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Smooth Hughes planes are classical

By

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Abstract. We prove that the only compact projective Hughes planes which are smooth projective planes are the classical planes over the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the Caley numbers $\mathbb{O}$. As a by-product this shows that an 8-dimensional smooth projective plane which admits a collineation group of dimension $d \geqq 17$ is isomorphic to the quaternion projective plane $\mathscr{P}_2\mathbb{H}$. For topological compact projective planes this is true if $d \geqq 19$, and this bound is sharp.

The object of this paper is to prove the following theorem.

Theorem A. *Every smooth Hughes plane is isomorphic as a smooth plane to one of the Moufang planes* $\mathscr{P}_2\mathbb{F}$ *with* $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

The theory of compact *topological* projective planes is presented in the recent book [24] of Salzmann, Betten, Grundhöfer, Hähl, Löwen and Stroppel. A main theme of this theory is the classification of *sufficiently homogeneous* planes, i.e. planes that admit a collineation group of sufficiently large dimension. For 8-dimensional compact projective planes we have the following theorem ([24], 84.28):

Theorem. *Let* $\mathscr{P}$ *be an 8-dimensional compact projective plane. If* $\dim \mathrm{Aut}(\mathscr{P}) > 18$, *then* $\mathscr{P}$ *is isomorphic to the projective plane* $\mathscr{P}_2\mathbb{H}$ *over the quaternions* $\mathbb{H}$ *and* $\mathrm{Aut}(\mathscr{P}) \cong \mathrm{PSL}_3\mathbb{H}$.

Note that the dimension bound 18 is sharp: there exist non-classical compact planes with an 18-dimensional (Lie) group of collineations, [24], 82.25. Such planes turn out to be translation planes. According to J. Otte, [17] and [18], there are only four *smooth* translation planes.

Otte's Theorem. *Every smooth projective translation plane is isomorphic (as a smooth projective plane) to one of the classical projective planes* $\mathscr{P}_\mathbb{F}$ *defined over an alternative field* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

Thus, in the smooth case, we immediately can lower the bound by one:

Theorem. *Let* $\mathscr{P}$ *be a smooth 8-dimensional compact projective plane. If* $\dim \mathrm{Aut}(\mathscr{P}) > 17$, *then* $\mathscr{P} \cong \mathscr{P}_2\mathbb{H}$.

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According to the classification of 8-dimensional compact projective planes $\mathscr{P}$ (see [24], 84.28), $\dim \operatorname{Aut}(\mathscr{P}) = 17$ implies that $\mathscr{P}$ is a translation plane or a so-called Hughes plane (for a definition of a Hughes plane see Section 3). Thus, as a consequence of the results proved in this paper we can sharpen the last theorem as follows.

Theorem B. *Let $\mathscr{P}$ be an 8-dimensional smooth projective plane with automorphism group Σ. Then $\mathscr{P}$ is either isomorphic to the quaternion plane $\mathscr{P}_2\mathbb{H}$ and $\Sigma \cong \mathrm{PGL}_3\mathbb{H}$ (and hence $\dim \Sigma = 35$), or $\dim \Sigma \leqq 16$ holds.*

1. Definitions and auxiliary results.

Definition. A projective plane $\mathscr{P} = (P, \mathscr{L})$ is called a *smooth plane* if the set P of points and the set $\mathscr{L}$ of lines are smooth ($= \mathrm{C}^\infty$) manifolds such that the geometric operations $\vee : P \times P \setminus \operatorname{diag}(P) \to \mathscr{L}$ of join and $\wedge : \mathscr{L} \times \mathscr{L} \setminus \operatorname{diag}(\mathscr{L}) \to P$ of intersection are smooth mappings.

