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Quantum communication with an accelerated partner

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An unsolved problem in relativistic quantum information research is how to model efficient, directional quantum communication between localized parties in a fully quantum field-theoretical framework. We propose a tractable approach to this problem based on calculating expectation values of localized field observables in the Heisenberg picture. We illustrate our approach by analyzing, and obtaining approximate analytical solutions to, the problem of communicating coherent states between an inertial sender, Alice, and an accelerated receiver, Rob. We use these results to determine the efficiency with which continuous variable quantum key distribution could be carried out over such a communication channel.

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I. INTRODUCTION

The study of how information can be carried and processed by systems described by relativistic quantum mechanics is referred to as relativistic quantum information [1]. This rapidly growing body of work studies how quantum information tasks and resources are altered by the relativistic treatment of space-time. A key topic is the sharing of quantum information between inertial and noninertial parties, as in this case the quantum ground states of the two observers will differ [2] and novel phenomena such as Unruh-Davies [3,4] and Hawking radiation [5] can emerge. An obvious place in which such effects could play a role is in quantum communication tasks, such as quantum key distribution [6]. However, a rigorous, quantum field-theoretic description of such protocols has not been developed. Previous analysis has been restricted to toy models that are ultimately inconsistent with the properties of real communication systems.

In this paper we introduce a rigorous and tractable framework for studying optical quantum communication between inertial and noninertial observers. We specifically apply our approach to the problem of continuous variable quantum communication [7]. We obtain approximate analytical results for conditions typical of such communication systems and use them to analyze quantum key distribution between an inertial Alice and a uniformly accelerating Rob. We find the secret key rate is limited in a time-dependent manner.

Traditional detection schemes analyzed with acceleration have utilized the Unruh-Dewitt detector [3,8]. The Unruh-Dewitt detector is a single two-level quantum system weakly coupled to the field over 4π steradians. It is not a good model for the efficient, unidirectional macroscopic detectors commonly employed in quantum communication experiments. The other method used to describe quantum information protocols in the presence of noninertial motion does so directly in terms of the field modes [9,10]. This approach suffers from two major problems: (i) the restriction of the description to a particular set of unphysical modes, the Unruh modes [3], in order to simplify the problem and (ii) the use of nonlocal states defined on single frequency global modes and the subsequent unfounded interpretation of these nonlocal results in terms of local observers. Although some work has been done on avoiding the latter problem [11] the reliance on the Unruh modes remained. The method developed here avoids both of these problems and leads to a richer and more realistic description of the physics. We note that another approach to this problem is described in Ref. [12].

We use a $(3+1)$-dimensional massless scalar field description to model a localized, directional, inertial source using Minkowski modes and, similarly, a localized, directional, uniformly accelerating detector using Rindler modes. Minkowski coordinates, (x_1,x_2,x_3,t), are the standard ones for describing inertial observers. Rindler coordinates, (ξ,x_2,x_3,τ), can be used to describe accelerated observers. The two coordinate systems are related within the right-hand sector (the right Rindler wedge—see Fig.1) via [2]:

$$t = a^{-1}e^{a\xi}\sinh(a\tau), \quad x_1 = a^{-1}e^{a\xi}\cosh(a\tau). \tag{1}$$

A stationary observer in Rindler coordinates, sufficiently well localized around $\xi = 0$, follows a uniformly accelerated trajectory in Minkowski coordinates, such as is depicted in Fig. 1. The rate of acceleration is given by the parameter a. Throughout this paper we work in units for which $c = 1$.

The quantum source is held by Alice, who is stationary in Minkowski coordinates. The quantum detector is held by Rob, who is stationary in Rindler coordinates. We quantify the communication channel between Alice and Rob in terms of expectation values of localized observables, calculated in the Heisenberg picture, rather than analyzing the quantum states. It is due to the difficulties of transforming the states that Unruh modes have been exclusively employed previously. By analyzing the observables we are able to avoid this difficulty and transform between arbitrary modes with ease, leading to the key advantage of this method.

The paper is arranged in the following way: in the next section we introduce our basic approach by solving the simpler problem of homodyne detection of the inertial vacuum state by the uniformly accelerating observer, Rob. Thermal radiation, as predicted by the Unruh effect, is observed. In Sec. III we move to the more complicated scenario of quantum communication between Alice and Rob. In Sec. IV we consider a specific quantum communication protocol, quantum key distribution, carried out between Alice and Rob, and then conclude in Sec. V.

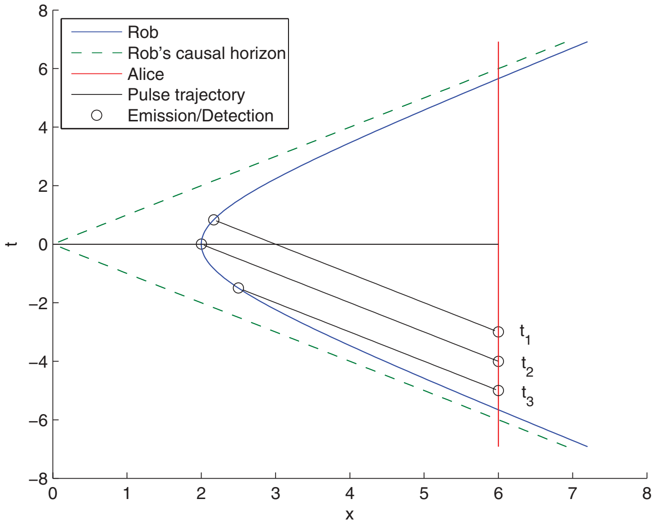

FIG. 1. (Color online) Geometry of the quantum communication scenario considered in Sec. III. Alice is stationary, while Rob is uniformly accelerating. Alice prepares coherent state pulses and sends them along with local-oscillator pulses to Rob at various times (e.g., t_1,t_2,t_3). Rob's detector is a broadband, time-integrated homodyne receiver.

