	255	0.52	13.53	0.60	4.04

Table 6.XII: Comparison of solution quality for CPLEX and the Lagrangian heuristic, relaxing demand and relocation linking constraints, considering instances for the DFLP_RPC where CPLEX found feasible solutions.

q	Instance size	Integr. Gap % compared to best known UB < 0.1% from opt # inst.	Integr. Gap % RPC-2i	RPC-GMC	Integr. Gap % compared to best known UB # inst.	Integr. Gap* % RPC-2i	RPC-GMC
3	50/50	13	4.97	0.60	18	5.42	0.97
	50/200	18	1.52	0.05	18	1.52	0.05
	100/100	17	3.93	0.09	18	3.97	0.12
	100/400	18	1.45	0.01	18	1.45	0.01
	150/150	17	3.46	0.05	18	3.49	0.07
	150/600	14	1.46	0.01	15	1.53	0.07
	200/200	17	3.16	0.03	18	3.19	0.04
	200/800	9	1.40	2.77	12	1.62	2.29
	250/250	17	2.97	0.02	18	3.05	0.06
	250/1000	6	1.32	0.03	9	1.50	0.19
	Avg All	146	2.66	0.26	162	2.80	0.33
5	50/50	5	6.76	0.98	18	8.59	2.40
	50/200	11	2.30	0.11	18	2.51	0.30
	100/100	9	4.86	0.22	18	5.95	0.78
	100/400	11	2.07	0.01	17	2.20	0.16
	150/150	6	4.55	0.07	18	5.05	0.44
	150/600	5	2.11	0.01	15	2.32	0.30
	200/200	7	3.95	0.02	18	4.55	0.32
	200/800	3	2.13	0.02	11	2.33	0.35
	250/250	6	3.79	0.02	15	4.20	0.16
	250/1000	2	2.16	0.03	5	1.89	0.08
	Avg All	65	3.46	0.14	153	4.25	0.59
10	50/50	0	-	-	18	13.54	4.18
	50/200	0	-	-	18	6.11	1.83
	100/100	0	-	-	17	11.65	3.13
	100/400	1	3.14	0.04	6	3.56	0.21
	150/150	0	-	-	11	9.10	2.06
	150/600	0	-	-	0	-	-
	200/200	0	-	-	6	6.57	1.17
	200/800	0	-	-	0	-	-
	250/250	0	-	-	5	5.88	1.00
	250/1000	0	-	-	0	-	-
	Avg All	1	3.14	0.04	81	9.16	2.44
All	**Avg All**	212	2.91	0.22	396	4.66	0.86

Table 6.XIII: Deviations of LP relaxation values from best known upper bounds of the RPC-2i and RPC-GMC formulations for the DFLP_RPC with RUC constraints.

Here, the RPC-GMC formulation finds feasible solutions for significantly more instances (251 vs. 131).

Table 6.XV reports a direct comparison between both formulations for the instances where both formulations found feasible solutions. The RPC-GMC formulation confirms its advantage: computing times and solution quality are significantly better.

Performance of the Lagrangian Heuristic

When using RUC constraints, the Lagrangian subproblem is more difficult to solve, since the demand allocation has to take into consideration the integer rounding and is solved by a multiple-choice integer knapsack. The solution of the subproblem therefore consumes significantly more time. Even though the problem variant with RUC constraints may select different opening schedules than the problem variant without RUC constraints, it is likely that both variants select similar facility locations in their optimal solutions.

The Lagrangian heuristics presented in this section are therefore based on a hybrid solution strategy combining both problem variants. In a first step, the Lagrange multipliers are initialized by solving a certain number of iterations solving the problem variant without RUC constraints. Then, the problem variant with RUC constraints is solved.

The initialization phase is terminated after a maximum of 300 iterations without RUC constraints or when the best found upper bound lays within 1% of the best known lower bound. Note that, in the initialization phase, we solve the subproblem for the problem variant without RUC constraints, but generate upper bounds for the problem variant with RUC constraints (as described in Section 6.3.3). Furthermore, note that the lower bound founds from the initialization phase are also valid for the problem variant with RUC constraints. The Lagrangian dual is solved by the bundle method, limited to a maximum of 500 iterations (including the iterations performed in the initialization phase). In a final optimization phase, we use a restricted MIP model, taking into consideration the RUC constraints, to improve the solution quality.

Given the difficulty to solve the problem with RUC constraints, the following experiments allow for a total of 180 minutes computing time, instead of 120 minutes as it

q	Instance size	RPC-2i formulation # inst.	Avg % Gap*	Max % Gap*	RPC-GMC formulation # inst.	Avg % Gap*	Max % Gap*
3	50/50	16	1.48	0.20	18	1.50	0.25
	50/200	18	0.01	0.00	18	0.01	0.00
	100/100	16	1.22	0.08	18	0.21	0.02
	100/400	12	1.92	0.16	18	0.01	0.00
	150/150	13	0.02	0.01	18	1.06	0.06
	150/600	6	4.75	1.65	17	0.04	0.01
	200/200	13	0.20	0.04	18	2.79	0.16
	200/800	-	-	-	9	3.71	0.43
	250/250	7	1.29	0.33	18	2.71	0.16
	250/1000	-	-	-	2	0.01	0.01
	Avg All	101	4.75	0.19	154	3.71	0.10
5	50/50	6	2.32	0.52	15	8.39	2.31
	50/200	8	0.01	0.00	16	0.92	0.19
	100/100	5	0.01	0.01	14	9.44	0.94
	100/400	3	0.35	0.12	11	0.03	0.01
	150/150	5	1.20	0.29	10	6.60	0.82
	150/600	-	-	-	5	0.19	0.04
	200/200	2	0.00	0.00	8	0.30	0.05
	200/800	-	-	-	1	0.01	0.01
	250/250	-	-	-	6	13.59	2.29
	250/1000	-	-	-	-	-	-
	Avg All	29	2.32	0.17	86	13.59	0.85
10	50/50	-	-	-	6	3.26	1.69
	50/200	1	0.28	0.28	2	0.51	0.42
	100/100	-	-	-	3	3.33	2.16
	100/400	-	-	-	-	-	-
	150/150	-	-	-	-	-	-
	150/600	-	-	-	-	-	-
	200/200	-	-	-	-	-	-
	200/800	-	-	-	-	-	-
	250/250	-	-	-	-	-	-
	250/1000	-	-	-	-	-	-
	Avg All	1	0.28	0.28	11	3.33	1.59
All	**Avg All**	131	4.75	0.19	251	13.59	0.43

Table 6.XIV: CPLEX optimization, using the RPC-2i and RPC-GMC formulations for the DFLP_RPC with RUC constraints.

q	Instance size	# inst.	RPC-2i formulation Avg % Gap*	Max % Gap*	Time (sec)	RPC-GMC formulation Avg % Gap*	Max % Gap*	Time (sec)
3	50/50	16	0.20	1.48	2,561.5	0.17	1.50	2,393.1
	50/200	18	0.00	0.01	1,398.4	0.00	0.01	733.3
	100/100	16	0.08	1.22	1,795.3	0.02	0.21	854.9
	100/400	12	0.16	1.92	2,968.2	0.00	0.01	529.3
	150/150	13	0.01	0.02	2,324.5	0.00	0.01	459.8
	150/600	6	1.65	4.75	7,200.5	0.01	0.02	2,682.5
	200/200	13	0.04	0.20	3,547.5	0.22	2.79	1,721.2
	200/800	0	-	-	-	-	-	-
	250/250	7	0.33	1.29	5,403.0	0.00	0.01	1,443.4
	250/1000	0	-	-	-	-	-	-
	Avg All	101	0.19	4.75	2,850.1	0.06	2.79	1,248.2
5	50/50	6	0.52	2.32	3,819.7	0.19	0.61	2,607.5
	50/200	8	0.00	0.01	1,762.8	0.00	0.01	347.1
	100/100	5	0.01	0.01	1,972.6	0.01	0.01	369.6
	100/400	3	0.12	0.35	3,744.3	0.00	0.00	745.0
	150/150	4	0.06	0.22	4,819.3	0.04	0.15	2,395.5
	150/600	0	-	-	-	-	-	-
	200/200	2	0.00	0.00	3,705.0	0.00	0.01	1,245.5
	200/800	0	-	-	-	-	-	-
	250/250	0	-	-	-	-	-	-
	250/1000	0	-	-	-	-	-	-
	Avg All	28	0.13	2.32	3,028.7	0.05	0.61	1,234.9
10	50/50	0	-	-	-	-	-	-
	50/200	1	0.28	0.28	7,201.0	0.34	0.34	7,200.0
	100/100	0	-	-	-	-	-	-
	100/400	0	-	-	-	-	-	-
	150/150	0	-	-	-	-	-	-
	150/600	0	-	-	-	-	-	-
	200/200	0	-	-	-	-	-	-
	200/800	0	-	-	-	-	-	-
	250/250	0	-	-	-	-	-	-
	250/1000	0	-	-	-	-	-	-
	Avg All	1	0.28	0.28	7,201.0	0.34	0.34	7,200.0
All	**Avg All**	130	0.18	4.75	2,922.0	0.06	2.79	1,291.1

Table 6.XV: CPLEX results comparing the two formulations for the DFLP_RPC with RUC constraints, considering instances where both formulations found feasible solutions.

has been the case in the previous experiments. For all experiments, a 0.01% optimality stopping criteria has been used.

Table 6.XVI compares the performance of three different configurations of the Lagrangian heuristic for all 540 instances. For the first configuration ("LH-RUC1") the restricted MIP has been used with parameter $n^S = 10$ and does not apply decision fixing. The second configuration ("LH-RUC2") uses a restricted MIP with $n^S = 3$ and $pFix = 0.7$. Both configurations have a time limit of 180 minutes and the restricted MIP is started after 120 minutes at the latest. Finally, the third configuration ("LH-RUC3") is identical to LH-RUC2, but we allow for a total of 360 minutes, also starting the restricted MIP after 120 minutes at the latest.

We observed that the lower bound provided by the bundle method is very close to the LP relaxation bound. Finding high quality upper bounds is difficult in particular for $q = 10$. Decision fixing in the restricted MIP only slightly improves the final solution quality and even allowing 4 hours to solve the restricted MIP is often not sufficient to solve the problem. An analysis showed that, for most of the instances with high optimality gaps, the MIP model has been too large to be reasonably solved. Often, the available computing time was not sufficient to solve the LP relaxation of the restricted MIP.

Table 6.XVII compares the performance of CPLEX and the Lagrangian heuristic for the instances for which CPLEX found feasible solutions within 180 minutes computing time. Note that, even though LH-RUC2 performed slightly better than LH-RUC1 for the entire set of 540 instances, LH-RUC1 performed better for the instances where CPLEX found feasible solutions. We therefore report the results for LH-RUC1. The table shows the number of instances where CPLEX does not find feasible solutions (*"# ns"*), as well as the average and maximum gaps from the best known lower bounds. The results indicate that the Lagrangian heuristic provides competitive results with a maximum optimality gap from the best known lower bounds significantly lower than CPLEX. Furthermore, we emphasize that the Lagrangian heuristic finds feasible solution for all instances, even for those where CPLEX does not find feasible solutions, as shown in Table 6.XVI.

		LH-RUC1 (3hs)		LH-RUC2 (3hs)		LH-RUC3 (6hs)	
q	Instance size	Avg Gap %	Max Gap %	Avg Gap %	Max Gap %	Avg Gap %	Max Gap %
3	50/50	0.31	1.22	0.44	1.31	0.44	1.31
	50/200	0.53	0.96	0.52	0.96	0.52	0.96
	100/100	0.40	0.87	0.40	0.87	0.40	0.87
	100/400	0.59	0.92	0.60	0.92	0.60	0.92
	150/150	0.29	0.99	0.31	0.99	0.31	0.99
	150/600	0.70	0.98	0.69	0.98	0.69	0.98
	200/200	0.42	0.94	0.41	0.94	0.41	0.94
	200/800	0.51	1.00	0.53	1.00	0.54	1.00
	250/250	0.40	0.93	0.39	0.93	0.39	0.93
	250/1000	0.29	1.00	0.37	1.00	0.36	1.00
	Avg All	0.44	1.22	0.47	1.31	0.47	1.31
5	50/50	1.70	4.57	1.99	5.27	1.87	4.91
	50/200	0.49	1.32	0.55	1.76	0.55	1.76
	100/100	0.73	2.26	0.88	3.31	0.88	3.31
	100/400	0.50	0.99	0.49	0.99	0.50	0.99
	150/150	0.60	1.50	0.65	1.65	0.65	1.65
	150/600	0.43	0.95	0.45	0.95	0.44	0.95
	200/200	0.45	1.20	0.51	1.35	0.51	1.35
	200/800	0.53	0.98	0.53	0.98	0.53	0.98
	250/250	0.35	0.98	0.46	0.98	0.46	0.99
	250/1000	0.30	0.97	0.32	0.97	0.31	0.97
	Avg All	0.60	4.57	0.68	5.27	0.66	4.91
10	50/50	6.38	20.19	6.38	23.78	5.84	23.78
	50/200	3.45	18.18	2.90	10.61	2.79	10.61
	100/100	4.99	20.86	4.50	11.38	4.30	11.63
	100/400	2.24	13.43	1.23	3.64	1.08	3.08
	150/150	6.04	29.05	3.84	8.10	3.67	8.34
	150/600	2.88	10.94	1.67	10.94	0.81	2.52
	200/200	5.16	26.26	4.00	19.65	3.03	7.46
	200/800	2.96	11.91	1.59	11.91	1.02	7.84
	250/250	6.26	27.29	3.86	27.29	2.43	4.98
	250/1000	0.86	5.68	0.28	0.74	0.29	0.81
	Avg All	4.11	29.05	3.00	27.29	2.37	23.78
All	**Avg All**	1.72	29.05	1.37	27.29	1.13	23.78

Table 6.XVI: Results for different configurations of the Lagrangian heuristic, relaxing demand and relocation linking constraints, for all 540 instances for the DFLP_RPC with RUC constraints.

		CPLEX (3hs)			LH-RUC1 (3hs)	
q	Instance size	# ns	Avg Gap %	Max Gap %	Avg Gap %	Max Gap %
3	50/50	0	0.23	1.21	0.31	1.22
	50/200	0	0.00	0.01	0.53	0.96
	100/100	0	0.01	0.18	0.40	0.87
	100/400	0	0.00	0.01	0.59	0.92
	150/150	0	0.02	0.26	0.29	0.99
	150/600	1	0.01	0.02	0.69	0.98
	200/200	0	0.12	2.04	0.42	0.94
	200/800	9	0.40	3.48	0.22	0.81
	250/250	0	0.10	1.79	0.40	0.93
	250/1000	15	0.84	2.51	0.05	0.11
	Avg All	25	0.10	3.48	0.43	1.22
5	50/50	3	2.15	8.39	1.19	3.28
	50/200	0	0.45	3.43	0.49	1.32
	100/100	4	0.36	1.48	0.42	1.29
	100/400	7	0.01	0.03	0.48	0.97
	150/150	7	0.92	7.56	0.34	0.89
	150/600	12	0.04	0.19	0.20	0.77
	200/200	10	0.09	0.39	0.16	0.50
	200/800	17	0.01	0.01	0.84	0.84
	250/250	12	0.04	0.14	0.18	0.73
	250/1000	18	-	-	-	-
	Avg All	90	0.63	8.39	0.50	3.28
10	50/50	12	1.78	4.07	1.85	2.81
	50/200	15	0.71	1.51	0.60	1.21
	100/100	13	2.88	8.82	1.92	3.14
	100/400	18	-	-	-	-
	150/150	18	-	-	-	-
	150/600	18	-	-	-	-
	200/200	18	-	-	-	-
	200/800	18	-	-	-	-
	250/250	18	-	-	-	-
	250/1000	18	-	-	-	-
	Avg All	166	1.94	8.82	1.61	3.14
All	**Avg All**	281	0.38	8.82	0.52	3.28

Table 6.XVII: Comparison of solution quality for CPLEX and the Lagrangian heuristic, relaxing demand and relocation linking constraints, considering instances for the DFLP_RPC with RUC constraints where CPLEX found feasible solutions.

6.5 Conclusions and Future Research

In this chapter, we investigated alternative solution approaches for the CSLP. We proposed an alternative model for the CSLP with and without relocation, based on the GMC modeling technique. The GMC based models have shown to provide significantly lower integrality gaps. On average, these gaps are more than 10 times lower than those of the 2i formulation based on the formulation presented in Chapter 3. However, the resulting models are too large to be solved by generic MIP solvers such as CPLEX, exceeding the available memory resources. The low integrality gaps for the GMC based formulations suggest that this formulation is an appropriate candidate for Lagrangian relaxation, resulting in strong lower bounds for the problem. Lagrangian heuristics have therefore been proposed, based on similar concepts as presented in Chapter 5.

For the problem variant without relocation, the relaxation of the demand constraints results in a Lagrangian subproblem that can be decomposed and solved in a similar manner as for the DFLPG. The maximum deviation of the final solution value from the best known lower bound is 3.72% and the average deviation is 0.77% among all 540 instances, whereas CPLEX does not find feasible solutions for about half of the instances.

For the problem variant with relocation, two relaxations are proposed: one that relaxes only the demand constraints, and one that relaxes the demand constraints and the relocation constraints. Computational results for heuristics based on both relaxations indicate that relaxing both the demand and the relocation linking constraints results in a higher solution quality. Taking into consideration the relocation decisions from the Lagrangian solutions to generate upper bounds at each iteration, as well as the use of the restricted MIP in a second optimization phase, yield competitive results for all instances with a maximum deviation of less than 5.5% and an average deviation of 0.96%.

Finally, adding the RUC constraints to the DFLP_RPC significantly complicates the solution of the problem. CPLEX does not find feasible solutions for more than half of the instances, in particular when a high number of capacity levels is used (i.e., $q = 10$). For the instances where CPLEX finds feasible solutions within the given time limit, the

Lagrangian heuristics provides a significantly lower maximum optimality gap. Furthermore, the heuristic provides robust results for all other instances. The average deviation of 1.37% is quite low. For some instances, the maximum deviation from the best known lower bounds is about 27% when a time limit of 3 hours is used. To decrease the optimality gaps for those instances, it may be promising to tune the MIP parameters better. Furthermore, it may be possible to solve the subproblem in a more efficient way.

CHAPTER 7

CONCLUSIONS

In this chapter, the contributions made in the thesis are summarized. Then, promising future research directions are discussed.

7.1 Summary

This thesis has investigated dynamic facility location problems that involve complex cost structures for the adjustment of capacities over time. Chapter 3 reported on an industrial application found in the forestry sector. The multi-period facility location problem considered in this application contains three interesting extensions of classical facility location problems: partial facility closing and reopening, economies of scale for capacity changes that depend on the total capacity involved in the operation and on the current capacity level, and capacity constraints that involve rounding of the total demand allocated to a facility. We have shown how to model the problem as a MIP model and demonstrated its usefulness on two case studies based on real data from a Canadian logging company.

Chapter 4 presented a MIP model that is capable to represent detailed cost structures as the one mentioned above. This model, referred to as the GMC, is very versatile and generalizes several classical facility location problems. It is shown that its LP relaxation is stronger than those of specialized formulations, derived from the existing literature. As the proposed modeling technique also performs better to find optimal solutions using generic MIP solvers, it is a promising alternative to model any location problem that involves modular capacities and their dynamic adjustment along time.

