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Automorphism groups of locally compact connected double loops are locally compact

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It is a well-known fact that the group Σ of all continuous collineations of a compact projective plane is a locally compact transformation group, see [4, §2], [1, Thm. 1], or [2, XI.3.1] (all topological groups in this paper are provided with the compact-open topology). This implies that the automorphism group Γ of a locally compact connected ternary field is locally compact, since that ternary field coordinatizes some compact projective plane [2, XI.3.2]. For "good" ternary fields, e.g. ternary fields having an associative addition or a diassociative multiplication, the automorphism group Γ is even a compact group. However, it is unknown whether the group Γ is compact in general. Considering the more general algebraic structure of a double loop instead of a ternary field, it was also an open problem whether the automorphism group of a locally compact connected double loop $\mathscr{D}$ is locally compact. In this paper, we shall give a positive answer to this question. The proof of the following theorem is based on the ideas of the proof in [4], which shows that the automorphism group of a compact connected projective plane is locally compact.

Theorem. *The automorphism group of a locally compact connected double loop is a locally compact transformation group with respect to the compact-open topology.*

Proof. Let $\mathscr{D} = (\mathscr{D}, +, \circ)$ be a locally compact connected double loop and let $\hat{\mathscr{D}} := \mathscr{D} \cup \{\infty\}$ denote its one-point compactification. Since $\mathscr{D}$ is metrizable and $\hat{\mathscr{D}}$ is homogeneous by [2, XI.1.2], the space $\hat{\mathscr{D}}$ is also metrizable by the metrization criterion of Urysohn. So let d be a bounded metric for $\hat{\mathscr{D}}$. The compact-open topology on $\Gamma = \mathrm{Aut}(\mathscr{D})$ is induced by the supremum metric

$$d^*(\gamma, \lambda) = \sup_{x \in \mathscr{D}} d(x^\gamma, x^\lambda).$$

For convenience we may choose the metric d on $\hat{\mathscr{D}}$ such that $d(0, \infty) \geqq 1$ and $d(0, 1) \geqq 1$. For $\varepsilon > 0$ and $x_0 \in \hat{\mathscr{D}}$ we define closed neighborhoods $U_\varepsilon(x_0) := \{x \in \hat{\mathscr{D}};\, d(x, x_0) \leqq \varepsilon\}$ in $\hat{\mathscr{D}}$ and $U_\varepsilon^* := \{\gamma \in \Gamma;\, d^*(\mathbb{1}, \gamma) \leqq \varepsilon\}$ in Γ. In the sequel we shall show that the neighborhood U_ε^* is compact if $\varepsilon < 1/4$. By the Arzela-Ascoli theorem we have to verify the equicontinuity of the set U_ε^*.

Assume that the set U_ε^* is not equicontinuous for some $\varepsilon < 1/4$. Then there are sequences $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{D}$ and $(\gamma_n)_{n\in\mathbb{N}}$ in U_ε^* such that $\lim_{n\to\infty} a_n = a$ and $d(a^{\gamma_n}, a_n^{\gamma_n}) \geqq \varepsilon$ hold for all $n\in\mathbb{N}$. Passing to subsequences we may assume by the compactness of $\hat{\mathscr{D}}$ that $\lim_{n\to\infty} a^{\gamma_n} = b \in \hat{\mathscr{D}}$ and $\lim_{n\to\infty} a_n^{\gamma_n} = u \in \hat{\mathscr{D}}$. Since the map $x \mapsto x - b$ is a homeomorphism of $\hat{\mathscr{D}}$, there is a real number ε' with $0 < \varepsilon' \leqq \varepsilon$ such that $d(a_n^{\gamma_n} - b, a^{\gamma_n} - b) \geqq \varepsilon'$ holds for all $n\in\mathbb{N}$. From the inequality

$$\begin{aligned}\varepsilon' &\leqq d(a_n^{\gamma_n} - b, a^{\gamma_n} - b)\\ &\leqq d(a_n^{\gamma_n} - b, a_n^{\gamma_n} - a^{\gamma_n}) + d(a_n^{\gamma_n} - a^{\gamma_n}, 0) + d(0, a^{\gamma_n} - b)\end{aligned}$$

we obtain that

$$d((a_n - a)^{\gamma_n}, (a-a)^{\gamma_n}) = d(a_n^{\gamma_n} - a^{\gamma_n}, 0) \geqq \varepsilon'/2$$

holds for almost all $n\in\mathbb{N}$. By renaming we thus may assume that $\lim_{n\to\infty} a_n = 0$ and $d(a_n^{\gamma_n}, 0) \geqq \varepsilon$ for all $n\in\mathbb{N}$. In particular, we have $\lim_{n\to\infty} a_n^{\gamma_n} = u \neq 0$.

