Decisive coalitions and coherence properties*

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This version: March 12, 2009

* The paper was presented at the 2008 Workshop on Social Decisions in Malaga and at Waseda University. We thank Susumu Cato for comments and suggestions. Financial support from a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan, the Fonds de Recherche sur la Société et la Culture of Québec and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

Abstract. In a seminal contribution, Hansson has demonstrated that the family of decisive coalitions associated with an Arrovian social welfare function forms an ultrafilter. If the population under consideration is infinite, his result implies the existence of non-dictatorial social welfare functions. He goes on to show that if transitivity is weakened to quasi-transitivity as the coherence property imposed on a social relation, the set of decisive coalitions is a filter. We examine the structure of decisive coalitions and analogous concepts with alternative coherence properties, namely, acyclicity and Suzumura consistency, and without assuming that the social relation is complete. *Journal of Economic Literature* Classification No.: D71.

Keywords: Infinite-Population Social Choice, Decisiveness, Suzumura Consistency.

1 Introduction

Arrow's (1951; 1963) theorem establishing the existence of a dictator as a consequence of a set of seemingly innocuous properties of a social welfare function is the most fundamental result in the theory of social choice. Its conclusion depends crucially on the assumption that the population under consideration is finite and alternative methods of proof provided by authors such as Fishburn (1970), Sen (1979) and Suzumura (2000) highlight the important role played by this finiteness property. Kirman and Sondermann (1972) and Hansson (1976) consider the structure of *decisive coalitions* in the Arrovian framework. A coalition (that is, a subset of the population) is decisive if its members can always guarantee a strict social preference for any alternative over any other if all coalition members have a strict preference for the former. Kirman and Sondermann (1972) and Hansson (1976) establish that, in the general case where the population may be finite or infinite, the set of decisive coalitions forms an *ultrafilter,* given that social relations are assumed to be orderings and Arrow's axioms unlimited domain, weak Pareto and independence of irrelevant alternatives are satisfied. In the finite-population case, all ultrafilters are *principal* ultrafilters, that is, they are generated by a singleton. This singleton, is, by definition of the decisiveness property, a dictator. Thus, the results of these two papers generate Arrow's theorem as a corollary. In contrast, if the population is infinite, there exist non-principal ultrafilters and these ultrafilters correspond to decisive coalition structures that are non-dictatorial. Kirman and Sondermann (1972) argue that sets of decisive coalitions that are non-principal ultrafilters still have a dictatorial flavor when expressed in a different space (leading to what they refer to as "invisible" dictators) but this does not make the underlying social welfare functions themselves dictatorial; see Hansson (1976) for a discussion.

There has been some renewed interest in specific applications of infinite-population Arrovian social choice, particularly in the context of infinite-horizon social choice problems where the unidirectional nature of the flow of time permits some natural domain restrictions; see, for instance, Ferejohn and Page (1978), Packel (1980) and Bossert and Suzumura (2008b). In this paper, we reexamine Hansson's (1976) approach from a different perspective by relaxing the properties imposed on social preferences. As in the original Arrovian setting, Hansson assumes in his first set of results that a collective choice rule always generates orderings. Moreover, he considers the case where social preferences are merely quasi-transitive but not necessarily transitive while retaining the richness properties of reflexivity and completeness. In this case, the family of decisive coalitions does not necessarily form an ultrafilter but it always is a *filter.* We will discuss these structures in more detail once we have introduced the requisite formal definitions.

Our first step towards a comprehensive analysis of the resulting decisiveness structures when the requirements imposed on social relations are relaxed consists of dropping reflexivity and completeness and exploring the consequences of transitivity and of quasi-transitivity by themselves. It turns out that, in the absence of any richness properties, the decisive coalition structures resulting from Arrow's axioms are *the same* for transitivity and for quasi-transitivity: the family of decisive coalitions is a filter in each of the two cases. Thus, unlike in the reflexive and complete case, transitivity and quasi-transitivity can be used interchangeably. Intuitively, this is the case because weak Pareto deals with strict preferences only and, without reflexivity and completeness, the absence of a strict preference for one alternative over another does not imply a weak preference in the other direction. As is the case for Hansson's results, the implications can be reversed in the sense that, for any given filter, there exists a collective choice rule that generates transitive (and thus quasi-transitive) social preferences that have this given filter as the family of decisive coalitions. A corollary of our observations is that reflexivity and completeness are redundant in the case of quasi-transitive social preferences: the decisiveness structures are unchanged if these two richness properties are added.

We then move on to an environment where social preferences are assumed to satisfy alternative weakenings of transitivity, namely, acyclicity or Suzumura consistency. Acyclicity rules out the existence of strict preference cycles, whereas Suzumura consistency (Suzumura, 1976) eliminates the existence of cycles with at least one strict preference. Transitivity implies Suzumura consistency which, in turn, implies acyclicity. Quasi-transitivity is intermediate in strength between transitivity and acyclicity as well and it is independent of Suzumura consistency. In the case of a reflexive and complete relation, transitivity and Suzumura consistency are equivalent. Suzumura consistency is an interesting property because it is *necessary and sufficient* for the existence of an *ordering extension* (Suzumura, 1976)—that is, a relation can be extended to an ordering respecting all weak and strict preferences of the underlying relation if and only if the original relation is Suzumura consistent. This fundamental insight represents a significant strengthening of the classical extension theorem established by Szpilrajn (1930) who showed that transitivity is sufficient for the existence of such an extension. Moreover, Suzumura consistency of a relation corresponds precisely to the requirement that an agent with such a relation is not a 'money pump' according to a well-known illustration of 'incoherent' preferences (Raiffa, 1968, p.78). See Bossert (2008) for a brief overview of recent applications of

Suzumura consistency.