In Chapter 5, we developed Lagrangian relaxation heuristics based on the GMC formulation. They outperform state-of-the-art MIP solvers and are capable to find high quality solutions for large instances with up to 250 candidate facility locations and 1000 customers. Due to the strength of the used model, the heuristics provide very good bounds on the optimal value. Given the generality of the GMC model, the proposed heuristic can handle an entire family of problems, consisting of all those that can be modeled by the DFLPG. This type of heuristics can also be used to evaluate "what-if" scenarios, which require repeatedly solving the problem under different scenarios, and perform re-optimization.

Finally, we have shown in Chapter 6 how the Lagrangian heuristics can be used in the context of the industrial application, a much more complex problem. The GMC modeling technique is extended to match the case where capacity changes are registered on two different levels: the existing capacity and the open capacity. Given that the resulting models are strong, but very large, they provided an ideal base to apply mathematical decomposition. Two different Lagrangian relaxations have been tested. Computational results demonstrate their usefulness in practice, as the heuristics provide high quality solutions for all instances, while generic MIP solvers fail on most of the tested problems. Adding the RUC constraints significantly complicates the problem. For test instances where generic MIP solvers find feasible solutions, the Lagrangian heuristic performs significantly better. The heuristic also provides very low average gaps for all other instances.

We believe that this dissertation has contributed to dynamic facility location and filled a gap in the landscape of problem variants by providing effective techniques to model and to solve problems with complex cost structures and capacity adjustment over time. We hope it will stimulate other researchers to build upon these results.

7.2 Future Research Directions

The results presented in this thesis open up future research directions on several levels. Those that seem most important are now discussed. On the level of the industrial application, the presented model assumed a few simplifications that can be addressed in future works. Camps may be composed by trailers of different capacities. Clearly, this increases the complexity of the cost structure. Furthermore, relocation has only been considered for entire facilities. Enabling the model to relocate parts of a camp to another location and even merge two camps may lead to solutions that are more cost efficient. However, a corresponding model has to be capable of supporting such a cost structure, which most probably has to simultaneously keep track of the number of hosting trailers and supporting trailers. These suggestions may be reformulated in the context of general facility location problems and therefore contribute interesting extensions to the classical problems, such as partial relocation of facilities under the consideration of a cost matrix.

The CSLP also involved capacity constraints with rounding of the total demand allocated to a facility. Such constraints may be found in other applications, but are currently ignored in their corresponding models. The valid inequalities presented in this thesis are very effective and therefore encouraging to use these types of capacity constraints in other models without expecting a significant increase in the difficulty to solve the new model.

Another promising direction for future research is the modeling technique used in the GMC formulation. This technique has resulted in very strong models and may be used for different types of location problems where modular capacity adjustments are performed. Two strong candidates for that are multi-period hub location problems where hubs may adjust their capacities, and multi-period network design problems where arcs may adjust their capacities.

Finally, much can be done to improve the solution of these problems. Other decomposition techniques such as Benders decomposition seem very promising for the DFLPG. The Lagrangian relaxation heuristics to solve the CSLP may yield better results by improving the way the restricted MIP model is constructed to find solutions of better quality. The presented heuristics have been capable to solve instances with 250 candidate facility locations, 1000 customers and 10 capacity levels. However, decomposition methods definitively hold the potential to go further. Generally speaking, making use of restricted MIP models and solving them by generic MIP solvers may be a successful strategy, since one can benefit from the constant improvement of the latter.

BIBLIOGRAPHY

Aardal, K. (1998a). Capacitated facility location: separation algorithms and computational experience. *Mathematical Programming*, 81(2):149–175.

Aardal, K. (1998b). Reformulation of capacitated facility location problems: How redundant information can help. *Annals of Operations Research*, 82(0):289 – 308.

Aardal, K., Pochet, Y., and Wolsey, L. a. (1995). Capacitated Facility Location: Valid Inequalities and Facets. *Mathematics of Operations Research*, 20(3):562–582.

Agar, M. C. and Salhi, S. (1998). Lagrangean heuristics applied to a variety of large capacitated plant location problems. *Journal of the Operational Research Society*, 49(10):1072–1084.

Albareda-Sambola, M., Fernandez, E., Hinojosa, Y., and Puerto, J. (2009). The multi-period incremental service facility location problem. *Computers & Operations Research*, 36(5):1356–1375.

Alumur, S. A., Nickel, S., Saldanha da Gama, F., and Verter, V. (2012). Multi-period reverse logistics network design. *European Journal of Operational Research*, 220(1):67–78.

Amiri-Aref, M., Javadian, N., Tavakkoli-Moghaddam, R., and Aryanezhad, M.-B. (2011). A multi-period facility location-relocation problem in the presence of a probabilistic line barrier. In *IEEM*, pages 1118–1122. IEEE.

Antunes, A. P. and Peeters, D. (2001). On solving complex multi-period location models using simulated annealing. *European Journal of Operational Research*, 130(1):190–201.

Arabani, A. B. and Zanjirani Farahani, R. (2011). Facility Location Dynamics: An Overview of Classifications and Applications. *Computers & Industrial Engineering*, 62(1):408–420.

Arostegui Jr., M. A., Kadipasaoglu, S. N., and Khumawala, B. M. (2006). An empirical comparison of Tabu Search, Simulated Annealing, and Genetic Algorithms for facilities location problems. *International Journal of Production Economics*, 103(2):742–754.

Audy, J. F., D'Amours, S., and Rönnqvist, M. (2012). Planning Methods and Decision Support Systems in Vehicle Routing Problems for Timber Transportation: A Review. Technical report, Centre interuniversitaire de recherche sur les réseaux d'entreprise, la logistique et le transport, CIRRELT-2012-38.

Avella, P. and Boccia, M. (2007). A cutting plane algorithm for the capacitated facility location problem. *Computational Optimization and Applications*, 43(1):39–65.

Balas, E. and Zemel, E. (1980). An algorithm for large zero-one knapsack problems. *operations Research*.

Ballou, R. H. (1968). Dynamic Warehouse Location Analysis. *Journal of Marketing Research*, 5(3):271–276.

Barcelo, J., Hallefjord, A., Fernandez, E., and Jörnsten, K. (1990). Lagrangean relaxation and constraint generation procedures for capacitated plant location problems with single sourcing. *OR Spektrum*, 12(2):79–88.

Beasley, J. (1993). Lagrangean heuristics for location problems. *European Journal of Operational Research*, 65(3):383–399.

Behmardi, B. and Lee, S. (2008). Dynamic Multi-commodity Capacitated Facility Location Problem in Supply Chain. In *Proceedings of the 2008 Industrial Engineering Research Conference*, pages 1914–1919.

Bellman, R. (1957). *Dynamic Programming*. Princeton University Press.

Bellman, R. (1966). Dynamic Programming. *Science*, 153:34–37.

Benders, J. F. (1962). Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252.

Berman, O. and LeBlanc, B. (1984). Location-Relocation of Mobile Facilities on a Stochastic Network. *Transportation Science*, 18(4).

Berman, O. and Wang, Q. (2008). Locating a semi-obnoxious facility with expropriation. *Computers & Operations Research*, 35(2):392–403.

Borghetti, A., Frangioni, A., Lacalandra, F., and Nucci, C. (2003). Lagrangian heuristics based on disaggregated bundle methods for hydrothermal unit commitment. *IEEE Transactions on Power Systems*, 18(1):313–323.

Boschetti, M. and Maniezzo, V. (2007). Benders decomposition, Lagrangean relaxation and metaheuristic design. *Journal of Heuristics*, 15(3):283–312.

Bredström, D., Jönsson, P., and Rönnqvist, M. (2010). Annual Planning of Harvesting Resources in the Forest Industry. *International Transactions in Operational Research*, 17(2):155–177.

Bredström, D., Lundgren, J. T., Rönnqvist, M., Carlsson, D., and Mason, A. (2004). Supply chain optimization in the pulp mill industry - IP models, column generation and novel constraint branches. *European Journal of Operational Research*, 156(1):2–22.

Broek, J. V., Schütz, P., Stougie, L., and Tomasgard, A. (2006). Location of slaughterhouses under economies of scale. *European Journal of Operational Research*, 175(2):740–750.

Brotcorne, L., Laporte, G., and Semet, F. (2003). Ambulance location and relocation models. *European Journal of Operational Research*, 147(3):451–463.

Campbell, J. F. and O'Kelly, M. E. (2012). Twenty-five years of hub location research. *Transportation Science*, 46(2):153–169.

Canel, C., Khumawala, B. M., Law, J., and Loh, A. (2001). An Algorithm for the Capacitated, Multi-Commodity Multi-Period Facility Location Problem. *Computers & Operations Research*, 28(5):411–427.

Carlsson, D., D'Amours, S., Martel, A., and Rönnqvist, M. (2009). Supply Chain Planning Models in the Pulp and Paper Industry. *INFOR: Information Systems and Operational Research*, 47(3):167–183.

Carlsson, D. and Rönnqvist, M. (2005). Supply chain management in forestry—case studies at Södra Cell AB. *European Journal of Operational Research*, 163(3):589–616.

Chan, T., Cordeau, J.-F., and Laporte, G. (2009). Locating Satellite Yards in Forestry Operations. *INFOR: Information Systems and Operational Research*, 47(3):223–234.

Chardaire, P., Sutter, A., and Costa, M.-C. (1996). Solving the dynamic facility location problem. *Networks*, 28(2):117–124.

COIN-OR (2014). COmputational INfrastructure for Operations Research.

Contardo, C., Cordeau, J.-F., and Gendron, B. (2013). A computational comparison of flow formulations for the capacitated location-routing problem. *Discrete Optimization*, 10(4):263–295.

Contreras, I., Cordeau, J.-F., and Laporte, G. (2011a). Benders Decomposition for Large-Scale Uncapacitated Hub Location. *Operations Research*, 59(6):1477–1490.

Contreras, I., Cordeau, J.-F., and Laporte, G. (2011b). The Dynamic Uncapacitated Hub Location Problem. *Transportation Science*, 45(1):18–32.

Cordeau, J.-F., Pasin, F., and Solomon, M. M. (2006). An integrated model for logistics network design. *Annals of Operations Research*, 144(1):59–82.

Cornuéjols, G. and Sridharan, R. (1991). A comparison of heuristics and relaxations for the Capacitated Plant Location Problem. *European Journal of Operational Research*, 50:280–297.

Correia, I. and Captivo, M. E. (2003). A Lagrangean heuristic for a modular capacitated location problem. *Annals of Operations Research*, 122:141–161.

Correia, I. and Captivo, M. E. (2006). Bounds for the single source modular capacitated plant location problem. *Computers & Operations Research*, 33(10):2991–3003.

Correia, I., Gouveia, L., and Saldanha da Gama, F. (2010). Discretized formulations for capacitated location problems with modular distribution costs. *European Journal of Operational Research*, 204(2):237–244.

D'Amours, S., Rönnqvist, M., and Weintraub, A. (2008). Using Operational Research for Supply Chain Planning in the Forest Products Industry. *INFOR: Information Systems and Operational Research*, 46(4):265–281.

Daskin, M. S. (2008). What you should know about location modeling. *Naval Research Logistics*, 55(4):283–294.

Dekker, R., Bloemhof, J., and Mallidis, I. (2012). Operations Research for green logistics – An overview of aspects, issues, contributions and challenges. *European Journal of Operational Research*, 219(3):671–679.

Diabat, A., Richard, J.-P., and Codrington, C. W. (2013). A Lagrangian relaxation approach to simultaneous strategic and tactical planning in supply chain design. *Annals of Operations Research*, 203(1):55–80.

Dias, J., Captivo, M. E., and Clímaco, J. (2006). Capacitated dynamic location problems with opening, closure and reopening of facilities. *IMA Journal of Management Mathematics*, 17(4):317–348.

Dias, J., Captivo, M. E., and Clímaco, J. (2007). Dynamic Location Problems with Discrete Expansion and Reduction Sizes of Available Capacities. *Investigação Operacional*, 27(2):107–130.

Du, D., Wang, X., and Xu, D. (2009). An approximation algorithm for the k-level capacitated facility location problem. *Journal of Combinatorial Optimization*, 20(4):361–368.

Easwaran, G. and Üster, H. (2009). Tabu Search and Benders Decomposition Approaches for a Capacitated Closed-Loop Supply Chain Network Design Problem. *Transportation Science*, 43(3):301–320.

Eiselt, H., Laporte, G., and Thisse, J. (1993). Competitive location models: A framework and bibliography. *Transportation Science*, 27(1):44–54.

Elhedhli, S. and Wu, H. (2010). A Lagrangean Heuristic for Hub-and-Spoke System Design with Capacity Selection and Congestion. *INFORMS Journal on Computing*, 22(2):282–296.

Epstein, R., Morales, R., Seron, J., and Weintraub, A. (1999). Use of OR Systems in the Chilean Forest Industries. *Interfaces*, 29(1):7–29.

Fisher, M. L. (1981). The Lagrangian Relaxation Method for Solving Integer Programming Problems. *Management Science*, 27(1):1–18.

Fisher, M. L. (1985). An Applications Oriented Guide to Lagrangian Relaxation. *Interfaces*, 15(2):10–21.

Fleischmann, B., Ferber, S., and Henrich, P. (2006). Strategic Planning of BMW's Global Production Network. *Interfaces*, 36(3):194–208.

Frangioni, A. (2002). Generalized bundle methods. *SIAM Journal on Optimization*, 13(1):117–156.

Frangioni, A. (2005). About Lagrangian Methods in Integer Optimization. *Annals of Operations Research*, 139(1):163–193.

Frangioni, A. and Gallo, G. (1999). A bundle type dual-ascent approach to linear multicommodity min-cost flow problems. *INFORMS Journal on Computing*, 11(4):370–393.

Gendron, B. and Crainic, T. G. (1994). Relaxations for multicommodity capacitated network design problems. Technical report, Publication CRT-945, Centre de recherche sur les transports, Université de Montréal.

Gendron, B., Khuong, P., and Semet, F. (2011). A Multilayer Variable Neighborhood Search Method for a Two-Echelon Adaptive Location-Distribution Problem.

Gendron, B., Khuong, P., and Semet, F. (2013). A Lagrangian-Based Branch-and-Bound Algorithm for the Two-Level Uncapacitated Facility Location Problem with Single-Assignment Constraints. Technical report, Centre interuniversitaire de recherche sur les reseaux d'entreprise, la logistique et le transport, CIRRELT-2013-21.

Gendron, B. and Larose, M. (2014). Branch-and-price-and-cut for large-scale multicommodity capacitated fixed-charge network design. *EURO Journal on Computational Optimization*, 2(1-2):55–75.

Gendron, B. and Semet, F. (2009). Formulations and relaxations for a multi-echelon capacitated location–distribution problem. *Computers & Operations Research*, 36(5):1335–1355.

Geoffrion, A. (1974). Lagrangean relaxation for integer programming. *Mathematical Programming Study 2*, 2:82–114.

Geoffrion, A. M. and Graves, G. W. (1974). Multicommodity Distribution System Design by Benders Decomposition. *Management Science*, 20(5):822–844.

Ghanmi, A. (2010). Canadian forces global reach support hubs: facility location and aircraft routing models. *Journal of the Operational Research Society*, 62(4):638–650.

Görtz, S. and Klose, A. (2012). A Simple but Usually Fast Branch-and-Bound Algorithm for the Capacitated Facility Location Problem. *INFORMS Journal on Computing*, 24(4):597–610.

Gourdin, E. and Klopfenstein, O. (2008). Multi-period capacitated location with modular equipments. *Computers & Operations Research*, 35(3):661–682.

Gouveia, L. and Saldanha da Gama, F. (2006). On the capacitated concentrator location problem: a reformulation by discretization. *Computers & Operations Research*, 33(5):1242–1258.

Gue, K. R. (2003). A dynamic distribution model for combat logistics. *Computers & Operations Research*, 30(3):367–381.

Guignard, M. (2003). Lagrangean Relaxation. *Top*, 11(2):151–228.

Guignard, M. and Kim, S. (1987). Lagrangean decomposition: a model yielding stronger Lagrangean bounds. *Mathematical Programming*, 39:215–228.

Gurobi Optimization, Inc. (2014). Gurobi Optimizer Reference Manual.

Haartveit, E. Y., Kozak, R. A., and Maness, T. C. (2004). Supply Chain Management Mapping for the Forest Products Industry : Three Cases from Western Canada. *Business*, 1(1):1–30.

Hamacher, H. and Nickel, S. (1998). Classification of location models. *Location Science*, 6(1-4):229–242.

Harkness, J. (2003). Facility location with increasing production costs. *European Journal of Operational Research*, 145(1):1–13.

Harris, I., Mumford, C. L., and Naim, M. M. (2014). A hybrid multi-objective approach to capacitated facility location with flexible store allocation for green logistics modeling. *Transportation Research Part E: Logistics and Transportation Review*, 66:1–22.

Held, M., Wolfe, P., and Crowder, H. P. (1974). Validation of Subgradient Optimization. *Mathematical Programming*, 6:62–88.

Hinojosa, Y., Kalcsics, J., Nickel, S., Puerto, J., and Velten, S. (2008). Dynamic supply chain design with inventory. *Computers & Operations Research*, 35(2):373–391.

Hochbaum, D. (1998). Locating centers in a dynamically changing network, and related problems. *Location Science*, 6(1-4):243–256.

Holmberg, K. (1994). Solving the staircase cost facility location problem with decomposition and piecewise linearization. *European Journal of Operational Research*, 75(1):41–61.

Holmberg, K. and Ling, J. (1997). A Lagrangean heuristic for the facility location problem with staircase costs. *European Journal of Operational Research*, 97(1):63–74.

Holmberg, K., Rönnqvist, M., and Yuan, D. (1999). An exact algorithm for the capacitated facility location problems with single sourcing. *European Journal of Operational Research*, 113(3):544–559.

Holmberg, K. and Yuan, D. (2000). A Lagrangian heuristic based branch-and-bound approach for the capacitated network design problem. *Operations Research*, 48(3):461–481.

Hugo, A. and Pistikopoulos, E. (2005). Environmentally conscious long-range planning and design of supply chain networks. *Journal of Cleaner Production*, 13(15):1471–1491.

IBM (2010). IBM ILOG CPLEX Optimizer.

Jacobsen, S. K. (1990). Multiperiod Capacitated Location Models. In Francis, R. and Mirchandani, P., editors, *Discrete Location Theory*, chapter 4, pages 173–208. Wiley-Interscience.

Jena, S. D., Cordeau, J.-F., and Gendron, B. (2012). Modeling and solving a logging camp location problem. *Annals of Operations Research*, doi 10.1007/s10479-012-1278-z.

Jena, S. D., Cordeau, J.-F., and Gendron, B. (2013). Dynamic Facility Location with Generalized Modular Capacities. Technical report, CIRRELT-2013-18, Centre interuniversitaire de recherche sur les reseaux d'entreprise, la logistique et le transport.

Kariv, O. and Hakimi, S. (1979a). An algorithmic approach to network location problems. I: The p-centers. *SIAM Journal on Applied Mathematics*, 37(3):513–538.

Kariv, O. and Hakimi, S. (1979b). An algorithmic approach to network location problems. II: The p-medians. *SIAM Journal on Applied Mathematics*, 37(3):539–560.

Kim, D.-G. and Kim, Y.-D. (2013). A Lagrangian heuristic algorithm for a public healthcare facility location problem. *Annals of Operations Research*, 206(1):221–240.

Klose, A. and Drexl, A. (2005). Facility location models for distribution system design. *European Journal of Operational Research*, 162(1):4–29.

Krzyzanowski, W. (1927). Review of the Literature of the Location of Industries. *The Journal of Political Economy*, 35(2):278–291.

Lee, C. Y. (1991). An optimal algorithm for the multiproduct capacitated facility location problem with a choice of facility type. *Computers & Operations Research*, 18(2):167–182.

