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Special functions of Weyl groups and their continuous and discrete orthogonality

par

Lenka Motlochová

Département de mathématiques et de statistique
Faculté des arts et des sciences

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présentée par

Lenka Motlochová

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Véronique Hussin
(président-rapporteur)

Jiří Patera
(directeur de recherche)

Alfred Michel Grundland
(membre du jury)

Mark Walton
(examinateur externe)

(représentant du doyen de la FAS)

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RÉSUMÉ

Cette thèse s'intéresse à l'étude des propriétés et applications de quatre familles des fonctions spéciales associées aux groupes de Weyl et dénotées C, S, S^s et S^l. Ces fonctions peuvent être vues comme des généralisations des polynômes de Tchebyshev. Elles sont en lien avec des polynômes orthogonaux à plusieurs variables associés aux algèbres de Lie simples, par exemple les polynômes de Jacobi et de Macdonald. Elles ont plusieurs propriétés remarquables, dont l'orthogonalité continue et discrète. En particulier, il est prouvé dans la présente thèse que les fonctions S^s et S^l caractérisées par certains paramètres sont mutuellement orthogonales par rapport à une mesure discrète. Leur orthogonalité discrète permet de déduire deux types de transformées discrètes analogues aux transformées de Fourier pour chaque algèbre de Lie simple avec racines des longueurs différentes. Comme les polynômes de Tchebyshev, ces quatre familles des fonctions ont des applications en analyse numérique. On obtient dans cette thèse quelques formules de «cubature», pour des fonctions de plusieurs variables, en liaison avec les fonctions C, S^s et S^l. On fournit également une description complète des transformées en cosinus discrètes de types V–VIII à n dimensions en employant les fonctions spéciales associées aux algèbres de Lie simples B_n et C_n, appelées cosinus antisymétriques et symétriques. Enfin, on étudie quatre familles de polynômes orthogonaux à plusieurs variables, analogues aux polynômes de Tchebyshev, introduits en utilisant les cosinus (anti)symétriques.

Mots-clés: groupes de Weyl; fonctions spéciales C, S, S^s et S^l; polynômes orthogonaux; transformées discrètes; formules de «cubature».

ABSTRACT

This thesis presents several properties and applications of four families of Weyl group orbit functions called C-, S-, S^s- and S^l-functions. These functions may be viewed as generalizations of the well-known Chebyshev polynomials. They are related to orthogonal polynomials associated with simple Lie algebras, e.g. the multivariate Jacobi and Macdonald polynomials. They have numerous remarkable properties such as continuous and discrete orthogonality. In particular, it is shown that the S^s- and S^l-functions characterized by certain parameters are mutually orthogonal with respect to a discrete measure. Their discrete orthogonality allows to deduce two types of Fourier-like discrete transforms for each simple Lie algebra with two different lengths of roots. Similarly to the Chebyshev polynomials, these four families of functions have applications in numerical integration. We obtain in this thesis various cubature formulas, for functions of several variables, arising from C-, S^s- and S^l-functions. We also provide a complete description of discrete multivariate cosine transforms of types V–VIII involving the Weyl group orbit functions arising from simple Lie algebras C_n and B_n, called antisymmetric and symmetric cosine functions. Furthermore, we study four families of multivariate Chebyshev-like orthogonal polynomials introduced via (anti)symmetric cosine functions.

Keywords: Weyl groups; orbit functions C, S, S^s and S^l; orthogonal polynomials; discrete transforms; cubature formulas.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

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INTRODUCTION

The purpose of this work is to study properties and applications of special functions related to the Weyl groups corresponding to simple Lie algebras. We are motivated by three main topics: discrete Fourier-like analysis, multivariate orthogonal polynomials and numerical integration.

Lie algebras have been named after Norwegian mathematician Marcus Sophus Lie, whose work on continuous transformation groups, nowadays known as Lie groups, led to the creation of Lie theory [**55, 60**]. Lie algebras have become, together with Lie groups, a subject of interest in several domains of mathematics and theoretical physics. We restrict our attention to the special case of semi-simple Lie algebras. They can be expressed as direct sums of simple Lie algebras which are completely determined by n vectors, known as simple roots, spanning an Euclidean space isomorphic to $\mathbb{R}^n$ [**6, 19, 29, 54, 60**]. To each simple Lie algebra, we can uniquely associate its finite Weyl group of geometric symmetries generated by a set of reflections with respect to the hyperplanes orthogonal to simple roots and passing through the origin [**20**]. The Weyl groups are of primary importance since our special functions are induced from their sign homomorphisms.

Among various types of special functions associated with Weyl groups, we are interested in the so-called Weyl group orbit functions, including four different families of functions, C-, S-, S^s- and S^l-functions [**9, 24, 26, 40**]. The symmetric C-functions and antisymmetric S-functions are well known from the representation theory of simple Lie algebras [**54, 60**]. In particular, the S-functions appear in the Weyl character formula and every character of irreducible representations of simple Lie algebra can be written as a linear combination of C-functions. Moreover, it is possible to show that the C- and S-functions arising in connection with simple Lie algebras B_n and C_n become, up to a constant, symmetric multivariate cosine functions and antisymmetric multivariate sine functions [**25**] respectively. For we dispose of two other generalizations of the common cosine and sine functions of one variable, it is natural to try to find their analogues in terms of Weyl group orbit functions. This led to the definition of S^s- and S^l-functions based on hybrid sign homomorphisms on Weyl groups. However, in comparison with the C- and S-functions, the S^s- and S^l-functions exist only in the case of simple Lie algebras with two different lengths of simple roots.

A review of several pertinent properties of the C-, S-, S^s- and S^l-functions can be found in [**14, 24, 26, 40**]. They have, for example, symmetries with respect to the affine Weyl group, which is an infinite extension of the Weyl group by translations. Therefore, we can consider C-, S-, S^s- and S^l-functions only on some subsets of the fundamental domain F of the affine Weyl group. Within each family, the functions are continuously orthogonal when integrated over F [**40, 43**]. This allows us to introduce continuous Fourier-like transforms involving C-, S-, S^s- or S^l-functions. Motivated by the processing of multidimensional digital data, the discrete orthogonality of the functions, developed in [**14, 16, 43**], is crucial. In particular, the continuous extension of discrete Fourier-like transforms, derived from the discrete orthogonality of C-, S-, S^s- and S^l-functions, interpolates digital data in any dimension and for any lattice symmetry afforded by the underlying simple Lie algebra. Several special cases connected to the simple Lie algebras of rank two are studied in [**49, 50, 51**]. It is still under study to know in which cases this transforms provide a more efficient interpolation than the multidimensional discrete Fourier transform.

In the case of Weyl group orbit functions, their discrete orthogonality can be used to derive several numerical integration formulas which approximate some weighted integral of any function by a weighted sum of a finite number of function values. In general, the formulas are required to be exact for all polynomial functions up to a certain degree [**5**]. In particular, the extensively studied Chebyshev polynomials are often used in mathematical analysis as efficient tools for numerical integration and approximations [**39, 52**]. Since the C-functions and S-functions of simple Lie algebra A_1 coincide, up to a constant, with the common cosine and sine functions respectively, they can be related to the Chebyshev polynomials and ,consequently, to the integration formulas, quadratures, for functions of one variable. In [**35**], it is shown that there are analogous formulas for numerical integration, for multivariate functions, that depend on the Weyl group of the simple Lie algebra A_n and the corresponding C- and S-functions. The resulting rules for functions of several variables are known as cubature formulas. The idea of [**35**] is extended to any simple Lie algebra in [**15, 40, 44**].

This work consists of Chapters 1–5 and Appendix A. Chapter 1 is intended to motivate our investigation of Weyl group orbit functions. It gives a brief review of the relations of the Weyl group orbit functions with other special functions which may be associated with the Weyl groups and which have applications in

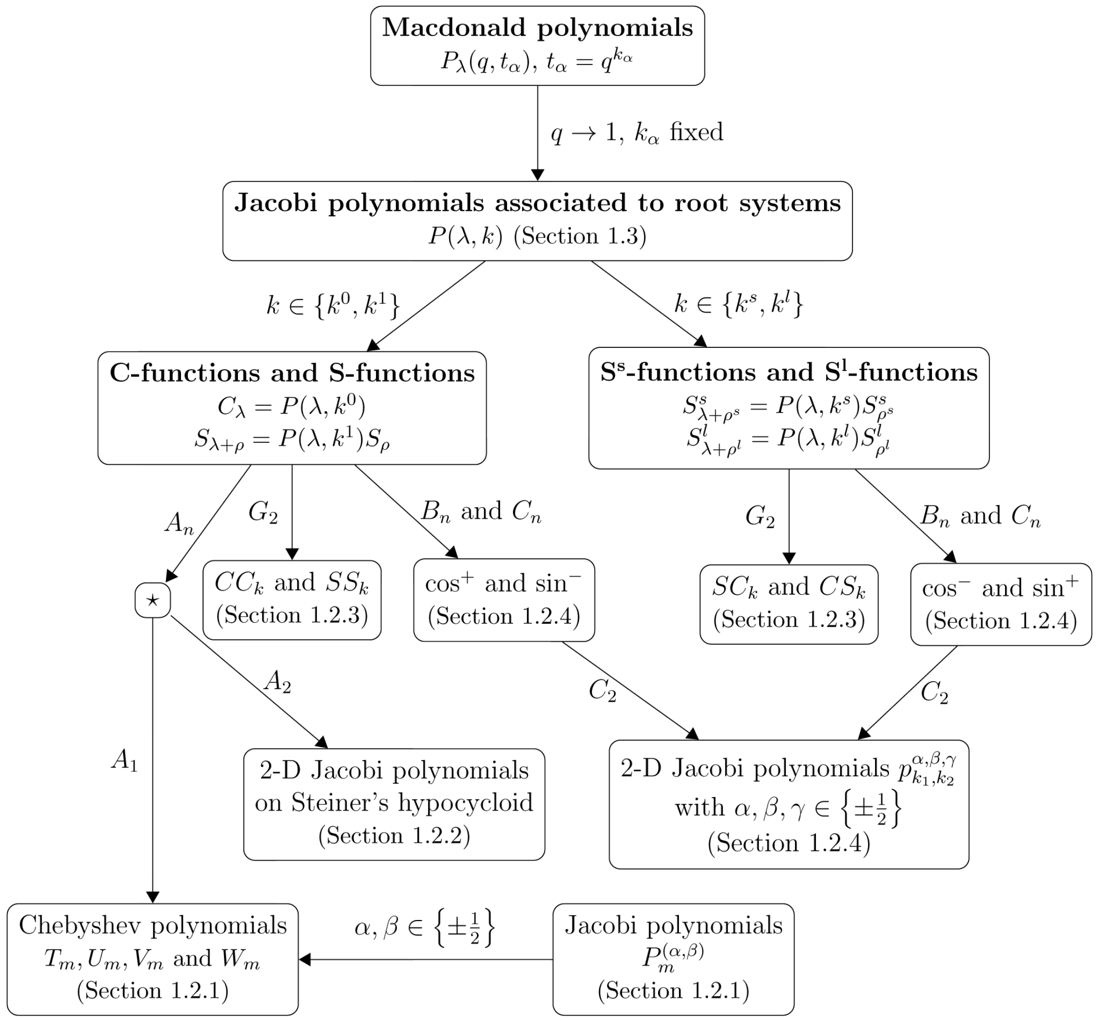

FIGURE 0.1. The brief sketch of relations among several functions associated with Weyl groups. More details can be found in the sections written in the boxes.

various scientific domains. For example, we have already mentioned the connection of the C- and S-functions of one variable with Chebyshev polynomials. In Chapter 1, we also show that each family of the Weyl group orbit functions corresponding to A_2 and C_2 can be viewed as a two-variable analogue of Jacobi polynomials [**30**]. Furthermore, we provide the exact connection with generalizations of trigonometric functions [**60**]. Thus, the Weyl group orbit functions may be considered as generalizations of named functions. On the other hand, it is shown that all four families are related to special cases of generalized Jacobi polynomials attached to root systems [**10**]. The Figure 0.1 presents a graphical realization of the dependence among different types of functions associated with Weyl groups.

Chapters 2–5 and Appendix A correspond to the articles [**12, 14, 15, 40**] and [**13**], each of them describing properties and applications of Weyl group orbit functions. In particular,

(1) *On discretization of tori of compact simple Lie groups II.* [**14**].
This article is devoted to the study of discrete transforms involving S^s- and S^l-functions. For any positive integer M, we consider multidimensional data sampled on finite lattice fragments F^s_M with variable density given by M. For each grid F^s_M, we determine a set of S^s-functions in such a way that it forms an orthogonal basis of the vector space of complex functions given on F^s_M with respect to the scalar product defined as a weighted sum on F^s_M. We show similar results connected to S^l-functions. Using the discrete orthogonality within each family of S^s- and S^l-functions, we perform discrete Fourier-like transforms in terms of those functions. We also provide formulas for the number of points in the finite grids.

(2) *Gaussian cubature arising from hybrid characters of simple Lie groups* [**40**].
In this article, we introduce applications of S^s- and S^l-functions in numerical analysis. These functions can be connected with two families of orthogonal polynomials. Instead of the standard grading of polynomials, we use the so-called m-degree of polynomials based on a set of Lie theoretical invariants. Together with continuous and discrete orthogonality of S^s- and S^l-functions, it allows us to deduce new cubature formulas. In particular, we have optimal Gaussian cubature formulas arising from S^s-functions of any simple Lie algebra with two different lengths of roots and slightly less efficient Radau cubature formulas connected to S^l-functions.

(3) *Cubature formulas of multivariate polynomials arising from symmetric orbit functions* [**15**].
In this article, we extend the results of [**40, 44**] to the family of C-functions. We obtain new cubature formulas arising in connection with any simple Lie algebra. We provide a detailed description of cubatures for simple Lie algebras of rank 2, e.g. we present explicit formulas for weight functions in terms of polynomial variables. We also indicate a few applications of cubatures such as polynomial approximations of any function. Considering simple Lie algebra C_2, we give an example of the approximation of a specific model function in terms of orthogonal polynomials related to C-functions.

(4) *Discrete transforms and orthogonal polynomials of (anti)symmetric multivariate cosine functions* (working title: *(Anti)symmetric discrete cosine*

transforms on fundamental domain of extended affine symmetric group and Chebyshev-like multivariate polynomials) [**12**].

This article consists of two main parts, both concerned with antisymmetric and symmetric cosine functions, i.e. with special cases of Weyl group orbit functions. In the first part, we establish (anti)symmetric discrete cosine transforms of type V–VIII using discrete cosine transforms of type V–VIII for functions of one variable. Inspired by the Chebyshev polynomials of the first and third kind, the second part introduces four families of multivariate orthogonal polynomials via (anti)symmetric cosine functions. We describe their properties such as continuous orthogonality with respect to a weighted integral. We show that there exist optimal Gaussian cubature formulas arising from each family of polynomials.

(5) *Two-dimensional symmetric and antisymmetric generalizations of sine functions* [**13**].

This article can be consider as a continuation of [**18**]. It describes properties and applications of two-dimensional symmetric and antisymmetric sine functions. We discuss eight types of multivariate discrete sine transforms derived from (anti)symmetric exponential transforms. We also show examples of two-dimensional sine interpolations.

Although this article was written and published before I started my doctoral studies at Université de Montréal, I decided to include it in my thesis in Appendix A because the study of two-dimensional (anti)symmetric sine functions led to the definition of S^s- and S^l-functions which are of main importance in my thesis.

My contribution to all five articles was essentially the same. I actively participated in solving the problems which are discussed in the articles, e.g. I proved most of the propositions and theorems contained in the articles. I did all needed calculations and I also effectively contributed by writing up the text and providing the figures.

Chapter 1

SPECIAL FUNCTIONS OF WEYL GROUPS

1.1. Weyl groups of simple Lie algebras

We wish to investigate special functions related to Weyl groups arising from simple Lie algebras. There are four series of simple Lie algebras $A_n (n \geq 1)$, $B_n (n \geq 3)$, $C_n (n \geq 2)$, $D_n (n \geq 4)$ and five exceptional simple Lie algebras E_6, E_7, E_8, F_4 and G_2, each connected with a Weyl group [**2, 19, 20, 29, 60**]. They are completely classified by Dynkin diagrams (see Figure 1.1). A Dynkin diagram

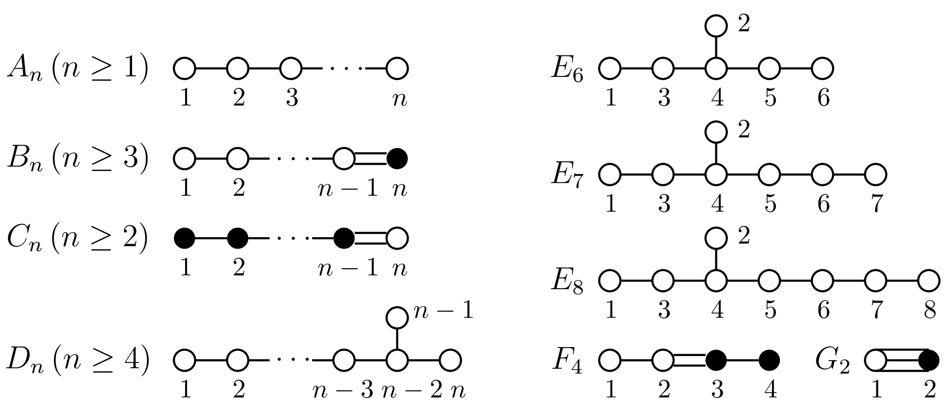

Figure 1.1. Dynkin diagrams of simple Lie algebras.

characterizes a set Δ of simple roots $\alpha_1, \dots, \alpha_n$ generating an Euclidean space isomorphic to $\mathbb{R}^n$ with the scalar product denoted by $\langle \cdot, \cdot \rangle$. Each node of the Dynkin diagram represents one simple root α_i. The number of links between two nodes corresponding to α_i and α_j respectively is equal to

$$\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle \,, \quad \text{where } \alpha_i^\vee \equiv \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \,.$$

The absence of direct link between two nodes indicates that the corresponding simple roots are orthogonal. One direct link means that the angle between the corresponding simple roots is $2\pi/3$, two and three links stay for the angle $3\pi/4$

and $5\pi/6$ respectively. The nodes are either of the same white color, i.e. all simple roots have the same length, or some of the nodes are black, i.e. there are two different lengths of simple roots. In the latter case, the black nodes denote short roots and the white nodes long roots. Note that we use the standard normalization for the lengths of roots, namely $\langle \alpha_i, \alpha_i \rangle = 2$ if α_i is a long simple root.

In addition to the basis of $\mathbb{R}^n$ consisting of the simple roots α_i, it is convenient for our purposes to introduce the basis of fundamental weights ω_j given by $\langle \omega_j, \alpha_i^\vee \rangle = \delta_{ij}$. It allows us to express the weight lattice P defined by

$$P \equiv \{\lambda \in \mathbb{R}^n \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i = 1, \dots, n\}$$

as $\mathbb{Z}$-linear combinations of ω_j. We consider the usual partial ordering on P given by $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is a sum of simple roots or $\lambda = \mu$.

To each simple root α_i corresponds a reflection r_i with respect to the hyperplane orthogonal to α_i,

$$r_i(a) \equiv r_{\alpha_i}(a) = a - \frac{2\langle a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \,, \qquad \text{for } a \in \mathbb{R}^n \,.$$

The finite group W generated by such reflections $r_i, i = 1, \dots, n$ is called the Weyl group. The properties of Weyl groups are fully described, for example, in [**20, 22**].

An action of the elements of W on simple roots leads to a finite set Π of vectors in $\mathbb{R}^n$ called a root system. In the case of simple Lie algebras with two different lengths of roots, Π can be written as the union of the set Π_s containing only short roots and Π_l containing only long roots. A function $k : \alpha \in \Pi \to k_\alpha \in \mathbb{R}^{\geq 0}$ such that

$$k_\alpha = k_{w(\alpha)} \text{ for all } w \in W$$

is known as a multiplicity function on Π. The trivial example is to take $k_\alpha = 1$ for all $\alpha \in \Pi$ which we denote by k^1. For simple Lie algebras with two different root lengths, it is natural to distinguish between short and long roots by defining

$$k^s_\alpha \equiv \begin{cases} 1 & \text{if } \alpha \in \Pi^s \\ 0 & \text{if } \alpha \in \Pi^l \end{cases} \,, \qquad k^l_\alpha \equiv \begin{cases} 0 & \text{if } \alpha \in \Pi^s \\ 1 & \text{if } \alpha \in \Pi^l \end{cases} \,. \tag{1.1.1}$$

Using the multiplicity function k, we also define

$$\varrho(k) \equiv \frac{1}{2} \sum_{\alpha \in \Pi_+} k_\alpha \alpha \,, \tag{1.1.2}$$

where Π_+ denotes the roots α of Π which satisfy $0 \preceq \alpha$. Using the fundamental weights, we obtain

$$\varrho \equiv \varrho(k^1) = \sum_{i=1}^{n} \omega_i \,, \quad \varrho^s \equiv \varrho(k^s) = \sum_{\alpha_i \in \Delta_s} \omega_i \,, \quad \varrho^l \equiv \varrho(k^l) = \sum_{\alpha_i \in \Delta_l} \omega_i \,, \tag{1.1.3}$$

where $\Delta_s = \Delta \cap \Pi_s$ and $\Delta_l = \Delta \cap \Pi_l$.

1.2. Weyl group orbit functions

Weyl group orbit functions arise from "sign" homomorphisms on Weyl groups, $\sigma : W \to \{\pm 1\}$. There exist only two different homomorphisms on W connected to simple Lie algebras with one length of roots, the well-known identity denoted by $\imath$ and the determinant denoted by det. They are given by their values on the generators r_i of W:

$$\begin{aligned} \imath : \quad & \imath(r_i) = 1 && \text{for all } \alpha_i \in \Delta \,, \\ \det : \quad & \det(r_i) = -1 && \text{for all } \alpha_i \in \Delta \,. \end{aligned}$$

In the case of simple Lie algebras with two different lengths of roots, i.e. B_n, C_n, F_4 and G_2, there are two additional available choices of homomorphisms varying for short and long roots, σ^s and σ^l, defined by

$$\begin{aligned} \sigma^s : \quad & \sigma^s(r_i) = -1 \quad \text{if } \alpha_i \in \Delta_s \,, \quad \sigma^s(r_i) = 1 \quad \text{if } \alpha_i \in \Delta_l \,, \\ \sigma^l : \quad & \sigma^l(r_i) = 1 \quad \text{if } \alpha_i \in \Delta_s \,, \quad \sigma^l(r_i) = -1 \quad \text{if } \alpha_i \in \Delta_l \,. \end{aligned}$$

Weyl group orbit functions are introduced using the following explicit formula.

$$\varphi_a^\sigma(b) = \sum_{w \in W} \sigma(w) e^{2\pi i \langle w(a), b \rangle} \,, \quad a, b \in \mathbb{R}^n \,.$$

Each homomorphism $\imath, \det, \sigma^s, \sigma^l$ induces one family of complex valued Weyl group orbit functions, called C, S, S^s and S^l respectively, labelled by the parameter $a \in \mathbb{R}^n$ and of the variable $b \in \mathbb{R}^n$,

$$\begin{aligned} & C\text{-functions:} && \sigma \equiv \imath \,, && \varphi^\sigma \equiv \Phi \,, \\ & S\text{-functions:} && \sigma \equiv \det \,, && \varphi^\sigma \equiv \varphi \,, \\ & S^s\text{-functions:} && \sigma \equiv \sigma^s \,, && \varphi^\sigma \equiv \varphi^s \,, \\ & S^l\text{-functions:} && \sigma \equiv \sigma^l \,, && \varphi^\sigma \equiv \varphi^l \,. \end{aligned}$$

To obtain the remarkable properties of the Weyl group orbit functions such as continuous and discrete orthogonality, we mostly restrict a to some subset of the weight lattice P (see for example [**14, 16, 24, 26, 43**]). Recall that the symmetric C-functions and antisymmetric S-functions are well-known from the theory of irreducible representations of simple Lie algebras [**54, 60**].

Sometimes, it is convenient to use an alternative definition of the Weyl group orbit functions via sums over the Weyl group orbits. If we consider

$$a \in P^+ \equiv \mathbb{Z}^{\geq 0}\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\omega_n\,,$$

then the following functions are well defined [**24, 26, 40**].

$$C_a(b) \equiv \sum_{\tilde{a}\in O(a)} e^{2\pi i\langle \tilde{a},b\rangle}\,, \qquad S_{a+\varrho}(b) \equiv \sum_{\tilde{a}\in O(a+\varrho)} \det(\tilde{a}) e^{2\pi i\langle \tilde{a},b\rangle}\,,$$

$$S^s_{a+\varrho^s}(b) \equiv \sum_{\tilde{a}\in O(a+\varrho^s)} \sigma^s(\tilde{a}) e^{2\pi i\langle \tilde{a},b\rangle}\,, \qquad S^l_{a+\varrho^l}(b) \equiv \sum_{\tilde{a}\in O(a+\varrho^l)} \sigma^l(\tilde{a}) e^{2\pi i\langle \tilde{a},b\rangle}\,,$$

where

- ϱ, ϱ^s, ϱ^l are determined by (1.1.3),
- $O(c)$ denotes the Weyl group orbit of c, i.e. $O(c) \equiv \{wc \mid w \in W\}$,
- $\det(\tilde{a}) \equiv \det(w)$ for any $w \in W$ such that $\tilde{a} = w(a+\varrho)$, $\sigma^s(\tilde{a})$ and $\sigma^l(\tilde{a})$ are defined similarly.

The functions $\Phi_a, \varphi_{a+\varrho}, \varphi^s_{a+\varrho^s}, \varphi^l_{a+\varrho^l}$ and $C_a, S_{a+\varrho}, S^s_{a+\varrho^s}, S^l_{a+\varrho^l}$ differ only by a constant,

$$C_a(b) = \frac{\Phi_a(b)}{h_a}\,, \quad S_{a+\varrho}(b) = \varphi_{a+\varrho}(b)\,, \quad S^s_{a+\varrho^s}(b) = \frac{\varphi^s_{a+\varrho^s}(b)}{h_{a+\varrho^s}}\,, \quad S^l_{a+\varrho^l}(b) = \frac{\varphi^l_{a+\varrho^l}(b)}{h_{a+\varrho^l}}$$

with h_c denoting the number of elements of W leaving c invariant. Therefore, the functions $C_a, S_{a+\varrho}, S^s_{a+\varrho^s}, S^l_{a+\varrho^l}$ are also called C-, S-, S^s- and S^l-functions. In what follows, it will always be specified what kind of functions are considered if necessary.

Furthermore, the C-, S-, S^s- and S^l-functions can be viewed as functional forms of elements from the algebra $\mathbb{C}[P]$ containing all complex linear combinations of formal exponentials $e^a, a \in P$, with multiplication defined by $e^a \cdot e^{\tilde{a}} = e^{a+\tilde{a}}$, the inverse given by $(e^a)^{-1} = e^{-a}$ and the identity $e^0 = 1$. The connection is based on the exponential mapping from Lie algebra to the corresponding Lie group [**2, 19, 44**].

1.2.1. The case A_1

The symmetric C-functions and antisymmetric S-functions of A_1 are, up to a constant, the common cosine and sine functions [**24, 26**],

$$C_a(b) = 2\cos(2\pi a_1 b_1)\,, \quad S_a(b) = 2i\sin(2\pi a_1 b_1)\,, \quad \text{where } a = a_1\omega_1\,, b = b_1\alpha_1^\vee\,.$$

It is well known that such functions appear in the definition of the extensively studied Chebyshev polynomials [**39, 52**]. There are several types of Chebyshev polynomials often used in mathematical analysis, in particular, as efficient tools for numerical integration and approximations. We introduce the Chebyshev polynomials of the first, second, third and fourth kind denoted respectively by $T_m(x)$, $U_m(x)$, $V_m(x)$ and $W_m(x)$. If $x = \cos(\theta)$, then we define, for any $m \in \mathbb{Z}^{\geq 0}$,

$$\begin{aligned} T_m(x) &\equiv \cos(m\theta)\,, & U_m(x) &\equiv \frac{\sin\left((m+1)\theta\right)}{\sin(\theta)}\,, \\ V_m(x) &\equiv \frac{\cos\left(\left(m+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}\,, & W_m(x) &\equiv \frac{\sin\left(\left(m+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}\,. \end{aligned}$$

Therefore, for specific choices of the parameter a_1 and $2\pi b_1 = \theta$, we can view the Weyl group orbit functions of A_1 as these Chebyshev polynomials.

Recall also that the Chebyshev polynomials are actually, up to a constant $c_{\alpha,\beta}$, special cases of Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$, $m \in \mathbb{Z}^{\geq 0}$, given as orthogonal polynomials with respect to the weight function

$$(1-x)^{\alpha}(1+x)^{\beta}\,, \quad -1 < x < 1\,,$$

where the parameters α, β are subjects to the condition $\alpha, \beta > -1$ [**7, 58**]. In particular, we have

$$\begin{aligned} T_m(x) &= c_{-\frac{1}{2},-\frac{1}{2}} P_m^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)\,, & U_m(x) &= c_{\frac{1}{2},\frac{1}{2}} P_m^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)\,, \\ V_m(x) &= c_{-\frac{1}{2},\frac{1}{2}} P_m^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x)\,, & W_m(x) &= c_{\frac{1}{2},-\frac{1}{2}} P_m^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x)\,. \end{aligned}$$

For both, the Chebyshev and Jacobi polynomials, we can find various multivariate generalizations, for example in [**5, 30, 34**]. In Sections 1.2.2, 1.2.3 and 1.2.4, we identify some of the two-variable analogous orthogonal polynomials with specific Weyl group orbit functions.

1.2.2. The case A_2

For A_2 has two simple roots of the same length, there are only two families of Weyl group orbit functions, C- and S-functions, arising from A_2. The explicit formulas of C-functions C_a and S-functions S_a are given in Section 4.3.1. We also show that C_a and S_a are related to the functions TC_k and TS_k defined in [**33**]. In particular, we have

$$C_a = \frac{6}{h_a} TC_k\,, \quad S_a = 6TS_k\,.$$

The discussion in Chapter 4 and in [**33**] guarantees that C_a and $S_{a+\varrho}/S_\varrho$ with $a \in P^+$ can be expressed as polynomials in C_{ω_1} and C_{ω_2}. Taking into account that $C_{\omega_1} = \overline{C_{\omega_2}}$, we can pass to real variables by making a natural change of variables,

$$X_1 = \frac{C_{\omega_1} + C_{\omega_2}}{2}\,, \quad X_2 = \frac{C_{\omega_1} - C_{\omega_2}}{2i}\,.$$

Since the C- and S-functions are continuously orthogonal, their polynomial versions inherit the orthogonality property. It can be proved that the corresponding polynomials are special cases of two-dimensional analogues of Jacobi polynomials orthogonal with respect to the weight function

$$w_\alpha(x,y) = [-(x^2+y^2+9)^2 + 8(x^3-3xy^2) + 108]^\alpha$$

on the region bounded by three-cusped deltoid called Steiner's hypocycloid [**30**]. More precisely, the polynomials C_a and S_a correspond to the choices $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$ respectively.

1.2.3. The case G_2

In the case G_2 with two simple roots of different lengths, we have all four families of Weyl group orbit functions, C_a, S_a, S^s_a and S^l_a [**24, 26, 40**]. They have been studied, for example, in [**34**], under the notation CC_k, SS_k, SC_k and CS_k. Indeed, performing the change of variables and parameters introduced in Section 4.3.3, we obtain

$$C_a = \frac{12}{h_a} CC_k\,, \quad S_a = -12 SS_k\,, \quad S^s_a = \frac{12i}{h_a} SC_k\,, \quad S^l_a = \frac{12i}{h_a} CS_k\,.$$

In [**34**], the above-mentioned functions are expressed as two-variables polynomials considered as another analogues of the Chebyshev polynomials.

1.2.4. The case B_n and C_n

We show that C-, S-, S^s- and S^l-functions arising from B_n and C_n are related to the symmetric and antisymmetric multivariate generalizations of trigonometric functions [**25**]. The symmetric cosine functions $\cos^+_\lambda(x)$ and antisymmetric cosine functions $\cos^-_\lambda(x)$ of the variable $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and labelled by the parameter $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$ are given by the following explicit formulas,

$$\cos^+_\lambda(x) \equiv \sum_{\sigma\in S_n} \prod_{k=1}^n \cos(\pi\lambda_{\sigma(k)}x_k)\,, \quad \cos^-_\lambda(x) \equiv \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^n \cos(\pi\lambda_{\sigma(k)}x_k)\,,$$

where S_n denotes the symmetric group consisting of all the permutations of numbers $1,\dots,n$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. Similarly, we define the symmetric sine functions $\sin^+_\lambda(x)$ and antisymmetric sine functions $\sin^-_\lambda(x)$,

$$\sin^+_\lambda(x) \equiv \sum_{\sigma\in S_n} \prod_{k=1}^{n} \sin(\pi\lambda_{\sigma(k)}x_k)\,, \quad \sin^-_\lambda(x) \equiv \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} \sin(\pi\lambda_{\sigma(k)}x_k)\,.$$

We start with the Lie algebra B_n and consider an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ such that

$$\alpha_i = e_i - e_{i+1} \;\text{ for } i=1,\dots,n-1 \qquad \text{and} \qquad \alpha_n = e_n\,.$$

If we determine any $a\in\mathbb{R}^n$ by its coordinates with respect to the basis $\{e_1,\dots,e_n\}$, $a=(a_1,\dots,a_n)=a_1e_1+\dots+a_ne_n$, then it holds for the generators r_i of the Weyl group $W(B_n)$ of B_n that

$$\begin{aligned} &r_i(a_1,\dots,a_i,a_{i+1},\dots,a_n) = (a_1,\dots,a_{i+1},a_i,\dots,a_n) \quad \text{ for } i=1,\dots,n-1\,,\\ &r_n(a_1,\dots,a_{n-1},a_n) = (a_1,\dots,a_{n-1},-a_n)\,. \end{aligned}$$

Therefore, $W(B_n)$ consists of all the permutations of the coordinates a_i with possible sign alternations of some of them, we have indeed that $W(B_n)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ [**20**]. This implies that

$$\begin{aligned} \Phi_a(b) &= \sum_{w\in W(B_n)} e^{2\pi i\langle w(a),b\rangle} = \sum_{\sigma\in S_n}\prod_{k=1}^{n}\sum_{l_k=\pm1} e^{2\pi i\left(l_k a_{\sigma(k)} b_k\right)}\\ &= \sum_{\sigma\in S_n}\prod_{k=1}^{n}\left(e^{2\pi i a_{\sigma(k)}b_k} + e^{-2\pi i a_{\sigma(k)}b_k}\right) = 2^n \sum_{\sigma\in S_n}\prod_{k=1}^{n}\cos(2\pi a_{\sigma(k)}b_k)\\ &= 2^n \cos^+_{2a}(b)\,. \end{aligned}$$

Since det is a homomorphism on $W(B_n)$, we obtain

$$\begin{aligned} \varphi_a(b) &= \sum_{w\in W(B_n)} \det(w) e^{2\pi i\langle w(a),b\rangle} = \sum_{\sigma\in S_n}\det(\sigma)\prod_{k=1}^{n}\sum_{l_k=\pm1} l_k e^{2\pi i\left(l_k a_{\sigma(k)} b_k\right)}\\ &= \sum_{\sigma\in S_n}\det(\sigma)\prod_{k=1}^{n}\left(e^{2\pi i a_{\sigma(k)}b_k} - e^{-2\pi i a_{\sigma(k)}b_k}\right)\\ &= (2i)^n \sum_{\sigma\in S_n}\det(\sigma)\prod_{k=1}^{n}\sin(2\pi a_{\sigma(k)}b_k) = (2i)^n \sin^-_{2a}(b)\,. \end{aligned}$$

A similar conclusion can be drawn for S^s- and S^l-functions,

$$\varphi^s_a(b) = (2i)^n \sin^+_{2a}(b)\,, \quad \varphi^l_a(b) = 2^n \cos^-_{2a}(b)\,.$$

Since the Lie algebras B_n and C_n are dual to each other, we can deduce that the symmetric and antisymmetric generalizations are connected to the Weyl group orbit functions of C_n as well. However, there are a few differences. To obtain the exact relations, we can proceed analogously to the case B_n and introduce an orthogonal basis $\{f_1, \ldots, f_n\}$ such that

$$\langle f_i, f_i \rangle = \frac{1}{2}\,, \quad \alpha_i = f_i - f_{i+1} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \quad \alpha_n = 2f_n\,.$$

We denote by $\tilde{a}_i$ the coordinates of any point $a \in \mathbb{R}^n$ with respect to the basis $\{f_1, \ldots, f_n\}$, i.e. $a = (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{a}_1 f_1 + \cdots + \tilde{a}_n f_n$. The generators r_i of the Weyl group $W(C_n)$ corresponding to C_n are also given by

$$r_i(\tilde{a}_1, \ldots, \tilde{a}_i, \tilde{a}_{i+1}, \ldots, \tilde{a}_n) = (\tilde{a}_1, \ldots, \tilde{a}_{i+1}, \tilde{a}_i, \ldots, \tilde{a}_n) \quad \text{for } i = 1, \ldots, n-1\,,$$
$$r_n(\tilde{a}_1, \ldots, \tilde{a}_{n-1}, \tilde{a}_n) = (\tilde{a}_1, \ldots, \tilde{a}_{n-1}, -\tilde{a}_n)\,.$$

Thus, proceeding as before, we derive the following.

$$\Phi_a(b) = 2^n \cos_a^+(b)\,, \quad \varphi_a(b) = (2i)^n \sin_a^-(b)\,,$$
$$\varphi_a^s(b) = 2^n \cos_a^-(b)\,, \quad \varphi_a^l(b) = (2i)^n \sin_a^+(b)\,.$$

Note that the S^s-functions are related to $\cos_a^-$ and the S^l-functions to $\sin_a^+$ in the case C_n, whereas the S^s-functions correspond to $\sin_{2a}^+$ and the S^l-functions to $\cos_{2a}^-$ if we consider simple Lie algebra B_n. This follows from the fact that the short (long) roots of C_n are dual to the long (short) roots of B_n.

The symmetric and antisymmetric cosine functions can be used to construct multivariate orthogonal polynomials analogous to the Chebyshev polynomials of the first and third kind. The method of construction is based on the decomposition of products of these functions and is fully described in Section 5.3.1. To build the polynomials analogous to the Chebyshev polynomials of the second and fourth kind, it seems that the symmetric and antisymmetric generalizations of sine functions have to be analysed. The theory is supported by the decomposition of products of two-dimensional sine functions which can be found in Section A.2.2.

Setting $n = 2$, the construction of the polynomials

$$\mathcal{P}^{I,+}_{(k_1,k_2)} \equiv \cos^+_{(k_1,k_2)}\,, \qquad \mathcal{P}^{I,-}_{(k_1,k_2)} \equiv \frac{\cos^-_{(k_1+1,k_2)}}{\cos^-_{(1,0)}}\,,$$
$$\mathcal{P}^{III,+}_{(k_1,k_2)} \equiv \frac{\cos^+_{(k_1+\frac{1}{2},k_2+\frac{1}{2})}}{\cos^+_{(\frac{1}{2},\frac{1}{2})}}\,, \quad \mathcal{P}^{III,-}_{(k_1,k_2)} \equiv \frac{\cos^-_{(k_1+\frac{3}{2},k_2+\frac{1}{2})}}{\cos^-_{(\frac{3}{2},\frac{1}{2})}}$$

labelled by $k_1 \geq k_2 \geq 0$ and in the variables

$$X_1 \equiv \cos^+_{(1,0)}(x_1, x_2) = \cos(x_1) + \cos(x_2)\,,$$
$$X_2 \equiv \cos^+_{(1,1)}(x_1, x_2) = 2\cos(x_1)\cos(x_2)$$

yields special cases of two-variable polynomials built in [**30, 31, 32**] by orthogonalization of monomials $1, u, v, u^2, uv, v^2, \ldots$ of generic variables u, v with respect to the weight function $(1-u+v)^\alpha (1+u+v)^\beta (u^2-4v)^\gamma$ in the domain bounded by the curves $1-u+v=0$, $1+u+v=0$ and $u^2-4v=0$. The parameters α,β,γ are required to satisfy the conditions $\alpha,\beta,\gamma>-1$, $\alpha+\gamma+\frac{3}{2}>0$ and $\beta+\gamma+\frac{3}{2}>0$. Resulting polynomials with the highest term $u^{m-k}v^k$ are denoted by $p^{\alpha,\beta,\gamma}_{m,k}(u,v)$, where $m\geq k\geq 0$. Indeed, the polynomial variables X_1 and X_2 are related to the variables u and v of [**30, 31, 32**] by

$$X_1=u,\qquad X_2=2v$$

and it can be easily shown that

- $\mathcal{P}^{I,+}_{(k_1,k_2)}$ coincides, up to a constant, with $p^{\alpha,\beta,\gamma}_{k_1,k_2}(u,v)$ for $\alpha=\beta=\gamma=-\frac{1}{2}$,
- $\mathcal{P}^{III,+}_{(k_1,k_2)}$ coincides, up to a constant, with $p^{\alpha,\beta,\gamma}_{k_1,k_2}(u,v)$ for $\alpha=\gamma=-\frac{1}{2}$ and $\beta=\frac{1}{2}$,
- $\mathcal{P}^{I,-}_{(k_1,k_2)}$ coincides, up to a constant, with $p^{\alpha,\beta,\gamma}_{k_1,k_2}(u,v)$ for $\alpha=\beta=-\frac{1}{2}$ and $\gamma=\frac{1}{2}$,
- $\mathcal{P}^{III,-}_{(k_1,k_2)}$ coincides, up to a constant, with $p^{\alpha,\beta,\gamma}_{k_1,k_2}(u,v)$ for $\alpha=-\frac{1}{2}$ and $\beta=\gamma=\frac{1}{2}$.

1.3. Jacobi polynomials associated to root systems

We assume that the multiplicity function k satisfies $k_\alpha\geq 0$. The Jacobi polynomial $P(\lambda,k)$ [**10, 11**] associated to the root system Π with highest weight λ and parameter k is defined by

$$P(\lambda,k)\equiv\sum_{\left\{\substack{\mu\in P^+\\ \mu\preceq\lambda}\right\}}c_{\lambda\mu}(k)C_\mu\,,\quad C_\mu=\sum_{\tilde{\mu}\in O(\mu)}e^{\tilde{\mu}}\,,\tag{1.3.1}$$

where the coefficients $c_{\lambda\mu}(k)$ are recursively given by the formula

$$\begin{gathered}(\langle\lambda+\varrho(k),\lambda+\varrho(k)\rangle-\langle\mu+\varrho(k),\mu+\varrho(k)\rangle)c_{\lambda\mu}(k)\\ =2\sum_{\alpha\in\Pi_+}k_\alpha\sum_{j=1}^{\infty}\langle\mu+j\alpha,\alpha\rangle c_{\lambda,\mu+j\alpha}(k)\end{gathered}$$

along with the initial value $c_{\lambda\lambda}=1$ and the assumption $c_{\lambda\mu}=c_{\lambda,w(\mu)}$ for all $w\in W$. Recall that $\varrho(k)$ denotes (1.1.2).

By setting $k_\alpha=0$ for all $\alpha\in\Pi$, the Jacobi polynomials leads to C-functions. We denote such k by k^0. In the case $k=k^1$, the formula for the calculation of coefficients becomes the Freudenthal's recurrence formula [**19**]. Therefore, each $P(\lambda,k^1)$ specializes to the character χ_λ of irreducible representation of simple Lie algebra of the highest weight λ, i.e.

$$P(\lambda,k^1)=\chi_\lambda=\frac{S_{\lambda+\varrho}}{S_\varrho}\,.$$

In addition, we show the following relations of Jacobi polynomials with S^s- and S^l-functions.

$$P(\lambda,k^s)=\frac{S^s_{\lambda+\varrho^s}}{S^s_{\varrho^s}}\quad\text{and}\quad P(\lambda,k^l)=\frac{S^l_{\lambda+\varrho^l}}{S^l_{\varrho^l}}\,.$$

We first observe that $S^s_{\lambda+\varrho^s}/S^s_{\varrho^s}$ are Weyl group invariant elements of $\mathbb{C}[P]$ (see Proposition 3.3.2). It is well known that the C-functions form a basis of the invariant elements of $\mathbb{C}[P]$ [**2**]. Therefore, each $S^s_{\lambda+\varrho^s}/S^s_{\varrho^s}$ can be expressed as a linear combination of C-functions. Moreover, since the unique maximal weight of $S^s_{\lambda+\varrho^s}$ is $\lambda+\varrho^s$ and the unique maximal weight of $S^s_{\varrho^s}$ is ϱ^s, we have

$$\frac{S^s_{\lambda+\varrho^s}}{S^s_{\varrho^s}}=\sum_{\left\{\substack{\mu\in P^+\\ \mu\preceq\lambda}\right\}} b_\mu C_\mu\,,\quad b_\lambda=1\,.$$

We proceed by using an equivalent definition of Jacobi polynomials with the multiplicity function satisfying $k_\alpha\in\mathbb{Z}^{\geq 0}$ [**11**] to prove $b_\mu=c_{\lambda\mu}(k^s)$. For any $f=\sum_\lambda a_\lambda e^\lambda$, we define

$$\overline{f}=\sum_\lambda \overline{a}_\lambda e^{-\lambda}\quad\text{and}\quad CT(f)=a_0\,.$$

If we introduce the scalar product $(\cdot,\cdot)$ on $\mathbb{C}[P]$ by

$$(f,g)\equiv CT(f\overline{g}\delta(k)^{\frac12}\overline{\delta(k)}^{\frac12})\,,\quad f,g\in\mathbb{C}[P]\,,\quad \delta(k)^{\frac12}\equiv\prod_{\alpha\in\Pi_+}\left(e^{\frac12\alpha}-e^{-\frac12\alpha}\right)^{k_\alpha}\,,$$

then the Jacobi polynomials $P(\lambda,k)$ (1.3.1) are the unique polynomials satisfying the requirement

$$(P(\lambda,k),P(\mu,k))=0\qquad\text{for all }\mu\in P^+\text{ such that }\mu\preceq\lambda\text{ and }\lambda\neq\mu$$

assuming $c_{\lambda\lambda}=1$.

Using Proposition 3.3.1, i.e. $\delta(k^s)^{\frac12}=S^s_{\varrho^s}$, we obtain

$$\left(\frac{S^s_{\lambda+\varrho^s}}{S^s_{\varrho^s}},\frac{S^s_{\mu+\varrho^s}}{S^s_{\varrho^s}}\right)=CT(S^s_{\lambda+\varrho^s}\overline{S^s_{\mu+\varrho^s}})=CT\left(\sum_{\tilde\lambda\in O(\lambda+\varrho^s)}\sum_{\tilde\mu\in O(\mu+\varrho^s)}\sigma^s(\tilde\lambda)\sigma^s(\tilde\mu)e^{\tilde\lambda-\tilde\mu}\right)\,.$$

Clearly $\tilde\lambda=\tilde\mu$ if and only if there exists $w\in W$ such that $\lambda+\varrho^s=w(\mu+\varrho^s)$. For we consider $\lambda\in P^+$ different from $\mu\in P^+$, it is not possible to have $\tilde\lambda=\tilde\mu$. This implies that

$$\left(\frac{S^s_{\lambda+\varrho^s}}{S^s_{\varrho^s}},\frac{S^s_{\mu+\varrho^s}}{S^s_{\varrho^s}}\right)=0\quad\text{and}\quad P(\lambda,k^s)=\frac{S^s_{\lambda+\varrho^s}}{S^s_{\varrho^s}}\,.$$

The proof of the relation for the long root case is similar.

Finally, note that the Jacobi polynomials can be viewed as the limiting case of the Macdonald polynomials $P_\lambda(q, t_\alpha)$ when $t_\alpha = q^{k_\alpha}$ with k_α fixed and $q \to 1$. See [**36, 37**] for more details.

Chapter 2

ON DISCRETIZATION OF TORI OF COMPACT SIMPLE LIE GROUPS II

Authors: Jiří Hrivnák, Lenka Motlochová and Jiří Patera.

Abstract: The discrete orthogonality of special function families, called C- and S-functions, which are derived from the characters of compact simple Lie groups, is described in [**16**]. Here, the results of [**16**] are extended to two additional recently discovered families of special functions, called S^s- and S^l-functions. The main result is an explicit description of their pairwise discrete orthogonality within each family, when the functions are sampled on finite fragments F^s_M and F^l_M of a lattice in any dimension $n \geq 2$ and of any density controlled by M, and of the symmetry of the weight lattice of any compact simple Lie group with two different lengths of roots.

Introduction

This paper focuses on the Fourier transform of data sampled on lattices of any dimension and any symmetry [**43, 47**]. The main problem is to find families of expansion functions that are complete in their space and orthogonal over finite fragments of the lattices. Generality of results is possible because the expansion function is built using properties that are uniformly valid over the series of semisimple Lie groups. Results of [**16**] on the discrete orthogonality of C- and S-functions of compact simple Lie groups are extended to the recently discovered families of S^s- and S^l-functions. The new families of functions add new possibilities of transforms for the same data.

Uniform discretization of tori of all semisimple Lie groups became possible after the classification of conjugacy classes of elements of finite order in compact simple Lie groups [**21**]. This was accomplished in [**41, 42**] for C-functions and extended to S-functions in [**43**]. Functions of C- and S-families are ingredients of irreducible characters of representations. They are uniformly defined for all semisimple Lie groups. Discretization here refers to their orthogonality when sampled on a fraction of a lattice F_M in the fundamental region F of the corresponding Lie group and summed up over all lattice points in F_M. This lattice is necessarily isomorphic to the weight lattice of the underlying Lie group, but its density is controlled by the choice of $M \in \mathbb{N}$.

C-functions are Weyl group invariant constituents of characters of irreducible representations. They are well known, even if infrequently used [**24**]. S-functions appear in the Weyl character formula. They are skew-invariant with respect to the Weyl group [**26**]. In the new families of S^s- and S^l-functions, the Weyl group acts differently when reflections are with respect to hyperplanes orthogonal to short and long roots of the Lie group. The functions are "half invariant and half skew-invariant" under the action of the Weyl group.

The key point of the discretization of S^s- and S^l-functions lies in finding appropriate subsets $F^s_M \subset F_M$ and $F^l_M \subset F_M$, which play the role of sampling points of a given data. The solution involves determining the sets of weights Λ^s_M and Λ^l_M, which label the discretely orthogonal S^s- and S^l-functions over the sets F^s_M and F^l_M. In order to verify the completeness of the found sets of functions, the last step involves comparing the number of points in F^s_M, F^l_M to the number of weights in Λ^s_M, Λ^l_M.

The pertinent standard properties of affine Weyl groups and their dual versions are recalled in Section 2. Two types of sign homomorphisms and the corresponding fundamental domains are defined in Section 3. The S^s- and S^l-functions and their behaviour on the given discrete grids are studied in Section 4. In Section 5, the number of points in F^s_M, F^l_M are shown to be equal to the number of weights in Λ^s_M, Λ^l_M. Explicit formulas for these numbers are also given. Section 6 contains the detailed description of the discrete orthogonality and discrete transforms of S^s- and S^l-functions. Comments and follow-up questions are in the last section.

2.1. Pertinent properties of affine Weyl groups

2.1.1. Roots and reflections

We use the notation established in [**16**]. Recall that, to the Lie algebra of the compact simple Lie group G of rank n, corresponds the set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ [**1, 20, 61**]. The set Δ spans the Euclidean space $\mathbb{R}^n$, with the scalar product denoted by $\langle\, , \,\rangle$. We consider here only simple algebras with two different lengths of roots, namely B_n, $n \geq 3$, C_n, $n \geq 2$, G_2 and F_4. For these algebras, the set of simple roots consists of short simple roots Δ_s and long simple roots Δ_l. Thus, we have the disjoint decomposition

$$\Delta = \Delta_s \cup \Delta_l. \tag{2.1.1}$$

We then use the following well-known objects related to the set Δ.

- The marks $m_1, \ldots, m_n$ of the highest root

$$\xi \equiv -\alpha_0 = m_1\alpha_1 + \cdots + m_n\alpha_n\,.$$

- The Coxeter number $m = 1 + m_1 + \cdots + m_n$ of G.
- The Cartan matrix C and its determinant

$$c = \det C. \tag{2.1.2}$$

- The root lattice $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$.
- The $\mathbb{Z}$-dual lattice to Q,

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee,\, \alpha\rangle \in \mathbb{Z},\, \forall \alpha \in \Delta\} = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee\,.$$

- The dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$, where $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i,\, \alpha_i\rangle$.
- The dual marks $m_1^\vee, \ldots, m_n^\vee$ of the highest dual root

$$\eta \equiv -\alpha_0^\vee = m_1^\vee\alpha_1^\vee + \cdots + m_n^\vee\alpha_n^\vee\,.$$

  The marks and the dual marks are summarized in Table 1 in [**16**].
- The $\mathbb{Z}$-dual lattice to $Q^\vee$

$$P = \{\omega \in \mathbb{R}^n \mid \langle \omega,\, \alpha^\vee\rangle \in \mathbb{Z},\, \forall \alpha^\vee \in Q^\vee\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n.$$

Recall that n reflections r_α, $\alpha \in \Delta$ in $(n-1)$-dimensional "mirrors" orthogonal to simple roots intersecting at the origin are given explicitly by

$$r_\alpha a = a - \frac{2\langle a,\, \alpha\rangle}{\langle \alpha,\, \alpha\rangle}\alpha\,, \qquad a \in \mathbb{R}^n\,.$$

The affine reflection r_0 with respect to the highest root ξ is given by

$$r_0 a = r_\xi a + \frac{2\xi}{\langle \xi,\, \xi\rangle}\,, \qquad r_\xi a = a - \frac{2\langle a,\, \xi\rangle}{\langle \xi,\, \xi\rangle}\xi\,, \qquad a \in \mathbb{R}^n\,.$$

We denote the set of reflections $r_1 \equiv r_{\alpha_1}, \ldots, r_n \equiv r_{\alpha_n}$, together with the affine reflection r_0, by R, i.e.

$$R = \{r_0, r_1, \ldots, r_n\}. \tag{2.1.3}$$

Analogously to (2.1.1), we divide the reflections of R into two subsets:

$$R^s = \{r_\alpha \mid \alpha \in \Delta_s\}\,,$$
$$R^l = \{r_\alpha \mid \alpha \in \Delta_l\} \cup \{r_0\}.$$

We then obtain the disjoint decomposition

$$R = R^s \cup R^l. \tag{2.1.4}$$

We can also call the sums of marks corresponding to the long or short roots, i.e. the numbers

$$\begin{aligned} m^s &= \sum_{\alpha_i \in \Delta_s} m_i, \\ m^l &= \sum_{\alpha_i \in \Delta_l} m_i + 1, \end{aligned}$$

the short and the long Coxeter numbers. Their sum gives the Coxeter number $m = m^s + m^l$.

The dual affine reflection $r_0^\vee$, with respect to the dual highest root η, is given by

$$r_0^\vee a = r_\eta a + \frac{2\eta}{\langle \eta, \eta \rangle}, \quad r_\eta a = a - \frac{2\langle a, \eta \rangle}{\langle \eta, \eta \rangle}\eta, \quad a \in \mathbb{R}^n.$$

We denote the set of reflections $r_1^\vee \equiv r_{\alpha_1}, \dots, r_n^\vee \equiv r_{\alpha_n}$, together with the dual affine reflection $r_0^\vee$, by $R^\vee$, i.e.

$$R^\vee = \{r_0^\vee, r_1^\vee, \dots, r_n^\vee\}. \tag{2.1.5}$$

Analogously to (2.1.1), (2.1.4), we divide the reflections of $R^\vee$ into two subsets:

$$\begin{aligned} R^{s\vee} &= \{r_\alpha \mid \alpha \in \Delta_s\} \cup \{r_0^\vee\}, \\ R^{l\vee} &= \{r_\alpha \mid \alpha \in \Delta_l\}. \end{aligned}$$

The disjoint decomposition of $R^\vee$ is then

$$R^\vee = R^{s\vee} \cup R^{l\vee}. \tag{2.1.6}$$

We can also call the sums of the dual marks corresponding to generators from $R^{s\vee}$, $R^{l\vee}$, i.e. the numbers

$$\begin{aligned} m^{s\vee} &= \sum_{\alpha_i \in \Delta_s} m_i^\vee + 1, \\ m^{l\vee} &= \sum_{\alpha_i \in \Delta_l} m_i^\vee, \end{aligned}$$

the short and the long dual Coxeter numbers. Again, their sum gives the Coxeter number $m = m^{s\vee} + m^{l\vee}$. Direct calculation of these numbers yields the following crucial result.