Both acyclicity and Suzumura consistency are too weak to allow for the standard notion of decisive coalitions. The reason for this observation is that they are not sufficient to establish results analogous to Sen's (1995) *field expansion lemma.* Loosely speaking, the field expansion lemma, variants of which can be established for transitive and for quasi-transitive social preference relations, states that decisiveness properties of a coalition can be expanded to all pairs of alternatives as soon as decisiveness is established for a specific pair. Thus, the best one can hope for is a notion of decisiveness that is alternative-dependent. Moreover, a coalition cannot be decisive in the usual sense in this environment because its power to enforce a strict preference for an alternative over another may depend on the preferences of the remaining members of society. Thus, a notion of decisiveness that applies to pairs of coalitions has to be employed. This property relates to the absence of cycles in chains of pairs of coalitions with the above-mentioned alternative-dependent definition of decisiveness. Because of the above-described dependence on the alternatives to be compared, the results are considerably more complex than in the transitive (or, equivalently, the quasi-transitive) case. On the other hand, the results obtained for acyclicity and Suzumura consistency parallel those for the pair consisting of transitivity and quasi-transitivity in one respect: acyclicity and Suzumura consistency lead to the same structure of alternative-dependent decisiveness structures, just as is the case for transitivity and quasi-transitivity.

In order to obtain a notion of decisiveness that is not alternative-dependent (and thus easier to express), we strengthen independence of irrelevant alternatives to neutrality. This is not necessary in the case of (quasi-)transitive social preferences due to the applicability of suitable versions of the field expansion lemma. We show that alternative-independent versions of the requisite decisiveness properties can be established and characterized in this case.

In the following section, we introduce our basic definitions. A brief review of Hansson's (1976) results is given in Section 3. Section 4 generalizes these observations by examining transitive and quasi-transitive social relations without reflexivity and without completeness. Acyclicity and Suzumura consistency are analyzed in the Arrovian setting in Section 5, and Section 6 explores the consequences of strengthening independence of irrelevant alternatives to neutrality. Section 7 concludes.

2 Collective choice rules

Suppose there is a (finite or infinite) set of alternatives X containing at least three elements. We identify the population with the set N, where N could be finite or infinite and contains at least two individuals. A (binary) relation on X is a subset R of the Cartesian product $X \times X$. The set of all relations on X is denoted by $\mathcal{B}$. For notational convenience, we write xRy instead of $(x, y) \in R$ whenever possible without creating ambiguities. The *asymmetric part* P of a relation R is defined by

$$xPy \Leftrightarrow [xRy \text{ and } \neg\, yRx]$$

for all $x, y \in X$. The *symmetric part* I of R is defined by

$$xIy \Leftrightarrow [xRy \text{ and } yRx]$$

for all $x, y \in X$. Analogously, the asymmetric and symmetric parts of a relation R' are denoted by P' and I' etc. If R is interpreted as a *weak preference relation,* P is the *strict preference relation* corresponding to R and I is the *indifference relation* corresponding to R.

A relation R is *reflexive* if and only if xRx for all $x \in X$. R is *complete* if and only if

$$xRy \text{ or } yRx$$

for all $x, y \in X$ such that $x \neq y$. We refer to reflexivity and completeness as *richness* conditions for binary relations because they require that certain pairs be in a relation.

R is *transitive* if and only if, for all $x, y, z \in X$,

$$[xRy \text{ and } yRz] \Rightarrow xRz$$

and R is *quasi-transitive* if and only if P is transitive. R is *acyclical* if and only if, for all $x, y \in X$,

$$\left.\begin{array}{c}\text{there exist } K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that} \\ x = x^0, x^{k-1}Px^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\end{array}\right\} \Rightarrow \neg\, yPx$$

and, finally, R is *Suzumura consistent* if and only if, for all $x, y \in X$,

$$\left.\begin{array}{c}\text{there exist } K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that} \\ x = x^0, x^{k-1}Rx^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\end{array}\right\} \Rightarrow \neg\, yPx.$$

Transitivity, quasi-transitivity, acyclicity and Suzumura consistency are referred to as *coherence* properties because they demand that if certain pairs are in R, then others must be in R as well (as is the case for transitivity and quasi-transitivity), or that certain others cannot be in R (as is the case for acyclicity and Suzumura consistency).

A reflexive, complete and transitive relation is an *ordering.* Transitivity, quasi-transitivity and acyclicity are well-established coherence properties. Suzumura consistency was introduced by Suzumura (1976) who showed that it is equivalent to the existence of an *ordering extension* for a relation R, thereby generalizing Szpilrajn's (1930) fundamental extension result. A relation R' is an extension of a relation R if

$$R \subseteq R' \text{ and } P \subseteq P'. \tag{1}$$

R' is an ordering extension of R if and only if (1) is satisfied and R' is an ordering.

Transitivity implies quasi-transitivity which, in turn, implies acyclicity. Analogously, transitivity is stronger than Suzumura consistency which is stronger than acyclicity. Quasi-transitivity and Suzumura consistency are independent. If a relation is reflexive and complete, transitivity and Suzumura consistency are equivalent.

We use $\mathcal{T}$, $\mathcal{Q}$, $\mathcal{A}$, $\mathcal{C}$ to denote the set of all transitive, quasi-transitive, acyclical, Suzumura consistent relations on X, respectively. Furthermore, the set of all orderings on X is denoted by $\mathcal{R}$. A social relation is an element R of $\mathcal{B}$. We assume that each individual $n \in N$ ranks the elements of X by means of an ordering $R_n \in \mathcal{R}$ with asymmetric part P_n and symmetric part I_n. A *profile* is a vector $\mathbf{R} = \langle R_n \rangle_{n \in N}$ of orderings on X, one for each member of society. The set of all such profiles is denoted by $\mathcal{R}^N$.

A *collective choice rule* is a mapping $W \colon \mathcal{R}^N \to \mathcal{B}$, that is, an unlimited domain assumption is built into the definition of W. The interpretation is that, for a profile $\mathbf{R} \in \mathcal{R}^N$, $W(\mathbf{R})$ is the social ranking of alternatives in X. We use $R = W(\mathbf{R})$ to denote the social preference associated with the profile $\mathbf{R}$ with the strict preference relation P and the indifference relation I. If $W(\mathcal{R}^N) \subseteq \mathcal{T}$, W is a *transitive collective choice rule;* if $W(\mathcal{R}^N) \subseteq \mathcal{Q}$, W is a *quasi-transitive collective choice rule;* if $W(\mathcal{R}^N) \subseteq \mathcal{A}$, W is an *acyclical collective choice rule;* if $W(\mathcal{R}^N) \subseteq \mathcal{C}$, W is a *Suzumura consistent collective choice rule.* Finally, if $W(\mathcal{R}^N) \subseteq \mathcal{R}$, W is a *social welfare function.*

Arrow (1951; 1963) imposed the axioms of *unlimited domain, weak Pareto* and *independence of irrelevant alternatives* and showed that, in the case of a finite population, the resulting social welfare functions are *dictatorial:* there exists an individual such that, whenever this individual strictly prefers one alternative over another, this strict preference is reproduced in the social ranking, irrespective of the preferences of other members of society. As mentioned above, unlimited domain is already imposed by assuming that the domain of W is given by $\mathcal{R}^N$. The remaining two Arrow axioms are defined as follows.