II. DETECTION OF UNRUH-DAVIES RADIATION WITH A HOMODYNE DETECTOR

As a first example of our approach we consider a simple example in which the signal that Rob detects is the Minkowski (inertial) vacuum. Rob performs homodyne detection on this signal, as seen in his reference frame, using, as a local-oscillator mode, a coherent state of amplitude β, where β is real and $\beta \gg 1$. The homodyne detector is formed from two identical photodetectors, that detect distinct modes S and L after they have been mixed on a beam splitter. The photocurrents from the photodetectors are subtracted to give the output signal. As a result the output of Rob's homodyne detector at some time τ (as measured in Rob's frame) is represented by the following operator [7]:

$$\hat{O}(\tau) = \hat{b}_S(\tau)\hat{b}_L^\dagger(\tau)e^{i\phi} + \hat{b}_S^\dagger(\tau)\hat{b}_L(\tau)e^{-i\phi}, \tag{2}$$

where $\hat{b}_K$ ($\hat{b}_K^\dagger$) are boson annihilation (creation) field operators with $K = S,L$. The subscripts S,L refer to the signal and local-oscillator modes, respectively. The relative phase ϕ determines the quadrature angle detected.

Creation of a coherent state can be modeled as a unitary displacement of the vacuum. Physically, a coherent state is an excellent approximation to the state produced by a well-stabilized laser. Rob's displacement operator can be written: $\hat{D}(\beta) = \exp[\beta(b_D^\dagger - b_D)]$, where the subscript D labels the mode to which the displacement is perfectly matched. The mode operators can be spectrally decomposed as

$$\hat{b}_K = \int dk_d\, f_K(k_d,\tau)\hat{b}_{k_d}. \tag{3}$$

For these distributions the $k_d = (k_{d1},k_{d2},k_{d3})$ refers to Rob's detector wave vector with the first component (corresponding to the direction of acceleration) being the Rindler frequency and the other two components being Minkowski. The integral $\int dk_d$ is over the whole wave-vector space. In this case it is $\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty dk_{d1}dk_{d2}dk_{d3}$ as the first component of this wave vector is a right Rindler mode which is strictly positive [13]. The operators $\hat{b}_{k_d}$ are the plane-wave Rindler operators, obeying the usual boson commutation relation

$$\big[\hat{b}_{k_d},\hat{b}_{k_d'}^\dagger\big] = \delta(k_{d1}-k_{d1}')\delta(k_{d2}-k_{d2}')\delta(k_{d3}-k_{d3}'). \tag{4}$$

They describe plane waves, oscillating at frequency $k_{d1}a$ as perceived by a collection of nonlocalized accelerating observers. The functions $f_K(k_d)$ localize these modes in some region of space-time and hence the mode operators $\hat{b}_K$ describe modes detected by a local accelerating observer. Rob will integrate the photocurrent from his detector over a time long compared to the inverse of the frequency being analyzed (as will be determined by the frequency of his local oscillator). The average value of the signal received by Rob will be given by the expectation value:

$$X = \left\langle \int d\tau\, \hat{O}(\tau) \right\rangle. \tag{5}$$

We will also be interested in the variance of the integrated signal

$$V = \left\langle \left(\int d\tau\, \hat{O}(\tau) \right)^2 \right\rangle - \left\langle \int d\tau\, \hat{O}(\tau) \right\rangle^2. \tag{6}$$

The initial state for both the signal and the local oscillator is the Minkowski vacuum state, $|0\rangle_M$. Formally, the expression for X becomes

$$X = \int d\tau \langle 0|_M \hat{D}^\dagger(\beta)(\hat{b}_S\hat{b}_L^\dagger + \hat{b}_S^\dagger\hat{b}_L)\hat{D}(\beta)|0\rangle_M, \tag{7}$$

where for simplicity we have taken $\phi = 0$. This expression can be expanded using the identity

$$\hat{D}^\dagger(\beta)\hat{b}_L\hat{D}(\beta) = \hat{b}_L + \beta \int dk_d\, f_L(k_d,\tau)f_D^*(k_d,\tau), \tag{8}$$

where the second term on the right-hand side (RHS) of Eq. (8) quantifies the coupling of the displacement into the local-oscillator mode via the overlap of their respective mode functions. We obtain

$$\begin{aligned} X &= \int d\tau \langle 0|_M \bigg(\hat{b}_S^\dagger \bigg(\hat{b}_L + \beta \int dk_d\, f_L(k_d,\tau) f_D^*(k_d,\tau) \bigg) \\ &\quad + \hat{b}_S \bigg(\hat{b}_L^\dagger + \beta \int dk_d\, f_L^*(k_d,\tau) f_D(k_d,\tau) \bigg) \bigg) |0\rangle_M \\ &\cong \beta \int d\tau \langle 0|_M \int dk_d (\hat{b}_S^\dagger f_L(k_d,\tau) f_D^*(k_d,\tau) \\ &\quad + \hat{b}_S f_L^*(k_d,\tau) f_D(k_d,\tau))|0\rangle_M \end{aligned} \tag{9}$$

where we use $\beta \gg 1$ to drop small terms in the second line.