Proposition 2.1.1. *For the numbers m^s, m^l and $m^{s\vee}$, $m^{l\vee}$, it holds that*

$$m^s = m^{s\vee}, \quad m^l = m^{l\vee}. \tag{2.1.7}$$

Type	R^s	R^l	$R^{s\vee}$	$R^{l\vee}$
B_n $(n \geq 3)$	r_n	$r_0,\, r_1, \dots, r_{n-1}$	$r_0^\vee,\, r_n^\vee$	$r_1^\vee, \dots, r_{n-1}^\vee$
C_n $(n \geq 2)$	$r_1, \dots, r_{n-1}$	$r_0,\, r_n$	$r_0^\vee, r_1^\vee, \dots, r_{n-1}^\vee$	$r_n^\vee$
G_2	r_2	$r_0,\, r_1$	$r_0^\vee,\, r_2^\vee$	$r_1^\vee$
F_4	r_3, r_4	$r_0,\, r_1,\, r_2$	$r_0^\vee,\, r_3^\vee, r_4^\vee$	$r_1^\vee,\, r_2^\vee$

Table 2.1. The decomposition of the sets of generators R, $R^\vee$ and the Coxeter numbers m^s, m^l. Numbering of the simple roots is standard (see e.g. Figure 1 in [**16**]).

Type	m^s	m^l
B_n $(n \geq 3)$	2	$2n-2$
C_n $(n \geq 2)$	$2n-2$	2
G_2	3	3
F_4	6	6

Table 2.2. The Coxeter numbers m^s, m^l.

The explicit form of decompositions (2.1.4) and (2.1.6) of sets R and $R^\vee$ is given in Table 2.1. The Coxeter numbers m^s, m^l can be found in Table 2.2.

2.1.2. Weyl group and affine Weyl group

Weyl group W is generated by n reflections r_α, $\alpha \in \Delta$. Applying the action of W on the set of simple roots Δ, we obtain the entire root system $W\Delta$. The root system $W\Delta$ contains two subsystems $W\Delta_s$ and $W\Delta_l$, i.e. we have the disjoint decomposition

$$W\Delta = W\Delta_s \cup W\Delta_l. \tag{2.1.8}$$

The set of $n+1$ generators R generates the affine Weyl group W^{aff}. The affine Weyl group W^{aff} can be viewed as the semidirect product of the Abelian group of translations $Q^\vee$ and of the Weyl group W:

$$W^{\mathrm{aff}} = Q^\vee \rtimes W. \tag{2.1.9}$$

Thus, for any $w^{\mathrm{aff}} \in W^{\mathrm{aff}}$, there exist a unique $w \in W$ and a unique shift $T(q^\vee)$ such that $w^{\mathrm{aff}} = T(q^\vee)w$. The retraction homomorphism $\psi : W^{\mathrm{aff}} \to W$ for $w^{\mathrm{aff}} \in W^{\mathrm{aff}}$ is given by

$$\psi(w^{\mathrm{aff}}) = \psi(T(q^\vee)w) = w. \tag{2.1.10}$$

The fundamental domain F of W^{aff}, which consists of precisely one point of each W^{aff}-orbit, is the convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}$. Considering

$n+1$ real parameters $y_0, \ldots, y_n \geq 0$, we have

$$F = \{y_1\omega_1^\vee + \cdots + y_n\omega_n^\vee \mid y_0 + y_1 m_1 + \cdots + y_n m_n = 1\}. \tag{2.1.11}$$

Recall that the stabilizer

$$\mathrm{Stab}_{W^{\mathrm{aff}}}(a) = \left\{w^{\mathrm{aff}} \in W^{\mathrm{aff}} \mid w^{\mathrm{aff}} a = a\right\} \tag{2.1.12}$$

of a point $a = y_1\omega_1^\vee + \cdots + y_n\omega_n^\vee \in F$ is trivial, $\mathrm{Stab}_{W^{\mathrm{aff}}}(a) = 1$, if the point a is in the interior of F, $a \in \mathrm{int}(F)$. Otherwise the group $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ is generated by such r_i for which $y_i = 0$, $i = 0, \ldots, n$.

Considering the standard action of W on the torus $\mathbb{R}^n/Q^\vee$, we denote for $x \in \mathbb{R}^n/Q^\vee$ the isotropy group and its order by

$$\mathrm{Stab}(x) = \{w \in W \mid wx = x\}, \quad h_x \equiv |\mathrm{Stab}(x)|$$

and denote the orbit and its order by

$$Wx = \{wx \in \mathbb{R}^n/Q^\vee \mid w \in W\}, \quad \varepsilon(x) \equiv |Wx|.$$

Then we have

$$\varepsilon(x) = \frac{|W|}{h_x}. \tag{2.1.13}$$

Recall the following three properties from Proposition 2.2 in [**16**] of the action of W on the torus $\mathbb{R}^n/Q^\vee$.

(1) For any $x \in \mathbb{R}^n/Q^\vee$, there exists $x' \in F \cap \mathbb{R}^n/Q^\vee$ and $w \in W$ such that

$$x = wx'. \tag{2.1.14}$$

(2) If $x, x' \in F \cap \mathbb{R}^n/Q^\vee$ and $x' = wx$, $w \in W$, then

$$x' = x = wx. \tag{2.1.15}$$

(3) If $x \in F \cap \mathbb{R}^n/Q^\vee$, i.e. $x = a + Q^\vee$, $a \in F$, then $\psi(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = \mathrm{Stab}(x)$ and

$$\mathrm{Stab}(x) \cong \mathrm{Stab}_{W^{\mathrm{aff}}}(a). \tag{2.1.16}$$

2.1.3. Dual affine Weyl group

The dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ is generated by the set $R^\vee$. Moreover, $\widehat{W}^{\mathrm{aff}}$ is a semidirect product of the group of shifts Q and the Weyl group W:

$$\widehat{W}^{\mathrm{aff}} = Q \rtimes W. \tag{2.1.17}$$

Thus, for any $w^{\mathrm{aff}} \in \widehat{W}^{\mathrm{aff}}$, there exist a unique $w \in W$ and a unique shift $T(q)$ such that $w^{\mathrm{aff}} = T(q)w$. The dual retraction homomorphism $\widehat{\psi} : \widehat{W}^{\mathrm{aff}} \to W$ for

$w^{\mathrm{aff}} \in \widehat{W}^{\mathrm{aff}}$ is given by

$$\widehat{\psi}(w^{\mathrm{aff}}) = \widehat{\psi}(T(q)w) = w. \tag{2.1.18}$$

The dual fundamental domain $F^\vee$ of $\widehat{W}^{\mathrm{aff}}$ is the convex hull of vertices

$$\left\{0, \frac{\omega_1}{m_1^\vee}, \dots, \frac{\omega_n}{m_n^\vee}\right\}.$$

Considering $n+1$ real parameters $z_0, \dots, z_n \geq 0$, we have

$$F^\vee = \left\{z_1\omega_1 + \dots + z_n\omega_n \mid z_0 + z_1 m_1^\vee + \dots + z_n m_n^\vee = 1\right\}. \tag{2.1.19}$$

Consider the point $a = z_1\omega_1 + \dots + z_n\omega_n \in F^\vee$ such that $z_0 + z_1 m_1^\vee + \dots + z_n m_n^\vee = 1$. The isotropy group

$$\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a) = \left\{w^{\mathrm{aff}} \in \widehat{W}^{\mathrm{aff}} \;\middle|\; w^{\mathrm{aff}} a = a\right\} \tag{2.1.20}$$

of point a is trivial, $\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a) = 1$, if $a \in \mathrm{int}(F^\vee)$, i.e. all $z_i > 0$, $i = 0, \dots, n$. Otherwise the group $\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a)$ is generated by such $r_i^\vee$ for which $z_i = 0$, $i = 0, \dots, n$.

Recall from [**16**] that, for an arbitrary $M \in \mathbb{N}$, the grid Λ_M is defined as cosets from the W-invariant group P/MQ with a representative element in $MF^\vee$, i.e.

$$\Lambda_M \equiv MF^\vee \cap P/MQ.$$

Considering a natural action of W on the quotient group $\mathbb{R}^n/MQ$, we denote for $\lambda \in \mathbb{R}^n/MQ$ the isotropy group and its order by

$$\mathrm{Stab}^\vee(\lambda) = \left\{w \in W \mid w\lambda = \lambda\right\}, \quad h_\lambda^\vee \equiv |\mathrm{Stab}^\vee(\lambda)|. \tag{2.1.21}$$

Recall the following three properties from Proposition 3.6 in [**16**] of the action of W on the quotient group $\mathbb{R}^n/MQ$.

(1) For any $\lambda \in P/MQ$, there exists $\lambda' \in \Lambda_M$ and $w \in W$ such that

$$\lambda = w\lambda'. \tag{2.1.22}$$

(2) If $\lambda, \lambda' \in \Lambda_M$ and $\lambda' = w\lambda$, $w \in W$, then

$$\lambda' = \lambda = w\lambda. \tag{2.1.23}$$

(3) If $\lambda \in MF^\vee \cap \mathbb{R}^n/MQ$, i.e. $\lambda = b + MQ$, $b \in MF^\vee$, then

$$\widehat{\psi}(\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(b/M)) = \mathrm{Stab}^\vee(\lambda)$$

and

$$\mathrm{Stab}^\vee(\lambda) \cong \mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(b/M). \tag{2.1.24}$$

2.2. Sign homomorphisms and orbit functions

2.2.1. Sign homomorphisms

The Weyl group W has the following abstract presentation [**1, 20**]:

$$r_i^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1, \quad i, j = 1, \dots, n, \tag{2.2.1}$$

where integers m_{ij} are elements of the Coxeter matrix. To introduce various classes of orbit functions, we consider "sign" homomorphisms $\sigma : W \to \{\pm 1\}$. An admissible mapping σ must satisfy the presentation condition (2.2.1)

$$\sigma(r_i)^2 = 1, \quad (\sigma(r_i)\sigma(r_j))^{m_{ij}} = 1, \quad i, j = 1, \dots, n. \tag{2.2.2}$$

If condition (2.2.2) is satisfied, then it follows from the universality property (see e.g. [**1**]) that σ is a well-defined homomorphism and its values on any $w \in W$ are given as products of generator values. The following two choices of homomorphism values of generators r_α, $\alpha \in \Delta$, obviously satisfying (2.2.2), lead to the well-known homomorphisms:

$$\mathfrak{1}(r_\alpha) = 1, \tag{2.2.3}$$

$$\sigma^e(r_\alpha) = -1, \tag{2.2.4}$$

which yield for any $w \in W$

$$\mathfrak{1}(w) = 1, \tag{2.2.5}$$

$$\sigma^e(w) = \det w. \tag{2.2.6}$$

It is shown in [**40**] that, for root systems with two different lengths of roots, there are two other available choices. Using the decomposition (2.1.1), these two new homomorphisms are given as follows [**40**]:

$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l \\ -1, & \alpha \in \Delta_s, \end{cases} \tag{2.2.7}$$

$$\sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s \\ -1, & \alpha \in \Delta_l. \end{cases} \tag{2.2.8}$$

Since the highest root $\xi \in W\Delta_l$, there exist $w \in W$ and $\alpha \in \Delta_l$ such that $\xi = w\alpha$. Then, from the relation $r_\xi = w r_\alpha w^{-1}$, we obtain for any sign homomorphism that $\sigma(r_\xi) = \sigma(r_\alpha)$ holds. Thus, we have

$$\sigma^s(r_\xi) = 1, \quad \sigma^l(r_\xi) = -1. \tag{2.2.9}$$

Similarly, for the highest dual root η there exists a root $\beta \in W\Delta_s$ such that $\eta = 2\beta/\langle\beta,\, \beta\rangle$, and we obtain

$$\sigma^s(r_\eta) = -1, \quad \sigma^l(r_\eta) = 1. \tag{2.2.10}$$

2.2.2. Fundamental domains

Each of the sign homomorphisms σ^s and σ^l determines a decomposition of the fundamental domain F. The factors of this decomposition will be crucial for the study of the orbit functions. We introduce two subsets of F:

$$\begin{aligned} F^s &= \left\{ a \in F \;\middle|\; \sigma^s \circ \psi\left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{1\} \right\}, \\ F^l &= \left\{ a \in F \;\middle|\; \sigma^l \circ \psi\left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{1\} \right\}, \end{aligned}$$

where ψ is the retraction homomorphism (2.1.10). Since for all points of the interior of F the stabilizer is trivial, i.e. $\mathrm{Stab}_{W^{\mathrm{aff}}}(a) = 1$, $a \in \mathrm{int}(F)$, the interior $\mathrm{int}(F)$ is a subset of both F^s and F^l. In order to determine the analytic form of the sets F^s and F^l, we define two subsets of the boundaries of F:

$$\begin{aligned} H^s &= \left\{ a \in F \mid (\exists r \in R^s)(ra = a) \right\}, \\ H^l &= \left\{ a \in F \mid (\exists r \in R^l)(ra = a) \right\}. \end{aligned}$$

Note that, since for the affine reflection $r_0 \in R^l$ it holds that $\psi(r_0) = r_\xi$, we have from (2.2.9) that $\sigma^s \circ \psi(r_0) = 1$ and $\sigma^l \circ \psi(r_0) = -1$. Taking into account the disjoint decomposition (2.1.4), we obtain for any $r \in R$ the following two exclusive choices:

$$\begin{aligned} &\sigma^s \circ \psi(r) = -1, \quad \sigma^l \circ \psi(r) = 1, \qquad r \in R^s, \\ &\sigma^s \circ \psi(r) = 1, \qquad \sigma^l \circ \psi(r) = -1, \quad r \in R^l. \end{aligned} \tag{2.2.11}$$

Proposition 2.2.1. *For the sets F^s and F^l, the following holds:*

(1) $F^s = F \setminus H^s$.

(2) $F^l = F \setminus H^l$.

Proof. Let $a \in F$.

(1) If $a \notin F \setminus H^s$, then $a \in H^s$, and there exists $r \in R^s$ such that $r \in \mathrm{Stab}_{W^{\mathrm{aff}}}(a)$. Then according to (2.2.11), we have $\sigma^s \circ \psi(r) = -1$. Thus, $\sigma^s \circ \psi\left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{\pm 1\}$ and consequently $a \notin F^s$. Conversely, if $a \in F \setminus H^s$, then the stabilizer $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ is either trivial or generated by generators from R^l only. Then, since for any generator $r \in R^l$ it follows from (2.2.11) that $\sigma^s \circ \psi(r) = 1$, we obtain $\sigma^s \circ \psi\left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{1\}$, i.e. $a \in F^s$.

(2) This case is completely analogous to case (1).

□

The explicit description of domains F^s and F^l now follows from (2.1.11) and Proposition 2.2.1. We introduce the symbols $y_i^s,\ y_i^l \in \mathbb{R},\ i = 0,\dots,n$ in the following way:

$$\begin{aligned} y_i^s > 0, \quad y_i^l \geq 0, \quad r_i \in R^s, \\ y_i^s \geq 0, \quad y_i^l > 0, \quad r_i \in R^l. \end{aligned} \tag{2.2.12}$$

Thus, the explicit form of F^s and F^l is given by

$$\begin{aligned} F^s &= \{y_1^s\omega_1^\vee + \cdots + y_n^s\omega_n^\vee \mid y_0^s + y_1^s m_1 + \cdots + y_n^s m_n = 1\}\,, \\ F^l &= \left\{y_1^l\omega_1^\vee + \cdots + y_n^l\omega_n^\vee \mid y_0^l + y_1^l m_1 + \cdots + y_n^l m_n = 1\right\}. \end{aligned} \tag{2.2.13}$$

2.2.3. Dual fundamental domains

The sign homomorphisms σ^s and σ^l also determine a decomposition of the dual fundamental domain $F^\vee$. The factors of this decomposition will be needed for the study of the discretized orbit functions. We introduce two subsets of $F^\vee$:

$$\begin{aligned} F^{s\vee} &= \left\{a \in F^\vee \,\middle|\, \sigma^s \circ \widehat{\psi}\left(\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a)\right) = \{1\}\right\}, \\ F^{l\vee} &= \left\{a \in F^\vee \,\middle|\, \sigma^l \circ \widehat{\psi}\left(\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a)\right) = \{1\}\right\}, \end{aligned} \tag{2.2.14}$$

where $\widehat{\psi}$ is the dual retraction homomorphism (2.1.18). Since for all points of the interior of $F^\vee$ the stabilizer is trivial, i.e. $\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a) = 1,\ a \in \mathrm{int}(F^\vee)$, the interior $\mathrm{int}(F^\vee)$ is a subset of both $F^{s\vee}$ and $F^{l\vee}$. In order to determine the analytic form of sets $F^{s\vee}$ and $F^{l\vee}$, we define two subsets of the boundaries of $F^\vee$:

$$\begin{aligned} H^{s\vee} &= \{a \in F^\vee \mid (\exists r \in R^{s\vee})(ra = a)\}\,, \\ H^{l\vee} &= \left\{a \in F^\vee \mid (\exists r \in R^{l\vee})(ra = a)\right\}. \end{aligned}$$

Note that, since for the affine reflection $r_0^\vee \in R^{s\vee}$ it holds that $\widehat{\psi}(r_0^\vee) = r_\eta$, we have from (2.2.10) that $\sigma^s \circ \widehat{\psi}(r_0^\vee) = -1$ and $\sigma^l \circ \widehat{\psi}(r_0^\vee) = 1$. Taking into account the disjoint decomposition (2.1.6), we obtain for any $r \in R^\vee$ the following two exclusive choices:

$$\begin{aligned} \sigma^s \circ \widehat{\psi}(r) = -1, \quad \sigma^l \circ \widehat{\psi}(r) = 1, \qquad r \in R^{s\vee}, \\ \sigma^s \circ \widehat{\psi}(r) = 1, \qquad \sigma^l \circ \widehat{\psi}(r) = -1, \quad r \in R^{l\vee}. \end{aligned} \tag{2.2.15}$$

Similarly to Proposition 2.2.1, we obtain the following.

Proposition 2.2.2. *For sets $F^{s\vee}$ and $F^{l\vee}$, the following holds:*

(1) $F^{s\vee} = F^\vee \setminus H^{s\vee}$.

(2) $F^{l\vee} = F^\vee \setminus H^{l\vee}$.

The explicit description of domains $F^{s\vee}$ and $F^{l\vee}$ now follows from (2.1.19) and Proposition 2.2.2. We introduce the symbols $z_i^s,\, z_i^l \in \mathbb{R}$, $i = 0, \dots, n$ in the following way:

$$\begin{aligned} z_i^s > 0, \quad z_i^l \geq 0, \quad r_i \in R^{s\vee}, \\ z_i^s \geq 0, \quad z_i^l > 0, \quad r_i \in R^{l\vee}. \end{aligned} \tag{2.2.16}$$

Thus, the explicit form of $F^{s\vee}$ and $F^{l\vee}$ is given by

$$\begin{aligned} F^{s\vee} &= \left\{ z_1^s \omega_1 + \dots + z_n^s \omega_n \mid z_0^s + z_1^s m_1^\vee + \dots + z_n^s m_n^\vee = 1 \right\}, \\ F^{l\vee} &= \left\{ z_1^l \omega_1 + \dots + z_n^l \omega_n \mid z_0^l + z_1^l m_1^\vee + \dots + z_n^l m_n^\vee = 1 \right\}. \end{aligned} \tag{2.2.17}$$

2.3. S^s- and S^l-functions

Four sign homomorphisms $\mathfrak{1}$, σ^e, σ^s and σ^l induce four types of families of complex orbit functions. Within each family, determined by $\sigma \in \{\mathfrak{1}, \sigma^e, \sigma^s, \sigma^l\}$, are the complex functions $\varphi_b^\sigma : \mathbb{R}^n \to \mathbb{C}$ labelled by weights $b \in P$ and in the general form

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w)\, e^{2\pi i \langle wb,\, a\rangle}, \quad a \in \mathbb{R}^n. \tag{2.3.1}$$

The resulting functions for $\sigma = \mathfrak{1}$ in (2.3.1) are called C-functions; for their detailed review, see [**24**]. For $\sigma = \sigma^e$, we obtain the well-known S-functions [**26**]. The discretization properties of both C- and S-functions on a finite fragment of the grid $\frac{1}{M}P^\vee$ were described in [**16**]. The remaining two options of homomorphisms σ^s and σ^l and corresponding functions $\varphi_\lambda^{\sigma^s}$, $\varphi_\lambda^{\sigma^l}$, called S^l- and S^s-functions [**40**], were studied in detail for G_2 only [**57**]. In order to describe the discretization of functions $\varphi_\lambda^{\sigma^s}$ and $\varphi_\lambda^{\sigma^l}$ in full generality, we first review their basic properties.

2.3.1. S^s-functions

2.3.1.1. *Symmetries of S^s-functions*

Choosing $\sigma = \sigma^s$ in (2.3.1), we obtain S^s-functions $\varphi_b^{\sigma^s}$; we abbreviate the notation by denoting $\varphi_b^s \equiv \varphi_b^{\sigma^s}$, i.e.

$$\varphi_b^s(a) = \sum_{w \in W} \sigma^s(w)\, e^{2\pi i \langle wb,\, a\rangle}, \quad a \in \mathbb{R}^n,\, b \in P. \tag{2.3.2}$$

The following properties of S^s-functions are crucial:

- (anti)symmetry with respect to $w \in W$

$$\varphi_b^s(wa) = \sigma^s(w)\varphi_b^s(a), \tag{2.3.3}$$

$$\varphi_{wb}^s(a) = \sigma^s(w)\varphi_b^s(a); \tag{2.3.4}$$

- invariance with respect to shifts from $q^\vee \in Q^\vee$

$$\varphi_b^s(a+q^\vee) = \varphi_b^s(a). \tag{2.3.5}$$

Thus, the S^s-functions are (anti)symmetric with respect to the affine Weyl group W^{aff}. This allows us to consider the values $\varphi_b^s(a)$ only for points of the fundamental domain $a \in F$. Moreover, from (2.2.11), (2.3.3) we deduce that

$$\varphi_b^s(ra) = -\varphi_b^s(a), \quad r \in R^s. \tag{2.3.6}$$

This antisymmetry implies that the functions φ_b^s are for all $b \in P$ zero on part H^s of the boundary of F:

$$\varphi_b^s(a') = 0, \quad a' \in H^s, \tag{2.3.7}$$

and therefore *we consider the functions φ_b^s on the fundamental domain $F^s = F \setminus H^s$ only.*

2.3.1.2. *Discretization of S^s-functions*

In order to develop discrete calculus of S^s-functions, we investigate the behaviour of these functions on the grid $\frac{1}{M}P^\vee$. Suppose we have fixed $M \in \mathbb{N}$ and $u \in \frac{1}{M}P^\vee$. It follows from (2.3.5) that we can consider φ_b^s as a function on cosets from $\frac{1}{M}P^\vee/Q^\vee$. It follows from (2.3.7) that we can consider φ_b^s only on the set

$$F_M^s \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^s. \tag{2.3.8}$$

Next we have

$$\varphi_{b+MQ}^s(u) = \varphi_b^s(u), \quad u \in F_M^s$$

and thus we can consider the functions φ_λ^s on F_M^s parametrized by cosets from $\lambda \in P/MQ$. Moreover, it follows from (2.1.22) and (2.3.4) that we can consider φ_λ^s on F_M^s parametrized by classes from Λ_M. Taking any $\lambda \in \Lambda_M$ and any reflection $r^\vee \in R^{s\vee}$ we calculate directly using (2.2.10), (2.3.4) and (2.3.5) that

$$\varphi^s_{Mr^\vee(\frac{\lambda}{M})}(u) = -\varphi_\lambda^s(u), \quad u \in F_M^s.$$

This implies that, for $\lambda \in MH^{s\vee} \cap \Lambda_M$, the functions φ_λ^s are zero on F_M^s, i.e.

$$\varphi_\lambda^s(u) = 0, \quad \lambda \in MH^{s\vee} \cap \Lambda_M,\ u \in F_M^s.$$

Defining the set

$$\Lambda_M^s \equiv P/MQ \cap MF^{s\vee}, \tag{2.3.9}$$

we conclude that *we can consider S^s-functions φ_λ^s on F_M^s parametrized by $\lambda \in \Lambda_M^s$ only.*

2.3.2. S^l-functions

2.3.2.1. *Symmetries of S^l-functions*

Choosing $\sigma = \sigma^l$ in (2.3.1), we obtain S^l-functions $\varphi_b^{\sigma^l}$; we abbreviate the notation by denoting $\varphi_b^l \equiv \varphi_b^{\sigma^l}$, i.e.

$$\varphi_b^l(a) = \sum_{w\in W} \sigma^l(w)\, e^{2\pi i\langle wb,\, a\rangle}, \quad a \in \mathbb{R}^n,\, b \in P. \tag{2.3.10}$$

The following properties of S^l-functions are crucial:

- (anti)symmetry with respect to $w \in W$

$$\varphi_b^l(wa) = \sigma^l(w)\varphi_b^l(a), \tag{2.3.11}$$

$$\varphi_{wb}^l(a) = \sigma^l(w)\varphi_b^l(a); \tag{2.3.12}$$

- invariance with respect to shifts from $q^\vee \in Q^\vee$

$$\varphi_b^l(a + q^\vee) = \varphi_b^l(a). \tag{2.3.13}$$

Thus, the S^l-functions are (anti)symmetric with respect to the affine Weyl group W^{aff}. This allows us to consider the values $\varphi_b^l(a)$ only for points of the fundamental domain $a \in F$. Moreover, from (2.2.11), (2.3.11) and (2.3.13) we deduce that

$$\varphi_b^l(ra) = -\varphi_b^l(a), \quad r \in R^l. \tag{2.3.14}$$

This antisymmetry implies that the functions φ_b^l are for all $b \in P$ zero on part H^l of the boundary of F:

$$\varphi_b^l(a') = 0, \quad a' \in H^l, \tag{2.3.15}$$

and therefore *we consider the functions φ_b^l on the fundamental domain $F^l = F\backslash H^l$ only.*

2.3.2.2. *Discretization of S^l-functions*

In order to develop discrete calculus of S^l-functions, we investigate the behaviour of these functions on the grid $\frac{1}{M}P^\vee$. Suppose we have fixed $M \in \mathbb{N}$ and $u \in \frac{1}{M}P^\vee$. It follows from (2.3.13) that we can consider φ_b^l as a function on cosets from $\frac{1}{M}P^\vee/Q^\vee$. It follows from (2.3.15) that we can consider φ_b^l only on the set

$$F_M^l \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^l. \tag{2.3.16}$$

Next we have

$$\varphi_{b+MQ}^l(u) = \varphi_b^l(u), \quad u \in F_M^l$$

and thus we can consider the functions φ^l_λ on F^l_M parametrized by cosets from $\lambda \in P/MQ$. Moreover, it follows from (2.1.22) and (2.3.12) that we can consider φ^l_λ on F^l_M parametrized by classes from Λ_M. Taking any $\lambda \in \Lambda_M$ and any reflection $r^\vee \in R^{l\vee}$, we calculate directly using (2.2.10), (2.3.12) that

$$\varphi^l_{r^\vee\lambda}(u) = -\varphi^l_\lambda(u), \quad u \in F^l_M.$$

This implies that for $\lambda \in MH^{l\vee} \cap \Lambda_M$, the functions φ^l_λ are zero on F^l_M, i.e.

$$\varphi^l_\lambda(u) = 0, \quad \lambda \in MH^{l\vee} \cap \Lambda_M,\, u \in F^l_M.$$

Defining the set

$$\Lambda^l_M \equiv P/MQ \cap MF^{l\vee}, \tag{2.3.17}$$

we conclude that *we can consider S^l-functions φ^l_λ on F^l_M parametrized by $\lambda \in \Lambda^l_M$ only.*

2.4. Number of grid elements

2.4.1. Number of elements of F^s_M and F^l_M

Recall from [**16**] that, for an arbitrary $M \in \mathbb{N}$, the grid F_M is given as cosets from the W-invariant group $\frac{1}{M}P^\vee/Q^\vee$ with a representative element in the fundamental domain F:

$$F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F,$$

and the following property holds:

$$WF_M = \frac{1}{M}P^\vee/Q^\vee. \tag{2.4.1}$$

The representative points of F_M can be explicitly written as

$$F_M = \left\{ \frac{u_1}{M}\omega^\vee_1 + \cdots + \frac{u_n}{M}\omega^\vee_n \mid u_0, u_1, \ldots, u_n \in \mathbb{Z}^{\geq 0},\, u_0 + u_1 m_1 + \cdots + u_n m_n = M \right\}. \tag{2.4.2}$$

The number of elements of F_M, denoted by $|F_M|$, are also calculated in [**16**] for all simple Lie algebras. Using these results, we derive the number of elements of F^s_M and F^l_M. Firstly, we describe explicitly the sets F^s_M and F^l_M. Similar to (2.2.12), we introduce the symbols u^s_i, $u^l_i \in \mathbb{R}$, $i = 0, \ldots, n$:

$$\begin{aligned} &u^s_i \in \mathbb{N}, && u^l_i \in \mathbb{Z}^{\geq 0}, && r_i \in R^s, \\ &u^s_i \in \mathbb{Z}^{\geq 0}, && u^l_i \in \mathbb{N}, && r_i \in R^l. \end{aligned} \tag{2.4.3}$$

The explicit form of F^s_M and F^l_M then follows from the explicit form of F^s and F^l in (2.2.13):

$$F^s_M = \left\{ \frac{u^s_1}{M}\omega^\vee_1 + \cdots + \frac{u^s_n}{M}\omega^\vee_n \mid u^s_0 + u^s_1 m_1 + \cdots + u^s_n m_n = M \right\}, \tag{2.4.4}$$

$$F^l_M = \left\{ \frac{u^l_1}{M}\omega^\vee_1 + \cdots + \frac{u^l_n}{M}\omega^\vee_n \mid u^l_0 + u^l_1 m_1 + \cdots + u^l_n m_n = M \right\}. \tag{2.4.5}$$

Using the following proposition, the number of elements of F^s_M and F^l_M can be obtained from the formulas for $|F_M|$.

Proposition 2.4.1. *Let m^s and m^l be the short and long Coxeter numbers, respectively. Then,*

$$|F^s_M| = \begin{cases} 0 & M < m^s \\ 1 & M = m^s \\ |F_{M-m^s}| & M > m^s, \end{cases} \qquad |F^l_M| = \begin{cases} 0 & M < m^l \\ 1 & M = m^l \\ |F_{M-m^l}| & M > m^l. \end{cases} \tag{2.4.6}$$

Proof. Taking non-negative numbers $u_i \in \mathbb{Z}^{\geq 0}$ and substituting the relations $u^s_i = 1 + u_i$ if $r_i \in R^s$ and $u^s_i = u_i$ if $r_i \in R^l$ into the defining relation (2.4.4), we obtain

$$u_0 + m_1 u_1 + \cdots + m_n u_n = M - m^s, \quad u_0, \ldots, u_n \in \mathbb{Z}^{\geq 0}.$$

This equation has one solution $[0, \ldots, 0]$ if $M = m^s$, no solution if $M < m^s$, and is equal to the defining relation (2.4.2) of F_{M-m^s} if $M > m^s$. The case of F^l_M is similar. □

Theorem 2.4.1. *The numbers of points of grids F^s_M and F^l_M of Lie algebras B_n, C_n, G_2 and F_4 are given by the following relations.*

(1) C_n, $n \geq 2$,

$$|F^s_{2k}(C_n)| = \binom{k+1}{n} + \binom{k}{n},$$

$$|F^s_{2k+1}(C_n)| = 2\binom{k+1}{n},$$

$$|F^l_{2k}(C_n)| = \binom{n+k-1}{n} + \binom{n+k-2}{n},$$

$$|F^l_{2k+1}(C_n)| = 2\binom{n+k-1}{n}.$$

(2) B_n, $n \geq 3$,

$$|F^s_M(B_n)| = |F^l_M(C_n)|, \quad |F^l_M(B_n)| = |F^s_M(C_n)|.$$

(3) G_2

$$\begin{aligned}
|F^s_{6k}(G_2)| &= 3k^2, & |F^s_{6k+1}(G_2)| &= 3k^2 + k,\\
|F^s_{6k+2}(G_2)| &= 3k^2 + 2k, & |F^s_{6k+3}(G_2)| &= 3k^2 + 3k + 1,\\
|F^s_{6k+4}(G_2)| &= 3k^2 + 4k + 1, & |F^s_{6k+5}(G_2)| &= 3k^2 + 5k + 2,
\end{aligned}$$

$$|F^l_M(G_2)| = |F^s_M(G_2)|.$$

(4) F_4

$$\begin{aligned}
&|F^s_{12k}(F_4)| = 18k^4 - k^2,\\
&|F^s_{12k+1}(F_4)| = 18k^4 + 6k^3 - \frac{5}{2}k^2 - \frac{1}{2}k,\\
&|F^s_{12k+2}(F_4)| = 18k^4 + 12k^3 + 2k^2,\\
&|F^s_{12k+3}(F_4)| = 18k^4 + 18k^3 + \frac{7}{2}k^2 - \frac{1}{2}k,\\
&|F^s_{12k+4}(F_4)| = 18k^4 + 24k^3 + 11k^2 + 2k,\\
&|F^s_{12k+5}(F_4)| = 18k^4 + 30k^3 + \frac{31}{2}k^2 + \frac{5}{2}k,\\
&|F^s_{12k+6}(F_4)| = 18k^4 + 36k^3 + 26k^2 + 8k + 1,\\
&|F^s_{12k+7}(F_4)| = 18k^4 + 42k^3 + \frac{67}{2}k^2 + \frac{21}{2}k + 1,\\
&|F^s_{12k+8}(F_4)| = 18k^4 + 48k^3 + 47k^2 + 20k + 3,\\
&|F^s_{12k+9}(F_4)| = 18k^4 + 54k^3 + \frac{115}{2}k^2 + \frac{51}{2}k + 4,\\
&|F^s_{12k+10}(F_4)| = 18k^4 + 60k^3 + 74k^2 + 40k + 8,\\
&|F^s_{12k+11}(F_4)| = 18k^4 + 66k^3 + \frac{175}{2}k^2 + \frac{99}{2}k + 10,\\
&|F^l_M(F_4)| = |F^s_M(F_4)|.
\end{aligned}$$

Proof. For the case C_n, we have that

$$|F_{2k}(C_n)| = \binom{n+k}{n} + \binom{n+k-1}{n}$$

from [**16**] and $m^s = 2n - 2$ from Table 2.2. It can be verified directly that the formula

$$|F^s_{2k}(C_n)| = \binom{k+1}{n} + \binom{k}{n}$$

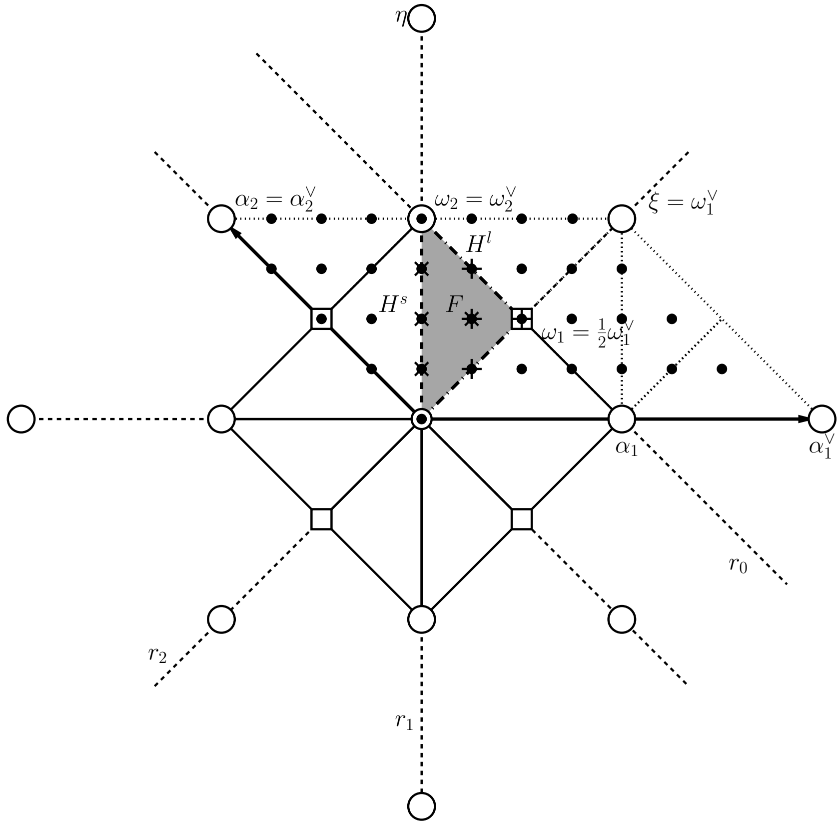

FIGURE 2.1. The fundamental domains F^s and F^l of C_2. The fundamental domain F is depicted as the grey triangle containing borders H^s and H^l, depicted as the thick dashed line and dot-and-dashed lines, respectively. The coset representatives of $\frac{1}{4}P^\vee/Q^\vee$ are shown as 32 black dots. The four representatives belonging to F_4^s and F_4^l are crossed with "+" and "×", respectively. The dashed lines represent "mirrors" r_0, r_1 and r_2. Circles are elements of the root lattice Q; together with the squares they are elements of the weight lattice P.

satisfies (2.4.6) for all values of $k \in \mathbb{N}$. Analogously, we obtain formulas for the remaining cases. $\square$

Example 2.4.1. *For the Lie algebra C_2, we have Coxeter number $m = 4$ and $c = 2$. For $M = 4$, the order of the group $\frac{1}{4}P^\vee/Q^\vee$ is equal to 32, and according to Theorem 2.4.1 we calculate*

$$|F_4^s(C_2)| = |F_4^l(C_2)| = \binom{3}{2} + \binom{2}{2} = 4.$$

The coset representatives of $\frac{1}{4}P^\vee/Q^\vee$ and the fundamental domains F^s and F^l are depicted in Figure 2.1.

2.4.2. Number of elements of Λ^s_M and Λ^l_M

In this section, we relate the numbers of elements of F^s_M, F^l_M to the numbers of elements Λ^s_M, Λ^l_M, defined by (2.3.9), (2.3.17). Firstly, we describe explicitly the sets Λ^s_M and Λ^l_M. Similarly to (2.2.16), we introduce the symbols t^s_i, $t^l_i \in \mathbb{R}$, $i = 0, \dots, n$:

$$\begin{aligned} &t^s_i \in \mathbb{N}, \quad && t^l_i \in \mathbb{Z}^{\geq 0}, \quad && r_i \in R^{s\vee}, \\ &t^s_i \in \mathbb{Z}^{\geq 0}, \quad && t^l_i \in \mathbb{N}, \quad && r_i \in R^{l\vee}. \end{aligned} \tag{2.4.7}$$

The explicit form of Λ^s_M and Λ^l_M then follows from the explicit form of $F^{s\vee}$ and $F^{l\vee}$ in (2.2.17):

$$\begin{aligned} \Lambda^s_M &= \left\{ t^s_1\omega_1 + \dots + t^s_n\omega_n \mid t^s_0 + t^s_1 m^\vee_1 + \dots + t^s_n m^\vee_n = M \right\}, \\ \Lambda^l_M &= \left\{ t^l_1\omega_1 + \dots + t^l_n\omega_n \mid t^l_0 + t^l_1 m^\vee_1 + \dots + t^l_n m^\vee_n = M \right\}. \end{aligned} \tag{2.4.8}$$

Similarly to Proposition 2.4.1, we obtain the following one.

Proposition 2.4.2. *Let $m^{s\vee}$ and $m^{l\vee}$ be the short and the long dual Coxeter numbers, respectively. Then*

$$|\Lambda^s_M| = \begin{cases} 0 & M < m^{s\vee} \\ 1 & M = m^{s\vee} \\ |F_{M-m^{s\vee}}| & M > m^{s\vee}, \end{cases} \qquad |\Lambda^l_M| = \begin{cases} 0 & M < m^{l\vee} \\ 1 & M = m^{l\vee} \\ |F_{M-m^{l\vee}}| & M > m^{l\vee}. \end{cases} \tag{2.4.9}$$

Combining Propositions 2.4.1, 2.4.2 and 2.1.1 and taking into account that $|F_M| = |\Lambda_M|$, we conclude with the following crucial result.

Corollary 2.4.1. *For the numbers of elements of the sets Λ^s_M and Λ^l_M it holds that*

$$\begin{aligned} |\Lambda^s_M| &= |F^s_M|, \\ |\Lambda^l_M| &= |F^l_M|. \end{aligned} \tag{2.4.10}$$

Example 2.4.2. *For the Lie algebra C_2 we have $|P/4Q| = 32$ and according to Theorem 2.4.1 and Corollary 2.4.1, we have*

$$|\Lambda^s_4(C_2)| = |\Lambda^l_4(C_2)| = 4.$$

The cosets representants of $P/4Q$ together with the grids of weights $\Lambda^s_4(C_2)$ and $\Lambda^l_4(C_2)$ are depicted in Figure 2.2.

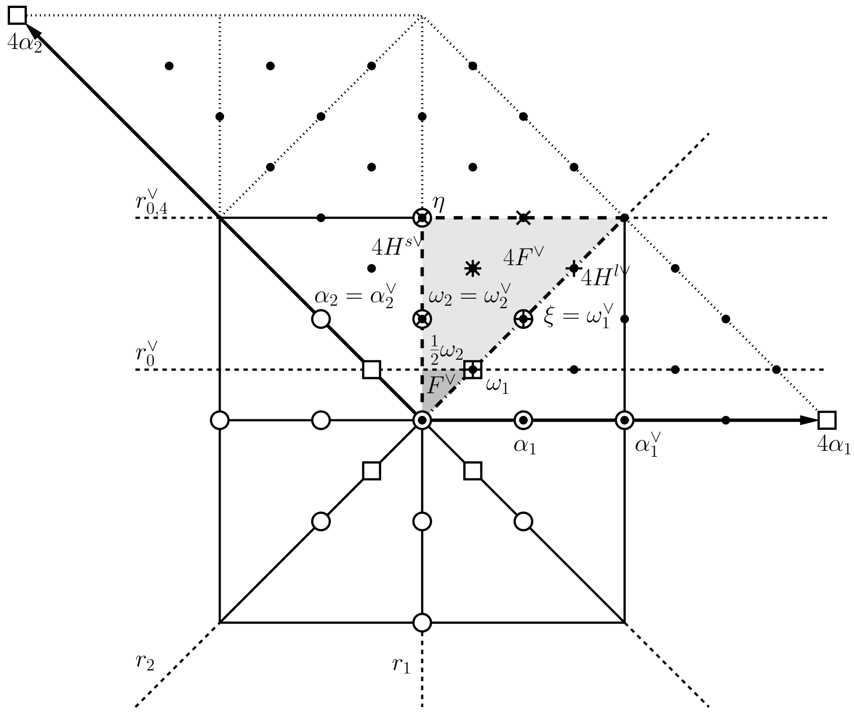

FIGURE 2.2. The grids of weights $\Lambda^s_4(C_2)$ and $\Lambda^l_4(C_2)$ of C_2. The darker grey triangle is the fundamental domain $F^\vee$ and the lighter grey triangle is the domain $4F^\vee$. The borders $4H^{s\vee}$ and $4H^{l\vee}$ are depicted as the thick dashed lines and dot-and-dashed lines, respectively. The cosets representants of $P/4Q$ of C_2 are shown as 32 black dots. The four representants belonging to $\Lambda^s_4(C_2)$ and $\Lambda^l_4(C_2)$ are crossed with "+" and "×", respectively. The dashed lines represent dual "mirrors" $r_0^\vee, r_1$, r_2 and the affine mirror $r^\vee_{0,4}$ is defined by $r^\vee_{0,4}\lambda = 4r_0^\vee(\lambda/4)$. The circles and squares coincide with those in Figure 2.1.

2.5. Discrete orthogonality and transforms of S^s- and S^l- functions

2.5.1. Discrete orthogonality of S^s- and S^l-functions

To describe the discrete orthogonality of the S^l- and S^s-functions, we use the ideas discussed in [**43**] and reformulated in [**16**]. Recall that basic orthogonality relations from [**16, 43**] are, for any $\lambda, \lambda' \in P/MQ$, of the following form:

$$\sum_{y\in \frac{1}{M}P^\vee/Q^\vee} e^{2\pi i\langle \lambda-\lambda',\, y\rangle} = cM^n\delta_{\lambda,\lambda'}. \tag{2.5.1}$$

We define the scalar product of two functions $f, g : F^s_M \to \mathbb{C}$ or $f, g : F^l_M \to \mathbb{C}$ by

$$\langle f,\, g\rangle_{F^s_M} = \sum_{x\in F^s_M} \varepsilon(x)f(x)\overline{g(x)}, \quad \langle f,\, g\rangle_{F^l_M} = \sum_{x\in F^l_M} \varepsilon(x)f(x)\overline{g(x)}, \tag{2.5.2}$$

where the numbers $\varepsilon(x)$ are determined by (2.1.13). We show that Λ^s_M and Λ^l_M are the lowest maximal sets of pairwise orthogonal S^s- and S^l-functions.

Theorem 2.5.1. *For $\lambda, \lambda' \in \Lambda^s_M$ it holds that*

$$\langle \varphi^s_\lambda,\, \varphi^s_{\lambda'} \rangle_{F^s_M} = c\,|W|\,M^n h^\vee_\lambda \delta_{\lambda,\lambda'}, \tag{2.5.3}$$

and for $\lambda, \lambda' \in \Lambda^l_M$ it holds that

$$\langle \varphi^l_\lambda,\, \varphi^l_{\lambda'} \rangle_{F^l_M} = c\,|W|\,M^n h^\vee_\lambda \delta_{\lambda,\lambda'}, \tag{2.5.4}$$

where c, $h^\vee_\lambda$ were defined by (2.1.2), (2.1.21), respectively, $|W|$ is the number of elements of the Weyl group W and n is the rank of G.

Proof. Since φ^s_λ vanishes on $F_M \setminus F^s_M$, we have

$$\langle \varphi^s_\lambda,\, \varphi^s_{\lambda'} \rangle_{F^s_M} = \sum_{x \in F^s_M} \varepsilon(x) \varphi^s_\lambda(x) \overline{\varphi^s_{\lambda'}(x)} = \sum_{x \in F_M} \varepsilon(x) \varphi^s_\lambda(x) \overline{\varphi^s_{\lambda'}(x)}.$$

The equality

$$\sum_{x \in F_M} \varepsilon(x) \varphi^s_\lambda(x) \overline{\varphi^s_{\lambda'}(x)} = \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} \varphi^s_\lambda(y) \overline{\varphi^s_{\lambda'}(y)}$$

follows from (2.1.15) and (2.4.1) and the W-invariance of the expression $\varphi^s_\lambda(x)\overline{\varphi^s_{\lambda'}(x)}$. Then, using the W-invariance of $\frac{1}{M}P^\vee/Q^\vee$ and (2.5.1), we have

$$\begin{aligned}\langle \varphi^s_\lambda,\, \varphi^s_{\lambda'} \rangle_{F^s_M} &= \sum_{w' \in W} \sum_{w \in W} \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} \sigma^s(ww') e^{2\pi i \langle w\lambda - w'\lambda',\, y \rangle} \\ &= |W| \sum_{w' \in W} \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} \sigma^s(w') e^{2\pi i \langle \lambda - w'\lambda',\, y \rangle} \\ &= c\,|W|\,M^n \sum_{w' \in W} \sigma^s(w') \delta_{w'\lambda',\lambda}.\end{aligned}$$

If $\lambda = w'\lambda'$, then we have from (2.1.23) that $w'\lambda = \lambda = \lambda'$, i.e. $w' \in \mathrm{Stab}^\vee(\lambda)$. Any $\lambda \in \Lambda^s_M$ is of the form $\lambda = b + MQ$ with $b \in MF^{s\vee}$. Then, considering (2.1.24) and (2.2.14), we have

$$\sigma^s(\mathrm{Stab}^\vee(\lambda)) = \sigma^s \circ \widehat{\psi}\left(\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(b/M)\right) = \{1\},$$

i.e. we obtain $\sigma^s(w') = 1$ for any $w' \in \mathrm{Stab}^\vee(\lambda)$, and consequently

$$\sum_{w' \in W} \sigma^s(w') \delta_{w'\lambda',\lambda} = \sum_{w' \in W} \delta_{w'\lambda',\lambda} = h^\vee_\lambda \delta_{\lambda',\lambda}.$$

The case of S^l-functions is similar. $\square$

Example 2.5.1. *The highest root ξ and the highest dual root η of C_2 are given by the formulas*

$$\xi = 2\alpha_1 + \alpha_2, \quad \eta = \alpha_1^\vee + 2\alpha_2^\vee.$$

The Weyl group of C_2 has eight elements, $|W| = 8$, and we calculate the determinant of the Cartan matrix $c = 2$. For a parameter with coordinates in ω-basis (a,b) and for a point with coordinates in $\alpha^\vee$-basis (x,y), we have the following explicit form of S^s- and S^l-functions of C_2:

$$\begin{aligned}\varphi^s_{(a,b)}(x,y) =& 2\{\cos(2\pi((a+2b)x-by)) + \cos(2\pi(ax+by)) \\ &- \cos(2\pi((a+2b)x-(a+b)y)) - \cos(2\pi(ax-(a+b)y))\},\\ \varphi^l_{(a,b)}(x,y) =& 2\{-\cos(2\pi((a+2b)x-by)) + \cos(2\pi(ax+by)) \\ &- \cos(2\pi((a+2b)x-(a+b)y)) + \cos(2\pi(ax-(a+b)y))\}.\end{aligned}$$

The grids F^s_M and F^l_M are given by

$$F^s_M(C_2) = \left\{ \frac{u^s_1}{M}\omega^\vee_1 + \frac{u^s_2}{M}\omega^\vee_2 \mid u^s_0,\, u^s_2 \in \mathbb{Z}^{\geq 0},\, u^s_1 \in \mathbb{N},\, u^s_0 + 2u^s_1 + u^s_2 = M \right\},$$
$$F^l_M(C_2) = \left\{ \frac{u^l_1}{M}\omega^\vee_1 + \frac{u^l_2}{M}\omega^\vee_2 \mid u^l_0,\, u^l_2 \in \mathbb{N},\, u^l_1 \in \mathbb{Z}^{\geq 0},\, u^l_0 + 2u^l_1 + u^l_2 = M \right\},$$

and the grids of weights Λ^s_M and Λ^l_M are determined by

$$\Lambda^s_M(C_2) = \left\{ t^s_1\omega_1 + t^s_2\omega_2 \mid t^s_0, t^s_1 \in \mathbb{N}, t^s_2 \in \mathbb{Z}^{\geq 0},\, t^s_0 + t^s_1 + 2t^s_2 = M \right\},$$
$$\Lambda^l_M(C_2) = \left\{ t^l_1\omega_1 + t^l_2\omega_2 \mid t^l_0, t^l_1 \in \mathbb{Z}^{\geq 0}, t^l_2 \in \mathbb{N},\, t^l_0 + t^l_1 + 2t^l_2 = M \right\}.$$

The discrete orthogonality relations of S^s- and S^l- functions of C_2, which hold for any two functions φ^s_λ, $\varphi^s_{\lambda'}$ labelled by $\lambda, \lambda' \in \Lambda^s_M(C_2)$, and φ^l_λ, $\varphi^l_{\lambda'}$ labelled by $\lambda, \lambda' \in \Lambda^l_M(C_2)$, are of the form (2.5.3) and (2.5.4), respectively. The calculation procedure of the coefficients $\varepsilon(x)$, $h^\vee_\lambda$, which appear in (2.5.2),(2.5.3) and (2.5.4), is detailed in Section 3.7 in [**16**]*. The values of the coefficients $\varepsilon(x)$, $h^\vee_\lambda$ for $x \in F^s_M(C_2)$, $\lambda \in \Lambda^s_M(C_2)$ and for $x \in F^l_M(C_2)$, $\lambda \in \Lambda^l_M(C_2)$ are listed in Table 2.3. We represent each point $x \in F^s_M(C_2)$ and each weight $\lambda^s \in \Lambda_M(C_2)$ by the coordinates $[u^s_0, u^s_1, u^s_2]$ and $[t^s_0, t^s_1, t^s_2]$. Similarly, we represent each point $x \in F^l_M(C_2)$ and each weight $\lambda^l \in \Lambda_M(C_2)$ by the coordinates $[u^l_0, u^l_1, u^l_2]$ and $[t^l_0, t^l_1, t^l_2]$.*

2.5.2. Discrete S^s- and S^l-transforms

Analogously to ordinary Fourier analysis, we define interpolating functions I^s_M and I^l_M

$$I^s_M(x) \equiv \sum_{\lambda\in\Lambda^s_M} c^s_\lambda \varphi^s_\lambda(x), \qquad I^l_M(x) \equiv \sum_{\lambda\in\Lambda^l_M} c^l_\lambda \varphi^l_\lambda(x), \quad x \in \mathbb{R}^n \tag{2.5.5}$$

which are given in terms of expansion functions φ^s_λ and φ^l_λ and expansion coefficients c^s_λ, c^l_λ, whose values need to be determined. These interpolating functions

$x \in F_M^s(C_2)$	$\varepsilon(x)$
$[u_0^s, u_1^s, u_2^s]$	8
$[0, u_1^s, u_2^s]$	4
$[u_0^s, u_1^s, 0]$	4
$[0, u_1^s, 0]$	2

$\lambda \in \Lambda_M^s(C_2)$	$h_\lambda^\vee$
$[t_0^s, t_1^s, t_2^s]$	1
$[t_0^s, t_1^s, 0]$	2

$x \in F_M^l(C_2)$	$\varepsilon(x)$
$[u_0^l, u_1^l, u_2^l]$	8
$[u_0^l, 0, u_2^l]$	4

$\lambda \in \Lambda_M^l(C_2)$	$h_\lambda^\vee$
$[t_0^l, t_1^l, t_2^l]$	1
$[0, t_1^l, t_2^l]$	2
$[t_0^l, 0, t_2^l]$	2
$[0, 0, t_2^l]$	4

TABLE 2.3. The coefficients $\varepsilon(x)$ and $h_\lambda^\vee$ of C_2. All variables $u_0^s, u_1^s, u_2^s, t_0^s, t_1^s, t_2^s$ and $u_0^l, u_1^l, u_2^l, t_0^l, t_1^l, t_2^l$ are assumed to be natural numbers.

can also be understood as finite cut-offs of infinite expansions. Suppose we have some function f sampled on the grid F_M^s or F_M^l. The interpolation of f consists in finding the coefficients c_λ^s or c_λ^l in the interpolating functions (2.5.5) such that

$$\begin{aligned} I_M^s(x) =& f(x), \quad x \in F_M^s, \\ I_M^l(x) =& f(x), \quad x \in F_M^l. \end{aligned} \tag{2.5.6}$$

Relations (2.4.10) and (2.5.3), (2.5.4) allow us to view the values $\varphi_\lambda^s(x)$ with $x \in F_M^s$, $\lambda \in \Lambda_M^s$ and the values $\varphi_\lambda^l(x)$ with $x \in F_M^l$, $\lambda \in \Lambda_M^l$ as elements of non-singular square matrices. These invertible matrices coincide with the matrices of the linear systems (2.5.6). Thus, the coefficients c_λ^s and c_λ^l can be uniquely determined. The formulas for calculation of c_λ^s and c_λ^l, which we call discrete S^s- and S^l-transforms, can obtained by means of calculation of standard Fourier coefficients

$$\begin{aligned} c_\lambda^s =& \frac{\langle f, \varphi_\lambda^s \rangle_{F_M^s}}{\langle \varphi_\lambda^s, \varphi_\lambda^s \rangle_{F_M^s}} = (c\,|W|\,M^n h_\lambda^\vee)^{-1} \sum_{x \in F_M^s} \varepsilon(x) f(x) \overline{\varphi_\lambda^s(x)}, \\ c_\lambda^l =& \frac{\langle f, \varphi_\lambda^l \rangle_{F_M^l}}{\langle \varphi_\lambda^l, \varphi_\lambda^l \rangle_{F_M^l}} = (c\,|W|\,M^n h_\lambda^\vee)^{-1} \sum_{x \in F_M^l} \varepsilon(x) f(x) \overline{\varphi_\lambda^l(x)}, \end{aligned} \tag{2.5.7}$$

and the corresponding Plancherel formulas also hold

$$\begin{aligned} \sum_{x \in F_M^s} \varepsilon(x)\, |f(x)|^2 =& c\,|W|\,M^n \sum_{\lambda \in \Lambda_M^s} h_\lambda^\vee |c_\lambda^s|^2, \\ \sum_{x \in F_M^l} \varepsilon(x)\, |f(x)|^2 =& c\,|W|\,M^n \sum_{\lambda \in \Lambda_M^l} h_\lambda^\vee |c_\lambda^l|^2. \end{aligned}$$

2.6. Concluding remarks

- In view of the ever-increasing amount of digital data, practically the most valuable property of the orbit functions of C-, S-, S^l- and S^s-families is their discrete orthogonality. The four families are distinguished most notably by their behaviour at the boundary of F. The functions of S^l- and S^s-families do not have an analogue in one variable, i.e. rank 1 simple Lie group.
- The product of two S^s-functions or two S^l-functions with the same underlying Lie group and the same arguments $x \in \mathbb{R}^n$ but different dominant weights, say λ and λ', decomposes into the sum of C-functions:

$$\varphi^s_\lambda(x) \cdot \varphi^s_{\lambda'}(x) = \sum_{w \in W} \sigma^s(w) \Phi_{\lambda + w\lambda'}(x),$$
$$\varphi^l_\lambda(x) \cdot \varphi^l_{\lambda'}(x) = \sum_{w \in W} \sigma^l(w) \Phi_{\lambda + w\lambda'}(x).$$

  where Φ_λ denotes the (normalized) C-function $\Phi_\lambda = \varphi^{\mathbf{1}}_\lambda$.
- The present work raises the question under which conditions converge the functional series $\{I^s_M\}_{M=1}^\infty$, $\{I^l_M\}_{M=1}^\infty$ assigned to a function $f : F \to \mathbb{C}$ by the relations (2.5.5) and (2.5.7).
- In addition to the C- and S-functions, which are multidimensional generalizations of common cosine and sine functions, the E-functions generalizing the exponential functions [**27**] is also defined [**47**]. The E-functions also admit discrete orthogonality [**17, 43**]. For these "standard" E-functions, the kernel of the homomorphism σ^e, given by (2.2.6), is crucial. It turns out that there are altogether six types of E-functions once the kernels of the sign homomorphisms σ^s and σ^l are included in the definition. So far, these six types have been studied in full detail only for rank 2 Lie groups [**8**].
- A general one-to-one link between the orbit functions and orthogonal polynomials in n variables was pointed out in [**46**]. Extensive literature exists about orthogonal polynomials, although most of it pertains to 2-variable polynomials. It cannot be assumed that our Lie group defined polynomials of rank two were not taken into account. For a greater number of variables, not all of the polynomials defined from the simple Lie groups have been noticed. Discrete orthogonality of the polynomials in more than one variable is outside the scope of traditional approaches.

