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Harsanyi's social aggregation theorem: a multi-profile approach with variable-population extensions*

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Abstract

This paper provides new versions of Harsanyi's social aggregation theorem that are formulated in terms of prospects rather than lotteries. Strengthening an earlier result, fixed-population ex-ante utilitarianism is characterized in a multi-profile setting with fixed probabilities. In addition, we extend the social aggregation theorem to social-evaluation problems under uncertainty with a variable population and generalize our approach to uncertain alternatives, which consist of compound vectors of probability distributions and prospects.

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1. Introduction

Harsanyi's [1955, 1977] social aggregation theorem establishes that if individuals and society are endowed with von Neumann – Morgenstern (vNM) functions (von Neumann and Morgenstern [1944, 1947]) defined on the set of lotteries over alternatives and the Pareto-indifference axiom formulated for lotteries is satisfied, then, under some regularity assumptions, the social vNM function can be written as an affine combination of the individual vNM functions. If, as Harsanyi assumes, there are individual vNM functions that measure actual (ex-post) utility, utilitarianism results. This result has been discussed extensively in the literature and variants of it have been established in a number of contributions; see, for example, Blackorby, Donaldson and Weymark [1980, 1999, 2003], Broome [1990, 1991], Coulhon and Mongin [1989], De Meyer and Mongin [1995], Domotor [1979], Fishburn [1984], Hammond [1981, 1983], Mongin [1994, 1995, 1998], Mongin and d'Aspremont [1998], Weymark [1991, 1993, 1994, 1995] and Zhou [1997]. An elegant proof due to Border [1981] is reproduced in expanded form in Weymark [1994].

Most of the literature focuses on the social ranking of lotteries. In this paper, however, we employ an alternative model in which probabilities are fixed but utility profiles are allowed to vary. In that case, prospects (vectors of alternatives, one for each of the possible states) rather than lotteries are ranked. An important feature of our approach is that it operates in a framework with multiple utility profiles.¹ This formulation, which follows the multi-profile of much of the traditional social-choice literature, allows us to proceed without the regularity conditions that are required in the lottery approach and permits the use of standard anonymity axioms. We prove our main theorems for a fixed probability distribution which is common to all individuals and the social evaluator. However, the results continue to apply if the probability distribution may vary, as long as probabilities are objectively known or agreed upon by all individuals and society.

We establish new variants of Harsanyi's social aggregation theorem. First, we generalize a fixed-population result due to Blackorby, Bossert and Donaldson [2002] by showing that its conclusion is valid under weaker assumptions: in particular, the minimal number of alternatives required can be reduced from four to three. The resulting characterization is a reformulation of a result due to Mongin [1994].² Second, we extend the model to variable-population comparisons. This generalization involves substantial additional complexities because the composition and the size of the population may vary across states. Thus, individuals may be alive in some states but not in others. We take the view that ex-ante assessment of prospects are meaningless if an individual is not alive in all possible

¹ Multi-profile models of social choice under uncertainty are also discussed, for example, in Blackorby, Donaldson and Weymark [1999, 2003], Hammond [1981, 1983], Mongin [1994] and Mongin and d'Aspremont [1998].

² Mongin [1994] uses a different domain assumption for some of his results.

states. As a consequence, we formulate all axioms that are concerned with individual ex-ante utilities so that they apply to *complete* prospects only—prospects where each person who is alive in one state is alive in all states.

Two classes of variable-population extensions of ex-ante utilitarianism are characterized. To do this, we employ axioms which ensure that, for every incomplete prospect, there is a complete prospect that is equally good. In addition, we discuss the generalization of our results to a setting in which both prospects and probabilities may vary.

The proof technique employed in the multi-profile fixed-population case is different from those that appear in the lottery framework. However, the most novel part of the paper is the variable-population extension: the consideration of population problems introduces substantial additional complexities.

The fixed-population model of social evaluation under uncertainty formulated in terms of prospects is introduced in Section 2. Section 3 contains a characterization of ex-ante utilitarianism and an impossibility result. Two classes of variable-population extensions of ex-ante utilitarianism are characterized in Section 4. Section 5 provides a generalization to uncertain alternatives, which consist of compound vectors of probability distributions and prospects. Section 6 concludes.

2. Ex-ante social evaluation with a fixed population

There are several ways of incorporating uncertainty into a model of social evaluation. Although much of the relevant literature—including Harsanyi’s [1955] original contribution—focuses on the social ranking of lotteries (probability distributions defined on the set of possible alternatives), we use a formulation in terms of prospects instead because it allows for a more natural extension of standard multi-profile social evaluation to situations involving uncertainty. In order to prove our theorems, it is not necessary to allow the probabilities to vary.

We use \mathcal{Z}_{++} to denote the set of positive integers. \mathcal{R} is the set of all real numbers, \mathcal{R}_{++} is the set of all positive real numbers and, for $n \in \mathcal{Z}_{++}$ and an arbitrary set S , S^n is the n -fold Cartesian product of S . $\mathbf{1}_n$ is the vector consisting of $n \in \mathcal{Z}_{++}$ ones.

In the fixed-population case, there are $n \in \mathcal{Z}_{++}$ individuals labelled $1, \dots, n$. Thus, the set of individuals is $N = \{1, \dots, n\}$. The universal set of alternatives is X and, in order to ensure that the standard welfarism results apply, we assume that X contains at least three elements.

Suppose there is a set $M = \{1, \dots, m\}$ of possible states, where $m \geq 2$, and an associated vector of fixed positive probabilities $p = (p_1, \dots, p_m) \in \mathcal{R}_{++}^m$ where, by definition of a probability distribution, $\sum_{j=1}^m p_j = 1$. Because probabilities are assumed to be fixed, any state with a probability of zero may be dropped and, therefore, the positivity requirement

on p involves no loss of generality as long as there are at least two states with positive probabilities. A prospect is a vector $\mathbf{x} \in X^m$ where, for all $j \in M$, \mathbf{x}_j is the alternative that materializes in state j . We use \mathbf{X} to denote the set of all prospects, that is, $\mathbf{X} = X^m$. For $x \in X$, the prospect $x\mathbf{1}_m$, defined as $(x, \dots, x) \in \mathbf{X}$, is one in which the alternative x occurs with certainty (that is, x is realized in all possible states).

For each individual $i \in N$, $U_i: X \rightarrow \mathcal{R}$ is i 's actual or ex-post utility function. That is, $U_i(x)$ measures i 's well-being in alternative $x \in X$. An ex-post utility profile is an n -tuple $U = (U_1, \dots, U_n)$, and the set of all possible profiles is \mathcal{U} . For an alternative $x \in X$, we let $U(x) = (U_1(x), \dots, U_n(x))$.

Individual i 's ex-ante utility function is $U_i^{EA}: \mathbf{X} \rightarrow \mathcal{R}$, that is, $U_i^{EA}(\mathbf{x})$ is the value of the prospect $\mathbf{x} \in \mathbf{X}$ to individual $i \in N$. A profile of ex-ante utility functions is an n -tuple $U^{EA} = (U_1^{EA}, \dots, U_n^{EA})$, and the set of all possible ex-ante profiles is \mathcal{U}^{EA} . For $\mathbf{x} \in \mathbf{X}$, we let $U^{EA}(\mathbf{x}) = (U_1^{EA}(\mathbf{x}), \dots, U_n^{EA}(\mathbf{x}))$. Note that, in contrast to ex-post utilities that assign an individual value to each alternative in X , ex-ante utility functions assign a value to each prospect in \mathbf{X} .

Throughout, we assume that ex-ante and ex-post utilities satisfy the fundamental consistency requirement that their assessments of certain outcomes are identical: if an alternative materializes in every possible state, ex-ante utility coincides with ex-post utility. That is, for all $i \in N$ and for all $x \in X$,

$$U_i^{EA}(x\mathbf{1}_m) = U_i(x). \quad (1)$$

(1) implies that the ex-ante utility function U_i^{EA} determines the ex-post utility function U_i , although the converse is not true.

To define an ex-ante criterion for social evaluation, we use the following terminology. An ex-ante ordering is an ordering defined on the set of prospects \mathbf{X} , and the set of all ex-ante orderings on \mathbf{X} is denoted by \mathcal{O}^{EA} . An ex-ante social-evaluation functional is a mapping $F^{EA}: \mathcal{D}^{EA} \rightarrow \mathcal{O}^{EA}$ with $\emptyset \neq \mathcal{D}^{EA} \subseteq \mathcal{U}^{EA}$. F^{EA} assigns a social ordering on \mathbf{X} to each profile of ex-ante utility functions in its domain \mathcal{D}^{EA} . We use the notation $R_{U^{EA}}^{EA} = F^{EA}(U^{EA})$ for all $U^{EA} \in \mathcal{D}^{EA}$. $P_{U^{EA}}^{EA}$ and $I_{U^{EA}}^{EA}$ denote the asymmetric and symmetric factors of $R_{U^{EA}}^{EA}$.

For an individual $i \in N$, a von Neumann – Morgenstern (vNM) function is a mapping $U_i^{vNM}: X \rightarrow \mathcal{R}$. A vNM profile is an n -tuple $U^{vNM} = (U_1^{vNM}, \dots, U_n^{vNM}) \in \mathcal{U}$. Note that, without further assumptions, vNM functions do not necessarily measure individual well-being and, conversely, ex-post utility functions can not necessarily be employed as vNM functions.

The individual expected-utility hypothesis requires that the individual ex-ante ranking of any two prospects \mathbf{x} and \mathbf{y} is determined by the expected values of i 's vNM function obtained for \mathbf{x} and \mathbf{y} , given the probability vector p . In our multi-profile approach,

we require the ex-ante social-evaluation functional to be consistent with the individual expected-utility hypothesis in the sense that it produces a social ordering for each utility profile that is composed of individual ex-ante utilities satisfying the hypothesis. Thus, expected-utility consistency is formulated as a domain assumption. The expected-utility domain $\mathcal{D}^{EU} \subseteq \mathcal{U}^{EA}$ is defined as follows. For all $U^{EA} \in \mathcal{U}^{EA}$, $U^{EA} \in \mathcal{D}^{EU}$ if and only if there exists a profile $U^{vNM} \in \mathcal{U}$ of vNM functions such that, for all $i \in N$ and for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$U_i^{EA}(\mathbf{x}) \geq U_i^{EA}(\mathbf{y}) \Leftrightarrow \sum_{j=1}^m p_j U_i^{vNM}(\mathbf{x}_j) \geq \sum_{j=1}^m p_j U_i^{vNM}(\mathbf{y}_j).$$

Clearly, this requirement is satisfied if and only if there exists an increasing function $\varphi_i: \mathcal{R} \rightarrow \mathcal{R}$ such that, for all $i \in N$ and for all $\mathbf{x} \in \mathbf{X}$,

$$U_i^{EA}(\mathbf{x}) = \varphi_i\left(\sum_{j=1}^m p_j U_i^{vNM}(\mathbf{x}_j)\right). \quad (2)$$

Equation (1) implies that $U_i(x) = \varphi_i \circ U_i^{vNM}$. If the social-evaluation functional is required to generate a social ordering for all profiles such that the individual ex-ante utilities are required to satisfy the expected-utility hypothesis, we obtain the following domain assumption.

Individual Expected-Utility Consistency: $\mathcal{D}^{EA} = \mathcal{D}^{EU}$.

The Bernoulli hypothesis (see Arrow [1972] and Broome [1991]) imposes a more stringent restriction on individual ex-ante utilities than the expected-utility hypothesis: it requires that the function φ_i in (2) is affine. Because, by assumption, an individual's ex-post utility of an alternative $x \in X$ is given by the ex-ante value of the prospect that yields x with certainty, the Bernoulli hypothesis implies

$$U_i(x) = U_i^{EA}(x\mathbf{1}_m) = a_i U_i^{vNM}(x) + b_i$$

where $a_i \in \mathcal{R}_{++}$ and $b_i \in \mathcal{R}$ are the parameters of the affine function φ_i . Thus, the Bernoulli hypothesis implies that the ex-post utility function U_i is an increasing affine transformation of the vNM function U_i^{vNM} and, therefore, is a particular vNM function itself. Equation (2) can therefore be written as

$$U_i^{EA}(\mathbf{x}) = a_i \sum_{j=1}^m p_j U_i^{vNM}(\mathbf{x}_j) + b_i = \sum_{j=1}^m p_j U_i(\mathbf{x}_j). \quad (3)$$

(3) shows that if the Bernoulli hypothesis is satisfied, the individual utility function U_i plays two roles: it is an indicator of actual well-being and, in addition, it is a vNM function.