Lee, C. Y. (1993a). A cross decomposition algorithm for a multiproduct-multitype facility location problem. *Computers & Operations Research*, 20(5):527–540.

Lee, C. Y. (1993b). The Multiproduct Warehouse Location Problem: Applying a Decomposition Algorithm. *International Journal of Physical Distribution & Logistics Management*, 23(6):3–13.

Lee, D.-H. and Dong, M. (2008). A heuristic approach to logistics network design for end-of-lease computer products recovery. *Transportation Research Part E: Logistics and Transportation Review*, 44(3):455–474.

Leung, J. M. Y. and Magnanti, T. L. (1989). Valid inequalities and facets of the capacitated plant location problem. *Mathematical Programming*, 44(1-3):271–291.

Li, J., Chu, F., and Prins, C. (2009). Lower and upper bounds for a capacitated plant location problem with multicommodity flow. *Computers & Operations Research*, 36(11):3019–3030.

Lim, G. J. and Sonmez, A. D. (2013). γ-Robust facility relocation problem. *European Journal of Operational Research*, 229(1):67–74.

Luss, H. (1982). Operations Research and Capacity Expansion Problems: A Survey. *Operations Research*, 30(5):907–947.

Martello, S. and Toth, P. (1990). *Knapsack problems: algorithms and computer implementations*. Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons.

Melachrinoudis, E. (2000). The dynamic relocation and phase-out of a hybrid, two-echelon plant/warehousing facility: A multiple objective approach. *European Journal of Operational Research*, 123(1):1–15.

Melkote, S. and Daskin, M. S. (2001). Capacitated facility location/network design problems. *European Journal of Operational Research*, 129(3):481–495.

Melo, M. T., Nickel, S., and Saldanha da Gama, F. (2006). Dynamic multi-commodity capacitated facility location: a mathematical modeling framework for strategic supply chain planning. *Computers & Operations Research*, 33(1):181–208.

Melo, M. T., Nickel, S., and Saldanha da Gama, F. (2009a). Facility location and supply chain management – A review. *European Journal of Operational Research*, 196(2):401–412.

Melo, M. T., Nickel, S., and Saldanha da Gama, F. (2011a). A tabu search heuristic for redesigning a multi-echelon supply chain network over a planning horizon. *International Journal of Production Economics*, 136(1):218–230.

Melo, M. T., Nickel, S., and Saldanha-da Gama, F. (2011b). An efficient heuristic approach for a multi-period logistics network redesign problem. *Top*, 22:80–108.

Melo, T., Nickel, S., and Saldanha da Gama, F. (2009b). An LP-rounding heuristic to solve a multi-period facility relocation problem. Technical Report 168, ITWM - Fraunhofer Institute for Industrial Mathematics, Kaiserslautern, Germany. Available at http://www.itwm.fraunhofer.de/fileadmin/ITWM-Media/Zentral/Pdf/Berichte_ITWM/2009/bericht_168.pdf.

Min, H. and Melachrinoudis, E. (1999). The relocation of a hybrid manufacturing/distribution facility from supply chain perspectives: a case study. *Omega*, 27(1):75–85.

O'Kelly, M. (1986). The location of interacting hub facilities. *Transportation Science*, 20(2):92–106.

Owen, S. H. and Daskin, M. S. (1998). Strategic facility location: A review. *European Journal of Operational Research*, 111(3):423–447.

Paquet, M., Martel, A., and Desaulniers, G. (2004). Including technology selection decisions in manufacturing network design models. *International Journal of Computer Integrated Manufacturing*, 17(2):117–125.

Pirkul, H. and Jayaraman, V. (1998). A multi-commodity, multi-plant, capacitated facility location problem: formulation and efficient heuristic solution. *Computers & Operations Research*, 25(10):869–878.

Prodhon, C. and Prins, C. (2014). A survey of recent research on location-routing problems. *European Journal of Operational Research*, pages 1–17.

Revelle, C. and Eiselt, H. (2005). Location analysis: A synthesis and survey. *European Journal of Operational Research*, 165(1):1–19.

Revelle, C., Eiselt, H., and Daskin, M. S. (2008). A bibliography for some fundamental problem categories in discrete location science. *European Journal of Operational Research*, 184(3):817–848.

Rönnqvist, M. (2003). Optimization in forestry. *Mathematical programming Series B*, 97(1-2):267–284.

Sankaran, J. K. and Raghavan, N. R. S. (1997). Locating and sizing plants for bottling propane in South India. *Interfaces*.

Schütz, P., Stougie, L., and Tomasgard, A. (2008). Stochastic facility location with general long-run costs and convex short-run costs. *Computers & Operations Research*, 35(9):2988–3000.

Shetty, B. (1990). Approximate solutions to large scale capacitated facility location problems. *Applied Mathematics and Computation*, 39(2):159–175.

Shmoys, D. B., Tardos, E., and Aardal, K. (1997). Approximation algorithms for facility location problems (extended abstract). *Proceedings of the twenty-ninth annual ACM symposium on Theory of computing - STOC '97*, 5:265–274.

Shulman, A. (1991). An Algorithm for Solving Dynamic Capacitated Plant Location Problems with Discrete Expansion Sizes. *Operations Research*, 39(3):423–436.

Smith, H. K., Laporte, G., and Harper, P. R. (2009). Locational analysis: highlights of growth to maturity. *Journal of the Operational Research Society*, 60:S140–S148.

Snyder, L. V. (2006). Facility location under uncertainty: a review. *IIE Transactions*, 38(7):547–564.

Sonmez, A. D. and Lim, G. J. (2012). A decomposition approach for facility location and relocation problem with uncertain number of future facilities. *European Journal of Operational Research*, 218(2):327–338.

Sridharan, R. (1991). A Lagrangian Heuristic for the Capacitated Plant Location Problem with Side Constraints. *Journal of the Operational Research Society*, 42(7):579–585.

Sridharan, R. (1995). The capacitated plant location problem. *European Journal of Operational Research*, 87(2):203–213.

Sweeney, D. J. and Tatham, R. L. (1976). An Improved Long-Run Model for Multiple Warehouse Location. *Management Science*, 22(7):748–758.

Thanh, P., Bostel, N., and Peton, O. (2008). A dynamic model for facility location in the design of complex supply chains. *International Journal of Production Economics*, 113(2):678–693.

Thomas, D. J. and Griffin, P. M. (1996). Coordinated supply chain management. *European Journal of Operational Research*, 94(1):1–15.

Troncoso, J. and Garrido, R. (2005). Forestry production and logistics planning: an analysis using mixed-integer programming. *Forest Policy and Economics*, 7(4):625–633.

Vahid, S. and Maness, T. (2010). Modelling customer demand in forest products industry supply chains: a review of the literature. *International Journal of Simulation and Process Modelling*, 6(2):103–114.

Vahidnia, M. H., Alesheikh, A. a., and Alimohammadi, A. (2009). Hospital site selection using fuzzy AHP and its derivatives. *Journal of environmental management*, 90(10):3048–56.

Van Roy, T. J. (1983). Cross decomposition for mixed integer programming. *Mathematical Programming*, 25:46–63.

Van Roy, T. J. (1986). A Cross Decomposition Algorithm for Capacitated Facility Location. *Operations Research*, 34(1):145–163.

Van Roy, T. J. and Erlenkotter, D. (1982). A Dual-Based Procedure for Dynamic Facility Location. *Management Science*, 28(10):1091–1105.

Vila, D., Martel, A., and Beauregard, R. (2006). Designing logistics networks in divergent process industries: A methodology and its application to the lumber industry. *International Journal of Production Economics*, 102(2):358–378.

Wang, Q., Batta, R., Bhadury, J., and Rump, C. M. (2003). Budget constrained location problem with opening and closing of facilities. *Computers & Operations Research*, 30:2047 – 2069.

Warszawski, A. (1973). Multi-Dimensional Location Problems. *Operational Research Quarterly (1970-1977)*, 24(2):165–179.

Weber, A. (1929). *Alfred Weber's theory of the location of industries (Translation of "Über den Standort der Industrien").* The University of Chicago Press.

Weintraub, A. and Romero, C. (2006). Operations Research Models and the Management of Agricultural and Forestry Resources: A Review and Comparison. *Interfaces*, 36(5):446–457.

Wentges, P. (1996). Accelerating Benders' decomposition for the capacitated facility location problem. *Mathematical Methods of Operations Research*, 44(2):267–290.

Wesolowsky, G. O. (1973). Dynamic facility location. *Management Science*, 19(11):1241–1248.

Wesolowsky, G. O. and Truscott, W. G. (1975). The Multiperiod Location-Allocation Problem with Relocation of Facilities. *Management Science*, 22(1):57–65.

Wilhelm, W., Han, X., and Lee, C. (2013). Computational comparison of two formulations for dynamic supply chain reconfiguration with capacity expansion and contraction. *Computers & Operations Research*, 40(10):2340–2356.

Wu, L., Zhang, X., and Zhang, J. (2006). Capacitated facility location problem with general setup cost. *Computers & Operations Research*, 33(5):1226–1241.

Xie, W. and Ouyang, Y. (2013). Dynamic Planning of Facility Locations with Benefits from Multitype Facility Colocation. *Computer-Aided Civil and Infrastructure Engineering*, 28(9):666–678.

Zanjirani Farahani, R., Drezner, Z., and Asgari, N. (2008). Single facility location and relocation problem with time dependent weights and discrete planning horizon. *Annals of Operations Research*, 167(1):353–368.

Zanjirani Farahani, R. and Hekmatfar, M., editors (2009). *Facility Location - Concepts, Models, Algorithms and Case Studies*. Contributions to Management Science. Physica-Verlag HD, Heidelberg.

Zanjirani Farahani, R., Hekmatfar, M., Fahimnia, B., and Kazemzadeh, N. (2014). Hierarchical facility location problem: Models, classifications, techniques, and applications. *Computers & Industrial Engineering*, 68:104–117.

APPENDIX A

SUPPLEMENT TO CHAPTER 3

In this section, we show how the relocation of facilities can be modeled in two different ways: by direct arcs, one for each direction between two locations, and by central hub nodes that redistribute facilities relocated from certain locations to other locations. We then prove that the models based on the two techniques are equally strong in the sense that their LP relaxation provides the same lower bound.

A.1 Relocation for the CSLP: Models

A.1.1 Relocation via Hub Nodes

The model for the CSLP, given by (3.12) - (3.36) (see Section 3.5), uses hub nodes to relocate facilities. In the model, flow variables w^O_{it} and w^I_{it} are used to relocate capacity from and to a location i, respectively. Binary variables w^{BO}_{ikt} are linked to the flow variables w^O_{it} and have value 1 if a facility of size k is relocated, from location i in the beginning of time period t. In the same way, binary variables w^{BI}_{ikt} are linked to the flow variables w^I_{it} and have value 1 if a facility of size k is relocated to location i in the beginning of time period t.

A.1.2 Relocation via Direct Arcs

An alternative to using hub nodes is to use direct arcs between the locations to perform facility relocation. An advantage of this technique is that the information of origin and destination is still available and may be used to better represent the costs involved in the facility location, e.g., by considering distance dependent costs.

To model relocation by the use of direct arcs, we replace the relocation variables mentioned above by the following two variables:

– $w_{i_1 i_2 t} \in \mathbb{Z}^+_0$ - number of trailers that are moved from location $i_1 \in I$ to location $i_2 \in I$ before time period $t \in T$.

– $w^B_{i_1 i_2 kt} \in \{0,1\}$ - 1, if a camp with k trailers is moved from location $i_1 \in I$ to location $i_2 \in I$ before time period $t \in T$.

In the objective function, the new term that represents the relocation costs is as follows:

$$\sum_{i_1 \in I} \sum_{i_2 \in I} \sum_{k \in K} \sum_{t \in T} c^R_k w^B_{i_1 i_2 kt}$$

The network flow structure using direct relocation arcs is illustrated in Figure A.1.

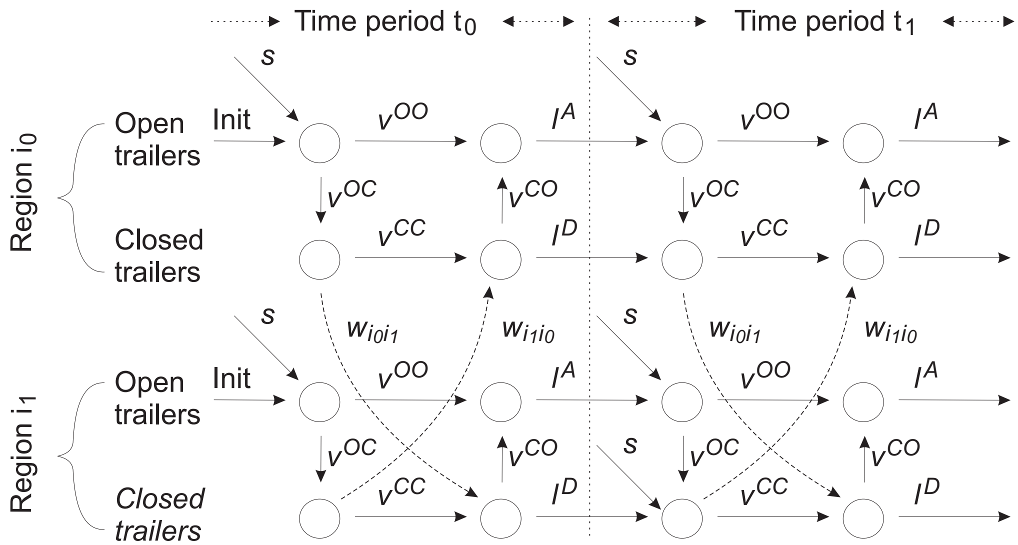

Figure A.1: Network flow structure for managing the number of open and closed trailers at each location as well as the camp relocation by direct arcs.

This results in the following model:

$$\min \sum_{i \in I} \sum_{k \in K} \sum_{t \in T} c^M_{kt} y_{ikt} + \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \sum_{p \in P} \sum_{t \in T} c^V_{ijkpt} x_{ijkpt} \tag{A.1}$$

$$+ \sum_{i \in I} \sum_{q \in K} \sum_{t \in T} c^C_{iq} s_{iqt} + \sum_{i_1 \in I} \sum_{i_2 \in I} \sum_{k \in K} \sum_{t \in T} c^R_k w^B_{i_1 i_2 kt}$$

$$+ \sum_{i \in I} \sum_{k \in K} \sum_{t \in T} c^{TO}_k v^{BCO}_{ikt} + \sum_{i \in I} \sum_{k \in K} \sum_{t \in T} c^{TC}_k v^{BOC}_{ikt}$$

$$s.t. \ (3.13) - (3.18), (3.20), (3.22), (3.23), (3.27), (3.28),$$

$$(3.31) - (3.34),$$

$$l^C_{i(t-1)} + v^{OC}_{it} = v^{CC}_{it} + \sum_{i_2 \in I} w_{ii_2t} \quad ;\forall i \in I \ ;\forall t \in T \tag{A.2}$$

$$v^{CC}_{it} + \sum_{i_1 \in I} w_{i_1it} = v^{CO}_{it} + l^C_{it} \quad ;\forall i \in I \ ;\forall t \in T \tag{A.3}$$

$$v^{CC}_{it} + v^{OO}_{it} \leq \overline{K}\left(1 - \sum_{i_2 \in I}\sum_{k \in K} w^B_{ii_2kt}\right) \quad ;\forall i \in I \ ;\forall t \in T \tag{A.4}$$

$$v^{CC}_{it} + v^{OO}_{it} - \sum_{q \in K} qs_{iqt} \leq \overline{K}\left(1 - \sum_{i_1 \in I}\sum_{k \in K} w^B_{i_1ikt}\right) \quad ;\forall i \in I \ ;\forall t \in T \tag{A.5}$$

$$\sum_{k \in K} kw^B_{i_1i_2kt} = w_{i_1i_2t} \quad ;\forall i_1 \in I \ ;\forall i_2 \in I \ ;\forall t \in T \tag{A.6}$$

$$w_{i_1i_2t} \in \mathbb{Z}^+_0 \quad ;\forall i_1 \in I \ ;\forall i_2 \in I \ ;\forall t \in T \tag{A.7}$$

$$l^O_{it}, l^C_{it}, v^{CC}_{it}, v^{CO}_{it}, v^{OO}_{it}, v^{OC}_{it} \in \mathbb{Z}^+ \quad ;\forall i \in I \ ;\forall t \in T \tag{A.8}$$

$$w^B_{i_1i_2kt} \in \{0,1\} \quad ;\forall i_1 \in I \ ;\forall i_2 \in I \ ;\forall k \in K \ ;\forall t \in T \tag{A.9}$$

$$v^{BCO}_{ikt}, v^{BOC}_{ikt} \in \{0,1\} \quad ;\forall i \in I \ ;\forall k \in K \ ;\forall t \in T \tag{A.10}$$

A.2 Relocation for the CSLP: Strength of the LP relaxations

We will now compare the strength of both formulations. We will prove that both formulations are equally strong in the sense that their LP relaxations provide the same bound. For any integer linear programming model P, let $\overline{P}$ be the corresponding LP relaxation. For any model P, we denote by $v(P)$ its optimal value.

We denote by DR the MIP formulation of the CSLP with relocation by direct arcs as indicated in Section A.1.2 and we denote by $\overline{DR}$ its linear programming relaxation. In the same way, we denote by HR the MIP formulation of the CSLP with relocation using hub nodes as stated in Section A.1.1 and we denote by $\overline{HR}$ its LP relaxation.

Theorem A.2.1. $v(\overline{DR}) = v(\overline{HR})$

Proof The proof consists of two parts: first, it is proven that all feasible solutions for $\overline{DR}$ have an equivalent solution that is feasible in $\overline{HR}$. Second, we show how to construct a feasible solution for $\overline{DR}$ from any feasible solution of $\overline{HR}$ and that both solutions have the same objective function value.

(A) Construction of a feasible $\overline{HR}$ solution from any $\overline{DR}$ solution

Consider any solution
$\left\{x_{ijkpt}, y_{ikt}, z_{ikpt}, s_{iqt}, l_{it}^{O}, l_{it}^{C}, v_{it}^{CC}, v_{it}^{CO}, v_{it}^{OC}, v_{it}^{OO}, w_{i_1 i_2 t}, v_{ikt}^{BCO}, v_{ikt}^{BOC}, w_{i_1 i_2 kt}^{B}\right\}$
that is feasible for $\overline{DR}$. We next show that an equivalent solution
$\{x_{ijkpt}, y_{ikt}, z_{ikpt}, s_{iqt}, l_{it}^{O}, l_{it}^{C}, v_{it}^{CC}, v_{it}^{CO}, v_{it}^{OC}, v_{it}^{OO}, w_{it}^{O}, w_{it}^{I}, v_{ikt}^{BCO}, v_{ikt}^{BOC}, w_{ikt}^{BO}, w_{ikt}^{BI}\}$
can be constructed that is feasible in $\overline{HR}$.