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Chapter 3

GAUSSIAN CUBATURE ARISING FROM HYBRID CHARACTERS OF SIMPLE LIE GROUPS

Authors: Robert V. Moody, Lenka Motlochová and Jiří Patera.

Abstract: Lie groups with two different root lengths allow two "mixed sign" homomorphisms on their corresponding Weyl groups, which in turn give rise to two families of hybrid Weyl group orbit functions and characters. In this paper we extend the ideas leading to the Gaussian cubature formulas for families of polynomials arising from the characters of irreducible representations of any simple Lie group, to new cubature formulas based on the corresponding hybrid characters. These formulas are new forms of Gaussian cubature in the short root length case and new forms of Radau cubature in the long root case. The nodes for the cubature arise quite naturally from the (computationally efficient) elements of finite order of the Lie group.

Introduction

It has long been known that the Chebyshev polynomials of the second kind are related to the representation theory of $SU(2)$, and of course to efficient methods of numerical quadrature. In [**35**] it was shown that there is a considerable generalization of this theory based on the series of lattices of type A_n (so that the original theory applied to the lattice of type A_1 and the representations of $SU(2)$). This generalization depended deeply on the Weyl groups of these lattices, but not particularly on the Lie groups associated with them. The resulting formulas, now for functions of n variables, went under the name of cubature formulas.

In [**44**] the idea that there is a genuine Lie theoretical connection here was extended to create a theory that works for every simple compact Lie group $\mathbb{G}$. The theory is again based on the root lattices but now also incorporates the representation theory of these groups in a deeper way, and more importantly uses the elements of finite order in the corresponding Lie group to define the nodes at which the cubature formulae are evaluated. The representations and the elements of finite order are in a sort of duality, and this duality plays a vital role in what happens. With a slight Lie-theoretical twist in the definition of the degrees of multi-variable polynomials, the crucial polynomials, their nodes and the cubature formulas appear completely naturally out of the theory and in fact are optimal (called Gaussian) in their efficiency.

The Weyl group W, which appears as a group of reflections in this theory, is of primary importance, notably its sign homomorphism $W \to \{\pm 1\}$ which takes the sign -1 for each of the reflections in the roots. It has long been known in the theory of orthogonal polynomials based on these reflection groups that in the cases where the simple Lie group has roots of two different lengths (namely for types B_n, C_n, F_4, G_2) there are, in addition, two hybrid sign functions which distinguish between reflections in long roots and reflections in short roots; that is, the sign function takes the value -1 for each reflection in a long root (respectively short root) and takes the value $+1$ on the reflections in the short (respectively long) roots.

In this paper we extend the ideas of Chebyshev polynomials, nodes, and cubature formulas to these hybrid situations. In principle the path should be straightforward, particularly since orthogonal polynomials and q-series based on this type of hybrid symmetry have been well studied, e.g. [**10**]. However, our theory depends on both the representations and the elements of finite order of the Lie group, and this somewhat intricate process requires making a number of correct decisions in how to define things to fit the new setting. In the end things work out as smoothly and as naturally as in [**44**], although for the long root case the cubature is slightly less efficient than in the Gaussian cubature of the standard and short root cases, being instead what is called Radau cubature.

The orientation of [**44**] was towards the approximation theory community since Gaussian formulas are rather rare and the Lie theoretical connections offer new and unexpected techniques for constructing them. In this paper, in addition to presenting the new results based on hybrid Weyl symmetry and simplifying the overall presentation of the ideas, the emphasis is more the other way around, aiming to introduce the Lie theoretical community to some new applications of simple Lie groups to approximation theory and cubature. It seems to us that there is more to be explored here, particularly the duality between elements of finite order and character theory.

3.1. Overview

We begin with a summary of the results of [**44**] and then introduce the ideas which lead to the new cubature formulas arising from the two new families of orbit functions.

Start with the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. This is given the structure of a graded ring by assigning a degree $d_j \in \mathbb{Z}^{>0}$ (called the m-degree, for reasons to be explained later) to each of the variables X_j. The degree of a monomial $X_1^{k_1} \cdots X_n^{k_n}$ is thus $k_1 d_1 + \cdots + k_n d_n$. Unlike the usual gradation, d_j need not be equal to 1. The value of n will ultimately be the rank of a compact simple Lie group $\mathbb{G}$ (or its complex simple Lie algebra $\mathfrak{g}$) and the degree structure will be given by the coefficients of its highest co-root.

The main result can be stated as a quadrature formula, called in this subject a cubature formula because it is not restricted to one dimension. Fix any non-negative integer M. Then for all $f \in \mathbb{C}[X_1, \dots X_n]$ of m-degree not exceeding $2M+1$,

$$(2\pi)^{-n} \int_\Omega f(X) K^{1/2}(X) dX = C \sum_{X \in \mathcal{F}_{M+h}} f(X) K(X)\,. \tag{3.1.1}$$

The main point is that integration is replaced by finite summing, and the elements of $\mathcal{F}_{M+h}$ over which the summation takes place are very easy to compute. Here $X = (X_1, \dots, X_n) \in \mathbb{C}^n$ and $\mathcal{F}_{M+h}$ is a finite subset of $\mathbb{C}^n$, C is a constant, K is a special polynomial in $\mathbb{C}[X_1, \dots, X_n]$ which is positive valued on $\Omega \subset \mathbb{C}^n$. All of these objects depend on the choice of $\mathbb{G}$. In the hybrid situation that we shall develop here, the variables $X^s = (X_1^s, \dots, X_n^s)$ and similarly $X^l = (X_1^l, \dots, X_n^l)$ are real valued.

The elements of $\mathcal{F}_{M+h}$ actually arise from elements of $\mathbb{G}$ finite order, but in this context they are called the nodes, and they have a number of special properties. Their number is exactly the dimension of the space of polynomials of m-degree at most M. Furthermore, an important part of the construction of this result is the introduction of special polynomials (related to characters and other $\mathbb{G}$-invariant functions on $\mathbb{G}$) $X_\lambda = X_{(\lambda_1, \dots, \lambda_n)}$ of m-degree $|\lambda|_m \equiv \lambda_1 d_1 + \cdots + \lambda_n d_n$, which form an orthogonal basis of $\mathbb{C}[X_1, \dots, X_n]$ with respect to the inner product

$$\langle f, \overline{g} \rangle_K \equiv (2\pi)^{-n} \int_\Omega f \overline{g} K^{1/2}\,, \tag{3.1.2}$$

which in view of (3.1.1) is $\sum_{X \in \mathcal{F}_{M+h}} f(X) \overline{g(X)} K(X)$ if the m-degrees of f, g do not exceed M. Now, the minimum number of nodes that could achieve such an orthogonal decomposition of these functions is the dimension of the space of polynomials of m-degree at most M, and that is exactly the number of elements in $\mathcal{F}_{M+h}$. This optimal situation is called Gaussian cubature [**35**].

The nodes are actually zeros of certain of these polynomials of degree $M+1$. The region Ω is the image of the interior of the fundamental region (or some modified version of it in the hybrid cases) under a certain polynomial map. In particular it is an open set with compact closure and boundary of measure 0.

If we move to the Hilbert space $L^2_K(\Omega)$ of square integrable functions on Ω with respect to the inner product $\langle\cdot,\cdot\rangle_K$ then every function $f \in L^2_K(\Omega)$ has a Fourier expansion $f = \sum_\lambda \langle f, X_\lambda\rangle_K X_\lambda$, equality here being in the usual L^2 sense. If the sum is truncated to

$$\sum_{|\lambda|_m \leq M} \langle f, X_\lambda\rangle X_\lambda$$

then this is the best approximation to f in the L^2_K-norm using only polynomials of m-degree at most M.

In essence what we have been describing arises from a duality that exists between the characters of the representations of $\mathbb{G}$ and the conjugacy classes of elements of finite order of $\mathbb{G}$. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. Since all the maximal tori are conjugate and every conjugacy class of $\mathbb{G}$ meets every one of them, every character of $\mathbb{G}$ is defined entirely by its restriction to $\mathbb{T}$ and every conjugacy class of elements of finite order has elements in $\mathbb{T}$. The relationship between $\mathbb{G}$ and its Lie algebra restricts to the relationship between $\mathbb{T}$ and its Lie algebra:

$$\exp 2\pi i(\cdot) : \mathfrak{t} \to \mathbb{T}\,. \tag{3.1.3}$$

Here it is more convenient to let $i\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$ because the Killing form is then positive definite on $\mathfrak{t} \simeq \mathbb{R}^n$, where n is the rank of $\mathbb{G}$. The kernel of this exponential mapping is the co-root lattice $Q^\vee$ of $\mathbb{G}$, so $\mathbb{T} \simeq \mathbb{R}^n/Q^\vee$. The $\mathbb{Z}$-dual of $Q^\vee$ in $\mathfrak{t}^*$ is the weight lattice P.

The normalizer N of $\mathbb{T}$ in $\mathbb{G}$ is always larger than $\mathbb{T}$ itself, and the Weyl group $W \equiv N/\mathbb{T}$ is the group that represents this excess. W acts on $\mathbb{T}$ via conjugation and then as linear transformations on $\mathfrak{t}$. The affine Weyl group is then the semi-direct product of $W_{\text{aff}} = W \ltimes Q^\vee$, which acts on $\mathfrak{t}$ with $Q^\vee$ acting as translations.

Let F be a standard simplicial fundamental region for W_{aff} in $\mathfrak{t}$, so that W_{aff} is generated by the reflections in the faces of F and W is generated by the reflections in the faces of F that pass through the origin, see [**2**]. The virtue of F is that it perfectly parametrizes the conjugacy classes of $\mathbb{G}$: for each such class there is a unique element of $x \in F$ for which $\exp(2\pi i x)$ lies in that class.

The characters on $\mathbb{G}$ restrict faithfully to W-invariant functions on $\mathbb{T}$, and the ring of all W-invariant functions on $\mathbb{T}$ is a polynomial ring in n-variables generated by the characters of a set of so-called fundamental representations. This is the ring $\mathbb{C}[X_1, \dots X_n]$ and the X_j can be viewed either abstractly as variables or as actual characters corresponding to a system of fundamental weights. One particularly important W-invariant function on $\mathbb{T}$ is $K \equiv |S_\varrho|^2$ where S_ϱ is the basic skew-symmetric function that appears as the denominator of Weyl's character formula. This is the K of the cubature formula.

Via the exponential mapping the characters can be viewed as W_{aff}-invariant functions on $\mathfrak{t}$. In this way we have the important mapping

$$\Xi : \mathfrak{t} \longrightarrow \mathbb{C}^n \qquad x \mapsto (X_1(\exp(2\pi i x)), \ldots, X_n(\exp(2\pi i x))) \tag{3.1.4}$$

The region Ω is the image of the interior F° of F under Ξ.

Remark 3.1.1. *There are several points of possible confusion regarding the many functions that appear in the paper. First of all there are many functions, like S_ϱ, which have interpretations as functions both on $\mathbb{T}$ and on $\mathfrak{t}$. This is not particularly troublesome since $\mathbb{T} \simeq \mathbb{R}^n/Q^\vee$ and all these functions are clearly periodic on $\mathfrak{t}$ with respect to $Q^\vee$. Thus interpreting $S_\varrho \overline{S_\varrho}$ as a function on $\mathfrak{t}$ or $\mathbb{T}$ is rather obvious.*

The second is the transition from exponential sums to new coordinates in $\mathbb{C}^n$ using characters (or hybrid characters) as new variables. This is the way in which the Lie theory translates over into a theory about polynomials where the cubature formulas are relevant. Rather than introduce new function names when we transition variables, we use different notation for the variables. Thus for functions on $\mathfrak{t}$ or $\mathbb{T}$ the generic variable name is $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, whereas for the new polynomial variables the generic variable name is $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$. When we deal with short and long root scenarios, as we mostly do in what follows, we use $X^s = (X_1^s, \ldots, X_n^s)$ in the short root case, and similarly for the long case.

There remains to briefly introduce the elements of finite order of $\mathbb{T}$. Each conjugacy class of an element of finite order has a unique representative in F. The set $\mathcal{F}_{M+h} \subset \Omega$ is the image under Ξ of the set of elements in F that have adjoint order $M+h$. Here h is the Coxeter number of $\mathbb{G}$ and by adjoint order we mean that the order of the element is $M+h$ in the adjoint representation of $\mathbb{G}$ on itself (i.e. by conjugation). The full order of an element is a finite multiple of the adjoint order.

This finishes our brief tour of the constituents of the basic cubature formula.

The Weyl group is a subgroup of the orthogonal group of $\mathfrak{t}$ with respect to its canonical Euclidean structure arising from the Killing form, and in particular there is the sign homomorphism

$$\sigma : W \longrightarrow \{\pm 1\} \qquad w \mapsto \sigma(w) = \det(w)\,,$$

with $\sigma(r) = -1$ for all reflections. The fact that W is generated by the reflections in the roots of the Lie algebra plays an essential role in elucidating the structure of simple Lie groups and their representations. Throughout, W-skew invariant functions and polynomials play a key role, Weyl's character formula being a typical example which expresses the characters (W-invariant exponential sums) as ratios of W-skew invariant exponential sums. In the case when the roots of the Lie algebra have two distinct lengths (called the short and long roots), there are two alternative hybrid sign homomorphisms: σ^s which is defined by taking the value -1 on the reflections in short roots and the value $+1$ on the reflections in long roots, and σ^l which does it the other way around. This gives rise to new hybrid invariants, skew invariant with respect to short reflections while being invariant with respect to long, or vice-versa. This leads to two new versions of each cubature formula, see (3.5.2) which say very much the same thing except that $\Omega, K, \mathcal{F}_{M+h}, C$ and a new function κ, all appear in short and long forms according to which hybrid symmetry is used. The effect is somewhat subtle: Ω is only altered along its boundary, the set $\mathcal{F}_{M+h}$ changes only by certain elements of finite order along the boundary of the fundamental region F, and the polynomial ring is still the space of W-invariant functions. However the interpretations of the variables X_j in terms of characters and the function K are significantly altered.

3.2. Basics

We establish the notation that we are using and recall some basic facts about simple Lie algebras. For more details, see for example [**23**].

3.2.1. Simple Lie algebras

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n with corresponding simple and simply connected compact Lie group $\mathbb{G}$. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$ and let $i\mathfrak{t}$ be its Lie algebra, so that we have the exponential map (3.1.3). Let $(\cdot \mid \cdot)$ on the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ be defined from the Killing form by duality. The natural pairing of $\mathfrak{t}^*$ and $\mathfrak{t}$ is denoted by $\langle \cdot , \cdot \rangle$.

Let Π denote the set of roots of $\mathfrak{g}$ and let $\Delta \equiv \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{t}^*$ be a set of simple roots, hence also a basis of $\mathfrak{t}^* \simeq \mathbb{R}^n$. We denote by C the corresponding Cartan matrix with entries

$$C_{ij} = \frac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)} .$$

Its determinant, denoted by $c_{\mathfrak{g}}$, is the order of the centre of $\mathbb{G}$ and is also the index of the root (co-root) lattice in side the weight (co-weight) lattice, see below.

We introduce the usual partial ordering on $\mathfrak{t}^*$: $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is non-negative integer sum of simple roots. The highest root in Π with respect to this ordering is denoted ξ. Its coordinates in the α-basis are called the marks:

$$\xi = m_1\alpha_1 + \cdots + m_n\alpha_n \,. \tag{3.2.1}$$

Let $Q, P \subset \mathfrak{t}^*$ be the root lattice and weight lattice respectively. Then

$$P = \left\{\lambda \in \mathfrak{t}^* \mid \langle \lambda, \alpha_j^\vee \rangle \in \mathbb{Z} \text{ for } \forall \alpha_j^\vee, j = 1, \dots, n\right\} ,$$

where $\Delta^\vee \equiv \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is the system of simple co-roots (which forms a basis in $\mathfrak{t}$) defined by

$$\langle \alpha_i, \alpha_j^\vee \rangle = C_{ij} \qquad \text{for } i, j = 1, \dots, n \,.$$

To these simple co-roots corresponds the system of co-roots $\Pi^\vee$, which is in fact the root system for the simple Lie algebra with Cartan matrix C^T (although this algebra never makes any real appearance in what follows). We have the highest co-root in $\eta \in \Pi^\vee$ and giving the co-marks $m_j^\vee$:

$$\eta = m_1^\vee\alpha_1^\vee + \cdots + m_n^\vee\alpha_n^\vee \,.$$

It is these co-marks that define the degree function on $\mathbb{C}[X_1, \dots, X_n]$ later.

The lattice P has as a basis the set of fundamental weights ω_i which is dual to the co-root basis in the sense that

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \qquad \text{for } i, j = 1, \dots, n \,.$$

This is so called ω-basis of $\mathfrak{t}^*$ that we will use.

We also have two lattices in $\mathfrak{t}$ denoted $Q^\vee$ and $P^\vee$. The co-root lattice $Q^\vee$ is kernel of the exponential map (3.1.3) with $\mathbb{Z}$-basis consisting of the $\alpha_i^\vee$. The co-weight lattice $P^\vee$ is the $\mathbb{Z}$-dual of Q in $\mathfrak{t}$ and has as a basis the set of fundamental co-weights $\omega_j^\vee$ defined by

$$\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij} \qquad \text{for } i, j = 1, \dots, n \,.$$

The relationships between the lattices and between the various root and weight bases and their co-equivalents are summarized in:

$$\begin{array}{ccccccc}
\{\alpha_1, \dots, \alpha_n\} & \subset & Q & & Q^\vee & \supset & \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \\
 & & \cap & \times & \cap & & \\
\{\omega_1, \dots, \omega_n\} & \subset & P & & P^\vee & \supset & \{\omega_1^\vee, \dots, \omega_n^\vee\} \\
 & & \cap & & \cap & & \\
 & & \mathfrak{t}^* & & \mathfrak{t} & &
\end{array}$$

Here the times symbol is meant to indicate that Q and $P^\vee$, as well as P and $Q^\vee$, are in $\mathbb{Z}$-duality with each other.

Finally we have the cone $P^+ \subset P$ of dominant weights:

$$P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\omega_n \,.$$

3.2.2. Affine Weyl group and its dual

The Weyl group acting on $\mathfrak{t}$ is generated by simple reflections $r_1, \ldots, r_n$ in the hyperplanes

$$H_i \equiv \{x \in \mathfrak{t} \mid \langle \alpha_i, x \rangle = 0\} \;, \qquad i = 1, \ldots, n$$

by

$$r_i(x) \equiv x - \langle \alpha_i, x \rangle \alpha_i^\vee \,.$$

By duality, we have the action of W on $\mathfrak{t}^*$ where the simple reflections on co-root side are given by

$$r_i(\lambda) \equiv \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \,.$$

The affine Weyl group is the semi-direct product of W and the translation group $Q^\vee$: $W_{\text{aff}} = W \ltimes Q^\vee$. Equivalently, W_{aff} can be defined as the group generated by the simple reflections r_i and the affine reflection r_0 given by

$$r_0(x) = r_\xi(x) + \xi^\vee \,, \qquad r_\xi(x) = x - \langle \xi, x \rangle \xi^\vee$$

where ξ is the highest root of Π.

The standard simplex F in $\mathbb{R}^n$ defined by

$$F = \{x \mid \langle \alpha_j, x \rangle \geq 0 \;\text{ for all }\; j = 1, \ldots, n, \quad \langle \xi, x \rangle \leq 1\} \,,$$

serves as a fundamental domain for the affine Weyl group. Its vertices are

$$F = \left\{0, \tfrac{1}{m_1}\omega_1^\vee, \ldots, \tfrac{1}{m_n}\omega_n^\vee\right\} \,, \tag{3.2.2}$$

where m_i, $i = 1, 2, \ldots, n$, are the marks (3.2.1). Note that r_0 is the reflection in the hyperplane H_0

$$H_0 \equiv \{x \in \mathfrak{t} \mid \langle \xi, x \rangle = 1\} \;.$$

3.2.3. Long and short roots

In dealing with the hybrid cases, we are only interested in the simple Lie algebras with two different lengths of roots:

$$B_n \; (n \geq 3) \,, \qquad C_n \; (n \geq 2) \,, \qquad F_4 \,, \qquad G_2 \,.$$

The root system Π of such algebras consists of short roots Π^s and long roots Π^l, so $\Pi = \Pi^l \cup \Pi^s$. Similarly, we decompose the set of simple roots Δ as $\Delta = \Delta^l \cup \Delta^s$

where $\Delta^l \equiv \Delta \cap \Pi^l$ and $\Delta^s \equiv \Delta \cap \Pi^s$. Our indexing of the simple roots is such that

$$\Delta^l(B_n) \ni \alpha_1, \ldots, \alpha_{n-1} \qquad \Delta^l(C_n) \ni \alpha_n,$$
$$\Delta^l(F_4) \ni \alpha_1, \alpha_2 \qquad \Delta^l(G_2) \ni \alpha_1 \,.$$

Since Π^s and Π^l are stabilized by W and span $\mathfrak{t}^*$, they both form root systems in $\mathfrak{t}^*$. Although we do not use the facts here, it is known that Π^l is the root system of a semisimple subalgebra of the simple Lie algebra $\mathfrak{g}$ belonging to Π and Π^s is the root system of a subjoined semisimple Lie algebra [**45, 48**], which is usually not a subalgebra of $\mathfrak{g}$.

$$\Pi^s \text{ is of type } \begin{cases} nA_1 \text{ in } B_n \\ D_n \text{ in } C_n \\ D_4 \text{ in } F_4 \\ A_2 \text{ in } G_2 \end{cases}, \qquad \Pi^l \text{ is of type } \begin{cases} D_n \text{ in } B_n \\ nA_1 \text{ in } C_n \\ D_4 \text{ in } F_4 \\ A_2 \text{ in } G_2 \end{cases}, \tag{3.2.3}$$

where nA_1 denotes the semisimple Lie algebra, $nA_1 = A_1 \times \cdots \times A_1$, ($n$ factors). In (3.2.3) we use the isomorphisms $D_2 \simeq A_1 \times A_1$ and $D_3 \simeq A_3$.

We define the set of positive short and positive long roots by

$$\Pi^s_+ \equiv \Pi^s \cap \Pi_+ \,, \quad \Pi^l_+ \equiv \Pi^l \cap \Pi_+$$

respectively, where Π_+ denotes the roots α of Π which satisfy $0 \preceq \alpha$.

Proposition 3.2.1. Π^t_+ *is a system of positive roots for* Π^t *where* $t \in \{s, l\}$.

Proof. All systems of positive roots in any root system Σ arise as

$$\Sigma_+ = \{\alpha \in \Sigma \mid (\nu \mid \alpha) > 0\}$$

for some ν in the span of Σ [**54**]. Now with ϱ being half the sum of the positive roots of Π, we have $\Pi_+ = \{\alpha \in \Pi \mid (\varrho \mid \alpha) > 0\}$. Then

$$\Pi^t_+ = \Big\{\alpha \in \Pi^t \mid \alpha \in \Pi_+\Big\} = \Big\{\alpha \in \Pi^t \mid (\varrho \mid \alpha) > 0\Big\}.$$

So Π^t_+ is a positive root system. $\square$

The highest long root γ^l of Π^l coincides with the highest root ξ of Π. So, the coefficients of γ^l written in α-basis are the marks m_i, $\gamma^l = m_1\alpha_1 + \cdots + m_n\alpha_n$, see Table 3.1. The highest short root of Π^s denoted γ^s is given by its coefficients m^s_i in α-basis, $\gamma^s = m^s_1\alpha_1 + \cdots + m^s_n\alpha_n$, see Table 3.1.

The dual root system $\Pi^\vee$ decomposes also as disjoint union of short co-roots $\Pi^{\vee s}$ and long co-roots $\Pi^{\vee l}$. The dual of γ^l is the highest short co-root $\gamma^{l\vee} = m^{l\vee}_1\alpha^\vee_1 + \cdots + m^{l\vee}_n\alpha^\vee_n$. Note: we label the highest short root with “l” to express the duality with the highest long root. Similarly, the dual of γ^s is the highest long co-root $\gamma^{s\vee} = m_1^\vee \alpha_1^\vee + \cdots + m_n^\vee \alpha_n^\vee$. The values of $m_i^\vee$ and $m_i^{l\vee}$ are written in Table 3.1.

Δ	$m_1, \dots, m_n$	$m_1^s, \dots, m_n^s$	$m_1^{l\vee}, \dots, m_n^{l\vee}$	$m_1^\vee, \dots, m_n^\vee$
B_n	$1, 2, \dots, 2$	$1, \dots, 1$	$1, 2, \dots, 2, 1$	$2, 2, \dots, 2, 1$
C_n	$2, \dots, 2, 1$	$1, 2, \dots, 2, 1$	$1, \dots, 1$	$1, 2, \dots, 2$
F_4	$2, 3, 4, 2$	$1, 2, 3, 2$	$2, 3, 2, 1$	$2, 4, 3, 2$
G_2	$2, 3$	$1, 2$	$2, 1$	$3, 2$

TABLE 3.1. The numbers m_i and m_i^s are the coefficients of the highest long root γ^l and highest short root γ^s, written in the standard basis of simple roots. Similarly, $m_i^{l\vee}$ and $m_i{}^\vee$ are the coefficients of the duals of the γ^l and γ^s, written in the basis of simple co-roots.

Δ	ϱ^l	ϱ^s	h^l	h^s
B_n	$1, \dots, 1, 0$	$0, \dots, 0, 1$	$2n-2$	2
C_n	$0, \dots, 0, 1$	$1, \dots, 1, 0$	2	$2n-2$
F_4	$1, 1, 0, 0$	$0, 0, 1, 1$	6	6
G_2	$1, 0$	$0, 1$	3	3

TABLE 3.2. The columns ϱ^l and ϱ^s are the coefficients of the half-sums of the positive long and short roots, written in the basis of fundamental weights. The numbers h^s and h^l denote the numbers (3.2.7).

A function

$$k : \alpha \in \Pi \to k_\alpha \in \mathbb{R}$$

for which $k_\alpha = k_{w(\alpha)}$ for $w \in W$ is called a multiplicity function [**10**]. The trivial example is $k_\alpha = 1$ for all $\alpha \in \Pi$ which we denote simply by k^1. Relevant for us are

$$\begin{aligned} k^l: &\quad k^l_\alpha \equiv 1 \quad \text{for } \alpha \in \Pi^l \quad \text{and} \quad k^l_\alpha \equiv 0 \quad \text{for } \alpha \in \Pi^s, \quad \text{and} \\ k^s: &\quad k^s_\alpha \equiv 0 \quad \text{for } \alpha \in \Pi^l \quad \text{and} \quad k^s_\alpha \equiv 1 \quad \text{for } \alpha \in \Pi^s\,. \end{aligned} \tag{3.2.4}$$

Defining

$$\varrho(k) \equiv \frac{1}{2} \sum_{\alpha \in \Pi_+} k_\alpha \alpha\,, \tag{3.2.5}$$

we see that in addition to the usual half-sum of the positive roots $\varrho = \varrho(k_0) = \frac{1}{2}\sum_{\alpha\in\Pi_+} \alpha = \sum_{i=1}^n \omega_i$ we have

$$\varrho^s \equiv \varrho(k^s) = \frac{1}{2} \sum_{\alpha \in \Pi^s_+} \alpha = \sum_{\alpha_i \in \Delta^s} \omega_i\,, \quad \varrho^l \equiv \varrho(k^l) = \frac{1}{2} \sum_{\alpha \in \Pi^l_+} \alpha = \sum_{\alpha_i \in \Delta^l} \omega_i\,. \tag{3.2.6}$$

To ϱ^s and ϱ^l correspond the important short and long Coxeter numbers h^s and h^l defined by

$$h^s \equiv \langle \varrho^s, \gamma^{s\vee} \rangle + 1\,, \qquad h^l \equiv \langle \varrho^l, \gamma^{l\vee} \rangle + 1\,. \tag{3.2.7}$$

The explicit calculations using the values in Table 3.2 imply that

$$h^s = 1 + \sum_{\alpha_i \in \Pi^s} m_i^\vee = \sum_{\alpha_i \in \Pi^s} m_i\,, \qquad h^l = \sum_{\alpha_i \in \Pi^l} m_i^\vee = 1 + \sum_{\alpha_i \in \Pi^l} m_i\,. \tag{3.2.8}$$

3.3. W-invariant and W-skew invariant functions on $\mathbb{T}$

3.3.1. Sign homomorphisms

In addition to the usual sign homomorphisms on the Weyl group W there are two others. This is well known, but since it is short we prove it. An abstract presentation determining W is

$$\langle r_1, \dots, r_n \mid r_i^2 = 1,\, (r_i r_j)^{a_{ij}} = 1,\, i,j = 1,\dots,n,\, i \neq j \rangle\,,$$

where $a_{ij} = 2,3,4,6$ according as nodes i and j in the Coxeter-Dynkin diagram are not joined, joined by a single bond, a double bond, or a triple bond. Any homomorphism $\sigma : W \to \{\pm 1\}$ is determined by the values on the generators r_i, $i = 1,\dots,n$. The necessary and sufficient condition for σ to be a homomorphism is that $(\sigma(r_i)\sigma(r_j))^{a_{ij}} = 1$ for all $i \neq j$. This is automatically satisfied if a_{ij} is even. When a_{ij} is odd, i.e. $a_{ij} = 3$, we need $\sigma(r_i) = \sigma(r_j)$. Looking at the Coxeter-Dynkin diagrams we see that this allows precisely one choice of sign for all the short reflections and one for all the long reflections, and no other. Note that it does not matter whether or not we have a reflection in simple root or in any root since for any two roots α, β of the same length there exists $w \in W$ such that $r_\alpha = w r_\beta w^{-1}$ which implies $\sigma(r_\alpha) = \sigma(r_\beta)$. Thus there are four homomorphisms σ:

$$\begin{aligned}
\mathrm{id}\ &: \quad \text{all signs equal to } 1 && \text{(the trivial homomorphism);}\\
\det\ &: \quad \text{all signs equal to } -1 && \text{(the determinant);}\\
\sigma^l\ &: \quad \text{all long signs equal to } -1, && \text{all short signs equal to } 1;\\
\sigma^s\ &: \quad \text{all short signs equal to } -1, && \text{all long signs equal to } 1.
\end{aligned} \tag{3.3.1}$$

We shall use all four homomorphisms to introduce various classes of W-orbit functions.

3.3.2. C-, S- , S^l- and S^s-functions

Let us fix the notation for the functions of the four families of W-orbit functions given by the homomorphisms (3.3.1). At first recall the definition of C- and S-functions which were studied in [**24, 26**].

$$\begin{aligned} C_\lambda(x) &= \sum_{\mu\in O(\lambda)} e^{2\pi i\langle\mu,x\rangle}\,, \\ S_{\lambda+\varrho}(x) &= \sum_{w\in W} \det(w) e^{2\pi i\langle w(\lambda+\varrho),x\rangle} = \sum_{\mu\in O(\lambda+\varrho)} \sigma(\mu) e^{2\pi i\langle\mu,x\rangle}\,. \end{aligned} \tag{3.3.2}$$

Here the parameter $\lambda\in P^+$ is a dominant weight, the variable $x\in\mathbb{R}^n$, $O(\lambda)$ is the W orbit of λ, and $\sigma(\mu)\equiv\sigma(w)$ where $\mu=w(\lambda+\varrho)$. Then $|O(\lambda)|=|W|/|\mathrm{stab}_W\lambda|$ is the number of points in $O(\lambda)$ where $|W|$ denotes the order of the Weyl group and $|\mathrm{stab}_W\lambda|$ is the number of points in the stabilizer in W of λ. For S-functions, the summation is in fact over the whole of W since $\lambda+\varrho$ has a trivial stabilizer.

When there are two different root lengths there are two other orbit functions, arising from the homomorphisms σ^s and σ^l:

$$S^s_{\lambda+\varrho^s}(x) = \sum_{\mu\in O(\lambda+\varrho^s)} \sigma^s(\mu) e^{2\pi i\langle\mu,x\rangle}\,, \qquad S^l_{\lambda+\varrho^l}(x) = \sum_{\mu\in O(\lambda+\varrho^l)} \sigma^l(\mu) e^{2\pi i\langle\mu,x\rangle}\,,$$

where ϱ^s,ϱ^l are given by (3.2.6). Here again we are defining $\sigma^s(\mu)\equiv\sigma^s(w)$ for $w\in W$ such that $\mu=w(\lambda+\varrho^s)$ and $\sigma^l(\mu)\equiv\sigma^l(w)$ for $w\in W$ such that $\mu=w(\lambda+\varrho^l)$. This makes sense because the stabilizer in W of ϱ^s is generated by long reflections r_i, so σ^s takes the constant value 1 on the stabilizer. Similarly, $\sigma(\mu)$ in (3.3.2) and $\sigma^l(\mu)$ are well defined.

Evidently the C-functions are W invariant while the S-functions (respectively S^s-, S^l-functions) are det (respectively σ^s, σ^l)-skew invariant.

All of these functions can be viewed as functional forms of formal exponential sums from $\mathbb{C}[P]$ of all linear combinations of formal exponentials e^μ with $\mu\in P$. In fact they are in $\mathbb{Z}[P]$ since all the coefficients are integers. We write $\mathbb{C}[P]^W$ (respectively $\mathbb{C}[P]^s$, $\mathbb{C}[P]^l$) for the W-invariant (respectively σ^s, σ^l-skew invariant) exponential sums, and similarly for the corresponding integral forms. More about the relationship between the formal exponentials and their use as functions may be found in [**44**].

The functions of $\mathbb{C}[P]$, as we have defined them are functions on $\mathbb{R}^n$. However, since they are periodic modulo $Q^\vee$, they may be considered as functions on $\mathbb{T}\simeq\mathbb{R}^n/Q^\vee$. This is the way in which we shall normally think of them. For integration purposes, an integral over $\mathbb{T}$ rewrites to an integral over a fundamental domain for the lattice $Q^\vee$, for instance $\{\sum_{j=1}^n x_j\alpha_j^\vee\ :\ 0\le x_j<1 \text{ for all } j\}$.

For notational convenience we use

$$\varphi_\mu : x \mapsto e^{2\pi i\langle \mu, x\rangle}\,, \tag{3.3.3}$$

which for each weight $\mu \in P$ combines the exponential mapping $x \mapsto \exp(2\pi i x)$ of $\mathfrak{t}$ to $\mathbb{T}$ and the $\mathbb{C}$-mapping $\exp(2\pi i x) \mapsto e^{2\pi i\langle \mu, x\rangle}$ on $\mathbb{T}$. As we have just said, we may think of φ_μ as a function on $\mathbb{T}$.

We note specially that the S^s- and S^l-functions are sums over orbits rather than sums over the entire Weyl group. Obviously they can be rewritten as Weyl group sums, but in general these are redundant and for what follows the orbit sums are what we need. They also may be interpreted as functions on $\mathbb{T}$ since they are invariant by $Q^\vee$-translations.

Proposition 3.3.1.

$$S^s_{\varrho^s}(x) = \prod_{\alpha\in\Pi^s_+}\left(e^{\pi i\langle\alpha,x\rangle} - e^{-\pi i\langle\alpha,x\rangle}\right), \quad S^l_{\varrho^l}(x) = \prod_{\alpha\in\Pi^l_+}\left(e^{\pi i\langle\alpha,x\rangle} - e^{-\pi i\langle\alpha,x\rangle}\right).$$

Proof. We show the result for $S^s_{\varrho^s}$, the proof for $S^l_{\varrho^l}$ is similar. Let W^s denote the Weyl group generated by short reflections and W^l the Weyl group generated by long reflections. Then W can be written as a semi-direct product $W \simeq V^l \ltimes W^s$ where V^l is a subgroup of W^l. We know that the stabilizer of ϱ^s is generated by long reflections, so $O(\varrho^s) = W^s(\varrho^s)$ and

$$S^s_{\varrho^s}(x) = \sum_{w\in W^s}\sigma^s(w)e^{2\pi i\langle w(\varrho^s),x\rangle}\,.$$

Thus the result is simply the usual formula that holds for all root systems. □

We are especially interested in the hybrid-characters:

$$\chi^l_\lambda(x) = \frac{S^l_{\lambda+\varrho^l}(x)}{S^l_{\varrho^l}(x)}\,, \qquad \chi^s_\lambda(x) = \frac{S^s_{\lambda+\varrho^s}(x)}{S^s_{\varrho^s}(x)}\,. \tag{3.3.4}$$

They are clearly W-invariant and we shall see that their linear span is $\mathbb{C}[P]^W$. In particular they are well defined functions on all of $\mathfrak{t}$ (and, of course, they can be considered as functions on $\mathbb{T}$). The hybrid characters for the fundamental weights $\omega_1, \dots \omega_n$ also generate $\mathbb{C}[P]^W$ as a ring, and the main point is that they will become the new variables $X^s_1, \dots, X^s_n$ and $X^l_1, \dots, X^l_n$. In fact these hybrid characters are in $\mathbb{Z}[P]^s$ and $\mathbb{Z}[P]^l$ and what we just said applies at the level of these rings. These facts are well known, but because of their central importance here we sketch out the proofs in what follows.

Proposition 3.3.2. $\mathbb{Z}[P]^s = \mathbb{Z}[P]^W S^s_{\varrho^s}, \quad \mathbb{Z}[P]^l = \mathbb{Z}[P]^W S^l_{\varrho^l}$.

PROOF. Inclusions in one direction are obvious. We show the reverse inclusion in the short case. Let $f \in \mathbb{Z}[P]^s$ and write $f = \sum_{\mu \in P} c_\mu e^\mu$. Let $\alpha \in \Pi_+^s$. Then $-f = r_\alpha f = \sum c_\mu e^{r_\alpha \mu}$ and so, $-f = \sum -c_{r_\alpha \mu} e^{r_\alpha \mu} = \sum -c_\mu e^\mu = \sum c_\mu e^{r_\alpha \mu}$.

Thus we can divide $\{\mu \mid c_\mu \neq 0\}$ into pairs $\{\mu_1, \mu_2\}$ where $\mu_2 = r_\alpha \mu_1$, $c_{\mu_2} = -c_{\mu_1}$, and $\mu_1 \preceq \mu_2$ (if $\mu_1 = \mu_2$ then $c_\mu = -c_\mu$, so $c_\mu = 0$). Thus

$$f = \sum_{\mu \in S} c_\mu \left(e^\mu - e^{\mu - z_\alpha \alpha}\right)$$

for some finite subset $S \subset P$.

Since $e^\mu - e^{\mu - z_\alpha \alpha} = e^\mu \left(1 - e^{-z_\alpha \alpha}\right)$ and $\left(1 - e^{-\alpha}\right)$ is always a factor of $\left(1 - e^{-z_\alpha \alpha}\right)$, we obtain $f = \left(1 - e^{-\alpha}\right) f_\alpha$ for some $f_\alpha \in \mathbb{Z}[P]$; and this statement is true for every $\alpha \in \Pi_+^s$. Now using [**2**] Ch.6, we have that $\{1 - e^{-\alpha} \mid \alpha \in \Pi_+\}$ are all relatively prime, and hence from $\left(1 - e^{-\alpha}\right) \mid f$ for each $\alpha \in \Pi_+^s$ we obtain $\Pi_{\alpha \in \Pi_+^s} \left(e^{\alpha/2} - e^{-\alpha/2}\right) \mid f$. The result now follows. $\square$

3.3.3. Domains F^s and F^l

The S^s-functions are σ^s-skew invariant and are also translationally invariant with respect to $Q^\vee$. As such they are determined entirely by their restriction to the fundamental region F. Because of Propositions. 3.3.1 and 3.3.2, the S^s-functions vanish on the root hyperplanes of F that correspond to the short roots, namely on $H^s \equiv \bigcup_{\alpha_j \in \Pi^s} H_j$. Define $F^s \equiv F \setminus H^s$. We shall be interested in the S^s-functions and their corresponding hybrid characters on this new domain.

All this can be done for the S^l-functions too, and we define

$$H^l \equiv H_0 \cup \bigcup_{\alpha_j \in \Pi^l} H_j \quad \text{and} \quad F^l \equiv F \setminus H^l\,.$$

Note that the hyperplane H_0 appears in this case, since it is always associated with reflection in a long root.

Using (3.2.2) and requiring

$$\begin{aligned} &y_i^s \in \mathbb{R}^{>0} \text{ if } \alpha_i \in \Delta^s &&\text{and}\quad y_0^s, y_i^s \in \mathbb{R}^{\geq 0} \text{ if } \alpha_i \in \Delta^l\,,\\ &y_0^l, y_i^l \in \mathbb{R}^{>0} \text{ if } \alpha_i \in \Delta^l &&\text{and}\quad y_i^l \in \mathbb{R}^{\geq 0} \text{ if } \alpha_i \in \Delta^s \end{aligned}$$

the domains F^s and F^l can be described by

$$\begin{aligned} F^s &= \left\{ y_1^s \omega_1^\vee + \cdots + y_n^s \omega_n^\vee \mid y_0^s + \sum_{i=1}^n m_i y_i^s = 1 \right\}\,,\\ F^l &= \left\{ y_1^l \omega_1^\vee + \cdots + y_n^l \omega_n^\vee \mid y_0^l + \sum_{i=1}^n m_i y_i^l = 1 \right\}\,. \end{aligned} \tag{3.3.5}$$

Although F^s and F^l are proper subsets of F, it is more relevant that each of them is a proper superset of F°. The original domain $\Omega \subset \mathbb{C}^n$ arises as a continuous image of F° via the mapping Ξ (3.1.4). The corresponding domains in the hybrid cases arise in a similar way from these two supersets:

$$\Omega^s \equiv \Xi^s(F^s) \supset \Omega \qquad \Omega^l \equiv \Xi^l(F^l) \supset \Omega\,.$$

These will appear when we switch from variables x to variables X.

3.3.4. Jacobi polynomials

All the characters χ_λ, the hybrid characters χ^s_λ, χ^l_λ, and the C-functions C_λ, $\lambda \in P^+$ lie in $\mathbb{Z}[P]^W$. Furthermore each set forms a $\mathbb{Z}$-basis for it and in each case the characters or hybrid characters indexed by the fundamental weights ω_j, $j = 1, \ldots, n$, generate $\mathbb{Z}[P]^W$ as a polynomial ring. Of course these facts apply to $\mathbb{C}[P]^W$ as well. This is quite easy to see because it is obvious that the C-functions C_λ, $\lambda \in P^+$, are a $\mathbb{Z}$-basis for $\mathbb{Z}[P]^W$ and the others can be written as sums of the form

$$C_\lambda + \sum_{\left\{\substack{\mu\in P^+\\ \mu\prec\lambda}\right\}} a_{\lambda,\mu} C_\mu$$

where the $a_{\lambda,\mu} \in \mathbb{Z}$. This triangular form with unit diagonal coefficients can be inverted in $\mathbb{Z}[P]^W$, showing that each of the other sets is a basis too. Similarly each $C_\lambda = C_{k_1\omega_1+\cdots+k_n\omega_n}$ can be written in the form

$$C^{k_1}_{\omega_1} \cdots C^{k_n}_{\omega_n} + \sum_{\left\{\substack{\mu\in P^+\\ \mu\prec\lambda}\right\}} a_{\lambda,\mu} C_\mu$$

with integer coefficients, and this provides the recursive step to write any element of $\mathbb{Z}[P]^W$ as a polynomial in the C_{ω_j}. The same thing can be done with the fundamental characters or hybrid characters.

Although we have no need for the specific values of the coefficients in these expressions, there are ways to compute them. As a specific example there are the Jacobi polynomials $P(\lambda, k)$, defined for any multiplicity function k, see [**10**], and any $\lambda \in P^+$ by

$$P(\lambda, k) = \sum_{\left\{\substack{\mu\in P^+\\ \mu\preceq\lambda}\right\}} c_{\lambda\mu}(k) C_\mu, \tag{3.3.6}$$

where the coefficients $c_{\lambda\mu}(k)$ are defined recursively by:

$$\begin{aligned} &\{(\lambda + \varrho(k) \mid \lambda + \varrho(k)) - (\mu + \varrho(k) \mid \mu + \varrho(k))\} c_{\lambda\mu}(k) \\ &\qquad = 2 \sum_{\alpha\in\Pi_+} k_\alpha \sum_{j=1}^{\infty} (\mu + j\alpha \mid \alpha) c_{\lambda,\mu+j\alpha}(k) \end{aligned} \tag{3.3.7}$$

along with the initial value $c_{\lambda\lambda} = 1$ and the assumption $c_{\lambda\mu} = c_{\lambda,w(\mu)}$ for all $w \in W$. Recall that $\varrho(k)$ is given by (3.2.5).

For $k = k^1$ the relation (3.3.7) is the Freudenthal recurrence relation used to find the coefficients of decomposition of characters $\chi_\lambda = \frac{S_{\lambda+\varrho}}{S_\varrho}$ of irreducible representations of simple Lie algebras into C-functions. In other words,

$$\chi_\lambda = P(\lambda, k^1) = \sum_{\left\{ \substack{\mu \in P^+ \\ \mu \preceq \lambda} \right\}} c_{\lambda\mu}(k^1) C_\mu \,.$$

Furthermore (3.3.6), for k^s and k^l be given by (3.2.4) and $\lambda \in P^+$, we have

$$\chi^s_\lambda = P(\lambda, k^s) \quad \text{and} \quad \chi^l_\lambda = P(\lambda, k^l)\,.$$

3.3.5. An inner product on $\mathbb{C}[P]^W$

The standard inner product on $\mathbb{C}[P]$ is defined by

$$\langle f, g\rangle_{\mathbb{T}} = \int_{\mathbb{T}} f\overline{g} d\theta_{\mathbb{T}}\,,$$

where $d\theta_{\mathbb{T}}$ is the normalized Haar measure on the torus $\mathbb{T}$. Relative to this, the functions φ_λ (3.3.3) form an orthogonal basis of $\mathbb{C}[P]$. Its completion is the Hilbert space $L^2(\mathbb{T}, \theta_{\mathbb{T}})$. We let $L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$ be the subspace of all W-invariant elements of $L^2(\mathbb{T}, \theta_{\mathbb{T}})$, which is in fact the closure of $\mathbb{C}[P]^W$ in $L^2(\mathbb{T}, \theta_{\mathbb{T}})$.

We now modify this inner product in a natural way so that the hybrid-characters χ^s_λ (or χ^l_λ) form an orthogonal basis for $L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$. Notice here that we are interpreting functions as functions on $\mathbb{T}$.

For any element $f \in L^2(\mathbb{T}, \theta_{\mathbb{T}})^W$, we have $fS^s_{\varrho^s} \in L^2(\mathbb{T}, \theta_{\mathbb{T}})$. One can form its Fourier expansion

$$fS^s_{\varrho^s} = \sum_{\mu \in P} \langle fS^s_{\varrho^s}, \varphi_\mu\rangle_{\mathbb{T}} \varphi_\mu\,,$$

and since $fS^s_{\varrho^s}$ is σ^s-skew-invariant with respect to W, this can be rewritten as

$$fS^s_{\varrho^s} = \sum_{\lambda \in P^+} \langle fS^s_{\varrho^s}, \varphi_{\lambda+\varrho^s}\rangle_{\mathbb{T}} \sum_{\mu' \in O(\lambda+\varrho^s)} \sigma^s(\mu')\varphi_{\mu'} = \sum_{\lambda \in P^+} \langle fS^s_{\varrho^s}, \varphi_{\lambda+\varrho^s}\rangle_{\mathbb{T}} S^s_{\lambda+\varrho^s}\,.$$

Dividing by $S^s_{\varrho^s}$ we have

$$f = \sum_{\lambda \in P^+} \langle fS^s_{\varrho^s}, \varphi_{\lambda+\varrho^s}\rangle_{\mathbb{T}}\, \chi^s_\lambda\,,$$

and then by the W-invariance of $\theta_{\mathbb{T}}$ and σ^s-skew-invariance of $fS^s_{\varrho^s}$, we obtain

$$\begin{aligned}
\langle fS^s_{\varrho^s}, \varphi_{\lambda+\varrho^s}\rangle_{\mathbb{T}} &= \int_{\mathbb{T}} fS^s_{\varrho^s}\overline{\varphi_{\lambda+\varrho^s}} d\theta_{\mathbb{T}} = \frac{1}{|W|}\int_{\mathbb{T}} \sum_{w \in W} \sigma^s(w) fS^s_{\varrho^s}\overline{\varphi_{w(\lambda+\varrho^s)}} d\theta_{\mathbb{T}} \\
&= \frac{|\mathrm{stab}_W(\lambda+\varrho^s)|}{|W|}\int_{\mathbb{T}} fS^s_{\varrho^s}\overline{S^s_{\lambda+\varrho^s}} d\theta_{\mathbb{T}} \\
&= |\mathrm{stab}_W(\lambda+\varrho^s)| \int_{F^s} f\overline{\chi^s_\lambda} S^s_{\varrho^s}\overline{S^s_{\varrho^s}} d\theta_{\mathbb{T}}\,.
\end{aligned}$$

This suggests the new inner product on $L^2(\mathbb{T},\theta_{\mathbb{T}})^W$ as

$$(f,g)_s = \int_{F^s} f\overline{g} S^s_{\varrho^s}\overline{S^s_{\varrho^s}} d\theta_{\mathbb{T}}.$$

Then, we can write

$$f = \sum_{\lambda\in P^+} |\mathrm{stab}_W(\lambda+\varrho^s)|(f,\chi^s_\lambda)_s\chi^s_\lambda\,.$$

In particular, with $f=\chi^s_\mu$ we have

$$\chi^s_\mu = \sum_{\lambda\in P^+} |\mathrm{stab}_W(\lambda+\varrho^s)|(\chi^s_\mu,\chi^s_\lambda)_s\chi^s_\lambda\,,$$

from which we have the orthogonality relations

$$(\chi^s_\mu,\chi^s_\lambda)_s = \frac{1}{|\mathrm{stab}_W(\mu+\varrho^s)|}\delta_{\mu\lambda}\,.$$

Writing this out, we have

Proposition 3.3.3. *For $\lambda,\mu\in P^+$,*

$$\int_{F^s} S^s_{\lambda+\varrho^s}(x)\overline{S^s_{\mu+\varrho^s}(x)}\,d\theta_{\mathbb{T}}(x) = (\chi^s_\lambda,\chi^s_\mu)_s = \frac{1}{|stab_W(\lambda+\varrho^s)|}\delta_{\lambda\mu}\,,$$

where $|stab_W(\lambda+\varrho^s)|$ denotes the number of elements in stabilizer of $\lambda+\varrho^s$ in W. The parallel result holds for the long root case.

3.4. Polynomial variables and elements of finite order

The cubature formulas rely on being able to identify the ring $\mathbb{C}[x_1,\dots,x_n]^W$ as a polynomial ring and then forming the connection between the variables X_j and characters on $\mathbb{G}$ (treated as functions on $\mathfrak{t}$). In the usual case, the characters are the characters of the fundamental representations with highest weight ω_j. In the hybrid cases we use hybrid characters instead. As we shall see, they all generate essentially the same ring, but the explicit mappings between the natural variables of $\mathfrak{t}$ and the variables X_j are different. We shall work specifically with the short case, the long case being in every way parallel to it.

3.4.1. Polynomial variables for the hybrid cases

Let $X^s_1,\dots,X^s_n$ denote the polynomial variables defined by

$$X^s_1 \equiv \chi^s_{\omega_1}(x),\dots,X^s_n\equiv\chi^s_{\omega_n}(x),\quad x\in F^s\,,$$

where $\chi^s_{\omega_j}$ are the fundamental hybrid-characters (3.3.4).

As in [**44**], we define the m-degree of the variables $X_1^s, \ldots, X_n^s$ by assigning degree $m_i^\vee$ to X_i^s. Thus the monomial $(X_1^s)^{\lambda_1} \ldots (X_n^s)^{\lambda_n}$ has m-degree

$$\lambda_1 m_1^\vee + \cdots + \lambda_n m_n^\vee$$

and the dimension of the space of polynomials of m-degree at most M is the cardinality of the set

$$\left\{(\lambda_1, \ldots, \lambda_n) \mid \sum_{i=1}^{n} m_i^\vee \lambda_i \leq M, \lambda_i \in \mathbb{Z}^{\geq 0}\right\}. \tag{3.4.1}$$

In addition, we say that $\lambda = (\lambda_1, \ldots, \lambda_n) = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$ has m-degree equal to

$$\langle \lambda, \gamma^{s\vee} \rangle = \lambda_1 m_1^\vee + \cdots + \lambda_n m_n^\vee. \tag{3.4.2}$$

The new variables give rise to the mapping

$$\Xi^s : x \mapsto (X_1^s(x), \ldots, X_n^s(x)) \in \mathbb{C}^n, \quad x \in F^s$$

and similarly Ξ^l. These mappings are injective since the values of these fundamental hybrid characters determine the values of all the characters (hybrid or otherwise), hence a specific conjugacy class in $\mathbb{G}$, and finally, then, a unique point in F. Then we have the domain

$$\Omega^s \equiv \Xi^s(F^s) = \{(X_1^s(x), \ldots, X_n^s(x)) \mid x \in F^s\}.$$

Evidently this is a subset of $\mathbb{C}^n$, but in fact $\Omega^s \subset \mathbb{R}^n$. By Section 3.3.4, we see that each variable X_i^s can be written as a polynomial in fundamental characters χ_{ω_k} with integer coefficients. As discussed in [**44**], we know that $\overline{\chi_{\omega_k}} = \chi_{\omega_k}$ for algebras with two roots lengths. Therefore, we also have $\overline{X_i^s} = X_i^s$ and thus $\Omega^s \subset \mathbb{R}^n$.

We define

$$K^s \equiv \frac{S^s_{\varrho^s}\overline{S^s_{\varrho^s}}}{S^l_{\varrho^l}\overline{S^l_{\varrho^l}}}.$$

This function arises as a kernel in the integral of the cubature formulas for the short root case. The denominator of K^s does not vanish anywhere on the interior F° of the fundamental domain F, so K^s is defined on this region. K^s is a W-invariant rational function and can be rewritten as a function in terms of the fundamental hybrid-characters χ_i^s. We can regard K^s as a strictly positive function on F° or as a function in the variables X_i^s on the interior of Ω^s,

$$K^s = K^s(X_1^s, \ldots, X_n^s) = \frac{S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}(x)}}{S^l_{\varrho^l}(x)\overline{S^l_{\varrho^l}(x)}}, \qquad x \in F^\circ,$$

see Remark 3.1.1

Along with K^s we define κ^s on Ω^s by

$$\kappa^s(X^s) = \frac{|Wx|}{|W|} S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}(x)}\,. \tag{3.4.3}$$

Note that $|Wx|$ is just a number of points in W-orbit of x in $\mathfrak{t}/Q^\vee$ and its value is uniquely associated with X^s since Ξ^s is injective.

The S^l-functions are handled in the same way. Just interchange s and l in the discussion above. In particular notice that $K^l = (K^s)^{-1}$ on F°. We emphasize here that the m-degree of λ for the long root case is equal to

$$\langle \lambda, \gamma^{l\vee}\rangle = \lambda_1 m_1^{l\vee} + \cdots + \lambda_n m_n^{l\vee}\,.$$

and thus is not the same as m-degree (3.4.2) of λ for the short root case or the same as the m-degree of polynomials (see Table 3.1).

3.4.2. The Jacobian

Although the cubature formulas that we are aiming to prove are set within the context of the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$, what underlies them is the realization of the variables X_j as functions, actually characters χ_{ω_j} (or hybrid characters $\chi^s_{\omega_j}, \chi^l_{\omega_j}$), on $\mathbb{T}$. These characters are first of all functions on $\mathbb{T}$, but are treated also as functions on $\mathfrak{t}$ via the exponential map – indeed they are exponential sums. As functions on $\mathfrak{t}$ they become functions of n variables in terms of the standard basis $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$. In order to make transitions from the $\alpha^\vee$-variables to the X_j-variables we require the Jacobian J with matrix entries $J_{jk} = D_{\alpha_j^\vee}\chi_{\omega_k}$, see definition below. This is written for the case of the characters, and in this case the Jacobian was determined in [**44**]. Since the transition from characters to hybrid characters is made through a unipotent transformation, the determinant of the Jacobian is not altered for the hybrid characters.