Thus, the value of a prospect $\mathbf{x} \in \mathbf{X}$ to $i \in N$ according to U_i^{EA} is the expected utility of \mathbf{x} given the probabilities p and the ex-post utility function U_i .

As is the case for the individual expected-utility hypothesis, consistency with the individual Bernoulli hypothesis is a domain restriction. The Bernoulli domain $\mathcal{D}^B \subseteq \mathcal{U}^{EA}$ is defined as follows. For all $U^{EA} \in \mathcal{U}^{EA}$, $U^{EA} \in \mathcal{D}^B$ if and only if there exists a profile $U \in \mathcal{U}$ such that, for all $i \in N$ and for all $\mathbf{x} \in \mathbf{X}$,

$$U_i^{EA}(\mathbf{x}) = \sum_{j=1}^m p_j U_i(\mathbf{x}_j). \quad (4)$$

Individual Bernoulli consistency can now be formulated by specifying the domain of the ex-ante social-evaluation functional F^{EA} .

Individual Bernoulli Consistency: $\mathcal{D}^{EA} = \mathcal{D}^B$.

In our multi-profile setting, consistency with the individual Bernoulli hypothesis requires the social-evaluation functional to produce a social ordering on a smaller domain than consistency with the individual expected-utility hypothesis does. Consequently, consistency with the individual Bernoulli hypothesis is a weaker requirement than consistency with the individual expected-utility hypothesis. In contrast, the individual Bernoulli hypothesis is the stronger assumption in the single-profile case because each axiom merely requires the single profile to belong to the appropriate domain.

Following Harsanyi [1955, 1977], we assume that individual ex-ante utility functions satisfy the Bernoulli hypothesis (the consequences of strengthening individual Bernoulli consistency to individual expected-utility consistency are examined later). Social orderings are assumed to satisfy the expected-utility hypothesis. This requires the existence of a social vNM function such that the social ranking of two prospects is obtained by comparing their social expected vNM values.

Social Expected-Utility Hypothesis: There exists a function $U_0: X \times \mathcal{D}^{EA} \rightarrow \mathcal{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^{EA}$,

$$\mathbf{x} R_{U^{EA}} \mathbf{y} \Leftrightarrow \sum_{j=1}^m p_j U_0(\mathbf{x}_j, U^{EA}) \geq \sum_{j=1}^m p_j U_0(\mathbf{y}_j, U^{EA}).$$

The social vNM function U_0 is allowed be profile-dependent. In our multi-profile setting, if this function were not allowed to depend on U^{EA} , an imposed social ranking would result. In Harsanyi's [1955] lottery framework, there is only a single profile of ex-ante utility functions and the second argument is not needed.

3. The social aggregation theorem for prospects

We first state a version of the welfarism theorem for ex-ante utilities. The first part is a straightforward reformulation of the standard welfarism theorem: on the Bernoulli domain, ex-ante versions of Pareto indifference and binary independence of irrelevant alternatives together are equivalent to the existence of an ex-ante social-evaluation ordering R on \mathcal{R}^n such that, for any ex-ante profile in the domain of F^{EA} and for any two prospects \mathbf{x} and \mathbf{y} , the ranking of \mathbf{x} and \mathbf{y} according to $R_{U^{EA}}^{EA}$ is determined by the ranking of the associated ex-ante utility vectors according to R . The second part shows that comparisons of prospects in which alternatives occur with certainty must be performed according to the ex-ante social-evaluation ordering R as well.

The ex-ante versions of the welfarism axioms are defined as follows.

Ex-Ante Pareto Indifference: For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^{EA}$, if $U^{EA}(\mathbf{x}) = U^{EA}(\mathbf{y})$, then $\mathbf{x} I_{U^{EA}}^{EA} \mathbf{y}$.

Ex-Ante Binary Independence of Irrelevant Alternatives: For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA}, \bar{U}^{EA} \in \mathcal{D}^{EA}$, if $U^{EA}(\mathbf{x}) = \bar{U}^{EA}(\mathbf{x})$ and $U^{EA}(\mathbf{y}) = \bar{U}^{EA}(\mathbf{y})$, then

$$\mathbf{x} R_{U^{EA}}^{EA} \mathbf{y} \Leftrightarrow \mathbf{x} R_{\bar{U}^{EA}}^{EA} \mathbf{y}.$$

We obtain the following result, the proof of which can be found in Blackorby, Bossert and Donaldson [2002] (see also Blackorby, Donaldson and Weymark [2003], Mongin [1994] and Mongin and d'Aspremont [1998]).

Theorem 1: *If F^{EA} satisfies individual Bernoulli consistency, ex-ante Pareto indifference and ex-ante binary independence of irrelevant alternatives, then there exists a social-evaluation ordering R on \mathcal{R}^n such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^B$,*

$$\mathbf{x} R_{U^{EA}}^{EA} \mathbf{y} \Leftrightarrow U^{EA}(\mathbf{x}) R U_E^n(\mathbf{y}) \tag{5}$$

and, for all $x, y \in X$ and for all $U \in \mathcal{U}$,

$$x \mathbf{1}_m R_{U^{EA}}^{EA} \mathbf{y} \mathbf{1}_m \Leftrightarrow U(x) R U(y)$$

where U is the profile corresponding to U^{EA} according to (4).

Now we show that any ex-ante social-evaluation functional satisfying the axioms of Theorem 1 and the social expected-utility hypothesis must possess a property that is equivalent to the requirement that R satisfy information invariance with respect to translation-scale non-comparability.³

³ In Blackorby, Bossert and Donaldson [2002], a weaker version of this theorem that requires X to contain at least four alternatives is proven. See Mongin [1994] and Mongin and d'Aspremont [1998] for a similar theorem, establishing that the independence axiom of expected-utility theory is equivalent to cardinal unit comparability formulated for an ex-ante social-evaluation ordering.

Theorem 2: *If F^{EA} satisfies individual Bernoulli consistency, the social expected-utility hypothesis, ex-ante Pareto indifference and ex-ante binary independence of irrelevant alternatives, then, for all $u, v, b \in \mathcal{R}^n$,*

$$uRv \Leftrightarrow (u + b)R(v + b) \quad (6)$$

where R is the ex-ante social-evaluation ordering corresponding to F^{EA} .

Proof. Let $u, v, b \in \mathcal{R}^n$. By individual Bernoulli consistency, $\mathcal{D}^{EA} = \mathcal{D}^B$. The social expected-utility hypothesis implies that there exists a function $U_0: X \times \mathcal{D}^B \rightarrow \mathcal{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^B$,

$$\mathbf{x}R_{U^{EA}}\mathbf{y} \Leftrightarrow \sum_{j=1}^m p_j U_0(\mathbf{x}_j, U^{EA}) \geq \sum_{j=1}^m p_j U_0(\mathbf{y}_j, U^{EA}).$$

Combined with (5), this yields

$$U^{EA}(\mathbf{x})RU^{EA}(\mathbf{y}) \Leftrightarrow \sum_{j=1}^m p_j U_0(\mathbf{x}_j, U^{EA}) \geq \sum_{j=1}^m p_j U_0(\mathbf{y}_j, U^{EA}) \quad (7)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^B$.

Because X contains at least three alternatives, we can choose $x, y, z \in X$ and $U \in \mathcal{U}$ so that

$$U(x) = \frac{1}{p_1}u - \frac{\sum_{j=2}^m p_j}{p_1}v + b,$$

$$U(y) = v + b$$

and

$$U(z) = v - \frac{p_1}{\sum_{j=2}^m p_j}b.$$

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in \mathbf{X}$ be such that $\mathbf{x}(1) = \mathbf{z}(1) = x$, $\mathbf{y}(1) = \mathbf{w}(1) = y$, $\mathbf{x}_j = \mathbf{y}_j = z$ for all $j \in M \setminus \{1\}$ and $\mathbf{z}_j = \mathbf{w}_j = y$ for all $j \in M \setminus \{1\}$. By individual Bernoulli consistency, the profile $U^{EA} \in \mathcal{D}^B$ that corresponds to U satisfies

$$U^{EA}(\mathbf{x}) = \sum_{j=1}^m p_j U(\mathbf{x}_j) = u,$$

$$U^{EA}(\mathbf{y}) = \sum_{j=1}^m p_j U(\mathbf{y}_j) = v,$$

$$U^{EA}(\mathbf{z}) = \sum_{j=1}^m p_j U(\mathbf{z}_j) = u + b$$

and

$$U^{EA}(\mathbf{w}) = \sum_{j=1}^m p_j U(\mathbf{w}_j) = v + b.$$

Substituting into (7), we obtain

$$\begin{aligned} uRv &\Leftrightarrow p_1 U_0(x, U^{EA}) + \sum_{j=2}^m p_j U_0(z, U^{EA}) \geq p_1 U_0(y, U^{EA}) + \sum_{j=2}^m p_j U_0(z, U^{EA}) \\ &\Leftrightarrow p_1 U_0(x, U^{EA}) \geq p_1 U_0(y, U^{EA}) \end{aligned} \quad (8)$$

and, using (7) with $\mathbf{x} = \mathbf{z}$ and $\mathbf{y} = \mathbf{w}$,

$$\begin{aligned} (u + b)R(v + b) &\Leftrightarrow p_1 U_0(x, U^{EA}) + \sum_{j=2}^m p_j U_0(y, U^{EA}) \geq p_1 U_0(y, U^{EA}) + \sum_{j=2}^m p_j U_0(y, U^{EA}) \\ &\Leftrightarrow p_1 U_0(x, U^{EA}) \geq p_1 U_0(y, U^{EA}). \end{aligned} \quad (9)$$

Because the second lines of (8) and (9) are identical, (6) follows. ■

The property of R established in Theorem 2 is identical to information invariance with respect to translation-scale non-comparability defined for the ex-ante social-evaluation ordering R . Therefore, we can apply a well-known result from the theory of social choice under certainty to characterize utilitarianism in the present framework. To do so, we introduce ex-ante versions of the axioms minimal increasingness and same-people anonymity.

Ex-ante minimal increasingness is implied by the ex-ante weak Pareto principle. Ex-ante weak Pareto requires that if each person's ex-ante utility is higher in prospect \mathbf{x} than in prospect \mathbf{y} , then \mathbf{x} is declared better than \mathbf{y} . Ex-ante minimal increasingness requires this conclusion to obtain only if the ex-ante utilities are equally distributed in both \mathbf{x} and in \mathbf{y} . If a single utility level increases, utility inequality may increase and ex-ante minimal increasingness permits a ranking in which the original situation is better.

Ex-Ante Minimal Increasingness: For all $a, b \in \mathcal{R}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^{EA}$, if $U^{EA}(\mathbf{x}) = a\mathbf{1}_n \gg b\mathbf{1}_n = U^{EA}(\mathbf{y})$, then $\mathbf{x}P_{U^{EA}}^{EA}\mathbf{y}$.

Ex-ante same-people anonymity is an impartiality condition requiring that the identities of the individuals are irrelevant.

Ex-Ante Same-People Anonymity: For all $U^{EA}, \bar{U}^{EA} \in \mathcal{D}^{EA}$, if there exists a bijection $\rho: N \rightarrow N$ such that $U_i^{EA} = \bar{U}_{\rho(i)}^{EA}$ for all $i \in N$, then $R_{U^{EA}}^{EA} = R_{\bar{U}^{EA}}^{EA}$.

A well-known result in social-choice theory states that the ex-post versions of weak Pareto, same-people anonymity and translation-scale non-comparability characterize the

utilitarian (ex-post) social-evaluation functional; see, for example, Blackorby, Bossert and Donaldson [2002], Blackorby, Donaldson and Weymark [1984], Blackwell and Girshick [1954], Bossert and Weymark [2003], d’Aspremont and Gevers [1977], Milnor [1954] and Roberts [1980]. Though the above contributions state the result using weak Pareto instead of minimal increasingness, it is clear from the proof employed in Blackorby, Bossert and Donaldson [2002] that the result can be strengthened by employing the weaker axiom. Translated into the uncertainty framework considered here, the axioms can be used to characterize an ex-ante version of utilitarianism. We call the corresponding ex-ante social-evaluation functional ex-ante utilitarianism and it is defined as follows. For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^{EA}$,

$$\begin{aligned} \mathbf{x}R_{U^{EA}}\mathbf{y} &\Leftrightarrow \sum_{i=1}^n U_i^{EA}(\mathbf{x}) \geq \sum_{i=1}^n U_i^{EA}(\mathbf{y}) \\ &\Leftrightarrow \sum_{j=1}^m p_j \sum_{i=1}^n U_i(\mathbf{x}_j) \geq \sum_{j=1}^m p_j \sum_{i=1}^n U_i(\mathbf{y}_j) \end{aligned}$$

where U is the profile corresponding to U^{EA} according to (4).