In order to calculate the expectation value of Eq. (9) against the Minkowski vacuum we need to rewrite Rob's measurement operators in terms of Minkowski modes. The transformation relations between Rindler and Minkowski spectral modes is given by [13]

$$\hat{b}_{k_d} = \int dk_s \big(A_{k_d k_s}\hat{a}_{k_s} + B_{k_d k_s}\hat{a}_{k_s}^\dagger \big), \tag{10}$$

where the operators $\hat{a}_{k_s}$ are the plane-wave Minkowski operators, obeying the usual boson commutation relation

$$[\hat{a}_{k_s},\hat{a}^\dagger_{k'_s}] = \delta(k_{s1}-k'_{s1})\delta(k_{s2}-k'_{s2})\delta(k_{s3}-k'_{s3}), \quad (11)$$

and

$$A_{k_d k_s} = \frac{\delta(\vec{k}_d-\vec{k}_s)}{\sqrt{2\pi\omega_s(1-e^{-2\pi k_{d1}})}}\left(\frac{\omega_s+k_{s1}}{\omega_s-k_{s1}}\right)^{i\frac{1}{2}k_{d1}},$$
$$B_{k_d k_s} = \frac{\delta(\vec{k}_d+\vec{k}_s)}{\sqrt{2\pi\omega_s(e^{2\pi k_{d1}}-1)}}\left(\frac{\omega_s+k_{s1}}{\omega_s-k_{s1}}\right)^{i\frac{1}{2}k_{d1}} \quad (12)$$

are the Bogolyubov coefficients, where $\vec{k}_d=(k_{d2},k_{d3})$, $\vec{k}_s=(k_{s2},k_{s3})$, and ω_s is the signal frequency $\omega_s=\sqrt{k_{s1}^2+k_{s2}^2+k_{s3}^2}$. Given that the transformation of Eq. (10) is linear in the creation and annihilation operators and that, by definition, the Minkowski annihilation operator annihilates the Minkowski vacuum, i.e.,

$$\hat{a}_{k_s}|0\rangle_M = 0, \quad (13)$$

it follows that $\langle 0|_M\hat{b}_K|0\rangle_M=0$, and hence $X=0$—the average value of the homodyne signal is zero.

More interesting is to calculate the variance of the signal. Formally, from Eq. (9), we have

$$V = \beta^2\langle 0|_M\left(\int d\tau\int dk_d[\hat{b}^\dagger_S f_L(k_d)f^*_D(k_d) + \hat{b}_S f^*_L(k_d)f_D(k_d)]\right)^2|0\rangle_M. \quad (14)$$

In order to proceed we need to introduce an explicit form for the detector wave function. We assume first that Rob looks in the ξ direction (i.e., the same direction as the acceleration) and focuses the signal and local-oscillator modes onto his detector such that we can make the paraxial approximation that the detector wave function can be factored into its transverse and longitudinal components as

$$f_K(k_d,\tau) = e^{-ik_{d1}a\tau}f_K(k_{d1})g_K(k_{d2})h_K(k_{d3}), \quad (15)$$

where the detector is centered on the space-time point ($\xi=0,x_2,x_3,\tau$). It is important that the longitudinal component of the detector wave function is well localized; otherwise, its interpretation as a detector following a particular space-time trajectory is compromised. Thus we consider a detector wave function that is very broad in k_{d1} such that it is well localized spatiotemporally. In particular we take $f_K(k_{d1})\approx 1/\sqrt{2\pi a}$ for $k_{d1}>0$ and zero otherwise. Using these definitions we can simplify terms in Eq. (14) such as

$$\int d\tau\,\hat{b}_S\int dk_d\,f^*_L(k_d)f_D(k_d)$$
$$= \int d\tau\int dk'_d\int dk_{d1}\frac{e^{-i(k'_{d1}-k_{d1})a\tau}}{2\pi a}\hat{b}_{k'_d}g_S(\vec{k}'_d)f_D(k_{d1})$$
$$= \int dk_{d1}\hat{b}_{k_{d1},S}\,f_D(k_{d1}), \quad (16)$$

where we have defined new boson annihilation operators:

$$\hat{b}_{k_{d1},K} \equiv \int d\vec{k}_d g_K(\vec{k}_d)\hat{b}_{k_d}, \quad (17)$$

with the shorthand $g_K(\vec{k}_d)=g_K(k_{d2})h_K(k_{d3})$. We assume that the transverse-mode functions, g_S and g_L are orthonormal, such that $[\hat{b}_{k_{d1},K},\hat{b}^\dagger_{k_{d1},K}]=1$, but $[\hat{b}_{k_{d1},S},\hat{b}^\dagger_{k_{d1},L}]=0$. In obtaining Eq. (16) we have also assumed that the integral over τ is sufficiently long that $\int d\tau\frac{1}{2\pi a}e^{-i(k_{d1}-k'_{d1})a\tau}\approx\delta(k_{d1}-k'_{d1})$ and that the transverse part of $f_D(k_d)$ perfectly matches $g_L(\vec{k}_d)$.