Weak Pareto. For all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^N$,

$$[xP_n y \text{ for all } n \in N] \;\Rightarrow\; xPy.$$

Independence of irrelevant alternatives. For all $x, y \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$,

$$[xR_n y \Leftrightarrow xR'_n y \text{ and } yR_n x \Leftrightarrow yR'_n x] \;\text{ for all } n \in N \;\Rightarrow\; [xRy \Leftrightarrow xR'y \text{ and } yRx \Leftrightarrow yR'x].$$

Let $x, y \in X$ be distinct. A set of individuals $M \subseteq N$ (also referred to as a *coalition*) is *decisive for x over y for a collective choice rule W* (in short, M is $d_W(x, y)$) if and only if, for all $\mathbf{R} \in \mathcal{R}^N$,

$$[xP_m y \;\text{ for all } m \in M] \;\Rightarrow\; xPy.$$

Furthermore, a set $M \subseteq N$ is *decisive for W* if and only if M is $d_W(x, y)$ for all distinct $x, y \in X$. Clearly, N is decisive for any collective choice rule satisfying weak Pareto. If there is an individual $n \in N$ such that $\{n\}$ is decisive for W, individual n is a *dictator for W*. Let $\mathcal{D}_W$ denote the set of all decisive coalitions for a collective choice rule W.

3 Hansson's results

Before summarizing Hansson's (1976) contributions, we require some observations regarding *filters* and *ultrafilters.* A filter on N is a collection $\mathcal{F}$ of subsets of N such that

- f.1. $N \in \mathcal{F}$;
- f.2. $\emptyset \notin \mathcal{F}$;
- f.3. for all $M, M' \subseteq N$, $([M \in \mathcal{F} \text{ and } M \subseteq M'] \;\Rightarrow\; M' \in \mathcal{F})$;
- f.4. for all $M, M' \in \mathcal{F}$, $M \cap M' \in \mathcal{F}$.

An ultrafilter on N is a *maximal* filter $\mathcal{U}$ on N in the sense that, within the class of all filters on N, $\mathcal{U}$ is undominated in terms of set inclusion. That is, a filter $\mathcal{U}$ on N is an ultrafilter on N if there does not exist a filter $\mathcal{F}$ on N such that $\mathcal{U}$ is a strict subset of $\mathcal{F}$. It is well-known that an equivalent definition of an ultrafilter is the following. An ultrafilter on N is a collection $\mathcal{U}$ of subsets of N such that

- u.1. $\emptyset \notin \mathcal{U}$;
- u.2. for all $M, M' \subseteq N$, $([M \in \mathcal{U} \text{ and } M \subseteq M'] \;\Rightarrow\; M' \in \mathcal{U})$;
- u.3. for all $M, M' \in \mathcal{U}$, $M \cap M' \in \mathcal{U}$;
- u.4. for all $M \subseteq N$, $[M \in \mathcal{U} \;\text{ or }\; N \setminus M \in \mathcal{U}]$.

Note that the conjunction of properties $u.1$ and $u.4$ implies that $N \in \mathcal{U}$ and, furthermore, the conjunction of properties $u.1$ and $u.3$ implies that the disjunction in property $u.4$ is exclusive—that is, M and $N \setminus M$ cannot both be in $\mathcal{U}$. Moreover, the conjunction of $u.3$ and $u.4$ implies that

$$\text{for all } M \in \mathcal{U} \text{ and for all } M' \subseteq M,\ [M' \in \mathcal{U} \ \text{ or } \ M \setminus M' \in \mathcal{U}]. \tag{2}$$

To see this, let $M \in \mathcal{U}$ and $M' \subseteq M$. If $M' \in \mathcal{U}$, we are done. If not, $u.4$ implies $N \setminus M' = (N \setminus M) \cup (M \setminus M') \in \mathcal{U}$. By $u.3$, $M \cap [(N \setminus M) \cup (M \setminus M')] = M \setminus M' \in \mathcal{U}$ and (2) is established.

To prove the equivalence of the two above definitions of an ultrafilter, suppose first that $\mathcal{U}$ is a collection of subsets of N satisfying $u.1$ through $u.4$. Clearly, $\mathcal{U}$ is a filter on N because the conjunction of $u.1$ through $u.4$ implies the conjunction of $f.1$ through $f.4$. By way of contradiction, suppose $\mathcal{U}$ is not undominated in terms of set inclusion. Then there exists a filter $\mathcal{F}$ on N such that $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{F} \setminus \mathcal{U} \neq \emptyset$. Let $M \in \mathcal{F} \setminus \mathcal{U}$. Because $M \notin \mathcal{U}$, $N \setminus M \in \mathcal{U}$ by $u.4$. Because $\mathcal{U} \subseteq \mathcal{F}$, $N \setminus M \in \mathcal{F}$. By $u.3$, $M \cap (N \setminus M) = \emptyset \in \mathcal{F}$, contradicting $f.2$.

Now let $\mathcal{U}$ be a maximal filter on N and, by way of contradiction, suppose that $\mathcal{U}$ does not satisfy $u.1$ through $u.4$. $u.1$ through $u.3$ are satisfied because they are equivalent to $f.2$ through $f.4$ and $\mathcal{U}$ is assumed to be a filter on N. Thus, $u.4$ must be violated and there exists $M \subseteq N$ such that $M \notin \mathcal{U}$ and $N \setminus M \notin \mathcal{U}$. Because $\mathcal{U}$ is a filter, neither M nor $N \setminus M$ can be empty (and, thus, neither M nor $N \setminus M$ can be equal to N), neither M nor $N \setminus M$ can be a strict superset of an element of $\mathcal{U}$ and neither M nor $N \setminus M$ can be the intersection of two elements in $\mathcal{U}$. Define $\mathcal{F} = \mathcal{U} \cup \{M\}$. Then $\mathcal{F}$ is a filter because the addition of M to $\mathcal{U}$ does not require any other additions to $\mathcal{U}$ as we have just established. But because $M \notin \mathcal{U}$ by assumption, this means that $\mathcal{U}$ is not maximal, a contradiction.