The proof of the following result is identical to the proof of Theorem 22 in Blackorby, Bossert and Donaldson [2002, p. 586]. Though the result reported there assumes that X contains at least four alternatives, the argument is the same given that Theorem 2 applies to the case of three alternatives as well. We therefore do not provide a proof here. Mongin [1994] proves a version of the theorem with a more structured universal set of alternatives.

Theorem 3: *Suppose F^{EA} satisfies individual Bernoulli consistency. F^{EA} satisfies the social expected-utility hypothesis, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives, ex-ante minimal increasingness and ex-ante same-people anonymity if and only if F^{EA} is ex-ante utilitarian.*

Note that, if the ex-post utilities in any two alternatives differ in a single state only, they can be ranked with the sum of ex-ante utilities or with the sum of ex-post utilities in that state. Thus, there is a utilitarian ex-post social-evaluation functional which applies to each state. Consequently, the social-evaluation functional is both ex-ante and ex-post welfarist. Note that ex-post welfarism is implied; it need not be assumed.

The characterization result of Theorem 3 requires the social-evaluation functional to produce a social ordering only for profiles of ex-ante utilities satisfying the Bernoulli hypothesis.⁴ An immediate question asks how the result is affected by requiring consistency with the individual expected-utility hypothesis instead. In contrast to the single-profile

⁴ See Broome [1991] for arguments in favor of the individual Bernoulli hypothesis.

setting, requiring consistency with the individual expected-utility hypothesis represents a stronger condition on the ex-ante social-evaluation functional because it is required to produce a social ordering on a larger domain. Thus, the scope of the other axioms is widened. In that case, we obtain an impossibility result. Again, the proof of the corresponding result in Blackorby, Bossert and Donaldson [2002, Theorem 23] for universal sets with at least four alternatives is easily adapted to the case where X may contain three alternatives only. However, a more direct proof is available, and we present it here.

Theorem 4: *There exists no ex-ante social-evaluation functional that satisfies individual expected-utility consistency, the social expected-utility hypothesis, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives, ex-ante minimal increasingness and ex-ante same-people anonymity.*

Proof. Suppose F^{EA} satisfies the axioms in the theorem statement. Clearly, $\mathcal{D}^B \subseteq \mathcal{D}^{EU}$. Theorem 3 implies that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^B$,

$$\mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} \Leftrightarrow U^{EA}(\mathbf{x})RU^{EA}(\mathbf{y})$$

or, equivalently,

$$\begin{aligned} \mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} \Leftrightarrow & \left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{x}_j), \dots, \sum_{j=1}^m p_j U_n^{vNM}(\mathbf{x}_j) \right) \\ & R \left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{y}_j), \dots, \sum_{j=1}^m p_j U_n^{vNM}(\mathbf{y}_j) \right) \end{aligned}$$

for some $U^{vNM} \in \mathcal{U}$, where R satisfies

$$uRv \Leftrightarrow \sum_{i=1}^n u_i \geq \sum_{i=1}^n v_i \tag{10}$$

for all $u, v \in \mathcal{R}^n$.

Let $\varphi: \mathcal{R} \rightarrow \mathcal{R}$ be an increasing, surjective and non-affine function, and define the subset \mathcal{D}^φ of \mathcal{D}^{EU} as follows. For all $U^{EA} \in \mathcal{D}^{EU}$, $U^{EA} \in \mathcal{D}^\varphi$ if and only if there exists a profile $U^{vNM} \in \mathcal{U}$ such that, for all $i \in N$ and for all $\mathbf{x} \in \mathbf{X}$,

$$U_i^{EA}(\mathbf{x}) = \varphi \left(\sum_{j=1}^m p_j U_i^{vNM}(\mathbf{x}_j) \right). \tag{11}$$

Now define the ordering R^φ on \mathcal{R}^n by

$$uR^\varphi v \Leftrightarrow (\varphi(u_1), \dots, \varphi(u_n))R(\varphi(v_1), \dots, \varphi(v_n)) \tag{12}$$

for all $u, v \in \mathcal{R}^n$. Because φ is a bijection, its inverse φ^{-1} exists and (12) is equivalent to

$$uRv \Leftrightarrow (\varphi^{-1}(u_1), \dots, \varphi^{-1}(u_n))R^\varphi(\varphi^{-1}(v_1), \dots, \varphi^{-1}(v_n)) \quad (13)$$

for all $u, v \in \mathcal{R}^n$. Let $U^{EA} \in \mathcal{D}^\varphi$. By definition, there exists a vNM profile $U^{vNM} \in \mathcal{U}$ such that (11) is satisfied for all $\mathbf{x} \in \mathbf{X}$. Therefore, for any two prospects $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, it follows that

$$\begin{aligned} U^{EA}(\mathbf{x})RU^{EA}(\mathbf{y}) &\Leftrightarrow \left(\varphi\left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{x}_j)\right), \dots, \varphi\left(\sum_{j=1}^m p_j U_n^{vNM}(\mathbf{x}_j)\right) \right) \\ &\quad R\left(\varphi\left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{y}_j)\right), \dots, \varphi\left(\sum_{j=1}^m p_j U_n^{vNM}(\mathbf{y}_j)\right) \right) \end{aligned}$$

and, by (12),

$$\begin{aligned} U^{EA}(\mathbf{x})RU^{EA}(\mathbf{y}) &\Leftrightarrow \left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{x}_j), \dots, \sum_{j=1}^m p_j U_n^{vNM}(\mathbf{x}_j) \right) \\ &\quad R^\varphi\left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{y}_j), \dots, \sum_{j=1}^m p_j U_n^{vNM}(\mathbf{y}_j) \right). \end{aligned}$$

Letting $\bar{U}_i^{EA}(\mathbf{x}) = \sum_{j=1}^m p_j U_i^{vNM}(\mathbf{x}_j)$ for all $i \in \{1, \dots, n\}$ and for all $\mathbf{x} \in \mathbf{X}$, this implies

$$\begin{aligned} \bar{U}^{EA}(\mathbf{x})R^\varphi\bar{U}^{EA}(\mathbf{y}) &\Leftrightarrow \left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{x}_j), \dots, \sum_{j=1}^m p_j U_n^{vNM}(\mathbf{x}_j) \right) \\ &\quad R^\varphi\left(\sum_{j=1}^m p_j U_1^{vNM}(\mathbf{y}_j), \dots, \sum_{j=1}^m p_j U_n^{vNM}(\mathbf{y}_j) \right). \end{aligned}$$

Because, by definition, $\bar{U}^{EA} \in \mathcal{D}^B$, it follows that

$$\bar{U}^{EA}(\mathbf{x})R^\varphi\bar{U}^{EA}(\mathbf{y}) \Leftrightarrow \mathbf{x}R_{\bar{U}^{EA}}^{EA}\mathbf{y}.$$

Applying Theorem 3, we obtain

$$uR^\varphi v \Leftrightarrow \sum_{i=1}^n u_i \geq \sum_{i=1}^n v_i$$

for all $u, v \in \mathcal{R}^n$. Together with (10) and (13), it follows that

$$\sum_{i=1}^n u_i \geq \sum_{i=1}^n v_i \Leftrightarrow \sum_{i=1}^n \varphi^{-1}(u_i) \geq \sum_{i=1}^n \varphi^{-1}(v_i) \quad (14)$$

for all $u, v \in \mathcal{R}^n$. (14) is equivalent to the existence of an increasing function $H: \mathcal{R} \rightarrow \mathcal{R}$ such that

$$\sum_{i=1}^n \varphi^{-1}(u_i) = H\left(\sum_{i=1}^n u_i\right) \quad (15)$$

for all $u, v \in \mathcal{R}^n$. (15) is a Pexider equation and it follows that φ^{-1} must be affine (see Aczél [1966, Chapter 3] for Pexider equations and their solutions). This implies that φ is affine as well, a contradiction. ■

The proof of Theorem 4 shows that, when the increasing transformations $\varphi_1, \dots, \varphi_n$ are identical, prospects must be ranked according to their sums of expected utilities. This means, however, that R must depend on the common transformation φ , contradicting welfarism. A variant of Theorem 4 is obtained if anonymity is replaced by continuity.⁵

In a single-profile setting, the impossibility is avoided because, with a single ex-ante profile, consistency with the individual expected-utility hypothesis is a weaker axiom than consistency with the individual Bernoulli hypothesis and ex-ante social-evaluation principles other than ex-ante utilitarianism become available. Blackorby, Donaldson and Weymark [2003] prove that, in the single-profile case, prospects must be ranked by their respective transformed sums of individual ex-ante utilities if the individual expected-utility hypothesis (but not necessarily the individual Bernoulli hypothesis) is satisfied. As in our multi-profile approach, utilitarianism obtains if the individual Bernoulli hypothesis is satisfied.⁶ We believe that the multi-profile framework employed here represents a suitable way of formulating social-choice problems under uncertainty and we consider Theorems 3 and 4 to provide a strong argument in favour of utilitarianism.

One possible relaxation of the axioms in Harsanyi's result (and our Theorem 3) is to drop the social expected-utility hypothesis, a move that has been suggested by Diamond [1967] and by Sen [1976, 1977, 1986]. They argue that ex-ante social-evaluation functionals that satisfy the social expected-utility hypothesis cannot take account of the fairness of procedures by which outcomes are generated (see also Weymark [1991]).

An interesting alternative, however, is to relax the ex-ante welfarism axioms. The form of ex-ante welfarism of Theorem 1 is not applied to actual well-being, and this suggests that ex-post welfarism may be more appropriate and ethically easier to defend than ex-ante welfarism. Given individual Bernoulli consistency, ex-ante welfarism implies ex-post welfarism but the converse implication is not true. A second way to relax the assumptions of Theorem 3, therefore, is provided by requiring ex-post welfarism only.

⁵ See also Blackorby, Donaldson and Weymark [1999, 2003], Roemer [1996], Sen [1976] and Weymark [1991].

⁶ A regularity condition is required in these single-profile characterizations. This is not necessary in our multi-profile approach.

Suppose, for example, that when the ex-post profile is $U \in \mathcal{U}$, such a principle produces the ex-post ordering R_U^{EP} of prospects, which is given by

$$\mathbf{x}R_U^{EP}\mathbf{y} \Leftrightarrow \sum_{j=1}^m p_j \frac{1}{n^2} \sum_{i=1}^n (2i-1)U_{(i)}(\mathbf{x}_j) \geq \sum_{j=1}^m p_j \frac{1}{n^2} \sum_{i=1}^n (2i-1)U_{(i)}(\mathbf{y}_j)$$

where, for all $x \in X$, $(U_{(1)}(x), \dots, U_{(n)}(x))$ is a permutation of $(U_1(x), \dots, U_n(x))$ such that $U_k(x) \geq U_{k+1}(x)$ for all $k \in \{1, \dots, n-1\}$. In this case, alternatives are ranked, ex post, with a social-evaluation functional that expresses aversion to utility inequality. This principle is not consistent with ex-ante Pareto indifference if individual ex-ante utilities satisfy the Bernoulli hypothesis. This means that a prospect \mathbf{x} may be regarded as better than a prospect \mathbf{y} although \mathbf{x} and \mathbf{y} are equally good for each person. With such a principle, therefore, social rationality trumps individual rationality.

4. Ex-ante population principles for prospects

We now extend the fixed-population model to cover situations where the population may vary within and among prospects.⁷ In a variable-population framework, the universal set X contains alternatives with different sets (and numbers) of individuals alive. We use the same notation as in the fixed-population model for simplicity; to avoid ambiguities, we rephrase all definitions in terms of the variable-population setting of this section.

There is a set $M = \{1, \dots, m\}$ of $m \geq 2$ possible states with fixed positive probabilities $p = (p_1, \dots, p_m) \in \mathcal{R}_{++}^m$, where $\sum_{j=1}^m p_j = 1$. As before, a prospect is a vector $\mathbf{x} \in \mathbf{X}$ where, for all $j \in M$, $\mathbf{x}_j \in X$ is the alternative that occurs in state j . For $x \in X$, the prospect $x\mathbf{1}_m = (x, \dots, x) \in \mathbf{X}$ is the prospect in which the alternative x occurs with certainty.