Combining the result of Eq. (16) (and its conjugate) with Eq. (14) we obtain

$$V \approx \beta^2\langle 0|_M\int dk_{d1}\int dk'_{d1}\big(f_D(k_{d1})f^*_D(k'_{d1})\hat{b}_{k_{d1},S}\hat{b}^\dagger_{k'_{d1},S} + f^*_D(k_{d1})f_D(k'_{d1})\hat{b}^\dagger_{k_{d1},S}\hat{b}_{k'_{d1},S} + f_D(k_{d1})f_D(k'_{d1})\hat{b}_{k_{d1},S}\hat{b}_{k'_{d1},S} + f^*_D(k_{d1})f^*_D(k'_{d1})\hat{b}^\dagger_{k_{d1},S}\hat{b}^\dagger_{k'_{d1},S}\big)|0\rangle_M \quad (18)$$

We now transform to Minkowski modes. Using Eq. (10) we find

$$\hat{b}_{k_{d1},S} = \int d\vec{k}_d g_S(\vec{k}_d)\int dk_s\big(A_{k_dk_s}\hat{a}_{k_s}+B_{k_dk_s}\hat{a}^\dagger_{k_s}\big)$$
$$= \int dk_s\frac{1}{\sqrt{2\pi\omega_s(e^{2\pi k_{d1}}-1)}}\left(\frac{\omega_s+k_{s1}}{\omega_s-k_{s1}}\right)^{i\frac{1}{2}k_{d1}} \times\big[e^{\pi k_{d1}}\hat{a}_{k_s}g_S(\vec{k}_s)+\hat{a}^\dagger_{k_s}g_S(-\vec{k}_s)\big]. \quad (19)$$

Substituting this into Eq. (18), using the properties of the Minkowski modes, Eqs. (11) and (13), and the identity

$$\int dk_s\frac{1}{2\pi\omega_s}\left(\frac{\omega_s+k_{s1}}{\omega_s-k_{s1}}\right)^{i\frac{1}{2}(x-x')} = \delta(x-x'), \quad (20)$$

we find

$$V = \beta^2\int dk_{d1}|f_D(k_{d1})|^2\frac{e^{2\pi k_{d1}}+1}{e^{2\pi k_{d1}}-1}. \quad (21)$$

The full mode function for the local oscillator close to the detector, i.e., for $\xi\approx 0$, can be written

$$f_D(k_d,\tau) = e^{-i\sqrt{a^2k_{d1}^2-k_{d2}^2-k_{d3}^2}\xi+ik_{d2}x_2+ik_{d3}x_3} \times e^{-ik_{d1}a\tau}f_D(k_{d1})g_D(k_{d2})h_D(k_{d3}). \quad (22)$$

As we are considering propagation in the $-\xi$ direction, we have that the average values of k_{d2} and k_{d3} are zero. This indicates that the average value of the local-oscillator frequency is equal to that of its wave vector in the ξ direction. If Rob's local-oscillator mode function is a Gaussian, strongly peaked around the frequency k_{do} with respect to his proper time, τ, then we can make the approximation

$$|f_D(k_{d1})|^2 \approx \delta(k_{d1}-k_{do}/a), \quad (23)$$

and finally (dividing out the local-oscillator amplitude) we obtain the expression for the variance of the signal mode,

$$\langle\Delta X_S(0)^2\rangle = V/\beta^2 = \frac{e^{2\pi k_{do}/a}+1}{e^{2\pi k_{do}/a}-1}. \quad (24)$$

This expression is identical to that obtained for homodyne detection, by an inertial observer, of a thermal bath at temperature $T=a\hbar/(2\pi k)$, with k Boltzmann's constant, as expected from the Unruh effect.

Notice we get the same result if Rob looks in, say, the $-x_2$ direction. Now Rob's local oscillator is propagating in the x_2 direction and we expect the average value of k_{d2} to be equal to the average local-oscillator frequency, with the average value of k_{d3} and $\sqrt{a^2k_{d1}^2 - k_{d2}^2 - k_{d3}^2}$ equal to zero. This still occurs when the k_{d1} component of the mode function is strongly peaked around the wave number k_{do}/a. Thus we again get the solution of Eq. (24), indicating the expected isotropic nature of the Unruh-Davies radiation.

III. QUANTUM COMMUNICATION BETWEEN ALICE AND ROB

Having confirmed that our model of localized, directional homodyne detection of the Minkowski vacuum by a Rindler observer is consistent with the Unruh-DeWitt detector results, we now consider the more interesting case of quantum communication. To illustrate our approach we consider the following protocol to transmit quantum information from Alice to Rob. Alice sends an optical pulse prepared in a coherent state of amplitude α, where α is an arbitrary complex number, as her signal state. She also produces another coherent state of amplitude β, where again β is real and now $\beta \gg |\alpha|$, as a local-oscillator mode. She sends both to Rob. The preparation of the local oscillator by Alice is now a practical necessity given that the local oscillator and signal states need to be phase locked. Example trajectories of Alice and Rob and the signals sent are depicted on a space-time diagram in Fig. 1. Rob performs homodyne detection on the signal and local-oscillator mode, as seen in his reference frame which again can be modeled by Eq. (2).