It is often convenient to regard a line as a *point row*, i.e. as the set of points incident with the line. We will use, however, the same symbol for the line and for its associated point row. By [1], point rows and line pencils are compact connected submanifolds of P and $\mathscr{L}$, respectively. Moreover, the set $\mathscr{F}$ of flags is a compact connected submanifold of the product space $P \times \mathscr{L}$. When speaking about submanifolds, we always mean smoothly embedded submanifolds, i.e. the inclusion map is a smooth embedding.

The group $\Gamma := \operatorname{Aut}(\mathscr{P})$ of continuous automorphisms of $\mathscr{P}$ is a Lie transformation group with respect to the compact-open topology on both the point space P and the line space $\mathscr{L}$, see [2], (2.4). Moreover, every continuous collineation of $\mathscr{P}$ is, in fact, a smooth map on P and on $\mathscr{L}$, [2], (2.3), cf. [5]. If a group G acts on a set X, we denote by $G_{[X]}$ the kernel of this action and we put $G^X := G/G_{[X]}$. For $x \in X$ the stabilizer of x in G is abbreviated by G_x. For a subset Y of X we set $G_Y = \{g \in G \mid Y^g \subseteq Y\}$ and $G_{[Y]} = \{g \in G \mid \forall y \in Y:\ y^g = y\}$. The connected component of a topological group G is written as G^1.

The integer $n = \dim P$, which is called the *dimension* of the projective plane $\mathscr{P}$, is always a power 2^k, where $k = 1, 2, 3, 4$, see [24], § 54. Moreover, for $n = 2l$, we have $\dim L = \dim \mathscr{L}_p = l$ for any line L and any line pencil $\mathscr{L}_p$.

By [1], for any point $p \in P$ the tangent space T_pP together with the *tangent spread* $\mathscr{S}_p = \{\mathrm{T}_pK \mid K \in \mathscr{L}_p\}$ induced by the lines of the line pencil $\mathscr{L}_p$ forms a locally compact affine translation plane $\mathscr{A}_p$. These affine planes are called *tangent translation planes*. Their projective closures are denoted by $\mathscr{P}_p$, and we write L_∞ for the line at infinity (which is also the translation line) of $\mathscr{A}_p$. Dually, we can define translation planes $\mathscr{A}_L$ for any line L. We denote by $\mathscr{A}_2\mathbb{F}$ the classical affine translation plane over the division ring $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and by $\mathscr{P}_2\mathbb{F}$ the projective closure of $\mathscr{A}_2\mathbb{F}$. Since every continuous collineation of a smooth projective plane $\mathscr{P}$ is smooth, we are able to compare the results of the topological situation with those in the smooth case. The stabilizer Γ_p of some point $p \in P$ induces an action on the tangent translation plane $\mathscr{A}_p$ via the *derivation mapping*

$$\mathrm{D}_p : \Gamma_p \to \Sigma_o := \operatorname{Aut}(\mathscr{A}_p)_o \leqq \mathrm{GL}(\mathrm{T}_pP) : \gamma \mapsto \mathrm{D}\gamma(p),$$

where $\operatorname{Aut}(\mathscr{A}_p)_o$ is the stabilizer of $\operatorname{Aut}(\mathscr{A}_p)$ at the origin $o \in \mathrm{T}_pP$. By [2], (3.3) and (3.9), the map D_p is a continuous homomorphism and $\ker \mathrm{D}_p = \Gamma_{[p,p]}$ is the subgroup of all elations of Γ having p as their center.

2. Subplanes of smooth projective planes. Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a projective plane and let $\mathscr{P}' = (P', \mathscr{L}', \mathscr{F}')$ be a subplane of $\mathscr{P}$, i.e. $P' \subseteqq P$, $L \in \mathscr{L}' \subseteqq \mathscr{L}$ if and only if L is incident with at least two points of P', and $\mathscr{F}' \subseteqq \mathscr{F}$. We will call elements (i.e., points, lines and flags) of $\mathscr{P}'$ *inner elements* of $\mathscr{P}$. An element of $\mathscr{P}$ which is not inner is called an *outer element*. A subplane $\mathscr{P}'$ of a topological (smooth) projective plane $\mathscr{P}$ is called *closed (smooth)* if P' is a closed subset (a submanifold) of P. Clearly, a subplane $\mathscr{P}'$ of a topological plane $\mathscr{P}$ is topological with respect to the induced topologies. Thus, by [24], Prop. 41.7, we have the following lemma.