Proposition 3.4.1.

$$\det(J) = \det(J^s) = \det(J^l) = S_\varrho = S^s_{\varrho^s} S^l_{\varrho^l}\,.$$

Note that from this we have

$$(K^s)^{1/2}|\det(J)| = |S^s_{\varrho_s}|^2\,. \tag{3.4.4}$$

With $x = (x_1, \dots, x_n) = x_1\alpha_1^\vee + \cdots + x_n\alpha_n^\vee$ as variables on $\mathfrak{t}$ and the derivation mapping $D_{\alpha_i^\vee}$ defined by

$$D_{\alpha_j^\vee} e^{\langle \lambda, 2\pi i x\rangle} = \langle \lambda, \alpha_j^\vee\rangle e^{\langle \lambda, 2\pi i \sum_{k=1}^n x_k \alpha_k^\vee\rangle} = \frac{1}{2\pi i}\frac{d}{dx_j} e^{\langle \lambda, 2\pi i x\rangle}\,,$$

we compute

$$D_{\alpha_j^\vee}\chi^s_{\omega_k} = \frac{1}{2\pi i}\frac{d}{dx_j}\chi^s_{\omega_k}\,.$$

Then Proposition 3.4.1 implies that the Jacobian of the transformation from the variables x to variables X^s or X^l is

$$|(2\pi i)^n S_\varrho(x)| = (2\pi)^n |S_\varrho(x)|\,.$$

So, by (3.4.4), we have

$$\begin{aligned}&\int_{\Omega^s} f\overline{g}(K^s)^{1/2}\,dX^s\\ &\qquad = \int_{\Omega^s} f(X^s_1,\dots,X^s_n)\overline{g(X^s_1,\dots,X^s_n)}(K^s)^{1/2}(X^s_1,\dots,X^s_n)\,dX^s_1\dots dX^s_n\\ &\qquad = (2\pi)^n\int_{F^s} f(\chi^s_{\omega_1}(x),\dots,\chi^s_{\omega_n}(x))\overline{g(\chi^s_{\omega_1}(x),\dots,\chi^s_{\omega_n}(x))}S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}(x)}\,dx\,.\end{aligned}$$

Particularly note the special case of this when $f = \chi^s_\lambda$ and $g = \chi^s_\mu$ when, along with Proposition 3.3.3, it becomes

$$(2\pi)^{-n}\int_{\Omega^s}\chi^s_\lambda\overline{\chi^s_\mu}(K^s)^{1/2}\,dX = \int_{F^s} S^s_{\lambda+\varrho^s}(x)\overline{S^s_{\mu+\varrho^s}(x)}\,dx = (\chi_\lambda,\chi_\mu)_s\,. \tag{3.4.5}$$

Note that the integrals over Ω^s are well defined since $(K^s)^{1/2}\,dX^s$ is defined over the interior of Ω^s and is zero on its boundary.

3.4.3. Cones of elements of finite order

Every element of $\mathbb{G}$ is conjugate to $\exp 2\pi i x$ for a unique $x \in F$. The elements of finite order (EFO) are particularly interesting because they provide a way of discretization that is intuitive, natural, and computationally efficient. The conjugacy classes of elements of finite order N (this includes all elements whose order divides N) are precisely given by $\frac{1}{N}Q^\vee \cap F$ and those of adjoint order N, i.e. of order N in the adjoint representation of G on itself, are given by $\frac{1}{N}P^\vee \cap F$ [**41**]. It is these latter elements that will define the nodes for the cubature formula. More particularly, having chosen some positive integer M, we wish to use

$$F^s_{M+h^s} \equiv \frac{1}{M+h^s}P^\vee \cap F^s\,, \qquad F^l_{M+h^l} \equiv \frac{1}{M+h^l}P^\vee \cap F^l\,,$$

where h^s, h^l are defined by (3.2.7). Using (3.3.5), the elements of the fragments can be represented as follows.

$$\begin{aligned} x \in F^s_{M+h^s} \quad \Longleftrightarrow \quad & x = \frac{1}{M+h^s}(s^s_1\omega^\vee_1 + \dots + s^s_n\omega^\vee_n) \text{ with } (s^s_1,\dots,s^s_n) \text{ satisfying}\\ & s^s_0 + \sum_{i=1}^n m_i s^s_i = M + h^s\,, \text{ where } s^s_i \in \mathbb{N} \text{ if } \alpha_i \in \Pi^s \text{ otherwise}\\ & s^s_i \in \mathbb{Z}^{\geq 0}\,,\end{aligned}$$

$$x \in F^l_{M+h^l} \quad \Longleftrightarrow \quad x = \frac{1}{M+h^l}(s^l_1\omega^\vee_1 + \cdots + s^l_n\omega^\vee_n) \text{ with } (s^l_1, \ldots, s^l_n) \text{ satisfying}$$

$$s^l_0 + \sum_{i=1}^n m_i s^l_i = M + h^l\,, \text{ where } s^l_i \in \mathbb{Z}^{\geq 0} \text{ if } \alpha_i \in \Pi^s \text{ otherwise}$$

$$s^l_0, s^l_i \in \mathbb{N}\,,$$

The coordinates $[s^s_0, s^s_1, \ldots, s^s_n]$ and $[s^l_0, s^l_1, \ldots, s^l_n]$ are called the Kac coordinates of x, [**41**].

Since $h^s = \sum_{\alpha_i \in \Pi^s} m_i$ and $h^l = 1 + \sum_{\alpha_i \in \Pi^l} m_i$, each of the sets F_{M+h^s} and F_{M+h^l} has the same cardinality as the set:

$$\left\{(t_1, \ldots, t_n) \mid \sum_{i=1}^n m_i t_i \leq M, t_i \in \mathbb{Z}^{\geq 0}\right\}\,. \tag{3.4.6}$$

The explicit formulas for the cardinality of $F^s_{M+h^s}$ and $F^l_{M+h^l}$ have been calculated for all M and for all simple Lie algebras in [**14**].

Comparing (3.4.1) and (3.4.6), and using the the fact that the marks and co-marks are just permutations of each other (see Table 3.1), we see the important fact:

Theorem 3.4.1. *The number of monomials in $\mathbb{C}[X^s_1, \ldots, X^s_n]$ of m-degree at most M is equal to the number of points in $F^s_{M+h^s}$. The parallel result holds for long root case.*

3.4.4. Points of $F^s_{M+h^s}$ as zeros of S^s-functions

It is very interesting that the points that will be the nodes for the cubature formulas are also distinguished by being zeros of certain S^s-functions.

Proposition 3.4.2. *Let $M \in \mathbb{Z}^{\geq 0}$. The functions $S^s_{\lambda+\varrho^s}$ and the hybrid-characters χ^s_λ with λ of m-degree $= M+1$ vanish at all points of $F^s_{M+h^s}$. The same is true with s replaced by l throughout.*

Proof. We denote by r the reflection in the highest short root γ^s, on the root and co-root side, given respectively by

$$r(\lambda) \equiv r\lambda = \lambda - \langle \lambda, \gamma^{s\vee}\rangle \gamma^s\,,$$

$$r(x) \equiv rx = x - \langle \gamma^s, x\rangle \gamma^{s\vee}\,.$$

Let $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n \in P^+$. Divide the orbit $O = O(\lambda + \varrho^s)$ into O_+ on which σ^s takes the value 1, and O_- on which it takes value -1, and note that $O_- = rO_+$. Then we can write

$$S^s_{\lambda+\varrho^s}(x) = \sum_{\mu \in O_+} (e^{2\pi i\langle \mu, x\rangle} - e^{2\pi i\langle r\mu, x\rangle}) = \sum_{\mu \in O_+} (e^{2\pi i\langle \mu, x\rangle} - e^{2\pi i\langle \mu, rx\rangle})\,.$$

Now, $S^s_{\lambda+\varrho^s}(x)$ will vanish for all $x \in F^s_{M+h^s}$ if each term

$$e^{2\pi i\langle\mu,x\rangle} - e^{2\pi i\langle\mu,rx\rangle} = 0\,,$$

or equivalently

$$\langle\mu,x\rangle - \langle\mu,rx\rangle \in \mathbb{Z}$$

for all $x \in \frac{1}{M+h^s}P^\vee$. Since $x \in \frac{1}{M+h^s}P^\vee$ is W-invariant, this amounts to

$$\langle\lambda+\varrho^s,x\rangle - \langle\lambda+\varrho^s,rx\rangle \in \mathbb{Z} \qquad \text{for all } x \in \frac{1}{M+h^s}P^\vee\,,$$

or equivalently

$$\langle\gamma^s,x\rangle\langle\lambda+\varrho^s,\gamma^{s\vee}\rangle \in \mathbb{Z} \qquad \text{for all } x \in \frac{1}{M+h^s}P^\vee\,.$$

Since $\langle\gamma^s,P^\vee\rangle \subset \mathbb{Z}$, we have $\langle\gamma^s,x\rangle \in \frac{1}{M+h^s}\mathbb{Z}$, and it is sufficient that

$$\langle\lambda+\varrho^s,\gamma^{s\vee}\rangle \in (M+h^s)\mathbb{Z}\,.$$

Requiring $\langle\lambda+\varrho^s,\gamma^{s\vee}\rangle = M+h^s$ leads to the condition

$$\langle\lambda,\gamma^{s\vee}\rangle = M+1$$

by definition of h^s (3.2.7). This is the condition of the hypothesis of the proposition and proves the result for the S^s-functions.

To get to the characters χ^s we have to divide by $S^s_{\varrho^s}$. The latter vanishes only on the walls of H^s and these are not part of F^s, and so this division does not affect the outcome.

The proof for the long root case is parallel. □

Recall that $|\frac{1}{M+h^s}P^\vee/Q^\vee| = c_{\mathfrak{g}}(M+h^s)^n$, where $c_{\mathfrak{g}}$ is the determinant of C (which is the value of the index $[P^\vee : Q^\vee]$). Of course there is a parallel formula for the long root case.

3.4.5. Discrete orthogonality of S^s- and S^l-functions

Proposition 3.4.3. *Let $M \in \mathbb{Z}^{\geq 0}$ and $\lambda,\mu \in P^+$ and suppose that for all $w,w' \in W$ we have $w(\lambda+\varrho^s) - w'(\mu+\varrho^s) \notin (M+h^s)Q$ unless $\lambda=\mu$ and $w(\lambda+\varrho^s) = w'(\lambda+\varrho^s)$. Then*

$$\frac{1}{c_{\mathfrak{g}}|W|(M+h^s)^n}\sum_{x\in F^s_{M+h^s}}|Wx|S^s_{\lambda+\varrho^s}(x)\overline{S^s_{\mu+\varrho^s}(x)} = \frac{1}{|stab_W(\lambda+\varrho^s)|}\delta_{\lambda\mu}\,. \quad (3.4.7)$$

The parallel result holds for the long root case. We recall that Wx is the W-orbit of x in $\mathfrak{t}/Q^\vee$.

PROOF. The summands appearing in (3.4.7) are dependent only on the values of $x \mod Q^\vee$, so we can reduce $\mod Q^\vee$ (see Remark 3.1.1). The set $F^s_{M+h^s}$ is mapped faithfully by the S^s-functions in this process.

We begin by replacing the sum over $F^s_{M+h^s}$ by a sum over the group

$$\frac{1}{M+h^s} P^\vee / Q^\vee \,.$$

For each representative element $x \in F^s_{M+h^s}$ we can form its W-orbit Wx. If we had all of $\frac{1}{M+h^s} P^\vee \cap F$ we would get all of $\frac{1}{M+h^s} P^\vee / Q^\vee$. As it is, we are missing the orbits of points $F \setminus F^s$ and these are all in H^s on which the S^s-functions vanish. So we can add them without changing anything. Thus

$$\sum_{x \in F^s_{M+h^s}} |Wx| \, S^s_{\lambda+\varrho^s}(x) \overline{S^s_{\mu+\varrho^s}(x)} = \sum_{x \in \frac{1}{M+h^s} P^\vee / Q^\vee} S^s_{\lambda+\varrho^s}(x) \overline{S^s_{\mu+\varrho^s}(x)} \,.$$

The two S^s terms when expanded are sums of exponential functions $\exp(2\pi i \langle \nu, x \rangle)$ (which are well defined as functions on $\frac{1}{M+h^s} P^\vee / Q^\vee$), where each ν is of the form $\nu = w(\lambda+\varrho^s) - w'(\mu+\varrho^s)$. Fixing ν and and summing over x, we get their sum over the group is zero as long as $\langle \nu, x \rangle \notin \mathbb{Z}$ for at least one x. This requirement is just the same as saying $\nu \notin (M+h^s)Q$. In view of our hypothesis this fails only if $\lambda = \mu$ and $w(\lambda+\varrho^s) = w'(\lambda+\varrho^s)$. In that case the sum is $c_{\mathfrak{g}}(M+h^s)^n$. This happens once for each element in $O(\lambda+\varrho^s)$. Since $|O(\lambda+\varrho^s)| = |W|/|\mathrm{stab}_W(\lambda+\varrho^s)|$, we are done. $\square$

For a slight different point of view on discrete orthogonality, as well as an algorithm for calculation of $|Wx|$, see [**14**].

3.5. Integration formulas

Our aim is to create cubature formulas for the integrals of the form

$$\int_{\Omega^s} f^s \overline{g^s} (K^s)^{1/2} dX^s_1 \dots dX^s_n \,, \qquad \int_{\Omega^l} f^l \overline{g^l} (K^l)^{1/2} dX^l_1 \dots dX^l_n \,.$$

where f^s, g^s are functions in the variables $X^s_1, \dots, X^s_n$ defined on Ω^s and f^l, g^l are functions in the variables $X^l_1, \dots, X^l_n$ defined on Ω^l. These cubature formulas depend on the two orthogonality results that we have shown, namely Proposition 3.3.3 and Proposition 3.4.3, the first involving an integral over Ω^s and the second a finite sum over $F^s_{M+h^s}$, which yield identical results. The discrete orthogonality relations require specific separation hypotheses on the weights, so to make use of the equalities we need only to guarantee that these hold. The same applies to the long root case too. The image of $F^s_{M+h^s}$ in Ω^s under Ξ^s is written as $\mathcal{F}^s_{M+h^s}$, and similarly for the long root case.

3.5.1. The key integration formulas

Theorem 3.5.1. *We use the notation $X^s = (X^s_1, \dots, X^s_n)$ and $X^l = (X^l_1, \dots, X^l_n)$.*

(i) Let $M \in \mathbb{Z}^{\geq 0}$ and f, g be any polynomials in $\mathbb{C}[X^s]$ with m-deg$(f) \leq M+1$ and m-deg$(g) \leq M$. Then

$$\int_{\Omega^s} f\overline{g}K^{1/2}dX^s = \frac{1}{c_{\mathfrak{g}}}\left(\frac{2\pi}{M+h^s}\right)^n \sum_{X^s \in \mathcal{F}^s_{M+h^s}} f(X^s)\overline{g(X^s)}\kappa^s(X^s) \tag{3.5.1}$$

with $dX^s = dX^s_1 \dots dX^s_n$.

(ii) Let $M \in \mathbb{N}$ and f, g be any polynomials in $\mathbb{C}[X^l]$ with m-deg$(f) \leq M$ and m-deg$(g) \leq M-1$. Then

$$\int_{\Omega^l} f\overline{g}(K^l)^{1/2}dX^l = \frac{1}{c_{\mathfrak{g}}}\left(\frac{2\pi}{M+h^l}\right)^n \sum_{X^l \in \mathcal{F}^l_{M+h^l}} f(X^l)\overline{g(X^l)}\kappa^l(X^l) \tag{3.5.2}$$

with $dX^l = dX^l_1 \dots dX^l_n$.

Proof. By Section 3.4.2, we obtain that the left-hand side of (3.5.1) is equal to

$$(2\pi)^n \int_{F^s} f(\chi^s_{\omega_1}(x), \dots, \chi^s_{\omega_n}(x))\overline{g(\chi^s_{\omega_1}(x), \dots, \chi^s_{\omega_n}(x))}S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}}(x)dx\,.$$

Using the definition (3.4.3) of κ^s we rewrite the right-hand side of (3.5.1) as

$$\sum_{x \in F^s_{M+h^s}} f(\chi^s_{\omega_1}(x), \dots, \chi^s_{\omega_n}(x))\overline{g(\chi^s_{\omega_1}(x), \dots, \chi^s_{\omega_n}(x))}|Wx|S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}}(x)$$

multiplied by $\frac{1}{c_{\mathfrak{g}}|W|}\left(\frac{2\pi}{M+h^s}\right)^n$.

By linearity of (3.5.1), we can only consider the monomials

$$(\chi^s_{\omega_1})^{\nu_1}, \dots, (\chi^s_{\omega_n})^{\nu_n}\,, \qquad \text{where } \nu_1 m^\vee_1 + \dots + \nu_n m^\vee_n \leq N$$

with $N = M+1$ for f and $N = M$ for g.

By Section 3.3.4, we see that such monomial decomposes as a linear combination of χ^s_λ with $\lambda \preceq \nu$ (see Section 3.2.1) and the coefficient of χ^s_ν is equal to 1.

Thus it is sufficient to prove that

$$\begin{aligned}\int_{F^s} &\chi^s_\lambda(x)\overline{\chi^s_\mu(x)}S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}}(x)dx \\ &= \frac{1}{c_{\mathfrak{g}}|W|(M+h^s)^n} \sum_{x \in F^s_{M+h^s}} \chi^s_\lambda(x)\overline{\chi^s_\mu(x)}|Wx|S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}}(x)\end{aligned}$$

for $\lambda, \mu \in P^+$ such that $m\text{-}deg(\lambda) \leq M+1$ and $m\text{-}deg(\mu) \leq M$. This is true from Proposition 3.3.3 and Proposition 3.4.3, provided the weight separation conditions of Proposition 3.4.3 apply, that is, whenever $\lambda \neq \mu$, it never happens that

$$w(\lambda + \varrho^s) - w'(\mu + \varrho^s) \in (M + h^s)Q$$

for any $w, w' \in W$. This follows line for line the proof of Theorem 7.1 of [**44**] since it does not change anything if we consider h^s instead of h.

We can prove the result for the long root case similarly. However, there is one difference which arises because $h^s = 1 + \sum_{\alpha_i \in \Pi^s} m_i^\vee$ whereas $h^l = \sum_{\alpha_i \in \Pi^l} m_i^\vee$. This difference appears in the validation of the separation conditions, which hold only for $\sum m_i^\vee \lambda_i \leq M$ and $\sum m_i^\vee \mu_i \leq M - 1$ in the long case. □

3.5.2. The cubature formulas

The following theorem can be proved in the same way as Theorem 3.5.1 since $\mu = 0$ and λ with $\sum \lambda_i m_i^\vee \leq 2M + 1$ ($2M - 1$ respectively) satisfy the separation conditions of Proposition 3.4.3.

Theorem 3.5.2.

i) Let $M \in \mathbb{Z}^{\geq 0}$ and f be any polynomial in $\mathbb{C}[X^s]$ with m-deg$(f) \leq 2M + 1$, then

$$\int_{\Omega^s} f(K^s)^{1/2} dX^s = \frac{1}{c_{\mathfrak{g}}} \left(\frac{2\pi}{M + h^s} \right)^n \sum_{X^s \in \mathcal{F}^s_{M+h^s}} f(X^s) \kappa^s(X^s) \,,$$

where κ^s is defined by (3.4.3).

ii) Let $M \in \mathbb{N}$ and f be any polynomial in $\mathbb{C}[X^l]$ with m-deg$(f) \leq 2M - 1$, then

$$\int_{\Omega^l} f(K^l)^{1/2} dX^l = \frac{1}{c_{\mathfrak{g}}} \left(\frac{2\pi}{M + h^l} \right)^n \sum_{X^l \in \mathcal{F}^l_{M+h^l}} f(X^l) \kappa^l(X^l) \,.$$

Remark 3.5.1. *One notes here that the short root case (i) is Gaussian cubature, with maximal efficiency in terms of the number of nodal points required, while the long root case (ii) fits into the Radau cubature class and is slightly less efficient.*

3.6. Approximating functions on Ω^s and Ω^l

In this section we just point out a few things that are direct consequences of the Fourier analysis that has been developed here. As usual, we write this down for the short root length case, the long root case being entirely parallel.

3.6.1. Polynomial expansion in terms of χ_λ^s

Let $L^2_{K^s}(\Omega^s)$ denote the space of all complex valued functions f on Ω^s such that $\int_{\Omega^s} |f|^2\,(K^s)^{1/2} dX^s < \infty$. We recall the inner product of (3.4.5) on $L^2_{K^s}(\Omega^s)$

$$\begin{aligned}(f,g)_s &\equiv (2\pi)^{-n}\int_{\Omega^s} f(X^s)\overline{g(X^s)}(K^s(X^s))^{1/2}dX^s\\ &= \int_{F^s} f(X^s(x))\overline{g(X^s(x))}S^s_{\varrho^s}(x)\overline{S^s_{\varrho^s}(x)}d\theta_{\mathbb{T}}(x)\,.\end{aligned}$$

We write $f \doteq g$ if $f = g$ almost everywhere in Ω^s. Since $(K^s)^{1/2}$ is continuous and strictly positive on interior of Ω^s, we have for any f that $(f,f)_s \geq 0$ with equality if and only if $f \doteq 0$. Thus, we can regard $L^2_{K^s}(\Omega^s)$ as a Hilbert space with $L^2_{K^s}$-norm of f equal to $(f,f)_s^{1/2}$.

By Proposition 3.3.3, the polynomials $X^s_\lambda \equiv \chi^s_\lambda(x)$, $x \in F^s$ with $\lambda \in P^+$ form an orthogonal set in $L^2_{K^s}(\Omega^s)$:

$$(X^s_\lambda, X^s_\mu)_s = |\mathrm{stab}_W(\lambda+\varrho^s)|^{-1}\delta_{\lambda\mu}\,,$$

and, in fact, they form a Hilbert basis in $L^2_{K^s}(\Omega^s)$. We can see this by relating $f(X^s)$ on Ω^s with $f(x)$ on F^s and using the discussion in Section 3.3.5 to make its Fourier expansion. Rewriting this back in Ω^s we obtain the basic expansion formulas

$$f \doteq \sum_{\lambda\in P^+} a_\lambda X^s_\lambda\,, \qquad \text{where } a_\lambda = |\mathrm{stab}_W(\lambda+\varrho^s)|(f, X^s_\lambda)_s\,.$$

3.6.2. Optimality

If $|\lambda|_m \equiv \sum_i m_i^\vee \lambda_i$, then the sums

$$\sum_{|\lambda|_m \leq M} |\mathrm{stab}_W(\lambda+\varrho^s)|(f, X^s_\lambda)_s X^s_\lambda$$

are the polynomials of m-degree at most M in the variables $X^s_1, \ldots, X^s_n$.

Proposition 3.6.1. *Let $f \in L^2_{K^s}(\Omega^s)$. Amongst all polynomials $p(X^s_1, \ldots, X^s_n)$ of m-degree less than or equal to M, the polynomial*

$$q = \sum_{|\lambda|_m \leq M} |stab_W(\lambda+\varrho^s)|(f, X^s_\lambda)_s X^s_\lambda$$

is the best approximation to f relative to the $L^2_{K^s}$-norm.

Proof. Let $p = \sum_{|\lambda|_m \le M} b_\lambda X^s_\lambda$ be any polynomial of m-degree at most M and $a_\lambda = |\mathrm{stab}_W(\lambda + \varrho^s)|(f, X^s_\lambda)_s$, then

$$\begin{aligned}(f-p, f-p)_s &= (f,f)_s - \sum_{|\lambda|_m \le M} |\mathrm{stab}_W(\lambda+\varrho^s)|^{-1} a_\lambda \overline{b_\lambda} \\ &\quad - \sum_{|\lambda|_m \le M} |\mathrm{stab}_W(\lambda+\varrho^s)|^{-1} b_\lambda \overline{a_\lambda} + \sum_{|\lambda|_m \le M} |\mathrm{stab}_W(\lambda+\varrho^s)|^{-1} |b_\lambda|^2 \\ &= (f-q, f-q)_s + \sum_{|\lambda|_m \le M} |\mathrm{stab}_W(\lambda+\varrho^s)|^{-1} |b_\lambda - a_\lambda|^2 \\ &\ge (f-q, f-q)_s\end{aligned}$$

with equality if and only if $b_\lambda = a_\lambda$. □

3.7. Example: Cubature formulas for G_2

In this section we illustrate briefly how the main constituents of the paper look in the case of the Lie group G_2 when $M = 15$.

3.7.1. S^s- and S^l-functions of G_2

Let us recall some basic facts about Lie group G_2. The simple roots α_1, α_2 and co-roots $\alpha_1^\vee, \alpha_2^\vee$ are determined by the Cartan matrices C and C^T,

$$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \quad C^T = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

We also have the following relations between the bases:

$$\begin{aligned} &\alpha_1 = 2\omega_1 - 3\omega_2\,, \quad \alpha_2 = -\omega_1 + 2\omega_2\,, \quad \omega_1 = 2\alpha_1 + 3\alpha_2\,, \quad \omega_2 = \alpha_1 + 2\alpha_2\,; \\ &\alpha_1^\vee = 2\omega_1^\vee - \omega_2^\vee\,, \quad \alpha_2^\vee = -3\omega_1^\vee + 2\omega_2^\vee\,, \quad \omega_1^\vee = 2\alpha_1^\vee + \alpha_2^\vee\,, \quad \omega_2^\vee = 3\alpha_1^\vee + 2\alpha_2^\vee\,. \end{aligned}$$

Using (3.2.8), $\varrho^s = \omega_2$, $\varrho^l = \omega_1$, $h^s = h^l = 3$.

The defining relations for the Weyl group are $r_1^2 = r_2^2 = (r_1 r_2)^6 = 1$. Defining $r_{opp} \equiv r_1r_2r_1r_2r_1r_2 = r_2r_1r_2r_1r_2r_1$, the Weyl group consists of

$$1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\,,$$

together with the product of r_{opp} with each of these elements. The corresponding values of σ^s are $1, 1, -1, -1, -1, -1$ and $\sigma^s(r_{opp}) = -1$; and for σ^l they are $1, -1, 1, -1, -1, 1$ and $\sigma^l(r_{opp}) = -1$.

Let $\lambda = (\lambda_1, \lambda_2) = \lambda_1\omega_1 + \lambda_2\omega_2$ and $x = (x_1, x_2) = x_1\alpha_1^\vee + x_2\alpha_2^\vee$. Any Weyl group orbit of a generic point λ consists of

$$\begin{aligned}\{&\pm(\lambda_1, \lambda_2)\,, \pm(-\lambda_1, 3\lambda_1 + \lambda_2)\,, \pm(\lambda_1 + \lambda_2, -\lambda_2)\,, \pm(2\lambda_1 + \lambda_2, -3\lambda_1 - \lambda_2)\,, \\ &\pm(-\lambda_1 - \lambda_2, 3\lambda_1 + 2\lambda_2)\,, \pm(-2\lambda_1 - \lambda_2, 3\lambda_1 + 2\lambda_2)\}\,.\end{aligned}$$

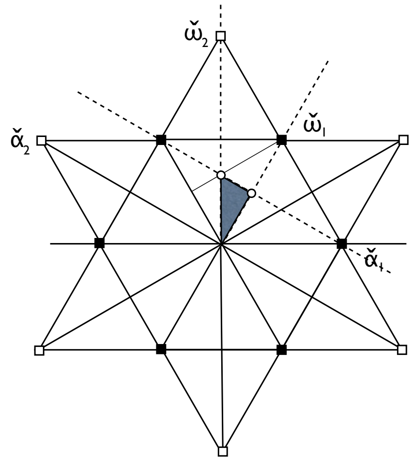

FIGURE 3.1. A schematic view of the co-root system of G_2. The shaded triangle is the fundamental region F. The dotted lines are the mirrors which define its boundaries, the reflections in which generate the affine Weyl group. The action of the affine Weyl group on F tiles the plane. A few tiles of this tiling are shown. Filled (respectively open) squares are the short (respectively long) co-roots of G_2.

Therefore the explicit formulas for the S^s- and S^l-functions are:

$$\begin{aligned}
S^s_{\lambda+\omega_2}(x) = \frac{2i}{|\mathrm{stab}_W(\lambda+\omega_2)|}&(\sin 2\pi(\lambda_1x_1+(\lambda_2+1)x_2)\\
&+\sin 2\pi(-\lambda_1x_1+(3\lambda_1+\lambda_2+1)x_2)\\
&-\sin 2\pi((\lambda_1+\lambda_2+1)x_1-(\lambda_2+1)x_2)\\
&-\sin 2\pi((2\lambda_1+\lambda_2+1)x_1+(-3\lambda_1-\lambda_2-1)x_2)\\
&-\sin 2\pi((-\lambda_1-\lambda_2-1)x_1+(3\lambda_1+2\lambda_2+2)x_2)\\
&-\sin 2\pi((-2\lambda_1-\lambda_2-1)x_1+(3\lambda_1+2\lambda_2+2)x_2)\,,\\
S^l_{\lambda+\omega_1}(x) = \frac{2i}{|\mathrm{stab}_W(\lambda+\omega_1)|}&(\sin 2\pi((\lambda_1+1)x_1+\lambda_2x_2)\\
&-\sin 2\pi(-(\lambda_1+1)x_1+(3\lambda_1+\lambda_2+3)x_2)\\
&+\sin 2\pi((\lambda_1+\lambda_2+1)x_1-\lambda_2x_2)\\
&-\sin 2\pi((2\lambda_1+\lambda_2+2)x_1+(-3\lambda_1-\lambda_2-3)x_2)\\
&-\sin 2\pi((-\lambda_1-\lambda_2-1)x_1+(3\lambda_1+2\lambda_2+3)x_2)\\
&+\sin 2\pi((-2\lambda_1-\lambda_2-2)x_1+(3\lambda_1+2\lambda_2+3)x_2)\,.
\end{aligned}$$

By definition the polynomial variables X_1^s, X_2^s and X_1^l, X_2^l are given by

$$\begin{aligned}
X_1^s &= \frac{S^s_{(1,1)}(x)}{S^s_{(0,1)}(x)} = 2(1+\cos 2\pi x_1 + \cos 2\pi(x_1-3x_2) + 2\cos 2\pi(x_1-2x_2) \\
&\qquad + 2\cos 2\pi(x_1-x_2) + 2\cos 2\pi x_2 + \cos 2\pi(2x_1-3x_2))\,, \\
X_2^s &= \frac{S^s_{(0,2)}(x)}{S^s_{(0,1)}(x)} = 2(1+\cos 2\pi x_2 + \cos 2\pi(x_1-2x_2) + \cos 2\pi(x_1-x_2))\,; \\
X_1^l &= \frac{S^l_{(2,0)}(x)}{S^l_{(1,0)}(x)} = 2(1+\cos 2\pi x_1 + \cos 2\pi(x_1-3x_2) + \cos 2\pi(2x_1-3x_2))\,, \\
X_2^l &= \frac{S^l_{(1,1)}(x)}{S^l_{(1,0)}(x)} = 2(\cos 2\pi x_2 + \cos 2\pi(x_1-2x_2) + \cos 2\pi(x_1-x_2))\,.
\end{aligned} \tag{3.7.1}$$

3.7.2. Integration regions Ω^s, Ω^l and grids $\mathcal{F}^s_{M+3}, \mathcal{F}^l_{M+3}$

Using the explicit formulas (3.7.1) for polynomial variables as functions of x_1, x_2, one can determine the integration regions Ω^s, Ω^l (see Figure 3.2 and 3.3), namely, Ω^s consists of points (X_1^s, X_2^s) satisfying

$$X_1^s > \frac{(X_2^s)^2}{4} + X_2^s - 4\,, -2-4X_2^s - 2(X_2^s+1)^{\frac{3}{2}} \leq X_1^s \leq -2-4X_2^s + 2(X_2^s+1)^{\frac{3}{2}}\,;$$

Ω^l contains only the points (X_1^l, X_2^l) such that

$$X_1^l \geq \frac{(X_2^l)^2}{4} - 1\,, -10-6X_2^l - 2(X_2^l+3)^{\frac{3}{2}} < X_1^l < -10-6X_2^l + 2(X_2^l+3)^{\frac{3}{2}}\,.$$

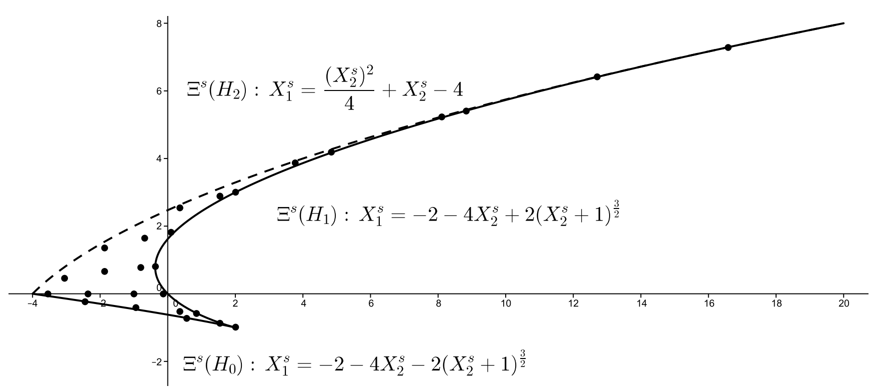

FIGURE 3.2. The region Ω^s along with the equations of its boundaries. Inside we see the points of $\mathcal{F}^s_{18}$. The dashed boundary is not included in Ω^s.

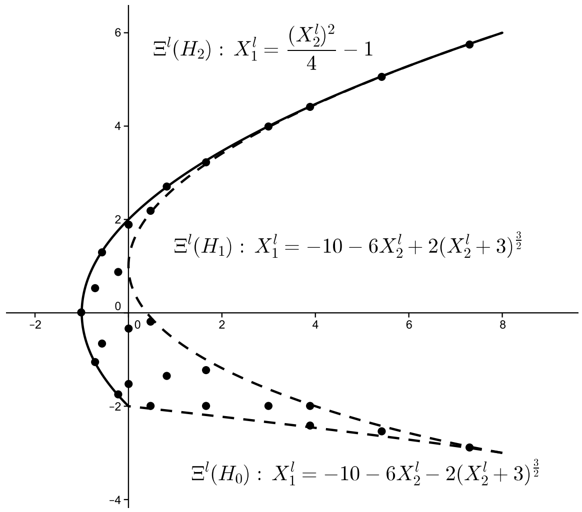

FIGURE 3.3. The region Ω^l along with the equations of its boundaries. Inside we see the points of $\mathcal{F}_{18}^l$. The dashed boundaries are not included in Ω^l.

The grids $\mathcal{F}_{M+3}^s, \mathcal{F}_{M+3}^l$ are the following finite sets of points in Ω^s and Ω^l respectively.

$$\mathcal{F}_{M+3}^s = \left\{ \left(X_1^s\left(\frac{2s_1+3s_2}{M+3}, \frac{s_1+2s_2}{M+3}\right), X_2^s\left(\frac{2s_1+3s_2}{M+3}, \frac{s_1+2s_2}{M+3}\right)\right)\right\},$$
$$\text{where } s_1 = 0,\dots,\left\lfloor \frac{M+3}{2}\right\rfloor, s_2 = 1,\dots,\left\lfloor\frac{M+3-2s_1}{3}\right\rfloor;$$
$$\mathcal{F}_{M+3}^l = \left\{ \left(X_1^l\left(\frac{2s_1+3s_2}{M+3}, \frac{s_1+2s_2}{M+3}\right), X_2^l\left(\frac{2s_1+3s_2}{M+3}, \frac{s_1+2s_2}{M+3}\right)\right)\right\},$$
$$\text{where } s_1 = 1,\dots,\left\lfloor \frac{M+2}{2}\right\rfloor, s_2 = 0,\dots,\left\lfloor\frac{M+2-2s_1}{3}\right\rfloor.$$

The list of EFOs for $M = 15$ is given in Table 3.3.

3.7.3. Cubature formulas

The functions K^s and K^l are given by the expressions:

$$K^s(X_1^s, X_2^s) = \frac{S_{\omega_2}^s\overline{S_{\omega_2}^s}}{S_{\omega_1}^l\overline{S_{\omega_1}^l}} = \frac{-(X_2^s)^2 - 4X_2^s + 4X_1^s + 16}{4(X_2^s)^3 - (X_1^s)^2 - 4(X_2^s)^2 - 8X_1^sX_2^s - 4X_1^s - 4X_2^s};$$
$$K^l(X_1^l, X_2^l) = \frac{S_{\omega_1}^l\overline{S_{\omega_1}^l}}{S_{\omega_2}^s\overline{S_{\omega_2}^s}} = \frac{4(X_2^l)^3 - (X_1^l)^2 - 12X_1^lX_2^l - 20X_1^l - 12X_2^l + 8}{-(X_2^l)^2 + 4X_1^l + 4}.$$

Thus, the explicit cubature formulas of G_2 are

$$\int_{\Omega^s} f(X_1^s, X_2^s)\sqrt{\frac{-(X_2^s)^2 - 4X_2^s + 4X_1^s + 16}{4(X_2^s)^3 - (X_1^s)^2 - 4(X_2^s)^2 - 8X_1^sX_2^s - 4X_1^s - 4X_2^s}}\, dX_1^s\, dX_2^s$$

(s_0, s_1, s_2)	F^s_{18}	F^l_{18}	(X^s_1, X^s_2)	(X^l_1, X^l_2)
$(0,0,6)$	✓	×	$(2,-1)$	×
$(0,3,4)$	✓	×	$(0.5662,-0.7169)$	×
$(0,6,2)$	✓	×	$(-2.4534,-0.2267)$	×
$(0,9,0)$	×	×	×	×
$(1,1,5)$	✓	✓	$(1.5321,-0.8794)$	$(7.2909,-2.8794)$
$(1,4,3)$	✓	✓	$(-0.9436,-0.4115)$	$(3.8794,-2.4115)$
$(1,7,1)$	✓	✓	$(-3.5321,0)$	$(0.4679,-2)$
$(2,2,4)$	✓	✓	$(0.3473,-0.5321)$	$(5.4115,-2.5321)$
$(2,5,2)$	✓	✓	$(-2.3473,0)$	$(1.6527,-2)$
$(2,8,0)$	×	✓	×	$(-0.2267,-1.7588)$
$(3,0,5)$	✓	×	$(0.852,-0.574)$	×
$(3,3,3)$	✓	✓	$(-1,0)$	$(3,-2)$
$(3,6,1)$	✓	✓	$(-3.0642,0.4679)$	$(0,-1.5321)$
$(4,1,4)$	✓	✓	$(-0.1206,0)$	$(3.8794,-2)$
$(4,4,2)$	✓	✓	$(-1.8794,0.6527)$	$(0.8152,-1.3473)$
$(4,7,0)$	×	✓	×	$(-0.7169,-1.0642)$
$(5,2,3)$	✓	✓	$(-0.8007,0.7733)$	$(1.6527,-1.2267)$
$(5,5,1)$	✓	✓	$(-1.8794,1.3473)$	$(-0.574,-0.6527)$
$(6,0,4)$	✓	×	$(-0.3696,0.8152)$	×
$(6,3,2)$	✓	✓	$(-0.6946,1.6527)$	$(0,-0.3473)$
$(6,6,0)$	×	✓	×	$(-1,0)$
$(7,1,3)$	✓	✓	$(0.0983,1.8152)$	$(0.4679,-0.1848)$
$(7,4,1)$	✓	✓	$(0.3473,2.5321)$	$(-0.7169,0.5321)$
$(8,2,2)$	✓	✓	$(1.5321,2.8794)$	$(-0.2267,0.8794)$
$(8,5,0)$	×	✓	×	$(-0.574,1.3054)$
$(9,0,3)$	✓	×	$(2,3)$	×
$(9,3,1)$	✓	✓	$(3.7588,3.8794)$	$(0,1.8794)$
$(10,1,2)$	✓	✓	$(4.8375,4.1848)$	$(0.4679,2.1848)$
$(10,4,0)$	×	✓	×	$(0.8152,2.6946)$
$(11,2,1)$	✓	✓	$(8.1061,5.2267)$	$(1.6527,3.2267)$
$(12,0,2)$	✓	×	$(8.823,5.4115)$	×
$(12,3,0)$	×	✓	×	$(3,4)$
$(13,1,1)$	✓	✓	$(12.7023,6.4115)$	$(3.8794,4.4115)$
$(14,2,0)$	×	✓	×	$(5.4115,5.0642)$
$(15,0,1)$	✓	×	$(16.5817,7.2909)$	×
$(16,1,0)$	×	✓	×	$(7.2909,5.7588)$
$(18,0,0)$	×	×	×	×

Table 3.3. A list of the EFOs for $M+3=18$, along with their coordinates in the domains Ω^s and Ω^l. Since F^s is missing the boundary defined by the fixed hyperplane for the short reflection r_2, EFOs falling on this boundary are not part of the short root scenario. For F^l it is EFOs on the hyperplanes for r_1 and r_0 that are not included.

(s_0, s_1, s_2)	$\lvert Wx\rvert$
$(\star, 0, 0)$	1
$(0, 0, \star)$	2
$(0, \star, 0)$	3
$(0, \star, \star)$	6
$(\star, 0, \star)$	6
$(\star, \star, 0)$	6
$(\star, \star, \star)$	12

TABLE 3.4. A table of values of $|Wx|$ for the group G_2 based on the form of the coordinates of $x \in F$. Recall that this is a count of the W orbit of x taken modulo $Q^\vee$. The values can be worked out using Figure 3.1. The cases $(\star, 0, 0), (0, \star, 0)$ do not appear in this context, but we include them to complete the table.

$$= \frac{1}{12}\left(\frac{2\pi}{M+3}\right)^2 \sum_{(X_1^s, X_2^s) \in \mathcal{F}^s_{M+3}} f(X_1^s, X_2^s)|Wx|(-(X_2^s)^2 - 4X_2^s + 4X_1^s + 16)\,;$$

$$\int_{\Omega^l} f(X_1^l, X_2^l)\sqrt{\frac{4(X_2^l)^3 - (X_1^l)^2 - 12X_1^lX_2^l - 20X_1^l - 12X_2^l + 8}{-(X_2^l)^2 + 4X_1^l + 4}}\,dX_1^l\,dX_2^l$$
$$= \frac{1}{12}\left(\frac{2\pi}{M+3}\right)^2 \sum_{(X_1^l, X_2^l) \in \mathcal{F}^l_{M+3}} f(X_1^l, X_2^l)|Wx|(4(X_2^l)^3 - (X_1^l)^2 - 12X_1^lX_2^l$$
$$- 20X_1^l - 12X_2^l + 8)\,.$$

The values $|Wx|$ are written in Table 3.4.

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Chapter 4

CUBATURE FORMULAS OF MULTIVARIATE POLYNOMIALS ARISING FROM SYMMETRIC ORBIT FUNCTIONS

Authors: Jiří Hrivnák, Lenka Motlochová and Jiří Patera.

Abstract: This paper develops applications of symmetric orbit functions, known from irreducible representations of simple Lie groups, in numerical analysis. It is shown that these functions have remarkable properties which lead to cubature formulas, approximating a weighted integral of any function by a weighted finite sum of function values, in connection with any simple Lie group. The cubature formulas are specialized for simple Lie groups of rank two. An optimal approximation of any function by multivariate polynomials arising from symmetric orbit functions is discussed.

Introduction

The purpose of this paper is to extend the results of [**40, 44**] to the family of symmetric Weyl group orbit functions, known from the theory of irreducible representations of simple Lie groups. These functions have several remarkable properties such as continuous and discrete orthogonality described, for example, in [**24, 43**]. The simplest symmetric orbit function arising from the simple Lie group A_1 coincides, up to a constant, with the common cosine function of one variable. Therefore, it is related with the Chebyshev polynomials of the first kind [**52**]. Motivated by various applications of Chebyshev polynomials in numerical analysis, the symmetric orbit functions are studied in order to develop similar applications for functions of several variables. Firstly, a cubature formula arising in connection with any simple Lie group is derived. Secondly, a multivariate polynomial approximation is presented.

A cubature formula is a multivariate generalization of a quadrature formula useful for numerical integration of one-variable functions. It is a remarkable mathematical result approximating a weighted integral of any function of several variables by a finite weighted sum of values of the same function on a specific set of points, called nodes, in the real Euclidean space $\mathbb{R}^n$. It is required that a cubature formula is an exact equality for polynomial functions up to a certain degree. There exist different types of cubature formulas with various efficiencies in terms of the number of nodes required. For example, the extensively studied Chebyshev polynomials of the first kind are connected to the numerical integration of the maximal efficiency known as the Gauss-Chebyshev quadrature using the least number of nodes possible [**39, 52**], i.e. it equates a weighted integral of any polynomial of degree at most $2M-1$ with a linear combination of its values at zeros of the Chebyshev polynomials of the first kind of degree M [**39, 52**]. The optimal cubature formulas for functions of several variables are called Gaussian cubatures, see e.g. [**53**].

An optimal cubature formula is derived in [**35**] from antisymmetric orbit functions arising from the simple Lie group A_n. In [**35**], it is also shown that the study of the symmetric orbit functions of A_n lead to slightly less efficient cubature formulas. The idea of [**35**] is generalized in [**44**] to obtain Gaussian cubature formulas arising from antisymmetric orbit functions of simple Lie groups of any type and rank. The crucial step, that make the extension of the A_n result to any simple Lie group possible, is an uncommon definition of the degree of polynomials based on Lie-theoretical invariants. In [**40**], the set of cubature formulas is enriched using two families of additional hybrid orbit functions existing only for Lie groups with two different lengths of roots, B_n, C_n, F_4 and G_2. In this paper, we complete and extend [**35, 40, 44**] by exploiting the remaining family of orbit functions. The resulting formula is not optimal unlike the Gauss-Chebyshev quadrature in the case of the Chebyshev polynomials, however it is derived for any simple Lie group and not restricted only to the case A_1.

The second part is devoted to the investigation of multivariate polynomial approximations in a Hilbert space of complex-valued measurable functions with respect to some weighted integral. There exists a Hilbert basis of orthogonal multivariate polynomials related to the symmetric orbit functions. Therefore, any function from the Hilbert space is expressed as a series involving these polynomials. Its specific truncated sum is actually the best approximation of the function by polynomials not exceeding certain degree.

In Section 4.1, we set up notation and terminology concerning simple Lie groups and affine Weyl groups. We also review some of the standard facts on symmetric orbit functions such as continuous and discrete orthogonality. In Section 4.2, we deduce cubature formulas arising from symmetric orbit functions of any simple Lie algebra. In particular, we introduce the X-transform which modifies the fundamental domain of affine Weyl group to the region over which it is integrated. Similarly, the X-transform defines sets of nodes as images of specific lattice fragments of any density controlled by M. The greater M, the denser the lattice. The symmetry of the lattice is not affected by different values of M, only its scale is changed. Finally, the weight function is connected to the antisymmetric orbit functions. In Section 4.3, we establish the explicit cubature formulas of the rank two cases, A_2, C_2, G_2. Specifically, the case A_2 is connected to two-variable analogues of Jacobi polynomials on Steiner's hypocycloid [**30**], the case C_2 is related to two-variable analogues of Jacobi polynomials on a domain bounded by two lines and a parabola [**30, 31, 32**] (Gaussian cubature formulas connected to these polynomials are studied for example in [**53**]) and the cubature formula arising from G_2 is also described in [**34**]. Section 4.4 discusses an approximation by polynomials.

4.1. Root systems and polynomials

4.1.1. Pertinent properties of root systems and weight lattices

We use the notation established in [**16**]. Recall that, to the Lie algebra of the compact, connected, simply connected simple Lie group G of rank n, corresponds the set of simple roots $\Delta = (\alpha_1, \dots, \alpha_n)$ [**1, 2, 20, 61**]. The set Δ spans the Euclidean space $\mathbb{R}^n$, with the scalar product denoted by $\langle\, ,\, \rangle$. The following standard objects related to the set of simple roots Δ are used.

- The marks $m_1, \dots, m_n$ of the highest root $\xi \equiv -\alpha_0 = m_1\alpha_1 + \dots + m_n\alpha_n$.
- The Coxeter number $m = 1 + m_1 + \dots + m_n$ of G.
- The Cartan matrix C and its determinant

$$c = \det C. \tag{4.1.1}$$

- The root lattice $Q = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$.
- The $\mathbb{Z}$-dual lattice to Q,

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee, \alpha\rangle \in \mathbb{Z}, \forall \alpha \in \Delta\} = \mathbb{Z}\omega_1^\vee + \dots + \mathbb{Z}\omega_n^\vee$$

with the vectors $\omega_i^\vee$ given by

$$\langle \omega_i^\vee, \alpha_j\rangle = \delta_{ij}.$$

- The dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$, where $\alpha_i^\vee = 2\alpha_i/\langle\alpha_i,\, \alpha_i\rangle$.
- The dual marks $m_1^\vee, \ldots, m_n^\vee$ of the highest dual root $\eta \equiv -\alpha_0^\vee = m_1^\vee\alpha_1^\vee + \cdots + m_n^\vee\alpha_n^\vee$. The marks and the dual marks are summarized in Table 1 in [**16**]. The highest dual root η satisfies for all $i = 1, \ldots, n$

$$\langle\eta,\, \alpha_i\rangle \geq 0. \tag{4.1.2}$$

- The $\mathbb{Z}$-dual weight lattice to $Q^\vee$

$$P = \{\omega \in \mathbb{R}^n \mid \langle\omega,\, \alpha^\vee\rangle \in \mathbb{Z},\, \forall\alpha^\vee \in Q^\vee\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n,$$

with the vectors ω_i given by $\langle\omega_i,\, \alpha_j^\vee\rangle = \delta_{ij}$. For $\lambda \in P$ the following notation is used,

$$\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n = (\lambda_1, \ldots, \lambda_n). \tag{4.1.3}$$

- The partial ordering on P is given: for $\lambda, \nu \in P$ it holds that $\nu \leq \lambda$ if and only if $\lambda - \nu = k_1\alpha_1 + \cdots + k_n\alpha_n$ with $k_i \geq 0$ for all $i \in \{1, \ldots, n\}$.
- The half of the sum of the positive roots

$$\varrho = \omega_1 + \cdots + \omega_n.$$

- The cone of positive weights P^+ and the cone of strictly positive weights $P^{++} = \varrho + P^+$

$$P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\omega_n, \quad P^{++} = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n.$$

- n reflections r_α, $\alpha \in \Delta$ in $(n-1)$-dimensional "mirrors" orthogonal to simple roots intersecting at the origin denoted by

$$r_1 \equiv r_{\alpha_1}, \ldots, r_n \equiv r_{\alpha_n}.$$

Following [**44**], we define so called m-degree of $\lambda \in P^+$ as the scalar product of λ with the highest dual root η, i.e. by the relation

$$|\lambda|_m = \langle\lambda,\, \eta\rangle = \lambda_1 m_1^\vee + \cdots + \lambda_n m_n^\vee.$$

Let us denote a finite subset of the cone of the positive weights P^+ consisting of the weights of the m-degree not exceeding M by P_M^+, i.e.

$$P_M^+ = \left\{\lambda \in P^+ \mid |\lambda|_m \leq M\right\}.$$

Recall also the separation lemma which asserts for $\lambda \in P^+$, $\lambda \neq 0$ and any $M \in \mathbb{N}$ that

$$|\lambda|_m < 2M \quad \Rightarrow \quad \lambda \notin MQ. \tag{4.1.4}$$

Note that this lemma is proved in [**44**] for $M > m$ only — the proof, however, can be repeated verbatim with any $M \in \mathbb{N}$.

For two dominant weights $\lambda, \nu \in P^+$ for which $\nu \leq \lambda$ we have for their m-degrees

$$|\lambda|_m - |\nu|_m = \langle \lambda - \nu, \eta \rangle = \sum_{i=1}^{n} k_i \langle \alpha_i, \eta \rangle, \quad k_i \geq 0. \tag{4.1.5}$$

Taking into account equation (4.1.2), we have the following proposition.

Proposition 4.1.1. *For two dominant weights $\lambda, \nu \in P^+$ with $\nu \leq \lambda$ it holds that $|\nu|_m \leq |\lambda|_m$.*

4.1.2. Affine Weyl groups

The Weyl group W is generated by n reflections $r_1, \dots, r_n$ and its order $|W|$ can be calculated using the formula

$$|W| = n!\, m_1 \dots m_n\, c. \tag{4.1.6}$$

The affine Weyl group W^{aff} is the semidirect product of the Abelian group of translations $Q^\vee$ and of the Weyl group W,

$$W^{\mathrm{aff}} = Q^\vee \rtimes W. \tag{4.1.7}$$

The fundamental domain F of W^{aff}, which consists of precisely one point of each W^{aff}-orbit, is the convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}$. Considering $n+1$ real parameters $y_0, \dots, y_n \geq 0$, we have

$$F = \left\{ y_1\omega_1^\vee + \dots + y_n\omega_n^\vee \mid y_0 + y_1 m_1 + \dots + y_n m_n = 1 \right\}. \tag{4.1.8}$$

The volumes $\mathrm{vol}(F) \equiv |F|$ of the simplices F are calculated in [**16**].

Considering the standard action of W on $\mathbb{R}^n$, we denote for $\lambda \in \mathbb{R}^n$ the isotropy group and its order by

$$\mathrm{Stab}(\lambda) = \{ w \in W \mid w\lambda = \lambda \}, \quad h_\lambda \equiv |\mathrm{Stab}(\lambda)|,$$

and denote the orbit by

$$W\lambda = \{ w\lambda \in \mathbb{R}^n \mid w \in W \}.$$

Then the orbit-stabilizer theorem gives for the orders

$$|W\lambda| = \frac{|W|}{h_\lambda}. \tag{4.1.9}$$

Considering the standard action of W on the torus $\mathbb{R}^n/Q^\vee$, we denote for $x \in \mathbb{R}^n/Q^\vee$ the order of its orbit by $\varepsilon(x)$, i.e.

$$\varepsilon(x) = |\{ wx \in \mathbb{R}^n/Q^\vee \mid w \in W \}|. \tag{4.1.10}$$

For an arbitrary $M \in \mathbb{N}$, the grid F_M is given as cosets from the W-invariant group $\frac{1}{M}P^\vee/Q^\vee$ with a representative element in the fundamental domain F

$$F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F.$$

The representative points of F_M can be explicitly written as

$$F_M = \left\{ \frac{u_1}{M}\omega_1^\vee + \cdots + \frac{u_n}{M}\omega_n^\vee \mid u_0, u_1, \ldots, u_n \in \mathbb{Z}^{\geq 0},\, u_0 + u_1 m_1 + \cdots + u_n m_n = M \right\}. \tag{4.1.11}$$

The numbers of elements of F_M, denoted by $|F_M|$, are also calculated in [**16**] for all simple Lie algebras.

4.1.3. Orbit functions

Symmetric orbit functions [**24**] are defined as complex functions $C_\lambda : \mathbb{R}^n \to \mathbb{C}$ with the labels $\lambda \in P^+$,

$$C_\lambda(x) = \sum_{\nu \in W\lambda} e^{2\pi i \langle \nu, x \rangle}, \quad x \in \mathbb{R}^n. \tag{4.1.12}$$

Note that in [**16**] the results for C-functions are formulated for the normalized C-functions Φ_λ which are related to the orbit sums (4.1.12) as

$$\Phi_\lambda = h_\lambda \, C_\lambda.$$

Due to the symmetries with respect to the Weyl group W as well as with respect to the shifts from $Q^\vee$

$$C_\lambda(wx) = C_\lambda(x), \quad C_\lambda(x + q^\vee) = C_\lambda(x), \quad w \in W,\, q^\vee \in Q^\vee, \tag{4.1.13}$$

it is sufficient to consider C-functions restricted to the fundamental domain of the affine Weyl group F. Moreover, the C-functions are continuously orthogonal on F,

$$\int_F C_\lambda(x)\overline{C_{\lambda'}(x)}\, dx = \frac{|F||W|}{h_\lambda}\, \delta_{\lambda,\lambda'}\,. \tag{4.1.14}$$

and form a Hilbert basis of the space $\mathcal{L}^2(F)$ [**24**], i.e. any function $\widetilde{f} \in \mathcal{L}^2(F)$ can be expanded into the series of C-functions

$$\widetilde{f} = \sum_{\lambda \in P^+} c_\lambda C_\lambda, \quad c_\lambda = \frac{h_\lambda}{|F||W|} \int_F \widetilde{f}(x)\overline{C_\lambda(x)}\, dx. \tag{4.1.15}$$

Special case of the orthogonality relations (4.1.14) is when one of the weights is equal to zero,

$$\int_F C_\lambda(x)\, dx = |F|\, \delta_{\lambda,0}\,. \tag{4.1.16}$$

For any $M \in N$, the C-functions from a certain subset of P^+ are also discretely orthogonal on F_M and form a basis of the space of discretized functions $\mathbb{C}^{F_M}$ of dimension $|F_M|$ [**16**]; special case of these orthogonality relations is when one of the weights is equal to zero modulo the lattice MQ,

$$\sum_{x\in F_M} \varepsilon(x)C_\lambda(x) = \begin{cases} cM^n & \lambda \in MQ, \\ 0 & \lambda \notin MQ. \end{cases} \tag{4.1.17}$$

The key point in developing the cubature formulas is comparison of formulas (4.1.16) and (4.1.17) in the following proposition.