An ultrafilter $\mathcal{U}$ is *principal* if there exists an $n \in N$ such that, for all $M \subseteq N$, $M \in \mathcal{U}$ if and only if $n \in M$. Otherwise, $\mathcal{U}$ is a *free* ultrafilter. If N is a finite set, then *all* ultrafilters are principal: starting with the finite set N which is an element of $\mathcal{U}$, we can apply (2) repeatedly until we reach an $n \in N$ such that $\{n\} \in \mathcal{U}$. By $u.2$, this immediately implies that $\mathcal{U}$ is principal. If N is infinite, however, there also exist free ultrafilters but they cannot be defined explicitly; the proof of their existence relies on non-constructive methods such as the axiom of choice.

Hansson (1976) has shown that if a social welfare function W satisfies unlimited domain, weak Pareto and independence of irrelevant alternatives, then $\mathcal{D}_W$ must be an ultrafilter on N. Conversely, if $\mathcal{U}$ is an ultrafilter on N, then there exists a social welfare function W satisfying weak Pareto and independence of irrelevant alternatives such that $\mathcal{D}_W = \mathcal{U}$.

Using the above observations on principal ultrafilters on finite sets, we can use Hansson's results to obtain Arrow's (1951; 1963) theorem for the special case of a finite population: if the set of decisive coalitions contains a singleton $\{n\}$, this singleton is a dictator. Because the set of decisive coalitions is an ultrafilter and all ultrafilters are principal if N is finite, there exists an individual n who is a dictator.

In the infinite-population case, a set of decisive coalitions that is a principal ultrafilter corresponds to a dictatorship just as in the finite case. However, because not all ultrafilters are principal in this case, Arrow's axioms allow for non-dictatorial social welfare functions—namely, those whose sets of decisive coalitions correspond to free ultrafilters.

Hansson (1976) also considered the case where the social relation is merely required to be quasi-transitive rather than transitive (but retaining the reflexivity and completeness assumptions). In that case, the set of decisive coalitions $\mathcal{D}_W$ is not necessarily an ultrafilter but it still is a filter whenever W satisfies weak Pareto and independence of irrelevant alternatives. In analogy to the corresponding observation for ultrafilters, for any filter $\mathcal{F}$, there exists a collective choice rule W that yields reflexive, complete and quasi-transitive social relations and satisfies weak Pareto and independence of irrelevant alternatives such that $\mathcal{D}_W = \mathcal{F}$.

4 Hansson's theorems without richness properties

As a first step in our analysis of the structure of decisive coalitions in the absence of the richness properties of reflexivity and completeness, we reexamine Hansson's (1976) observations involving transitive and quasi-transitive social relations satisfying the Arrow axioms. Although the resulting decisiveness structures can be recovered following steps analogous to those employed by Hansson himself, there is an interesting difference: once reflexivity and completeness are dropped, the families of decisive coalitions associated with transitive collective choice rules and with quasi-transitive collective choice rules can no longer be distinguished. Intuitively, this is the case because only strict preferences are necessarily imposed by weak Pareto and, in the absence of completeness, an absence of strict preference does not imply a weak preference in the reverse direction. Moreover, as a corollary to the results of this section, it will become clear that quasi-transitivity with reflexivity and completeness is equivalent to quasi-transitivity without reflexivity and completeness, whereas transitivity without reflexivity and completeness results in weaker structural properties of the family of decisive coalitions—namely, the same structure that obtains for quasi-transitivity. It is worth noting at this stage that the notion of a decisive coalition continues to be well-defined; this is in contrast with the structures to be uncovered in the following section where we consider alternative coherence properties.

Our first two results strengthen Hansson's (1976) observations regarding reflexive, complete and quasi-transitive collective choice rules. We show that reflexivity and completeness can be dispensed with and yet the family of decisive coalitions is well-defined and forms a filter in the presence of weak Pareto and independence of irrelevant alternatives. As a preliminary result, we show that the conclusion of Sen's *field expansion lemma* (Sen, 1995, p.4) remains valid without reflexivity and completeness.

Lemma 1 *Let W be a quasi-transitive collective choice rule that satisfies weak Pareto and independence of irrelevant alternatives. Let $x, y \in X$ be distinct and let $M \subseteq N$. If M is $d_W(x, y)$, then $M \in \mathcal{D}_W$.*

Proof. Let W be a quasi-transitive collective choice rule that satisfies the two Arrow axioms, let $x, y \in X$ be distinct and let $M \subseteq N$ be $d_W(x, y)$. We have to establish that M is $d_W(z, w)$ for any choice of distinct alternatives z and w. Thus, we have to show that M is:

(i) $d_W(z, w)$ for all distinct $z, w \in X \setminus \{x, y\}$;

(ii) $d_W(x, z)$ for all $z \in X \setminus \{x, y\}$;

(iii) $d_W(z, y)$ for all $z \in X \setminus \{x, y\}$;

(iv) $d_W(z, x)$ for all $z \in X \setminus \{x, y\}$;

(v) $d_W(y, z)$ for all $z \in X \setminus \{x, y\}$;

(vi) $d_W(y, x)$.

(i) Because W has an unlimited domain, we can consider a profile $\mathbf{R} \in \mathcal{R}^N$ such that

$$\begin{aligned} zP_m x P_m y P_m w \qquad & \text{for all } m \in M, \\ zP_m x \ \text{ and } \ yP_m w \qquad & \text{for all } m \in N \setminus M. \end{aligned}$$

By weak Pareto, zPx and yPw. Because M is $d_W(x, y)$, we have xPy. By quasi-transitivity, zPw. Because of independence of irrelevant alternatives, this social preference cannot depend on individual preferences over pairs of alternatives other than z

and w. The ranking of z and w is not specified for individuals outside of M and, thus, M is $d_W(z, w)$.