In contrast to the fixed-population case analyzed in the earlier sections, the composition and the size of the population may differ from one alternative in X to another. To keep track of the population associated with each alternative, we use a function $\mathbf{N}: X \rightarrow \mathcal{P}(\mathcal{Z}_{++})$, where $\mathcal{P}(\mathcal{Z}_{++})$ is the set of all non-empty and finite subsets of \mathcal{Z}_{++} . Thus, for an alternative $x \in X$, $\mathbf{N}(x)$ is the set of individuals alive in x . Furthermore, we employ a function $\mathbf{n}: X \rightarrow \mathcal{P}(\mathcal{Z}_{++})$ to denote the number of people alive in each alternative; that is, $\mathbf{n}(x) = |\mathbf{N}(x)|$ for all $x \in X$. For a non-empty and finite set $N \subseteq \mathcal{Z}_{++}$, $X^N \subseteq X$ is the set of all alternatives $x \in X$ such that $\mathbf{N}(x) = N$. Analogously to the corresponding assumption in the fixed-population setting, we assume that X^N contains at least three elements for all non-empty and finite $N \subseteq \mathcal{Z}_{++}$. For $i \in \mathcal{Z}_{++}$, we define

$$X_i = \{x \in X \mid i \in \mathbf{N}(x)\},$$

⁷ A variable-population model of social choice under uncertainty in a lottery setting is discussed in Blackorby, Bossert and Donaldson [1998].

that is, $X_i \subseteq X$ is the set of alternatives in which i is alive.

We introduce analogous definitions for prospects. Let $N \subseteq \mathcal{Z}_{++}$ be non-empty and finite. \mathbf{X}^N is the set of all prospects where the individuals in N are alive in *all* states. That is,

$$\mathbf{X}^N = \{\mathbf{x} \in \mathbf{X} \mid \mathbf{N}(\mathbf{x}_j) = N \text{ for all } j \in M\}.$$

The functions $\mathbf{N}^{EA}: X \rightarrow \mathcal{P}(\mathcal{Z}_{++})$ and $\mathbf{n}^{EA}: X \rightarrow \mathcal{P}(\mathcal{Z}_{++})$ are defined by

$$\mathbf{N}^{EA}(\mathbf{x}) = \cup_{j=1}^m \mathbf{N}(\mathbf{x}_j) \quad \text{and} \quad \mathbf{n}^{EA}(\mathbf{x}) = |\mathbf{N}^{EA}(\mathbf{x})|$$

for all $\mathbf{x} \in \mathbf{X}$, that is, $\mathbf{N}^{EA}(\mathbf{x})$ is the set of individuals alive in at least one state of prospect \mathbf{x} and $\mathbf{n}^{EA}(\mathbf{x})$ is their number. Furthermore, let

$$\mathbf{X}_\theta = \{\mathbf{x} \in \mathbf{X} \mid \mathbf{N}(\mathbf{x}_j) = \mathbf{N}(\mathbf{x}_k) \text{ for all } j, k \in M\}.$$

The set \mathbf{X}_θ is the set of complete prospects. In them, everyone who is alive in at least one state is alive in all states. For $\mathbf{x} \in \mathbf{X}_\theta$, we let $\mathbf{N}_\theta(\mathbf{x}) = \mathbf{N}(\mathbf{x}_j)$ for some $j \in M$. Clearly, by definition of \mathbf{X}_θ , any state $j \in M$ can be used in this definition.

Consider an individual $i \in \mathcal{Z}_{++}$, and let

$$\mathbf{X}_i = \{\mathbf{x} \in \mathbf{X} \mid i \in \mathbf{N}(\mathbf{x}_j) \text{ for all } j \in M\}.$$

The set \mathbf{X}_i contains all prospects such that i is alive in all states. $U_i: X_i \rightarrow \mathcal{R}$ is i 's actual or ex-post utility function. An ex-post utility profile is an infinite-dimensional vector $U = (U_i)_{i \in \mathcal{Z}_{++}}$, and the set of all possible profiles is \mathcal{U} . For an alternative $x \in X$, we let $U(x) = (U_i(x))_{i \in \mathbf{N}(x)}$. Individual ex-post utilities are interpreted as lifetime utilities to avoid counter-intuitive recommendations regarding the termination of lives. We use the standard normalization employed in the literature and identify a neutral life with a lifetime-utility level of zero. A neutral life is a life that is as good as a life without any experiences from the viewpoint of the individual leading it. Consequently, a fully informed, self-interested and rational individual whose lifetime utility is below neutrality considers her or his life to be worse than a life with no experiences.⁸

Individual i 's ex-ante utility function is $U_i^{EA}: \mathbf{X}_i \rightarrow \mathcal{R}$. A profile of ex-ante utility functions is denoted by $U^{EA} = (U_i^{EA})_{i \in \mathcal{Z}_{++}}$, and the set of all logically possible ex-ante profiles is \mathcal{U}^{EA} . For $\mathbf{x} \in \mathbf{X}$, we define $U^{EA}(\mathbf{x}) = (U_i^{EA}(\mathbf{x}))_{i \in \mathbf{N}^{EA}(\mathbf{x})}$. Note that U_i^{EA} is defined on the domain \mathbf{X}_i , that is, on the set of prospects in which i is alive in all states. As is the case in a certainty framework, there is no reasonable interpretation of individual utility values for situations in which the individual does not exist. Again, we assume that

⁸ See, for example, Blackorby, Bossert and Donaldson [1997] and Broome [1993, 1999] for discussions of neutrality and its normalization in population ethics.

ex-ante utility and ex-post utility coincide in the presence of certainty. That is, for all $i \in \mathcal{Z}_{++}$ and for all $x \in X_i$,

$$U_i^{EA}(x\mathbf{1}_m) = U_i(x).$$

As in the fixed-population case, ex-post utility functions are determined by ex-ante utility functions even though the ex-ante functions are not defined for all prospects.

The set of all ex-ante orderings on \mathbf{X} is denoted by \mathcal{O}^{EA} , and an ex-ante social-evaluation functional is a mapping $F^{EA}: \mathcal{D}^{EA} \rightarrow \mathcal{O}^{EA}$ with $\emptyset \neq \mathcal{D}^{EA} \subseteq \mathcal{U}^{EA}$. F^{EA} assigns a social ordering on \mathbf{X} to each admissible profile of ex-ante utility functions. We define $R_{U^{EA}}^{EA} = F^{EA}(U^{EA})$ for all $U^{EA} \in \mathcal{D}^{EA}$, and we use $P_{U^{EA}}^{EA}$ and $I_{U^{EA}}^{EA}$ to denote the asymmetric and symmetric factors of $R_{U^{EA}}^{EA}$.

The formulation of the Bernoulli hypothesis is easily extended to our variable-population setting. The extended Bernoulli domain $\mathcal{D}^B \subseteq \mathcal{U}^{EA}$ is defined as follows. For all $U^{EA} \in \mathcal{U}^{EA}$, $U^{EA} \in \mathcal{D}^B$ if and only if there exists a profile $U \in \mathcal{U}$ such that, for all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, for all $i \in N$ and for all $\mathbf{x} \in \mathbf{X}^N$,

$$U_i^{EA}(\mathbf{x}) = \sum_{j=1}^m p_j U_i(\mathbf{x}_j). \quad (16)$$

As in the certainty case, U_i is the ex-post utility function for person i . Extended individual Bernoulli consistency is, again, expressed in the form of a domain restriction. Note that restrictions are only imposed on complete prospects.

Extended Individual Bernoulli Consistency: $\mathcal{D}^{EA} = \mathcal{D}^B$.

The only change that is required in order to extend the social expected-utility hypothesis to the variable-population setting is to require the existence of a social expected-utility function that can be used to rank the prospects in \mathbf{X}^N for all non-empty and finite $N \subseteq \mathcal{Z}_{++}$.

Extended Social Expected-Utility Hypothesis: For all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, there exists a function $U_0^N: X^N \times \mathcal{D}^{EA} \rightarrow \mathcal{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^N$ and for all $U^{EA} \in \mathcal{D}^{EA}$,

$$\mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} \Leftrightarrow \sum_{j=1}^m p_j U_0^N(\mathbf{x}_j, U^{EA}) \geq \sum_{j=1}^m p_j U_0^N(\mathbf{y}_j, U^{EA}).$$

The variable-population versions of the welfarism axioms are defined as follows.

Ex-Ante Pareto Indifference: For all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^N$ and for all $U^{EA} \in \mathcal{D}^{EA}$, if $U^{EA}(\mathbf{x}) = U^{EA}(\mathbf{y})$, then $\mathbf{x}I_{U^{EA}}^{EA}\mathbf{y}$.

Ex-Ante Binary Independence of Irrelevant Alternatives: For all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^N$ and for all $U^{EA}, \bar{U}^{EA} \in \mathcal{D}^{EA}$, if $U^{EA}(\mathbf{x}) = \bar{U}^{EA}(\mathbf{x})$ and $U^{EA}(\mathbf{y}) = \bar{U}^{EA}(\mathbf{y})$, then

$$\mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} \Leftrightarrow \mathbf{x}R_{\bar{U}^{EA}}^{EA}\mathbf{y}.$$

Ex-ante minimal increasingness translates into the variable-population framework analogously.

Ex-Ante Minimal Increasingness: For all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, for all $a, b \in \mathcal{R}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^N$ and for all $U^{EA} \in \mathcal{D}^{EA}$, if $U^{EA}(\mathbf{x}) = a\mathbf{1}_n \gg b\mathbf{1}_n = U^{EA}(\mathbf{y})$, then $\mathbf{x}P_{U^{EA}}^{EA}\mathbf{y}$.

We use the same names for the last three axioms as for their same-number counterparts because they merely require the corresponding same-number axiom to be satisfied for every population. Finally, we define an extended version of ex-ante anonymity suitable for our variable-population model.

Extended Ex-Ante Anonymity: For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\theta$ and for all $U^{EA} \in \mathcal{D}^{EA}$, if there exists a bijection $\rho: \mathbf{N}_\theta(\mathbf{x}) \rightarrow \mathbf{N}_\theta(\mathbf{y})$ such that $U_i^{EA}(\mathbf{x}) = U_{\rho(i)}^{EA}(\mathbf{y})$ for all $i \in \mathbf{N}_\theta(\mathbf{x})$, then $\mathbf{x}I_{U^{EA}}^{EA}\mathbf{y}$.

If an ex-ante social-evaluation functional F^{EA} satisfies extended individual Bernoulli consistency, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives and extended ex-ante anonymity, then complete prospects are ranked by a single anonymous ordering of vectors of ex-ante utilities. The result follows from the variable-population version of the welfarism theorem without uncertainty and is omitted.⁹ An ordering R^n on \mathcal{R}^n is anonymous if and only if, for all bijections $\rho: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ and for all $u, v \in \mathcal{R}^n$, if $v_i = u_{\rho(i)}$ for all $i \in \{1, \dots, n\}$, then $uI^n v$. Furthermore, we say that an ordering R on $\cup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n$ is anonymous if and only if its restriction to \mathcal{R}^n is anonymous for each $n \in \mathcal{Z}_{++}$.

Theorem 5: *If F^{EA} satisfies extended individual Bernoulli consistency, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives and extended ex-ante anonymity, then there exists an anonymous ordering R^E on $\cup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\theta$ and for all $U^{EA} \in \mathcal{D}^B$,*

$$\mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} \Leftrightarrow U^{EA}(\mathbf{x})R^E U^{EA}(\mathbf{y}).$$

⁹ See Blackorby, Bossert and Donaldson [1999] or Blackorby and Donaldson [1984].

Theorems 3 and 5 together imply that the same-number sub-orderings of R^E must be utilitarian. We call any principle with this property a same-number utilitarian principle.

Theorem 6: *If F^{EA} satisfies extended individual Bernoulli consistency, the extended social expected-utility hypothesis, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives, ex-ante minimal increasingness and extended ex-ante anonymity, then, for all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^N$ and for all $U^{EA} \in \mathcal{D}^B$,*

$$\begin{aligned} \mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} &\Leftrightarrow \sum_{i \in N} U^{EA}(\mathbf{x}) \geq \sum_{i \in N} U^{EA}(\mathbf{y}) \\ &\Leftrightarrow \sum_{i \in N} \sum_{j=1}^m p_j U^i(\mathbf{x}_j) \geq \sum_{i \in N} \sum_{j=1}^m p_j U^i(\mathbf{y}_j) \\ &\Leftrightarrow \sum_{j=1}^m p_j \sum_{i \in N} U^i(\mathbf{x}_j) \geq \sum_{j=1}^m p_j \sum_{i \in N} U^i(\mathbf{y}_j) \end{aligned}$$

where $U \in \mathcal{U}$ is the profile corresponding to U^{EA} according to (16).