Proposition 4.1.2. *For any $M \in \mathbb{N}$ and $\lambda \in P^+_{2M-1}$ it holds that*

$$\frac{1}{|F|}\int_F C_\lambda(x)\,dx = \frac{1}{cM^n}\sum_{x\in F_M} \varepsilon(x)C_\lambda(x). \tag{4.1.18}$$

Proof. Suppose first that $\lambda = 0$. Then from (4.1.16) and (4.1.17) we obtain

$$\frac{1}{|F|}\int_F C_0(x)\,dx = 1 = \frac{1}{cM^n}\sum_{x\in F_M} \varepsilon(x)C_0(x)$$

Secondly let $\lambda \neq 0$ and $|\lambda|_m < 2M$. The from the separation lemma (4.1.4) we have that $\lambda \notin MQ$ and thus

$$\frac{1}{|F|}\int_F C_\lambda(x)\,dx = 0 = \frac{1}{cM^n}\sum_{x\in F_M} \varepsilon(x)C_\lambda(x).$$

□

Let us denote for convenience the C-functions corresponding to the basic dominant weights ω_j by Z_j, i.e.

$$Z_j \equiv C_{\omega_j}.$$

Recall from [**2**], Chapter VI, §4 that any W-invariant sum of the exponential functions $e^{2\pi i\langle \nu, a\rangle}$ can be expressed as a linear combination of some functions C_λ with $\lambda \in P^+$. Also for any $\lambda \in P^+$ a function of the monomial type $Z_1^{\lambda_1}Z_2^{\lambda_2}\dots Z_n^{\lambda_n}$ can be expressed as the sum of C-functions by less or equal dominant weights than λ, i.e.

$$Z_1^{\lambda_1}Z_2^{\lambda_2}\dots Z_n^{\lambda_n} = \sum_{\nu\le\lambda,\,\nu\in P^+} c_\nu C_\nu, \quad c_\nu \in \mathbb{C}, \quad c_\lambda = 1. \tag{4.1.19}$$

Conversely, any function C_λ, $\lambda \in P^+$ can be expressed as a polynomial in variables $Z_1,\dots,Z_n$, i.e. there exist a multivariate polynomial $\widetilde{p}_\lambda \in \mathbb{C}[y_1,\dots,y_n]$ such that

$$C_\lambda = \widetilde{p}_\lambda(Z_1,\dots,Z_n) = \sum_{\nu\le\lambda,\,\nu\in P^+} d_\nu Z_1^{\nu_1}Z_2^{\nu_2}\dots Z_n^{\nu_n}, \quad d_\nu \in \mathbb{C}, \quad d_\lambda = 1. \tag{4.1.20}$$

Antisymmetric orbit functions [**26**] are defined as complex functions $S_\lambda : \mathbb{R}^n \to \mathbb{C}$ with the labels $\lambda \in P^{++}$,

$$S_\lambda(x) = \sum_{w\in W} \det(w)\, e^{2\pi i \langle w\lambda,\, x\rangle}, \quad x \in \mathbb{R}^n. \tag{4.1.21}$$

The antisymmetry with respect to the Weyl group W and the symmetry with respect to the shifts from $Q^\vee$ holds,

$$S_\lambda(wx) = (\det w)\, S_\lambda(x), \quad S_\lambda(x+q^\vee) = S_\lambda(x), \quad w \in W,\, q^\vee \in Q^\vee,$$

Recall that Proposition 9 in [**26**] states that for the lowest S-function S_ϱ it holds that

$$S_\varrho(x) = 0, \quad x \in F \setminus F^\circ, \tag{4.1.22}$$

$$S_\varrho(x) \neq 0, \quad x \in F^\circ. \tag{4.1.23}$$

Since the square of the absolute value $|S_\varrho|^2 = S_\varrho \overline{S_\varrho}$ is a W-invariant sum of exponentials it can be expressed as a linear combinations of C-functions. Each C-function in this combination is moreover a polynomial of the form (4.1.20). Thus there exist a unique polynomial $\widetilde{K} \in \mathbb{C}[y_1,\dots,y_n]$ such that

$$|S_\varrho|^2 = \widetilde{K}(Z_1,\dots,Z_n). \tag{4.1.24}$$

4.2. Cubature formulas

4.2.1. The X-transform

The key component in the development of the cubature formulas is the integration by substitution. The X-transform transforms the fundamental $F \subset \mathbb{R}^n$ domain to the domain $\Omega \subset \mathbb{R}^n$ on which are the cubature rules defined. In order to obtain a real valued transform we first need to examine the values of the C-functions.

The C-functions of the algebras

$$A_1, B_n(n\geq 3), C_n(n\geq 2), D_{2k}(k\geq 2), E_7, E_8, F_4, G_2 \tag{4.2.1}$$

are real-valued [**24**]. Using the notation (4.1.3), for the remaining cases it holds that

$$\begin{aligned} A_n(n\geq 2):&\quad C_{(\lambda_1,\lambda_2,\dots,\lambda_n)}(x) = \overline{C_{(\lambda_n,\lambda_{n-1},\dots,\lambda_1)}(x)}\,,\\ D_{2k+1}(k\geq 2):&\quad C_{(\lambda_1,\lambda_2,\dots,\lambda_{2k-1},\lambda_{2k},\lambda_{2k+1})}(x) = \overline{C_{(\lambda_1,\lambda_2,\dots,\lambda_{2k-1},\lambda_{2k+1},\lambda_{2k})}(x)}\,,\\ E_6:&\quad C_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)}(x) = \overline{C_{(\lambda_5,\lambda_4,\lambda_3,\lambda_2,\lambda_1,\lambda_6)}(x)}\,. \end{aligned} \tag{4.2.2}$$

Specializing the relations (4.2.2) for the C-functions corresponding to the fundamental dominant weights Z_j, we obtain that the functions Z_j are real valued, except for the following cases for which it holds that

$$\begin{aligned} A_{2k}(k \geq 1): &\quad Z_j = \overline{Z_{2k-j+1}},\ j = 1, \dots, k, \\ A_{2k+1}(k \geq 1): &\quad Z_j = \overline{Z_{2k-j+2}},\ j = 1, \dots, k, \\ D_{2k+1}(k \geq 2): &\quad Z_{2k} = \overline{Z_{2k+1}}, \\ E_6: &\quad Z_2 = \overline{Z_4}, Z_1 = \overline{Z_5}. \end{aligned} \tag{4.2.3}$$

Taking into account (4.2.3), we introduce the real-valued functions X_j, $j \in \{1, \dots, n\}$ as follows. For the cases (4.2.1) we set

$$X_j \equiv Z_j, \tag{4.2.4}$$

and for the remaining cases (4.2.3) we define

$$\begin{aligned} A_{2k}: &\quad X_j = \frac{Z_j + Z_{2k-j+1}}{2},\ X_{2k-j+1} = \frac{Z_j - Z_{2k-j+1}}{2i},\ j = 1, \dots, k; \\ A_{2k+1}: &\quad X_j = \frac{Z_j + Z_{2k-j+2}}{2},\ X_{k+1} = Z_{k+1},\ X_{2k-j+2} = \frac{Z_j - Z_{2k-j+2}}{2i}, \\ &\quad j = 1, \dots, k; \\ D_{2k+1}: &\quad X_j = Z_j,\ X_{2k} = \frac{Z_{2k} + Z_{2k+1}}{2},\ X_{2k+1} = \frac{Z_{2k} - Z_{2k+1}}{2i}, \\ &\quad j = 1, \dots, 2k-1; \\ E_6: &\quad X_1 = \frac{Z_1 + Z_5}{2},\ X_2 = \frac{Z_2 + Z_4}{2},\ X_3 = Z_3,\ X_4 = \frac{Z_2 - Z_4}{2i}, \\ &\quad X_5 = \frac{Z_1 - Z_5}{2i}, X_6 = Z_6. \end{aligned} \tag{4.2.5}$$

Thus, we obtain a crucial mapping $X : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$X(x) \equiv (X_1(x), \dots, X_n(x)). \tag{4.2.6}$$

The image $\Omega \subset \mathbb{R}^n$ of the fundamental domain F under the mapping X forms the integration domain on which the cubature rules will be formulated, i.e.

$$\Omega \equiv X(F). \tag{4.2.7}$$

In order to use the mapping X for an integration by substitution we need to know that it is one-to-one except for possibly some set of zero measure. Since the image $\Omega_M \subset \mathbb{R}^n$ of the set of points F_M under the mapping X forms the set of nodes for the cubature rules, i.e.

$$\Omega_M \equiv X(F_M), \tag{4.2.8}$$

a discretized version of the one-to-one correspondence of the restricted mapping X_M of X to F_M, i.e.

$$X_M \equiv X \restriction_{F_M} \tag{4.2.9}$$

is also essential. Note that due to the periodicity of C-functions (4.1.13), the restriction (4.2.9) is well-defined for the cosets from F_M.

Proposition 4.2.1. *The mapping $X : F \to \Omega$, given by* (4.2.6), *is one-to-one correspondence except for some set of zero measure. For any $M \in \mathbb{N}$ is the restriction mapping $X_M : F_M \to \Omega_M$, given by* (4.2.9), *one-to-one correspondence and thus it holds that*

$$|\Omega_M| = |F_M|. \tag{4.2.10}$$

Proof. Let us assume that there exists a set $F' \subset F$ of non-zero measure such that $X(x) = X(y)$ with $x, y \in F'$. Since the transforms (4.2.4), (4.2.5) are as regular linear mappings one-to-one correspondences, this fact implies that $Z_1(x) = Z_1(y), \dots, Z_n(x) = Z_n(y)$ with $x, y \in F'$. Then from the polynomial expression (4.1.20) we obtain for all $\lambda \in P^+$ that it holds that $C_\lambda(x) = C_\lambda(y)$. Since the C-functions C_λ, $\lambda \in P^+$ form a Hilbert basis of the space $\mathcal{L}^2(F)$ we conclude that for any $f \in \mathcal{L}^2(F)$ is valid that $f(x) = f(y)$, $x, y \in F'$ which is contradiction.

Retracing the steps of the continuous case above, let us assume that there exist two distinct points $x, y \in F_M$, $x \neq y$ such that $X(x) = X(y)$. Since the transforms (4.2.4), (4.2.5) are as regular linear mappings one-to-one correspondences, this fact again implies that $Z_1(x) = Z_1(y), \dots, Z_n(x) = Z_n(y)$. Then from the polynomial expression (4.1.20) we obtain for all $\lambda \in P^+$ that it holds that $C_\lambda(x) = C_\lambda(y)$. The same equality has hold for those C-functions C_λ which form a basis of the space $\mathbb{C}^{F_M}$. We conclude that for any $f \in \mathbb{C}^{F_M}$ is valid that $f(x) = f(y)$, $x \neq y$ which is contradiction. $\square$

The absolute value of the determinant of the Jacobian matrix of the X-transform (4.2.6) is essential for construction of the cubature formulas — its value is determined in the following proposition.

Proposition 4.2.2. *The absolute value of the Jacobian determinant $|J_x(X)|$ of the X-transform* (4.2.6) *is given by*

$$|J_x(X)| = \frac{\kappa(2\pi)^n}{|F||W|}|S_\varrho(x)|, \tag{4.2.11}$$

where κ is defined as

$$\kappa = \begin{cases} 2^{-\lfloor \frac{n}{2} \rfloor} & \textit{for } A_n \\ \frac{1}{2} & \textit{for } D_{2k+1} \\ \frac{1}{4} & \textit{for } E_6 \\ 1 & \textit{otherwise.} \end{cases} \tag{4.2.12}$$

PROOF. Note that the X transform can be composed of the the following two transforms: the transform $\zeta : x \mapsto (Z_1(x), \dots, Z_n(x))$ and the transform $R : (Z_1, \dots, Z_n) \mapsto (X_1, \dots, X_n)$ via relations (4.2.4), (4.2.5). To calculate the Jacobian of the transform ζ, let us denote by $\alpha^\vee$ the matrix of the coordinates (in columns) of the vectors $\alpha_1^\vee, \dots, \alpha_n^\vee$ in the standard orthonormal basis of $\mathbb{R}^n$ and by $a_1, \dots, a_n$ the coordinates of a point $x \in \mathbb{R}^n$ in $\alpha^\vee$-basis, i.e. $x = a_1\alpha_1^\vee + \dots + a_n\alpha_n^\vee$. If a denotes the coordinates $a_1, \dots, a_n$ arranged in a column vector then it holds that $x = \alpha^\vee a$. The absolute value of the Jacobian of the mapping $a \mapsto (Z_1(\alpha^\vee a), \dots, Z_n(\alpha^\vee a))$ is according to equation (32) in [**44**] given by $(2\pi)^n |S_\varrho(\alpha^\vee a)|$. Using the chain rule, this implies for the absolute value of the Jacobian $|J_x(\zeta)|$ of the map ζ that

$$|J_x(\zeta)| = |\det \alpha^\vee|^{-1} (2\pi)^n |S_\varrho(x)|.$$

It can be seen directly from formula (4.1.6) and Proposition 2.1 in [**16**] that

$$|\det \alpha^\vee| = |W||F|.$$

The calculation of the absolute value of the Jacobian determinant $\kappa = |J_x(R)|$ is straightforward from definitions (4.2.4), (4.2.5). $\square$

4.2.2. The cubature formula

We attach to any $\lambda \in P^+$ a monomial $y^\lambda \equiv y_1^{\lambda_1} \dots y_n^{\lambda_n} \in \mathbb{C}[y_1, \dots, y_n]$ and assign to this monomial the m-degree $|\lambda|_m$ of λ. The m-degree of a polynomials $p \in \mathbb{C}[y_1, \dots, y_n]$, denoted by $\deg_m p$, is defined as the largest m-degree of a monomial occurring in $p(y) \equiv p(y_1, \dots, y_n)$. For instance we observe from Proposition 4.1.1 and (4.1.20) that the m-degree of the C-polynomials $\widetilde{p}_\lambda$ coincides with the m-degree of λ,

$$\deg_m \widetilde{p}_\lambda = |\lambda|_m. \tag{4.2.13}$$

The subspace $\Pi_M \subset \mathbb{C}[y_1, \dots, y_n]$ is formed by the polynomials of m-degree at most M, i.e.

$$\Pi_M \equiv \{p \in \mathbb{C}[y_1, \dots, y_n] \mid \deg_m p \leq M\} . \tag{4.2.14}$$

In order to investigate how the m-degree of a polynomial changes under the substitution of the type (4.2.5) we formulate the following proposition.

Proposition 4.2.3. *Let $j,k \in \{1,\dots,n\}$ be two distinct indices $j<k$ such that $m_j^\vee = m_k^\vee$ and $p,\widetilde{p} \in \mathbb{C}[y_1,\dots,y_n]$ two polynomials such that*

$$\widetilde{p}(y_1,\dots,y_n) = p(y_1,\dots,y_{j-1},\frac{y_j+y_k}{2},\dots,y_{k-1},\frac{y_j-y_k}{2i},\dots y_n)$$

holds. Then $\deg_m\widetilde{p} = \deg_m p$.

Proof. Since any polynomial p is a linear combination of monomials y^λ, it is sufficient to prove $\deg_m\widetilde{p} = \deg_m p$ for all monomials p. If p is a monomial y^λ, then

$$\widetilde{p}(y_1,\dots,y_n) = \left(\frac{y_j+y_k}{2}\right)^{\lambda_j}\left(\frac{y_j-y_k}{2i}\right)^{\lambda_k}\prod_{l\in\{1,\dots,n\}\setminus\{j,k\}} y_l^{\lambda_l}\,.$$

Using the binomial expansion, we obtain that $\widetilde{p}(y_1,\dots,y_n)$ is equal to

$$\frac{1}{i^{\lambda_k}2^{\lambda_j+\lambda_k}}\sum_{r=0}^{\lambda_j}\sum_{s=0}^{\lambda_k}(-1)^{\lambda_k-s}\binom{\lambda_j}{r}\binom{\lambda_k}{s}y_j^{r+s}y_k^{\lambda_j+\lambda_k-(r+s)}\prod_{l\in\{1,\dots,n\}\setminus\{j,k\}} y_l^{\lambda_l}\,.$$

Therefore, the m-degree of the polynomial $\widetilde{p}$ is given by

$$\deg_m\widetilde{p} = \max_{r,s}\{(r+s)m_j^\vee + (\lambda_j+\lambda_k-(r+s))m_k^\vee + \sum_{l\in\{1,\dots,n\}\setminus\{j,k\}}\lambda_l m_l^\vee\}\,.$$

Since we assume that $m_j^\vee = m_k^\vee$, we conclude that $\deg_m\widetilde{p} = \sum_{l=1}^n \lambda_l m_l^\vee = \deg_m p$. $\square$

Having the X_M transform (4.2.9), it is possible to transfer uniquely the values (4.1.10) of $\varepsilon(x)$, $x\in F_M$ to the points of Ω_M, i.e. by the relation

$$\widetilde{\varepsilon}(y) \equiv \varepsilon(X_M^{-1}y), \quad y\in\Omega_M. \tag{4.2.15}$$

Taking the inverse transforms of (4.2.4), (4.2.5) and substituting them into the polynomial (4.1.24) we obtain the polynomial $K\in\mathbb{C}[y_1,\dots,y_n]$ such that

$$|S_\varrho|^2 = K(X_1,\dots,X_n), \tag{4.2.16}$$

Theorem 4.2.1 (Cubature formula)**.** *For any $M\in\mathbb{N}$ and any $p\in\Pi_{2M-1}$ it holds that*

$$\int_\Omega p(y)K^{-\frac{1}{2}}(y)\,dy = \frac{\kappa}{c|W|}\left(\frac{2\pi}{M}\right)^n\sum_{y\in\Omega_M}\widetilde{\varepsilon}(y)p(y)\,. \tag{4.2.17}$$

Proof. Proposition 4.2.1 guarantees that the X-transform is one-to-one except for some set of measure zero and Proposition 4.2.2 together with (4.1.23) gives that the Jacobian determinant is non-zero except for the boundary of F. Thus using the integration by substitution $y = X(x)$ we obtain

$$\int_\Omega p(y) K^{-\frac{1}{2}}(y)\, dy = \frac{\kappa(2\pi)^n}{|F||W|} \int_F p(X(x))\, dx.$$

The one-to-one correspondence for the points F_M and Ω_M from Proposition 4.2.1 enables us to rewrite the finite sum in (4.2.17) as

$$\frac{\kappa}{c|W|}\left(\frac{2\pi}{M}\right)^n \sum_{y\in\Omega_M} \widetilde{\varepsilon}(y) p(y) = \frac{\kappa}{c|W|}\left(\frac{2\pi}{M}\right)^n \sum_{x\in F_M} \varepsilon(x) p(X(x)). \tag{4.2.18}$$

Successively applying Proposition 4.2.3 to perform the substitutions (4.2.5) in p we conclude that there exist a polynomial $\widetilde{p} \in \Pi_{2M-1}$ such that $\widetilde{p}(Z_1,\dots,Z_n) = p(X_1,\dots,X_n)$. Due to (4.1.19) we obtain for the polynomial $\widetilde{p}$ that

$$\begin{aligned} p(X_1,\dots,X_n) = \widetilde{p}(Z_1,\dots,Z_n) &= \sum_{\lambda\in P^+_{2M-1}} \widetilde{c}_\lambda Z_1^{\lambda_1} Z_2^{\lambda_2} \dots Z_n^{\lambda_n} \\ &= \sum_{\lambda\in P^+_{2M-1}} \widetilde{c}_\lambda \sum_{\nu\leq\lambda,\, \nu\in P^+} c_\nu C_\nu \end{aligned}$$

and therefore it holds that

$$\frac{1}{|F|}\int_F p(X(x))\, dx = \sum_{\lambda\in P^+_{2M-1}} \widetilde{c}_\lambda \sum_{\nu\leq\lambda,\, \nu\in P^+} c_\nu \frac{1}{|F|}\int_F C_\nu(x)\, dx \tag{4.2.19}$$

and

$$\frac{1}{cM^n}\sum_{x\in F_M} \varepsilon(x) p(X(x)) = \sum_{\lambda\in P^+_{2M-1}} \widetilde{c}_\lambda \sum_{\nu\leq\lambda,\, \nu\in P^+} c_\nu \frac{1}{cM^n}\sum_{x\in F_M} \varepsilon(x) C_\nu(x). \tag{4.2.20}$$

Since Proposition 4.1.1 states that for all $\nu \leq \lambda$ it holds that $|\nu|_m \leq |\lambda|_m \leq 2M-1$, we connect equations (4.2.19) and (4.2.20) by Proposition 4.1.2. □

Note that for practical purposes it may be more convenient to use the cubature formula (4.2.17) in its less developed form resulting from (4.2.18),

$$\int_\Omega p(y) K^{-\frac{1}{2}}(y)\, dy = \frac{\kappa}{c|W|}\left(\frac{2\pi}{M}\right)^n \sum_{x\in F_M} \varepsilon(x) p(X(x))\,. \tag{4.2.21}$$

This form may be more practical since the explicit inverse transform to X_M is usually not available and, on the contrary, the calculation of the coefficients $\varepsilon(x)$ and the points $X(x)$, $x \in F_M$ is straightforward.

4.3. Cubature formulas of rank two

In this section we specialize the cubature formula for the irreducible root systems of rank two. Let us firstly recall some basic facts about root systems of rank 2, i.e. A_2, C_2 and G_2. They are characterized by two simple roots $\Delta =$

(α_1, α_2) which satisfy

$$\begin{aligned} A_2: \quad & \langle \alpha_1, \alpha_1 \rangle = 2\,, \quad \langle \alpha_2, \alpha_2 \rangle = 2\,, \quad \langle \alpha_1, \alpha_2 \rangle = -1, \\ C_2: \quad & \langle \alpha_1, \alpha_1 \rangle = 1\,, \quad \langle \alpha_2, \alpha_2 \rangle = 2\,, \quad \langle \alpha_1, \alpha_2 \rangle = -1, \\ G_2: \quad & \langle \alpha_1, \alpha_1 \rangle = 2\,, \quad \langle \alpha_2, \alpha_2 \rangle = \frac{2}{3}\,, \quad \langle \alpha_1, \alpha_2 \rangle = -1\,. \end{aligned} \tag{4.3.1}$$

The transformation rules among the root system and the remaining three bases are given as follows,

$$\begin{aligned} A_2: \quad & \alpha_1 = \alpha_1^\vee = 2\omega_1 - \omega_2\,, \quad \alpha_2 = \alpha_2^\vee = -\omega_1 + 2\omega_2\,, \quad \omega_1 = \omega_1^\vee\,, \quad \omega_2 = \omega_2^\vee\,, \\ C_2: \quad & \alpha_1 = \frac{1}{2}\alpha_1^\vee = 2\omega_1 - \omega_2\,, \quad \alpha_2 = \alpha_2^\vee = -2\omega_1 + 2\omega_2\,, \quad \omega_1 = \frac{1}{2}\omega_1^\vee\,, \quad \omega_2 = \omega_2^\vee\,, \\ G_2: \quad & \alpha_1 = \alpha_1^\vee = 2\omega_1 - 3\omega_2\,, \quad \alpha_2 = \frac{1}{3}\alpha_2^\vee = -\omega_1 + 2\omega_2\,, \quad \omega_1 = \omega_1^\vee\,, \quad \omega_2 = \frac{1}{3}\omega_2^\vee\,. \end{aligned} \tag{4.3.2}$$

Taking the weights in the standard form $\lambda = (\lambda_1, \lambda_2) = \lambda_1\omega_1 + \lambda_2\omega_2$, the corresponding Weyl group is generated by reflections r_1 and r_2 of the explicit form

$$\begin{aligned} A_2: \quad & r_1(\lambda_1, \lambda_2) = (-\lambda_1, \lambda_1 + \lambda_2)\,, \quad r_2(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2, -\lambda_2)\,, \\ C_2: \quad & r_1(\lambda_1, \lambda_2) = (-\lambda_1, \lambda_1 + \lambda_2)\,, \quad r_2(\lambda_1, \lambda_2) = (\lambda_1 + 2\lambda_2, -\lambda_2)\,, \\ G_2: \quad & r_1(\lambda_1, \lambda_2) = (-\lambda_1, 3\lambda_1 + \lambda_2)\,, \quad r_2(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2, -\lambda_2)\,. \end{aligned} \tag{4.3.3}$$

Any Weyl group orbit of a generic point $\lambda \in P$ consists of

$$\begin{aligned} A_2: \quad & \{(\lambda_1, \lambda_2)\,, (-\lambda_1, \lambda_1 + \lambda_2)\,, (\lambda_1 + \lambda_2, -\lambda_2)\,, (-\lambda_2, -\lambda_1)\,, (-\lambda_1 - \lambda_2, \lambda_1)\,, \\ & (\lambda_2, -\lambda_1 - \lambda_2)\}\,, \\ C_2: \quad & \{\pm(\lambda_1, \lambda_2)\,, \pm(-\lambda_1, \lambda_1 + \lambda_2)\,, \pm(\lambda_1 + 2\lambda_2, -\lambda_2)\,, \pm(\lambda_1 + 2\lambda_2, -\lambda_1 - \lambda_2)\}\,, \\ G_2: \quad & \{\pm(\lambda_1, \lambda_2)\,, \pm(-\lambda_1, 3\lambda_1 + \lambda_2)\,, \pm(\lambda_1 + \lambda_2, -\lambda_2)\,, \pm(2\lambda_1 + \lambda_2, -3\lambda_1 - \lambda_2)\,, \\ & \pm(-\lambda_1 - \lambda_2, 3\lambda_1 + 2\lambda_2)\,, \pm(-2\lambda_1 - \lambda_2, 3\lambda_1 + 2\lambda_2)\}\,. \end{aligned} \tag{4.3.4}$$

4.3.1. The case A_2

If the points are considered in the $\alpha^\vee$-basis, $x = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, the symmetric C-functions (4.3.4) and antisymmetric S-functions of A_2 are explicitly given by

$$\begin{aligned} C_{(\lambda_1,\lambda_2)}(a_1, a_2) = \frac{1}{h_\lambda} \Big(& e^{2\pi i(\lambda_1 a_1 + \lambda_2 a_2)} + e^{2\pi i(-\lambda_1 a_1 + (\lambda_1 + \lambda_2) a_2)} + e^{2\pi i((\lambda_1 + \lambda_2) a_1 - \lambda_2 a_2)} \\ & + e^{2\pi i(\lambda_2 a_1 - (\lambda_1 + \lambda_2) a_2)} + e^{2\pi i((-\lambda_1 - \lambda_2) a_1 + \lambda_1 a_2)} + e^{2\pi i(-\lambda_2 a_1 - \lambda_1 a_2)} \Big)\,, \\ S_{(\lambda_1,\lambda_2)}(a_1, a_2) = \; & e^{2\pi i(\lambda_1 a_1 + \lambda_2 a_2)} - e^{2\pi i(-\lambda_1 a_1 + (\lambda_1 + \lambda_2) a_2)} - e^{2\pi i((\lambda_1 + \lambda_2) a_1 - \lambda_2 a_2)} \\ & + e^{2\pi i(\lambda_2 a_1 - (\lambda_1 + \lambda_2) a_2)} + e^{2\pi i((-\lambda_1 - \lambda_2) a_1 + \lambda_1 a_2)} - e^{2\pi i(-\lambda_2 a_1 - \lambda_1 a_2)}\,, \end{aligned}$$

		h_λ	
$\lambda \in P^+$	A_2	C_2	G_2
$(0,0)$	6	8	12
$(\star,0)$	2	2	2
$(0,\star)$	2	2	2
$(\star,\star)$	1	1	1

TABLE 4.1. The values of h_λ are shown for A_2, C_2 and G_2 with $\star$ denoting the corresponding coordinate different from 0.

where the values h_λ can be found in Table 4.1. Performing the transform (4.2.5), the resulting real-valued functions X_1, X_2 are given by

$$\begin{aligned} X_1(a_1,a_2) &= \cos 2\pi a_1 + \cos 2\pi a_2 + \cos 2\pi(a_1-a_2)\,, \\ X_2(a_1,a_2) &= \sin 2\pi a_1 - \sin 2\pi a_2 - \sin 2\pi(a_1-a_2)\,. \end{aligned}$$

The integral domain Ω can be described explicitly as

$$\Omega = \left\{(y_1,y_2)\in\mathbb{R}^2 \mid -(y_1^2+y_2^2+9)^2 + 8(y_1^3-3y_1y_2^2)+108 \geq 0\right\}.$$

The weight K-polynomial (4.2.16) is given explicitly as

$$K(y_1,y_2) = -(y_1^2+y_2^2+9)^2 + 8(y_1^3-3y_1y_2^2)+108\,.$$

The index set which will label the sets of points F_M and Ω_M is introduced via

$$I_M = \left\{[s_0,s_1,s_2]\in(\mathbb{Z}^{\geq 0})^3 \mid s_0+s_1+s_2 = M\right\}.$$

Thus the grid F_M consists of the points

$$F_M = \left\{\frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \mid [s_0,s_1,s_2]\in I_M\right\}.$$

If for $j=[s_0,s_1,s_2]\in I_M$ we denote

$$(y_1^{(j)},y_2^{(j)}) = \left(X_1\left(\frac{2s_1+s_2}{3M},\frac{s_1+2s_2}{3M}\right), X_2\left(\frac{2s_1+s_2}{3M},\frac{s_1+2s_2}{3M}\right)\right)$$

then the set of nodes Ω_M consists of the points

$$\Omega_M = \left\{(y_1^{(j)},y_2^{(j)})\in\mathbb{R}^2 \mid j\in I_M\right\}.$$

The integration domain Ω is together with the set of nodes Ω_{15} depicted in Figure 4.1. Each point of F_M as well as of Ω_M is labelled by the index set I_M and it is convenient for the point $x\in F_M$ and its image in Ω_M labelled by $j\in I_M$ to denote

$$\varepsilon_j = \varepsilon(x) = \widetilde{\varepsilon}(X_M(x)).$$

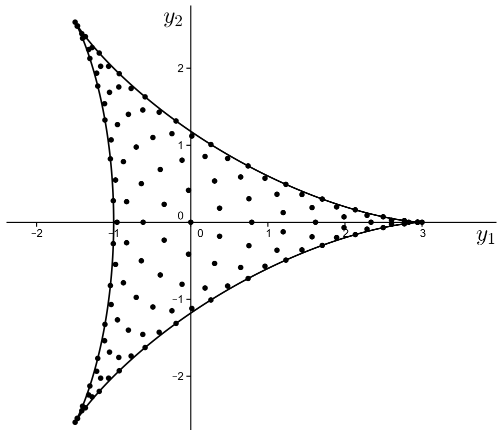

FIGURE 4.1. The region Ω of A_2 together with the points of Ω_{15}. The boundary of Ω is defined by the equation $K(y_1, y_2) = 0$.

The values of ε_j can be found in Table 4.2. The cubature rule for any $p \in \Pi_{2M-1}$ is of the form

$$\int_\Omega p(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2)\, dy_1\, dy_2 = \frac{\pi^2}{9M^2} \sum_{j \in I_M} \varepsilon_j p(y_1^{(j)}, y_2^{(j)}). \tag{4.3.5}$$

The cubature rule (4.3.5) is an analogue of the formula deduced in [**33**] using the generalized cosine functions TC_k and generalized sine functions TS_k defined by

$$\begin{aligned} TC_k(t) = \frac{1}{3} \Big[& e^{\frac{i\pi}{3}(k_2-k_3)(t_2-t_3)} \cos k_1 \pi t_1 + e^{\frac{i\pi}{3}(k_2-k_3)(t_3-t_1)} \cos k_1 \pi t_2 \\ & + e^{\frac{i\pi}{3}(k_2-k_3)(t_1-t_2)} \cos k_1 \pi t_3 \Big] \,, \\ TS_k(t) = \frac{1}{3} \Big[& e^{\frac{i\pi}{3}(k_2-k_3)(t_2-t_3)} \sin k_1 \pi t_1 + e^{\frac{i\pi}{3}(k_2-k_3)(t_3-t_1)} \sin k_1 \pi t_2 \\ & + e^{\frac{i\pi}{3}(k_2-k_3)(t_1-t_2)} \sin k_1 \pi t_3 \Big] \,, \end{aligned}$$

where $t = (t_1, t_2, t_3) \in \mathbb{R}^3$ with $t_1 + t_2 + t_3 = 0$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_1 + k_2 + k_3 = 0$. It can be shown by performing the following change of variables and parameters

$$\begin{aligned} & t_1 = 2a_1 - a_2 \,, \quad t_2 = -a_1 + 2a_2 \,, \quad t_3 = -a_1 - a_2 \,, \\ & k_1 = \lambda_1 \,, \quad k_2 = \lambda_2 \,, \quad k_3 = -\lambda_1 - \lambda_2 \,, \end{aligned}$$

that generalized cosine and sine functions actually coincide (up to scalar multiplication) with the symmetric C-functions and antisymmetric S-functions of A_2.

More precisely, we obtain

$$TC_k(t) = \frac{h_\lambda}{6} C_\lambda(x)\,, \qquad TS_k(t) = \frac{1}{6} S_\lambda(t)\,.$$

Example 4.3.1. *The cubature formula* (4.3.5) *is the exact equality of a weighted integral of any polynomial function of m-degree up to M with a weighted sum of finite number of polynomial values. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.*

If we choose the function $f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)$ as our test function, then we can estimate the integral of 1 *over* Ω

$$\int_\Omega f(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2)\, dy_1\, dy_2 = \int_\Omega 1\, dy_1\, dy_2$$

by finite weighted sums with different M's and compare the obtained results with the exact value of the integral of 1 *which is* $2\pi \doteq 6.2832$. *Table 4.3 shows the values of the finite weighted sums for* $M = 10, 20, 30, 50, 100$.

4.3.2. The case C_2

If the points are considered in the $\alpha^\vee$-basis, $x = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, the symmetric C-functions (4.3.4) and antisymmetric S-functions of C_2 are explicitly given by

$$\begin{aligned} C_{(\lambda_1,\lambda_2)}(a_1, a_2) &= \frac{2}{h_\lambda}\left(\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) + \cos 2\pi(-\lambda_1 a_1 + (\lambda_1+\lambda_2)a_2)\right. \\ &\quad + \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - \lambda_2 a_2) \\ &\quad \left.+ \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - (\lambda_1+\lambda_2)a_2)\right)\,, \\ S_{(\lambda_1,\lambda_2)}(a_1, a_2) &= 2\left(\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) - \cos 2\pi(-\lambda_1 a_1 + (\lambda_1+\lambda_2)a_2)\right. \\ &\quad - \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - \lambda_2 a_2) \\ &\quad \left.+ \cos 2\pi((\lambda_1 + 2\lambda_2)a_1 - (\lambda_1+\lambda_2)a_2)\right)\,. \end{aligned}$$

Performing the transform (4.2.4), the resulting real-valued functions X_1, X_2 are given by

$$X_1 = 2(\cos 2\pi a_1 + \cos 2\pi(a_1 - a_2))\,, \quad X_2 = 2(\cos 2\pi a_2 + \cos 2\pi(2a_1 - a_2))\,.$$

The integral domain Ω can be described explicitly as

$$\Omega = \left\{(y_1, y_2) \in \mathbb{R}^2 \mid -2y_1 - 4 \le y_2\,,\ 2y_1 - 4 \le y_2\,,\ \frac{1}{4}y_1^2 \ge y_2\right\}.$$

The weight K-polynomial (4.2.16) is given explicitly as

$$K(y_1, y_2) = (y_1^2 - 4y_2)((y_2+4)^2 - 4y_1^2)\,.$$

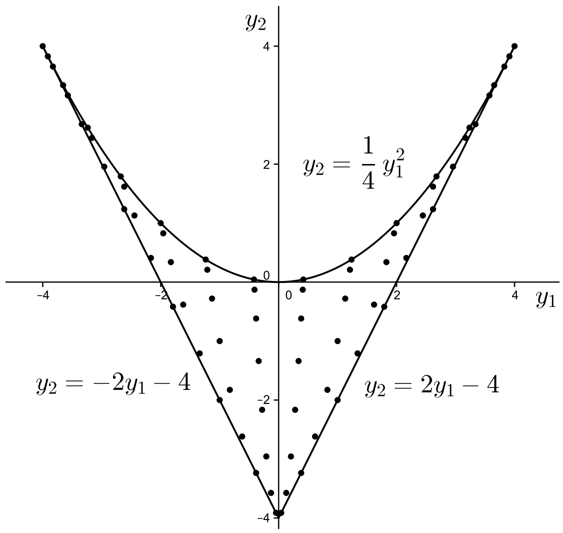

FIGURE 4.2. The region Ω of C_2 together with with the points Ω_{15}. The boundary is described three equations $y_2 = -2y_1 - 4$, $y_2 = 2y_1 - 4$ and $y_2 = \frac{1}{4}y_1^2$.

The index set which will label the sets of points F_M and Ω_M is introduced via

$$I_M = \left\{[s_0, s_1, s_2] \in (\mathbb{Z}^{\geq 0})^3 \mid s_0 + 2s_1 + s_2 = M\right\} .$$

Thus the grid F_M consists of the points

$$F_M = \left\{\frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \mid [s_0, s_1, s_2] \in I_M\right\} .$$

If for $j = [s_0, s_1, s_2] \in I_M$ we denote

$$(y_1^{(j)}, y_2^{(j)}) = \left(X_1\left(\frac{2s_1 + s_2}{2M}, \frac{s_1 + s_2}{M}\right), X_2\left(\frac{2s_1 + s_2}{2M}, \frac{s_1 + s_2}{M}\right)\right)$$

then the set of nodes Ω_M consists of the points

$$\Omega_M = \left\{(y_1^{(j)}, y_2^{(j)}) \in \mathbb{R}^2 \mid j \in I_M\right\} .$$

The integration domain Ω is together with the set of nodes Ω_{15} depicted in Figure 4.2. Similarly to the case A_2, each point of F_M as well as of Ω_M is labelled by the index set I_M and it is convenient for the point $x \in F_M$ and its image in Ω_M labelled by $j \in I_M$ to denote

$$\varepsilon_j = \varepsilon(x) = \tilde{\varepsilon}(X_M(x)) .$$

The values of ε_j can be found in Table 4.2. The cubature rule for any $p \in \Pi_{2M-1}$

	ε_j		
$j \in I_M$	A_2	C_2	G_2
$[\star, 0, 0]$	1	1	1
$[0, \star, 0]$	1	2	3
$[0, 0, \star]$	1	1	2
$[\star, \star, 0]$	3	4	6
$[\star, 0, \star]$	3	4	6
$[\star, \star, 0]$	3	4	6
$[\star, \star, \star]$	6	8	12

TABLE 4.2. The values of ε_j are shown for A_2, C_2 and G_2 with $\star$ denoting the corresponding coordinate different from 0.

takes the form

$$\int_\Omega p(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1\, dy_2 = \frac{\pi^2}{4M^2} \sum_{j \in I_M} \varepsilon_j p(y_1^{(j)}, y_2^{(j)}))\,. \tag{4.3.6}$$

Example 4.3.2. *The cubature formula* (4.3.6) *is the exact equality of a weighted integral of any polynomial function of m-degree up to M with a weighted sum of finite number of polynomial values. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.*

Similarly to Example 4.3.1, if we choose the function $f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)$ as our test function, then we can estimate the integral of 1 *over* Ω

$$\int_\Omega f(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2)\, dy_1\, dy_2 = \int_\Omega 1\, dy_1\, dy_2$$

by finite weighted sums with different M's and compare the obtained results with the exact value of the integral of 1 *which is* $\frac{32}{3} = 10,666\bar{6}$*. Table 4.3 shows the values of the finite weighted sums for* $M = 10, 20, 30, 50, 100$*.*

4.3.3. The case G_2

If the points are considered in the $\alpha^\vee$-basis, $x = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, the symmetric C-functions (4.3.4) and antisymmetric S-functions of G_2 are explicitly given by

$$\begin{aligned}
C_{(\lambda_1,\lambda_2)}(a_1,a_2) = \frac{2}{h_\lambda}&\left(\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) + \cos 2\pi(-\lambda_1 a_1 + (3\lambda_1+\lambda_2)a_2)\right.\\
&+ \cos 2\pi((\lambda_1+\lambda_2)a_1 - \lambda_2 a_2)\\
&+ \cos 2\pi((2\lambda_1+\lambda_2)a_1 - (3\lambda_1+\lambda_2)a_2)\\
&+ \cos 2\pi((-\lambda_1-\lambda_2)a_1 + (3\lambda_1+2\lambda_2)a_2)\\
&\left.+ \cos 2\pi((-2\lambda_1-\lambda_2)a_1 + (3\lambda_1+2\lambda_2)a_2)\right) ,\\
S_{(\lambda_1,\lambda_2)}(a_1,a_2) = 2&\left(\cos 2\pi(\lambda_1 a_1 + \lambda_2 a_2) - \cos 2\pi(-\lambda_1 a_1 + (3\lambda_1+\lambda_2)a_2)\right.\\
&- \cos 2\pi((\lambda_1+\lambda_2)a_1 - \lambda_2 a_2)\\
&+ \cos 2\pi((2\lambda_1+\lambda_2)a_1 - (3\lambda_1+\lambda_2)a_2)\\
&+ \cos 2\pi((-\lambda_1-\lambda_2)a_1 + (3\lambda_1+2\lambda_2)a_2)\\
&\left.- \cos 2\pi((-2\lambda_1-\lambda_2)a_1 + (3\lambda_1+2\lambda_2)a_2)\right) .
\end{aligned}$$

Performing the transform (4.2.4), the resulting real-valued functions X_1, X_2 are given by

$$\begin{aligned}
X_1 &= 2(\cos 2\pi a_1 + \cos 2\pi(a_1 - 3a_2) + \cos 2\pi(2a_1 - 3a_2))\,,\\
X_2 &= 2(\cos 2\pi a_2 + \cos 2\pi(a_1 - a_2) + \cos 2\pi(a_1 - 2a_2))\,.
\end{aligned}$$

The integral domain Ω consists of all points $(y_1,y_2)\in\mathbb{R}^2$ such that

$$-2((y_2+3)^{\frac{3}{2}} + 3y_2 + 6) \le y_1 \le 2((y_2+3)^{\frac{3}{2}} - 3y_2 - 6), \quad y_1 \ge \frac{1}{4}y_2^2 - 3\,.$$

The weight K-polynomial (4.2.16) is given explicitly as

$$K(y_1,y_2) = (y_2^2 - 4y_1 - 12)(y_1^2 - 4y_2^3 + 12y_1y_2 + 24y_1 + 36y_2 + 36)\,.$$

The index set which will label the sets of points F_M and Ω_M is introduced via

$$I_M = \left\{[s_0,s_1,s_2]\in(\mathbb{Z}^{\ge 0})^3 \mid s_0 + 2s_1 + 3s_2 = M\right\}\,.$$

Thus the grid F_M consists of the points

$$F_M = \left\{\frac{s_1}{M}\omega_1^\vee + \frac{s_1}{M}\omega_2^\vee \mid [s_0,s_1,s_2]\in I_M\right\}$$

If for $j=[s_0,s_1,s_2]\in I_M$ we denote

$$(y_1^{(j)},y_2^{(j)}) = \left(X_1\left(\frac{2s_1+3s_2}{M},\frac{s_1+2s_2}{M}\right), X_2\left(\frac{2s_1+3s_2}{M},\frac{s_1+2s_2}{M}\right)\right)$$

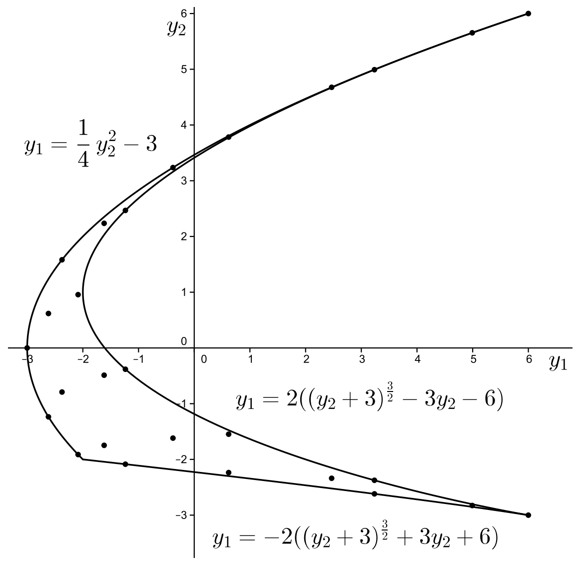

FIGURE 4.3. The region Ω of G_2 together with the points of Ω_{15}. The boundary of Ω is described by three equations $y_1 = \frac{1}{4}y_2^2 - 3$, $y_1 = 2((y_2+3)^{\frac{3}{2}} - 3y_2 - 6)$ and $y_1 = -2((y_2+3)^{\frac{3}{2}} + 3y_2 + 6)$.

then the set of nodes Ω_M consists of the points

$$\Omega_M = \left\{ (y_1^{(j)}, y_2^{(j)}) \in \mathbb{R}^2 \mid j \in I_M \right\} .$$

The integration domain Ω is together with the set of nodes Ω_{15} depicted in Figure 4.3. Similarly to the case A_2, each point of F_M as well as of Ω_M is labelled by the index set I_M and it is convenient for the point $x \in F_M$ and its image in Ω_M labelled by $j \in I_M$ to denote

$$\varepsilon_j = \varepsilon(x) = \tilde{\varepsilon}(X_M(x)) .$$

The values of ε_j can be found in Table 4.2. The cubature rule for any $p \in \Pi_{2M-1}$ is of the form

$$\int_\Omega p(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2) dy_1\, dy_2 = \frac{\pi^2}{3M^2} \sum_{j \in I_M} \varepsilon_j p(y_1^{(j)}, y_2^{(j)}) . \tag{4.3.7}$$

As in the case A_2, the cubature rule (4.3.7) is similar to the Gauss-Lobatto cubature formula derived in [**34**], where Xu et al study four types of functions $CC_k(t), SC_k(t), CS_k(t)$ and $SS_k(t)$ closely related to the orbit functions over Weyl groups. Concretely, the functions $CC_k(t)$ and $SS_k(t)$ are given by

$$CC_k(t) = \frac{1}{3}\Bigg[\cos\frac{\pi(k_1-k_3)(t_1-t_3)}{3}\cos\pi k_2 t_2 + \cos\frac{\pi(k_1-k_3)(t_2-t_1)}{3}\cos\pi k_2 t_3 + \cos\frac{\pi(k_1-k_3)(t_3-t_2)}{3}\cos\pi k_2 t_1\Bigg],$$

$$SS_k(t) = \frac{1}{3}\Bigg[\sin\frac{\pi(k_1-k_3)(t_1-t_3)}{3}\sin\pi k_2 t_2 + \sin\frac{\pi(k_1-k_3)(t_2-t_1)}{3}\sin\pi k_2 t_3 + \sin\frac{\pi(k_1-k_3)(t_3-t_2)}{3}\sin\pi k_2 t_1\Bigg],$$

where the variable t is given by homogeneous coordinates, i.e.

$$t = (t_1, t_2, t_3) \in \mathbb{R}^3_H = \left\{t \in \mathbb{R}^3 \mid t_1 + t_2 + t_3 = 0\right\}$$

and parameter $k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \cap \mathbb{R}^3_H$. It can be verified that the linear transformations of variable and parameter

$$t_1 = -a_1 + 3a_2\,, \quad t_2 = 2a_1 - 3a_2\,, \quad t_3 = -a_1,$$
$$k_1 = \lambda_1 + \lambda_2\,, \quad k_2 = \lambda_1\,, \quad k_3 = -2\lambda_1 - \lambda_2\,,$$

give the connection with C-functions and S-functions of G_2:

$$C_\lambda(x) = \frac{12}{h_\lambda} CC_k(t)\,, \qquad S_\lambda(x) = -12 SS_k(t)\,.$$

Example 4.3.3. *The cubature formula* (4.3.7) *is the exact equality of a weighted integral of any polynomial function of m-degree up to M with a weighted sum of finite number of polynomial values. It can be used in numerical integration to approximate a weighted integral of any function by finite summing.*

Similarly to Example 4.3.1, if we choose the function $f(y_1, y_2) = K^{\frac{1}{2}}(y_1, y_2)$ as our test function, then we can estimate the integral of 1 *over* Ω

$$\int_\Omega f(y_1, y_2) K^{-\frac{1}{2}}(y_1, y_2)\, dy_1\, dy_2 = \int_\Omega 1\, dy_1\, dy_2$$

by finite weighted sums with different M's and compare the obtained results with the exact value of the integral of 1 *which is* $\frac{128}{15} = 8.533\bar{3}$. *Table 4.3 shows the values of the finite weighted sums for* $M = 10, 20, 30, 50, 100$.

4.4. Polynomial approximations

4.4.1. The optimal polynomial approximation

Since the polynomial function (4.2.16) is continuous and strictly positive in Ω°, its square root $K^{-\frac{1}{2}}$ can serve as a weight function for the weighted Hilbert

M	10	20	30	50	100
A_2	$6,0751$	$6,2314$	$6,2602$	$6,2749$	$6,2811$
C_2	$10,056$	$10,5133$	$10,5985$	$10,6421$	$10,6605$
G_2	$7,4789$	$8,2561$	$8,4092$	$8,4885$	$8,5221$

Table 4.3. It shows the estimations of the integrals of 1 over the regions Ω of A_2, C_2 and G_2 respectively by finite weighted sums of the right-hand side of (4.3.5),(4.3.6) and (4.3.7) respectively for $M = 10, 20, 30, 50, 100$.

space $\mathcal{L}^2_K(\Omega)$, i.e. a space of complex-valued cosets of measurable functions f such that $\int_\Omega |f|^2 K^{-\frac{1}{2}} < \infty$ with an inner product defined by

$$(f, g)_K = \frac{1}{\kappa(2\pi)^n} \int_\Omega f(y)\overline{g(y)} K^{-\frac{1}{2}}(y)\, dy. \tag{4.4.1}$$

Our aim is to construct a suitable Hilbert basis of $\mathcal{L}^2_K(\Omega)$. Taking the inverse transforms of (4.2.4), (4.2.5) and substituting them into the polynomials (4.1.20) we obtain the polynomials $p_\lambda \in \mathbb{C}[y_1, \dots, y_n]$ such that

$$C_\lambda = p_\lambda(X_1, \dots, X_n). \tag{4.4.2}$$

Moreover, successively applying Proposition 4.2.3 to perform the substitutions (4.2.5) in p_λ and taking into account (4.2.13) we obtain that

$$\deg_m p_\lambda = |\lambda|_m. \tag{4.4.3}$$

Calculating the scalar product (4.4.1) for the p_λ polynomials (4.4.2), we obtain that the continuous orthogonality of the C-functions (4.1.14) is inherited, i.e.

$$(p_\lambda, p_{\lambda'})_K = h_\lambda^{-1} \delta_{\lambda,\lambda'}, \quad \lambda, \lambda' \in P^+. \tag{4.4.4}$$

Assigning to any function $f \in \mathcal{L}^2_K(\Omega)$ a function $\widetilde{f} \in \mathcal{L}^2(F)$ by the relation $\widetilde{f}(x) = f(X(x))$ and taking into account the expansion (4.1.15) we obtain for its expansion coefficients c_λ that

$$c_\lambda = \frac{h_\lambda}{|W||F|} \int_F \widetilde{f}(x)\overline{C_\lambda(x)}\, dx = \frac{h_\lambda}{\kappa(2\pi)^n} \int_\Omega f(y)\overline{p_\lambda(y)} K^{-\frac{1}{2}}(y)\, dy. \tag{4.4.5}$$

Therefore any $f \in \mathcal{L}^2_K(\Omega)$ can be expanded in terms of p_λ,

$$f = \sum_{\lambda \in P^+} a_\lambda p_\lambda\,, \quad a_\lambda = h_\lambda\, (f, p_\lambda)_K \tag{4.4.6}$$

and the set of C-polynomials p_λ, $\lambda \in P^+$ is a Hilbert basis of $\mathcal{L}^2_K(\Omega)$.

To construct a basis of the space of multivariate polynomials Π_M suffices to note that equation (4.1.19) guarantees each monomial $Z_1^{\lambda_1} Z_2^{\lambda_2} \dots Z_n^{\lambda_n}$ can be expanded in terms of C_λ with $\lambda \in P^+_M$; the same can be said about the transformed monomials $X_1^{\lambda_1} X_2^{\lambda_2} \dots X_n^{\lambda_n}$. Thus by the same argument as above we obtain for any $p \in \Pi_M$ the expansion

$$p = \sum_{\lambda \in P_M^+} b_\lambda p_\lambda \,, \quad b_\lambda = h_\lambda \, (p, p_\lambda)_K. \tag{4.4.7}$$

Truncating the series (4.4.6) to the finite set P_M^+ we obtain a polynomial approximation $u_M[f] \in \Pi_M$ of the functions $f \in \mathcal{L}_K^2(\Omega)$,

$$u_M[f] = \sum_{\lambda \in P_M^+} a_\lambda p_\lambda \,, \quad a_\lambda = h_\lambda \, (f, p_\lambda)_K. \tag{4.4.8}$$

Relative to the $\mathcal{L}_K^2(\Omega)$ norm is this approximation indeed optimal among all polynomials from Π_M as states the following proposition.

Proposition 4.4.1. *For any $f \in \mathcal{L}_K^2(\Omega)$ is the $u_M[f]$ polynomial* (4.4.8) *the best approximation of f, relative to the $\mathcal{L}_K^2(\Omega)$-norm, by any polynomial from Π_M.*

Proof. Consider any $p \in \Pi_M$ a polynomial of the form (4.4.7), any $f \in \mathcal{L}_K^2(\Omega)$ expanded by (4.4.6) and $u_M[f]$ an approximation polynomial (4.4.8). Then we calculate that

$$\begin{aligned}
(f-p, f-p)_K &= (f,f)_K - (f,p)_K - (p,f)_K + (p,p)_K \\
&= (f,f)_K - \sum_{\lambda \in P_M^+} h_\lambda^{-1} a_\lambda \overline{b_\lambda} - \sum_{\lambda \in P_M^+} h_\lambda^{-1} b_\lambda \overline{a_\lambda} + \sum_{\lambda \in P_M^+} h_\lambda^{-1} |b_\lambda|^2 \\
&= (f - u_M[f], f - u_M[f])_K + \sum_{\lambda \in P_M^+} h_\lambda^{-1} |b_\lambda - a_\lambda|^2 \\
&\geq (f - u_M[f], f - u_M[f])_K \,.
\end{aligned}$$

□

4.4.2. The cubature polynomial approximation

Rather than the optimal polynomial approximation (4.4.8) one may consider for practical applications its weakened version. Such a weaker version is obtained by using the cubature formula for an approximate calculation of $(f, p_\lambda)_K$, i.e. we set

$$v_M[f] = \sum_{\lambda \in P_M^+} a_\lambda p_\lambda \,, \quad a_\lambda = \frac{h_\lambda}{c|W|M^n} \sum_{y \in \Omega_M} \widetilde{\varepsilon}(y) f(y) \overline{p_\lambda(y)}. \tag{4.4.9}$$

Since for $f \in \Pi_{M-1}$ and p_λ, $\lambda \in P_M^+$ it holds that $f \overline{p_\lambda} \in \Pi_{2M-1}$and the cubature formula is thus valid, we obtain that the optimal approximation coincides with $v_M[f]$,

$$v_M[f] = u_M[f], \quad f \in \Pi_{M-1}. \tag{4.4.10}$$

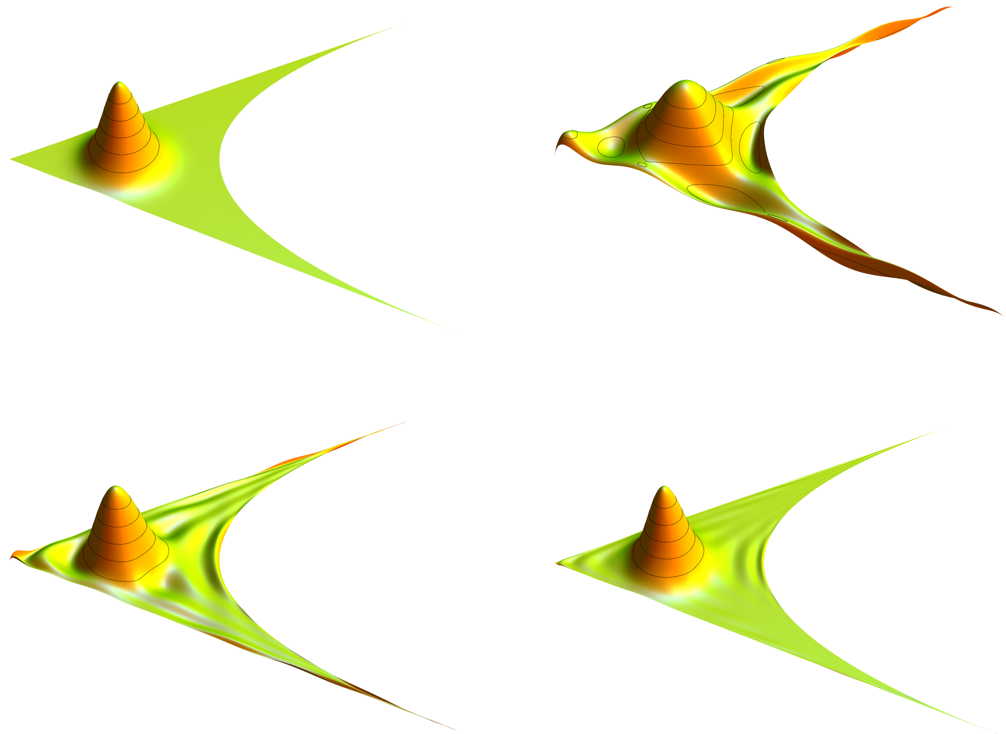

FIGURE 4.4. The figure shows the model function f and its approximations $v_M[f]$ for $M = 10, 20, 30$ on Ω.