In Minkowski coordinates, the state Alice produces can be represented by displacement of the Minkowski vacuum state, $|0\rangle_M$, as

$$|\alpha,\beta,t\rangle_j = D_S(\alpha)D_L(\beta)|0\rangle_M, \tag{25}$$

where now the displacement operators are given by $D_K(\gamma) = \exp[\gamma a_K^\dagger - \gamma^* a_K]$, with $\gamma = \alpha,\beta$ and

$$\hat{a}_K = \int dk_s\, f_K(k_s)\hat{a}_{k_s}. \tag{26}$$

Formally, the expression for X becomes

$$X = \int d\tau \langle 0|_M \hat{D}_L^\dagger(\beta)\hat{D}_S^\dagger(\alpha) \times (\hat{b}_S\hat{b}_L^\dagger + \hat{b}_S^\dagger\hat{b}_L)\hat{D}_S(\alpha)\hat{D}_L(\beta)|0\rangle_M. \tag{27}$$

A generalization of Eq. (8) to allow for the transformation between Rindler and Minkowski modes gives the identity

$$D_K^\dagger(\gamma)\hat{b}_K D_K(\gamma) = \hat{b}_K + \gamma\int dk_d \int dk_s\, f_K(k_d) \times \left(A_{k_dk_s} f_{D_K}^*(k_s) + B_{k_dk_s} f_{D_K}(k_s)\right) \tag{28}$$

The expressions for X (and V) can be expanded via Eq. (28). The resulting expressions comprise expectation values of Heisenberg picture operators over the initial Minkowski vacuum state. Hence we can obtain exact formal solutions for the average quadrature values and their variances. For example, the expression for X becomes

$$X = \beta\alpha^* e^{-i\phi}\int d\tau \int dk_d \int dk_s\, f_L(k_d)\big[A_{k_dk_s} f_{D_L}^*(k_s) + B_{k_dk_s} f_{D_L}(k_s)\big]\int dk_d' \int dk_s'\, f_S^*(k_d')\big[A_{k_d'k_s'} f_{D_S}(k_s') + B_{k_d'k_s'} f_{D_S}^*(k_s')\big] + \text{c.c.}, \tag{29}$$

where we have used $\langle 0|_M \hat{b}_K|0\rangle_M = 0$ [see discussion around Eq. (13)] and that $\beta \gg |\alpha|$ to discard small terms. Expressions such as Eq. (29) can be numerically solved for specific localized detection and signal wave functions. To obtain analytical solutions we need to make some approximations based on the form of the wave functions.

We assume that the communication between Alice and Rob is "beamlike" in the sense that Alice sends a well-directed Gaussian mode to Rob who focuses it down to perfectly match the transverse spatial profile of his detector. For simplicity, we assume the communication is aligned with the acceleration. We can again make the paraxial approximation and factor the signal wave function into transverse and longitudinal components, i.e., $f_{D_K}(k_s) = e^{-i(\omega_s t - k_{s1}x)} f_D(k_{s1})g_{D_K}(\vec{k}_s)$, where the origin of Alice's signal pulse is centered on the space-time point (x,x_2,x_3,t). We make the assumption that the transverse components of the source displacement match those of the detector wave functions which are taken to be the same as in the previous section [see Eq. (15) and discussion following]. Initially, we make the impractical assumption that this is achieved with unit efficiency but relax this in our final discussion.

We assume that the longitudinal part of the signal wave function is peaked at a large wave number k_{so} such that, for the region of wave numbers for which the wave function is nonzero, $|k_{s1}| \gg |k_{s2}|,|k_{s3}|$. We also assume that the standard deviation of the longitudinal part of the wave function, though broad on the wavelength scale, is small compared to k_{so}. Hence we write $k_{s1} = k_{so} + \bar{k}$, where $|k_{so}| \gg |\bar{k}|$ for the region of wave numbers for which the wave function is nonzero. These are typical approximations used for nonrelativistic quantum communication systems. Given this, the longitudinal part of the signal wave function becomes $e^{i|k_{s1}|(\pm x - t)} f_D(k_{s1})$, where $+$ $(-)$ corresponds to positive (negative) k_{so}, i.e., propagation in the positive (negative) x direction. We can approximate the signal frequency dependent term in Eqs. (12) as

$$\left(\frac{\omega_s + k_{s1}}{\omega_s - k_{s1}}\right)^{i\frac{1}{2}k_{d1}} \approx e^{\pm i\frac{1}{2}k_{d1}[2\ln(2|k_{s1}|) - \ln(k_{s2}^2 + k_{s3}^2)]} \approx e^{\pm i|\frac{k_{s1}}{k_{so}}|k_{d1}} e^{\pm ik_{d1}[\ln(2|k_{so}|) - \frac{1}{2}\ln(k_{s2}^2 + k_{s3}^2) - 1]}.$$

As a specific case we take $k_{so} < 0$ in the following as per the example of Fig. 1.

Substituting our approximate forms into Eq. (29) we obtain

$$X = \beta\alpha^* e^{-i\phi}\int d\tau\left(\int dk_d \int dk_{s1} \frac{e^{-ik_{d1}a\tau}}{\sqrt{2\pi}} \frac{g_L(\vec{k}_d)}{\sqrt{2\pi|k_{so}|(1 - e^{-2\pi k_{d1}})}}\Big[f_D(k_{s1})^* g_{D_L}(\vec{k}_d)^* e^{-i|\frac{k_{s1}}{k_{so}}|k_{d1}} e^{-ik_{d1}[\ln 2|k_{so}| - \frac{1}{2}\ln(k_{d2}^2 + k_{d3}^2) - 1]} e^{i|k_{s1}|(x+t)} + f_D(k_{s1})g_{D_L}(-\vec{k}_d)e^{-i|\frac{k_{s1}}{k_{so}}|k_{d1}} e^{-ik_{d1}[\ln 2|k_{so}| - \frac{1}{2}\ln(k_{d2}^2 + k_{d3}^2) - 1]} e^{-2\pi k_{d1}} e^{-i|k_{s1}|(x+t)}\Big]\right)\left(\int dk_d' \int dk_{s1}' \frac{e^{ik_{d1}'a\tau}}{\sqrt{2\pi}} \frac{g_S^*(\vec{k}_d')}{\sqrt{2\pi|k_{so}|(1 - e^{-2\pi k_{d1}'})}}\right.$$