The fixed-population characterization of Theorem 3 can now be extended to all prospects in a variable-population setting by employing, in addition to the variable-population versions of the axioms introduced above, the critical-level population principle, which is the uncertainty analogue of Blackorby and Donaldson's [1984] corresponding axiom. It requires the existence of a fixed critical level $c \in \mathcal{R}_+$ with the following property. Consider two prospects $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and a profile $U^{EA} \in \mathcal{D}^B$, and suppose there is a person $k \in \mathcal{Z}_{++}$ who is not alive in state j for some $j \in M$; that is, $k \notin \mathbf{N}(\mathbf{x}_j)$. Now consider a prospect $\bar{\mathbf{x}} \in \mathbf{X}$ and a profile $\bar{U}^{EA} \in \mathcal{D}^B$ such that, in $\bar{\mathbf{x}}$, individual k is alive in state j with utility level $\bar{U}_k(\bar{\mathbf{x}}_j) = c$, other things the same. The critical-level population principle requires that the ranking of \mathbf{x} and \mathbf{y} according to $R_{U^{EA}}^{EA}$ is the same as the ranking of $\bar{\mathbf{x}}$ and \mathbf{y} according to $R_{\bar{U}^{EA}}^{EA}$. Note that the axiom applies to the extended Bernoulli domain only—the individual ex-post utility functions U_i rather than the ex-ante utility functions U_i^{EA} are referred to. We require the fixed critical level to be non-negative so that additions to utility-unaffected populations of individuals with negative utility levels are never ranked as social improvements. We could impose this property separately but we incorporate it into the critical-level axiom to simplify our exposition.

Critical-Level Population Principle: If $\mathcal{D}^{EA} = \mathcal{D}^B$, then there exists $c \in \mathcal{R}_+$ such that, for all $\mathbf{x}, \mathbf{y}, \bar{\mathbf{x}} \in \mathbf{X}$, for all $U^{EA}, \bar{U}^{EA} \in \mathcal{D}^{EA}$, for all $j \in M$ and for all $k \in \mathcal{Z}_{++} \setminus \mathbf{N}(\mathbf{x}_j)$, if

$$\bar{U}(\mathbf{y}_\ell) = U(\mathbf{y}_\ell)$$

for all $\ell \in M$,

$$\mathbf{N}(\bar{\mathbf{x}}_\ell) = \mathbf{N}(\mathbf{x}_\ell)$$

for all $\ell \in M \setminus \{j\}$,

$$\bar{U}_i(\bar{\mathbf{x}}_\ell) = U_i(\mathbf{x}_\ell)$$

for all $\ell \in M$ and for all $i \in \mathbf{N}(\mathbf{x}_\ell)$,

$$\mathbf{N}(\bar{\mathbf{x}}_j) = \mathbf{N}(\mathbf{x}_j) \cup \{k\}$$

and

$$\bar{U}_k(\bar{\mathbf{x}}_j) = c,$$

then

$$\mathbf{x}R_{U^{EA}\mathbf{y}}^{EA} \Leftrightarrow \bar{\mathbf{x}}R_{\bar{U}^{EA}\mathbf{y}}^{EA}.$$

As an example, consider the two prospects which are outlined in Table 1. There are two states. In prospect \mathbf{x} , person 1 is alive in both states with ex-post utility levels 10 and 20, but person 2 is alive in state 1 only with a utility level of 12. Consequently, an ex-ante utility level is not defined for person 2.

Table 1

	Prospect \mathbf{x}		Prospect $\bar{\mathbf{x}}$	
	State 1	State 2	State 1	State 2
Person 1	10	20	10	20
Person 2	12		12	c

The critical-level population principle asserts that there exists a non-negative utility level c such that prospects \mathbf{x} and $\bar{\mathbf{x}}$ are equally good. Prospect $\bar{\mathbf{x}}$ is complete and ex-ante utilities are defined for both individuals. c is independent of all other information and it can be used to convert any prospect to a complete prospect that is equally good.

A variable-population extension of ex-ante utilitarianism is given by ex-ante critical-level utilitarianism which is defined as follows. There exists $\alpha \in \mathcal{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^{EA}$,

$$\mathbf{x}R_{U^{EA}\mathbf{y}}^{EA} \Leftrightarrow \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{x}_j)} [U_i(\mathbf{x}_j) - \alpha] \geq \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{y}_j)} [U_i(\mathbf{y}_j) - \alpha] \quad (17)$$

where U is the profile corresponding to U_E according to (16).¹⁰ For complete prospects, $\mathbf{N}(\mathbf{x}_j) = \mathbf{N}(\mathbf{x}_k) = \mathbf{N}_\theta(\mathbf{x})$ for all $j, k \in \{1, \dots, m\}$ and all $\mathbf{x} \in \mathbf{X}_\theta$ and, in that case, (17) can be written as

$$\begin{aligned} \mathbf{x}R_{U^{EA}\mathbf{y}}^{EA} &\Leftrightarrow \sum_{i \in \mathbf{N}_\theta(\mathbf{x})} \left[\left(\sum_{j=1}^m p_j U_i(\mathbf{x}_j) \right) - \alpha \right] \geq \sum_{i \in \mathbf{N}_\theta(\mathbf{y})} \left[\left(\sum_{j=1}^m p_j U_i(\mathbf{y}_j) \right) - \alpha \right] \\ &\Leftrightarrow \sum_{i \in \mathbf{N}_\theta(\mathbf{x})} \left[U_i^{EA}(\mathbf{x}) - \alpha \right] \geq \sum_{i \in \mathbf{N}_\theta(\mathbf{y})} \left[U_i^{EA}(\mathbf{y}) - \alpha \right] \end{aligned} \quad (18)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\theta$. The entries on the second line of (18) are given by the critical-level-utilitarian value function applied to ex-ante utilities.

We obtain the following characterization of ex-ante critical-level utilitarianism with a non-negative critical level α .

Theorem 7: *Suppose F^{EA} satisfies extended individual Bernoulli consistency. F^{EA} satisfies the extended social expected-utility hypothesis, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives, ex-ante minimal increasingness, extended ex-ante anonymity and the critical-level population principle if and only if F^{EA} is ex-ante critical-level utilitarian with $\alpha \geq 0$.*

Proof. That ex-ante critical-level utilitarianism with a non-negative critical level satisfies the required axioms is straightforward to verify. Now suppose F^{EA} satisfies the axioms. By Theorem 6, for all non-empty and finite $N \subseteq \mathcal{Z}_{++}$, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^N$ and for all $U^{EA} \in \mathcal{D}^B$,

$$\mathbf{x}R_{U^{EA}\mathbf{y}}^{EA} \Leftrightarrow \sum_{j=1}^m p_j \sum_{i \in N} U_i(\mathbf{x}_j) \geq \sum_{j=1}^m p_j \sum_{i \in N} U_i(\mathbf{y}_j) \quad (19)$$

where $U \in \mathcal{U}$ is the profile corresponding to U^{EA} according to (16). By the critical-level population principle, there exists $c \in \mathcal{R}_+$ with the properties described in the axiom. Let $\alpha = c$ and consider two prospects $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Let $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbf{X}$ and $\bar{U}^{EA} \in \mathcal{D}^B$ be such that

$$\mathbf{N}(\bar{\mathbf{x}}_j) = \mathbf{N}^{EA}(\mathbf{x}) \quad \text{and} \quad \mathbf{N}(\bar{\mathbf{y}}_j) = \mathbf{N}^{EA}(\mathbf{y})$$

for all $j \in M$,

$$\bar{U}_i(\bar{\mathbf{x}}_j) = U_i(\mathbf{x}_j) \quad \text{and} \quad \bar{U}_k(\bar{\mathbf{y}}_j) = U_k(\mathbf{y}_j)$$

for all $j \in M$, for all $i \in \mathbf{N}(\mathbf{x}_j)$ and for all $k \in \mathbf{N}(\mathbf{y}_j)$,

$$\bar{U}_i(\bar{\mathbf{x}}_j) = \alpha \quad \text{and} \quad \bar{U}_k(\bar{\mathbf{y}}_j) = \alpha$$

¹⁰ See Blackorby, Bossert and Donaldson [1995] and Blackorby and Donaldson [1984] for a discussion of critical-level utilitarianism in the context of social evaluation under certainty.

for all $j \in M$, for all $i \in \mathbf{N}^{EA}(\mathbf{x}) \setminus \mathbf{N}(\mathbf{x}_j)$ and for all $k \in \mathbf{N}^{EA}(\mathbf{y}) \setminus \mathbf{N}(\mathbf{y}_j)$. By repeated application of the critical-level population principle,

$$\mathbf{x}R_{\bar{U}^{EA}\bar{\mathbf{y}}}^{EA} \Leftrightarrow \bar{\mathbf{x}}R_{\bar{U}^{EA}\bar{\mathbf{y}}}^{EA} \bar{\mathbf{y}}. \quad (20)$$

Suppose that $\mathbf{n}^{EA}(\bar{\mathbf{x}}) = \mathbf{n}^{EA}(\bar{\mathbf{y}})$. By extended ex-ante anonymity, we can, without loss of generality, assume that $\mathbf{N}^{EA}(\bar{\mathbf{x}}) = \mathbf{N}^{EA}(\bar{\mathbf{y}})$, and we denote this common set of individuals by N^{EA} . By (19), it follows that

$$\begin{aligned} \bar{\mathbf{x}}R_{\bar{U}^{EA}\bar{\mathbf{y}}}^{EA} \Leftrightarrow & \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) + \sum_{i \in N^{EA} \setminus \mathbf{N}(\mathbf{x}_j)} \alpha \right] \\ & \geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) + \sum_{i \in N^{EA} \setminus \mathbf{N}(\mathbf{y}_j)} \alpha \right]. \end{aligned}$$

This inequality is equivalent to

$$\begin{aligned} & \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) + \sum_{i \in N^{EA}} \alpha - \sum_{i \in \mathbf{N}(\mathbf{x}_j)} \alpha \right] \\ & \geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) + \sum_{i \in N^{EA}} \alpha - \sum_{i \in \mathbf{N}(\mathbf{y}_j)} \alpha \right]. \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} & \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) - \sum_{i \in \mathbf{N}(\mathbf{x}_j)} \alpha \right] \\ & \geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) - \sum_{i \in \mathbf{N}(\mathbf{y}_j)} \alpha \right] \end{aligned}$$

and, therefore,

$$\bar{\mathbf{x}}R_{\bar{U}^{EA}\bar{\mathbf{y}}}^{EA} \Leftrightarrow \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{x}_j)} [U_i(\mathbf{x}_j) - \alpha] \geq \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{y}_j)} [U_i(\mathbf{y}_j) - \alpha].$$

Together with (20), this implies (17).