(ii) Consider a profile $\mathbf{R} \in \mathcal{R}^N$ such that

$$\begin{aligned} xP_m yP_m z \qquad & \text{for all } m \in M, \\ yP_m z \qquad & \text{for all } m \in N \setminus M. \end{aligned}$$

Because M is $d_W(x, y)$, we have xPy. By weak Pareto, yPz. By quasi-transitivity, xPz and it follows as in the proof of (i) that M is $d_W(x, z)$.

(iii) Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\begin{aligned} zP_m xP_m y \qquad & \text{for all } m \in M, \\ zP_m x \qquad & \text{for all } m \in N \setminus M. \end{aligned}$$

By weak Pareto, zPx. Because M is $d_W(x, y)$, we have xPy. By quasi-transitivity, zPy and it follows as in the proof of (i) and (ii) that M is $d_W(z, y)$.

(iv) Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\begin{aligned} zP_m yP_m x \qquad & \text{for all } m \in M, \\ yP_m x \qquad & \text{for all } m \in N \setminus M. \end{aligned}$$

By (iii), zPy. Weak Pareto implies yPx and, by quasi-transitivity, we obtain zPx. As in the earlier cases, it follows that M is $d_W(z, x)$.

(v) Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\begin{aligned} yP_m xP_m z \qquad & \text{for all } m \in M, \\ yP_m x \qquad & \text{for all } m \in N \setminus M. \end{aligned}$$

By weak Pareto, yPx. By (ii), we have xPz. By quasi-transitivity, yPz and it follows that M is $d_W(y, z)$.

(vi) Let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$yP_m zP_m x \qquad \text{for all } m \in M.$$

By (v), yPz and by (iv), zPx. By quasi-transitivity, yPx and it follows that M is $d_W(y, x)$.
■

Hansson's (1976) theorem remains valid even in the absence of reflexivity and completeness. Thus, we obtain the following result.

Theorem 1 *If a quasi-transitive collective choice rule W satisfies weak Pareto and independence of irrelevant alternatives, then $\mathcal{D}_W$ is a filter on N.*

Proof. Suppose W is a quasi-transitive collective choice rule that satisfies weak Pareto and independence of irrelevant alternatives. We need to show that $\mathcal{D}_W$ has the four properties of a filter.

f.1. This property is an immediate consequence of weak Pareto.

f.2. If $\emptyset \in \mathcal{D}_W$, we obtain xPy and yPx for any two distinct alternatives $x, y \in X$ and for any profile $\mathbf{R} \in \mathcal{R}^N$ such that all individuals are indifferent between x and y, which is impossible. Thus, $\emptyset \notin \mathcal{D}_W$.

f.3. This property follows immediately from the definition of decisiveness.

f.4. Suppose $M, M' \in \mathcal{D}_W$. Let $x, y, z \in X$ be pairwise distinct and let $\mathbf{R} \in \mathcal{R}^N$ be such that

$$\begin{aligned} yP_m x \text{ and } zP_m x \qquad & \text{for all } m \in M \setminus M', \\ zP_m x P_m y \qquad & \text{for all } m \in M \cap M', \\ xP_m y \text{ and } xP_m z \qquad & \text{for all } m \in M' \setminus M. \end{aligned}$$

Because M is decisive, we have zPx. Because M' is decisive, we have xPy. By quasi-transitivity, zPy. This implies that $M \cap M'$ is $d_W(z, y)$ because the preferences of individuals outside of $M \cap M'$ over z and y are not specified. By Lemma 1, $M \cap M' \in \mathcal{D}_W$. ■

As an immediate corollary of Theorem 1, it follows that if W is a transitive rather than a quasi-transitive collective choice rule, the same conclusion holds: the family of decisive coalitions must form a filter in the presence of the two Arrow axioms.

We now examine the possibility of establishing the reverse implication of that in Theorem 1 in order to see whether the filter structure exhausts all implications of the requisite axioms. This is indeed the case, and an even stronger result is valid: given any filter $\mathcal{F}$ on N, it follows that there exists a *transitive* (and not merely a quasi-transitive) collective choice rule W satisfying the axioms such that $\mathcal{D}_W = \mathcal{F}$.

Theorem 2 *For any filter $\mathcal{F}$ on N, there exists a transitive collective choice rule W satisfying weak Pareto and independence of irrelevant alternatives such that $\mathcal{D}_W = \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a filter on N. Define a collective choice rule W by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$xRy \;\Leftrightarrow\; \{m \in N \mid xP_m y\} \in \mathcal{F}.$$

We first prove that R is transitive for all possible profiles $\mathbf{R}$. Suppose $\mathbf{R} \in \mathcal{R}^N$ and $x, y, z \in X$ are such that xRy and yRz. By definition of W,

$$\{m \in N \mid xP_m y\} \in \mathcal{F} \;\text{ and }\; \{m \in N \mid yP_m z\} \in \mathcal{F}.$$

By f.4,

$$\{m \in N \mid xP_m y\} \cap \{m \in N \mid yP_m z\} \in \mathcal{F}.$$

Because individual preferences are transitive, it follows that

$$\{m \in N \mid xP_m y\} \cap \{m \in N \mid yP_m z\} \subseteq \{m \in N \mid xP_m z\}$$

and, by f.3, $\{m \in N \mid xP_m z\} \in \mathcal{F}$. Thus, by definition of W, xRz as was to be established.

That weak Pareto is satisfied follows immediately from f.1 and f.2. Independence of irrelevant alternatives is satisfied because the social ranking of any two alternatives x and y is defined exclusively in terms of the individual rankings of x and y.

It remains to show that $\mathcal{D}_W = \mathcal{F}$. As a first step, we show that R is asymmetric for all possible profiles. By way of contradiction, suppose $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ are such that xRy and yRx. By definition of W, this means that

$$\{m \in N \mid xP_m y\} \in \mathcal{F} \;\text{ and }\; \{m \in N \mid yP_m x\} \in \mathcal{F}.$$

By f.3,

$$\{m \in N \mid xP_m y\} \cap \{m \in N \mid yP_m x\} = \emptyset \in \mathcal{F},$$

contradicting f.2.