Example 4.4.1. *As a specific example of a continuous model function in the case* C_2*, we consider*

$$f(y_1, y_2) = e^{-(y_1^2+(y_2+1.8)^2)/(2\times 0.35^2)}$$

defined on Ω*. The graph of* f *together with its approximations* $v_M[f]$ *for* $M = 10, 20, 30$ *is shown in Figure 4.4. Integral error estimates of the polynomial approximations* $v_M[f]$

$$\int_\Omega |f(y_1, y_2) - v_M[f](y_1, y_2)|^2 K^{-\frac{1}{2}}(y_1, y_2) dy_1\, dy_2$$

can be found in Table 4.4.

M	10	20	30
$\int_\Omega \lvert f - v_M[f]\rvert^2 K^{-\frac{1}{2}} dy_1\, dy_2$	$0,0636842$	$0,0035217$	$0,0000636$

TABLE 4.4. The table shows the values of integral error estimates of the polynomial approximations $v_M[f]$ for $M = 10, 20, 30$.

Acknowledgements

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Chapter 5

DISCRETE TRANSFORMS AND ORTHOGONAL POLYNOMIALS OF (ANTI)SYMMETRIC MULTIVARIATE COSINE FUNCTIONS

Authors: Jiří Hrivnák and Lenka Motlochová.

Abstract: The discrete cosine transforms of types V–VIII are generalized to the antisymmetric and symmetric multivariate discrete cosine transforms. Four families of discretely and continuously orthogonal Chebyshev-like polynomials corresponding to the antisymmetric and symmetric generalizations of cosine functions are introduced. Each family forms an orthogonal basis of the space of all polynomials with respect to some weighted integral. Cubature formulas, which correspond to these families of polynomials and which stem from the developed discrete cosine transforms, are derived. Examples of three-dimensional interpolation formulas and three-dimensional explicit forms of the polynomials are presented.

Introduction

This paper aims to complete and extend the study of antisymmetric and symmetric multivariate generalizations of common cosine functions of one variable from [**25**]. Firstly, the set of four symmetric and four antisymmetric discrete Fourier-like transforms from [**25**] is extended. Then four families of Chebyshev-like orthogonal polynomials are introduced and the entire collection of the sixteen discrete cosine transforms is used to derive the corresponding numerical integration formulas.

The antisymmetric and symmetric cosine functions of n variables are introduced in [**25**] as determinants and permanents of matrices whose entries are cosine functions of one variable. The lowest dimensional non-trivial case $n = 2$ is detailed in [**18**]. These real-valued functions have several remarkable properties such as continuous and discrete orthogonality which lead to continuous and discrete analogues of Fourier transforms. The discrete orthogonality relations of these functions are a consequence of ubiquitous discrete cosine transforms (DCTs) and their Cartesian product multidimensional generalizations. There are eight known different types of DCTs based on various boundary conditions [**3**]. Only the first four transforms are generalized to the multidimensional symmetric cosine functions [**25**]. Therefore, the remaining four transforms of the type V-VIII need to be developed to obtain the full collections of antisymmetric and symmetric cosine transforms with all possible boundary conditions. The resulting sixteen transforms are then available for similar applications as multidimensional DCTs. Besides a straightforward utilization of these transforms to interpolation methods, they may also serve as a starting point for Chebyshev-like polynomial analysis.

The Chebyshev polynomials of one variable are well–known and extensively studied orthogonal polynomials connected to efficient methods of numerical integration and approximations. The Chebyshev polynomials are of the four basic kinds [**39**], the first and third kind related to the cosine functions, the second and fourth kind related to the sine functions. Since both families of antisymmetric and symmetric cosine functions are based on the one-dimensional cosine functions, they admit a multidimensional generalization of the one-dimensional Chebyshev polynomials of the first and third kind. The results are four families of orthogonal polynomials, which can be viewed as the symmetric and antisymmetric Chebyshev-like polynomials of the first and third kind. Several generalizations of Chebyshev polynomials to higher dimensions are known – in fact for the two-dimensional case the resulting polynomials become, up to a multiplication by constant, special cases of two-variable analogues of Jacobi polynomials [**30, 31, 32**]. As is the case of the Chebyshev polynomials of one variable, the multivariate Chebyshev-like polynomials inherit properties from the generalized cosine functions. This link provides tools to generalize efficient numerical integration formulas of the classical Chebyshev polynomials.

One of the most important application of the classical Chebyshev polynomials is the calculation of numerical quadratures — formulas equating a weighted integral of a polynomial function not exceeding a specific degree with a linear combination of polynomial values at some points called nodes [**39, 52**]. The main point of quadrature formulas, which are in a multidimensional setting known as cubature formulas, is to replace integration by finite summing. The specific degree, to which cubature formulas hold exactly for polynomials, represents a degree of precision of a given formula.

If a cubature formula holds only for polynomials of degree at most $2d-1$ then the least number of nodes is equal to the dimension of the space of multivariate polynomials of degree at most $d-1$. Any cubature formulas, which satisfy the lowest bound of the number of nodes, have the maximal degree of precision and are called the Gaussian cubature formulas [**5**]. In addition, it is known that such formulas exist only if the number of real distinct common zeros of the corresponding orthogonal polynomials of degree d is also exactly the dimension of the space of the multivariate polynomials of degree at most $d-1$ [**5**]. It appears that among the sixteen types of cubature formulas, resulting from sixteen discrete transforms, four are Gaussian. The remaining twelve, even though they are not optimal, extend the options for numerical calculation of multivariate integrals.

In Section 5.1, notation and terminology as well as relevant facts from [**25**] are reviewed. Simplified forms of three special cases of the generalized cosine functions, which appear as denominators in the definition of the Chebyshev-like multivariate polynomials, are deduced. The continuous orthogonality relations of the generalized cosine functions are presented. In Section 5.2, the one-dimensional DCTs of types V–VIII are reviewed and generalized to the antisymmetric and symmetric multivariate discrete cosine transforms. The interpolation formulas in terms of antisymmetric and symmetric cosine functions are developed. In Section 5.3, the four families of the multivariate Chebyshev-like polynomials are introduced and three-dimensional examples presented. It is proved that each family of polynomials forms an orthogonal basis of the space of all polynomials with respect to the scalar product given by a weighted integral. In section 5.4, the cubature formulas for each family of polynomials are derived. The last section contains concluding remarks and addresses follow-up questions.

5.1. Symmetric and antisymmetric multivariate cosine functions

The symmetric and antisymmetric multivariate generalizations of the cosine functions are defined and their properties detailed in [**25**]. The antisymmetric cosine functions $\cos^-_\lambda(x)$ and the symmetric cosine functions $\cos^+_\lambda(x)$ of variable $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and labelled by parameter $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ are defined as determinants and permanents, respectively, of the matrices with the entries $\cos(\pi\lambda_i x_j)$, i.e. taking a permutation $\sigma \in S_n$ with its sign $\operatorname{sgn}(\sigma)$ one has

$$\begin{aligned}\cos^-_\lambda(x) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cos(\pi\lambda_{\sigma(1)}x_1)\cos(\pi\lambda_{\sigma(2)}x_2)\cdots\cos(\pi\lambda_{\sigma(n)}x_n)\,,\\ \cos^+_\lambda(x) &= \sum_{\sigma\in S_n} \cos(\pi\lambda_{\sigma(1)}x_1)\cos(\pi\lambda_{\sigma(2)}x_2)\cdots\cos(\pi\lambda_{\sigma(n)}x_n)\,.\end{aligned} \tag{5.1.1}$$

The $\cos^\pm_\lambda(x)$ functions are continuous and have continuous derivatives of all degrees in $\mathbb{R}^n$. Moreover, they are invariant or anti-invariant under all permutations in $\sigma \in S_n$, i.e.

$$\cos^-_\lambda(\sigma(x)) = \operatorname{sgn}(\sigma)\cos^-_\lambda(x)\,,\quad \cos^-_{\sigma(\lambda)}(x) = \operatorname{sgn}(\sigma)\cos^-_\lambda(x)\,, \tag{5.1.2}$$

$$\cos^+_\lambda(\sigma(x)) = \cos^+_\lambda(x)\,,\qquad \cos^+_{\sigma(\lambda)}(x) = \cos^+_\lambda(x)\,, \tag{5.1.3}$$

where $\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ and $\sigma(\lambda) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(n)})$. Additionally, they are symmetric with respect to the alternations of signs — denoting τ_i the change of sign of x_i, i.e. $\tau_i(x) = \tau_i(x_1,\dots,x_i,\dots,x_n) \equiv (x_1,\dots,-x_i,\dots,x_n)$ it holds that

$$\begin{aligned}\cos^-_\lambda(\tau_i(x)) &= \cos^-_\lambda(x)\,,\quad \cos^-_{\tau_i(\lambda)}(x) = \cos^-_\lambda(x)\,,\\ \cos^+_\lambda(\tau_i(x)) &= \cos^+_\lambda(x)\,,\quad \cos^+_{\tau_i(\lambda)}(x) = \cos^+_\lambda(x)\,.\end{aligned} \tag{5.1.4}$$

Considering the functions $\cos^\pm_k(x)$ with their parameter having only integer values $k \in \mathbb{Z}^n$ and denoting

$$\varrho \equiv \left(\frac{1}{2}, \dots, \frac{1}{2}\right), \tag{5.1.5}$$

the symmetry related to the periodicity of the one-dimensional cosine functions, i.e. for $t = (t_1,\dots,t_n) \in \mathbb{Z}^n$ we obtain

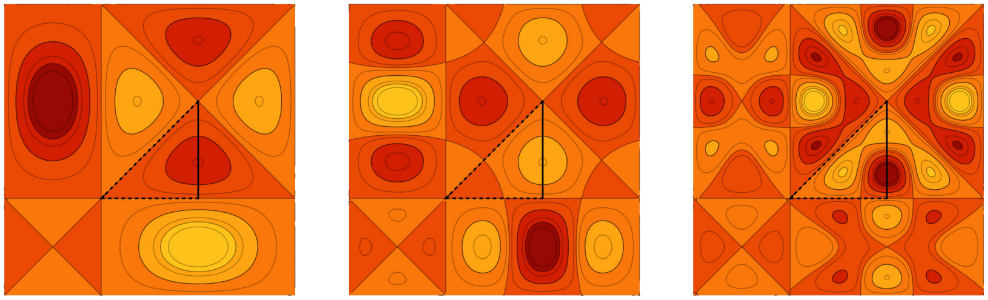
$$\begin{aligned}\cos^-_k(x+2t) &= \cos^-_k(x)\,,\quad \cos^-_{k+\varrho}(x+2t) = (-1)^{t_1+\dots+t_n}\cos^-_{k+\varrho}(x),\\ \cos^+_k(x+2t) &= \cos^+_k(x)\,,\quad \cos^+_{k+\varrho}(x+2t) = (-1)^{t_1+\dots+t_n}\cos^+_{k+\varrho}(x).\end{aligned} \tag{5.1.6}$$

The relations (5.1.2), (5.1.3) and (5.1.4) imply that we consider only the following restricted values of $k \in \mathbb{Z}^n$,

$$\begin{aligned}\cos^-_k(x)\,, \cos^-_{k+\varrho}(x) &: \quad k_1 > k_2 > \dots > k_n \geq 0\,,\\ \cos^+_k(x)\,, \cos^+_{k+\varrho}(x) &: \quad k_1 \geq k_2 \geq \dots \geq k_n \geq 0\,.\end{aligned} \tag{5.1.7}$$

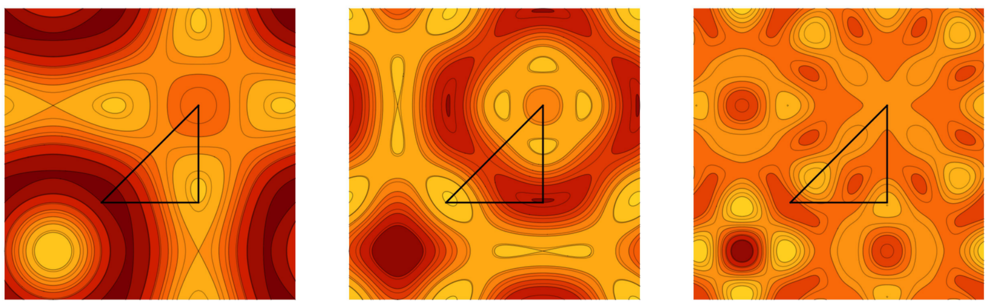
By the relations (5.1.2)–(5.1.6) we consider the antisymmetric and symmetric cosine functions $\cos^\pm_k(x)$ labelled by $k \in \mathbb{Z}^n$ on the closure of the fundamental domain $F(\widetilde{S}^{\text{aff}}_n)$ of the form

$$F(\widetilde{S}^{\text{aff}}_n) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid 1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}\,. \tag{5.1.8}$$



$\cos^-_{(2,\,1,\,0)}(x_1, x_2, \frac{1}{3})$ $\cos^-_{(3,\,2,\,1)}(x_1, x_2, \frac{1}{3})$ $\cos^-_{(4,\,3,\,2)}(x_1, x_2, \frac{1}{3})$

FIGURE 5.1. The contour plots of the graph cuts ($x_3 = \frac{1}{3}$) of the antisymmetric cosine functions $\cos^-_k(x)$. The cut of the boundary of the fundamental domain $F(\widetilde{S}_3^{\mathrm{aff}})$ is depicted as the black triangle with the dashed lines representing the part of the boundary for which all the antisymmetric cosine functions vanish.



$\cos^+_{(2,\,1,\,0)}(x_1, x_2, \frac{1}{3})$ $\cos^+_{(3,\,2,\,1)}(x_1, x_2, \frac{1}{3})$ $\cos^+_{(4,\,3,\,2)}(x_1, x_2, \frac{1}{3})$

FIGURE 5.2. The contour plots of the graph cuts ($x_3 = \frac{1}{3}$) of the symmetric cosine functions $\cos^+_k(x)$. The cut of the boundary of the fundamental domain $F(\widetilde{S}_3^{\mathrm{aff}})$ is depicted as the black triangle.

Due to equations (5.1.2), (5.1.4) and the identity $\cos \pi \left(k_i + \frac{1}{2}\right) = 0$, valid for $k_i \in \mathbb{Z}$ and $i \in \{1, \dots, n\}$, we omit those boundaries of $F(\widetilde{S}_n^{\mathrm{aff}})$ for which

- $x_i = x_{i+1}$, $i \in \{1, \dots, n-1\}$ in the case of $\cos^-_k(x)$,
- $x_i = x_{i+1}$, $i \in \{1, \dots, n-1\}$ or $x_1 = 1$ in the case of $\cos^-_{k+\varrho}(x)$,
- $x_i = 1$, $i \in \{1, \dots, n\}$ in the case of $\cos^+_{k+\varrho}(x)$.

The contour plots of the cuts of the three-dimensional antisymmetric and symmetric cosine functions are depicted in Fig. 5.1 and Figure 5.2.

For several special choices of $k \in \mathbb{Z}^n$, the functions (5.1.1) are expressed as products of the one-dimensional cosine and sine functions [**25**]. In addition to the formulas from [**25**], we calculate this form for the special cases needed for the analysis of the related orthogonal polynomials. Denoting

$$\varrho_1 \equiv (n-1, n-2, \dots, 0), \tag{5.1.9}$$

$$\varrho_2 \equiv \left(n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right), \tag{5.1.10}$$

we derive the form of the functions $\cos^+_\varrho(x)$, $\cos^-_{\varrho_1}(x)$ and $\cos^-_{\varrho_2}(x)$ in the following proposition.

Proposition 5.1.1. *Let $k \in \mathbb{N}$ is given by*

$$k = \begin{cases} \frac{n-1}{2} & \text{for } n \text{ odd,} \\ \frac{n}{2} & \text{for } n \text{ even.} \end{cases} \tag{5.1.11}$$

Then it holds that

$$\cos^-_{\varrho_1}(x_1, \ldots, x_n) = (-1)^k 2^{(n-1)^2} \prod_{1 \le i < j \le n} \sin\left(\frac{\pi}{2}(x_i + x_j)\right) \sin\left(\frac{\pi}{2}(x_i - x_j)\right), \tag{5.1.12}$$

$$\cos^-_{\varrho_2}(x_1, \ldots, x_n) = (-1)^k 2^{n(n-1)} \prod_{i=1}^{n} \cos\left(\frac{\pi}{2}x_i\right) \prod_{1 \le i < j \le n} \sin\left(\frac{\pi}{2}(x_i + x_j)\right) \sin\left(\frac{\pi}{2}(x_i - x_j)\right), \tag{5.1.13}$$

$$\cos^+_\varrho(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} \cos\left(\frac{\pi}{2}x_i\right). \tag{5.1.14}$$

where ϱ_1, ϱ_2 and ϱ are given by (5.1.9), (5.1.10) *and* (5.1.5), *respectively.*

Proof. We have from definition (5.1.1) that

$$\cos^-_{\varrho_1}(x_1, \ldots, x_n) = \det \begin{pmatrix} \cos(\pi(n-1)x_1) & \cos(\pi(n-2)x_1) & \cdots & \cos(\pi x_1) & 1 \\ \cos(\pi(n-1)x_2) & \cos(\pi(n-2)x_2) & \cdots & \cos(\pi x_2) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \cos(\pi(n-1)x_n) & \cos(\pi(n-2)x_n) & \cdots & \cos(\pi x_n) & 1 \end{pmatrix}.$$

Using the trigonometric identity for the powers of cosine function

$$2^{m-1}\cos^m\theta = \begin{cases} \cos(m\theta) + \sum_{k=1}^{\frac{m-1}{2}} \binom{m}{k} \cos(m-2k)\theta & \text{if } m \text{ is odd,} \\ \cos(m\theta) + \frac{1}{2}\binom{m}{\frac{m}{2}} + \sum_{k=1}^{\frac{m}{2}-1} \binom{m}{k} \cos(m-2k)\theta & \text{if } m \text{ is even,} \end{cases}$$

we bring the determinant to the following form

$$\begin{aligned} \cos^-_{\varrho_1}(x_1, \ldots, x_n) &= \det \begin{pmatrix} 2^{n-2}\cos^{n-1}(\pi x_1) & 2^{n-3}\cos^{n-2}(\pi x_1) & \cdots & \cos(\pi x_1) & 1 \\ 2^{n-2}\cos^{n-1}(\pi x_2) & 2^{n-3}\cos^{n-2}(\pi x_2) & \cdots & \cos(\pi x_2) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 2^{n-2}\cos^{n-1}(\pi x_n) & 2^{n-3}\cos^{n-2}(\pi x_n) & \cdots & \cos(\pi x_n) & 1 \end{pmatrix} \\ &= (-1)^k 2^{\frac{(n-2)(n-1)}{2}} \det \begin{pmatrix} 1 & \cos(\pi x_1) & \cdots & \cos^{n-2}(\pi x_1) & \cos^{n-1}(\pi x_1) \\ 1 & \cos(\pi x_2) & \cdots & \cos^{n-2}(\pi x_2) & \cos^{n-1}(\pi x_2) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \cos(\pi x_n) & \cdots & \cos^{n-2}(\pi x_n) & \cos^{n-1}(\pi x_n) \end{pmatrix}. \end{aligned}$$

Taking into account that the last determinant is of the Vandermonde type and that the trigonometric identity

$$\cos(\pi x_j) - \cos(\pi x_i) = 2\sin\left(\frac{\pi}{2}(x_i + x_j)\right)\sin\left(\frac{\pi}{2}(x_i - x_j)\right),$$

holds, we obtain that

$$\cos^-_{\varrho_1}(x_1, \dots, x_n) = (-1)^k 2^{\frac{(n-2)(n-1)}{2}} \prod_{1\le i<j\le n} \cos(\pi x_j) - \cos(\pi x_i),$$

and the identity (5.1.12) follows. The proof of formula (5.1.13) similar and relation (5.1.14) follows directly from definition (5.1.1). □

Note that the equalities (5.1.12)–(5.1.13) allow us to analyze the zeros of the functions $\cos^-_{\varrho_1}$, $\cos^-_{\varrho_2}$ and $\cos^+_{\varrho}$,

- $\cos^-_{\varrho_1}(x) = 0$ if and only if $x = (x_1, \dots, x_n) \in F(\widetilde{S}^{\mathrm{aff}}_n)$ satisfies $x_i = x_{i+1}$,
- $\cos^-_{\varrho_2}(x) = 0$ if and only if $x = (x_1, \dots, x_n) \in F(\widetilde{S}^{\mathrm{aff}}_n)$ satisfies $x_i = x_{i+1}$ or $x_1 = 1$, and
- $\cos^+_{\varrho}(x) = 0$ if and only if $x = (x_1, \dots, x_n) \in F(\widetilde{S}^{\mathrm{aff}}_n)$ satisfies $x_1 = 1$.

Observing that all zero values are located on the boundaries of $F(\widetilde{S}^{\mathrm{aff}}_n)$ we conclude that

Corollary 5.1.1. *The functions $\cos^-_{\varrho_1}$, $\cos^-_{\varrho_2}$ and $\cos^+_{\varrho}$ are non-zero in the interior of the fundamental domain $F(\widetilde{S}^{\mathrm{aff}}_n)$.*

5.1.1. Continuous orthogonality

The antisymmetric and symmetric cosine functions (5.1.7) are mutually continuously orthogonal within each family. Denoting the order of the stabilizer of the point $k \in \mathbb{R}^n$ under the action of the permutation group S_n by H_k, i.e.

$$H_k = \#\{\sigma k = k \mid \sigma \in S_n\}, \tag{5.1.15}$$

and introducing the symbol $h_k = h_{k_1} \dots h_{k_n}$ by

$$h_{k_i} = \begin{cases} 1 & \text{if } k_i = 0, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

we obtain

$$\int_{F(\widetilde{S}^{\mathrm{aff}}_n)} \cos^-_k(x)\cos^-_{k'}(x)\,dx = h_k\delta_{kk'}, \qquad k_1 > k_2 > \dots > k_n \ge 0, \tag{5.1.16}$$

$$\int_{F(\widetilde{S}^{\mathrm{aff}}_n)} \cos^-_{k+\varrho}(x)\cos^-_{k'+\varrho}(x)\,dx = 2^{-n}\delta_{kk'}, \qquad k_1 > k_2 > \dots > k_n \ge 0, \tag{5.1.17}$$

$$\int_{F(\widetilde{S}^{\mathrm{aff}}_n)} \cos^+_k(x)\cos^+_{k'}(x)\,dx = h_kH_k\delta_{kk'}, \qquad k_1 \ge k_2 \ge \dots \ge k_n \ge 0, \tag{5.1.18}$$

$$\int_{F(\widetilde{S}_n^{\rm aff})} \cos^+_{k+\varrho}(x)\cos^+_{k'+\varrho}(x)\,dx = 2^{-n} H_k \delta_{kk'}\,,\quad k_1 \geq k_2 \geq \cdots \geq k_n \geq 0\,. \quad (5.1.19)$$

The orthogonality relations (5.1.16), (5.1.18) are deduced in [**25**] from the continuous orthogonality of the ordinary cosine functions $\cos(\pi m\theta)$,

$$\int_0^1 \cos(\pi m\theta)\cos(\pi m'\theta) = h_m\delta_{mm'}, \qquad m,m' \in \mathbb{Z}\,.$$

The remaining two orthogonality relations follow from the continuous orthogonality of cosine functions $\cos\left(\pi\left(m+\frac{1}{2}\right)\theta\right)$,

$$\int_0^1 \cos\left(\pi\left(m+\frac{1}{2}\right)\theta\right)\cos\left(\pi\left(m'+\frac{1}{2}\right)\theta\right) = \frac{1}{2}\delta_{mm'}\,. \quad (5.1.20)$$

Since the antisymmetric cosine functions are anti-invariant with respect to all elements of S_n and the group S_n applied on $F(\widetilde{S}_n^{\rm aff})$ gives the cube $[0,1]^n$, we have

$$\int_{F(\widetilde{S}_n^{\rm aff})} \cos^-_{k+\varrho}(x)\cos^-_{k'+\varrho}(x)\,dx = \frac{1}{n!}\int_{[0,1]^n} \cos^-_{k+\varrho}(x)\cos^-_{k'+\varrho}(x)\,dx\,.$$

Using the property $\cos^-_{k+\varrho}(x) = \cos^-_{\sigma'(k+\varrho)}(\sigma'(x))$ for $\sigma' \in S_n$, which follows from (5.1.2), together with definition (5.1.1) we have

$$\cos^-_{k+\varrho}(x)\cos^-_{k'+\varrho}(x)$$
$$= \sum_{\sigma,\sigma'\in S_n} \operatorname{sgn}(\sigma\sigma')\prod_{i=1}^n \cos\left(\pi\left(k_{\sigma\sigma'(i)}+\frac{1}{2}\right)x_{\sigma'(i)}\right)\cos\left(\pi\left(k'_i+\frac{1}{2}\right)x_{\sigma'(i)}\right).$$

Therefore, we rewrite the integral (5.1.17) as follows,

$$\int_{F(\widetilde{S}_n^{\rm aff})} \cos^-_{k+\varrho}(x)\cos^-_{k'+\varrho}(x)\,dx$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{i=1}^n \int_0^1 \cos\left(\pi\left(k_{\sigma(i)}+\frac{1}{2}\right)x_i\right)\cos\left(\pi\left(k'_i+\frac{1}{2}\right)x_i\right)dx_i,$$

where we made the change of variables from $x_{\sigma'(i)}$ to x_i. Using the relation (5.1.20), we obtain

$$\int_{F(\widetilde{S}_n^{\rm aff})} \cos^-_{k+\varrho}(x)\cos^-_{k'+\varrho}(x)\,dx = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{i=1}^n \frac{1}{2}\delta_{k_{\sigma(i)}k'_i}\,.$$

Since $k_{\sigma(i)} = k'_i$ for all $i \in \{1,\ldots,n\}$ if and only if σ is the identity permutation and $k_i = k'_i$, the orthogonality relation (5.1.17) is derived. Similarly is obtained the orthogonality relation (5.1.19).

5.2. Discrete multivariate cosine transforms

5.2.1. Discrete cosine transforms of types V–VIII

The one-dimensional discrete cosine transforms (DCTs) and their cartesian product generalizations have many applications in mathematics, physics and engineering. They are of efficient use in various domains of information processing, e.g., image, video or audio processing. There are several types of DCTs arising from discretized solutions of harmonic oscillator equation with different choices of boundary conditions — see [**3, 56**]. They express a function defined on a finite grid of points as a sum of cosine functions with different frequencies and amplitudes. They have numerous useful properties as the unitary property and convolution properties [**3**]. The multivariate symmetric and antisymmetric generalizations of the cosine transforms of types I–IV are found in [**25, 18**]. In order to derive the symmetric and antisymmetric DCTs of types V–VIII we first review their one-dimensional versions.

5.2.1.1. *DCT V*

For $N \in \mathbb{N}$ are the cosine functions $\cos(\pi k s)$, $k = 0, \dots, N-1$, defined on the finite grid of points

$$s \in \left\{ \frac{2r}{2N-1} \mid r = 0, \dots, N-1 \right\} \tag{5.2.1}$$

pairwise discretely orthogonal,

$$\sum_{r=0}^{N-1} c_r \cos\left(\frac{2\pi k r}{2N-1}\right) \cos\left(\frac{2\pi k' r}{2N-1}\right) = \frac{2N-1}{4c_k} \delta_{kk'}\,, \tag{5.2.2}$$

where

$$c_r = \begin{cases} \frac{1}{2} & \text{if } r = 0 \text{ or } r = N \\ 1 & \text{otherwise.} \end{cases} \tag{5.2.3}$$

Therefore, any discrete function f given on the finite grid (5.2.1) is expressed in terms of cosine functions as

$$\begin{aligned} f(s) &= \sum_{k=0}^{N-1} A_k \cos(\pi k s), \\ A_k &= \frac{4c_k}{2N-1} \sum_{r=0}^{N-1} c_r f\left(\frac{2r}{2N-1}\right) \cos\left(\frac{2\pi k r}{2N-1}\right). \end{aligned} \tag{5.2.4}$$

The formulas (5.2.4) determine the transform DCT V.

5.2.1.2. *DCT VI*

For $N \in \mathbb{N}$ are the cosine functions $\cos(\pi k s)$, $k = 0, \dots, N-1$, defined on the finite grid of points

$$s \in \left\{ \frac{2\left(r+\frac{1}{2}\right)}{2N-1} \mid r = 0, \dots, N-1 \right\} \tag{5.2.5}$$

pairwise orthogonal,

$$\sum_{r=0}^{N-1} c_{r+1} \cos\left(\frac{2\pi k\left(r+\frac{1}{2}\right)}{2N-1}\right) \cos\left(\frac{2\pi k'\left(r+\frac{1}{2}\right)}{2N-1}\right) = \frac{2N-1}{4c_k}\delta_{kk'}\,, \tag{5.2.6}$$

where c_r are determined by (5.2.3). Therefore, any discrete function f given on the finite grid (5.2.5) is expressed in terms of cosine functions as

$$\begin{aligned} f(s) &= \sum_{k=0}^{N-1} A_k \cos(\pi k s), \\ A_k &= \frac{4c_k}{2N-1} \sum_{r=0}^{N-1} c_{r+1} f\left(\frac{2\left(r+\frac{1}{2}\right)}{2N-1}\right) \cos\left(\frac{2\pi k\left(r+\frac{1}{2}\right)}{2N-1}\right). \end{aligned} \tag{5.2.7}$$

The formulas (5.2.7) determine the transform DCT VI.

5.2.1.3. *DCT VII*

For $N \in \mathbb{N}$ are the cosine functions $\cos\left(\pi\left(k+\frac{1}{2}\right)s\right)$, $k = 0, \dots, N-1$, defined on the finite grid of points (5.2.1), pairwise discretely orthogonal,

$$\sum_{r=0}^{N-1} c_r \cos\left(\frac{2\pi\left(k+\frac{1}{2}\right)r}{2N-1}\right) \cos\left(\frac{2\pi\left(k'+\frac{1}{2}\right)r}{2N-1}\right) = \frac{2N-1}{4c_{k+1}}\delta_{kk'}\,, \tag{5.2.8}$$

where c_r, d_{k+1} is determined by (5.2.3). Therefore, any discrete function f given on the finite grid (5.2.1) is expressed in terms of cosine functions as

$$\begin{aligned} f(s) &= \sum_{k=0}^{N-1} A_k \cos\left(\pi\left(k+\frac{1}{2}\right)s\right), \\ A_k &= \frac{4c_{k+1}}{2N-1} \sum_{r=0}^{N-1} c_r f\left(\frac{2r}{2N-1}\right) \cos\left(\frac{2\pi\left(k+\frac{1}{2}\right)r}{2N-1}\right). \end{aligned} \tag{5.2.9}$$

The formulas (5.2.9) determine the transform DCT VII.

5.2.1.4. *DCT VIII*

For $N \in \mathbb{N}$ are the cosine functions $\cos\left(\pi\left(k+\frac{1}{2}\right)s\right)$, $k=0,\dots,N-1$, defined on the finite grid of points

$$s \in \left\{ \frac{2\left(r+\frac{1}{2}\right)}{2N+1} \mid r=0,\dots,N-1 \right\} \tag{5.2.10}$$

pairwise orthogonal,

$$\sum_{r=0}^{N-1} \cos\left(\frac{2\pi\left(k+\frac{1}{2}\right)\left(r+\frac{1}{2}\right)}{2N+1}\right) \cos\left(\frac{2\pi\left(k'+\frac{1}{2}\right)\left(r+\frac{1}{2}\right)}{2N+1}\right) = \frac{2N+1}{4}\delta_{kk'}\,. \tag{5.2.11}$$

Therefore, any discrete function f given on the finite grid (5.2.10) is expressed in terms of cosine functions as

$$\begin{aligned} f(s) &= \sum_{k=0}^{N-1} A_k \cos\left(\pi\left(k+\frac{1}{2}\right)s\right), \\ A_k &= \frac{4}{2N+1}\sum_{r=0}^{N-1} f\left(\frac{2\left(r+\frac{1}{2}\right)}{2N+1}\right)\cos\left(\frac{2\pi\left(k+\frac{1}{2}\right)\left(r+\frac{1}{2}\right)}{2N+1}\right). \end{aligned} \tag{5.2.12}$$

The formulas (5.2.12) determine the transform DCT VIII.

5.2.2. Antisymmetric discrete multivariate cosine transforms

The antisymmetric discrete multivariate cosine transforms (AMDCTs), which can be viewed as antisymmetric multivariate generalizations of DCTs, are derived using the one-dimensional DCTs from Section 5.2.1. The four types of AMDCT, connected to DCT I–IV, are contained in [**25**]. Our goal is to complete the list of AMDCTs by developing the remaining four transforms of types V–VIII. Firstly, we introduce the set of labels

$$D_N \equiv \{(k_1,\dots,k_n) \in \mathbb{Z}^n \mid N-1 \geq k_i \geq 0,\ i=1,\dots,n\} \tag{5.2.13}$$

and to any point $(k_1,\dots,k_n) \in D_N$ we assign two values

$$d_k = c_{k_1}\dots c_{k_n}, \tag{5.2.14}$$

$$\widetilde{d}_k = c_{k_1+1}\dots c_{k_n+1}. \tag{5.2.15}$$

where c_{k_i} are determined by (5.2.3). The discrete calculus is performed on three types of grids inside $F(\widetilde{S}_n^{\mathrm{aff}})$; two of them are subsets of two cubic grids

$$C_N \equiv \left\{ \left(\frac{2r_1}{2N-1},\dots,\frac{2r_n}{2N-1}\right) \mid (r_1,\dots,r_n) \in D_N \right\}, \tag{5.2.16}$$

$$\widetilde{C}_N \equiv \left\{ \left(\frac{2\left(r_1+\frac{1}{2}\right)}{2N-1}, \dots, \frac{2\left(r_n+\frac{1}{2}\right)}{2N-1} \right) \mid (r_1,\dots,r_n) \in D_N \right\}. \tag{5.2.17}$$

To any point $s \in C_N$, which is according to (5.2.16) labelled by the point $(r_1,\dots,r_n) \in D_N$, we assign the value

$$\varepsilon_s \equiv c_{r_1} \dots c_{r_n}, \tag{5.2.18}$$

and to any point $s \in \widetilde{C}_N$, which is according to (5.2.17) labelled by the point $(r_1,\dots,r_n) \in D_N$, we assign the value

$$\widetilde{\varepsilon}_s \equiv c_{r_1+1} \dots c_{r_n+1}. \tag{5.2.19}$$

5.2.2.1. *AMDCT V*

For $N \in \mathbb{N}$ we consider the antisymmetric cosine functions $\cos^-_k(s)$ labelled by the index set

$$D_N^- \equiv \{(k_1,\dots,k_n) \in D_N \mid k_1 > k_2 > \dots > k_n\} \tag{5.2.20}$$

and restricted to the finite grid of points contained in $C_N \subset F(\widetilde{S}_n^{\mathrm{aff}})$

$$F_N^{\mathrm{V},-} \equiv \left\{ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) \mid (r_1,\dots,r_n) \in D_N^- \right\}. \tag{5.2.21}$$

The scalar product of any two functions $f, g : F_N^{\mathrm{V},-} \to \mathbb{R}$ given on the points of the grid $F_N^{\mathrm{V},-}$ is defined by

$$\langle f, g \rangle = \sum_{s \in F_N^{\mathrm{V},-}} \varepsilon_s f(s) g(s), \tag{5.2.22}$$

where ε_s is given by (5.2.18). Using the orthogonality relation of one-dimensional cosine functions (5.2.2), we show that the antisymmetric cosine functions labelled by parameters in $k, k' \in D_N^-$, are pairwise discretely orthogonal, i.e.

$$\langle \cos^-_k, \cos^-_{k'} \rangle = \sum_{s \in F_N^{\mathrm{V},-}} \varepsilon_s \cos^-_k(s) \cos^-_{k'}(s) = d_k^{-1} \left(\frac{2N-1}{4} \right)^n \delta_{kk'}, \tag{5.2.23}$$

where d_k is given by (5.2.14). Denoting the set of points

$$F_N \equiv \left\{ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) \in C_N \mid r_1 \geq r_2 \geq \dots \geq r_n \right\}. \tag{5.2.24}$$

we observe that the functions $\cos^-_k(s)$, $k \in D_N^-$ vanish on the points $s \in F_N \setminus F_N^{\mathrm{V},-}$ and thus

$$\langle \cos^-_k, \cos^-_{k'} \rangle = \sum_{s \in F_N} \varepsilon_s H_s^{-1} \cos^-_k(s) \cos^-_{k'}(s),$$

with the order of the stabilizer H_s determined by (5.1.15). Acting by all permutations of S_n on the grid F_N we obtain the entire cube C_N, i.e. $S_n F_N = C_N$ and therefore we have that $\langle \cos^-_k, \cos^-_{k'} \rangle$ is equal to

$$\frac{1}{n!}\sum_{r_1,\dots,r_n=0}^{N-1} c_{r_1}\dots c_{r_n}\cos^-_k\left(\frac{2r_1}{2N-1},\dots,\frac{2r_n}{2N-1}\right)\cos^-_{k'}\left(\frac{2r_1}{2N-1},\dots,\frac{2r_n}{2N-1}\right).$$

Moreover, the product of two antisymmetric cosine functions labelled by $k, k' \in D^-_N$ is rewritten due to the anti-invariance under permutations as

$$\cos^-_k(x)\cos^-_{k'}(x) = \sum_{\sigma,\sigma'\in S_n}\operatorname{sgn}(\sigma\sigma')\prod_{i=1}^{n}\cos\left(\pi k_{\sigma\sigma'(i)}x_{\sigma'(i)}\right)\cos\left(\pi k'_i x_{\sigma'(i)}\right).$$

Together with the S_n-invariance of the set D_N this implies that

$$\langle \cos^-_k, \cos^-_{k'}\rangle = \sum_{\sigma\in S_n}\operatorname{sgn}\sigma\prod_{i=1}^{n}\sum_{r_i=0}^{N-1} c_{r_i}\cos\left(\frac{2\pi k_{\sigma(i)}r_i}{2N-1}\right)\cos\left(\frac{2\pi k'_i r_i}{2N-1}\right).$$

Finally, we apply the orthogonality relation of one-dimensional cosine functions (5.2.2) to obtain (5.2.23),

$$\langle \cos^-_k, \cos^-_{k'}\rangle = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}\frac{2N-1}{4c_{k_{\sigma(i)}}}\delta_{k_{\sigma(i)}k'_i} = d_k^{-1}\left(\frac{2N-1}{4}\right)^n\delta_{kk'}.$$

Due to the relation (5.2.23), we expand any function $f: F^{\mathrm{V},-}_N \to \mathbb{R}$ in terms of antisymmetric cosine functions as follows.

$$f(s) = \sum_{k\in D^-_N} A_k\cos^-_k(s) \quad \text{with} \quad A_k = d_k\left(\frac{4}{2N-1}\right)^n\sum_{s\in F^{\mathrm{V},-}_N}\varepsilon_s f(s)\cos^-_k(s).$$

Validity of the expansion follows from the fact that the number of points in D^-_N is equal to the number of points in $F^{\mathrm{V},-}_N$ and thus the functions $\cos^-_k$ with $k \in D^-_N$ form an orthogonal basis of the space of all functions $f: F^{\mathrm{V},-}_N \to \mathbb{R}$ with the scalar product (5.2.22). The remaining transforms AMDCT VI–VIII are deduced similarly.

5.2.2.2. *AMDCT VI*

For $N \in \mathbb{N}$ we consider the antisymmetric cosine functions $\cos^-_k(s)$ labelled by the index set $k \in D^-_N$ and restricted to the finite grid of points contained in $\widetilde{C}_N \subset F(\widetilde{S}^{\mathrm{aff}}_n)$,

$$F^{\mathrm{VI},-}_N \equiv \left\{\left(\frac{2\left(r_1+\frac{1}{2}\right)}{2N-1},\dots,\frac{2\left(r_n+\frac{1}{2}\right)}{2N-1}\right) \mid (r_1,\dots,r_n)\in D^-_N\right\}. \qquad (5.2.25)$$

The antisymmetric cosine functions labelled by parameters in $k, k' \in D_N^-$, are pairwise discretely orthogonal on the grid $F_N^{\mathrm{VI},-}$, i.e.

$$\sum_{s\in F_N^{\mathrm{VI},-}} \widetilde{\varepsilon}_s \cos_k^-(s)\cos_{k'}^-(s) = d_k^{-1}\left(\frac{2N-1}{4}\right)^n \delta_{kk'}, \tag{5.2.26}$$

where $\widetilde{\varepsilon}_s$ is given by (5.2.19). Therefore, we expand any function $f : F_N^{\mathrm{VI},-} \to \mathbb{R}$ in terms of antisymmetric cosine functions as

$$f(s) = \sum_{k\in D_N^-} A_k \cos_k^-(s) \quad \text{with} \quad A_k = d_k \left(\frac{4}{2N-1}\right)^n \sum_{s\in F_N^{\mathrm{VI},-}} \widetilde{\varepsilon}_s f(s)\cos_k^-(s)\,.$$

5.2.2.3. *AMDCT VII*

For $N \in \mathbb{N}$ are the antisymmetric cosine functions $\cos_{k+\varrho}^-(s)$, $k \in D_N^-$, and $\varrho = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ restricted to the finite grid of points $F_N^{\mathrm{VII},-} \equiv F_N^{\mathrm{V},-} \subset C_N$ pairwise discretely orthogonal with respect to the scalar product (5.2.22), i.e.

$$\sum_{s\in F_N^{\mathrm{VII},-}} \varepsilon_s \cos_{k+\varrho}^-(s)\cos_{k'+\varrho}^-(s) = \widetilde{d}_k^{-1}\left(\frac{2N-1}{4}\right)^n \delta_{kk'}\,, \tag{5.2.27}$$

where $\widetilde{d}_k$ is given by (5.2.15). Therefore, we expand any function $f : F_N^{\mathrm{VII},-} \to \mathbb{R}$ in terms of antisymmetric cosine functions as follows.

$$f(s) = \sum_{k\in D_N^-} A_k \cos_{k+\varrho}^-(s) \quad \text{with} \quad A_k = \widetilde{d}_k \left(\frac{4}{2N-1}\right)^n \sum_{s\in F_N^{\mathrm{VII},-}} \varepsilon_s f(s)\cos_{k+\varrho}^-(s)\,.$$

5.2.2.4. *AMDCT VIII*

For $N \in \mathbb{N}$ are the antisymmetric cosine functions $\cos_{k+\varrho}^-(s)$, $k \in D_N^-$, and $\varrho = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$, restricted to the finite grid of points

$$F_N^{\mathrm{VIII},-} \equiv \left\{ \left(\frac{2\left(r_1+\frac{1}{2}\right)}{2N+1}, \dots, \frac{2\left(r_n+\frac{1}{2}\right)}{2N+1}\right) \mid (r_1,\dots,r_n) \in D_N^- \right\} \tag{5.2.28}$$

pairwise discretely orthogonal, i.e. for any $k, k' \in D_N^-$ it holds that

$$\sum_{s\in F_N^{\mathrm{VIII},-}} \cos_{k+\varrho}^-(s)\cos_{k'+\varrho}^-(s) = \left(\frac{2N+1}{4}\right)^n \delta_{kk'}\,. \tag{5.2.29}$$

Therefore, we expand any function $f : F_N^{\mathrm{VIII},-} \to \mathbb{R}$ in terms of antisymmetric cosine functions as follows.

$$f(s) = \sum_{k\in D_N^-} A_k \cos_{k+\varrho}^-(s) \quad \text{with} \quad A_k = \left(\frac{4}{2N+1}\right)^n \sum_{s\in F_N^{\mathrm{VIII},-}} f(s)\cos_{k+\varrho}^-(s)\,.$$

5.2.3. Interpolations by antisymmetric multivariate cosine functions

Suppose we have a real-valued function f given on $F(\widetilde{S}_n^{\mathrm{aff}})$. In Section 5.2.2 we defined three finite grids in $F(\widetilde{S}_n^{\mathrm{aff}})$, namely $F_N^{\mathrm{VI},-}, F_N^{\mathrm{V},-} \equiv F_N^{\mathrm{VII},-}$ and $F_N^{\mathrm{VIII},-}$. We are interested in finding the interpolating polynomial of f in the form of a finite sum of antisymmetric multivariate cosine functions labelled by k or by $k+\varrho$, $k \in D_N^-$ in such a way that it coincides with f on one of the grid in $F(\widetilde{S}_n^{\mathrm{aff}})$. We distinguish between four types of interpolating polynomials defined for $x \in F(\widetilde{S}_n^{\mathrm{aff}})$ and satisfying different conditions,

$$
\begin{aligned}
\psi_N^{\mathrm{V},-}(x) &= \sum_{k \in D_N^-} B_k \cos_k^-(x)\,, && \psi_N^{\mathrm{V},-}(s) = f(s)\,, \quad s \in F_N^{\mathrm{V},-}\,,\\
\psi_N^{\mathrm{VI},-}(x) &= \sum_{k \in D_N^-} B_k \cos_{k+\varrho}^-(x)\,, && \psi_N^{\mathrm{VI},-}(s) = f(s)\,, \quad s \in F_N^{\mathrm{VI},-}\,,\\
\psi_N^{\mathrm{VII},-}(x) &= \sum_{k \in D_N^-} B_k \cos_k^-(x)\,, && \psi_N^{\mathrm{VII},-}(s) = f(s)\,, \quad s \in F_N^{\mathrm{VII},-}\,,\\
\psi_N^{\mathrm{VIII},-}(x) &= \sum_{k \in D_N^-} B_k \cos_{k+\varrho}^-(x)\,, && \psi_N^{\mathrm{VIII},-}(s) = f(s)\,, \quad s \in F_N^{\mathrm{VIII},-}\,.
\end{aligned}
\tag{5.2.30}
$$

According to Section 5.2.2, the coefficients B_k are chosen to be equal to corresponding A_k. In fact, it is not possible to have other values of the coefficients B_k since it would contradict the fact that the antisymmetric cosine functions labelled by k or by $k+\varrho$ with $k \in D_N^-$ are basis vectors of the space of functions given on the corresponding grid.

Example 5.2.1. *For $n = 3$ we choose a model function f and interpolate it by $\psi_N^{\mathrm{V},-}(x,y,z)$ and $\psi_N^{\mathrm{VII},-}(x,y,z)$ for various values of N. The model function f is a smooth characteristic function given by*

$$
f_{\alpha,\beta,(x_0,y_0,z_0)}(x,y,z) := \begin{cases} 1 & r < \alpha \\ 0 & r > \beta \\ \mathrm{e} \cdot \exp\left(\left(\frac{r-\alpha}{\beta-\alpha}\right)^2 - 1\right)^{-1} & \text{otherwise,} \end{cases} \tag{5.2.31}
$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, with fixed values of the parameters $\alpha, \beta, (x_0, y_0, z_0)$ as

$$
\alpha = 1/10\,, \quad \beta = 1/6\,, \quad (x_0, y_0, z_0) = (0.8, 0.53, 0.25)\,. \tag{5.2.32}
$$

Figure 5.3 contains the function $f_{\alpha,\beta,(x_0,y_0,z_0)}$ for fixed parameters (5.2.32) in the fundamental domain $F(\widetilde{S}_3^{\mathrm{aff}})$ and cut by the plane $z = \frac{1}{3}$. The antisymmetric interpolating polynomials of type V. and VII. for $N = 5, 10, 20$ are depicted in Figures 5.4 and 5.5.

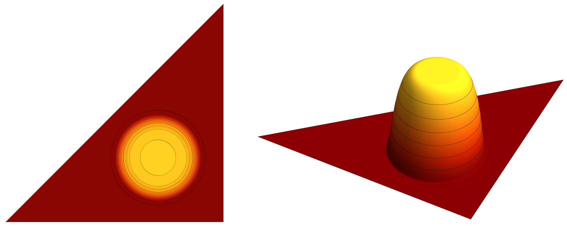


FIGURE 5.3. The cut of the characteristic function (5.2.31) with fixed parameters (5.2.32) and $z = \frac{1}{3}$.

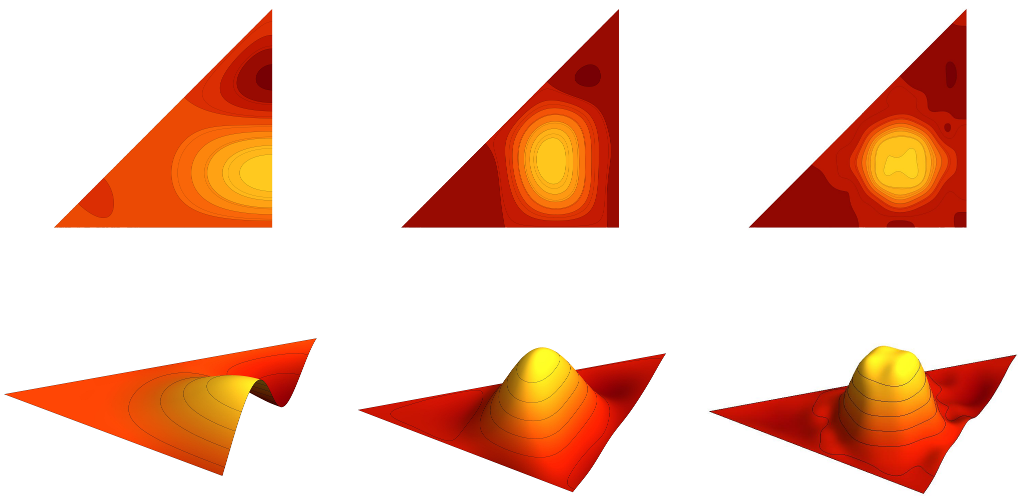


FIGURE 5.4. The antisymmetric cosine interpolating polynomials $\psi_N^{\mathrm{V},-}(x, y, \frac{1}{3})$ of the characteristic function (5.2.31) shown in Figure 5.3 with $N = 5, 10, 20$.

5.2.4. Symmetric discrete multivariate cosine transforms

The symmetric discrete multivariate cosine transforms (SMDCTs), which can be viewed as symmetric multivariate generalizations of DCTs, are derived using the one-dimensional DCTs from Section 5.2.1. The four types of SMDCT, connected to DCT I–IV, are contained in [**25**]. Our goal is to complete the list of SMDCTs by developing the remaining four transforms of types V–VIII. Note that the coefficients $\varepsilon_s, \widetilde{\varepsilon}_s$, and $d_k, \widetilde{d}_k$ are given by (5.2.18), (5.2.19) and (5.2.14), (5.2.15), respectively.

5.2.4.1. *SMDCT V*

For $N \in \mathbb{N}$ we consider the symmetric cosine functions $\cos_k^+(s)$ labelled by the index set

$$D_N^+ \equiv \{(k_1, \dots, k_n) \in D_N \mid k_1 \geq k_2 \geq \dots \geq k_n\} \tag{5.2.33}$$

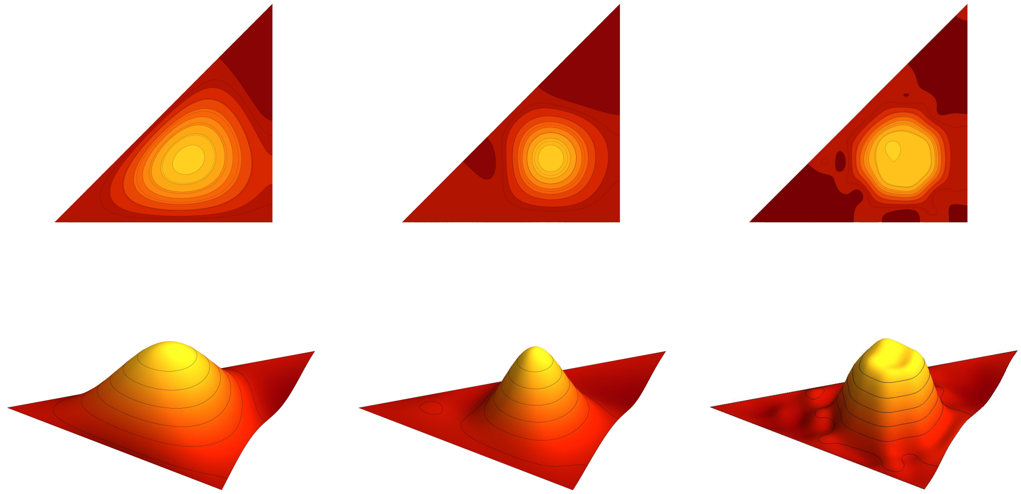


FIGURE 5.5. The antisymmetric cosine interpolating polynomials $\psi_N^{\mathrm{VII},-}(x,y,\frac{1}{3})$ of the characteristic function (5.2.31) shown in Figure 5.3 with $N=5,10,20$.

and restricted to the finite grid of points contained in $C_N \subset F(\widetilde{S}_n^{\mathrm{aff}})$

$$F_N^{\mathrm{V},+} \equiv \left\{ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) \mid (r_1,\dots,r_n) \in D_N^+ \right\} . \tag{5.2.34}$$

The scalar product of any two functions $f,g : F_N^{\mathrm{V},+} \to \mathbb{R}$ given on the points of the grid $F_N^{\mathrm{V},+}$ is defined by

$$\langle f,g \rangle \equiv \sum_{s\in F_N^{\mathrm{V},+}} \varepsilon_s H_s^{-1} f(s) g(s) \,, \quad f,g : F_N^{\mathrm{V},+} \to \mathbb{R} \,, \tag{5.2.35}$$

where H_s is given by (5.1.15). Using the discrete orthogonality of one-dimensional cosine functions (5.2.2) we show that the symmetric cosine functions labelled by parameters in $k,k' \in D_N^+$ are pairwise discretely orthogonal, i.e.

$$\langle \cos_k^+, \cos_{k'}^+ \rangle = \sum_{s\in F_N^{\mathrm{V},+}} \varepsilon_s H_s^{-1} \cos_k^+(s) \cos_{k'}^+(s) = \frac{H_k}{d_k} \left(\frac{2N-1}{4} \right)^n \delta_{kk'} \,. \tag{5.2.36}$$

Note that acting by all permutations in S_n on the grid F_N^+ we obtain the whole finite grid C_N, where the points $s \in F_N^+$, which are invariant with respect to some non-trivial permutation in S_n, emerge exactly H_s times. Since the symmetric cosine functions are symmetric with respect to S_n, we obtain that $\langle \cos_k^+, \cos_{k'}^+ \rangle$ is equal to

$$\frac{1}{n!} \sum_{r_1,\dots,r_n=0}^{N-1} c_{r_1} \dots c_{r_n} \cos_k^+ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) \cos_{k'}^+ \left(\frac{2r_1}{2N-1}, \dots, \frac{2r_n}{2N-1} \right) ,$$

where $n!$ represents the order of the group S_n. Moreover, the product of two symmetric cosine functions labelled by $k,k' \in D_N^+$ is rewritten due to the invariance under permutations as

$$\cos^+_k(x)\cos^+_{k'}(x) = \sum_{\sigma,\sigma'\in S_n}\prod_{i=1}^n \cos\left(\pi k_{\sigma\sigma'(i)} x_{\sigma'(i)}\right)\cos\left(\pi k'_i x_{\sigma'(i)}\right).$$

Together with the S_n-invariance of the set D_N this implies that

$$\langle \cos^+_k, \cos^+_{k'}\rangle = \sum_{\sigma\in S_n}\prod_{i=1}^n \sum_{r_i=0}^{N-1} c_{r_i}\cos\left(\frac{2\pi k_{\sigma(i)} r_i}{2N-1}\right)\cos\left(\frac{2\pi k'_i r_i}{2N-1}\right).$$

Finally, we apply the one-dimensional orthogonality relation (5.2.2) to obtain (5.2.36),

$$\langle \cos^+_k, \cos^+_{k'}\rangle = \sum_{\sigma\in S_n}\prod_{i=1}^n \frac{2N-1}{4c_{k_{\sigma(i)}}}\delta_{k_{\sigma(i)}k'_i} = \frac{H_k}{d_k}\left(\frac{2N-1}{4}\right)^n \delta_{kk'}\,.$$

Due to the relation (5.2.36), we expand any function $f : F_N^{\mathrm{V},+} \to \mathbb{R}$ in terms of symmetric cosine functions as

$$f(s) = \sum_{k\in D_N^+} A_k \cos^+_k(s), \qquad A_k = \frac{d_k}{H_k}\left(\frac{4}{2N-1}\right)^n \sum_{s\in F_N^{\mathrm{V},+}} \varepsilon_s H_s^{-1} f(s)\cos^+_k(s)\,.$$

Validity of the expansion follows from the fact that the number of points in D_N^+ is equal to the number of points in $F_N^{\mathrm{V},+}$ and thus the functions $\cos^+_k$ with $k\in D_N^+$ form an orthogonal basis of the space of all functions $f : F_N^{\mathrm{V},+} \to \mathbb{R}$ with the scalar product (5.2.35). The remaining transforms SMDCT VI–VIII are deduced similarly.