For all variables in $\overline{HR}$ except w_{it}^{O}, w_{it}^{I}, w_{ikt}^{BO} and w_{ikt}^{BI}, we set the values as in the corresponding variables for the $\overline{DR}$ solution. The relocation variables are set as follows:

$$\sum_{i_2 \in I} w_{ii_2t} = w_{it}^{O} \;\; ;\forall i \in I \;\; ;\forall t \in T \tag{R1}$$

$$\sum_{i_1 \in I} w_{i_1it} = w_{it}^{I} \;\; ;\forall i \in I \;\; ;\forall t \in T \tag{R2}$$

$$\sum_{i_2 \in I} w_{ii_2kt}^{B} = w_{ikt}^{BO} \;\; ;\forall i \in I \;\; ;\forall k \in K \;\; ;\forall t \in T \tag{R3}$$

$$\sum_{i_1 \in I} w_{i_1ikt}^{B} = w_{ikt}^{BI} \;\; ;\forall i \in I \;\; ;\forall k \in K \;\; ;\forall t \in T \tag{R4}$$

The constraints (3.19), (3.21), (3.24) and (3.25) are satisfied by the use of (R1), (R2), (R3) and (R4), as they will equal constraints (A.2), (A.3), (A.4) and (A.5), respectively. Equalities (3.26) are satisfied as can be seen by replacing the relocation variables using (R3) and (R4). Finally, to show that (3.29) and (3.30) are also satisfied, we transform equality (A.6) of the *DR* model:

$$\sum_{k\in K} kw^{B}_{i_1 i_2 kt} = w_{i_1 i_2 t} \tag{A.6}$$

$$\overset{\forall i_1\in I}{\Rightarrow} \quad \sum_{i_1\in I}\sum_{k\in K} kw^{B}_{i_1 i_2 kt} = \sum_{i_1\in I} w_{i_1 i_2 t}$$

$$\Leftrightarrow \quad \sum_{k\in K} k \sum_{i_1\in I} w^{B}_{i_1 i_2 kt} = \sum_{i_1\in I} w_{i_1 i_2 t}$$

$$\overset{(R2) and (R4)}{\Leftrightarrow} \quad \sum_{k\in K} kw^{BI}_{i_2 kt} = w^{I}_{i_2 t} \tag{3.30}$$

which then equal equalities (3.30). We proceed in a similar manner to transform equality (A.6) into equality (3.29):

$$\sum_{k\in K} kw^{B}_{i_1 i_2 kt} = w_{i_1 i_2 t} \tag{A.6}$$

$$\overset{\forall i_2\in I}{\Rightarrow} \quad \sum_{i_2\in I}\sum_{k\in K} kw^{B}_{i_1 i_2 kt} = \sum_{i_2\in I} w_{i_1 i_2 t}$$

$$\Leftrightarrow \quad \sum_{k\in K} k \sum_{i_2\in I} w^{B}_{i_1 i_2 kt} = \sum_{i_2\in I} w_{i_1 i_2 t}$$

$$\overset{(R1) and (R3)}{\Leftrightarrow} \quad \sum_{k\in K} kw^{BO}_{i_1 kt} = w^{O}_{i_1 t} \tag{3.29}$$

which then equal equalities (3.29). All constraints are thus satisfied. As the solution value in the objective function is the same in both solutions (this can be verified by using (R3) and (R4) in the objective function), we can conclude that $v(\overline{DR}) \geq v(\overline{HR})$.

(B) Construction of a feasible $\overline{DR}$ solution from any $\overline{HR}$ solution

Consider any solution $\left\{x_{ijkpt}, y_{ikt}, z_{ikpt}, s_{iqt}, l^{O}_{it}, l^{C}_{it}, v^{CC}_{it}, v^{CO}_{it}, v^{OC}_{it}, v^{OO}_{it}, w^{O}_{it}, w^{I}_{it}, v^{BCO}_{ikt}, v^{BOC}_{ikt}, w^{BO}_{ikt}, w^{BI}_{ikt}\right\}$ that is feasible in $\overline{\text{HR}}$. We now construct an equivalent solution $\left\{x_{ijkpt}, y_{ikt}, z_{ikpt}, s_{iqt}, l^{O}_{it}, l^{C}_{it}, v^{CC}_{it}, v^{CO}_{it}, v^{OC}_{it}, v^{OO}_{it}, w_{i_1 i_2 t}, v^{BCO}_{ikt}, v^{BOC}_{ikt}, w^{B}_{i_1 i_2 kt}\right\}$ that is feasible in $\overline{\text{DR}}$ and has the same objective function value.

For all variables in $\overline{DR}$ except $w_{i_1 i_2 t}$ and $w^B_{i_1 i_2 kt}$, we set the values as in the corresponding variables for the $\overline{HR}$ solution. The solution values for the $w^B_{i_1 i_2 kt}$ variables are computed as outlined by Algorithm A.1, which proceeds as follows for each $k \in K$ and $t \in T$. All $w^B_{i_1 i_2 kt}$ variables are initialized with 0 and execute the following steps for each pair of $k \in K$ and $t \in T$. For each arc $w^B_{i_1 i_2 kt}$, the corresponding flow based on the solution values of $w^{BO}_{i_1 kt}$ and $w^{BI}_{i_2 kt}$ are computed. Clearly, the maximum flow possible from i_1 to i_2 is $\alpha = \min(w^{BO}_{i_1 kt}, w^{BI}_{i_2 kt})$. We thus set $w^B_{i_1 i_2 kt} = \alpha$. If we subtract α from $w^{BO}_{i_1 kt}$ and $w^{BI}_{i_2 kt}$, one of the two variables will become 0. If $w^{BO}_{i_1 kt} = 0$, then we proceed to the next variable $w^{BO}_{i' kt}$ and we set the next variable $w^B_{i' i_2 kt}$ to $\alpha = \min(w^{BO}_{i' kt}, w^{BI}_{i_2 kt})$. In the same way, if $w^{BI}_{i_2 kt} = 0$, then we proceed to the next w^{BI} variable $w^{BI}_{i' kt}$ and set the next variable $w^B_{i_1 i' kt}$ to $\alpha = \min(w^{BO}_{i_1 kt}, w^{BI}_{i' kt})$. We proceed in this way until all w^{BO} and w^{BI} variables have been considered. At the end of the algorithm, the entire flow from the w^{BO} and w^{BI} variables has been attributed to the w^B variables.

Algorithm A.1 Flow Distribution

Input: $k \in K, t \in T$

Variables: $w^{BO}_{ikt}, w^{BI}_{ikt} \quad, \forall i \in I$

$w^{B}_{i_1 i_2 kt} = 0 \quad ; \forall i_1 \in I \quad , \forall i_2 \in I.$

$i_1 = 1.$

$i_2 = 1.$

$r^O = w^{BO}_{i_1 kt}.$

$r^I = w^{BO}_{i_2 kt}.$

repeat

  if $r^O = 0$ **then**

    $i_1 = i_1 + 1.$

    $r^O = w^{BO}_{i_1 kt}.$

  end if

  if $r^I = 0$ **then**

    $i_2 = i_2 + 1.$

    $r^I = w^{BI}_{i_2 kt}.$

  end if

  $\alpha = \min(r^O, r^I).$

  $w^{B}_{i_1 i_2 kt} = \alpha.$

  $r^O = r^O - \alpha.$

  $r^I = r^I - \alpha.$

until $i_1 = |I|$ & $i_2 = |I|$

return $w^{B}_{i_1 i_2 kt}.$

Constraints (3.26) ensure that both r^O and r^I are 0 by the end of the algorithm, i.e., all flow has been distributed to the w^B arcs. Equalities (R3) and (R4) hold because flow from variables w^{BO}_{it} is only distributed to the variables $w^{B}_{ii_2kt}$ ($i_2 \in I$) and flow from w^{BI}_{it} variables is only distributed to the variables $w^{B}_{i_1ikt}$ ($i_1 \in I$). As (R3) and (R4) hold, constraints (A.4) and (A.5) may be transformed to equivalent constraints (3.24) and (3.25) and are therefore also satisfied.

The solution values for the $w_{i_1 i_2 t}$ variables are computed according to equalities (A.6), which are therefore satisfied. Finally, we need to show that constraints (A.2) and (A.3) are satisfied. This is true if (R1) and (R2) are valid, which is proved by considering the following equations:

$$\begin{aligned}\sum_{i_2 \in I} w_{i_1 i_2 t} &\overset{(A.6)}{=} \sum_{i_2 \in I} \sum_{k \in K} k w^B_{i_1 i_2 k t} \\ &= \sum_{k \in K} k \sum_{i_2 \in I} w^B_{i_1 i_2 k t} \\ &\overset{(R3)}{=} \sum_{k \in K} k w^{BO}_{i_1 k t} \\ &\overset{(3.29)}{=} w^O_{i_1 t},\end{aligned}$$

which then equal equalities (R1). In the same way, we show that equalities (R2) hold:

$$\begin{aligned}\sum_{i_1 \in I} w_{i_1 i_2 t} &\overset{(A.6)}{=} \sum_{i_1 \in I} \sum_{k \in K} k w^B_{i_1 i_2 k t} \\ &= \sum_{k \in K} k \sum_{i_1 \in I} w^B_{i_1 i_2 k t} \\ &\overset{(R4)}{=} \sum_{k \in K} k w^{BI}_{i_2 k t} \\ &\overset{(3.30)}{=} w^I_{i_2 t},\end{aligned}$$

which then equal (R2). Hence, constraints (A.2) and (A.3) are satisfied. The solution is therefore feasible in $\overline{DR}$. As equalities (R3) and (R4) hold, it can be verified that both solutions have the same objective function value. Therefore, $v(\overline{HR}) \geq v(\overline{DR})$.

From the two parts (A) and (B) above, it follows that $v(\overline{DR}) = v(\overline{HR})$. □

A.3 Relocation for the CSLP: Computational Experiments

Computational experiments have been performed to assess the performance of the two modeling techniques when using a generic MIP solver. We used CPLEX v12.4 to solve the LP of both formulations for a total of 288 instances. These instances include the 216 instances reported in Chapter 3, as well as 72 instances that correspond to variations of the real-world instance used in the second case study of the chapter.

The results are summarized in Table A.I. For each of the two formulations (relocation by the use of *direct arcs* and relocation by the use of *hub nodes*), two values are reported: the average time to solve the LP relaxation and the number of instances where the LP relaxation has not been solved (*# ns*) in the given time limit of 6 hours. In addition, the last columns of the table indicate the average and standard deviation of the relative improvement of the solution time.

	Direct arcs		Hub nodes		Sol time improvement	
Instance set	time (sec)	# ns	time (sec)	# ns	%	Std dev
10/20	35	0	30	0	11.5	25.1
10/50	3836	5	3205	3	-1.8	48.4
29/29	108	0	72	0	29.5	19.5
All	1936	5	1617	3	11.0	36.7

Table A.I: Computing times to solve the LP relaxation: direct arcs vs. hub nodes.

The results indicate that the LP relaxation of the formulation using hub nodes is in particular effective for the instance sets (10/20) and (29/29). However, the solution time improvement strongly varies, as can be observed in the high standard deviation. Figure A.2 illustrates the improvements for each of the 288 instances. For most of the instances, the solution time decreased.

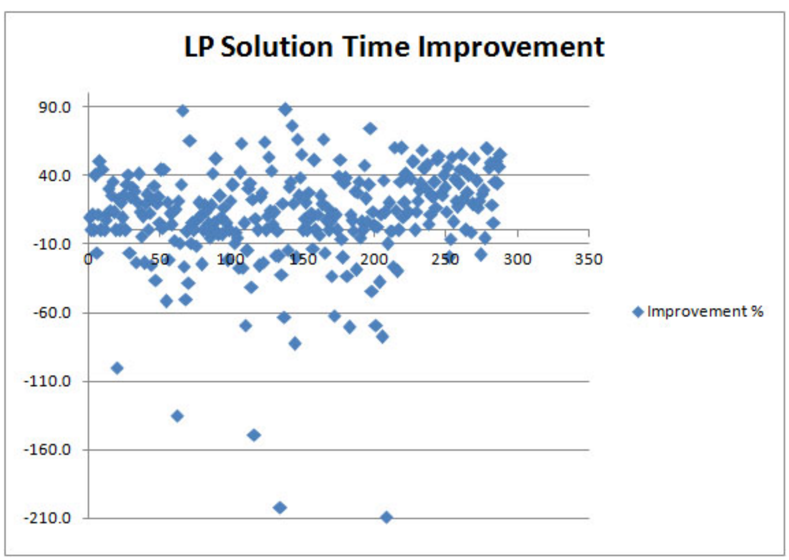

Figure A.2: Relative improvement of time to solve the LP relaxation for each of the instances when hub node relocation is used instead of direct arcs.

Table A.II compares the average optimality gaps, solution times and number of instances where no feasible solutions have been found when solving the problem with CPLEX, limited to one hour of computing time. Note that the averages are computed over all instances for which a feasible solution has been found. One observes that the model using hub nodes performs significantly better than the model using direct arcs, given that the number of instances where no feasible integer solution has been found is much lower.

	Direct arcs			Hub nodes		
Instance set	Opt gap %	# ns	time (sec)	Opt gap %	# ns	time (sec)
10/20	27.0	8	3,178	28.5	0	3,165
10/50	11.8	101	3,600	39.6	67	3,600
29/29	26.6	21	3,209	28.7	1	3,311
All	26,2	131	3,206	30,6	68	3,293

Table A.II: Computing time to solve problems: direct arcs vs. hub nodes.

APPENDIX B

SUPPLEMENT TO CHAPTER 4

B.1 Theoretical Results

B.1.1 Theoretical Results for the DMCFLP_CR formulations

We now prove the dominance relationships between the three formulations presented for the DMCFLP_CR. For any integer linear programming model P, let $\overline{P}$ denote the corresponding LP relaxation. For any model P, we denote by $v(P)$ its optimal value.

B.1.1.1 CR-GMC and CR-1I are equally strong

We prove that the LP relaxations of the formulations CR-GMC and CR-1I provide the same lower bound.

Theorem B.1.1. $v(\overline{CR\text{-}GMC}) = v(\overline{CR\text{-}1I})$.

Proof. The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{CR-1I}}$ from any feasible solution for $\overline{\text{CR-GMC}}$ and that both solutions have the same objective function value. Then, we show the same, constructing an equivalent and feasible $\overline{\text{CR-GMC}}$ solution from any feasible $\overline{\text{CR-1I}}$ solution.

To facilitate the proof, we first write the CR-GMC in its explicit form as it is defined in Section 4.4.2. As previously defined, we let $L = \{0, 1, 2, .., q\}$ be the set of available capacity levels to define the facility size. For each open capacity level $\ell \in L \backslash \{0\}$, we let

$\overline{\ell}$ denote a closed facility of capacity level ℓ. The model is:

$$\text{(CR-GMC)} \quad \min \sum_{j\in J}\sum_{\ell_2\in L}\sum_{t\in T}(f^o_{j\ell_2}+F^o_{j\ell_2})y_{j0\ell_2 t}+\sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}g_{ij\ell t}d_{it}x_{ij\ell t}$$
$$+\sum_{j\in J}\sum_{\ell_1\in L\setminus\{0\}}\sum_{t\in T}c^c_{j\ell_1}y_{j\ell_1\overline{\ell}_1 t}+\sum_{j\in J}\sum_{\ell_1\in L\setminus\{0\}}\sum_{t\in T}(c^o_{j\ell_1}+F^o_{j\ell_1})y_{j\overline{\ell}_1\ell_1 t}$$

$$s.t.\ \sum_{j\in J}\sum_{\ell\in L}x_{ij\ell t}=1 \ \ \forall i\in I,\ \ \forall t\in T \tag{B.1}$$

$$\sum_{i\in I}d_{it}x_{ij\ell t}\le u_{j\ell}(y_{j\ell\ell t}+y_{j\overline{\ell}\ell t}+y_{j0\ell t}) \ \ \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{B.2}$$

$$y_{j0\ell(t-1)}+y_{j\ell\ell(t-1)}+y_{j\overline{\ell}\ell(t-1)}=y_{j\ell\ell t}+y_{j\ell\overline{\ell}t} \ \ \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T\setminus\{1\} \tag{B.3}$$

$$y_{j\ell\overline{\ell}(t-1)}+y_{j\overline{\ell}\overline{\ell}(t-1)}=y_{j\overline{\ell}\overline{\ell}t}+y_{j\overline{\ell}\ell t} \ \ \forall j\in J,\ \ \forall \ell\in L\setminus\{0\},\ \ \forall t\in T\setminus\{1\} \tag{B.4}$$

$$\sum_{\ell_2\in L}y_{j0\ell_2 1}=1 \ \ \forall j\in J \tag{B.5}$$

$$x_{ij\ell t}\ge 0 \ \ \forall i\in I,\ \ \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{B.6}$$

$$y_{j0\ell_2 t}\in\{0,1\} \ \ \forall j\in J,\ \ \forall \ell_2\in L,\ \ \forall t\in T \tag{B.7}$$

$$y_{j\ell_1\ell_1 t}\in\{0,1\} \ \ \forall j\in J,\ \ \forall \ell_1\in L\setminus\{0\},\ \ \forall t\in T \tag{B.8}$$

$$y_{j\overline{\ell}_1\overline{\ell}_1 t}\in\{0,1\} \ \ \forall j\in J,\ \ \forall \ell_1\in L\setminus\{0\},\ \ \forall t\in T \tag{B.9}$$

$$y_{j\ell_1\overline{\ell}_1 t}\in\{0,1\} \ \ \forall j\in J,\ \ \forall \ell_1\in L\setminus\{0\},\ \ \forall t\in T \tag{B.10}$$

$$y_{j\overline{\ell}_1\ell_1 t}\in\{0,1\} \ \ \forall j\in J,\ \ \forall \ell_1\in L\setminus\{0\},\ \ \forall t\in T. \tag{B.11}$$

(A) Construction of a feasible $\overline{\text{CR-1I}}$ solution from any $\overline{\text{CR-GMC}}$ solution

Consider any solution $\left\{x_{ij\ell t}, y_{j\ell_1\ell_2 t}\right\}$ that is feasible in $\overline{\text{CR-GMC}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CR-1I}}$ and has the same objective function value.