To prove that $\mathcal{D}_W \subseteq \mathcal{F}$, suppose that $M \in \mathcal{D}_W$. Let $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ be such that $xP_m y$ for all $m \in M$. By definition of a decisive coalition, this implies xPy and, thus, xRy. By definition of W, this implies $M \in \mathcal{F}$.

To complete the proof, suppose that $M \in \mathcal{F}$. Let $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ be such that $xP_m y$ for all $m \in M$. By definition of W, we obtain xRy. Because R is asymmetric, it follows that xPy. Thus, M is decisive and hence $M \in \mathcal{D}_W$. ■

Combining Theorems 1 and 2, it follows immediately that the class of transitive collective choice rules and the class of quasi-transitive collective choice rules have identical characterizations in terms of decisive coalitions. This contrasts with the case of collective choice rules that always generate reflexive and complete social relations where the two concepts are distinct.

5 Acyclicity and Suzumura consistency

If, in the absence of reflexivity and completeness, merely one of the properties acyclicity or Suzumura consistency is imposed on social preferences, the approach involving decisive coalitions and filters employed in the previous section must be amended. This is the case because the existence of such coalitions is not guaranteed and there is no equivalent of the field expansion lemma (Lemma 1) with acyclicity or with Suzumura consistency. Intuitively, this difference emerges because, unlike transitivity and quasi-transitivity, acyclicity and Suzumura consistency do not *force* certain additional pairs to be in a relation as a consequence of the presence of others but, rather, *prevent* certain pairs of alternatives to appear in the relation in order to avoid the respective cycles to be excluded. For that reason, property $f.4$ of a filter cannot be established because it relies on the transitivity of R or P. Moreover, even property $f.3$ must be abandoned. To see that this is the case, consider the following example which extends an example due to Bossert and Suzumura (2008a, p.316). Suppose $X = \{x, y, z\}$ and $N = \{1, \ldots, 7\}$. Define a collective choice rule by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$xRy \;\Leftrightarrow\; (|\{n \in N \mid yP_nx\}| = 0 \;\text{ or }\; [|\{n \in N \mid xP_ny\}| = 5 \;\text{ and }\; |\{n \in N \mid yP_nx\}| = 2]).$$

It can be verified that this collective choice rule is Suzumura consistent and satisfies weak Pareto and independence of irrelevant alternatives. However, even though xP_my for all $m \in \{1, \ldots, 5\}$ and yP_mx for all $m \in \{6, 7\}$ implies xPy, we obtain non-comparability of x and y rather than a strict preference for x over y if xP_my for all $m \in \{1, \ldots, 6\}$ and yP_7x. Thus, an expansion of a coalition that can force x to be preferred to y does *not* generate a coalition with that power. In addition, even the coalition $\{1, \ldots, 5\}$ cannot be considered decisive according to our earlier definition: if xP_my for all $m \in \{1, \ldots, 5\}$, xI_6y and yP_7x, non-comparability results again and the coalition $\{1, \ldots, 5\}$ is not decisive. Thus, the framework based on decisiveness and filters as families of decisive coalitions cannot be employed. In this context, it is interesting to compare our results to those of Banks (1995). Banks employs a monotonicity property that strengthens independence of irrelevant alternatives. This property is akin to non-negative responsiveness, requiring that a strict preference for x over y must be preserved if we move to a profile where the set of those preferring x to y and the set of those considering x at least as good as y weakly expand. Clearly, this property rules out examples of the above nature. See also Blair and Pollak (1982) for discussions of acyclical collective choice in the presence of the richness properties reflexivity and completeness. As is the case for Banks (1995), Blair and Pollak (1982) restrict attention to the finite-population case.

We now introduce a notion of decisiveness that is not a property of a coalition M but a property of a *pair* of coalitions (M, M') with the interpretation that M is decisive against M' if the social preference strictly prefers x to y whenever all members of M strictly prefer x to y *and* all members of M' strictly prefer y to x. In addition, because no result analogous to the field expansion lemma is valid, these pairs of coalitions may depend on the alternatives x and y to be compared. Note that such a dependence is ruled out in the transitive and quasi-transitive cases. However, if the social relation merely has to satisfy Suzumura consistency or acyclicity, it is straightforward to define examples that illustrate the dependence of the notion of decisiveness of one coalition against another on the alternatives under consideration. For example, suppose $x^0, y^0 \in X$ are distinct alternatives, and define a collective choice rule as follows. For all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$\begin{aligned} xRy \;\Leftrightarrow\; & ([xR_n y \text{ for all } n \in N] \text{ or } [\text{there exist } \ell, m \in N \text{ such that} \\ & xP_\ell y \text{ and } yP_m x \text{ and } (x,y) = (x^0, y^0)]). \end{aligned}$$

This collective choice rule generates Suzumura consistent (and, therefore, acyclical) social preferences and satisfies weak Pareto and independence of irrelevant alternatives. Furthermore, any coalition M consisting of at least one member can guarantee a strict social preference of x^0 over y^0 by expressing a strict preference for x^0 over y^0 against any other non-empty coalition M' the members of which all strictly prefer y^0 to x^0. But that coalition M cannot force a strict social preference of y^0 over x^0 against M'. See Bossert and Suzumura (2008a, p.319) for a discussion of this example in a related but different context.

Let W be a collective choice rule, let $M, M' \subseteq N$ be non-empty and let $x, y \in X$ be distinct. We say that *M is decisive against M' for W for x over y* (in symbols, $(M, M') \in \mathcal{O}_W(x,y)$) if and only if, for all $\mathbf{R} \in \mathcal{R}^N$,

$$[xP_m y \text{ for all } m \in M \text{ and } yP_m x \text{ for all } m \in M'] \;\Rightarrow\; xPy.$$

Clearly, $(M, M') \in \mathcal{O}_W(x,y)$ implies that $M \cap M' = \emptyset$ but it is not necessarily the case that $M \cup M' = N$.