5.2.4.2. *SMDCT VI*

For $N\in\mathbb{N}$ we consider the antisymmetric cosine functions $\cos^+_k(s)$ labelled by the index set $k\in D_N^+$ and restricted to the finite grid of points contained in $\widetilde{C}_N \subset F(\widetilde{S}_n^{\mathrm{aff}})$,

$$F_N^{\mathrm{VI},+} \equiv \left\{ \left(\frac{2\left(r_1+\frac{1}{2}\right)}{2N-1}, \dots, \frac{2\left(r_n+\frac{1}{2}\right)}{2N-1}\right) \mid (r_1,\dots,r_n)\in D_N^+ \right\} \tag{5.2.37}$$

The antisymmetric cosine functions labelled by parameters in $k,k'\in D_N^+$, are pairwise discretely orthogonal on the grid $F_N^{\mathrm{VI},+}$, i.e.

$$\sum_{s\in F_N^{\mathrm{VI},+}} \widetilde{\varepsilon}_s H_s^{-1}\cos^+_k(s)\cos^+_{k'}(s) = \frac{H_k}{d_k}\left(\frac{2N-1}{4}\right)^n \delta_{kk'}\,. \tag{5.2.38}$$

Therefore, we expand any function $f : F_N^{\mathrm{VI},+} \to \mathbb{R}$ in terms of symmetric cosine functions as follows.

$$f(s) = \sum_{k\in D_N^+} A_k \cos_k^+(s) \quad \text{with} \quad A_k = \frac{d_k}{H_k}\left(\frac{4}{2N-1}\right)^n \sum_{s\in F_N^{\mathrm{VI},+}} \widetilde{\varepsilon}_s H_s^{-1} f(s) \cos_k^-(s)\,.$$

5.2.4.3. *SMDCT VII*

For $N \in \mathbb{N}$ are the symmetric cosine functions $\cos_{k+\varrho}^+(s)$, $k \in D_N^+$, and $\varrho = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ restricted to the finite grid of points $F_N^{\mathrm{VII},+} \equiv F_N^{\mathrm{V},+} \subset C_N$ pairwise discretely orthogonal with respect to the scalar product (5.2.35), i.e.

$$\sum_{s\in F_N^{\mathrm{VII},+}} \varepsilon_s H_s^{-1} \cos_{k+\varrho}^+(s) \cos_{k'+\varrho}^+(s) = \frac{H_k}{\widetilde{d}_k}\left(\frac{2N-1}{4}\right)^n \delta_{kk'}\,. \tag{5.2.39}$$

Therefore, we expand any function $f : F_N^{\mathrm{VII},+} \to \mathbb{R}$ in terms of symmetric cosine functions as

$$f(s) = \sum_{k\in D_N^+}^{N-1} A_k \cos_{k+\varrho}^+(s), \quad A_k = \frac{\widetilde{d}_k}{H_k}\left(\frac{4}{2N-1}\right)^n \sum_{s\in F_N^{\mathrm{VII},+}} \varepsilon_s H_s^{-1} f(s) \cos_{k+\varrho}^+(s)\,.$$

5.2.4.4. *SMDCT VIII*

For $N \in \mathbb{N}$ are the symmetric cosine functions $\cos_{k+\varrho}^+(s)$, $k \in D_N^+$, and $\varrho = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ restricted to the finite grid of points

$$F_N^{\mathrm{VIII},+} \equiv \left\{ \left(\frac{2\left(r_1+\frac{1}{2}\right)}{2N+1}, \ldots, \frac{2\left(r_n+\frac{1}{2}\right)}{2N+1} \right) \mid (r_1,\ldots,r_n) \in D_N^+ \right\} \tag{5.2.40}$$

pairwise discretely orthogonal, i.e. for any $k, k' \in D_N^+$ it holds that

$$\sum_{s\in F_N^{\mathrm{VIII},+}} H_s^{-1} \cos_{k+\varrho}^+(s) \cos_{k'+\varrho}^+(s) = H_k \left(\frac{2N+1}{4}\right)^n \delta_{kk'}\,. \tag{5.2.41}$$

Therefore, we expand any function $f : F_N^{\mathrm{VIII},+} \to \mathbb{R}$ in terms of symmetric cosine functions as

$$f(s) = \sum_{k\in D_N^+} A_k \cos_{k+\varrho}^+(s), \quad A_k = \frac{1}{H_k}\left(\frac{4}{2N+1}\right)^n \sum_{s\in F_N^{\mathrm{VIII},+}} H_s^{-1} f(s) \cos_{k+\varrho}^+(s)\,.$$

5.2.5. Interpolations by symmetric multivariate cosine functions

Suppose we have a real-valued function f given on $F(\widetilde{S}_n^{\mathrm{aff}})$. In Section 5.2.4 we defined three finite grids in $F(\widetilde{S}_n^{\mathrm{aff}})$, namely $F_N^{\mathrm{VI},+}$, $F_N^{\mathrm{V},+} \equiv F_N^{\mathrm{VII},+}$ and $F_N^{\mathrm{VIII},+}$. We are interested in finding the interpolating polynomial of f in the form of a finite sum of symmetric multivariate cosine functions labelled by k or by $k+\varrho$, $k \in D_N^+$

in such a way that it coincides with f on one of the grid in $F(\widetilde{S}_n^{\mathrm{aff}})$. We distinguish between four types of interpolating polynomials defined for $x \in F(\widetilde{S}_n^{\mathrm{aff}})$ and satisfying different conditions,

$$\begin{aligned}
\psi_N^{\mathrm{V},+}(x) &= \sum_{k\in D_N^+} B_k \cos_k^+(x)\,, & \psi_N^{\mathrm{V},+}(s) &= f(s)\,, \quad s \in F_N^{\mathrm{V},+}\,,\\
\psi_N^{\mathrm{VI},+}(x) &= \sum_{k\in D_N^+} B_k \cos_{k+\varrho}^+(x)\,, & \psi_N^{\mathrm{VI},+}(s) &= f(s)\,, \quad s \in F_N^{\mathrm{VI},+}\,,\\
\psi_N^{\mathrm{VII},+}(x) &= \sum_{k\in D_N^+} B_k \cos_k^+(x)\,, & \psi_N^{\mathrm{VII},+}(s) &= f(s)\,, \quad s \in F_N^{\mathrm{VII},+}\,,\\
\psi_N^{\mathrm{VIII},+}(x) &= \sum_{k\in D_N^+} B_k \cos_{k+\varrho}^+(x)\,, & \psi_N^{\mathrm{VIII},+}(s) &= f(s)\,, \quad s \in F_N^{\mathrm{VIII},+}\,.
\end{aligned} \tag{5.2.42}$$

According to Section 5.2.4, the coefficients B_k are chosen to be equal to corresponding A_k. In fact, it is not possible to have other values of the coefficients B_k since it would contradict the fact that the antisymmetric cosine functions labelled by k or by $k+\varrho$ with $k \in D_N^+$ are basis vectors of the space of functions given on the corresponding grid.

Example 5.2.2. *For $n = 3$ is the characteristic function* (5.2.31) *with values of parameters given by* (5.2.32) *chosen as a model function. We interpolate this function by polynomials of symmetric cosine functions $\psi_N^{\mathrm{V},+}(x,y,z)$ and $\psi_N^{\mathrm{VII},+}(x,y,z)$ with $N = 5, 10, 20$. Plots of this symmetric cosine interpolating polynomials are depicted in Figures 5.6 and 5.7.*

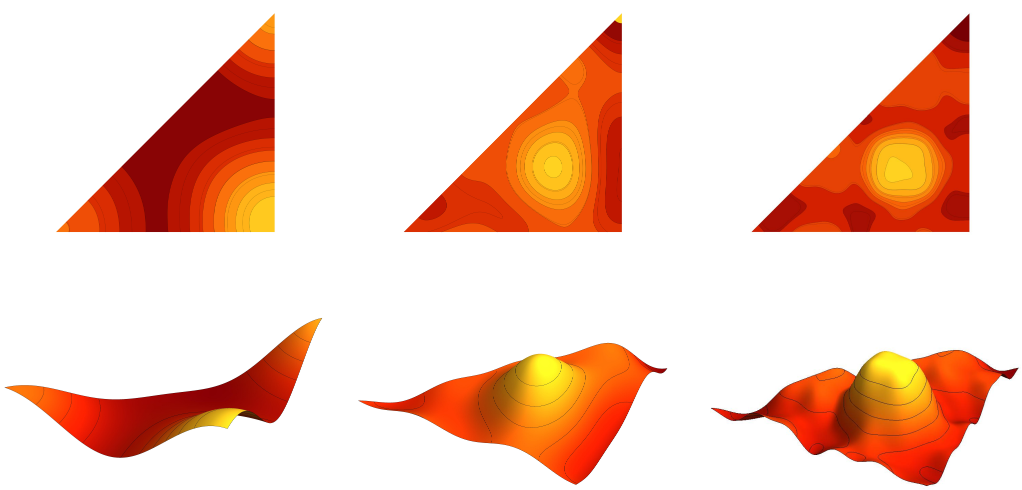


FIGURE 5.6. The symmetric cosine interpolating polynomials $\psi_N^{\mathrm{V},+}(x,y,\frac{1}{3})$ of the characteristic function (5.2.31) shown in Figure 5.3 with $N = 5, 10, 20$.

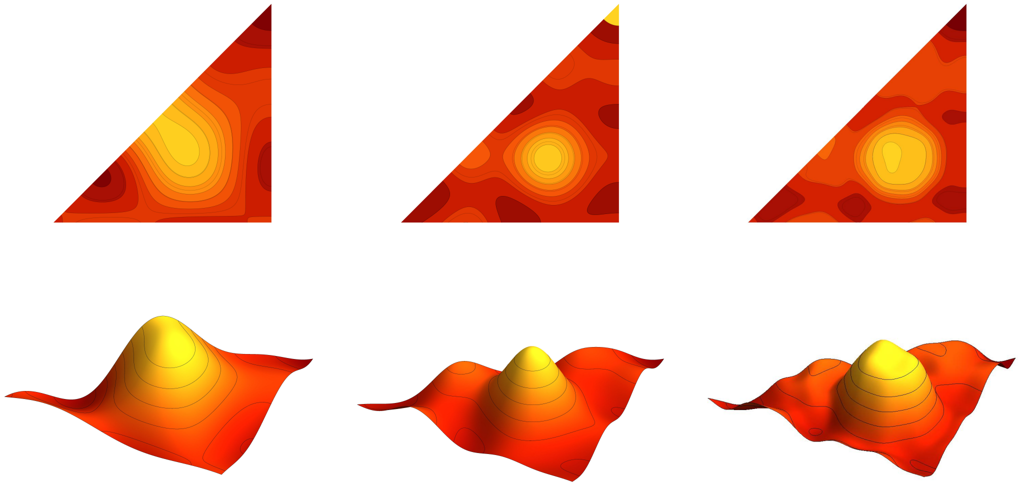


FIGURE 5.7. The symmetric cosine interpolating polynomials $\psi_N^{\mathrm{VII},+}(x, y, \frac{1}{3})$ of the characteristic function (5.2.31) shown in Figure 5.3 with $N = 5, 10, 20$.

5.3. Chebyshev-like multivariate orthogonal polynomials

Recall that the vectors ϱ, ϱ_1 and ϱ_2 are defined by (5.1.5), (5.1.9) and (5.1.10), respectively, and let us introduce the index set

$$P^+ \equiv \{(k_1, \dots, k_n) \in \mathbb{Z}^n \mid k_1 \geq k_2 \geq \dots \geq k_n \geq 0\} \tag{5.3.1}$$

together with n functions $X_1, X_2, \dots, X_n$ defined by

$$X_1 = \cos^+_{(1,0,\dots,0)},\ X_2 = \cos^+_{(1,1,0,\dots,0)},\ X_3 = \cos^+_{(1,1,1,0,\dots,0)}, \dots, X_n = \cos^+_{(1,1,\dots,1)}\,. \tag{5.3.2}$$

We demonstrate that the following defining relations, valid for all points from the interior of $F(\widetilde{S}_n^{\mathrm{aff}})$, $x \in F(\widetilde{S}_n^{\mathrm{aff}})^\circ$, $k \in P^+$

$$\begin{aligned}
\mathcal{P}_k^{I,+}(X_1(x), \dots, X_n(x)) &= \cos^+_k(x)\,,\\
\mathcal{P}_k^{I,-}(X_1(x), \dots, X_n(x)) &= \frac{\cos^-_{k+\varrho_1}(x)}{\cos^-_{\varrho_1}(x)}\,,\\
\mathcal{P}_k^{III,+}(X_1(x), \dots, X_n(x)) &= \frac{\cos^+_{k+\varrho}(x)}{\cos^+_{\varrho}(x)}\,,\\
\mathcal{P}_k^{III,-}(X_1(x), \dots, X_n(x)) &= \frac{\cos^-_{k+\varrho_2}(x)}{\cos^-_{\varrho_2}(x)}\,,
\end{aligned} \tag{5.3.3}$$

determine four classes of orthogonal polynomials

$$\mathcal{P}_k^{I,+}, \mathcal{P}_k^{I,-}, \mathcal{P}_k^{III,-}, \mathcal{P}_k^{III,-} \in \mathbb{R}[X_1, \dots, X_n]$$

of degree k_1. Since from Corollary 5.1.1 follows that the three functions in the denominators of (5.3.3) have non-zero values inside $F(\widetilde{S}_n^{\mathrm{aff}})$, the functions (5.3.3) are well defined for any point in the interior of $F(\widetilde{S}_n^{\mathrm{aff}})$. Moreover, whenever one of the denominators is zero at some points of the boundary of $F(\widetilde{S}_n^{\mathrm{aff}})$ the corresponding nominator is zero as well. We introduce an ordering within each family of polynomials (5.3.3). We say that a polynomial p_1 depending on $k = (k_1, \dots, k_n) \in P^+$ is greater than any polynomial p_2 depending on $k' = (k'_1, \dots, k'_n) \neq k$ if for all $i \in \{1, \dots, n\}$ it holds that $k_i \geq k'_i$; equally, we state that p_2 is lower than p_1. Note that the functions (5.3.3) can be viewed as generalizations of Chebyshev polynomials of the first and third kind [**39**].

5.3.1. Recurrence relations

The construction of polynomials is based on the decomposition of products of symmetric and antisymmetric cosine functions. There are three types of products which decompose to a sum of either symmetric or antisymmetric cosine functions. Such a decomposition is obtained by using classical trigonometric identities and is completely described by

$$\begin{aligned}
\cos^+_{(\lambda_1,\dots,\lambda_n)} \cdot \cos^+_{(\mu_1,\dots,\mu_n)} &= \frac{1}{2^n} \sum_{\sigma \in S_n} \sum_{\left\{\substack{a_i=\pm 1 \\ i=1,\dots,n}\right\}} \cos^+_{(\lambda_1+a_1\mu_{\sigma(1)},\dots,\lambda_n+a_n\mu_{\sigma(n)})} \,, \\
\cos^-_{(\lambda_1,\dots,\lambda_n)} \cdot \cos^+_{(\mu_1,\dots,\mu_n)} &= \frac{1}{2^n} \sum_{\sigma \in S_n} \sum_{\left\{\substack{a_i=\pm 1 \\ i=1,\dots,n}\right\}} \cos^-_{(\lambda_1+a_1\mu_{\sigma(1)},\dots,\lambda_n+a_n\mu_{\sigma(n)})} \,, \\
\cos^-_{(\lambda_1,\dots,\lambda_n)} \cdot \cos^-_{(\mu_1,\dots,\mu_n)} &= \frac{1}{2^n} \sum_{\sigma \in S_n} \sum_{\left\{\substack{a_i=\pm 1 \\ i=1,\dots,n}\right\}} \operatorname{sgn}(\sigma) \cos^+_{(\lambda_1+a_1\mu_{\sigma(1)},\dots,\lambda_n+a_n\mu_{\sigma(n)})} \,.
\end{aligned} \tag{5.3.4}$$

Using (5.3.4), we obtain the following pertinent recurrence relations. Let $\lambda = (\lambda_1, \dots, \lambda_n)$, l_i be a vector in $\mathbb{R}^n$ with $i-$th component equal to 1 and others to 0, then

$$\begin{aligned}
\cos^\pm_\lambda &= \frac{2^1}{1!(n-1)!} \cos^\pm_{\lambda-l_1} X_1 - \cos^\pm_{\lambda-2l_1} - \sum_{i=2}^{n} \left(\cos^\pm_{\lambda-l_1+l_i} + \cos^\pm_{\lambda-l_1-l_i} \right) \,, \\
\cos^\pm_\lambda &= \frac{2^2}{2!(n-2)!} \cos^\pm_{\lambda-l_1-l_2} X_2 - \cos^\pm_{\lambda-2l_1-2l_2} - \cos^\pm_{\lambda-2l_1} - \cos^\pm_{\lambda-2l_2} \\
&\quad - \sum_{i=3}^{n} \left(\cos^\pm_{\lambda-l_2+l_i} + \cos^\pm_{\lambda-2l_1-l_2-l_i} + \cos^\pm_{\lambda-l_2-l_i} + \cos^\pm_{\lambda-2l_1-l_2+l_i} \right)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\left\{ \substack{i,j=2 \\ i<j} \right\}}^{n} \Big(\cos^{\pm}_{\lambda-l_1-l_2+l_i+l_j} + \cos^{\pm}_{\lambda-l_1-l_2-l_i-l_j} + \cos^{\pm}_{\lambda-l_1-l_2+l_i-l_j} \\
&+ \cos^{\pm}_{\lambda-l_1-l_2-l_i+l_j} \Big) \,, \\
&\vdots \\
\cos^{\pm}_{\lambda} = \frac{2^n}{n!} \cos^{\pm}_{\lambda-l_1-l_2-\dots-l_n} X_n - \sum_{i=1}^{n} \cos^{\pm}_{\lambda-2l_i} - \sum_{\left\{ \substack{i,j=1 \\ i<j} \right\}}^{n} \cos^{\pm}_{\lambda-2l_i-2l_j} - \dots \\
&- \cos^{\pm}_{\lambda-2l_1-2l_2-\dots-2l_n} \,. \qquad (5.3.5)
\end{aligned}
$$

Using the relations (5.3.5) and the symmetry properties of $\cos^{\pm}_{\lambda}$ each polynomial of (5.3.3) is expressed as a linear combination of lower polynomials and a product of some lower polynomial with some X_i. Therefore, all polynomials (5.3.3) are built recursively.

Proposition 5.3.1. *Let $k \in P^+$. The functions $\mathcal{P}^{I,\pm}_k$ and $\mathcal{P}^{III,\pm}_k$ are expressed as polynomials of degree k_1 in variables $X_1, \dots, X_n$. The number of $\mathcal{P}^{I,\pm}_k$ or $\mathcal{P}^{III,\pm}_k$ with $k_1 = d$ is equal to the number of monomials of degree d, i.e.,*

$$
\binom{d+n-1}{d} . \qquad (5.3.6)
$$

Proof. At first, we proceed by induction on k_1 to show that any $\mathcal{P}^{I,+}_k$ is expressed as a polynomial of degree k_1 in the variables (5.3.2).

- If $k_1 = 0$, $\mathcal{P}^{I,+}_{(0,\dots,0)}$ is trivially the constant polynomial $n!$ of degree 0.
- If $k_1 = 1$, the polynomials $X_1, X_2, \dots, X_n$ of degree 1 correspond by definition to the set of functions $\mathcal{P}^{I,+}_{(1,k_2,\dots,k_n)}$, where $1 \geq k_2 \geq \dots \geq k_n \geq 0$.
- If $k_1 = 2$, using relations (5.3.5) and the basic properties of symmetric cosine functions, we deduce that any $\mathcal{P}^{I,+}_{(2,k_2,\dots,k_n)}$ is constructed from the decomposition of the products $X_i X_j$. Indeed, we start by obtaining the lowest polynomial $\mathcal{P}^{I,+}_{(2,0,\dots,0)}$ from the product X_1^2 which decompose into the linear combination of $\mathcal{P}^{I,+}_{(2,0,\dots,0)}$ and polynomials of degree 1. We continue by the decomposition of the product $X_1 X_2$ to have the polynomial expression of $\mathcal{P}^{I,+}_{(2,1,\dots,0)}$ and so on. Finally, we obtain that each $\mathcal{P}^{I,+}_{(2,k_2,\dots,k_n)}$ is expressed as a polynomial of degree 2 in variables (5.3.2).
- We next suppose that every $\mathcal{P}^{I,+}_{(d-1,k_2,\dots,k_n)}$ is a polynomial of degree $d-1$ in variables (5.3.2), then again using relations (5.3.5) the polynomial $\mathcal{P}^{I,+}_{(d,k_2,\dots,k_n)}$ is a linear combination of lower polynomials of degree at most d and $\mathcal{P}^{I,+}_{(d-1,\widetilde{k}_2,\dots,\widetilde{k}_n)} X_i$ of degree d.

By induction, this results to the fact that any $\mathcal{P}_k^{I,+}$ is expressed as a polynomial of degree k_1 in the variables (5.3.2). By similar arguments, we obtain the same statement for the functions $\mathcal{P}_k^{I,-}$ and $\mathcal{P}_k^{III,\pm}$. Note that we start with the products of $\cos^-_{\varrho_1}$, $\cos^+_{\varrho}$, $\cos^-_{\varrho_2}$ respectively with the variables X_i.

To prove the second statement, we observe that the number of polynomials of type $\mathcal{P}_k^{I,\pm}$ or $\mathcal{P}_k^{III,\pm}$ with $k_1 = d$ is equal to the number of elements in $\{(k_2, k_3, \dots, k_n) \in \mathbb{Z}^{n-1} \mid d \geq k_2 \geq k_3 \geq \cdots \geq k_n \geq 0\}$ which is the same as

$$\#\left\{(l_n + l_{n-1} + \cdots + l_2, \dots, l_n + l_{n-1}, l_n) \mid l_i, l_0 \in \mathbb{Z}^{\geq 0}, l_0 + \sum_{i=2}^{n} l_i = d\right\}$$

or equivalently

$$\#\left\{X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} \mid \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = d\right\} = \binom{d+n-1}{d}.$$

The proof of the last equality is found in [**5**]. □

Example 5.3.1. *In particular, the relations* (5.3.5) *together with the symmetry properties of symmetric cosine functions imply the following set of recurrence relations for* $P^{I,+}_{(k_1,k_2,k_3)}$.

$$\mathcal{P}^{I,+}_{(0,0,0)} = 6\,, \quad \mathcal{P}^{I,+}_{(1,0,0)} = X_1\,, \quad \mathcal{P}^{I,+}_{(1,1,0)} = X_2\,, \quad \mathcal{P}^{I,+}_{(1,1,1)} = X_3\,, \tag{5.3.7}$$

$$\begin{aligned}
k_1 \geq 2,\ k_2 = k_3 = 0: \quad & \mathcal{P}^{I,+}_{(k_1,0,0)} = \mathcal{P}^{I,+}_{(k_1-1,0,0)} X_1 - \mathcal{P}^{I,+}_{(k_1-2,0,0)} - 4\mathcal{P}^{I,+}_{(k_1-1,1,0)}\,,\\
k_1 - 1 > k_2 > k_3 = 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_2,0)} = \mathcal{P}^{I,+}_{(k_1-1,k_2,0)} X_1 - \mathcal{P}^{I,+}_{(k_1-2,k_2,0)} - 2\mathcal{P}^{I,+}_{(k_1-1,k_2,1)}\\
& \qquad - \mathcal{P}^{I,+}_{(k_1-1,k_2+1,0)} - \mathcal{P}^{I,+}_{(k_1-1,k_2-1,0)}\,,\\
k_1 - 1 > k_2 = k_3 > 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_2,k_2)} = \mathcal{P}^{I,+}_{(k_1-1,k_2,k_2)} X_1 - \mathcal{P}^{I,+}_{(k_1-2,k_2,k_2)}\\
& \qquad - 2\mathcal{P}^{I,+}_{(k_1-1,k_2+1,k_2)} - 2\mathcal{P}^{I,+}_{(k_1-1,k_2,k_2-1)}\,,\\
k_1 - 1 > k_2 > k_3 > 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_2,k_3)} = \mathcal{P}^{I,+}_{(k_1-1,k_2,k_3)} X_1 - \mathcal{P}^{I,+}_{(k_1-2,k_2,k_3)} - \mathcal{P}^{I,+}_{(k_1-1,k_2,k_3+1)}\\
& \qquad - \mathcal{P}^{I,+}_{(k_1-1,k_2,k_3-1)} - \mathcal{P}^{I,+}_{(k_1-1,k_2+1,k_3)} - \mathcal{P}^{I,+}_{(k_1-1,k_2-1,k_3)}\,,\\
k_1 - 1 = k_2 > k_3 = 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_1-1,0)} = \frac{1}{2}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,0)} X_1 - \mathcal{P}^{I,+}_{(k_1-1,k_1-2,0)}\\
& \qquad - \mathcal{P}^{I,+}_{(k_1-1,k_1-1,1)}\,,\\
k_1 - 1 = k_2 > k_3 > 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_1-1,k_3)} = \frac{1}{2}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3)} X_1 - \mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_3)}\\
& \qquad - \frac{1}{2}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3+1)} - \frac{1}{2}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3-1)}\,,\\
k_1 - 1 = k_2 = k_3 > 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_1-1,k_1-1)} = \frac{1}{3}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-1)} X_1 - \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-2)}\,,
\end{aligned}$$

$$
\begin{aligned}
k_1 = k_2 > 2, k_3 = 0: \quad & \mathcal{P}^{I,+}_{(k_1,k_1,0)} = 2\mathcal{P}^{I,+}_{(k_1-1,k_1-1,0)}X_2 - 2\mathcal{P}^{I,+}_{(k_1-1,k_1-2,0)}X_1 \\
& \quad - 2\mathcal{P}^{I,+}_{(k_1-1,k_1-1,1)}X_1 + \mathcal{P}^{I,+}_{(k_1-2,k_1-2,0)} + 4\mathcal{P}^{I,+}_{(k_1-1,k_1-1,0)} \\
& \quad + 4\mathcal{P}^{I,+}_{(k_1-1,k_1-2,1)} + 2\mathcal{P}^{I,+}_{(k_1-1,k_1-3,0)} + 2\mathcal{P}^{I,+}_{(k_1-1,k_1-1,2)}\,, \\
k_1 = k_2 > k_3 + 2 > 2: \quad & \mathcal{P}^{I,+}_{(k_1,k_1,k_3)} = 2\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3)}X_2 - 2\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_3)}X_1 \\
& \quad - \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3+1)}X_1 - \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3-1)}X_1 \\
& \quad + \mathcal{P}^{I,+}_{(k_1-2,k_1-2,k_3)} + 2\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_3+1)} \\
& \quad + 2\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_3-1)} + 4\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3)} \\
& \quad + 2\mathcal{P}^{I,+}_{(k_1-1,k_1-3,k_3)} + \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3+2)} \\
& \quad + \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_3-2)}\,, \\
k_1 = k_2 = k_3 + 2 > 3: \quad & \mathcal{P}^{I,+}_{(k_1,k_1,k_1-2)} = 2\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-2)}X_2 - 2\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_1-2)}X_1 \\
& \quad - \frac{2}{3}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-1)}X_1 - \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-3)}X_1 \\
& \quad + \mathcal{P}^{I,+}_{(k_1-2,k_1-2,k_1-2)} + 5\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-2)} \\
& \quad + 4\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_1-3)} + \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-4)}\,, \\
k_1 = k_2 = k_3 + 1 > 2: \quad & \mathcal{P}^{I,+}_{(k_1,k_1,k_1-1)} = \frac{2}{3}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-1)}X_2 - \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-2)}X_1 \\
& \quad + \mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_1-2)} + \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-1)} \\
& \quad + \mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-3)}\,, \\
k_1 = k_2 = k_3 > 3: \quad & \mathcal{P}^{I,+}_{(k_1,k_1,k_1)} = \frac{4}{3}\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-1)}X_3 - 6\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-2)}X_2 \\
& \quad + 3\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_1-2)}X_1 + 2\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-1)}X_1 \\
& \quad + 3\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-3)}X_1 - \mathcal{P}^{I,+}_{(k_1-2,k_1-2,k_1-2)} \\
& \quad - 9\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-2)} - 6\mathcal{P}^{I,+}_{(k_1-1,k_1-2,k_1-3)} \\
& \quad - 3\mathcal{P}^{I,+}_{(k_1-1,k_1-1,k_1-4)}
\end{aligned} \tag{5.3.8}
$$

together with additional polynomial expressions

$$
\begin{aligned}
\mathcal{P}^{I,+}_{(2,2,0)} &= 2X_2^2 - 2X_1^2 - \frac{4}{3}X_1X_3 + 8X_2 + 6\,, \\
\mathcal{P}^{I,+}_{(2,2,1)} &= \frac{2}{3}X_2X_3 - X_1X_2 + 2X_3 + X_1\,, \\
\mathcal{P}^{I,+}_{(2,2,2)} &= \frac{4}{3}X_3^2 - 6X_2^2 + 3X_1^2 + 4X_1X_2 - 12X_2 - 6\,, \\
\mathcal{P}^{I,+}_{(3,3,1)} &= \frac{4}{3}X_2^2X_3 - \frac{8}{9}X_1X_3^2 - 2X_1^2X_3 + 8X_2X_3 + 9X_3\,, \\
\mathcal{P}^{I,+}_{(3,3,3)} &= \frac{16}{9}X_3^3 - 12X_2^2X_3 + 8X_1X_3^2 + 9X_1^2X_3 - 36X_2X_3 - 27X_3\,.
\end{aligned} \tag{5.3.9}
$$

	1	X_2	X_1^2	X_1X_3	X_2^2	X_3^2
$\mathcal{P}^{I,+}_{(0,0,0)}$	6					
$\mathcal{P}^{I,+}_{(1,1,0)}$	0	1				
$\mathcal{P}^{I,+}_{(2,0,0)}$	-6	-4	1			
$\mathcal{P}^{I,+}_{(2,1,1)}$	0	-1	0	$\frac{1}{3}$		
$\mathcal{P}^{I,+}_{(2,2,0)}$	6	8	-2	$-\frac{4}{3}$	2	
$\mathcal{P}^{I,+}_{(2,2,2)}$	-6	-12	3	4	-6	$\frac{4}{3}$

Table 5.1. The coefficients of the polynomials $P^{I,+}_{(k_1,k_2,k_3)}$ with $k_1 \leq 2$ and $k_1+k_2+k_3$ even.

	1	X_2	X_1^2	X_1X_3	X_2^2	X_3^2
$\mathcal{P}^{I,-}_{(0,0,0)}$	1					
$\mathcal{P}^{I,-}_{(1,1,0)}$	3	2				
$\mathcal{P}^{I,-}_{(2,0,0)}$	-4	-2	1			
$\mathcal{P}^{I,-}_{(2,1,1)}$	-3	-2	1	$\frac{2}{3}$		
$\mathcal{P}^{I,-}_{(2,2,0)}$	12	14	-3	$-\frac{4}{3}$	4	
$\mathcal{P}^{I,-}_{(2,2,2)}$	-9	-12	3	$\frac{10}{3}$	-4	$\frac{8}{9}$

Table 5.2. The coefficients of the polynomials $P^{I,-}_{(k_1,k_2,k_3)}$ with $k_1 \leq 2$ and $k_1+k_2+k_3$ even.

	X_1	X_3	X_1X_2	X_2X_3
$\mathcal{P}^{I,+}_{(1,0,0)}$	1			
$\mathcal{P}^{I,+}_{(1,1,1)}$	0	1		
$\mathcal{P}^{I,+}_{(2,1,0)}$	-1	-1	$\frac{1}{2}$	
$\mathcal{P}^{I,+}_{(2,2,1)}$	1	2	-1	$\frac{2}{3}$

	X_1	X_3	X_1X_2	X_2X_3
$\mathcal{P}^{I,-}_{(1,0,0)}$	1			
$\mathcal{P}^{I,-}_{(1,1,1)}$	1	$\frac{2}{3}$		
$\mathcal{P}^{I,-}_{(2,1,0)}$	0	$-\frac{4}{3}$	2	
$\mathcal{P}^{I,-}_{(2,2,1)}$	1	$\frac{8}{3}$	0	$\frac{4}{3}$

Table 5.3. The coefficients of the polynomials $\mathcal{P}^{I,\pm}_{(k_1,k_2,k_3)}$ with $k_1 \leq 2$ and $k_1+k_2+k_3$ odd.

Similarly, one may find recurrence relations for $\mathcal{P}^{I,-}_{(k_1,k_2,k_3)}$ and $\mathcal{P}^{III,\pm}_{(k_1,k_2,k_3)}$. The polynomials $\mathcal{P}^{I,\pm}_{(k_1,k_2,k_3)}$, $\mathcal{P}^{III,\pm}_{(k_1,k_2,k_3)}$ of degree at most 2 are shown in Tables 5.1–5.5.

5.3.2. Continuous orthogonality

Continuous orthogonality of antisymmetric and symmetric cosine functions within each family is detailed in Section 5.1.1. Our goal is to reformulate the orthogonality relations (5.1.16) – (5.1.19) after the change of variables $(x_1,\dots,x_n)$ to polynomial variables $(X_1,\dots,X_n)$ and obtain the orthogonality relations for polynomials $\mathcal{P}^{I,\pm}_k$ and $\mathcal{P}^{III,\pm}_k$. In order to determine the corresponding weight

	1	X_1	X_2	X_3	X_1^2	X_1X_2	X_1X_3	X_2^2	X_2X_3	X_3^2
$\mathcal{P}^{III,+}_{(0,0,0)}$	1									
$\mathcal{P}^{III,+}_{(1,0,0)}$	-1	$\frac{1}{3}$								
$\mathcal{P}^{III,+}_{(1,1,0)}$	1	$-\frac{2}{3}$	$\frac{2}{3}$							
$\mathcal{P}^{III,+}_{(1,1,1)}$	-1	1	-2	$\frac{4}{3}$						
$\mathcal{P}^{III,+}_{(2,0,0)}$	-1	$-\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{1}{3}$					
$\mathcal{P}^{III,+}_{(2,1,0)}$	1	0	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$				
$\mathcal{P}^{III,+}_{(2,1,1)}$	-1	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{4}{9}$			
$\mathcal{P}^{III,+}_{(2,2,0)}$	1	$\frac{2}{3}$	$\frac{10}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{8}{9}$	$\frac{4}{3}$		
$\mathcal{P}^{III,+}_{(2,2,1)}$	-1	$-\frac{1}{3}$	-2	$\frac{4}{3}$	$\frac{2}{3}$	0	0	$-\frac{4}{3}$	$\frac{8}{9}$	
$\mathcal{P}^{III,+}_{(2,2,2)}$	-1	-1	-6	$-\frac{16}{3}$	1	2	4	-4	$-\frac{8}{3}$	$\frac{16}{9}$

TABLE 5.4. The coefficients of the polynomials $\mathcal{P}^{III,+}_{(k_1,k_2,k_3)}$ with $k_1 \leq 2$.

	1	X_1	X_2	X_3	X_1^2	X_1X_2	X_1X_3	X_2^2	X_2X_3	X_3^2
$\mathcal{P}^{III,-}_{(0,0,0)}$	1									
$\mathcal{P}^{III,-}_{(1,0,0)}$	-1	1								
$\mathcal{P}^{III,-}_{(1,1,0)}$	3	-1	2							
$\mathcal{P}^{III,-}_{(1,1,1)}$	-3	2	-2	$\frac{4}{3}$						
$\mathcal{P}^{III,-}_{(2,0,0)}$	-3	-1	-2	0	1					
$\mathcal{P}^{III,-}_{(2,1,0)}$	1	1	0	$-\frac{4}{3}$	-1	2				
$\mathcal{P}^{III,-}_{(2,1,1)}$	-3	-2	-2	0	2	-2	$\frac{4}{3}$			
$\mathcal{P}^{III,-}_{(2,2,0)}$	8	0	12	$\frac{4}{3}$	-2	-2	$-\frac{4}{3}$	4		
$\mathcal{P}^{III,-}_{(2,2,1)}$	-6	2	-10	4	1	2	0	-4	$\frac{8}{3}$	
$\mathcal{P}^{III,-}_{(2,2,2)}$	-6	-2	-10	$-\frac{16}{3}$	3	0	$\frac{16}{3}$	-4	$-\frac{8}{3}$	$\frac{16}{9}$

TABLE 5.5. The coefficients of the polynomials $\mathcal{P}^{III,-}_{(k_1,k_2,k_3)}$ with $k_1 \leq 2$.

functions in the integrals defining continuous orthogonality, we calculate the Jacobian of the change of variables $(X_1(x_1,\dots,x_n),\dots,X_n(x_1,\dots,x_n))$ to variables $(x_1,\dots,x_n)$.

Proposition 5.3.2. *The determinant of the Jacobian matrix*

$$J \equiv J(x_1,\dots,x_n) \equiv \det \frac{\partial(X_1,\dots,X_n)}{\partial(x_1,\dots,x_n)}$$

for the coordinate change from $(X_1,\dots,X_n)$ *to* $(x_1,\dots,x_n)$ *is given by*

$$J = (-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}} \pi^n \left(\prod_{i=1}^{n}(n-i)!i!\right) \sin^{-}_{(1,2,\dots,n)}(x_1,\dots,x_n)\,, \qquad (5.3.10)$$

where

$$
\begin{aligned}
&\sin^-_{(1,2,\dots,n)}(x_1,\dots,x_n)\\
&\qquad\equiv \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\sin(\pi\sigma(1)x_1)\sin(\pi\sigma(2)x_2)\cdots\sin(\pi\sigma(n)x_n)\\
&\qquad= 2^{n(n-1)}\prod_{i=1}^{n}\sin(\pi x_i)\prod_{1\le i<j\le n}\sin\left(\frac{\pi}{2}(x_i+x_j)\right)\sin\left(\frac{\pi}{2}(x_i-x_j)\right).
\end{aligned}
\tag{5.3.11}
$$

PROOF. We prove the formulas (5.3.10), (5.3.11) by direct calculation. By the definition of the variables X_j via symmetric cosine functions, we have

$$X_j=(n-j)!j!\sum_{\left\{\substack{i_1,\dots,i_j=1\\ i_1<\dots<i_j}\right\}}^{n}\cos(\pi x_{i_1})\dots\cos(\pi x_{i_j}).$$

Thus, the partial derivatives of X_j with respect to x_k are given by

$$j=1:\quad \frac{\partial X_1}{\partial x_k}=-\pi(n-1)!1!\sin(\pi x_k),$$

$$
\begin{aligned}
j\ge 2:\quad \frac{\partial X_j}{\partial x_k}&=-\pi(n-j)!j!\sin(\pi x_k)\Biggl(\sum_{\left\{\substack{i_1,\dots,i_{j-1}=1\\ i_1<\dots<i_{j-1}}\right\}}^{n}\cos(\pi x_{i_1})\dots\cos(\pi x_{i_{j-1}})\\
&\quad-\cos(\pi x_k)\Biggl(\sum_{\left\{\substack{i_1,\dots,i_{j-2}=1\\ i_1<\dots<i_{j-2}}\right\}}^{n}\cos(\pi x_{i_1})\dots\cos(\pi x_{i_{j-2}})\\
&\quad-\cos(\pi x_k)\left(\cdots-\cos(\pi x_k)\left(\sum_{i_1=1}^{n}\cos(\pi x_{i_1})-\cos(\pi x_k)\right)\dots\right)\Biggr)\Biggr).
\end{aligned}
$$

Therefore, using the properties of determinants, we rewrite the Jacobian as

$$J=\det\begin{pmatrix}
-\pi(n-1)!1!\sin(\pi x_1) & \pi(n-2)!2!\sin(\pi x_1)\cos(\pi x_1) & \dots & (-1)^n\pi 0!n!\sin(\pi x_1)\cos^{n-1}(\pi x_1)\\
-\pi(n-1)!1!\sin(\pi x_2) & \pi(n-2)!2!\sin(\pi x_2)\cos(\pi x_2) & \dots & (-1)^n\pi 0!n!\sin(\pi x_2)\cos^{n-1}(\pi x_2)\\
\vdots & \vdots & & \vdots\\
-\pi(n-1)!1!\sin(\pi x_n) & \pi(n-2)!2!\sin(\pi x_n)\cos(\pi x_n) & \dots & (-1)^n\pi 0!n!\sin(\pi x_n)\cos^{n-1}(\pi x_n)
\end{pmatrix}.$$

For any $m\ge 1$, $\sin(\pi x_i)\cos^m(\pi x_i)$ is equal either to

$$\frac{1}{2^m}\sum_{j=0}^{\frac{m-1}{2}}\binom{m}{j}(\sin((m+1-2j)\pi x_i)+\sin((m-1-2j)\pi x_i)),$$

if m is odd, or to

$$\frac{1}{2^m}\binom{m}{\frac{m}{2}}\sin(\pi x_i)+\frac{1}{2^m}\sum_{j=0}^{\frac{m}{2}-1}\binom{m}{j}(\sin((m+1-2j)\pi x_i)+\sin((m-1-2j)\pi x_i)),$$

if m is even. Thus, we finally obtain that

$$J=(-1)^{\frac{n(n+1)}{2}}\left(\frac{1}{2}\right)^{\frac{n(n-1)}{2}}\pi^n\left(\prod_{i=1}^{n}(n-i)!i!\right)\det\begin{pmatrix}\sin(\pi x_1) & \sin(2\pi x_1) & \dots & \sin(n\pi x_1)\\ \sin(\pi x_2) & \sin(2\pi x_2) & \dots & \sin(n\pi x_2)\\ \vdots & \vdots & & \vdots\\ \sin(\pi x_n) & \sin(2\pi x_n) & \dots & \sin(n\pi x_n)\end{pmatrix}.$$

Comparing the final form of the Jacobian with the preceding one, we observe that

$$\sin^-_{(1,2,\dots,n)}(x_1,\dots,x_n)=2^{\frac{n(n-1)}{2}}\det\begin{pmatrix}\sin(\pi x_1) & \sin(\pi x_1)\cos(\pi x_1) & \dots & \sin(\pi x_1)\cos^{n-1}(\pi x_1)\\ \sin(\pi x_2) & \sin(\pi x_2)\cos(\pi x_2) & \dots & \sin(\pi x_2)\cos^{n-1}(\pi x_2)\\ \vdots & \vdots & & \vdots\\ \sin(\pi x_n) & \sin(\pi x_n)\cos(\pi x_n) & \dots & \sin(\pi x_n)\cos^{n-1}(\pi x_n)\end{pmatrix}.$$

Therefore, the equality (5.3.11) follows directly from the proof of Proposition 5.1.1. □

Note that due to the equality (5.3.11) the Jacobian J does not vanish for any point in the interior of $F(\widetilde{S}_n^{\mathrm{aff}})$. One verifies that $\sin^-_{(1,2,\dots,n)}\cdot\sin^-_{(1,2,\dots,n)}$ decomposes into a sum of symmetric cosine functions labelled by integer parameters

$$\begin{aligned}&\sin^-_{(1,2,\dots,n)}\cdot\sin^-_{(1,2,\dots,n)}\\&\qquad=\frac{1}{2^n}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\sum_{\left\{\substack{a_i=\pm1\\ i=1,\dots,n}\right\}}(-1)^{\alpha(a_1,\dots,a_n)}\cos^+_{(1+a_1\sigma(1),\dots,n+a_n\sigma(n))},\end{aligned}$$

where $\alpha(a_1,\dots,a_n)$ is the number of positive numbers among $a_1,\dots,a_n$. Therefore due to Proposition 5.3.1 it can be expressed as a polynomial $p^{I,+}$ in $X_1,\dots,X_n$ of degree $2n$. Thus, we use this polynomial $p^{I,+}$ to rewrite the absolute value of the Jacobian J as a function $\mathcal{J}$ of $X_1,\dots,X_n$,

$$|J(x)|=\mathcal{J}(X_1(x),\dots,X_n(x)) \tag{5.3.12}$$

Similarly, it follows from (5.3.4) that the products $\cos^-_{\varrho_1}\cdot\cos^-_{\varrho_1}$, $\cos^+_{\varrho}\cdot\cos^+_{\varrho}$ and $\cos^-_{\varrho_2}\cdot\cos^-_{\varrho_2}$ can be expressed as polynomials $\mathcal{J}^{I,-}$, $\mathcal{J}^{III,+}$ and $\mathcal{J}^{III,-}$ in $X_1,\dots,X_n$ such that

$$\mathcal{J}^{I,-}(X_1(x),\dots,X_n(x))=\cos^-_{\varrho_1}(x)\cos^-_{\varrho_1}(x) \tag{5.3.13}$$

$$\mathcal{J}^{III,+}(X_1(x),\dots,X_n(x))=\cos^+_{\varrho}(x)\cos^+_{\varrho}(x) \tag{5.3.14}$$

$$\mathcal{J}^{III,-}(X_1(x),\dots,X_n(x))=\cos^-_{\varrho_2}(x)\cos^-_{\varrho_2}(x). \tag{5.3.15}$$

Example 5.3.2. *For $n=3$ is the explicit form of the function $\mathcal{J}$ determined by*

$$\pi^3\sqrt{\left(-8X_2^3+X_1^2X_2^2-12X_3^2+12X_1X_2X_3-\frac{4}{3}X_1^3X_3\right)((3X_2+6)^2-(X_3+3X_1)^2)}$$

and the polynomials $\mathcal{J}^{I,-}$, $\mathcal{J}^{III,+}$ and $\mathcal{J}^{III,-}$ are given by

$$\mathcal{J}^{I,-}=\frac{1}{4}\left(-8X_2^3+X_1^2X_2^2-12X_3^2+12X_1X_2X_3-\frac{4}{3}X_1^3X_3\right),$$

$$\mathcal{J}^{III,+} = \frac{3}{4}(X_3 + 3X_2 + 3X_1 + 6),$$
$$\mathcal{J}^{III,-} = \frac{1}{8}(X_3 + 3X_2 + 3X_1 + 6)\left(-8X_2^3 + X_1^2X_2^2 - 12X_3^2 + 12X_1X_2X_3 - \frac{4}{3}X_1^3X_3\right).$$

The domain on which the polynomials are continuously orthogonal $\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})$ is the transformed domain $F(\widetilde{S}_n^{\mathrm{aff}})$, i.e.

$$\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}}) = \left\{(X_1(x), \dots, X_n(x)) \mid x \in F(\widetilde{S}_n^{\mathrm{aff}})\right\}.$$

In order to use the integration by substitution, we show that the transform

$$\varphi : x \in F(\widetilde{S}_n^{\mathrm{aff}}) \to (X_1(x), \dots, X_n(x)) \in \mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}}) \tag{5.3.16}$$

is in fact one-to-one correspondence. Any continuous function f on $F(\widetilde{S}_n^{\mathrm{aff}})$ can be expanded in terms of symmetric cosine functions [**25**] and thus also in terms of polynomials $\mathcal{P}_k^{I,+}$ as

$$f(x) = \sum_k B_k \mathcal{P}_k^{I,+}(X_1(x), \dots, X_n(x)).$$

Assume that there are $y, \widetilde{y} \in F(\widetilde{S}_n^{\mathrm{aff}})$ such that $y \neq \widetilde{y}$ and $(X_1(y), \dots, X_n(y)) = (X_1(\widetilde{y}), \dots, X_n(\widetilde{y}))$. Let us define a continuous function $f(x) = x_i$ where the index i is chosen in such a way that $y_i \neq \widetilde{y}_i$. Using the expansion in polynomials, we obtain that $f(y) = f(\widetilde{y})$ which is in contradiction with $y_i \neq \widetilde{y}_i$. Therefore, we conclude that the transform φ is injective.

Due to the orthogonality relations (5.1.16)–(5.1.19), Proposition 5.3.1 and integration by substitution (5.3.16), we deduce the continuous orthogonality relations for polynomials $\mathcal{P}_k^{I,\pm}, \mathcal{P}_k^{III,\pm}$ in the following statement.

Proposition 5.3.3. *Let $X = (X_1, \dots, X_n)$ and $dX = dX_1 \dots dX_n$. Each family of polynomials $\mathcal{P}_k^{I,\pm}$, $\mathcal{P}_k^{III,\pm}$ forms an orthogonal basis of the vector space of all polynomials in $\mathbb{R}[X]$ with the scalar product defined by weighted integral of two polynomials f, g,*

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(X)g(X)w(X)\, dX\,,$$

where $w(X)$ denotes the weight function for each family of polynomials,

$$w(X) = \begin{cases} w^{I,+}(X) = \frac{1}{\mathcal{J}(X)} & \text{for } \mathcal{P}_k^{I,+}\,, \\ w^{I,-}(X) = \frac{\mathcal{J}^{I,-}(X)}{\mathcal{J}(X)} & \text{for } \mathcal{P}_k^{I,-}\,, \\ w^{III,+}(X) = \frac{\mathcal{J}^{III,+}(X)}{\mathcal{J}(X)} & \text{for } \mathcal{P}_k^{III,+}\,, \\ w^{III,-}(X) = \frac{\mathcal{J}^{III,-}(X)}{\mathcal{J}(X)} & \text{for } \mathcal{P}_k^{III,-}\,. \end{cases}$$

The continuous orthogonality relations are of the form

$$\begin{aligned}
\int\limits_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{I,+}(X)\mathcal{P}_{k'}^{I,+}(X)w^{I,+}(X)\,dX &= h_k H_k \delta_{kk'}\,, \\
\int\limits_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{I,-}(X)\mathcal{P}_{k'}^{I,-}(X)w^{I,-}(X)\,dX &= h_k \delta_{kk'}\,, \\
\int\limits_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{III,+}(X)\mathcal{P}_{k'}^{III,+}(X)w^{III,+}(X)\,dX &= 2^{-n} H_k \delta_{kk'}\,, \\
\int\limits_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{III,-}(X)\mathcal{P}_{k'}^{III,-}(X)w^{III,-}(X)\,dX &= 2^{-n} \delta_{kk'}\,.
\end{aligned} \tag{5.3.17}$$

5.4. Cubature formulas

5.4.1. Sets of nodes

In Sections 5.2.2 and 5.2.4, the discrete orthogonality relations of the four symmetric cosine functions and the four antisymmetric cosine transforms over finite sets $F_N^{\mathrm{V},\pm},\dots,F_N^{\mathrm{VIII},\pm}$ are detailed. The orthogonality relations of the types I–IV, which are described in [**25**], are defined over the sets $F_N^{\mathrm{I},\pm},\dots,F_N^{\mathrm{IV},\pm}$ given by

$$F_N^{\mathrm{I},\pm} = \left\{ \left(\frac{r_1}{N},\dots,\frac{r_n}{N}\right) \mid (r_1,\dots,r_n)\in D_{N+1}^{\pm} \right\}\,, \tag{5.4.1}$$

$$F_N^{\mathrm{II},\pm} = \left\{ \left(\frac{r_1+\frac{1}{2}}{N},\dots,\frac{r_n+\frac{1}{2}}{N}\right) \mid (r_1,\dots,r_n)\in D_{N}^{\pm} \right\}\,, \tag{5.4.2}$$

$$F_N^{\mathrm{III},\pm} = \left\{ \left(\frac{r_1}{N},\dots,\frac{r_n}{N}\right) \mid (r_1,\dots,r_n)\in D_{N}^{\pm} \right\}\,, \tag{5.4.3}$$

$$F_N^{\mathrm{IV},\pm} = F_N^{\mathrm{II},\pm}\,. \tag{5.4.4}$$

The families of polynomials $\mathcal{P}_k^{I,\pm}$ and $\mathcal{P}_k^{III,\pm}$ inherit these orthogonality relations which are crucial for the validity of cubature formulas. The sets of nodes arise from the finite sets $F_N^{\mathrm{I},\pm},\dots,F_N^{\mathrm{VIII},\pm}$ as images of the transform φ, given by (5.3.16). Denoting these images of the sets $F_N^{\mathrm{I},\pm},\dots,F_N^{\mathrm{VIII},\pm}$ as $\mathfrak{F}_N^{\mathrm{I},\pm},\dots,\mathfrak{F}_N^{\mathrm{VIII},\pm}$, i.e.

$$\mathfrak{F}_N^{t,\pm} \equiv \left\{ \varphi(s) \mid s\in F_N^{t,\pm} \right\}\,, \quad t\in\{\mathrm{I},\dots,\mathrm{VIII}\}\,, \tag{5.4.5}$$

we prove that the restriction of φ on any set $F_N^{t,\pm}$ is injective in the following Proposition.

Proposition 5.4.1.

$$\left|\mathfrak{F}_N^{t,\pm}\right| = \left|F_N^{t,\pm}\right|\,, \quad t\in\{\mathrm{I},\dots,\mathrm{VIII}\}\,.$$

Proof. Suppose that there exist $s, \widetilde{s} \in F_N^{\mathrm{I},+}$ such that $s \neq \widetilde{s}$ and $\varphi(s) = \varphi(\widetilde{s})$, i.e.,

$$(X_1(s), \dots, X_n(s)) = (X_1(\widetilde{s}), \dots, X_n(\widetilde{s})).$$

According to the expansion resulting from SMDCT I in [**25**], any function f given on $F_N^{\mathrm{I},+}$ is written as a linear combination of finite number of functions $\cos_k^+$ and therefore as a polynomial in $X_1, \dots, X_n$. Using this expansion in polynomials, we have for the function $f(s) = s_i$ (with i chosen in such a way that $s_i \neq \widetilde{s}_i$) that $f(s) = f(\widetilde{s})$ — which contradicts $s \neq \widetilde{s}$. Thus, we obtain $\left|\mathfrak{F}_N^{\mathrm{I},+}\right| = \left|F_N^{\mathrm{I},+}\right|$. Using the corresponding discrete orthogonality relations, the proof for other grids is similar. □

Due to the one-to-one correspondence given by φ restricted to the grids $F_N^{t,\pm}$, the symbols $\mathcal{H}_Y, \mathcal{E}_Y$ and $\widetilde{\mathcal{E}}_Y$ are for $\varphi(s) = Y$ well-defined by the relations

$$\mathcal{H}_Y \equiv H_s\,, \quad \mathcal{E}_Y \equiv \varepsilon_s\,, \quad \widetilde{\mathcal{E}}_Y \equiv \widetilde{\varepsilon}_s. \tag{5.4.6}$$

5.4.2. Gaussian cubature formulas

Each family of polynomials $\mathcal{P}_k^{I,\pm}, \mathcal{P}_k^{III,\pm}$ has properties which yield to optimal Gaussian cubature formulas.

Theorem 5.4.1.