We set the values for the $x_{ij\ell t}$ variables identical to those in the CR-GMC solution. The values for the variables $y_{j\ell t}$, $s_{j\ell t}$, $v^o_{j\ell t}$ and $v^c_{j\ell t}$ are set by establishing the following relations $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$s_{j\ell t} = y_{j0\ell t} \tag{CR.R1}$$

$$v^{o}_{j\ell t} = y_{j\bar{\ell}\ell t} \tag{CR.R2}$$

$$v^{c}_{j\ell t} = y_{j\ell\bar{\ell}t} \tag{CR.R3}$$

$$y_{j\ell t} = y_{j0\ell t} + y_{j\ell\ell t} + y_{j\bar{\ell}\ell t}. \tag{CR.R4}$$

According to these relations and the way the objective function coefficients are composed (see Section 4.4.2), it can easily be verified that both solutions have the same objective function value. Constraints (4.11) are satisfied, as they contain the same variables with the same values in both models. Replacing the right-hand side in constraints (4.12) by (CR.R4) results in constraints (B.2). Therefore, constraints (4.12) are also satisfied. We show that constraints (4.13) hold by using the relationships defined above:

$$y_{j\ell t} = y_{j\ell(t-1)} + s_{j\ell t} + v^{o}_{j\ell t} - v^{c}_{j\ell t} \tag{4.13}$$

$$\overset{\text{(CR.R1) - (CR.R4)}}{\Leftrightarrow} y_{j0\ell t} + y_{j\ell\ell t} + y_{j\bar{\ell}\ell t} = y_{j0\ell(t-1)} + y_{j\ell\ell(t-1)} + y_{j\bar{\ell}\ell(t-1)} + y_{j0\ell t} + y_{j\bar{\ell}\ell t} - y_{j\ell\bar{\ell}t}$$

$$\overset{\text{cancel } y_{j0\ell t} \,\&\, y_{j\bar{\ell}\ell t}}{\Leftrightarrow} y_{j\ell\ell t} = y_{j0\ell(t-1)} + y_{j\ell\ell(t-1)} + y_{j\bar{\ell}\ell(t-1)} - y_{j\ell\bar{\ell}t}. \tag{B.3}$$

As equalities (B.3) necessarily hold, constraints (4.13) are also satisfied. In a similar way, we show that constraints (4.14) hold:

$$\sum_{t'=1}^{t} v^{o}_{j\ell t'} \leq \sum_{t'=1}^{t} v^{c}_{j\ell t'} \tag{4.14}$$

$$\overset{\text{(CR.R2) \& (CR.R3)}}{\Leftrightarrow} \sum_{t'=1}^{t} y_{j\bar{\ell}\ell t'} \leq \sum_{t'=1}^{t} y_{j\ell\bar{\ell}t'}$$

$$\overset{\text{replace LHS by (B.4)}}{\Leftrightarrow} \sum_{t'=1}^{t} y_{j\ell\bar{\ell}(t'-1)} + \sum_{t'=1}^{t} y_{j\bar{\ell}\bar{\ell}(t'-1)} - \sum_{t'=1}^{t} y_{j\bar{\ell}\bar{\ell}t'} \leq \sum_{t'=1}^{t} y_{j\ell\bar{\ell}t'}$$

$$\overset{\text{cancel } y_{j\bar{\ell}\bar{\ell}t'} \,\&\, y_{j\ell\bar{\ell}t'}}{\Leftrightarrow} -y_{j\bar{\ell}\bar{\ell}t} \leq y_{j\ell\bar{\ell}t},$$

which is true, since the y variables are non-negative. Finally, to show that constraints (4.15) are also satisfied, note that constraints (B.3) - (B.5) ensure that the $y_{j\ell_1\ell_2t}$ variables sum to 1 at each location j and time period t. This flow also contains the case where no facility exits, i.e., $\ell = 0$. Furthermore, once a facility is constructed, the flow in the GMC network cannot return to capacity level 0, since, according to (B.7) - (B.11), $y_{j\ell_1\ell_2t}$ is not defined for $\ell_2 = 0$. Therefore, $\sum_{t=1}^{|T|} y_{j\ell_1\ell_2t} \leq 1$ which proves that constraints (4.15) are satisfied.

If the SIs are used, they are also feasible in the CR-1I model. They can be deduced by replacing (CR.R4) in the SIs of the CR-GMC.

(B) Construction of a feasible $\overline{\text{CR-GMC}}$ solution from any $\overline{\text{CR-1I}}$ solution

Consider any solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CR-1I}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell_1\ell_2t}\right\}$ that is feasible in $\overline{\text{CR-GMC}}$ and has the same objective function value.

The values for the $x_{ij\ell t}$ variables are set identical to those in the CR-1I solution. The arcs for constructing a facility ($y_{j0\ell t}$), closing an open facility ($y_{j\ell\bar{\ell}t}$) and reopening a closed facility ($y_{j\bar{\ell}\ell t}$) are set by using the equalities (CR.R1) - (CR.R3) and therefore satisfy their domain constraints. The solution values for the arcs to keep a facility open ($y_{j\ell\ell t}$) are set by replacing (CR.R1) and (CR.R2) in equality (CR.R4):

$$
\begin{aligned}
& & y_{j\ell t} &= y_{j0\ell t} + y_{j\ell\ell t} + y_{j\bar{\ell}\ell t} & \text{(CR.R4)}\\
&\overset{(CR.R1),(CR.R2)}{\Leftrightarrow} & y_{j\ell t} &= s_{j\ell t} + y_{j\ell\ell t} + v^o_{j\ell t} &\\
&\Leftrightarrow & y_{j\ell\ell t} &= y_{j\ell t} - s_{j\ell t} - v^o_{j\ell t}. &
\end{aligned}
$$

The variables are non-negative, as can be verified in equalities (4.13). According to inequalities (4.15), the total flow for capacity construction does not exceed 1 for the entire planning horizon. For the first time period, $y_{j0\ell 1}$ is set to $s_{j\ell 1}$, resulting in $\sum_{\ell_2 \in L\setminus\{0\}} y_{j0\ell_2 1} \leq 1$ for each location j. Furthermore, we set $y_{j001} = 1 - \sum_{\ell_2 \in L\setminus\{0\}} y_{j0\ell_2 1}$. Therefore, the flow initialization constraints (B.5) are satisfied.

Constraints (B.1) and (B.2) are satisfied, as they contain the same variables with the same values in both models. As shown above in part (A), we can transform equalities (4.13) into equalities (B.3), and vice-versa, by using (CR.R1)-(CR.R2). This proves the feasibility of constraints (B.3). Finally, we compute the values for $y_{j\overline{\ell\ell}t}$ by using equalities (B.4), sequentially from time period 1 to $|T|$:

$$\begin{aligned} & y_{j\overline{\ell\ell}t} = y_{j\overline{\ell\ell}(t-1)} + y_{j\ell\bar{\ell}(t-1)} - y_{j\bar{\ell}\ell t} \\ \overset{(CR.R2),(CR.R3)}{\Leftrightarrow} \quad & y_{j\overline{\ell\ell}t} = y_{j\overline{\ell\ell}(t-1)} + v^c_{j\ell(t-1)} - v^o_{j\ell t}. \end{aligned} \tag{B.4}$$

Note that, due to (4.14), the variables have non-negative values. Furthermore, their sum never exceeds 1, because the only way how to insert flow into the v^o and v^c variables is by using the s variables, whose total sum is strictly limited to 1 by inequalities (4.15).

We note that the constructed solution has the same value, as can be verified by the used relationships (CR.R1)-(CR.R4) as well as the way the variables' coefficients are composed (see Section 4.4.2).

From the two parts (A) and (B) above, it follows that $v(\overline{CR-GMC}) = v(\overline{CR-1I})$. □

B.1.1.2 CR-GMC and CR-1I are stronger than CR-2I

We next prove that the CR-GMC and CR-1I formulations provide stronger LP bounds than the CR-2I formulation.

Theorem B.1.2. $v(\overline{\textit{CR-1I}}) \geq v(\overline{\textit{CR-2I}})$.

Proof. The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{CR-2I}}$ from any feasible solution of $\overline{\text{CR-1I}}$ and that both solutions have the same objective function value. Then, we describe a small problem instance where the $\overline{\text{CR-1I}}$ formulation provides a better LP relaxation bound than the $\overline{\text{CR-2I}}$ formulation.

(A) Construction of a feasible $\overline{\text{CR-2I}}$ solution from any feasible $\overline{\text{CR-1I}}$ solution

We first set the solution values for the $s_{j\ell t_1 t_2}$ and $y_{j\ell t_1 t_2}$ variables. For each j and ℓ, we consider the diagram that describes the opening schedule of a facility of size ℓ in the CR-1I solution. We separate the opening schedules for each capacity level ℓ into blocks, as described by the following algorithm:

Algorithm 1.

Input: A facility opening schedule, consisting of a value between 0 and 1, indicating the fraction at which the facility is open for each of the $|T|$ time periods (indicated by the value of $y_{j\ell t}$).

Output: The opening schedule horizontally cut into blocks. Each block is defined by a starting and ending period as well as a value between 0 and 1, indicating the fraction at which the block represents the open facility.

Description: The opening schedule, as shown in Figure 4.1 (a), is horizontally cut into blocks whenever the value of the opening fraction increases or decreases. Doing this, the increase and/or decrease of capacity may be split into several increases and/or decreases, respectively. This results in a representation as in Figure 4.1 (b). In this example, the capacity increase at the beginning of period 3 is split into two capacity increases of half size each, while the capacity decrease at the beginning of period 6 is split into two capacity decreases. To be precise, the algorithm separates the opening schedules into three capacity blocks: the first block spans periods 2 to 5, the second block spans periods 3 to 5 and the third block spans periods 3 to 4. It is easy to see that this kind of division is unambiguous, i.e., there is only one way to separate into blocks. The design of an algorithm to find this division is straightforward. We therefore do not explicitly state such an algorithm.

Note that, in the opening schedule, a capacity increase at time period t is always caused by the use of the variables $s_{j\ell t}$ or $v^o_{j\ell t}$. A capacity decrease at time period t is caused by the use of variable $v^c_{j\ell t}$.

After division, we have a number of separate blocks (each spanning one or more time periods). We divide these blocks into two groups: blocks where the capacity increase is originated from a variable $s_{j\ell t}$ and blocks where the capacity increase is originated from reopening variables $v^o_{j\ell t}$. Each block originated from a variable $s_{j\ell t}$ represents a $s_{j\ell t_1 t_2}$ variable and each block originated from a $v^o_{j\ell t}$ variable represents a variable $y_{j\ell t_1 t_2}$. The value for these variables is set equal to the fraction of value represented by the corresponding variables $s_{j\ell t}$ and $v^o_{j\ell t}$.

The following relationships then hold, since the solution value of $y_{j\ell t}$ is the sum of all capacity blocks at time period t:

$$\sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} \left(s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}\right) = y_{j\ell t} \quad \forall j \in J,\ \ \forall \ell \in L,\ \ \forall t \in T. \tag{CR.R5}$$

Furthermore, the following relationships hold, since the solution value of $s_{j\ell t}$ is distributed over all $s_{j\ell t_1 t_2}$ variables that originate from $t_1 = t$. The same relation is valid between the variables $v^o_{j\ell t}$ and $y_{j\ell t_1 t_2}$:

$$\sum_{t_2=t}^{|T|} s_{j\ell t t_2} = s_{j\ell t} \quad \forall j \in J,\ \ \forall \ell \in L,\ \ \forall t \in T. \tag{CR.R6}$$

All $x_{ij\ell t}$ variables are set as given in the CR-1I solution. Thus, constraints (4.19) are satisfied. Inequalities (4.14) guarantee that, at any time period t, variable $v^o_{j\ell t}$ does not hold more capacity than has been previously constructed. Thus, constraints (4.20) also hold. Using (CR.R6) in (4.15) shows that constraints (4.21) are satisfied. Inequalities (4.22) are also satisfied. To show this, first replace the terms by (CR.R5). Then, recognize that (4.15) limits the entire facility construction to 1. As $y_{j\ell t}$ is linked to the facility construction in equalities (4.13), its sum over all capacity levels can never exceed 1. Finally, the capacity constraints (4.23) are feasible. This is shown by replacing (CR.R5) in constraints (4.23), which then equal the capacity constraints of the CR-1I. If SIs are used, the feasibility of the SIs in the CR-2I formulation can be shown by replacing its RHS terms by (CR.R5).

We note that the constructed solution has the same value as the CR-1I solution. This can be seen by recognizing that the $s_{j\ell t_1 t_2}$ and $y_{j\ell t_1 t_2}$ blocks in the CR-2I solution have been constructed following the corresponding solution values of $s_{j\ell t}$ and $y_{j\ell t}$ and considering how the cost coefficients are set as described in Section 4.5.1.2.

(B) Problem instance where CR-1I is stronger

Consider the following example instance. We consider a planning over three time periods. A single customer exists with demands of 15, 15 and 20 units for each of the time periods, respectively. Two locations can be used to construct facilities. A single capacity level is available, providing a capacity of 10 units. The construction costs are 100\$ and the maintenance costs for an open facility are set to 500\$ for one time period. Facility closing and reopening is free. The same holds for the production and transportation of the commodity. For the given instance, the CR-1I provides a better bound than the CR-2I formulation. The solution of the CR-1I model is $y_{j_0\ell_1 t_0} = 1.0$, $y_{j_0\ell_1 t_1} = 0.5$, $y_{j_0\ell_1 t_2} = 1.0$, $y_{j_1\ell_1 t_0} = 0.5$, $y_{j_1\ell_1 t_1} = 1.0$, $y_{j_1\ell_1 t_2} = 1.0$, $s_{j_0\ell_1 t_0} = 1.0$, $s_{j_1\ell_1 t_0} = 0.5$, $s_{j_1\ell_1 t_1} = 0.5$, $v^c_{j_0\ell_1 t_1} = 0.5$, $v^o_{j_0\ell_1 t_2} = 0.5$. The demand allocation variables have the values $x_{i_0 j_0\ell_1 t_0} = 0.66$, $x_{i_0 j_1\ell_1 t_0} = 0.33$, $x_{i_0 j_0\ell_1 t_1} = 0.33$, $x_{i_0 j_1\ell_1 t_1} = 0.66$, $x_{i_0 j_0\ell_1 t_2} = 0.5$, $x_{i_0 j_1\ell_1 t_2} = 0.5$. The cost of this solution is 2700\$.

In the solution of the CR-2I model, the binary decision variables have the following values: $s_{j_0\ell_1 t_0 t_2} = 1.0$, $s_{j_1\ell_1 t_0 t_0} = 0.5$, $y_{j_1\ell_1 t_1 t_2} = 0.5$, $y_{j_1\ell_1 t_2 t_2} = 0.5$. The demand allocation variables are as follows: $x_{i_0 j_0\ell_1 t_0} = 0.66$, $x_{i_0 j_1\ell_1 t_0} = 0.33$, $x_{i_0 j_0\ell_1 t_1} = 0.66$, $x_{i_0 j_1\ell_1 t_1} = 0.33$, $x_{i_0 j_0\ell_1 t_2} = 0.5$, $x_{i_0 j_1\ell_1 t_2} = 0.5$. The cost of this solution is 2650\$.

From the two parts (A) and (B) above it follows that $v(\overline{\text{CR-1I}}) \geq v(\overline{\text{CR-2I}})$. □

Theorem B.1.3. $v(\overline{\textit{CR-GMC}}) \geq v(\overline{\textit{CR-2I}})$.

Proof. The result follows by transitivity from Theorems B.1.1 and B.1.2. □

B.1.1.3 CR-2I+ is equally strong as CR-GMC and CR-1I

Theorem B.1.4. $v(\overline{\textit{CR-1I}}) = v(\overline{\textit{CR-2I+}})$.

Proof. It has already been shown that we can construct an equivalent and feasible $\overline{\text{CR-2I}}$ solution from any $\overline{\text{CR-1I}}$ solution. Due to the way the described algorithm assigns the values to the $s_{j\ell t_1 t_2}$ and $y_{j\ell t_1 t_2}$ variables as well as the direct relationship between these variables and the v^c and v^o variables, it can be shown that the new constraints (4.26) are also satisfied.

(A) Construction of a feasible $\overline{\text{CR-1I}}$ solution from any feasible $\overline{\text{CR-2I+}}$ solution

Consider any solution $\{x_{ij\ell t}, s_{j\ell_1 t_1 t_2}, y_{j\ell_1 t_1 t_2}\}$ that is feasible in $\overline{\text{CR-2I+}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CR-1I}}$ and that has the same value.

We set the values for the $x_{ij\ell t}$ variables identical to those in the CR-2I+ solution. The values for the variables $y_{j\ell t}$, $s_{j\ell t}$, $v^o_{j\ell t}$ and $v^c_{j\ell t}$ are set by establishing the following relations $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$s_{j\ell t} = \sum_{t_2=t}^{|T|} s_{j\ell t t_2} \tag{CR.R7}$$

$$y_{j\ell t} = \sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} s_{j\ell t_1 t_2} + \sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} y_{j\ell t_1 t_2} \tag{CR.R8}$$

$$v^o_{j\ell t} = \sum_{t_2=t}^{|T|} y_{j\ell t t_2} \tag{CR.R9}$$

$$v^c_{j\ell t} = \sum_{t_1=1}^{t-1} s_{j\ell t_1 (t-1)} + \sum_{t_1=1}^{t-1} y_{j\ell t_1 (t-1)} \tag{CR.R10a}$$

$$\Leftrightarrow v^c_{j\ell (t+1)} = \sum_{t_1=1}^{t} s_{j\ell t_1 t} + \sum_{t_1=1}^{t} y_{j\ell t_1 t} \tag{CR.R10}$$

Constraints (4.11) are equivalent to constraints (4.19) and are thus satisfied. By using (CR.R8), constraints (4.12) correspond to constraints (4.23). Constraints (4.15) correspond to constraints (4.21) by using (CR.R7).

Replacing (CR.R9) and (CR.R10) in constraints (4.14) gives, $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$\sum_{t'=1}^{t} v^{o}_{j\ell t'} \leq \sum_{t'=1}^{t} v^{c}_{j\ell t'} \tag{4.14}$$

$$\Leftrightarrow \sum_{t'=1}^{t} v^{o}_{j\ell t'} \leq \sum_{t'=0}^{t-1} v^{c}_{j\ell (t'+1)}$$

$$\overset{(CR.R9)\&(CR.R10)}{\Leftrightarrow} \sum_{t'=1}^{t} \sum_{t_2=t'}^{|T|} y_{j\ell t' t_2} \leq \sum_{t'=0}^{t-1} \sum_{t_1=1}^{t'} s_{j\ell t_1 t'} + \sum_{t'=0}^{t-1} \sum_{t_1=1}^{t'} y_{j\ell t_1 t'}$$

$$\Leftrightarrow \sum_{t'=1}^{t} \sum_{t_2=t'}^{|T|} y_{j\ell t' t_2} \leq \sum_{t_1=1}^{t-1} \sum_{t'=t_1}^{t-1} s_{j\ell t_1 t'} + \sum_{t_1=1}^{t-1} \sum_{t'=t_1}^{t-1} y_{j\ell t_1 t'},$$

which is true due to constraints (4.26). Thus, constraints (4.14) also hold. The feasibility of the flow conservation constraints (4.13) can be shown by replacing the variables by the terms given in the relations (CR.R7), (CR.R9) and (CR.R10a). By doing so, all terms on the LHS and RHS will cancel each other.

Given the relations (CR.R7)-(CR.10a) and the way the variable coefficients are composed in both formulations, it can easily be verified that both solutions have the same value. Both formulations are thus equally strong. □

Theorem B.1.5. $v(\overline{CR\text{-}GMC}) = v(\overline{CR\text{-}2I\text{+}})$.

Proof. The result follows by transitivity from Theorems B.1.1 and B.1.4. □

B.1.2 Theoretical Results for the DMCFLP_ER formulations

We now prove the dominance relationships for the three formulations presented for the DMCFLP_ER. Let $\overline{\text{ER-GMC}}$ be the linear programming relaxation of ER-GMC. In the same way, we denote $\overline{\text{ER-1I}}$ the linear programming relaxation of ER-1I and $\overline{\text{ER-2I}}$ the linear programming relaxation of ER-2I.

B.1.2.1 ER-1I and ER-2I are equally strong

We first prove that the LP relaxations of the formulations ER-1I and ER-2I provide the same lower bound.

Theorem B.1.6. $v(\overline{ER\text{-}1I}) = v(\overline{ER\text{-}2I})$.

Proof. The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{ER-1I}}$ from any feasible solution of $\overline{\text{ER-2I}}$ and that both solutions have the same objective function value. Then, we show the same, constructing an equivalent $\overline{\text{ER-2I}}$ solution based on any feasible $\overline{\text{ER-1I}}$ solution.

(A) Construction of a feasible $\overline{\text{ER-1I}}$ solution from any $\overline{\text{ER-2I}}$ solution

Consider any solution $\left\{x_{ij\ell t}, y'_{j\ell t_1 t_2}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\right\}$ that is feasible in $\overline{\text{ER-2I}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\right\}$ that is feasible in $\overline{\text{ER-1I}}$ and has the same objective function value.