Now suppose $\mathbf{n}^{EA}(\bar{\mathbf{x}}) \neq \mathbf{n}^{EA}(\bar{\mathbf{y}})$. Without loss of generality, suppose $\mathbf{n}^{EA}(\bar{\mathbf{x}}) < \mathbf{n}^{EA}(\bar{\mathbf{y}})$ and, by extended ex-ante anonymity, we can assume that $\mathbf{N}^{EA}(\bar{\mathbf{x}}) \subset \mathbf{N}^{EA}(\bar{\mathbf{y}}) = \mathbf{N}^{EA}(\mathbf{y})$. Let $\hat{\mathbf{x}} \in \mathbf{X}$ and $\hat{U}^{EA} \in \mathcal{D}^B$ be such that

$$\hat{U}(\bar{\mathbf{y}}_j) = \bar{U}(\bar{\mathbf{y}}_j)$$

for all $j \in M$,

$$\mathbf{N}(\hat{\mathbf{x}}_j) = \mathbf{N}_E(\mathbf{y})$$

for all $j \in M$,

$$\hat{U}_i(\hat{\mathbf{x}}_j) = \bar{U}_i(\bar{\mathbf{x}}_j)$$

for all $j \in M$ and for all $i \in \mathbf{N}^{EA}(\bar{\mathbf{x}})$ and

$$\hat{U}_i(\hat{\mathbf{x}}_j) = \alpha$$

for all $j \in M$ and for all $i \in \mathbf{N}^{EA}(\mathbf{y}) \setminus \mathbf{N}^{EA}(\bar{\mathbf{x}})$. Again applying the critical-level population principle repeatedly, we obtain

$$\bar{\mathbf{x}}R_{\bar{U}^{EA}\bar{\mathbf{y}}}^{EA} \Leftrightarrow \hat{\mathbf{x}}R_{\hat{U}^{EA}\bar{\mathbf{y}}}^{EA}. \quad (21)$$

By (19), it follows that

$$\begin{aligned} \hat{\mathbf{x}}R_{\hat{U}^{EA}\bar{\mathbf{y}}}^{EA} &\Leftrightarrow \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) + \sum_{i \in \mathbf{N}^{EA}(\mathbf{x}) \setminus \mathbf{N}(\mathbf{x}_j)} \alpha + \sum_{i \in \mathbf{N}^{EA}(\mathbf{y}) \setminus \mathbf{N}^{EA}(\mathbf{x})} \alpha \right] \\ &\geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) + \sum_{i \in \mathbf{N}^{EA}(\mathbf{y}) \setminus \mathbf{N}(\mathbf{y}_j)} \alpha \right]. \end{aligned}$$

Rewriting, the inequality becomes

$$\begin{aligned} &\sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) - \sum_{i \in \mathbf{N}(\mathbf{x}_j)} \alpha + \sum_{i \in \mathbf{N}^{EA}(\mathbf{y})} \alpha \right] \\ &\geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) - \sum_{i \in \mathbf{N}(\mathbf{y}_j)} \alpha + \sum_{i \in \mathbf{N}^{EA}(\mathbf{y})} \alpha \right] \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) - \sum_{i \in \mathbf{N}(\mathbf{x}_j)} \alpha \right] \\ &\geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) - \sum_{i \in \mathbf{N}(\mathbf{y}_j)} \alpha \right]. \end{aligned}$$

Therefore, we obtain

$$\hat{\mathbf{x}}R_{\hat{U}^{EA}\bar{\mathbf{y}}}^{EA} \Leftrightarrow \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{x}_j)} [U_i(\mathbf{x}_j) - \alpha] \geq \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{y}_j)} [U_i(\mathbf{y}_j) - \alpha]$$

and, together with (20) and (21), this implies (17). ■

It is natural to ask what additional principles become available if the critical levels in the critical-level population principle are allowed to be different for different utility vectors and, therefore, population sizes. The following axiom is a weakening of the critical-level population principle.

Critical-Level Consistency: If $\mathcal{D}^{EA} = \mathcal{D}^B$, then there exists a function $C: \cup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n \rightarrow \mathcal{R}_+$ such that, for all $\mathbf{x}, \mathbf{y}, \bar{\mathbf{x}} \in \mathbf{X}$, for all $U^{EA}, \bar{U}^{EA} \in \mathcal{D}^{EA}$, for all $j \in M$ and for all $k \in \mathcal{Z}_{++} \setminus \mathbf{N}(\mathbf{x}_j)$, if

$$\bar{U}(\mathbf{y}(\ell)) = U(\mathbf{y}(\ell))$$

for all $\ell \in M$,

$$\mathbf{N}(\bar{\mathbf{x}}(\ell)) = \mathbf{N}(\mathbf{x}(\ell))$$

for all $\ell \in M \setminus \{j\}$,

$$\bar{U}_i(\bar{\mathbf{x}}(\ell)) = U_i(\mathbf{x}(\ell))$$

for all $\ell \in M$ and for all $i \in \mathbf{N}(\mathbf{x}(\ell))$,

$$\mathbf{N}(\bar{\mathbf{x}}_j) = \mathbf{N}(\mathbf{x}_j) \cup \{k\}$$

and

$$\bar{U}_k(\bar{\mathbf{x}}_j) = C\left(U(\mathbf{x}_j)\right),$$

then

$$\mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} \Leftrightarrow \bar{\mathbf{x}}R_{\bar{U}^{EA}}^{EA}\mathbf{y}.$$

Critical-level consistency asserts that, for each incomplete prospect, there is a complete prospect which is equally good. The (ex-post) critical level used in the expansion is, however, given by a function which can depend on the utilities of those alive (and their number) in the state in question. As in the critical-level population principle, non-negativity of the critical levels prevents the social-evaluation functional from ranking the ceteris paribus addition of a person whose utility level is below neutrality as a social improvement.

Replacing the critical-level population principle with the above axiom yields a characterization of a subclass of an ex-ante version of the number-sensitive critical-level utilitarian orderings.¹¹ F^{EA} is ex-ante number-sensitive critical-level utilitarian if and only if there exists a function $A: \mathcal{Z}_{++} \rightarrow \mathcal{R}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for all $U^{EA} \in \mathcal{D}^{EA}$,

$$\begin{aligned} \mathbf{x}R_{U^{EA}}^{EA}\mathbf{y} &\Leftrightarrow \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) - A(\mathbf{n}(\mathbf{x}_j)) \right] \\ &\geq \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) - A(\mathbf{n}(\mathbf{y}_j)) \right] \end{aligned}$$

¹¹ See Blackorby, Bossert and Donaldson [2001] for a discussion of the certainty version of these principles.

where U is the profile corresponding to U^{EA} according to (16). The function A can be written as

$$A(n) = \sum_{k=1}^n c_{k-1}$$

for all $n \in \mathcal{Z}_{++}$, where $c_0 \in \mathcal{R}$ is arbitrary and c_k is the ex-post critical level for population size $k \in \{1, \dots, n\}$. Note that the ex-post critical levels can depend on population size but not on utilities. We obtain

Theorem 8: *Suppose F^{EA} satisfies extended individual Bernoulli consistency. F^{EA} satisfies the extended social expected-utility hypothesis, ex-ante Pareto indifference, ex-ante binary independence of irrelevant alternatives, ex-ante minimal increasingness, extended ex-ante anonymity and critical-level consistency if and only if F^{EA} is ex-ante number-sensitive critical-level utilitarian with a non-decreasing function A .*

Proof. That the ex-ante number-sensitive critical-level utilitarian ex-ante social-evaluation functionals with a non-decreasing function A satisfy the required axioms is straightforward to verify. Now suppose F^{EA} satisfies the axioms. By critical-level consistency, there exists a function $C: \cup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n \rightarrow \mathcal{R}$ with the requisite properties. Let $n \in \mathcal{Z}_{++}$ be arbitrary and consider four prospects $\mathbf{x}, \mathbf{y}, \bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbf{X}_\theta$ and two profiles $U^{EA}, \bar{U}^{EA} \in \mathcal{D}^B$ such that

$$\mathbf{N}_\theta(\mathbf{x}) = \mathbf{N}_\theta(\mathbf{y}) = \{1, \dots, n\},$$

$$U_i(\mathbf{y}_j) = \sum_{k=1}^m p_k U_i(\mathbf{x}_k)$$

for all $i \in \{1, \dots, n\}$ and for all $j \in M$,

$$\mathbf{N}_\theta(\bar{\mathbf{x}}) = \mathbf{N}_\theta(\bar{\mathbf{y}}) = \{1, \dots, n+1\},$$

$$\bar{U}_i(\bar{\mathbf{x}}_j) = U_i(\mathbf{x}_j) \text{ and } \bar{U}_i(\bar{\mathbf{y}}_j) = U_i(\mathbf{y}_j)$$

for all $i \in \{1, \dots, n\}$ and for all $j \in M$, and

$$\bar{U}_{n+1}(\bar{\mathbf{x}}_j) = C(U(\mathbf{x}_j)) \text{ and } \bar{U}_{n+1}(\bar{\mathbf{y}}_j) = C(U(\mathbf{y}_j))$$

for all $j \in M$, where $U, \bar{U} \in \mathcal{U}$ are the profiles of ex-post utilities corresponding to U^{EA} and \bar{U}^{EA} . Using Theorem 6, it follows that $\mathbf{x} I_{U^{EA}}^A \mathbf{y}$. By repeated application of critical-level consistency, this implies $\bar{\mathbf{x}} I_{\bar{U}^{EA}}^A \bar{\mathbf{y}}$. Again using Theorem 6, it follows that

$$\sum_{j=1}^m p_j C(U(\mathbf{x}_j)) = C\left(\sum_{j=1}^m p_j U(\mathbf{x}_j)\right). \quad (22)$$

Because, for any m vectors $u^1, \dots, u^m \in \mathcal{R}^n$, a profile $U \in \mathcal{U}$ with the above properties can be chosen so that $U(\mathbf{x}_j) = u^j$ for all $j \in M$, the function C must satisfy

$$\sum_{j=1}^m p_j C(u^j) = C\left(\sum_{j=1}^m p_j u^j\right) \quad (23)$$

for all $u^1, \dots, u^m \in \mathcal{R}^n$. By Theorem 3 and critical-level consistency,

$$C(u^j) = C\left(\frac{1}{n} \sum_{i=1}^n u_i^j \mathbf{1}_n\right) \quad (24)$$

and

$$C\left(\sum_{j=1}^m p_j u^j\right) = C\left(\left[\sum_{j=1}^m p_j \frac{1}{n} \sum_{i=1}^n u_i^j\right] \mathbf{1}_n\right) \quad (25)$$

for all $j \in M$ and for all $u^j \in \mathcal{R}^n$. Fix $n \in \mathcal{Z}_{++}$ and define $\bar{C}^n(\tau) = C(\tau \mathbf{1}_n)$ for all $\tau \in \mathcal{R}$. Letting $t_j = (1/n) \sum_{i=1}^n u_i^j$ for all $j \in M$, (23), (24) and (25) together imply

$$\sum_{j=1}^m p_j \bar{C}^n(t_j) = \bar{C}^n\left(\sum_{j=1}^m p_j t_j\right) \quad (26)$$

for all $t \in \mathcal{R}^m$. Letting $z_j = p_j t_j$ for all $t \in \mathcal{R}^m$ and for all $j \in M$, this equation specializes to

$$\sum_{j=1}^m p_j \bar{C}^n(z_j/p_j) = \bar{C}^n\left(\sum_{j=1}^m z_j\right)$$

for all $z \in \mathcal{R}^m$. Defining $\hat{C}_j^n(z_j) = p_j \bar{C}^n(z_j/p_j)$ for all $z \in \mathcal{R}^m$ and for all $j \in M$, we obtain

$$\sum_{j=1}^m \hat{C}_j^n(z_j) = \bar{C}^n\left(\sum_{j=1}^m z_j\right) \quad (27)$$

for all $z \in \mathcal{R}^m$. This is a Pexider equation which has the solutions

$$\hat{C}_j^n(\tau) = d^n \tau + \bar{c}_j^n \quad (28)$$

and

$$\bar{C}^n(\tau) = d^n \tau + \sum_{j=1}^m \bar{c}_j^n \quad (29)$$

for all $\tau \in \mathcal{R}$ and for all $j \in M$, where $d^n \in \mathcal{R}$ and $\bar{c}_j^n \in \mathcal{R}$ for all $j \in M$. To establish that there are no further solutions, we show that the \hat{C}_j^n (and, thus, \bar{C}^n) must be bounded below on a non-degenerate interval (see Aczél [1966, p. 34 and p. 142]). Let $j \in M$ and

consider an arbitrary $z \in \mathcal{R}^m$. Given z , let $\bar{z} \in \mathcal{R}^m$ be such that $\bar{z}_k = z_k$ for all $k \in M \setminus \{j\}$ and $\bar{z}_j = z_0$ where $z_0 \in \mathcal{R}$ is fixed. Applying Theorem 3 and critical-level consistency, it follows that

$$\sum_{k=1}^m z_k + \sum_{k=1}^m \hat{C}_k^n(z_k) \geq \sum_{k=1}^m \bar{z}_k + \sum_{k=1}^m \hat{C}_k^n(\bar{z}_k) \Leftrightarrow \sum_{k=1}^m z_k \geq \sum_{k=1}^m \bar{z}_k. \quad (30)$$