Because the double loop $\mathscr{D}$ is separable by [2, XI.1.2], there exists a countable dense subset $M := \{x_n; n\in\mathbb{N}\}$ of $\mathscr{D}$ which contains the sequence $(a_n)_{n\in\mathbb{N}}$. Next, we inductively choose for every $m\in\mathbb{N}$ subsequences $(\gamma_{m,n})_{n\in\mathbb{N}}$ of $(\gamma_n)_{n\in\mathbb{N}}$ satisfying the conditions

(i) $\lim_{n\to\infty} x_m^{\gamma_{m,n}} = \tilde{x}_m$ exists in $\hat{\mathscr{D}}$,

(ii) $(\gamma_{m+1,n})_{n\in\mathbb{N}}$ is a subsequence of $(\gamma_{m,n})_{n\in\mathbb{N}}$.

Set $\gamma_n' := \gamma_{n,n} = \gamma_{k_n}$ and $a_n' := a_{k_n}$ for all $n\in\mathbb{N}$. Since for the rest of the proof we are only considering the diagonal sequence $(\gamma_n')_{n\in\mathbb{N}}$, no confusion will arise if we now write again $(\gamma_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ instead of $(\gamma_n')_{n\in\mathbb{N}}$ and $(a_n')_{n\in\mathbb{N}}$. Then the equality $\lim_{n\to\infty} a_n^{\gamma_n} = u$ still holds and furthermore we have

$$\lim_{n\to\infty} x_m^{\gamma_n} = \tilde{x}_m \in \hat{\mathscr{D}}$$

for any $m\in\mathbb{N}$. Moreover, condition (i) implies that the dense subset M is contained in

$$L := \left\{x\in\mathscr{D};\ \lim_{n\to\infty} x^{\gamma_n} \text{ exists in } \hat{\mathscr{D}}\right\}.$$

In particular, the set L is also dense in $\mathscr{D}$. Consider the map $\lambda: L \to \hat{\mathscr{D}}: x \mapsto \tilde{x}$. Then, for any elements $x, y \in L$ we have

$$(*) \qquad d(x, y^\lambda) \leqq d(x, y) + \varepsilon,$$

since $d(x, y^{\gamma_n}) \leqq d(x, y) + d(y, y^{\gamma_n}) \leqq d(x, y) + d(\mathbb{1}, \gamma_n) \leqq d(x, y) + \varepsilon$ holds for all $n\in\mathbb{N}$. Set $R := \left\{x\in L;\ \lim_{n\to\infty} x^{\gamma_n} \in \mathscr{D}\right\}$. It can easily be verified that R is a topological double loop with respect to the relative topology. Since all double loop operations in $\mathscr{D}$ are continuous, the closure $\bar{R}$ of R in $\mathscr{D}$ is also a (locally compact) topological double loop. Moreover, the double loop $\bar{R}$ contains the neighborhood $U_\varepsilon(0)$. To prove this, note that for any element $y \in L \cap U_\varepsilon(0)$ we have $d(0, y^\lambda) \leqq d(0, y) + \varepsilon \leqq 2\varepsilon < d(0, \infty)$ by inequality $(*)$. This implies that $y^\lambda \neq \infty$ and thus we have $y \in R$. Thus the set R is dense in $U_\varepsilon(0)$, i.e. the closure $\bar{R}$ contains $U_\varepsilon(0)$. Consequently, the sub-double-loop $\bar{R}$ is open in $\mathscr{D}$, because it contains an open subset (see [3, (3.2)]). Since $\mathscr{D}$ is connected, this implies $\bar{R} = \mathscr{D}$.

In the next step we want to show that $u \neq \infty$. For this, note that for almost all $n \in \mathbb{N}$ the inequalities

1) $d(a_n^{\gamma_n}, u) < \varepsilon$,
2) $d(a_n, \infty) > 1 - \varepsilon$

are satisfied. The first inequality follows from the relation $\lim_{n \to \infty} a_n^{\gamma_n} = u$ and the second can be deduced from $1 \leqq d(0, \infty) \leqq d(0, a_n) + d(a_n, \infty)$ and $\lim_{n \to \infty} a_n = 0$. Combining these two inequalities, we obtain from

$$d(a_n, \infty) \leqq d(a_n, a_n^{\gamma_n}) + d(a_n^{\gamma_n}, u) + d(u, \infty)$$

the estimate

$$d(u, \infty) \geqq (1 - \varepsilon) - \varepsilon - \varepsilon = 1 - 3\varepsilon > 0,$$

which proves that $u \neq \infty$.