We set all variables $x_{ij\ell t}, y_{j\ell t}$, $s_{j\ell t}$ and $w_{j\ell t}$ in the $\overline{\text{ER-1I}}$ formulation as given in the $\overline{\text{ER-2I}}$ solution. Given that both formulations have the same objective function, the solution value is also the same. Also observe that the formulations have the same constraints except for constraints (4.28) in the ER-1I and constraints (4.35) - (4.37) in the ER-2I formulation. The constraints that are part of both models (including the SIs) have the same variables with the same solution values in both solutions and are thus feasible. Therefore, we only have to show that constraints (4.28) are also feasible. We do so by replacing (4.35) - (4.37) in (4.28) for $\forall j \in J, \forall t \in T$:

$$\sum_{\ell\in L} \ell y_{j\ell t} = \sum_{\ell\in L} \ell y_{j\ell(t-1)} + \sum_{\ell\in L} \ell s_{j\ell t} - \sum_{\ell\in L} \ell w_{j\ell t} \tag{4.28}$$

$$\overset{(4.35)-(4.37)}{\Leftrightarrow} \sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t_1 t_2} = \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\sum_{t_2=t-1}^{|T|} \ell y'_{j\ell t_1 t_2} + \sum_{\ell\in L}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t t_2} - \sum_{\ell\in L}\sum_{t_1=1}^{t-1} \ell y'_{j\ell t_1 (t-1)}$$

$$\Leftrightarrow \quad \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t_1 t_2} + \sum_{\ell\in L}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t t_2}$$

$$= \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\sum_{t_2=t-1}^{|T|} \ell y'_{j\ell t_1 t_2} + \sum_{\ell\in L}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t t_2} - \sum_{\ell\in L}\sum_{t_1=1}^{t-1} \ell y'_{j\ell t_1 (t-1)}.$$

The remaining terms now cancel each other and therefore constraints (4.28) hold.

(B) Construction of a feasible $\overline{\text{ER-2I}}$ solution from any $\overline{\text{ER-1I}}$ solution

Consider any solution $\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\}$ that is feasible in $\overline{\text{ER-1I}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y'_{j\ell t_1 t_2}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\right\}$ that is feasible in $\overline{\text{ER-2I}}$ and has the same objective function value.

We set the values for the $x_{ij\ell t}, y_{j\ell t}$, $s_{j\ell t}$ and $w_{j\ell t}$ variables in the $\overline{\text{ER-2I}}$ formulation as given in the $\overline{\text{ER-1I}}$ solution. Given that both formulations have the same objective function, the solution value in the objective function is also the same. All constraints (including the SIs), except for constraints (4.35) - (4.37), are the same as in formulation ER-1I and are therefore feasible.

We now set the solution values for the $y'_{j\ell t_1 t_2}$ variables. For each $\ell \in L$, we consider the diagram that describes the opening schedule of a facility of size ℓ. Each opening schedule is horizontally cut into blocks as described by Algorithm 1. Note that in the optimal solution, due to (4.28), an increase in $y_{j\ell t}$ by an amount of α necessarily means that $s_{j\ell t} = \alpha$ and $w_{j\ell t} = 0$, whereas a decrease in $y_{j\ell t}$ by an amount of α necessarily means that $w_{j\ell t} = \alpha$ and $s_{j\ell t} = 0$. After separation, each of the separated blocks represents a variable $y'_{j\ell t_1 t_2}$ with a solution value greater than 0. The solution value of $s_{j\ell t}$ will be distributed over all $y'_{j\ell t_1 t_2}$ variables that start at $t_1 = t$, the solution value of $w_{j\ell t}$ will be distributed over all $y'_{j\ell t_1 t_2}$ variables that terminate at the end of $t_2 = t$ and the solution value of $y_{j\ell t}$ will be distributed over all $y'_{j\ell t_1 t_2}$ variables that start at or before t and terminate at or after t. Therefore, the following relationships hold:

$$\sum_{t_2=t}^{|T|} y'_{j\ell t t_2} = s_{j\ell t} \quad \forall j \in J,\ \forall \ell \in L,\ \forall t \in T$$

$$\sum_{t_1=1}^{t-1} y'_{j\ell t_1 (t-1)} = w_{j\ell t} \quad \forall j \in J,\ \forall \ell \in L,\ \forall t \in T$$

$$\sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} y'_{j\ell t_1 t_2} = y_{j\ell t} \quad \forall j \in J,\ \forall \ell \in L,\ \forall t \in T.$$

Replacing these relationships in the constraints (4.35) - (4.37), respectively, shows that these constraints remain feasible.

From the two parts (A) and (B) above it follows that $v(\overline{ER-1I}) = v(\overline{ER-2I})$. $\square$

B.1.2.2 ER-GMC is stronger than ER-1I and ER-2I

We now compare the strength of the ER-GMC and ER-1I formulations. We will prove that the ER-GMC formulation is at least as strong (strictly stronger for some instances) as the ER-1I formulation in the sense that its linear programming relaxations provides a better bound. By transitivity, the same result follows for the relation between the ER-GMC and ER-2I.

The ER-GMC formulation has the following form. Its objective function contains the costs for capacity expansion, capacity reduction and remaining at the same capacity level. Its constraints are the same as defined by the GMC:

$$\text{(ER-GMC)} \quad \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\substack{\ell_2\in L,\\ \ell_1>\ell_2}}\sum_{t\in T}\left(f^c_{j(\ell_1-\ell_2)}+F^o_{j\ell_2}\right)y_{j\ell_1\ell_2t}$$
$$\sum_{j\in J}\sum_{\ell_1\in L}\sum_{\substack{\ell_2\in L,\\ \ell_1<\ell_2}}\sum_{t\in T}\left(f^o_{j(\ell_2-\ell_1)}+F^o_{j\ell_2}\right)y_{j\ell_1\ell_2t}$$
$$\sum_{j\in J}\sum_{\ell_1\in L}\sum_{t\in T}F^o_{j\ell_1}y_{j\ell_1\ell_1t}+\sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}g_{ij\ell t}d_{it}x_{ij\ell t}$$
$$s.t.\ (4.2)-(4.7).$$

Theorem B.1.7. $v(\overline{ER\text{-}GMC}) \geq v(\overline{ER\text{-}1I})$.

Proof. The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{ER-1I}}$ from any feasible solution of $\overline{\text{ER-GMC}}$ and that both solutions have the same objective function value. Second, we provide a problem instance where $\overline{\text{ER-GMC}}$ provides a better bound than $\overline{\text{ER-1I}}$.

(A) Construction of a feasible $\overline{\text{ER-1I}}$ solution from any $\overline{\text{ER-GMC}}$ solution

Consider any solution $\left\{x_{ij\ell t}, y_{j\ell_1\ell_2t}\right\}$ that is feasible in $\overline{\text{ER-GMC}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\right\}$ that is feasible in $\overline{\text{ER-1I}}$ and has the same objective function value.

We deduce the values for the new variables from those of the existing solution variables $y_{j\ell_1\ell_2 t}$. Equalities (4.4) in the ER-GMC formulation conserve the flow for open facilities as it is found at the end of each planning period. They can be used to deduce the values for the y variables $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$y_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1 \ell t}. \tag{ER.R1}$$

The same equalities (4.4) also lead to the following result:

$$y_{j\ell(t-1)} \overset{(ER.R1)}{=} \sum_{\ell_1 \in L} y_{j\ell_1 \ell(t-1)} \overset{(4.4)}{=} \sum_{\ell_2 \in L} y_{j\ell\ell_2 t}. \tag{ER.R2}$$

Furthermore, we set $s_{j\ell t}$ and $w_{j\ell t}$ as follows:

$$s_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1(\ell_1+\ell)t} \tag{ER.R3}$$

$$w_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1(\ell_1-\ell)t}. \tag{ER.R4}$$

Having set the variables for the ER-1I formulation, we now show that the equalities (4.28) still hold. We replace the variables by the deduced values according to (ER.R1)-(ER.R4):

$$\sum_{\ell \in L} \ell y_{j\ell t} + \sum_{\ell \in L} \ell w_{j\ell t} = \sum_{\ell \in L} \ell y_{j\ell(t-1)} + \sum_{\ell \in L} \ell s_{j\ell t} \quad \forall j \in J, \ \forall t \in T \tag{4.28}$$

$$\overset{(ER.R1)-(ER.R4)}{\Leftrightarrow} \quad \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\ell_2 t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_1-\ell)t}$$
$$= \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_1 y_{j\ell_1\ell_2 t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_1+\ell)t} \quad \forall j \in J, \ \forall t \in T. \tag{B.12}$$

In the following, we prove that (B.12) is true by using the principle of induction:

Proposition: Equalities (B.12) are true for all sizes of L.

Basic cases: We start with the trivial case of $q = 1$, i.e., $L = \{0, 1\}$. Note that, for the sake of simplicity, we suppress the variable indices j and t, but indicate only the values for the indices ℓ_1 and ℓ_2:

$$LHS: \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\ell_2 t} \rightarrow 0y_{00} + 1y_{01} + 0y_{10} + 1y_{11}$$

$$\sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_1-\ell)t} \rightarrow 1y_{10}$$

$$RHS: \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_1 y_{j\ell_1\ell_2 t} \rightarrow 0y_{00} + 0y_{01} + 1y_{10} + 1y_{11}$$

$$\sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_1+\ell)t} \rightarrow 1y_{01}$$

It can be easily verified that the terms on the LHS equal the terms on the RHS. The proposition is thus true for $q = 1$.

Inductive step: We now show that the proposition also holds for $q = q + 1$. For $q + 1$, the LHS and RHS include the same terms as in the previous step. In addition, the following terms are added:

$$LHS: \sum_{\ell_2=0}^{q+1} \ell_2 y_{j(q+1)\ell_2 t} + \sum_{\ell_1=0}^{q} (q+1) y_{j\ell_1(q+1)t} + \sum_{\ell_2=0}^{q+1} (q+1-\ell_2) y_{j(q+1)\ell_2 t}$$

$$RHS: \sum_{\ell_2=0}^{q+1} (q+1) y_{j(q+1)\ell_2 t} + \sum_{\ell_1=0}^{q} \ell_1 y_{j\ell_1(q+1)t} + \sum_{\ell_1=0}^{q+1} (q+1-\ell_1) y_{j\ell_1(q+1)t}$$

Summing up all terms on the LHS and all terms on the RHS shows that both sides are equivalent. Hence the result follows by induction.

Therefore, constraints (4.28) are satisfied. For the $x_{ij\ell t}$ variables, we choose the same solution values as in the ER-GMC solution. Constraints (4.11) are therefore necessarily satisfied. In addition, the demand allocation contributes equally to the objective function in both formulations. Constraints (4.12) are also satisfied, as can be verified by replacing the $y_{j\ell_1\ell_2t}$ variables in constraints (4.3) by (ER.R1). The limiting constraints (see Section 4.5.2) are also satisfied by noting that each of the variables can be replaced by corresponding $y_{j\ell_1\ell_2t}$ variables and the sum of all $y_{j\ell_1\ell_2t}$ variables never exceeds 1. Finally, the SIs are feasible due to relationship (ER.R1).

The contribution of the variables $y_{j\ell t}$, $s_{j\ell t}$ and $w_{j\ell t}$ to the total solution costs is equivalent to that of the $y_{j\ell_1\ell_2t}$ variables. This can be easily shown by verifying the equalities (ER.R1)-(ER.R4) and the costs attributed to the $y_{j\ell_1\ell_2t}$ in Section 4.4.2.

(B) Problem instance where ER-GMC is stronger

We now explain, by the use of a small problem instance, under which circumstances the ER-GMC provides a better LP bound than the ER-1I and ER-2I formulations.

This instance contains one potential facility location and one client. The planning horizon contains one time period, in which the customer has a demand of 10 units. Production and transportation of the commodities is free. The maximum capacity level is 2. The capacity expansion costs 200\$ for one capacity level and 350\$ for two capacity levels. The capacity reduction costs are set to 20\$ for one capacity level and to 35\$ for two capacity levels. The maintenance costs for a facility is 300\$ at capacity level 1 and 500\$ at capacity level 2. The facility capacity is 10 at level 1 and 11 at level 2. Therefore, the costs to provide and maintain a certain amount of capacity do not follow the principle of economies of scale.

The ER-GMC formulation provides a better bound than the other formulations. The LP relaxation for the ER-GMC sets $y_{j_0\ell_0\ell_1t_0} = 1.0$ and $x_{i_0j_0\ell_1t_0} = 1.0$, resulting in a solution with a total cost of 500\$. For the ER-1I and ER-2I formulations, with and without the SIs, the optimal LP relaxation solution constructs half a level 2 facility, while allocating demand to a full level 1 facility. The decision variables linked to the objective function thus have the solution values $y_{j_0\ell_1t_0} = 1.0$ and $s_{j_0\ell_2t_0} = 0.5$. This solution has a total cost of 475$.

From the two parts (A) and (B) above, it follows: $v(\overline{\text{ER-GMC}}) \geq v(\overline{\text{ER-1I}})$. □

Theorem B.1.8. $v(\overline{ER\text{-}GMC}) \geq v(\overline{ER\text{-}2I})$.

Proof. The result follows by transitivity from Theorems B.1.6 and B.1.7. □

B.1.3 Theoretical Results for the DMCFLP_CRER formulations

Sections 4.6.3 and 5.3.2 referred to the special case DMCFLP_CRER. It has been outlined how this problem can be modeled by using the general DFLPG model, resulting in a model denoted to the *CRER-GMC*. The performance of this model has then been compared to an alternative model for the DMCFLP_CRER, using classical flow conservation constraints We now explicitly state this specialized formulation. To model the problem variant in which facilities can be closed and reopened, as well as expand or reduce their capacities, the flow conservation constraints are divided into two sets: one to manage the open facilities and one to manage the closed facilities. In the resulting network, nodes are thus given for open and closed capacities at each time period and each location (see Figure B.1). Note that the arcs which represent capacity closing link to the nodes for the subsequent time period to avoid cycles in the network structure. The network is therefore slightly different from the one used in the CSLP.

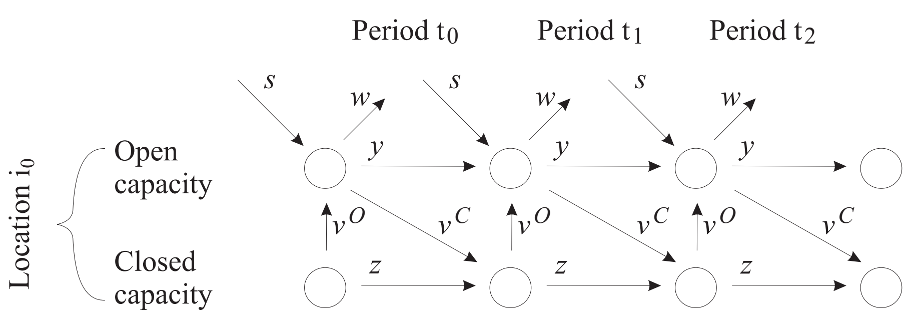

Figure B.1: Network flow to manage open and closed capacities at each facility for the DMCFLP_CRER.

Consider the variables $x_{ij\ell t}$, $y_{j\ell t}$, $s_{j\ell t}$, $w_{j\ell t}$, $v^o_{j\ell t}$ and $v^c_{j\ell t}$ as they are defined in Chapter 4. In addition, consider the binary variables $z_{j\ell t}$ which are 1, if a facility of size ℓ located at j is temporarily closed at period $t-1$ and remains closed during period t. The model, denoted as the *Single time-index flow formulation (CRER-1I)*, is given by:

$$(\text{CRER-1I}) \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t} + f^c_{j\ell}w_{j\ell t} + F^o_{j\ell}y_{j\ell t} + c^o_{j\ell}v^o_{j\ell t} + c^c_{j\ell}v^c_{j\ell t}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} d_{it}g_{ij\ell t}x_{ij\ell t} \tag{B.13}$$

$$s.t. \ \sum_{i\in I}\sum_{\ell\in L} x_{ij\ell t} = 1 \ \ ;\forall j\in J \ \ ;\forall t\in T \tag{B.14}$$

$$\sum_{i\in I} d_{it}x_{ij\ell t} \le u_{j\ell}y_{j\ell t} \ \ ;\forall j\in J \ \ ;\forall \ell\in L \ \ ;\forall t\in T \tag{B.15}$$

$$\sum_{\ell\in L}\left(\ell y_{j\ell(t-1)} + \ell s_{j\ell t} + \ell v^o_{j\ell t}\right) = \sum_{\ell\in L}\left(\ell y_{j\ell t} + \ell w_{j\ell t} + \ell v^c_{j\ell t}\right) \ \ ;\forall j\in J \ \ ;\forall t\in T \tag{B.16}$$

$$\sum_{\ell\in L}\left(\ell z_{j\ell(t-1)} + \ell v^c_{j\ell(t-1)}\right) = \sum_{\ell\in L}\left(\ell z_{j\ell t} + \ell v^o_{j\ell t}\right) \ \ ;\forall j\in J \ \ ;\forall t\in T \tag{B.17}$$

$$\sum_{\ell\in L}\left(y_{j\ell t} + z_{j\ell t} + v^c_{j\ell t}\right) \le 1 \ \ ;\forall j\in J \ \ ;\forall t\in T \tag{B.18}$$

$$w_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}, y_{j\ell t}, z_{j\ell t} \in \{0,1\}, \ x_{ij\ell t} \ge 0$$

Equalities (B.16) and (B.17) are the new flow conservation constraints for open and closed facilities, respectively. Inequalities (B.18) ensure that facilities can be closed and reopened only as a whole, but not partially. Note that these inequalities also imply that capacity can only be extended at open facilities. Finally, we also add the limiting constraints for the s and w variables as shown in Section 4.5.2 to ensure that feasible solutions only use one active variable of each type s and w at each location and time period (note that the y variables are already limited by constraints (B.18)).

Dominance Relationships. We now prove that the CRER-GMC formulation is at least as strong (strictly stronger for some instances) as the CRER-1I formulation in the sense that its LP relaxation provides a better bound.

Theorem B.1.9. $v(\overline{CRER\text{-}GMC}) \ge v(\overline{CRER\text{-}1I})$

Proof The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{CRER-1I}}$ from any feasible solution of $\overline{\text{CRER-GMC}}$ and that both solutions have the same objective function value. Second, we give a problem instance where $\overline{\text{CRER-GMC}}$ provides a better bound than $\overline{\text{CRER-1I}}$.

(A) Construction of feasible $\overline{\text{CRER-1I}}$ solution from any $\overline{\text{CRER-GMC}}$ solution

Consider any solution $\left\{x_{ij\ell t}, y_{j\ell_1\ell_2 t}\right\}$ that is feasible in $\overline{\text{CRER-GMC}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, z_{j\ell t}, s_{j\ell t}, w_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CRER-1I}}$ and has the same objective function value. Note that we denote the set of open capacity levels by L and the set of closed capacity levels by L'.