$$\times \Big[f_D(k'_{s1}) g_{D_S}(\vec{k}'_d) e^{i|\frac{k'_{s1}}{k_{so}}|k'_{d1}} e^{ik'_{d1}[\ln 2|k_{so}| - \frac{1}{2}\ln({k'_{d2}}^2 + {k'_{d3}}^2) - 1]} e^{-i|k'_{s1}|(x+t)}$$
$$+ f_D^*(k'_{s1}) g^*_{D_S}(-\vec{k}'_d) e^{i|\frac{k'_{s1}}{k_{so}}|k'_{d1}} e^{ik'_{d1}[\ln 2|k_{so}| - \frac{1}{2}\ln({k'_{d2}}^2 + {k'_{d3}}^2) - 1]} e^{-2\pi k_{d1}} e^{i|k'_{s1}|(x+t)} \Big] \Bigg) + \text{c.c.} \tag{30}$$

A major simplification of this expression is possible if we assume that $f_D(k_{s1})$ is sufficiently broad in frequency that

$$\int dk_{s1} \frac{1}{\sqrt{2\pi |k_{so}|}} f_D(k_{s1}) e^{-i|\frac{k_{s1}}{k_{so}}|k_{d1}} e^{\pm i|k_{s1}|(x+t)} \approx \bar{f}_D \delta(k_{d1} \mp |k_{so}|(x+t)), \tag{31}$$

where $\bar{f}_D$ is the average value of $f_D(k_{s1})$. First, the second term in each sum in Eq. (30) goes to zero as $(x+t) > 0$ for the range of Alice's source positions considered (see Fig. 1) but $f_i(k_{d1}) = 0$ for $k_{d1} < 0$ in the right Rindler wedge. Next, the integral over τ produces a delta function between the two integrals over k_{d1}. Inserting the assumption that the transverse components of the source and detector wave functions are matched such that $g_K(\vec{k}_d) = g_{D_K}(\vec{k}_d) e^{i|k_{so}|(x+t)\frac{1}{2}\ln(k_{d2}^2 + k_{d3}^2)}$, and using the normalization of the transverse wave function $\int dk_{d2} \int dk_{d3} |g_{D_K}(\vec{k}_d)|^2 = 1$, Eq. (30) reduces to

$$X = \beta \int dk'_{s1} \frac{\alpha e^{i\phi} f_D^*(k'_{s1}) \bar{f}_D + \alpha^* e^{-i\phi} f_D(k'_{s1}) \bar{f}_D^*}{(1 - e^{-2\pi |k_{so}|(x+t)})} \approx \frac{\beta(\alpha e^{i\phi} + \alpha^* e^{-i\phi})}{(1 - e^{-2\pi |k_{so}|(x+t)})}, \tag{32}$$

where we have also used an approximate normalization over the signal wave function. By dividing out the amplitude of the local oscillator, $\bar{\beta} = \beta\sqrt{(1 - e^{-2\pi |k_{so}|(x+t)})}$, we obtain the expectation value of the quadrature amplitude of Alice's signal as observed by Rob,

$$\langle X_B(\phi) \rangle_A = X/\bar{\beta} = \frac{\alpha e^{i\phi} + \alpha^* e^{-i\phi}}{\sqrt{(1 - e^{-2\pi |k_{so}|(x+t)})}}. \tag{33}$$

Using a similar sequence of approximations and the techniques used in the previous section, we obtain the variance of the signal quadrature as

$$\langle \Delta X_B(\phi)^2 \rangle_A = V/\bar{\beta}^2 = \frac{(e^{2\pi |k_{so}|(x+t)} + 1)}{(e^{2\pi |k_{so}|(x+t)} - 1)}. \tag{34}$$

Equations (33) and (34) are our main results, characterizing the quadrature signals observed by Rob in the ideal limit of unit efficiency, well-localized detection of coherent states sent by Alice. Generalization to the detection of other initial states—squeezed states, entangled states, etc.—is straightforward.

The solutions have the general form of linear amplification of the initial state, as anticipated from single-mode treatments [14]. However, the effective gain,

$$G = 1/(1 - e^{-2\pi |k_{so}|(x+t)}), \tag{35}$$

exhibits a dependence on the time at which the pulse is sent, and hence on Rob's position on his trajectory at which he receives the pulse [cf. Eq. (34) with Eq. (24)]. The behavior of the effective gain can be explained in the following way. Consider first the pulse path labeled t_2 in Fig. 1. This pulse is received by Rob around $t_R = 0$ in Minkowski coordinates. According to Eq. (1), $t = 0$ corresponds to $\tau = 0$ in Rindler coordinates and hence (with $\xi = 0$) $x_R = 1/a$. Given that $x + t$ is a constant for the path (i.e., the path is a geodesic) we have $x + t = x_R + t_R = 1/a$. So the effective gain is $G = 1/(1 - e^{-2\pi |k_{so}|/a})$. Linear amplification of the vacuum with this gain gives a thermal state distribution as a function of the detection frequency, k_{so}, and the acceleration on the detector trajectory, a, as expected from the Unruh effect [3] [Eqs. (34) and (24) are equivalent for this signal path].