Substituting the definition of \bar{z} and rearranging, (30) implies that

$$\hat{C}_j^n(z_j) \geq \hat{C}_j^n(z_0) + z_0 - z_j$$

for all $z_j \geq z_0$. This implies that \hat{C}_j^n is bounded below on any interval $[a, b]$ with $z_0 < a < b$ and, therefore, the only solutions of (27) are given by (28) and (29). Using (24), (26) and the definition of \bar{C}^n , substituting back yields

$$C(u^j) = C\left(\frac{1}{n} \sum_{i=1}^n u_i^j \mathbf{1}_n\right) = \bar{C}^n\left(\frac{1}{n} \sum_{i=1}^n u_i^j\right) = \frac{d^n}{n} \sum_{i=1}^n u_i^j + c_n$$

for all $j \in M$ and for all $u^j \in \mathcal{R}^n$, where $c_n = \sum_{j=1}^m \bar{c}_j^n$. By critical-level consistency, critical levels must be non-negative and, because average utility can be arbitrarily high or arbitrarily low for any given value of n , it follows that d^n must be equal to zero for all $n \in \mathcal{Z}_{++}$. Thus, there exists a sequence $(c_n)_{n \in \mathcal{Z}_{++}}$ such that the critical-level function C is given by $C(u) = c_n$ for all $n \in \mathcal{Z}_{++}$ and for all $u \in \mathcal{R}^n$. Because the range of C is \mathcal{R}_+ , it follows that $c_n \geq 0$ for all $n \in \mathcal{Z}_{++}$. Letting $c_0 \in \mathcal{R}$ be arbitrary, setting $A(n) = \sum_{i=1}^n c_{i-1}$ for all $n \in \mathcal{Z}_{++}$ and using the definition of critical levels as in the proof of Theorem 7, it follows that F^{EA} is ex-ante number-sensitive critical-level utilitarian. Because the c_n are non-negative for all $n \in \mathcal{Z}_{++}$, A is non-decreasing. ■

The proof of Theorem 8 can be illustrated by an example which is depicted in Table 2. As in the previous example, there are two states. In prospects \mathbf{x} and \mathbf{y} , the population is $\{1, \dots, n\}$ and, in prospects $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, the population is $\{1, \dots, n+1\}$. u_i^j is the utility level of person i in state j and $EU_i = \sum_{j=1}^2 p_j u_i^j$ is the expected utility of person i . Because each person has the same expected utility in prospects \mathbf{x} and \mathbf{y} , they are equally good.

Table 2

	Prospect \mathbf{x}		Prospect \mathbf{y}	
	State 1	State 2	State 1	State 2
Person 1	u_1^1	u_1^2	EU_1	EU_1
\vdots	\vdots	\vdots	\vdots	\vdots
Person n	u_n^1	u_n^2	EU_n	EU_n
	Prospect $\bar{\mathbf{x}}$		Prospect $\bar{\mathbf{y}}$	
	State 1	State 2	State 1	State 2
Person 1	u_1^1	u_1^2	EU_1	EU_1
\vdots	\vdots	\vdots	\vdots	\vdots
Person n	u_n^1	u_n^2	EU_n	EU_n
Person $n + 1$	$C(u_1^1, \dots, u_n^1)$	$C(u_1^2, \dots, u_n^2)$	$C(EU_1, \dots, EU_n)$	$C(EU_1, \dots, EU_n)$

In prospects $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, the utilities of persons $1, \dots, n$ are the same as in prospects \mathbf{x} and \mathbf{y} . In $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, the added person has a utility levels given by the critical-level function whose existence is guaranteed by critical-level consistency. That axiom implies that prospects \mathbf{x} and $\bar{\mathbf{x}}$ are equally good and that prospects \mathbf{y} and $\bar{\mathbf{y}}$ are equally good. Because \mathbf{x} and \mathbf{y} are equally good, transitivity implies that $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are equally good. Because, by Theorem 6, the social-evaluation functional is same-number utilitarian, it must therefore be true that expected utilities for person $n + 1$ are the same in prospects $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$. Thus,

$$C(EU_1, \dots, EU_n) = C\left(\sum_{j=1}^2 p_j u_1^j, \dots, \sum_{j=1}^2 p_j u_n^j\right) = \sum_{j=1}^2 p_j C(u_1^j, \dots, u_n^j).$$

The solution of this functional equation (equation (22) in the proof) allows C to depend on population size but not on utility levels. The non-negativity requirement for ex-post critical levels plays an important role in this result.

The conclusion of Theorem 8 is quite remarkable. Critical levels are *shown* to be independent of utilities (but may depend on population sizes) as a result of adding other axioms—critical-level consistency by itself allows for utility-dependent critical levels.

Analogous to the fixed-population case, there is a single variable-population ordering which, when applied to ex-ante utilities, can be used to rank complete prospects and, when

applied to ex-post utilities, can be used to rank prospects whenever they differ in a single state only.

5. Variable probability distributions and prospects

The theorems of Sections 3 and 4 can be extended so that combinations of lotteries and prospects are ranked. To do this, we let $\mathbf{P} = \{p \in \mathcal{R}_+^m \mid \sum_{j=1}^m p_j = 1\}$ and define an uncertain alternative (U-alternative) to be a compound vector consisting of a probability vector $p \in \mathbf{P}$ and a prospect $\mathbf{x} \in \mathbf{X}$. That is, $(p, \mathbf{x}) \in \mathbf{P} \times \mathbf{X}$ is a U-alternative in which x_j is realized with probability p_j for all $j \in M$. Because probabilities are allowed to be zero, the case in which the number of states is U-alternative dependent is implicitly covered.

We first turn to the fixed-population case and, as in Section 2, let the population be $N = \{1, \dots, n\}$. Individual ex-ante utility functions can depend on the probability vector p . $U_i^{UEA}: \mathbf{P} \times \mathbf{X} \rightarrow \mathcal{R}$ is the individual ex-ante utility function for person $i \in N$ and \mathcal{U}^{UEA} is the set of all profiles of such functions. If the individual Bernoulli hypothesis is satisfied,

$$U_i^{UEA}(p, \mathbf{x}) = \sum_{j=1}^m p_j U_i(\mathbf{x}_j)$$

where $U_i \in \mathcal{U}$ is person i 's ex-post utility function. \mathcal{U}^{UB} is the set of all profiles of ex-ante utility functions that satisfy the Bernoulli hypothesis.

An ex-ante social-evaluation functional is a mapping $F^{UEA}: \mathcal{D}^{UEA} \rightarrow \mathcal{O}^{UEA}$ where $\emptyset \neq \mathcal{D}^{UEA} \subseteq \mathbf{P} \times \mathcal{U}^{UEA}$ is the domain and \mathcal{O}^{UEA} is the set of all orderings on $\mathbf{P} \times \mathbf{X}$. We write $R_{\mathcal{U}^{UEA}}^{UEA} = F^{UEA}(\mathcal{U}^{UEA})$ and use $P_{\mathcal{U}^{UEA}}^{UEA}$ and $I_{\mathcal{U}^{UEA}}^{UEA}$ to denote the asymmetric and symmetric factors of $R_{\mathcal{U}^{UEA}}^{UEA}$.

The fixed-population axioms are straightforward generalizations of the fixed-population axioms of Section 2.

Uncertainty Individual Bernoulli Consistency: $\mathcal{D}^{UEA} = \mathbf{P} \times \mathcal{D}^{UB}$.

Uncertainty Social Expected-Utility Hypothesis: There exists a function $U_0: \mathbf{P} \times \mathbf{X} \times \mathcal{D}^{UEA} \rightarrow \mathcal{R}$ such that, for all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$,

$$(p, \mathbf{x}) R_{\mathcal{U}^{UEA}}^{UEA} (q, \mathbf{y}) \Leftrightarrow \sum_{j=1}^m p_j U_0(\mathbf{x}_j, U^{UEA}) \geq \sum_{j=1}^m q_j U_0(\mathbf{y}_j, U^{UEA}).$$

Uncertainty Ex-Ante Pareto Indifference: For all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$, if $U^{UEA}(p, \mathbf{x}) = U^{UEA}(q, \mathbf{y})$, then $(p, \mathbf{x}) I_{\mathcal{U}^{UEA}}^{UEA} (q, \mathbf{y})$.

Uncertainty Ex-Ante Binary Independence of Irrelevant Alternatives: For all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA}, \bar{U}^{UEA} \in \mathcal{D}^{UEA}$, if $U^{UEA}(p, \mathbf{x}) = \bar{U}^{UEA}(p, \mathbf{x})$ and $U^{UEA}(q, \mathbf{y}) = \bar{U}^{UEA}(q, \mathbf{y})$, then

$$(p, \mathbf{x})R_{U^{UEA}}^{UEA}(q, \mathbf{y}) \Leftrightarrow (p, \mathbf{x})R_{\bar{U}^{UEA}}^{UEA}(q, \mathbf{y}).$$

Uncertainty Ex-Ante Minimal Increasingness: For all $a, b \in \mathcal{R}$, for all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$, if $U^{UEA}(p, \mathbf{x}) = a\mathbf{1}_n \gg b\mathbf{1}_n = U^{UEA}(q, \mathbf{y})$, then $(p, \mathbf{x})P_{U^{UEA}}^{UEA}(q, \mathbf{y})$.

Uncertainty Ex-Ante Same-People Anonymity: For all $U^{UEA}, \bar{U}^{UEA} \in \mathcal{D}^{UEA}$, if there exists a bijection $\rho: N \rightarrow N$ such that $U_i^{UEA} = \bar{U}_{\rho(i)}^{UEA}$ for all $i \in N$, then $R_{U^{UEA}}^{UEA} = R_{\bar{U}^{UEA}}^{UEA}$.

The following theorem generalizes Theorem 1.

Theorem 9: *If F^{UEA} satisfies uncertainty individual Bernoulli consistency, uncertainty ex-ante Pareto indifference and uncertainty ex-ante binary independence of irrelevant alternatives, then there exists a social-evaluation ordering R^U on \mathcal{R}^n such that, for all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$,*

$$(p, \mathbf{x})R_{U^{UEA}}^{UEA}(q, \mathbf{y}) \Leftrightarrow U^{UEA}(p, \mathbf{x})R^U U^{UEA}(q, \mathbf{y}). \quad (31)$$

The social-evaluation functional F^{UEA} is ex-ante utilitarian if and only if, for all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$,

$$\begin{aligned} (p, \mathbf{x})R_{U^{UEA}}^{UEA}(q, \mathbf{y}) &\Leftrightarrow \sum_{i=1}^n U_i^{UEA}(p, \mathbf{x}) \geq \sum_{i=1}^n U_i^{UEA}(q, \mathbf{y}) \\ &\Leftrightarrow \sum_{i=1}^n \sum_{j=1}^m p_j U_i(\mathbf{x}_j) \geq \sum_{i=1}^n \sum_{j=1}^m q_j U_i(\mathbf{y}_j) \\ &\Leftrightarrow \sum_{j=1}^m p_j \sum_{i=1}^n U_i(\mathbf{x}_j) \geq \sum_{j=1}^m q_j \sum_{i=1}^n U_i(\mathbf{y}_j). \end{aligned}$$

The following theorem is a consequence of Theorems 3 and 9.

Theorem 10: *Suppose F^{UEA} satisfies uncertainty individual Bernoulli consistency. F^{UEA} satisfies the uncertainty social expected-utility hypothesis, uncertainty ex-ante Pareto indifference, uncertainty ex-ante binary independence of irrelevant alternatives, uncertainty ex-ante minimal increasingness and uncertainty ex-ante same-people anonymity if and only if F^{UEA} is ex-ante utilitarian.*

Proof. If F^{UEA} is ex-ante utilitarian, it is straightforward to show that all of the axioms are satisfied. Now suppose that F^{UEA} satisfies the axioms of the theorem statement. By Theorem 9, there exists an ordering R^U on \mathcal{R}^n such that (31) is satisfied. Because R^U is independent of probabilities, it can be found by examining the restriction of $R_{U^{UEA}}^{UEA}$ to $\{\bar{p}\} \times \mathbf{X}$ where $\bar{p} \in \mathbf{P} \cap \mathcal{R}_{++}^m$. Theorem 3 implies that, for all $u, v \in \mathcal{R}^n$,

$$uR^U v \Leftrightarrow \sum_{i=1}^n u_i \geq \sum_{i=1}^n v_i$$

and, as a consequence, F^{UEA} is ex-ante utilitarian. ■

Although notational complexities have persuaded us not to include a formal demonstration, the results of Theorems 7 and 8 can be extended to cover all U-alternatives. To do so, the axioms presented in Section 4 must be extended to the variable-population environment of U-alternatives. In addition, it is a simple matter to rewrite the critical-level population principle and critical-level consistency in a similar way. We call the resulting axioms the uncertainty critical-level population principle and uncertainty critical-level consistency.