We set the values for the $x_{ij\ell t}$ variables to those in the CRER-GMC solution. The values for the variables $y_{j\ell t}$, $s_{j\ell t}$, $w_{j\ell t}$, $v^o_{j\ell t}$, $v^c_{j\ell t}$ and $z_{j\ell t}$ are set by establishing the following relations. $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$y_{j\ell t} = \sum_{\overline{\ell_1} \in L'} y_{j\overline{\ell_1}\ell t} + \sum_{\ell_1 \in L} y_{j\ell_1\ell t} \tag{CRER.R1}$$

$$s_{j\ell t} = \sum_{\overline{\ell_1} \in L'} y_{j\overline{\ell_1}(\ell_2=\ell_1+\ell)t} + \sum_{\ell_1 \in L} y_{j\ell_1(\ell_2=\overline{\ell_1+\ell})t} + \sum_{\ell_1 \in L} y_{j\ell_1(\ell_2=\ell_1+\ell)t} \tag{CRER.R2}$$

$$w_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1(\ell_2=\overline{\ell_1-\ell})t} + \sum_{\overline{\ell_1} \in L'} y_{j\overline{\ell_1}(\ell_2=\ell_1-\ell)t} + \sum_{\ell_1 \in L} y_{j\ell_1(\ell_2=\ell_1-\ell)t} \tag{CRER.R3}$$

$$v^o_{j\ell t} = \sum_{\ell_2 \in L} y_{j\bar{\ell}\ell_2 t} \tag{CRER.R4}$$

$$v^c_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1\bar{\ell}t} \tag{CRER.R5}$$

$$z_{j\ell t} = y_{j\overline{\ell\ell}t} \tag{CRER.R6}$$

In other words, $y_{j\ell t}$ is the sum of capacity that reopens and changes from any level to level ℓ, plus the capacity that has already been open and changes from any level to level ℓ. $z_{j\ell t}$ is the capacity that has been closed at $t-1$ and remains closed at t. $s_{j\ell t}$ is the sum of all facility expansions of ℓ capacity levels for the following three cases: a facility has been open at $t-1$ and remains open at t, a facility reopens and expands its capacity, and a facility closes right after having expanded its capacity. $w_{j\ell t}$ is the sum of all facility reductions of ℓ capacity levels for the following three cases: a facility reduces its capacity and remains open, a facility reduces its capacity and closes, and a facility reopens and reduces its capacity. $v^o_{j\ell t}$ is set to the sum of facility reopenings at capacity level ℓ (and may have changed capacity afterwards). $v^c_{j\ell t}$ is set to the sum of facility closings at capacity level ℓ (and possibly changed capacity before).

Note that the flow initialization and conservation constraints (4.4) and (4.5) guarantee that all variables as set above respect their domains, i.e., they do not exceed value 1. Demand constraints (B.14) hold, as all x variables have the same solution values in the solutions for both models. Replacing relation (CRER.R1) in inequalities (4.3) shows that constraints (B.15) are respected.

Due to equalities (4.4) in the GMC formulation as well as the relationships above, the following relations also hold $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$y_{j\ell(t-1)} \overset{(CRER.R1)}{=} \sum_{\overline{\ell_1} \in L'} y_{j\overline{\ell_1}\ell(t-1)} + \sum_{\ell_1 \in L} y_{j\ell_1\ell(t-1)} \overset{(4.4)}{=} \sum_{\ell_2 \in L} y_{j\ell\overline{\ell_2}t} + \sum_{\ell_2 \in L} y_{j\ell\ell_2 t} \quad \text{(CRER.R7)}$$

$$l_{j\ell(t-1)} + v^c_{j\ell(t-1)} \overset{(CRER.R5)\&(CRER.R6)}{=} y_{j\overline{\ell\ell}(t-1)} + \sum_{\ell_1 \in L} y_{j\ell_1\overline{\ell}(t-1)} \overset{(4.4)}{=} y_{j\overline{\ell\ell}t} + \sum_{\ell_2 \in L} y_{j\overline{\ell}\ell_2 t} \quad \text{(CRER.R8)}$$

We can now show that the flow conservation constraints (B.16) and (B.17) hold, by using the relationships defined above. To prove that the former one is satisfied, we proceed as follows:

$$\sum_{\ell \in L} \ell y_{j\ell(t-1)} + \sum_{\ell \in L} \ell s_{j\ell t} + \sum_{\ell \in L} \ell v^{o}_{j\ell t} = \sum_{\ell \in L} \ell y_{j\ell t} + \sum_{\ell \in L} \ell w_{j\ell t} + \sum_{\ell \in L} \ell v^{c}_{j\ell t} \tag{B.16}$$

$$\begin{aligned}
\overset{(CRER.R1)-(CRER.R5)}{\Leftrightarrow} \quad & \sum_{\ell_1 \in L} \sum_{\overline{\ell_2} \in L'} \ell_1 y_{j\ell_1\overline{\ell_2}t} + \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_1 y_{j\ell_1\ell_2 t} \\
& + \sum_{\overline{\ell_1} \in L'} \sum_{\ell \in L} \ell y_{j\overline{\ell_1}(\ell_1+\ell)t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_1+\ell)t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\overline{\ell_1+\ell})t} \\
& + \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\overline{\ell_1}\ell_2 t} \\
& = \sum_{\overline{\ell_1} \in L'} \sum_{\ell_2 \in L} \ell_2 y_{j\overline{\ell_1}\ell_2 t} + \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\ell_2 t} \\
& + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\overline{\ell_1-\ell})t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_1-\ell)t} + \sum_{\overline{\ell_1} \in L'} \sum_{\ell \in L} \ell y_{j\overline{\ell_1}(\ell_1-\ell)t} \\
& + \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\overline{\ell_2}t} \quad ;\forall i \in I \quad ;\forall t \in T
\end{aligned} \tag{B.19}$$

The resulting equalities (B.19) contains all terms of equalities (B.12) (which have been proven to be true). We may therefore eliminate all terms of (B.12) from equalities (B.19), resulting in the following equalities:

$$\begin{aligned}
& \sum_{\ell_1 \in L} \sum_{\overline{\ell_2} \in L'} \ell_1 y_{j\ell_1\overline{\ell_2}t} \\
& + \sum_{\overline{\ell_1} \in L'} \sum_{\ell \in L} \ell y_{j\overline{\ell_1}(\ell_1+\ell)t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\overline{\ell_1+\ell})t} \\
& + \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\overline{\ell_1}\ell_2 t} \\
& = \sum_{\overline{\ell_1} \in L'} \sum_{\ell_2 \in L} \ell_2 y_{j\overline{\ell_1}\ell_2 t} \\
& + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\overline{\ell_1-\ell})t} + \sum_{\overline{\ell_1} \in L'} \sum_{\ell \in L} \ell y_{j\overline{\ell_1}(\ell_1-\ell)t} \\
& + \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\overline{\ell_2}t} \quad ;\forall i \in I \quad ;\forall t \in T
\end{aligned} \tag{B.20}$$

In the following, we prove that (B.20), and therefore (B.19) and (B.16) as well, are true by using the principle of induction:

Induction - Proposition: Equalities (B.20) are true for all sizes of L.

Induction - Basic cases: We start with the trivial case of $q=1$, i.e., $L=\{0,1\}$. Note that, for sake of simplicity, we suppress the variable indices j and t, but indicate only the values for the indices ℓ_1 and ℓ_2:

$$\begin{aligned}
LHS: &\sum_{\ell_1 \in L}\sum_{\overline{\ell_2} \in L'} \ell_1 y_{j\ell_1\overline{\ell_2}t} \to 0y_{0\bar{1}} + 1y_{1\bar{1}} \\
&\sum_{\overline{\ell_1} \in L'}\sum_{\ell \in L} \ell y_{j\overline{\ell_1}(\ell_1+\ell)t} \to 0y_{\bar{1}1} \\
&\sum_{\ell_1 \in L}\sum_{\ell \in L} \ell y_{j\ell_1(\overline{\ell_1+\ell})t} \to 0y_{1\bar{1}} + 1y_{0\bar{1}} \\
&\sum_{\ell_1 \in L}\sum_{\ell_2 \in L} \ell_2 y_{j\overline{\ell_1}\ell_2 t} \to 0y_{\bar{1}0} + 0y_{\bar{1}1} + 1y_{\bar{1}0} + 1y_{\bar{1}1} \\
RHS: &\sum_{\overline{\ell_1} \in L'}\sum_{\ell_2 \in L} \ell_2 y_{j\overline{\ell_1}\ell_2 t} \to 0y_{\bar{1}0} + 1y_{\bar{1}1} \\
&\sum_{\ell_1 \in L}\sum_{\ell \in L} \ell y_{j\ell_1(\overline{\ell_1-\ell})t} \to 0 \\
&\sum_{\overline{\ell_1} \in L'}\sum_{\ell \in L} \ell y_{j\overline{\ell_1}(\ell_1-\ell)t} \to 0y_{\bar{1}1} + 1y_{\bar{1}0} \\
&\sum_{\ell_1 \in L}\sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\overline{\ell_2}t} \to 1y_{0\bar{1}} + 1y_{1\bar{1}}
\end{aligned}$$

It can easily be verified that the terms on the LHS equal the terms on the RHS. The proposition is thus true for $q=1$.

Inductive step: We now show that the proposition also holds for $q=q+1$. For $q+1$, the LHS and RHS include the same terms as in the previous step. In addition, the following terms are added:

$$
\begin{aligned}
LHS: & \sum_{\ell_1=0}^{q} \ell_1 y_{j\ell_1(\overline{q+1})t} + \sum_{\ell_2=0}^{q} (q+1) y_{j(q+1)\overline{\ell_2}t} + (q+1) y_{j(q+1)(\overline{q+1})t} \\
& + \sum_{\ell_1=0}^{q} (q+1-\ell_1) y_{j\overline{\ell_1}(q+1)t} + \sum_{\ell_1=0}^{q} (q+1-\ell_1) y_{j\ell_1(\overline{q+1})t} \\
& \sum_{\ell_1=0}^{q} \ell_1 y_{j\overline{\ell_1}(q+1)t} + \sum_{\ell_2=0}^{q} (q+1) y_{j(\overline{q+1})\ell_2 t} + (q+1) y_{j(\overline{q+1})(q+1)t} \\
RHS: & \sum_{\ell_1=1}^{q} (q+1) y_{j\overline{\ell_1}(q+1)t} + \sum_{\ell_2=0}^{q} \ell_2 y_{j(\overline{q+1})\ell_2 t} + (q+1) y_{j(\overline{q+1})(q+1)t} \\
& + \sum_{\ell_2=0}^{q} (q+1-\ell_2) y_{j(q+1)\overline{\ell_2}t} + \sum_{\ell_2=0}^{q} (q+1-\ell_2) y_{j(\overline{q+1})\ell_2 t} \\
& \sum_{\ell_1=0}^{q} (q+1) y_{j\ell_1(\overline{q+1})t} + \sum_{\ell_2=0}^{q} \ell_2 y_{j(q+1)\overline{\ell_2}t} + (q+1) y_{j(q+1)(\overline{q+1})t}
\end{aligned}
$$

Summing up all terms on the LHS and all terms on the RHS shows that both sides are equivalent, namely:

$$
\begin{aligned}
LHS = RHS = & \sum_{\ell_2=1}^{q} (q+1) y_{j(q+1)\overline{\ell_2}t} + \sum_{\ell_1=0}^{q} (q+1) y_{j\ell_1(\overline{q+1})t} + (q+1) y_{j(q+1)(\overline{q+1})t} \\
& + \sum_{\ell_2=0}^{q} (q+1) y_{j(\overline{q+1})\ell_2 t} + \sum_{\ell_1=1}^{q} (q+1) y_{j\overline{\ell_1}(q+1)t} + (q+1) y_{j(\overline{q+1})(q+1)t}
\end{aligned}
$$

Hence the result follows by induction. □

The flow conservation constraints (B.16) are thus satisfied. The flow conservation constraints (B.17) are also satisfied. This can be easily verified by replacing the variables by relations (CRER.R4), (CRER.R6) and (CRER.R8).

Constraints (B.18) and the limiting constraints for the s and w variables (see Section 4.5.2) also hold. This can be verified by replacing the variables by relationships (CRER.R1), (CRER.R2), (CRER.R3), (CRER.R5) and (CRER.R6).

Finally, both solutions have the same value, as can be verified by the way the CRER-GMC (see Section 4.4.2) and CRER-1I variable OF coefficients are composed.

(B) Problem instance where CRER-GMC is stronger

Consider the problem instance presented in Section B.1.2.2. Additionally assume that the facility closing and reopening costs are sufficiently high such that they are not part of the optimal LP solution. For both formulations, the given problem instance thus produces the same results as already observed in Section B.1.2.2. The CRER-GMC formulation provides a better bound, since it combines the decisions of construction and maintenance in a single variable. In contrast, the CRER-1I formulation uses two different variables for construction and maintenance decisions and thus constructs half a facility of size 2 while maintaining an entire facility of size 1.

From the two parts (A) and (B) above, it follows that $v(\overline{\text{CRER-GMC}}) \geq v(\overline{\text{CRER-1I}})$.

□

B.2 Test Instances

Instances for multi-period facility location problems essentially contain information about the customer demand for each time period, the construction costs of the facilities and the costs to allocate demand between customers and facilities. The DFLPG and the three special cases additionally involve capacity changes. Due to the lack of openly available instance sets that include these properties, a large set of problem instances has been generated to test the proposed models. The instances can be divided into five different sets, each with a different number of time periods. Each of these sets contains a total of 288 instances, 96 for each capacity level.

In the following we present how the instance properties are generated and which parameters are used.

B.2.1 Number of time periods

Instances have been generated with different lengths of the planning horizon $|T|$, chosen such that $|T| \in \{6, 8, 10, 12, 14\}$. Most of the computational results in this work are based on the instances with $|T| = 12$, as this is a very common discretization in practice, for example for a monthly representation of an entire year.

B.2.2 Problem dimension

Instances have been generated with different numbers of customers $|I|$ and candidate facility locations $|J|$. We considered the following dimensions $(|J|/|I|)$, always assuming that $|J| \leq |I|$: *(10/20)*, *(10/50)*, *(50/50)*, *(50/100)*, *(50/250)*, *(100/250)*, *(100/500)* and *(100/1000)*.

B.2.3 Number of capacity levels

The number of capacity levels q also impacts on the size of the models. Instances are generated with a maximum of 3, 5 and 10 capacity levels, which are assumed to be reasonable values for a broad variety of different application contexts.

The capacities $u_{j\ell}$ are generated based on the total number of customers and are chosen such that a considerably large number of facilities (about half of the candidate locations) is selected, resulting in instances that tend to be difficult to solve. The larger the set of customers, the higher is the capacity of each level. To be precise, we set $u_{j1} = 150$ if the instance covers 20 customers, $u_{j1} = 300$ if the instance covers 50 customers, $u_{j1} = 600$ if the instance covers 100 customers, $u_{j1} = 1200$ if the instance covers 250 customers, $u_{j1} = 2500$ if the instance covers 500 customers and $u_{j1} = 5000$ if the instance covers 1000 customers. The capacities of higher capacity levels $\ell \geq 2$ are set as multiples of the first capacity level, i.e., $u_{j\ell} = \ell \cdot u_{j1}$.

Note that we assume that the problem instances do not contain initially existing facilities, i.e., the initial capacity level of each facility is 0.

B.2.4 Customer/facility locations

For each of the different problem sizes, $|I|$ customer demand points have been randomly generated following a continuous uniform distribution, rounding the x and y coordinates to the next lowest integer value. The first $|J|$ points of $|I|$ customer locations have additionally been defined as candidate facility locations and therefore coincide with the customer demand points. The networks were generated on squares of the following three sizes: 300*km*, 380*km* and 450*km*.

B.2.5 Demand allocation costs

Costs are divided into fixed and variable costs and are based on those given in an industrial application (Jena et al., 2012). Fixed costs are given by the construction of facilities and the change of their capacity levels. Variable costs are composed of the costs to produce and transport the commodities.

Transportation costs have been computed based on the Euclidean distance between the points, including a small modification that results in a slight clustering effect of the customers close to a facility. The transportation costs are composed of two components:

i. A cost that depends on the total distance, referred to as the vehicle cost. The vehicle cost is linear in function of the Euclidean distance between the two points on the network (5$/km).

ii. A cost that depends on the travel time, referred to as the driver's payment. The driver's payment is 0 if the two points are within one hour of transportation distance (assuming an average vehicle speed of 62km/h) and linear in function of the Euclidean distance if the two points are at more than one hour of driving distance (50$/h).

Let $dist_{ij}$ denote the distance between facility location j and customer i. The costs to transport one unit of demand from facility j to customer i is therefore set to:

$$g^T_{ij} = 5 \cdot dist_{ij} + 50 \cdot \max\left(0, \frac{dist_{ij}}{62} - 1\right)$$

The variable and fixed costs include economies of scale in function of the size of the facility. These costs are therefore described by concave cost functions, as explained in the following. The production costs for each unit served from a facility to a customer is defined as the cost to operate a facility and depends on the size of the facility. The cost to produce one commodity unit at capacity level 1 is set to 20.90$. At each higher capacity level, the production cost is 3% cheaper than at the previous level:

$$g^P_{j0} = 20.90$$

$$g^P_{j\ell} = 0.97 \cdot g^p_{j(\ell-1)}$$

Note that the production costs are added to the transportation costs to determine the total demand allocation costs $g_{ij\ell t}$ to serve the customer demands:

$$g_{ij\ell t} = g^T_{ij} + g^P_{j\ell}$$

In addition to the demand allocation costs as discussed above, a second set of instances was generated with five times higher transportation costs.

B.2.6 Fixed costs

The construction cost, also referred to as capacity expansion cost, is set to 100,000$ for a facility of level 1. Each additional capacity level is 10% cheaper than the previous one. The construction costs for facilities of different capacity levels are therefore computed according to the following formula:

$$f^o_{j0} = 0$$

$$f^o_{j1} = 100,000$$

$$f^o_{j2} = 190,000$$

$$f^o_{j\ell} = f^o_{j(\ell-1)} + 0.9 \cdot (f^o_{j(\ell-1)} - f^o_{j(\ell-2)})$$

The maintenance costs for a facility of a certain size are computed in a similar fashion. They are set relatively high to motivate capacity changes. The maintenance costs for a facility of capacity level 1 are set to 51,000$. The maintenance costs for each additional capacity level are 15% cheaper than the previous ones:

$$F^o_{j0} = 0$$

$$F^o_{j1} = 51,000$$

$$F^o_{j2} = 94,350$$

$$F^o_{j\ell} = F^o_{j(\ell-1)} + 0.85 \cdot (F^o_{j(\ell-1)} - F^o_{j(\ell-2)})$$

The cost to reduce the capacity of a facility by ℓ capacity levels is set to 10% of the costs to expand the capacity of a facility by ℓ capacity levels. Finally, the costs for reopening and closing existing facilities have been adopted from the input data the industrial application mentioned above (Jena et al., 2012). Although being strictly increasing, these costs do not necessarily represent economies of scale. The costs to reopen a closed facility of capacity level $1,\dots,10$ are 3,138.34$, 4,084.69$, 4,924.58$, 5,693.26\$, 7,085.07\$, 7,727.50\$, 8,342.34\$, 8,933.68\$, 10,057.70\$ and 10,594.80\$, respectively. The costs to temporarily close an open facility of capacity level $1, \ldots, 10$ are 8,624.93\$, 11,595.80\$, 14,305.60\$, 16,836.50\$, 21,524.10\$, 23,727.90\$, 25,858.30\$, 27,925.70\$, 31,901.10\$ and 33,820.70\$, respectively.

B.2.7 Demand distribution

We consider two different demand scenarios. In both scenarios, the demand for each of the customers is randomly generated and randomly distributed over time. The two scenarios differ in their total demand summed over all customers in each time period. In the first scenario (*regular*), the total demand is similar in each time period. We set the average demand for a customer to 12 units per time period. The total demand for all customers is therefore approximately $12 \cdot |I|$ units at each time period. The second scenario (*irregular*) assumes that the total demand follows strong variations along time and therefore varies at each time period. In this scenario, the total demand for all customers is multiplied by a random distortion factor at each time period. This random distortion factor is set to the absolute value of a normal random variable with mean value 1.0 and standard deviation 0.6 (note that this procedure produced distortion factors from 0.14 to 2.24). Let $totDem_t$ be the total customer demand for time period t, computed as explained above for one of the two scenarios.