Now consider signals sent at t_1 (t_3); see Fig. 1. The $t = 0$ intercepts for these signals are $1/a' < 1/a$ ($1/a'' > 1/a$). Therefore, the effective gain is lower (higher) for these signals. At first this seems surprising as thermalization due to the Unruh effect is predicted to be constant along the detector trajectory. The explanation is that the detection frequency of the homodyne detector is determined by Alice's local oscillator, as observed by Rob. Rob is instantaneously stationary at $t = 0$ and so observes the local oscillator at k_{so}. However, Rob receives the signals sent at t_1 (t_3) when moving away from (towards) Alice. Because of Rob's motion, the effective detection frequency is Doppler shifted to higher (lower) frequencies resulting in the lower (higher) effective gains. It is straightforward to show in general that for signals that intercept Rob's trajectory at a point where his instantaneous velocity is v, $x + t = 1/a\sqrt{(1+v)/(1-v)}$ as expected from the Doppler shift.

IV. QUANTUM KEY DISTRIBUTION BETWEEN ALICE AND ROB

A quantum communication protocol that can be implemented via the exchange of coherent states in the way described is continuous variable quantum key distribution [15]. The techniques for proving and quantifying the security are well established [16] but we will outline them here. We will consider a particular protocol where Alice draws a string of numbers from a bivariate Gaussian distribution of zero mean and variance V_S which she uses to choose the amplitudes of an ensemble of coherent states [17].

For each transmission, Rob is essentially trying to determine which coherent state was chosen from this ensemble which collectively looks like a thermal state. Loss between Alice and Rob of magnitude η can be modeled in the calculation of the previous section by assuming that the transverse mode overlap between Alice's signal displacement and Rob's detector mode is $\sqrt{1-\eta}$, i.e., $\int d\vec{k}_s g_S g^*_{K_S} = \sqrt{1-\eta}$. Then Eq. (33) is scaled such that $\langle X_B(\phi) \rangle_{A,\text{loss}} = \sqrt{1-\eta}\langle X_B(\phi) \rangle_A$, whilst Eq. (34) remains the same. Both passive loss and the Unruh thermalization itself are Gaussian processes so the whole situation can be regarded as a Gaussian channel with a combination of loss η and linear amplification of gain G.

The performance of a QKD protocol is quantified by the secret key rate, which is the difference between mutual information actually established by Alice and Rob and the maximum that could theoretically be attributed to the eavesdropper and is written

$$K = I(a:b) - S(a:E). \tag{36}$$

The first term is the classical mutual information established by correlations in the classical strings (denoted by lower case) of Alice's encoding (in this case the amplitude of coherent states) and Rob's homodyne measurement and is given by Shannon's formula

$$I(a:b) = s(b) - s(b|a), \tag{37}$$

where $s(b) = -\int p(b)\log_2 p(b)db$ is the Shannon entropy of a classical variable a with probability density $p(b)$ and $s(b|a)$ is the corresponding quantity conditioned on knowing a. For Gaussian distributions one can simply evaluate Shannon's formula to find

$$I(a:b) = \frac{1}{2}\log_2\left(\frac{V_R}{V_{R|a}}\right). \tag{38}$$

The conditional variance $V_{R|a}$ is that measured by Rob for an individual coherent state and is given directly by Eq. (34). The variance for the whole ensemble is the sum of the modulation variance and the noise of the individual states. Before transmission this would read $V_R = V_S + 1$, where the shot noise is unity. After transmission the losses and amplification scale the modulation, and the individual states are all thermalized such that

$$V_R = (1-\eta)GV_S + V_{B|a}. \tag{39}$$

Substituting Eqs. (34) and (35) yields

$$I(a:b) = \frac{1}{2}\log_2\left(\frac{V_S(1-\eta)}{(1+e^{-2\pi|k_{so}|(x+t)})}\right). \tag{40}$$

The second term in Eq. (36) is the maximum information an eavesdropper (Eve) could have obtained about a assuming she makes the optimal quantum measurement on her state E. For a given state this is bounded by the Holevo quantity,

$$S(a:E) = S(E) - S(E|a), \tag{41}$$

where $S(E) = -\mathrm{tr}(\rho_E \log_2 \rho_E)$ is understood to be the von Neumann entropy. In general, this quantity is extremely difficult to compute given only Alice's encoding and Rob's measurement results; however, for Gaussian states the calculation of Eve's information remains tractable.

The first step is to note that the ensemble of coherent states sent to Rob could equivalently have been generated by Alice creating EPR pairs and performing a heterodyne detection upon her arm, projecting Rob's into a coherent state. We can thus interpret Alice's encoding and Rob's measurement as being derived from a bipartite state ρ_{AB} that is shared at the end of the protocol [18] and turn our analysis to this equivalent entanglement based version. The worst case scenario would be to attribute all observed losses and noise as information that has leaked to Eve in which case the final state shared by Alice, Rob, and Eve is pure. This allows us to rewrite Eq. (41) as

$$S(a:E) = S(AB) - S(B|a). \tag{42}$$

The von Neumann entropy for a Gaussian state is solely a function of the covariance matrix, and thus our calculation of the first and second moments of the state received by Rob will be sufficient to find the secret key rate. For an N-mode Gaussian state, we have

$$S(\rho) = \sum_{i=1}^{N} \Phi(\lambda_i), \tag{43}$$

where

$$\Phi(x) = \left(\frac{x+1}{2}\right)\log_2\left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right)\log_2\left(\frac{x-1}{2}\right)$$

and λ_i are the symplectic eigenvalues.