For $x, y \in X$, let $\mathcal{G}(x, y)$ be a family of pairs of subsets of N. An *alternative-dependent cycle-free collection on* N is a collection $\langle \mathcal{G}(x, y) \rangle_{(x,y) \in X \times X}$ such that

$g.1.$ $(N, \emptyset) \in \mathcal{G}(x, y)$ for all $x, y \in X$;

$g.2.$ $(\emptyset, N) \notin \mathcal{G}(x, y)$ for all $x, y \in X$;

$g.3.$ for all $K \in \mathbb{N}$, for all non-empty $M^0, \ldots, M^K \subseteq N$ and for all $x^0, \ldots, x^K \in X$,

$$[(M^{k-1}, M^k) \in \mathcal{G}(x^{k-1}, x^k) \text{ for all } k \in \{1, \ldots, K\}] \Rightarrow (M^K, M^0) \notin \mathcal{G}(x^K, x^0).$$

The set of acyclical or Suzumura consistent collective choice rules can be identified in terms of an alternative-dependent cycle-free collection of pairs of coalitions. First, we obtain the following result.

Theorem 3 *If an acyclical collective choice rule W satisfies weak Pareto and independence of irrelevant alternatives, then $\langle \mathcal{O}_W(x, y) \rangle_{(x,y) \in X \times X}$ is an alternative-dependent cycle-free collection on N.*

Proof. Suppose W is an acyclical collective choice rule that satisfies weak Pareto and independence of irrelevant alternatives. That the collection $\langle \mathcal{O}_W(x, y) \rangle_{(x,y) \in X \times X}$ does not depend on any profile under consideration is an immediate consequence of independence of irrelevant alternatives. We need to prove that $\langle \mathcal{O}_W(x, y) \rangle_{(x,y) \in X \times X}$ is an alternative-dependent cycle-free collection.

The properties $g.1$ and $g.2$ follow from weak Pareto. To establish $g.3$, suppose, by way of contradiction, that there exist $K \in \mathbb{N}$, non-empty sets $M^0, \ldots, M^K \subseteq N$ and alternatives $x^0, \ldots, x^K \in X$ such that

$$[(M^{k-1}, M^k) \in \mathcal{O}_W(x^{k-1}, x^k) \text{ for all } k \in \{1, \ldots, K\}] \text{ and } (M^K, M^0) \in \mathcal{O}_W(x^K, x^0). \tag{3}$$

Define a profile $\mathbf{R} \in \mathcal{R}^N$ as follows. For all $k \in \{0, \ldots, K\}$, for all $m \in M^k$, for all $\ell \in \{0, \ldots, K\}$ such that $m \in M^k \cap M^\ell$ and for all $x, y \in X \setminus \bigcup_{\substack{j \in \{0, \ldots, K\}: \\ m \in M^k \cap M^j}} \{x^j\}$,

$$x^\ell I_m x^k P_m x I_m y.$$

Furthermore, for all $m \in N \setminus \bigcup_{k \in \{0, \ldots, K\}} M^k$ and for all $x, y \in X$,

$$x I_m y.$$

Clearly, $\mathbf{R} \in \mathcal{R}^N$. By definition, $M^{k-1} \cap M^k = \emptyset$ for all $k \in \{1, \ldots, K\}$ and $M^K \cap M^0 = \emptyset$. Thus, for all $k \in \{1, \ldots, K\}$,

$$[x^{k-1} P_m x^k \text{ for all } m \in M^{k-1}] \text{ and } [x^k P_m x^{k-1} \text{ for all } m \in M^k]$$

and

$$[x^K P_m x^0 \text{ for all } m \in M^K] \text{ and } [x^0 P_m x^K \text{ for all } m \in M^0].$$

By (3) and by definition of $\langle \mathcal{O}_W(x,y)\rangle_{(x,y)\in X\times X}$, it follows that

$$x^{k-1} P x^k \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K P x^0,$$

contradicting the acyclicity of R. ■

We conclude this section by showing that Theorem 3 is 'tight' in the sense that the conjunction of properties $g.1$, $g.2$ and $g.3$ is all that can be deduced from the assumptions in the theorem statement. Moreover, as is the case for transitive and quasi-transitive collective choice rules, Suzumura consistency and acyclicity cannot be distinguished in this framework.

Theorem 4 *For any alternative-dependent cycle-free collection $\langle \mathcal{G}(x,y)\rangle_{(x,y)\in X\times X}$ on N, there exists a Suzumura consistent collective choice rule W satisfying weak Pareto and independence of irrelevant alternatives such that $\langle \mathcal{O}_W(x,y)\rangle_{(x,y)\in X\times X} = \langle \mathcal{G}(x,y)\rangle_{(x,y)\in X\times X}$.*

Proof. Let $\langle \mathcal{G}(x,y)\rangle_{(x,y)\in X\times X}$ be an alternative-dependent cycle-free collection on N. Define a collective choice rule W by letting, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$xRy \Leftrightarrow (\{m \in N \mid xP_m y\}, \{m \in N \mid yP_m x\}) \in \mathcal{G}(x,y).$$

We now prove that R is Suzumura consistent for all possible profiles $\mathbf{R}$. Suppose this is not the case. Then there exist $\mathbf{R} \in \mathcal{R}^N$, $K \in \mathbb{N}$ and alternatives $x^0, \ldots, x^K \in X$ such that $x^{k-1} R x^k$ for all $k \in \{1,\ldots,K\}$ and $x^K P x^0$. By definition of W,

$$(\{m \in N \mid x^{k-1} P_m x^k\}, \{m \in N \mid x^k P_m x^{k-1}\}) \in \mathcal{G}(x^{k-1}, x^k) \text{ for all } k \in \{1,\ldots,K\}$$

and

$$(\{m \in N \mid x^K P_m x^0\}, \{m \in N \mid x^0 P_m x^K\}) \in \mathcal{G}(x^K, x^0).$$

This contradicts $g.3$.

That weak Pareto is satisfied follows immediately from $g.1$ and $g.2$. Independence of irrelevant alternatives is satisfied because the social ranking of any two alternatives x and y is defined exclusively in terms of the individual rankings of x and y.