Being the pointwise limit of double loop homomorphisms, the restriction $\lambda|_R : R \to R^\lambda$ is also a double loop homomorphism. Thus the sub-double-loop R^λ of $\mathscr{D}$ is a topological double loop with respect to the relative topology. Furthermore, the map $\lambda|_R$ is injective. To see this, fix an element $r \in R \setminus \{0\}$. Since R is a dense subset of $\mathscr{D}$ and since the mapping $x \mapsto r \circ x : R \to R$ is a bijection of R, there is an element $s \in R$ such that $r \circ s \in U_\varepsilon(1)$. Thus we obtain from (*)

$$d(1, (r \circ s)^\lambda) \leqq d(1, r \circ s) + \varepsilon \leqq 2\varepsilon < 1.$$

Since $d(0, 1) \geqq 1$, we infer that $(r \circ s)^\lambda \neq 0$. Because $\lambda|_R$ is a multiplicative map, this means that $r^\lambda \neq 0$. Hence, the double loop homomorphism $\lambda|_R$ is injective. The loop R is infinite, because R is dense in $\mathscr{D}$. So, the set R^λ is infinite, too, and thus the closure $\overline{R^\lambda}$ in $\hat{\mathscr{D}}$ contains an accumulation point v of R^λ. Let $(v_n)_{n \in \mathbb{N}}$ be a sequence in R^λ which converges to the accumulation point v. Then the sequence $(v_{n+1} - v_n)_{n \in \mathbb{N}}$ is also contained in R^λ and converges to the neutral element 0. Hence, there is an element $c \in R \setminus \{0\}$ such that the element $b \in R$ which is uniquely determined by the relation $u \circ b = c^\lambda$ is not contained in $U_{3\varepsilon}(\infty)$. Now let $b_n \in \mathscr{D}$ denote the (unique) solution of the equation $a_n \circ b_n = c$. Then we have $\lim_{n \to \infty} b_n = \infty$, because $\lim_{n \to \infty} a_n = 0$ and $c \neq 0$. Moreover, we have

$$c^\lambda = \lim_{n \to \infty} (a_n \circ b_n)^{\gamma_n} = \lim_{n \to \infty} a_n^{\gamma_n} \circ b_n^{\gamma_n}.$$

Thus the limit $\lim_{n \to \infty} b_n^{\gamma_n}$ exists in $\mathscr{D}$, since $\lim_{n \to \infty} a_n^{\gamma_n} = u$. Because the element b is the unique solution of the equation $u \circ b = c^\lambda$, we consequently obtain that

$$\lim_{n \to \infty} b_n^{\gamma_n} = b.$$

So we may choose an integer $n \in \mathbb{N}$ such that $d(b_n, \infty) \leqq \varepsilon$ and $d(b, b_n^{\gamma_n}) \leqq \varepsilon$. Inequality (*) now yields the contradiction

$$3\varepsilon < d(b, \infty) \leqq d(b, b_n^{\gamma_n}) + d(b_n^{\gamma_n}, b_n) + d(b_n, \infty) \leqq 3\varepsilon.$$

This contradiction shows that U_ε^* is an equicontinuous set. Hence, the closure $\overline{U_\varepsilon^*}$ of U_ε^* in the space $C(\hat{\mathscr{D}}, \hat{\mathscr{D}})$ of all continuous mapping of $\hat{\mathscr{D}}$ is compact by the Arzela-Ascoli theorem.

To finish the proof, we shall show that $\overline{U_\varepsilon^*} \subseteqq U_\varepsilon^*$. For that, fix an element $\gamma \in \overline{U_\varepsilon^*}$. We have to verify that γ is a double loop automorphism. Since there is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in U_ε^*, which converges to γ, the map γ is a double loop homomorphism. Moreover, the sequence $(\gamma_n^{-1})_{n \in \mathbb{N}}$ has an accumulation point δ in the compact set $\overline{U_\varepsilon^*}$, because $U_\varepsilon^{*-1} = U_\varepsilon^* \subseteqq \overline{U_\varepsilon^*}$. Hence, the sequence $(\gamma_n \gamma_n^{-1})_{n \in \mathbb{N}}$ accumulates at the point $\gamma\delta$. But as $\gamma_n \gamma_n^{-1} = \mathbb{1}$ holds for each $n \in \mathbb{N}$, this implies that $\gamma\delta = \mathbb{1}$, since composition $C(\hat{\mathscr{D}}, \hat{\mathscr{D}})$ is continuous. Similarly, the relation $\delta\gamma = \mathbb{1}$ is verified. Consequently, the map γ is a bijection and also a double loop automorphism.

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