In the entanglement-based scheme the state before transmission corresponding to the whole ensemble of coherent states is a pure EPR state with covariance matrix of the form

$$\gamma_{AB} = \begin{pmatrix} a\ \mathbb{I}_2 & c\ \sigma_z \\ c\ \sigma_z & b\ \mathbb{I}_2 \end{pmatrix}, \tag{44}$$

with $a = b = V_S + 1$, $c = \sqrt{a^2-1}$, and $\sigma_z = [0,1;0,-1]$. After transmission the covariance matrix characterizing the entangled version of the protocol for a channel combining losses with Unruh radiation will be given by the same form, but with

$$\begin{aligned} a &= V_S + 1, \\ b &= V_R, \\ c &= \sqrt{(1-\eta)G(a^2-1)} \end{aligned} \tag{45}$$

The state after Alice's measurement, or alternatively, given the knowledge of which coherent state was sent is, as shown

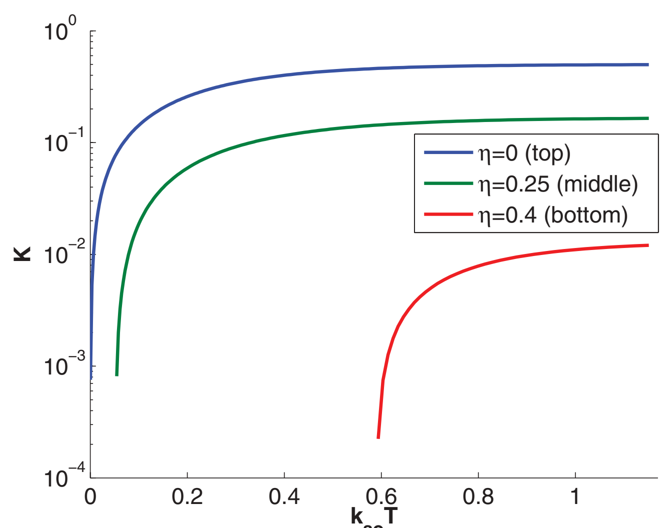

FIG. 2. (Color online) Secret key rates obtained for a continuous variable quantum key distribution protocol implemented between Alice and Rob. The key rates (K) are plotted as a function of a dimensionless quantity proportional to the emission time, $T = x + t$, and the center frequency of the pulse, k_{so}. Key rates are reduced by a thermal background due to the Unruh effect. Communication efficiency is $1-\eta$.

above, a noisy version of that state and has a covariance matrix,

$$\gamma_{B|a} = \begin{pmatrix} V_R & 0 \\ 0 & V_R \end{pmatrix}. \tag{46}$$

Thus Eve's information is given by

$$S(a:E) = \Phi(\lambda_{AB}^{+}) + \Phi(\lambda_{AB}^{-}) - \Phi(\lambda_{B|a}). \tag{47}$$

For a single-mode state the symplectic eigenvalue is simply the square root of the determinant so that $\lambda_{B|a} = \det(\gamma_{B|a}) = V_R$, whereas for a two-mode state of the form of Eq. (44), we have

$$\lambda_{AB}^{\pm} = \sqrt{\frac{1}{2}\Delta \pm \sqrt{\Delta^2 - 4\det(\gamma_{AB})}}, \tag{48}$$

where $\Delta = a^2 + b^2 - 2c^2$. Substituting in Eq. (45) and the channel characterization given by Eqs. (33), (34), (39), and (35), it is straightforward to calculate the secret key rates if Alice and Rob were to implement this protocol. The results of such a calculation are shown in Fig. 2:

$$\begin{aligned}\lambda_{AB}^{\pm} = {} & (1+V_S)^2 - GV_S(2+V_S)(1-\eta) \\ & \pm \{(1+V_S)(1-V_R+V_S)[(1+V_S)(1+V_R+V_S) \\ & + 2GV_S(2+V_S)(-1+\eta)]\}^{\frac{1}{2}}.\end{aligned}$$

The figure shows that even with the unrealistic assumption of unit efficiency, secret key rates are reduced by the Unruh effect. The reduction is most pronounced at earlier times, when the signals are Doppler shifted to lower frequencies that are more effected by the thermalization. If Rob's receiver is assumed to have nonunit efficiency via this technique, then Fig. 2 shows that quantum key distribution becomes impossible at sufficiently early times.

V. CONCLUSION

The techniques we have introduced allow the rigorous evaluation of relativistic quantum communication protocols in terms of the localized detectors and sources typically used for quantum communication. The results we have derived here directly apply to continuous variable protocols between an inertial and noninertial observer. They could straightforwardly be generalized to discrete variable protocols by considering localized number state detection by Rob, extending the description from scalar to vector fields, and considering number state creation by Alice. These techniques could also be adapted to treat quantum communication in curved space [2], or situations in which inertial detectors couple to Rindler modes due to rapid changes in their energy levels [19]. The latter case is of near term experimental interest.

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