Using the argument in the proof of Theorem 10, it is immediate that all complete U-alternatives (in which each person is alive in all states or in none) with the same number of people must be ranked with ex-ante utilitarianism.

Given the extended axioms and the uncertainty critical-level population principle, the uncertainty ex-ante critical-level utilitarian principles with a non-negative critical level are characterized. That is, there exists $\alpha \geq 0$ such that, for all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$,

$$(p, \mathbf{x})R_{U^{UEA}}^{EA}(q, \mathbf{y}) \Leftrightarrow \sum_{j=1}^m p_j \sum_{i \in \mathbf{N}(\mathbf{x}_j)} [U_i(\mathbf{x}_j) - \alpha] \geq \sum_{j=1}^m q_j \sum_{i \in \mathbf{N}(\mathbf{y}_j)} [U_i(\mathbf{y}_j) - \alpha]. \quad (32)$$

If uncertainty critical-level consistency is used instead of the uncertainty critical-level population principle, the argument in the proof of Theorem 8 establishes that ex-post critical levels are utility-independent but may depend on the number of people alive. We therefore obtain a characterization of uncertainty ex-ante number-sensitive critical-level utilitarianism with a non-decreasing function A . That is, there exists a non-decreasing function $A: \mathcal{Z}_{++} \rightarrow \mathcal{R}$ such that, for all $(p, \mathbf{x}), (q, \mathbf{y}) \in \mathbf{P} \times \mathbf{X}$ and for all $U^{UEA} \in \mathcal{D}^{UEA}$,

$$\begin{aligned} (p, \mathbf{x})R_{U^{UEA}}^{UEA}(q, \mathbf{y}) &\Leftrightarrow \sum_{j=1}^m p_j \left[\sum_{i \in \mathbf{N}(\mathbf{x}_j)} U_i(\mathbf{x}_j) - A(\mathbf{n}(\mathbf{x}_j)) \right] \\ &\geq \sum_{j=1}^m q_j \left[\sum_{i \in \mathbf{N}(\mathbf{y}_j)} U_i(\mathbf{y}_j) - A(\mathbf{n}(\mathbf{y}_j)) \right]. \end{aligned}$$

As in Section 4, the function A can be written as

$$A(n) = \sum_{k=1}^n c_{k-1}$$

for all $n \in \mathcal{Z}_{++}$, where $c_0 \in \mathcal{R}$ is arbitrary and c_k is the ex-post critical level for population size $k \in \{1, \dots, n\}$.

A simple example illustrates the application of the first of these classes of social-evaluation functionals. Suppose that, in the year 2100, astronomers discover that an asteroid is on a collision course with Earth and will obliterate life on the planet if nothing is done (alternative x). If, however, resources are committed to a very costly international program, the asteroid might be diverted and the planet saved. The probability of success (alternative y) is $1/2$. If the program fails, alternative z is realized. Population sizes and average utilities are given in Table 3.

Table 3

	Population size	Average utility
x	10 billion	50
y	20 billion	40
z	10 billion	35

If nothing is done (x), 10 billion people live with an average utility level of 50. If the asteroid is diverted (y), the number of people who ever live is doubled and their average utility is 40; average well-being is lower because of the resources used to divert the asteroid. If the attempt to divert the asteroid fails (z), 10 billion people live with an average utility of 35; average utility is still lower because of the sacrifice of the initial population.

The possibilities can be expressed by means of the two U-alternatives $((1, 0), (x, x))$ and $((1/2, 1/2), (y, z))$. According to (32), taking action is better than doing nothing if and only if

$$\frac{1}{2}[20(40 - \alpha)] + \frac{1}{2}[10(35 - \alpha)] > 10(50 - \alpha),$$

which obtains if and only if $\alpha < 15$.

6. Concluding remarks

Harsanyi's social aggregation theorem is one of the most fundamental results in social-choice theory. Its original formulation and most of the subsequent literature are phrased

in terms of variable lotteries and a single utility profile. The absence of a multi-profile framework necessitates some regularity assumptions and, as a consequence, the result lacks, to some extent, transparency. Moreover, the anonymity axioms of standard social-choice theory cannot be applied in the single-utility-profile setting. Our formulation in terms of prospects in a multi-profile setting allows us to proceed without regularity assumptions (not even continuity is required if anonymity is assumed) and fits naturally with the standard multi-profile social-choice framework.

The proof technique in the fixed-population case is novel: we prove a variant of Harsanyi's theorem by showing that translation-scale non-comparability is implied, which permits us to involve a classical result on utilitarian social-evaluation functionals under certainty. This differs substantially from the methods employed in the lottery setting.

The variable-population model represents a substantial generalization of the fixed-population case. Although the fixed-population results are used in our characterizations, there is a considerable amount of additional complexity. By using an axiom—critical-level consistency—that associates an equally-good complete prospect with every incomplete one, we are able to reduce the plethora of ethically acceptable social-evaluation functions to a single class, the ex-ante number-sensitive utilitarian functionals. The ex-ante critical-level utilitarian subclass results if the stronger critical-level population principle is employed.

References

- Aczél, J., 1966, *Lectures on Functional Equations and Their Applications*, Academic Press, New York.
- Arrow, K., 1972, Exposition of the theory of choice under uncertainty, in *Decision and Organization*, C. McGuire and R. Radner, eds., North-Holland, Amsterdam, 19–55.
- Blackorby, C., W. Bossert and D. Donaldson, 1995, Intertemporal population ethics: critical-level utilitarian principles, *Econometrica* **63**, 1303–1320.
- Blackorby, C., W. Bossert and D. Donaldson, 1997, Critical-level utilitarianism and the population-ethics dilemma, *Economics and Philosophy* **13**, 197–230.
- Blackorby, C., W. Bossert and D. Donaldson, 1998, Uncertainty and critical-level population principles, *Journal of Population Economics* **11**, 1–20.
- Blackorby, C., W. Bossert and D. Donaldson, 1999, Information invariance in variable-population social-choice problems, *International Economic Review* **40**, 403–422.
- Blackorby, C., W. Bossert and D. Donaldson, 2001, Population principles with number-dependent critical levels, *Journal of Public Economic Theory* **4**, 347–368.
- Blackorby, C., W. Bossert and D. Donaldson, 2002, Utilitarianism and the theory of justice, in *Handbook of Social Choice and Welfare*, K. Arrow, A. Sen and K. Suzumura, eds., Elsevier, Amsterdam, 543–596.
- Blackorby, C. and D. Donaldson, 1984, Social criteria for evaluating population change, *Journal of Public Economics* **25**, 13–33.

- Blackorby, C., D. Donaldson and J. Weymark, 1980, On John Harsanyi's defences of utilitarianism, Discussion Paper 80-04, University of British Columbia, Department of Economics.
- Blackorby, C., D. Donaldson and J. Weymark, 1984, Social choice with interpersonal utility comparisons: a diagrammatic introduction, *International Economic Review* **25**, 327–356.
- Blackorby, C., D. Donaldson and J. Weymark, 1999, Harsanyi's social aggregation theorem for state-contingent alternatives, *Journal of Mathematical Economics* **32**, 365–387.
- Blackorby, C., D. Donaldson and J. Weymark, 2003, Social aggregation and the expected utility hypothesis, in *Justice, Political Liberalism, and Utilitarianism: Themes from Harsanyi and Rawls*, M. Salles and J. Weymark, eds., Cambridge University Press, Cambridge, forthcoming.
- Blackwell, D. and M. Girshick, 1954, *Theory of Games and Statistical Decisions*, Wiley, New York.
- Border, K., 1981, Notes on von Neumann – Morgenstern social welfare functions, unpublished manuscript, California Institute of Technology, Division of the Humanities and Social Sciences.
- Bossert, W. and J. Weymark, 2003, Utility in social choice, in *Handbook of Utility Theory, vol. 2: Extensions*, S. Barberà, P. Hammond and C. Seidl, eds., Kluwer, Dordrecht, forthcoming.
- Broome, J., 1990, Bolker-Jeffrey expected utility theory and axiomatic utilitarianism, *Review of Economic Studies* **57**, 477–502.
- Broome, J., 1991, *Weighing Goods*, Basil Blackwell, Oxford.
- Broome, J., 1993, Goodness is reducible to betterness: the evil of death is the value of life, in *The Good and the Economical: Ethical Choices in Economics and Management*, P. Koslowski, ed., Springer, Berlin, 69–83.
- Broome, J., 1999, *Weighing Lives*, unpublished manuscript.
- Coulhon, T. and P. Mongin, 1989, Social choice theory in the case of von Neumann – Morgenstern utilities, *Social Choice and Welfare* **6**, 175–187.
- d'Aspremont, C. and L. Gevers, 1977, Equity and the informational basis of collective choice, *Review of Economic Studies* **44**, 199–209.
- De Meyer, B. and P. Mongin, 1995, A note on affine aggregation, *Economics Letters* **47**, 177–183.
- Diamond, P., 1967, Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility: comment, *Journal of Political Economy* **75**, 765–766.
- Domotor, Z., 1979, Ordered sum and tensor product of linear utility structures, *Theory and Decision* **11**, 375–399.
- Fishburn, P., 1984, On Harsanyi's utilitarian cardinal welfare theorem, *Theory and Decision* **17**, 21–28.
- Hammond, P., 1981, Ex-ante and ex-post welfare optimality under uncertainty, *Economica* **48**, 235–250.

- Hammond, P., 1983, Ex-post optimality as a dynamically consistent objective for collective choice under uncertainty, in *Social Choice and Welfare*, P. Pattanaik and M. Salles, eds., North-Holland, Amsterdam, 175–205.
- Harsanyi, J., 1955, Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility, *Journal of Political Economy* **63**, 309–321.
- Harsanyi, J., 1977, *Rational Behavior and Bargaining Equilibrium in Games and Social Situations*, Cambridge University Press, Cambridge.
- Milnor, J., 1954, Games against nature, in *Decision Processes*, R. Thrall, C. Coombs and R. Davis, eds., Wiley, New York, 49–59.
- Mongin, P., 1994, Harsanyi’s aggregation theorem: multi-profile version and unsettled questions, *Social Choice and Welfare* **11**, 331–354.
- Mongin, P., 1995, Consistent Bayesian aggregation, *Journal of Economic Theory* **66**, 313–351.
- Mongin, P., 1998, The paradox of the Bayesian experts and state-dependent utility theory, *Journal of Mathematical Economics* **29**, 331–361.
- Mongin, P. and C. d’Aspremont, 1998, Utility theory in ethics, in *Handbook of Utility Theory, vol. 1: Principles*, S. Barberà, P. Hammond and C. Seidl, eds., Kluwer, Dordrecht, 371–481.
- Roberts, K., 1980, Interpersonal comparability and social choice theory, *Review of Economic Studies* **47**, 421–439.
- Roemer, J., 1996, *Theories of Distributive Justice*, Harvard University Press, Cambridge.
- Sen, A., 1976, Welfare inequalities and Rawlsian axiomatics, *Theory and Decision* **7**, 243–262.
- Sen, A., 1977, Non-linear social welfare functions: a reply to Professor Harsanyi, in *Foundational Problems in the Special Sciences*, R. Butts and J. Hintikka, eds., Reidel, Dordrecht, 297–302.
- Sen, A., 1986, Social choice theory, in *Handbook of Mathematical Economics, vol. III*, K. Arrow and M. Intriligator, eds., North-Holland, Amsterdam, 1073–1181.
- von Neumann, J. and O. Morgenstern, 1944 (second ed. 1947), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton.
- Weymark, J., 1991, A reconsideration of the Harsanyi-Sen debate on utilitarianism, in *Interpersonal Comparisons of Well-Being*, J. Elster and J. Roemer, eds., Cambridge University Press, Cambridge, 255–320.
- Weymark, J., 1993, Harsanyi’s social aggregation theorem and the weak Pareto principle, *Social Choice and Welfare* **10**, 209–222.
- Weymark, J., 1994, Harsanyi’s social aggregation theorem with alternative Pareto principles, in *Models and Measurement of Welfare and Inequality*, W. Eichhorn, ed., Springer, Berlin, 869–887.
- Weymark, J., 1995, Further remarks on Harsanyi’s social aggregation theorem and the weak Pareto principle, *Social Choice and Welfare* **12**, 87–92.
- Zhou, L., 1997, Harsanyi’s utilitarianism theorems: general societies, *Journal of Economic Theory* **72**, 198–207.