We now explain how the individual demands for each of the customers are generated and distributed on the different time periods such that its total sum equals approximately the value of $totDem_t$ at each of the time periods. For all customers and all time periods, the total demand covers approximately $12 \cdot |I| \cdot |T|$ units. In a first step, this total demand is randomly distributed on each of the customers. In a second step, each customer demand is distributed on different time periods:

i. Let *totRemDem* denote the total demand for all customers and time periods that has not yet been allocated to any customer. Furthermore, let *numRemCust* indicate the number of customers that have not yet been allocated any demand. For each customer, its total demand for all time periods, denoted to $totJDem_j$, is computed as a random normal variable with a mean $\mu = totRemDemand/numRemCust$ and standard deviation $\sigma = \mu/2$. Note that this method did not produce any negative value in the course of our instance generation.

ii. The total demand for each customer, $totJDem_j$ is then divided into four equal parts. One part of the demand is allocated to a time period that is randomly selected following a uniform distribution. Each of the other three parts is allocated to the time period t that has the highest gap between the total demand yet allocated to period t and its value $totDem_t$.

Note that the choice of allocating demand to only a few of the time periods is motivated by the aforementioned industrial application in the forest industry, where each logging region is harvested, on average, about four seasons over a ten-period planning horizon. Furthermore, it results in a geographically more dispersed distribution of the demand which creates the need to adjust capacities at the facilities.

B.3 Model Sizes

This section summarizes the model sizes expected for each of the problem variants and formulations. Tables B.I, B.II and B.III show the model sizes for problem variants DMCFLP_CR, DMCFLP_ER and DMCFLP_CR_ER, respectively. All instances possess twelve time periods.

The number of continuous x variables (*Cont. vars / SIs*) is identical for all formulations of the same problem variant. This number also represents the number of Strong Inequalities, as there is exactly one SI for each x variable. For each of the formulations, the number of additional binary variables (*Bin. var.*) and the number of constraints (*Constr.*) is given.

When the instances are generated, the total demand for each customer is randomly distributed on four time periods. For some customers, a time period may be selected more than once. Therefore, some customers may have less than four time periods with non-zero demands and the total number of non-negative demands may not necessarily equal $4 \cdot |I| \cdot |T|$. The total number of non-negative customer demands directly impacts the number of continuous x variables and the number of demand constraints. Thus, for some of the instances, the number of continuous variables and constraints may be slightly smaller than the numbers indicated in the tables. As demands are generated independently for each of the three networks (see Section B.2.4), instances with the same numbers of customers may have different numbers of x variables and demand constraints for each network. An analysis showed that each of the instances possesses at least 90% of the $4 \cdot |I| \cdot |T|$ non-negative customer demands.

Finally, note that the number of Aggregated Demand Constraints is always equal to the number of time periods (i.e., in this case 12).

q	Instance size	Cont. vars / SIs	CR-GMC Bin. var.	CR-GMC Constr.	CR-1I Bin. var.	CR-1I Constr.	CR-2I+ Bin. var.	CR-2I+ Constr.
3	10/20	2,400	1,920	1,280	1,440	1,160	4,680	930
	10/50	6,000	1,920	1,400	1,440	1,280	4,680	1,050
	50/50	30,000	9,600	6,200	7,200	5,600	23,400	4,450
	50/100	60,000	9,600	6,400	7,200	5,800	23,400	4,650
	50/250	150,000	9,600	7,000	7,200	6,400	23,400	5,250
	100/250	300,000	19,200	13,000	14,400	11,800	46,800	9,500
	100/500	600,000	19,200	14,000	14,400	12,800	46,800	10,500
	100/1000	1,200,000	19,200	16,000	14,400	14,800	46,800	12,500
5	10/20	4,000	3,120	2,000	2,400	1,880	7,800	1,410
	10/50	10,000	3,120	2,120	2,400	2,000	7,800	1,530
	50/50	50,000	15,600	9,800	12,000	9,200	39,000	6,850
	50/100	100,000	15,600	10,000	12,000	9,400	39,000	7,050
	50/250	250,000	15,600	10,600	12,000	10,000	39,000	7,650
	100/250	500,000	31,200	20,200	24,000	19,000	78,000	14,300
	100/500	1,000,000	31,200	21,200	24,000	20,000	78,000	15,300
	100/1000	2,000,000	31,200	23,200	24,000	22,000	78,000	17,300
10	10/20	8,000	6,120	3,800	4,800	3,680	15,600	2,610
	10/50	20,000	6,120	3,920	4,800	3,800	15,600	2,730
	50/50	100,000	30,600	18,800	24,000	18,200	78,000	12,850
	50/100	200,000	30,600	19,000	24,000	18,400	78,000	13,050
	50/250	500,000	30,600	19,600	24,000	19,000	78,000	13,650
	100/250	1,000,000	61,200	38,200	48,000	37,000	156,000	26,300
	100/500	2,000,000	61,200	39,200	48,000	38,000	156,000	27,300
	100/1000	4,000,000	61,200	41,200	48,000	40,000	156,000	29,300

Table B.I: Model sizes for the formulations CR-GMC, CR-1I and CR-2I+.

q	Instance size	Cont. vars / SIs	ER-GMC Bin. var.	ER-GMC Constr.	ER-1I Bin. var.	ER-1I Constr.	ER-2I Bin. var.	ER-2I Constr.
3	10/20	2,400	1,920	920	1,080	920	3,420	1,160
	10/50	6,000	1,920	1,400	1,080	2,000	3,420	1,280
	50/50	30,000	9,600	6,200	5,400	9,200	17,100	5,600
	50/100	60,000	9,600	6,400	5,400	9,400	17,100	5,800
	50/250	150,000	9,600	7,000	5,400	10,000	17,100	6,400
	100/250	300,000	19,200	13,000	10,800	19,000	34,200	11,800
	100/500	600,000	19,200	14,000	10,800	20,000	34,200	12,800
	100/1000	1,200,000	19,200	16,000	10,800	22,000	34,200	14,800
5	10/20	4,000	4,320	2,000	1,800	1,160	5,700	1,400
	10/50	10,000	4,320	2,120	1,800	3,200	5,700	1,520
	50/50	50,000	21,600	9,800	9,000	15,200	28,500	6,800
	50/100	100,000	21,600	10,000	9,000	15,400	28,500	7,000
	50/250	250,000	21,600	10,600	9,000	16,000	28,500	7,600
	100/250	500,000	43,200	20,200	18,000	31,000	57,000	14,200
	100/500	1,000,000	43,200	21,200	18,000	32,000	57,000	15,200
	100/1000	2,000,000	43,200	23,200	18,000	34,000	57,000	17,200
10	10/20	8,000	6,120	3,800	3,600	1,760	11,400	2,000
	10/50	20,000	6,120	3,920	3,600	6,200	11,400	2,120
	50/50	100,000	30,600	18,800	18,000	30,200	57,000	9,800
	50/100	200,000	30,600	19,000	18,000	30,400	57,000	10,000
	50/250	500,000	30,600	19,600	18,000	31,000	57,000	10,600
	100/250	1,000,000	61,200	38,200	36,000	61,000	114,000	20,200
	100/500	2,000,000	61,200	39,200	36,000	62,000	114,000	21,200
	100/1000	4,000,000	61,200	41,200	36,000	64,000	114,000	23,200

Table B.II: Model sizes for the formulations ER-GMC, ER-1I and ER-2I.

q	Instance size	Cont. vars / SIs	CRER-GMC Bin. var.	CRER-GMC Constr.	CRER-1I Bin. var.	CRER-1I Constr.
3	10/20	2,400	3,720	1,280	2,160	1,040
	10/50	6,000	3,720	1,400	2,160	1,160
	50/50	30,000	18,600	6,200	10,800	5,000
	50/100	60,000	18,600	6,400	10,800	5,200
	50/250	150,000	18,600	7,000	10,800	5,800
	100/250	300,000	37,200	13,000	21,600	10,600
	100/500	600,000	37,200	14,000	21,600	11,600
	100/1000	1,200,000	37,200	16,000	21,600	13,600
5	10/20	4,000	8,520	2,000	3,600	1,280
	10/50	10,000	8,520	2,120	3,600	1,400
	50/50	50,000	42,600	9,800	18,000	6,200
	50/100	100,000	42,600	10,000	18,000	6,400
	50/250	250,000	42,600	10,600	18,000	7,000
	100/250	500,000	85,200	20,200	36,000	13,000
	100/500	1,000,000	85,200	21,200	36,000	14,000
	100/1000	2,000,000	85,200	23,200	36,000	16,000
10	10/20	8,000	28,920	3,800	7,200	1,880
	10/50	20,000	28,920	3,920	7,200	2,000
	50/50	100,000	144,600	18,800	36,000	9,200
	50/100	200,000	144,600	19,000	36,000	9,400
	50/250	500,000	144,600	19,600	36,000	10,000
	100/250	1,000,000	289,200	38,200	72,000	19,000
	100/500	2,000,000	289,200	39,200	72,000	20,000
	100/1000	4,000,000	289,200	41,200	72,000	22,000

Table B.III: Model sizes for the formulations CRER-GMC and CRER-1I.

APPENDIX C

SUPPLEMENT TO CHAPTER 5

C.1 Test Instances

Instances for multi-period facility location problems essentially contain information about the customer demand for each time period, construction costs of the facilities and the costs to allocate demand between customers and facilities. The DFLPG and the three special cases additionally involve a detailed cost structure for the capacity changes. Due to the lack of openly available instance sets that include these properties, we generated a total of 540 instances, 180 for each capacity level, to test the presented models. These essentially extend the instances used by Jena et al. (2013) by adding multiple commodities, the use of a cost matrix for capacity changes and a larger set of candidate facility locations. In the following we present how these instance properties are generated and which parameters are used.

C.1.1 Problem dimension

Instances were generated with different numbers of candidate facility locations $|J|$ and customers $|I|$, combining all pairs of $J \in \{50, 100, 150, 200, 250\}$ and $I \in \{|J|, 4 \cdot |J|\}$. To be precise, the instance dimensions are: *(10/20)*, *(50/50)*, *(50/200)*, *(100/100)*, *(100/400)*, *(150/150)*, *(150/600)*, *(200/200)*, *(200/800)*, *(250/250)* and *(250/1000)*.

C.1.2 Number of capacity levels

The number of capacity levels q also impacts on the size of the models. Instances are generated with a maximum of 3, 5 and 10 capacity levels, which are assumed to be reasonable values for a broad variety of different application contexts.

The capacities $u_{j\ell}$ are generated based on the total number of customers and are chosen such that a considerably large number of facilities (about half of the candidate locations) is selected. The larger the set of customers, the higher is the capacity of each level. To be precise, we set $u_{j1} = 300$ if the instance covers 50 customers, $u_{j1} = 600$ if the instance covers 100 customers, $u_{j1} = 800$ if the instance covers 150 customers, $u_{j1} = 1000$ if the instance covers 200 customers, $u_{j1} = 1200$ if the instance covers 250 customers, $u_{j1} = 2000$ if the instance covers 400 customers, $u_{j1} = 2500$ if the instance covers 600 customers, $u_{j1} = 3000$ if the instance covers 800 customers and $u_{j1} = 5000$ if the instance covers 1000 customers. The capacities of higher capacity levels $\ell \geq 2$ are set as multiples of the first capacity level, i.e., $u_{j\ell} = \ell \cdot u_{j1}$. Note that we assume that the problem instances do not contain initially existing facilities, i.e., the initial capacity level of each facility is 0.

C.1.2.1 Number of time periods

All generated instances contain ten time periods, which is found to be sufficient to demonstrate capacity changes along time and small enough to not increase the size of the models too much. It is also very close to a monthly discretization of an entire year, which is very common in practice.

C.1.3 Customer/facility locations

For each of the different problem sizes, $|I|$ customer demand points have been randomly generated following a continuous uniform distribution, rounding the x and y coordinates to the next lowest integer value. The first $|J|$ points of $|I|$ customer locations have additionally been defined as candidate facility locations and therefore coincide with the customer demand points. The networks were generated on squares of the following three sizes: 300*km*, 380*km* and 450*km*.

C.1.4 Demand allocation costs

Costs are divided into fixed and variable costs. Fixed costs are given by the construction of facilities and the change of their capacity levels. Variable costs are composed of the costs to produce and transport the commodities.

Transportation costs have been computed based on the Euclidean distance between the points, including a small modification that results in a slight clustering effect of the customers close to a facility. The transportation costs are composed of two components:

i. A cost that depends on the total distance, referred to as the vehicle cost. The vehicle cost is linear in function of the Euclidean distance between the two points on the network (5$/km).

ii. A cost that depends on the travel time, referred to as the driver's payment. The driver's payment is 0 if the two points are within one-hour of transportation distance (assuming an average vehicle speed of 62km/h) and linear in function of the Euclidean distance if the two points are at more than one hour of driving distance (50$/h).

Let $dist_{ij}$ denote the distance between facility location j and customer i. The costs to transport one unit of demand from facility j to customer i is therefore set to:

$$g_{ij}^{T} = 5 \cdot dist_{ij} + 50 \cdot \max\left(0, \frac{dist_{ij}}{62} - 1\right)$$

The variable and fixed costs include economies of scale in function of the size of the facility. These costs are therefore described by concave cost functions, as explained in the following. The production costs for each unit served from a facility to a customer is defined as the cost to operate a facility and depends on the size of the facility. The cost to produce one commodity unit at capacity level 1 is set to 20.90$. At each higher capacity level, the production cost is 3% cheaper than at the previous level:

$$g_{j0}^{P} = 20.90$$

$$g_{j\ell}^{P} = 0.97 \cdot g_{j(\ell-1)}^{p}$$

Note that the production costs are added to the transportation costs to determine the total demand allocation costs $g_{ij\ell t}$ to serve the customer demands:

$$g_{ij\ell t} = g_{ij}^{T} + g_{j\ell}^{P}$$

In addition to the demand allocation costs as discussed above, a second set of instances was generated with five times higher transportation costs.

C.1.5 Fixed costs

The construction cost, also referred to as capacity expansion cost, is set to 100,000$ for a facility of level 1. Each additional capacity level is 10% cheaper than the previous one. The construction costs for facilities of different capacity levels are therefore computed according to the following formula:

$$f^o_{j0} = 0$$

$$f^o_{j1} = 100,000$$

$$f^o_{j2} = 190,000$$

$$f^o_{j\ell} = f^o_{j(\ell-1)} + 0.9 \cdot (f^o_{j(\ell-1)} - f^o_{j(\ell-2)})$$

The maintenance costs for a facility of a certain size are computed in a similar fashion. They are set relatively high to motivate capacity changes. The maintenance costs for a facility of capacity level 1 are set to 51,000$. The maintenance costs for each additional capacity level are 15% cheaper than the previous ones:

$$F^o_{j0} = 0$$

$$F^o_{j1} = 51,000$$

$$F^o_{j2} = 94,350$$

$$F^o_{j\ell} = F^o_{j(\ell-1)} + 0.85 \cdot (F^o_{j(\ell-1)} - F^o_{j(\ell-2)})$$

Fixed Costs for the Special Cases. For the three special cases, i.e., the DMCFLP_CR, DMCFLP_ER and the DMCFLP_CR_ER, the cost to reduce the capacity of a facility by ℓ capacity levels is set to 10% of the costs to expand the capacity of a facility by ℓ

capacity levels.

Finally, the costs for reopening and closing existing facilities were taken from the input data of the previously mentioned industrial application introduced by Jena et al. (2012). Although being strictly increasing, these costs do not necessarily represent economies of scale. The costs to reopen a closed facility of capacity level $1, \ldots, 10$ are 3,138.34, 4,084.69, 4,924.58, 5,693.26, 7,085.07, 7,727.50, 8,342.34, 8,933.68, 10,057.70 and 10,594.80, respectively. The costs to close an open facility of capacity level $1, \ldots, 10$ are 8,624.93, 11,595.80, 14,305.60, 16,836.50, 21,524.10, 23,727.90, 25,858.30, 27,925.70, 31,901.10 and 33,820.70, respectively.

Fixed Costs for the DFLPG. For the DFLPG, the construction costs are as indicated above, i.e., the costs to construct a facility of size ℓ and its maintenance costs at time period t are set to: $f_{j0\ell t} = f^o_{j\ell} + F^o_{j\ell}$.

The costs to change capacity levels for this problem are based on a cost matrix, and, therefore, differ from the costs for capacity expansion and reduction shown above for the special cases. The cost to completely remove a facility are set to 25% of the construction costs of a facility of the same size: $f_{j\ell 0t} = f^o_{j\ell}/4$.

Finally, the cost to change the capacity level from $\ell_1 \geq 1$ to $\ell_2 \geq 1$ are set to the difference of their construction costs, scaled by 50%:

$$f_{j\ell_1\ell_2 t} = \begin{cases} 1.5 \cdot (f^o_{j\ell_2} - f^o_{j\ell_1}) \text{ , if } \ell_1 < \ell_2 \\ 1.5 \cdot (f^o_{j\ell_1} - f^o_{j\ell_2}) \text{ , if } \ell_1 > \ell_2. \end{cases}$$

C.1.6 Demand distribution

We consider two different demand scenarios. In both scenarios, the demand for each of the customers is randomly generated and randomly distributed over time. The two scenarios differ in their total demand summed over all customers in each time period. In the first scenario (*regular*), the total demand is similar in each time period. We set the average demand for a customer to 12 units per time period. The total demand for all customers is therefore approximately $10 \cdot |I|$ units at each time period. The second scenario (*irregular*) assumes that the total demand follows strong variations along time and therefore varies at each time period. In this scenario, the total demand for all customers is multiplied by a random distortion factor at each time period. This random distortion factor is set to the absolute value of a normal random variable with mean value 1.0 and standard deviation 0.6 (note that this procedure produced distortion factors from 0.14 to 2.24). Let $totDem_t$ be the total customer demand for time period t, computed as explained above for one of the two scenarios.

We now explain how the individual demands for each of the customers are generated and distributed on the different time periods such that its total sum equals approximately the value of $totDem_t$ at each of the time periods. For all customers and all time periods, the total demand covers approximately $12 \cdot |I| \cdot |T|$ units. In a first step, this total demand is randomly distributed on each of the customers. In a second step, each customer demand is distributed on different time periods:

i. Let $totRemDem$ denote the total demand for all customers and time periods that has not yet been allocated to any customer. Furthermore, let *numRemCust* indicate the number of customers that have not yet been allocated any demand. For each customer, its total demand for all time periods, denoted to $totJDem_j$, is computed as a random normal variable with a mean $\mu = totRemDemand/numRemCust$ and standard deviation $\sigma = \mu/2$. Note that, throughout our instance generation, this method did not produce any negative value.

ii. The total demand for each customer, $totJDem_j$ is then divided into four equal parts. One part of the demand is allocated to a time period that is randomly selected following a uniform distribution. Each of the other three parts is allocated to the time period t that has the highest gap between the total demand yet allocated to period t and its value $totDem_t$.

The demands for the second to fifth commodity are computed based on the demand of the first commodity. To be precise, the demand d_{ipt} for $p \geq 2$ is computed as $d_{ipt} = d_{i1t} \cdot rand(1.0, 0.2) \cdot avgDem_p/avgDem_1$, where $avgDem_1 = 10$, $avgDem_2 = 6$, $avgDem_3 =$

9, $avgDem_4 = 5$, $avgDem_5 = 8$ and $rand(1.0, 0.2)$ is a random variable with normal distribution, a mean of 1.0 and a standard deviation of 0.2.

Note that the choice of allocating demand to only a few of the time periods is motivated by the aforementioned industrial application in the forest industry, where each logging region is harvested, on average, about four seasons over the ten-period planning horizon. Furthermore, it results in a geographically more dispersed distribution of the demand which creates the need to adjust capacities at the facilities.