(1) For any $N \in \mathbb{N}$ and any polynomial f of degree at most $2N-1$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,+}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{II},+}} \mathcal{H}_Y^{-1} f(Y)\,, \tag{5.4.7}$$

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,+}(Y)\,dY = \left(\frac{2}{2N+1}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{VIII},+}} \mathcal{H}_Y^{-1} f(Y)\mathcal{J}^{III,+}(Y)\,. \tag{5.4.8}$$

(2) For any $N \in \mathbb{N}$ and any polynomial f of degree at most $2(N-n)+1$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,-}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{II},-}} f(Y)\mathcal{J}^{I,-}(Y)\,, \tag{5.4.9}$$

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,-}(Y)\,dY = \left(\frac{2}{2N+1}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{VIII},-}} f(Y)\mathcal{J}^{III,-}(Y)\,. \tag{5.4.10}$$

Proof. We show that the formula

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,+}(Y)\,dY = \left(\frac{2}{2N+1}\right)^n \sum_{Y \in \mathfrak{F}_N^{\mathrm{VIII},+}} \mathcal{H}_Y^{-1} f(Y)\mathcal{J}^{III,+}(Y)$$

holds. Due to the linearity of the integrals and the sums, it is sufficient to prove it only for monomials f of degree not exceeding $2N-1$. Such monomials are expressed as a product of two monomials — g_1 of degree at most N and g_2 of degree at most $N-1$. From Proposition 5.3.3 and Proposition 5.3.1, the polynomials g_1 and g_2 are rewritten as linear combinations of polynomials $\mathcal{P}_k^{III,+}$ with $k_1 \leq N$ and $\mathcal{P}_{k'}^{III,+}$ with $k'_1 \leq N-1$. Therefore, we only need to show the formula for those polynomials. Using the continuous orthogonality (5.3.17), we obtain

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} \mathcal{P}_k^{III,+}(Y)\mathcal{P}_{k'}^{III,+}(Y)\omega^{III,+}(Y)\,dY = \frac{1}{2^n} H_k \delta_{kk'}\,.$$

Similarly, from equality (5.2.41) follows that

$$\left(\frac{2}{2N+1}\right)^n \sum_{s\in F_N^{\mathrm{VIII},+}} H_s^{-1}\mathcal{P}_k^{III,+}(X(s))\mathcal{P}_{k'}^{III,+}(X(s))(\cos_\varrho^+(s))^2 = \frac{1}{2^n} H_k \delta_{kk'}\,,$$

if $k_1 \leq N-1$ and $k'_1 \leq N-1$. Note that if $k_1 = N$, then $\mathcal{P}_k^{III,+}(Y) = 0$ for all points Y from $\mathfrak{F}_N^{\mathrm{VIII},+}$. This implies that we extend the discrete orthogonality relation for any k with $k_1 \leq N$ and any k' with $k'_1 \leq N-1$. Consequently, the second cubature formula follows from the continuous and discrete orthogonality of polynomials $\mathcal{P}_k^{III,+}$. We prove the other formulas similarly. Note that SMDCT II and AMDCT II needed to derive two of the results are found in [**25**]. □

Theorem 5.4.2. *The cubature formulas* (5.4.7) – (5.4.10) *are optimal Gaussian cubature formulas. Moreover, it holds that*

- *the orthogonal polynomials $\mathcal{P}_k^{I,+}$ with $k_1 = N$ vanish for all points of the set $\mathfrak{F}_N^{\mathrm{II},+}$,*
- *the orthogonal polynomials $\mathcal{P}_k^{III,+}$ with $k_1 = N$ vanish for all points of the set $\mathfrak{F}_N^{\mathrm{VIII},+}$,*
- *the orthogonal polynomials $\mathcal{P}_k^{I,-}$ with $k_1 = N-n+1$ vanish for all points of the set $\mathfrak{F}_N^{\mathrm{II},-}$,*
- *the orthogonal polynomials $\mathcal{P}_k^{III,-}$ with $k_1 = N-n+1$ vanish for all points of the set $\mathfrak{F}_N^{\mathrm{VIII},-}$.*

Proof. The fact that the nodes are common zeros of the specific sets of orthogonal polynomials follows directly by substituting the grid points to the definition of the polynomials via symmetric and antisymmetric cosine functions (5.3.3). By Propositions 5.3.1 and 5.4.1 we obtain that the number of polynomials $\mathcal{P}_k^{I,+}$ of degree $N-1$ is equal to the number of nodes in $\mathfrak{F}_N^{\mathrm{II},+}$ and therefore, the cubature formula (5.4.7) is Gaussian. The proof is similar for the other formulas. □

5.4.3. Other cubature formulas

Similarly as in Theorem 5.4.1, one uses the remaining twelve discrete transforms to derive additional cubature formulas for the orthogonal polynomials. Note that these formulas are slightly less efficient than Gaussian cubature formulas.

5.4.3.1. *Formulas related to $\mathcal{P}_k^{I,+}$*

The transforms SMDCT I, V and VI are used to derive additional cubature formulas for the orthogonal polynomials $\mathcal{P}_k^{I,+}$,

(1) For any $N \in \mathbb{N}$ and any polynomial f of degree at most $2N-1$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,+}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{I},+}} \mathcal{E}_Y \mathcal{H}_Y^{-1} f(Y)\,.$$

(2) For any $N \in \mathbb{N}$, $N \geq 2$, and any polynomial f of degree at most $2(N-1)$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,+}(Y)\,dY = \left(\frac{2}{2N-1}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{V},+}} \mathcal{E}_Y \mathcal{H}_Y^{-1} f(Y)\,,$$

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,+}(Y)\,dY = \left(\frac{2}{2N-1}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{VI},+}} \widetilde{\mathcal{E}}_Y \mathcal{H}_Y^{-1} f(Y)\,.$$

5.4.3.2. *Formulas related to $\mathcal{P}_k^{I,-}$*

The transforms AMDCT I, V and VI are used to derive additional cubature formulas for the orthogonal polynomials $\mathcal{P}_k^{I,-}$,

(1) For any $N \in \mathbb{N}$, $N \geq n$, and any polynomial f of degree at most $2(N-n)+1$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,-}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{I},-}} \mathcal{E}_Y f(Y) \mathcal{J}^{I,-}(Y)\,.$$

(2) For any $N \in \mathbb{N}$, $N > n$, and any polynomial f of degree at most $2(N-n)$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,-}(Y)\,dY = \left(\frac{2}{2N-1}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{V},-}} \mathcal{E}_Y f(Y) \mathcal{J}^{I,-}(Y)\,,$$

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{I,-}(Y)\,dY = \left(\frac{2}{2N-1}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{VI},-}} \widetilde{\mathcal{E}}_Y f(Y) \mathcal{J}^{I,-}(Y)\,.$$

5.4.3.3. *Formulas related to* $\mathcal{P}_k^{III,+}$

The transforms SMDCT III, IV and VII are used to derive additional cubature formulas for the polynomials $\mathcal{P}_k^{III,+}$,

(1) For any $N \in \mathbb{N}$, $N \geq 2$, and any polynomial f of degree at most $2(N-1)$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,+}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{III},+}} \mathcal{E}_Y \mathcal{H}_Y^{-1} f(Y)\mathcal{J}^{III,+}(Y)\,,$$

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,+}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{IV},+}} \mathcal{H}_Y^{-1} f(Y)\mathcal{J}^{III,+}(Y)\,.$$

(2) For any $N \in \mathbb{N}$, $N \geq 2$, and any polynomial f of degree at most $2(N-1)-1$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,+}(Y)\,dY = \left(\frac{2}{2N-1}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{VII},+}} \mathcal{E}_Y \mathcal{H}_Y^{-1} f(Y)\mathcal{J}^{III,+}(Y)\,.$$

5.4.3.4. *Formulas related to* $\mathcal{P}_k^{III,-}$

The transforms AMDCT III, IV and VII are used to derive additional cubature formulas for the polynomials $\mathcal{P}_k^{III,-}$,

(1) For any $N \in \mathbb{N}$, $N > n$, and any polynomial f of degree at most $2(N-n)$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,-}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{III},-}} \mathcal{E}_Y f(Y)\mathcal{J}^{III,-}(Y)\,,$$

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,-}(Y)\,dY = \left(\frac{1}{N}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{IV},-}} f(Y)\mathcal{J}^{III,-}(Y)\,.$$

(2) For any $N \in \mathbb{N}$, $N > n$, and any polynomial f of degree at most $2(N-n)-1$, are the following cubature formulas exact

$$\int_{\mathfrak{F}(\widetilde{S}_n^{\mathrm{aff}})} f(Y)\omega^{III,-}(Y)\,dY = \left(\frac{2}{2N-1}\right)^n \sum_{Y\in\mathfrak{F}_N^{\mathrm{VII},-}} \mathcal{E}_Y f(Y)\mathcal{J}^{III,-}(Y)\,.$$

5.5. Concluding remarks

- In this paper, only the antisymmetric and symmetric generalizations of cosine functions are investigated. Similarly to the cosine functions, generalizations of the common sine functions of one variable are defined in [**25**] and these generalizations have several remarkable discretization properties. The discrete antisymmetric and symmetric sine transforms of types

I–IV are developed in [**13**] for the two-dimensional case only and the discrete multivariate sine transforms of type I are found in [**25**]. The generalization of the antisymmetric and symmetric discrete sine transforms, analogous to the discrete multivariate sine transforms of types II–VIII [**3**], have not yet been described.

- The sixteen discrete cosine transforms, which are described on the grids $F_N^{\mathrm{I},\pm}, \dots, F_N^{\mathrm{VIII},\pm}$, are straightforwardly translated into sixteen transforms of the corresponding polynomials on the grids $\mathfrak{F}_N^{\mathrm{I},\pm}, \dots, \mathfrak{F}_N^{\mathrm{VIII},\pm}$. Polynomial interpolation formulas, similar to formulas (5.2.42), can obviously be formulated. The interpolation properties of these polynomial formulas as well as the convergence of the corresponding polynomials series poses an open problem.
- The question whether one can introduce multivariate Chebyshev-like polynomials of the second and fourth kind in connection with the generalizations of the sine functions forms another open problem. The decomposition of products of two-variable antisymmetric and symmetric sine functions from [**13**] indicates the possibility of construction of such multivariate polynomials. Moreover, the continuous and discrete orthogonality of the antisymmetric and symmetric sine functions further indicates that the corresponding cubature formulas, useful in numerical analysis, can again be developed.
- Since the Weyl groups corresponding to the simple Lie algebras B_n and C_n are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ [**20**], it is possible to show that the antisymmetric and symmetric generalizations of trigonometric functions are related to the orbit functions, so called C-, S-, S^s- and S^l-functions, studied in [**24, 26, 40**]. For example, the symmetric cosine functions coincide, up to a multiplication by constant, with C-functions and the antisymmetric cosine functions become, up to a multiplication by constant, S^l-functions in the case B_n and S^s-functions in the case C_n. The grids considered in this paper, which allow the eight types of the transforms for each case, are, however, different than the grids on which is the discrete calculus of the orbit functions described. Thus, the collection of the resulting cubature formulas is richer and includes formulas of the Gaussian type.
- The exploration of the vast number of theoretical aspects as well as applications of the Chebyshev polynomials [**39, 52**] is beyond the scope of this work. However, since the basic properties of these polynomials — such as

the discrete and continuous orthogonality together with the cubature formulas — are replicated for the multivariate symmetric and antisymmetric cosine functions, one may expect that many other properties will find their corresponding multidimensional (anti)symmetric generalizations as well. The work presented in this paper may represent a starting point for these open problems and further research.

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Appendix A

TWO DIMENSIONAL SYMMETRIC AND ANTISYMMETRIC GENERALIZATIONS OF SINE FUNCTIONS

Authors: Jiří Hrivnák, Lenka Motlochová and Jiří Patera.

Abstract: Properties of 2-dimensional generalizations of sine functions that are symmetric or antisymmetric with respect to permutation of their two variables are described. It is shown that the functions are orthogonal when integrated over a finite region F of the real Euclidean space, and that they are discretely orthogonal when summed up over a lattice of any density in F. Decomposability of the products of functions into their sums is shown by explicitly decomposing products of all types. The formalism is set up for Fourier-like expansions of digital data over 2-dimensional lattices in F. Continuous interpolation of digital data is studied.

Introduction

The purpose of this paper is to complete and extend [**18**] by considering the remaining two families of special functions and their properties, namely the generalizations of sine functions of two variables, which are either symmetric or antisymmetric with respect to permutations of their two variables. They are denoted here by $\sin^{\pm}_{(\lambda,\mu)}(x,y)$, where $x, y \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{N}$. The functions are of independent interest. The paper [**18**] is devoted to the study of 2-dimensional symmetric and antisymmetric generalizations of the common exponential and cosine functions, namely $E^{\pm}_{(\lambda,\mu)}(x,y)$ and $\cos^{\pm}_{(\lambda,\mu)}(x,y)$.

In [**25**], the functions denoted $SIN^{\pm}$ and $COS^{\pm}$ were introduced for any number of real variables, and their properties were studied. Thus, the functions $\cos^{\pm}_{(\lambda,\mu)}(x,y)$ of [**18**], as well as the functions $\sin^{\pm}_{(\lambda,\mu)}(x,y)$ considered here, are the functions of [**25**] specialized to $2D$.

Standard trigonometric Fourier decompositions of functions (continuous or discrete) of two variables [**59**] use special functions formed as products of two trigonometric functions, each depending on one variable. The variables are measured along two orthogonal axes. The approach undertaken in [**18**], and extended here, appears to be the only $2D$ "trigonometric" alternative to the standard approach in the literature. Our expansion functions are also built as products of two trigonometric functions, but the two variables are intertwined within each trigonometric function, so that no substitution of variables can bring it to the form used in the standard approach.

Restriction of the functions of [**25**] to two variables allows us to be more specific about the details of their properties, most notable being their discretization and orthogonality, continuous and discrete. In particular, analogs of the four types of standard cosine transforms [**3, 38, 56**], developed in [**18**] and in here, would not be possible without specific description of discrete domains of orthogonality of the expansion functions applicable in the four cases.

Decomposition of all products of pairs of functions into their sums is described here. It was not considered elsewhere. There are three types of products of sine functions, $\sin^+\sin^+$, $\sin^+\sin^-$, and $\sin^-\sin^-$, with the same arguments x, y. For completeness, we also decompose all products $\sin^\pm\cos^\pm$, and $\cos^\pm\cos^\pm$. The general structure of decompositions of products is rather interesting. It is summarized symbolically in Table A.1.

product	terms	product	terms
$\sin^+\sin^+$	$\cos^+$	$\sin^-\cos^+$	$\sin^-$
$\sin^+\sin^-$	$\cos^-$	$\sin^-\cos^-$	$\sin^+$
$\sin^-\sin^-$	$\cos^+$	$\cos^+\cos^+$	$\cos^+$
$\sin^+\cos^+$	$\sin^+$	$\cos^+\cos^-$	$\cos^-$
$\sin^+\cos^-$	$\sin^-$	$\cos^-\cos^-$	$\cos^+$

TABLE A.1. Structure of the ten types of products decomposed in the paper. The second row shows the functions appearing in all the terms of a decomposition.

There are numerous reasons motivating the study of $\sin^\pm$ and $\cos^\pm$ functions in more than one variable. One such reason is the ever presence of trigonometric functions in applications of mathematics from the elementary to the most sophisticated. Our immediate motivation is to be able to use the functions in Fourier-like analysis and interpolation [**4**] of digital data in $2D$.

We describe four versions of the sine transforms of functions given on lattices. They differ by certain small shifts of their arguments which allow orthogonality of the expansion functions to be maintained. These transforms correspond to the well known cosine transforms, labelled as I, II, III, and IV, where the expansion functions have separated variables [**56**], namely $\cos(\pi m x)\cdot\cos(\pi n y)$. The four variants of the antisymmetric sine transforms and the four variants of the symmetric sine transforms correspond to different boundary conditions at the boundaries of the fundamental domain. Similarly, there are different types of ordinary cosine and sine transforms in $1D$ [**3**].

In Section 2 we consider the sine functions, followed by some general properties of sine and cosine functions in Section 3. Discretization of the antisymmetric and symmetric sine transforms are the subject of Section 4 together with their interpolations of types I, II, III, and IV. Three remarks are found in the last Section.

A.1. Continuous sine transforms in $\mathbb{R}^2$

First we consider the antisymmetric, then symmetric $2D$ sine transforms. The discretization of transforms is described in Section 4. Here the definitions of symmetric and antisymmetric functions $\cos^{\pm}(x,y)$ are recalled because they appear in decompositions of products of sine functions into their sums.

A.1.1. Antisymmetric sine functions

A.1.1.1. *Definitions, symmetries and general properties*

The antisymmetric sine functions $\sin^{-}_{(\lambda,\mu)}(x,y) : \mathbb{R}^2 \to \mathbb{R}$ are defined as follows:

$$\sin^{-}_{(\lambda,\mu)}(x,y) = \begin{vmatrix} \sin(\pi\lambda x) & \sin(\pi\lambda y) \\ \sin(\pi\mu x) & \sin(\pi\mu y) \end{vmatrix}^{-} = \sin(\pi\lambda x)\sin(\pi\mu y) - \sin(\pi\mu x)\sin(\pi\lambda y) \tag{A.1.1}$$

with $\lambda,\mu,x,y \in \mathbb{R}$. Clearly the functions are continuous and have continuous derivatives of all degrees in $\mathbb{R}^2$. A few examples of functions are shown in Figure A.1. Note that instead of the factor 2π, which was used in [**25**], we use the "half" argument π.

The following properties of the functions are verified directly from the definition (A.1.1):

$$\sin^{-}_{(\lambda,\lambda)}(x,y) = 0, \tag{A.1.2}$$

$$\sin^{-}_{(\lambda,0)}(x,y) = \sin^{-}_{(0,\mu)}(x,y) = 0, \tag{A.1.3}$$

$$\sin^{-}_{(\lambda,\mu)}(x,y) = -\sin^{-}_{(\mu,\lambda)}(x,y), \tag{A.1.4}$$

$$\sin^{-}_{(-\lambda,\mu)}(x,y) = \sin^{-}_{(\lambda,-\mu)}(x,y) = -\sin^{-}_{(-\lambda,-\mu)}(x,y) = -\sin^{-}_{(\lambda,\mu)}(x,y), \tag{A.1.5}$$

$$\sin^-_{(\lambda,\mu)}(x,x) = 0, \tag{A.1.6}$$

$$\sin^-_{(\lambda,\mu)}(x,0) = \sin^-_{(\lambda,\mu)}(0,y) = 0, \tag{A.1.7}$$

$$\sin^-_{(\lambda,\mu)}(x,y) = -\sin^-_{(\lambda,\mu)}(y,x), \tag{A.1.8}$$

$$\sin^-_{(\lambda,\mu)}(-x,y) = \sin^-_{(\lambda,\mu)}(x,-y) = -\sin^-_{(\lambda,\mu)}(-x,-y) = -\sin^-_{(\lambda,\mu)}(x,y). \tag{A.1.9}$$

Because of (A.1.2) – (A.1.5), we consider only $\sin^-_{(\lambda,\mu)}$ with $\lambda > \mu > 0$.

In addition, the functions $\sin^-_{(k,l)}$ with $k, l \in \mathbb{Z}$ have symmetries related to the periodicity of the sine function

$$\sin^-_{(k,l)}(x+2r, y+2s) = \sin^-_{(k,l)}(x,y), \quad r, s \in \mathbb{Z} \tag{A.1.10}$$

and

$$\sin^-_{(k,l)}(r,s) = 0, \quad r, s \in \mathbb{Z}. \tag{A.1.11}$$

The relations (A.1.8) – (A.1.10) imply that it is sufficient to consider the functions $\sin^-_{(k,l)}(x,y)$ on the closed triangle $F(\widetilde{S}_2^{\mathrm{aff}})$ given by its vertices in $\mathbb{R}^2$:

$$(x,y) \in F(\widetilde{S}_2^{\mathrm{aff}}) = \{(0,0),\ (1,0),\ (1,1)\}, \tag{A.1.12}$$

called the fundamental domain of the extended affine symmetric group [**28**]. The relations (A.1.6) and (A.1.11) imply that $\sin^-_{(k,l)}(x,y)$ vanishes on the boundary $\partial F(\widetilde{S}_2^{\mathrm{aff}})$ of the fundamental domain.

The graphs of a few lowest functions $\sin^-_{(k,l)}(x,y)$ in the fundamental domain are plotted in Figure A.1.

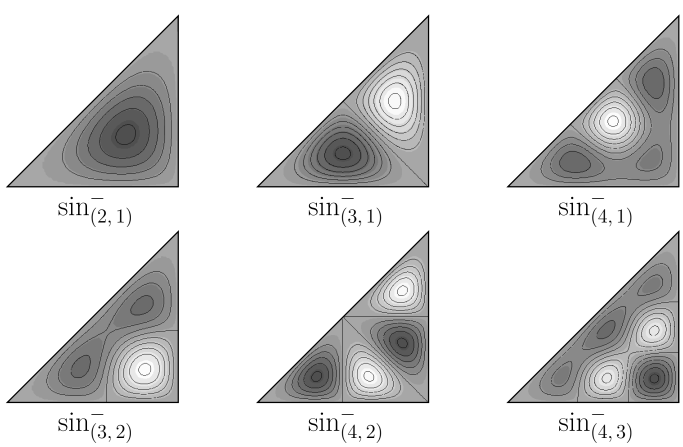

FIGURE A.1. The contour plots of examples of functions $\sin^-_{(k,l)}(x,y)$ in the fundamental domain.

A.1.1.2. *Continuous orthogonality*

The functions $\sin^-_{(k,l)}$ are pairwise orthogonal on the fundamental domain $F(\widetilde{S}_2^{\mathrm{aff}})$,

$$\int_{F(\tilde{S}_2^{\mathrm{aff}})} \sin^-_{(k,l)}(x,y)\sin^-_{(k',l')}(x,y)\,dx\,dy = \frac{1}{4}\delta_{kk'}\delta_{ll'}, \quad k,l,k',l' \in \mathbb{N},\ k>l, k'>l'. \tag{A.1.13}$$

Any function $f:\mathbb{R}^2 \to \mathbb{C}$ that is antisymmetric $f(x,y) = -f(y,x)$, periodic $f(x+2r, y+2s) = f(x,y)$, $r,s \in \mathbb{Z}$, has continuous derivatives and vanishes on $\partial F(\widetilde{S}_2^{\mathrm{aff}})$ can be expanded in the antisymmetric sine functions $\sin^-_{(k,l)}$:

$$f(x,y) = \sum_{\left\{\substack{k,l\in\mathbb{N}\\ k>l}\right\}} c_{kl}\sin^-_{(k,l)}(x,y), \quad c_{kl} = 4\int_{F(\tilde{S}_2^{\mathrm{aff}})} f(x,y)\sin^-_{(k,l)}(x,y)\,dx\,dy. \tag{A.1.14}$$

A.1.2. Symmetric sine functions

A.1.2.1. *Definitions, symmetries and general properties*

Two-dimensional symmetric sine functions $\sin^+_{(\lambda,\mu)}:\mathbb{R}^2 \to \mathbb{C}$ are defined for $\lambda,\mu \in \mathbb{R}$ in the following form:

$$\sin^+_{(\lambda,\mu)}(x,y) = \begin{vmatrix} \sin(\pi\lambda x) & \sin(\pi\lambda y) \\ \sin(\pi\mu x) & \sin(\pi\mu y)\end{vmatrix}^+ = \sin(\pi\lambda x)\sin(\pi\mu y) + \sin(\pi\mu x)\sin(\pi\lambda y). \tag{A.1.15}$$

A few examples of functions are shown in Figure A.2. Note that instead of the factor 2π used in [**25**], we use the "half" argument π.

The following properties of the functions are verified directly from the definition (A.1.15):

$$\sin^+_{(\lambda,0)}(x,y) = \sin^+_{(0,\mu)}(x,y) = 0, \tag{A.1.16}$$

$$\sin^+_{(\lambda,\mu)}(x,y) = \sin^+_{(\mu,\lambda)}(x,y), \tag{A.1.17}$$

$$\sin^+_{(-\lambda,\mu)}(y,x) = \sin^+_{(\lambda,-\mu)}(x,y) = -\sin^+_{(-\lambda,-\mu)}(x,y) = -\sin^+_{(\lambda,\mu)}(x,y), \tag{A.1.18}$$

$$\sin^+_{(\lambda,\mu)}(x,0) = \sin^+_{(\lambda,\mu)}(0,y) = 0, \tag{A.1.19}$$

$$\sin^+_{(\lambda,\mu)}(x,y) = \sin^+_{(\lambda,\mu)}(y,x), \tag{A.1.20}$$

$$\sin^+_{(\lambda,\mu)}(-x,y) = \sin^+_{(\lambda,\mu)}(x,-y) = -\sin^+_{(\lambda,\mu)}(-x,-y) = -\sin^+_{(\lambda,\mu)}(x,y). \tag{A.1.21}$$

Because of (A.1.16) – (A.1.18), we consider only such $\sin^+_{(\lambda,\mu)}$ with $\lambda \geq \mu > 0$.

The functions $\sin^+_{(k,l)}$ with $k,l \in \mathbb{Z}$ have symmetries related to the periodicity of sine function:

$$\sin^+_{(k,l)}(x+2r, y+2s) = \sin^+_{(k,l)}(x,y), \quad r,s \in \mathbb{Z}, \tag{A.1.22}$$

$$\sin^+_{(k,l)}(r,s) = 0, \quad r,s \in \mathbb{Z}. \tag{A.1.23}$$

The relations (A.1.20) – (A.1.22) imply that it is sufficient to consider the functions $\sin^+_{(k,l)}$, $k,l \in \mathbb{N}$, $k \geq l$ on the fundamental domain $F(\widetilde{S}_2^{\mathrm{aff}})$ [**28**].

The graphs of the lowest symmetric sine functions $\sin^+_{(k,l)}$, $k,l \in \{1,2,3\}$, $k \geq l$ in the fundamental domain are plotted in Figure A.2.

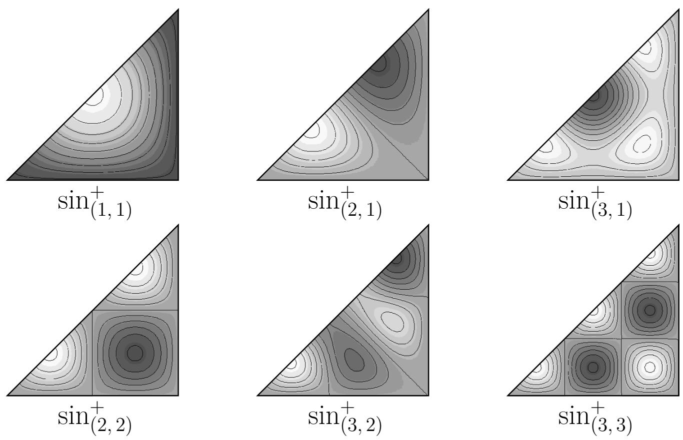

FIGURE A.2. The contour plots of examples of functions $\sin^+_{(k,l)}(x,y)$ in the fundamental domain.

A.1.2.2. *Continuous orthogonality*

The functions $\sin^+_{(k,l)}$ are mutually orthogonal on the fundamental domain $F(\widetilde{S}_2^{\mathrm{aff}})$,

$$\int_{F(\widetilde{S}_2^{\mathrm{aff}})} \sin^+_{(k,l)}(x,y)\sin^+_{(k',l')}(x,y)\,dx\,dy = \frac{G_{kl}}{4}\delta_{kk'}\delta_{ll'}, \quad k,l,k',l' \in \mathbb{N},\ k \geq l, k' \geq l', \tag{A.1.24}$$

where G_{kl} is defined by

$$G_{kl} = \begin{cases} 2 & k = l, \\ 1 & \text{otherwise.} \end{cases} \tag{A.1.25}$$

Any function $f : \mathbb{R}^2 \to \mathbb{C}$ that is symmetric $f(x,y) = f(y,x)$, periodic $f(x + 2r, y + 2s) = f(x,y)$, $r,s \in \mathbb{Z}$, has continuous derivatives and vanishes for $x = 1$ and $y = 0$ can be expanded in the symmetric sine functions $\sin^+_{(k,l)}$:

$$f(x,y) = \sum_{\left\{ \substack{k,l\in\mathbb{N} \\ k\geq l} \right\}} c_{kl}\sin^+_{(k,l)}(x,y), \quad c_{kl} = 4G_{kl}^{-1}\int_{F(S_2^{\mathrm{aff}})} f(x,y)\sin^+_{(k,l)}(x,y)\,dx\,dy. \tag{A.1.26}$$

A.2. Additional properties of $2D$ trigonometric functions

Properties of $2D$ generalizations of sine and cosine are closely interwoven. Let us first recall definitions of the functions $\cos^{\pm}_{(\lambda,\mu)}(x,y)$ according to [**18, 25**].

$$\cos^{\pm}_{(\lambda,\mu)}(x,y) = \begin{vmatrix} \cos(\pi\lambda x) & \cos(\pi\lambda y) \\ \cos(\pi\mu x) & \cos(\pi\mu y) \end{vmatrix}^{\pm} = \cos(\pi\lambda x)\cos(\pi\mu y) \pm \cos(\pi\mu x)\cos(\pi\lambda y)\,, \tag{A.2.1}$$

where the upper or lower signs should be taken simultaneously.

A.2.1. Laplace and other differential operators

An obvious relation between $2D$ sine and cosine functions arises from the second derivatives,

$$\frac{\partial^2}{\partial x \partial y} \sin^{\pm}_{(\lambda,\mu)}(x,y) = \pi^2 \lambda\mu \cos^{\pm}_{(\lambda,\mu)}(x,y)\,, \tag{A.2.2}$$

$$\frac{\partial^2}{\partial x \partial y} \cos^{\pm}_{(\lambda,\mu)}(x,y) = \pi^2 \lambda\mu \sin^{\pm}_{(\lambda,\mu)}(x,y)\,. \tag{A.2.3}$$

The functions $\sin^{\pm}_{(\lambda,\mu)}(x,y)$ are eigenfunctions of the Laplace operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \sin^{\pm}_{(\lambda,\mu)}(x,y) = -\pi^2(\lambda^2+\mu^2) \sin^{\pm}_{(\lambda,\mu)}(x,y)$$

as well as of the operator (A.2.2) applied twice:

$$\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} \sin^{\pm}_{(\lambda,\mu)}(x,y) = \pi^4\lambda^2\mu^2 \sin^{\pm}_{(\lambda,\mu)}(x,y).$$

Functions $\sin^+$ satisfy the equality

$$\frac{\partial}{\partial \mathbf{n}} \sin^{+}_{(\lambda,\mu)}(x,x) = 0,$$

where $\mathbf{n}$ is normal to the boundary $x = y$.

A.2.2. Product decompositions

Products of two $\sin^{\pm}_{(\lambda,\mu)}(x,y)$ functions decompose into the sum of cosine functions. Products of $\cos^{\pm}_{(\lambda,\mu)}(x,y)$ functions decompose into the sum of cosine functions, while mixed products, $\sin^{\pm}_{(\lambda,\mu)}(x,y)\cdot\cos^{\pm}_{(\lambda',\mu')}(x,y)$, decompose into the sum of sine functions, see Table A.1. Common trigonometrical identities,

$$\begin{aligned} 2\sin(pz)\sin(qz) &= -\cos((p+q)z) + \cos((p-q)z), \\ 2\cos(pz)\cos(qz) &= \cos((p+q)z) + \cos((p-q)z), \\ 2\sin(pz)\cos(qz) &= \sin((p+q)z) + \sin((p-q)z), \end{aligned} \tag{A.2.4}$$

		$(\lambda \pm \lambda', \mu \pm \mu')$				$(\lambda \pm \mu', \mu \pm \lambda')$			
		$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$\sin^+ \cdot \sin^+$	$\cos^+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$\sin^+ \cdot \sin^-$	$\cos^-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
$\sin^- \cdot \sin^-$	$\cos^+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
$\sin^+ \cdot \cos^+$	$\sin^+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\sin^+ \cdot \cos^-$	$\sin^-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\sin^- \cdot \cos^+$	$\sin^-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\sin^- \cdot \cos^-$	$\sin^+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\cos^+ \cdot \cos^+$	$\cos^+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\cos^+ \cdot \cos^-$	$\cos^-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\cos^- \cdot \cos^-$	$\cos^+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

TABLE A.2. Decomposition of all products of functions $\sin^{\pm}(x, y)$ and $\cos^{\pm}(x, y)$ into the sum of eight such functions. The first column shows the product. The second column contains the function appearing in the decomposition terms. Subsequent columns provide: (i) the subscripts labelling the decomposition terms (first line); (ii) the pair of signs applicable in the subscripts of each term; (iii) the signs in front of each term in the decomposition (remaining lines).

are used when decompositions of products of $\sin^{\pm}_{(\lambda,\mu)}(x, y)$ and $\cos^{\pm}_{(\lambda,\mu)}(x, y)$ are calculated.

There are altogether ten different types of products to consider. A concise presentation of the ten decompositions is found in Table A.2. The following examples are intended to illustrate how the actual decomposition is obtained from the Table A.2.

Example A.2.1. *The top left entry of the table identifies the line where the decomposition of products of the form* $\sin^+_{(\lambda,\mu)}(x, y) \cdot \sin^+_{(\lambda',\mu')}(x, y)$ *is given. For our example, we selected* $(\lambda, \mu) = (2, 1)$*,* $(\lambda', \mu') = (3, 2)$*. The second column entry on the same line indicates that all terms of the decomposition are the functions* $\cos^+_{(\alpha,\beta)}(x, y)$*. The subsequent eight entries on that line refer to individual terms of the decomposition. Each term has a coefficient* ± 1*. The actual sign of a term is shown on the line. The value of* α *and* β *is identified on the two top lines of the column. Thus for the first term of the example we find at the top line that* $\alpha = 2 \pm 3$ *and* $\beta = 1 \pm 2$*. Which of the two signs is applicable in* α *and* β *is specified at the head of the column of the first term as* $(+, +)$*, so that* $\cos^+_{(\alpha,\beta)}(x, y) = \cos^+_{(5,3)}$*. Subsequent terms are identified in a similar way. The symbol* (x, y) *is omitted to simplify the expression. From* (A.2.4) *follows the presence of the coefficient 4 multiplying each product below.*

In this way, some of the subscripts of the terms of the decomposition are negative. A convenient convention is to label the terms by non-negative subscripts. Therefore the symmetries of the functions need to be used in order to write the subscripts as positive. If a subscript of a term $\sin^{\pm}_{(\alpha,\beta)}$ *should be 0, the sine term vanishes.*

The four specific examples illustrate the decompositions:

$$\begin{aligned}4\sin^{+}_{(2,1)}\cdot\sin^{+}_{(3,2)} &= \cos^{+}_{(5,3)}-\cos^{+}_{(5,-1)}-\cos^{+}_{(-1,3)}+\cos^{+}_{(-1,-1)}+\cos^{+}_{(4,4)}-\cos^{+}_{(4,-2)}\\ &\quad-\cos^{+}_{(0,4)}+\cos^{+}_{(0,-2)}=\cos^{+}_{(5,3)}-\cos^{+}_{(5,1)}-\cos^{+}_{(3,1)}+\cos^{+}_{(1,1)}\\ &\quad+\cos^{+}_{(4,4)}-\cos^{+}_{(4,2)}-\cos^{+}_{(4,0)}+\cos^{+}_{(2,0)}\,.\end{aligned}$$

$$\begin{aligned}4\sin^{+}_{(2,1)}\cdot\sin^{-}_{(3,2)} &= \cos^{-}_{(5,3)}-\cos^{-}_{(5,-1)}-\cos^{-}_{(-1,3)}+\cos^{-}_{(-1,-1)}-\cos^{-}_{(4,4)}+\cos^{-}_{(4,-2)}\\ &\quad+\cos^{-}_{(0,4)}-\cos^{-}_{(0,-2)}=\cos^{-}_{(5,3)}-\cos^{-}_{(5,1)}+\cos^{-}_{(3,1)}+0-0\\ &\quad+\cos^{-}_{(4,2)}-\cos^{-}_{(4,0)}+\cos^{-}_{(2,0)}\,.\end{aligned}$$

$$\begin{aligned}4\sin^{+}_{(2,1)}\cdot\cos^{+}_{(3,2)} &= \sin^{+}_{(5,3)}+\sin^{+}_{(5,-1)}+\sin^{+}_{(-1,3)}+\sin^{+}_{(-1,-1)}+\sin^{+}_{(4,4)}+\sin^{+}_{(4,-2)}\\ &\quad+\sin^{+}_{(0,4)}+\sin^{+}_{(0,-2)}=\sin^{+}_{(5,3)}-\sin^{+}_{(5,1)}-\sin^{+}_{(3,1)}+\sin^{+}_{(1,1)}\\ &\quad+\sin^{+}_{(4,4)}-\sin^{+}_{(4,2)}+0+0.\end{aligned}$$

$$\begin{aligned}4\sin^{-}_{(2,1)}\cdot\cos^{+}_{(3,2)} &= \sin^{-}_{(5,3)}+\sin^{-}_{(5,-1)}+\sin^{-}_{(-1,3)}+\sin^{-}_{(-1,-1)}+\sin^{-}_{(4,4)}+\sin^{-}_{(4,-2)}\\ &\quad+\sin^{-}_{(0,4)}+\sin^{-}_{(0,-2)}=\sin^{-}_{(5,3)}-\sin^{-}_{(5,1)}+\sin^{-}_{(3,1)}+0+0\\ &\quad-\sin^{-}_{(4,2)}+0+0.\end{aligned}$$

A.3. Discrete transforms

The four versions of the discrete sine transforms introduced here use a grid of points (x_m, y_n) [**18**], extending over finite regions that differ by their boundaries. The grid is defined by the numbers N, T, and b. The number of points of the grid is N^2, constant $b \in [0,1]$ is the displacement of lattice points from their original position, and T determines argument of sine functions, so that the argument is equal to $\frac{2\pi}{T}$. To make the distinctions between the four versions of the antisymmetric and symmetric transforms easy to compare, we first describe the pertinent regions.

Define the “closed square” $K_{[a,a']}$ by

$$K_{[a,a']} \equiv [a,a'] \times [a,a'], \qquad a, a' \in \mathbb{Z}\,. \tag{A.3.1}$$

For the antisymmetric discrete sine transforms, it is necessary to define a "partly-open triangle" $K^-_{[0,1]}$:

$$K^-_{[0,1]} \equiv \Big\{(x,y) \in [0,1] \times [0,1] \mid x > y\Big\}. \tag{A.3.2}$$

Similarly, for the symmetric sine transforms, we introduce a "closed triangle" $K^+_{[0,1]}$, which contains one additional side, described by $x = y$, in comparison with the triangle for antisymmetric sine transforms:

$$K^+_{[0,1]} \equiv \{(x,y) \in [0,1] \times [0,1] \mid x \geq y\}\,. \tag{A.3.3}$$

A.3.1. Antisymmetric discrete sine transforms

The antisymmetric sine functions are closely related to the antisymmetric exponential functions [**18**]. Four types of discrete antisymmetric sine transforms can be derived from the antisymmetric exponential transforms. The idea is to suitably extend given functions and then apply the antisymmetric exponential transforms from [**18**].

In order to derive discrete antisymmetric sine transforms, we define the following three functional extension operators. They extend a complex function defined on the "partly-opened triangle" $K^-_{[0,1]}$ to functions defined on the "closed square" $K_{[-L,L]}$.

First, let $f : K_{[0,L]} \to \mathbb{C}$. We define its extension $E_L f : K_{[-L,L]} \to \mathbb{C}$ as follows:

$$E_L f(x,y) \equiv \begin{cases} f(x,y), & 0 \leq x \leq L, \quad 0 < y < L, \\ -f(-x,y), & -L \leq x < 0, \quad 0 \leq y \leq L, \\ -f(x,-y), & 0 \leq x \leq L, -L \leq y < 0, \\ f(-x,-y), & -L \leq x < 0, -L \leq y < 0. \end{cases} \tag{A.3.4}$$

Secondly, let $f : K_{[0,1]} \to \mathbb{C}$. We define its extension $Rf : \mathrm{K}_{[0,2]} \to \mathbb{C}$ to the square as follows:

$$Rf(x,y) \equiv \begin{cases} f(x,y), & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ f(2-x,y), & 1 < x \leq 2,\ 0 \leq y \leq 1, \\ f(x,2-y), & 0 \leq x \leq 1,\ 1 < y \leq 2, \\ f(2-x,2-y), & 1 < x \leq 2,\ 1 < y \leq 2. \end{cases} \tag{A.3.5}$$

For the function $f : K^-_{[0,1]} \to \mathbb{C}$, we define antisymmetric extensions $Af : K_{[0,1]} \to \mathbb{C}$ as follows:

$$Af(x,y) = \begin{cases} f(x,y), & x > y, \\ 0, & x = y, \\ -f(y,x), & x < y. \end{cases} \tag{A.3.6}$$

In the first two types of discrete antisymmetric transforms, we consider the given functions $f_1 : K^-_{[0,1]} \to \mathbb{C}$ that have zero values for the lines $x = 1$ and $y = 0$. In the last two types, we consider the given functions $f_2 : K^-_{[0,1]} \to \mathbb{C}$ that vanish on the line $y = 0$. The additional conditions on the boundary are derived from properties of sine functions on the regions in view.

The following extensions of a function f_1 or f_2 with corresponding values of N, T and b are substituted into the formula (56) from [**18**]:

$$\text{(I)} \quad E_1Af_1 \;:\; K_{[-1,1]} \to \mathbb{C}, \quad \text{where} \quad N = 2M, \quad T = 2, \quad b = 1, \tag{A.3.7}$$

$$\text{(II)} \quad E_1Af_1 \;:\; K_{[-1,1]} \to \mathbb{C}, \quad \text{where} \quad N = 2M, \quad T = 2, \quad b = \tfrac{1}{2}, \tag{A.3.8}$$

$$\text{(III)} \quad E_2RAf_2 : K_{[-2,2]} \to \mathbb{C}, \quad \text{where} \quad N = 4M, \quad T = 4, \quad b = 1, \tag{A.3.9}$$

$$\text{(IV)} \quad E_2RAf_2 : K_{[-2,2]} \to \mathbb{C}, \quad \text{where} \quad N = 4M, \quad T = 4, \quad b = \tfrac{1}{2}. \tag{A.3.10}$$

We notice that E_1Af_1 has zero values on the boundary of $K_{[-1,1]}$, on the axes x, y and on the lines $x = \pm y$, and E_2RAf_2 vanishes on the boundary of $K_{[-2,2]}$, on the axes x, y and on the lines $x = \pm y$. The different conditions of zero values are due to the fact that in the first and second type of antisymmetric sine transforms, we consider $\sin^-_{(k,l)}(x,y)$ in comparison with the third and fourth type, where we consider $\sin^-_{(k,l)}(x/2, x/2)$.

Due to (anti)symmetry and zero values of the extended functions on axes x, y, on the lines $x = \pm y$ and on borders of $K_{[-1,1]}$ or $K_{[-2,2]}$, we obtain the final explicit form of the four antisymmetric sine transforms defined as follows.

AMDST–I.

$$\psi^{\mathrm{I},-}_M(x,y) = \sum_{\left\{\substack{k,l=1 \\ k>l}\right\}}^{M-1} c^{\mathrm{I},-}_{k,l} \sin^-_{(k,l)}(x,y),$$

$$c^{\mathrm{I},-}_{k,l} = \frac{4}{M^2} \sum_{\left\{\substack{m,n=1 \\ m>n}\right\}}^{M-1} f_1\left(x_m, y_n\right) \sin^-_{(k,l)}\left(x_m, y_n\right),$$

where $\quad x_m = \frac{m}{M}, \; y_n = \frac{n}{M}.$

AMDST–II.

$$\psi_M^{\mathrm{II},-}(x,y) = \sum_{\left\{\substack{k,l=1\\ k>l}\right\}}^{M} c_{k,l}^{\mathrm{II},-} \sin^{-}_{(k,l)}(x,y),$$

$$c_{k,l}^{\mathrm{II},-} = \frac{4d_{k,M}d_{l,M}}{M^2} \sum_{\left\{\substack{m,n=0\\ m>n}\right\}}^{M-1} f_1\left(x_m,y_n\right) \sin^{-}_{(k,l)}\left(x_m,y_n\right),$$

where $\quad x_m = \frac{m+\frac{1}{2}}{M},\ y_n = \frac{n+\frac{1}{2}}{M}, \quad d_{M,M} = \frac{1}{2}$ and $d_{k,M} = 1$ for $k \neq M$.

AMDST–III.

$$\psi_M^{\mathrm{III},-}(x,y) = \sum_{\left\{\substack{k,l\ =0\\ k>l}\right\}}^{M-1} c_{k,l}^{\mathrm{III},-} \sin^{-}_{(k+\frac{1}{2},l+\frac{1}{2})}(x,y),$$

$$c_{k,l}^{\mathrm{III},-} = \frac{4}{M^2} \sum_{\left\{\substack{m,n=1\\ m>n}\right\}}^{M} d_{m,M}d_{n,M} f_2\left(x_m,y_n\right) \sin^{-}_{(k+\frac{1}{2},l+\frac{1}{2})}\left(x_m,y_n\right),$$

where $\quad x_m = \frac{m}{M},\ y_n = \frac{n}{M}.$

AMDST–IV.

$$\psi_M^{\mathrm{IV},-}(x,y) = \sum_{\left\{\substack{k,l=0\\ k>l}\right\}}^{M-1} c_{kl}^{\mathrm{IV},-} \sin^{-}_{(k+\frac{1}{2},l+\frac{1}{2})}(x,y),$$

$$c_{kl}^{\mathrm{IV},-} = \frac{4}{M^2} \sum_{\left\{\substack{m,n=0\\ m>n}\right\}}^{M-1} f_2\left(x_m,y_n\right) \sin^{-}_{(k+\frac{1}{2},l+\frac{1}{2})}\left(x_m,y_n\right),$$

where $\quad x_m = \frac{m+\frac{1}{2}}{M},\ y_n = \frac{n+\frac{1}{2}}{M}.$

A.3.1.1. *Example of antisymmetric sine interpolation*

We proceed as follows: Choose a continuous model function, sample its value on a lattice, develop the digital data into the finite series according to one of the transforms above, interpolate the digital data, and compare the resulting continuous function with the model function.

Our model function is the Gaussian distribution

$$f(x,y) = e^{-\frac{(x-x')^2+(y-y')^2}{2\sigma^2}}, \tag{A.3.11}$$

with the following parameters $(x', y') = (0.707, 0.293)$ and $\sigma = 0.079$. The function f, restricted to the domain $F(\tilde{S}_2^{\mathrm{aff}})$, is shown in Figure A.3.

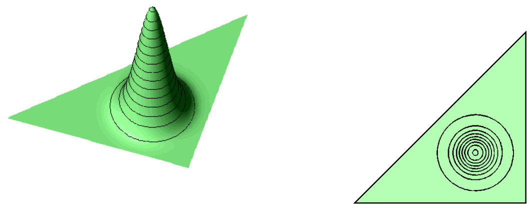

FIGURE A.3. The function f of (A.3.11) is plotted over the domain $F(\tilde{S}_2^{\mathrm{aff}})$.

We calculate the antisymmetric interpolating sine functions of type AMDST–II, namely $\psi_4^{\mathrm{II},-}$, $\psi_7^{\mathrm{II},-}$ and $\psi_{12}^{\mathrm{II},-}$, using grids of density given by $M = 4$, 7, and 12. The interpolating functions, together with the sampling grids, are presented in Figure A.4.

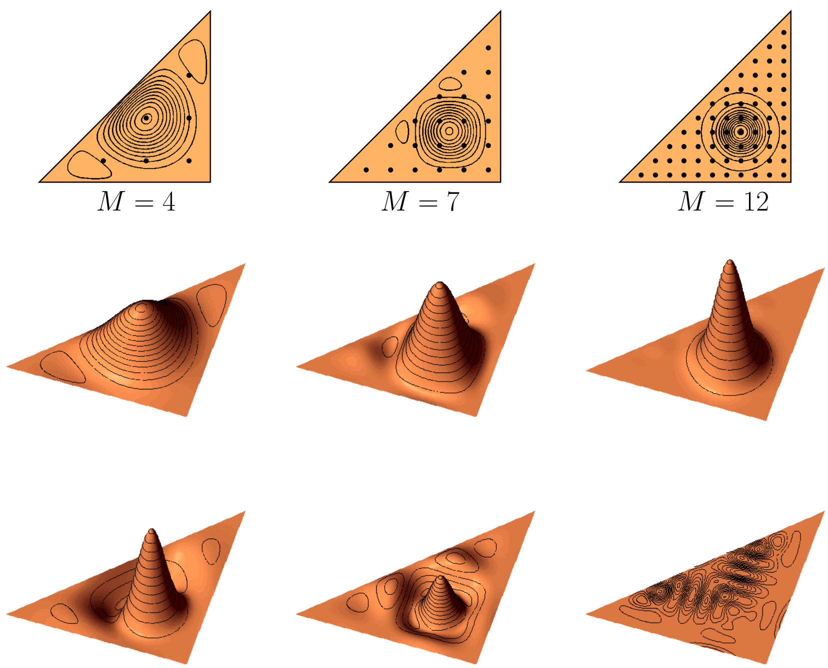

FIGURE A.4. Three examples $\psi_4^{\mathrm{II},-}$, $\psi_7^{\mathrm{II},-}$, and $\psi_{12}^{\mathrm{II},-}$ of antisymmetric sine functions of the type AMDST-II interpolating the function f in Figure A.3. Sampling points for the interpolation are shown as small black dots. Figures in the third line depict the difference $f - \psi_N^{\mathrm{II},-}$.

A.3.2. Symmetric discrete sine transforms

Symmetric sine functions are related to symmetric exponential functions [**18**]. Four types of discrete symmetric sine functions are derived from the four types of symmetric exponential transforms. We apply symmetric exponential transforms from [**18**] to the extensions of a given function.

In order to derive discrete symmetric sine transforms, we use functional operators E_L, R, and for the functions $f: K^+_{[0,1]} \to \mathbb{C}$, we define symmetric extension $Sf: K_{[0,1]} \to \mathbb{C}$ by the formula

$$Sf(x,y) = \begin{cases} f(x,y), & x \geq y, \\ f(y,x), & x < y. \end{cases} \tag{A.3.12}$$

In the first two types of discrete symmetric transforms, we consider the given functions $f_1 : K^+_{[0,1]} \to \mathbb{C}$ that have zero values for the lines $x = 1$ and $y = 0$. In the last two types, we consider the given functions $f_2 : K^+_{[0,1]} \to \mathbb{C}$ that vanish on the line $y = 0$. The additional conditions on the boundary are derived from properties of sine functions on the regions in view.

The following extension of a function f_1 or f_2 with the corresponding values of N, T and b are substituted into the formula (65) from [**18**]:

$$\text{(I)} \quad E_1Sf_1 : K_{[-1,1]} \to \mathbb{C}, \quad \text{where} \quad N = 2M, \quad T = 2, \quad b = 1, \tag{A.3.13}$$

$$\text{(II)} \quad E_1Sf_1 : K_{[-1,1]} \to \mathbb{C}, \quad \text{where} \quad N = 2M, \quad T = 2, \quad b = \tfrac{1}{2}, \tag{A.3.14}$$

$$\text{(III)} \quad E_2RSf_2 : K_{[-2,2]} \to \mathbb{C}, \quad \text{where} \quad N = 4M, \quad T = 4, \quad b = 1, \tag{A.3.15}$$

$$\text{(IV)} \quad E_2RSf_2 : K_{[-2,2]} \to \mathbb{C}, \quad \text{where} \quad N = 4M, \quad T = 4, \quad b = \tfrac{1}{2}. \tag{A.3.16}$$

We notice that E_1Sf_1 has zero values on the boundary of $K_{[-1,1]}$ and on the axes x, y, and E_2RSf_2 vanishes on the boundary of $K_{[-2,2]}$ and on the axes x, y. The different conditions of zero values are due to the fact that in the first and second type of symmetric sine transforms, we consider $\sin^-_{(k,l)}(x,y)$ in comparison with the third and fourth type, where we consider $\sin^-_{(k,l)}(\frac{x}{2}, \frac{x}{2})$.

Due to (anti)symmetry and zero values of the extended functions on the axes x, y and on borders of $K_{[-1,1]}$ or $K_{[-2,2]}$, we obtain the final explicit form of the four symmetric sine transforms:

SMDST–I.

$$\psi^{\mathrm{I},+}_M(x,y) = \sum_{\left\{ \substack{k,l=1 \\ k \geq l} \right\}}^{M-1} c^{\mathrm{I},+}_{kl} \sin^+_{(k,l)}(x,y),$$

$$c_{kl}^{\mathrm{I},+} = \frac{4}{M^2 G_{kl}} \sum_{\left\{ \substack{m,n=1 \\ m \geq n} \right\}}^{M-1} G_{mn}^{-1} f_1\left(x_m, y_n\right) \sin_{(k,l)}^{+}\left(x_m, y_n\right),$$

where $x_m = \frac{m}{M},\ y_n = \frac{n}{M}$.

SMDST–II.

$$\psi_M^{\mathrm{II},+}(x,y) = \sum_{\left\{ \substack{k,l=1 \\ k \geq l} \right\}}^{M} c_{k,l}^{\mathrm{II},+} \sin_{(k,l)}^{+}(x,y),$$

$$c_{k,l}^{\mathrm{II},+} = \frac{4}{M^2 G_{kl}} \sum_{\left\{ \substack{m,n=0 \\ m \geq n} \right\}}^{M-1} G_{mn}^{-1} f_1\left(x_m, y_n\right) \sin_{(k,l)}^{+}\left(x_m, y_n\right)$$

where $x_m = \frac{m+\frac{1}{2}}{M},\ y_n = \frac{n+\frac{1}{2}}{M}$.

SMDST–III.

$$\psi_M^{\mathrm{III},+}(x,y) = \sum_{\left\{ \substack{k,l=0 \\ k \geq l} \right\}}^{M-1} c_{k,l}^{\mathrm{III},+} \sin_{(k+\frac{1}{2},l+\frac{1}{2})}^{+}(x,y),$$

$$c_{k,l}^{\mathrm{III},+} = \frac{4}{M^2 G_{kl}} \sum_{\left\{ \substack{m,n=1 \\ m \geq n} \right\}}^{M} d_{m,M} d_{n,M} G_{mn}^{-1} f_2\left(x_m, y_n\right) \sin_{(k+\frac{1}{2},l+\frac{1}{2})}^{+}\left(x_m, y_n\right)$$

where $x_m = \frac{m}{M},\ y_n = \frac{n}{M}$.

SMDST–IV.

$$\psi_M^{\mathrm{IV},+}(x,y) = \sum_{\left\{ \substack{k,l=0 \\ k \geq l} \right\}}^{M-1} c_{kl}^{\mathrm{IV},+} \sin_{(k+\frac{1}{2},l+\frac{1}{2})}^{+}(x,y),$$

$$c_{kl}^{\mathrm{IV},+} = \frac{4}{M^2 G_{kl}} \sum_{\left\{ \substack{m,n=0 \\ m \geq n} \right\}}^{M-1} G_{mn}^{-1} f_2\left(x_m, y_n\right) \sin_{(k+\frac{1}{2},l+\frac{1}{2})}^{+}\left(x_m, y_n\right)$$

where $x_m = \frac{m+\frac{1}{2}}{M},\ y_n = \frac{n+\frac{1}{2}}{M}$.

A.3.2.1. *Example of symmetric sine interpolation*

Consider the Gaussian distribution f defined by (A.3.11). We calculate the symmetric sine interpolating functions of the type SMDST–II: $\psi_4^{\mathrm{II},+}$, $\psi_7^{\mathrm{II},+}$ and $\psi_{12}^{\mathrm{II},+}$. These interpolating functions, together with the interpolating grids, are depicted in Figure A.5.

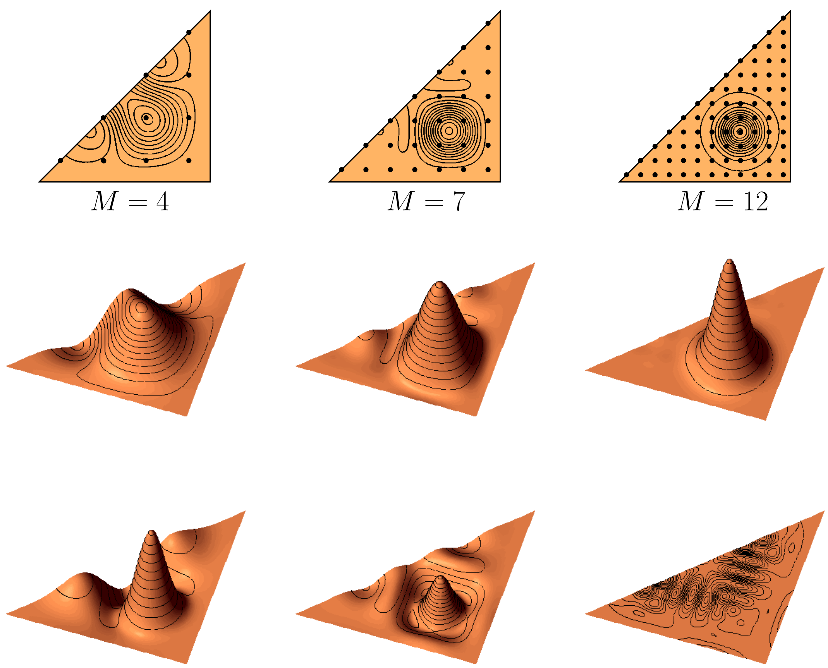

FIGURE A.5. Three examples $\psi_4^{\mathrm{II},+}$, $\psi_7^{\mathrm{II},+}$ and $\psi_{12}^{\mathrm{II},+}$ of symmetric sine functions of the type SMDST–II interpolating the function f in Figure A.3. Sampling points for the interpolation are shown as small black dots. The third line show the difference $f - \psi_N^{\mathrm{II},+}$.

A.4. Concluding remarks

Decomposition of products of functions described in the article for all pairs of functions $\sin^{\pm}_{(\lambda,\mu)}$ and $\cos^{\pm}_{(\lambda,\mu)}$ paves the way to a wealth of new properties of these functions, such as a large variety of trigonometric-like identities, representation of functions as orthogonal polynomials, and recursion relations for their construction.

In the paper, we make no use of a particular property of the four families of functions, which is proving useful elsewhere: The functions of each family split into two mutually exclusive congruence classes according to the value of the sum of their subscript $\lambda + \mu \mod 2$. For example, it can be seen in Table A.2 that all terms in the decomposition of one product belong to the same congruence class, and that the classes add up during multiplication of functions.

The well known Weyl formula for the character of an irreducible finite dimensional representation of semisimple Lie groups is a ratio of two S-functions studied in [**26**]. The character functions are endowed with many properties that are fundamental to the theory of representations in general. The $\sin^{\pm}$ functions in the paper, which depend on two variables, resemble the S-functions of the group $SU(3)$. In fact, in the case of one variable, these functions coincide. It is therefore interesting to explore properties of functions formed as an analogous ratio of $\sin^{\pm}$ functions.

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