It remains to show that $\langle \mathcal{O}_W(x,y)\rangle_{(x,y)\in X\times X} = \langle \mathcal{G}(x,y)\rangle_{(x,y)\in X\times X}$. As a first step, we show that R is asymmetric for all possible profiles. By way of contradiction, suppose $\mathbf{R} \in \mathcal{R}^N$ and $x, y \in X$ are such that xRy and yRx. By definition of W, this means that

$$(\{m \in N \mid xP_m y\}, \{m \in N \mid yP_m x\}) \in \mathcal{G}(x,y)$$

and

$$(\{m \in N \mid yP_m x\}, \{m \in N \mid xP_m y\}) \in \mathcal{G}(y,x).$$

Setting $K = 1$, $M^0 = \{m \in N \mid xP_m y\}$, $M^1 = \{m \in N \mid yP_m x\}$, $x^0 = x$ and $x^1 = y$ yields a contradiction to $g.3$.

To prove that $\mathcal{O}_W(x,y) \subseteq \mathcal{G}(x,y)$ for all $(x,y) \in X \times X$, suppose that $(x,y) \in X \times X$ and $(M, M') \in \mathcal{O}_W(x,y)$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that $xP_m y$ for all $m \in M$ and $yP_m x$ for all $m \in M'$. By definition of $\mathcal{O}_W(x,y)$, this implies xPy and, thus, xRy. By definition of W, this implies $(M, M') \in \mathcal{G}(x,y)$.

To complete the proof, suppose that $(M, M') \in \mathcal{G}(x,y)$. Let $\mathbf{R} \in \mathcal{R}^N$ be such that $xP_m y$ for all $m \in M$ and $yP_m x$ for all $m \in M'$. By definition of W, we obtain xRy. Because R is asymmetric, it follows that xPy. Thus, M is decisive against M' for W for x over y and hence $(M, M') \in \mathcal{O}_W(x,y)$. ■

6 Neutrality

The structure described in the results of the previous section is relatively complex because of the dependence of the relevant pairs of coalitions on the alternatives under consideration. To obtain a clearer picture, one possibility is to strengthen independence of irrelevant alternatives to *neutrality.*

Neutrality. For all $x, y, x', y' \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$,

$$[xR_n y \Leftrightarrow x'R'_n y' \text{ and } yR_n x \Leftrightarrow y'R'_n x'] \text{ for all } n \in N \Rightarrow [xRy \Leftrightarrow x'R'y' \text{ and } yRx \Leftrightarrow y'R'x'].$$

Let W be a collective choice rule and let $M, M' \subseteq N$ be non-empty. We say that M *is decisive against M' for W* (in symbols, $(M, M') \in \mathcal{S}_W$) if and only if, for all $\mathbf{R} \in \mathcal{R}^N$ and for all $x, y \in X$,

$$[xP_m y \text{ for all } m \in M \text{ and } yP_m x \text{ for all } m \in M'] \Rightarrow xPy.$$

Let $\mathcal{H}$ be a family of pairs of subsets of N. $\mathcal{H}$ is a *cycle-free collection* if it satisfies

$h.1.$ $(N, \emptyset) \in \mathcal{H}$;

$h.2.$ $(\emptyset, N) \notin \mathcal{H}$;

$h.3.$ for all $K \in \mathbb{N}$ and for all non-empty $M^0, \ldots, M^K \subseteq N$,

$$[(M^{k-1}, M^k) \in \mathcal{H} \text{ for all } k \in \{1, \ldots, K\}] \Rightarrow (M^K, M^0) \notin \mathcal{H}.$$

Replacing independence of irrelevant alternatives with neutrality, we obtain the following results. They show that a cycle-free structure regarding the relevant pairs obtains but, unlike in the previous section, this structure is no longer dependent on the alternatives under consideration. This can be seen easily by noting that neutrality (which strengthens independence of irrelevant alternatives) implies that any pair (M, M') such that M is decisive against M' for W cannot depend on the alternatives to be compared. The rest of the proofs of the following two theorems then follows as in the previous section.

Theorem 5 *If an acyclical collective choice rule W satisfies weak Pareto and neutrality, then $\mathcal{S}_W$ is a cycle-free collection on N.*

Theorem 6 *For any cycle-free collection $\mathcal{H}$ on N, there exists a Suzumura consistent collective choice rule W satisfying weak Pareto and neutrality such that $\mathcal{S}_W = \mathcal{H}$.*

7 Concluding remarks

A question that might arise in relation to the issues addressed in this paper is whether the concepts developed here could usefully be applied to single-profile social choice problems. We think this poses some conceptual difficulties which is why we do not pursue this matter in detail. Samuelson (1977) heavily criticized an assumption that is central to the Arrow-type impossibilities established by authors such as Kemp and Ng (1976) and Parks (1976)—namely, the axiom that combines independence with a notion of neutrality. Our concern regarding the single-profile approach in the context analyzed in this paper is the following. In addition to an axiom playing the role of multi-profile independence, the single-profile approach must rely on some form of preference-diversity assumption in order to construct the preferences over triples of alternatives required. These assumptions (for example, the axiom of unrestricted domain over triples employed by Pollak, 1979) tend to impose rather stringent restrictions on the number of alternatives relative to the number of agents; in particular, the set of alternatives must be rather large. The infinite-population case is of crucial importance in our setting and an analysis of single-profile analogues of our concepts would require to restrict attention to infinite sets of alternatives. When combined with Samuelson's (1977) criticism, it seems to us that there is little to be gained from embarking upon an investigation of that nature.

To the best of our knowledge, the notions of decisiveness developed in the previous two sections have not appeared in the earlier literature. Especially the alternative-dependent variant may be worthwhile to be explored further. For instance, it might be possible to link it to established mathematical structures just as those of filters and ultrafilters that emerge naturally in the transitive and quasi-transitive cases. Moreover, further properties of such families of sets could be studied. Some steps in this spirit have already been taken; see, for instance, Cato (2008) for alternative definitions of decisiveness and their properties.

The results of this paper are established in a general framework in the sense that (with the exception of those reported in the previous section) they do not go beyond the original Arrovian setting—that is, we restrict attention to the axioms employed in Arrow's impossibility theorem. There have been approaches that examine to what extent weakenings of the requirements imposed on social relations allow us to obtain collective choice rules that may have additional properties such as anonymity or compliance with the strong rather than merely the weak Pareto principle; see, in particular, Sen (1969; 1970), Weymark (1984), Bossert and Suzumura (2008a) and Cato and Hirata (2009). Especially in the acyclical and Suzumura consistent cases, there appears to be room for explorations in a setting involving additional properties and their consequences.

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