Lemma 2.1. *Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a compact projective plane and let $\mathscr{P}' = (P', \mathscr{L}', \mathscr{F}')$ be a subplane of $\mathscr{P}$. The following statements are equivalent.*

(i) *$\mathscr{P}'$ is a closed subplane of $\mathscr{P}$.*
(ii) *$\mathscr{L}'$ is a closed subset of $\mathscr{L}$.*
(iii) *$\mathscr{F}'$ is a closed subset of $\mathscr{F}$.*
(iv) *The set of inner points of an inner line L is closed in the point row L.*
(v) *The set of inner lines through an inner point p is closed in $\mathscr{L}_p$.*

An analoguous result for the smooth case reads as follows.

Lemma 2.2. *Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a smooth projective plane and let $\mathscr{P}' = (P', \mathscr{L}', \mathscr{F}')$ be a subplane of $\mathscr{P}$. The following statements are equivalent.*

(i) *$\mathscr{P}'$ is a smooth subplane of $\mathscr{P}$.*
(ii) *$\mathscr{L}'$ is a submanifold of $\mathscr{L}$.*
(iii) *$\mathscr{F}'$ is a submanifold of $\mathscr{F}$.*
(iv) *The set L' of inner points of an inner line L is a submanifold of the point row L.*
(v) *The set $\mathscr{L}'_p$ of inner lines through an inner point p is a submanifold of $\mathscr{L}_p$.*

Proof. Let $\mathscr{P}'$ be a smooth subplane of $\mathscr{P}$. Fix an inner line L and choose some inner point p not on L. Since P' is a submanifold of P, we may consider the smooth map $f : U' \to P' : x \mapsto (x \vee p) \wedge L$ defined on some open subset U' of P'. Since $p \notin L$, we may choose U' in such a way that it contains the set $P' \cap L = L'$ of inner points. Clearly, we have $f \circ f = f$. This implies that $L' = f(U')$ is a submanifold of P', see [1], (2.5). Since the point row L is a submanifold of P and L' is contained in L, we infer that statement (iv) holds. The same kind of argument shows that (ii) implies (v). Since the perspectivities mapping a point row to a line pencil are diffeomorphisms (see [1], (1.7)), statements (iv) and (v) are equivalent. Since the line space $\mathscr{L}'$ is locally diffeomorphic to the product of two line pencils (see [1], (1.12 (i))), assertion (v) implies statement (ii). Dually, assertion (i) follows from (iv). Assuming (i) and (ii), we get from [1], (1.14) that $\mathscr{F}'$ is a submanifold of the product manifold $P' \times \mathscr{L}'$. Thus $\mathscr{F}' = \mathscr{F} \cap (P' \times \mathscr{L}')$ is a submanifold of $\mathscr{F}$. Conversely, let p be an inner point and let $\mathscr{L}'_p \subseteqq \mathscr{L}_p$ be the set of inner lines. Since the projections $\pi_P : \mathscr{F} \to P$ and $\pi_{\mathscr{L}} : \mathscr{F} \to \mathscr{L}$ are submersions ([1], (1.15)), the inverse image $\mathscr{B} := \pi_P^{-1}(p) = \{p\} \times \mathscr{L}_p$ is a submanifold of $\mathscr{F}$ which is mapped diffeomorphically onto $\mathscr{L}_p$ by $\pi_{\mathscr{L}}$ ([1], (1.16)). The subset $\mathscr{B}' := \mathscr{B} \cap \mathscr{F}'$ is a submanifold of $\mathscr{B}$: fix some line K not incident with p and consider the smooth map

$$f_K : \mathscr{F}' \setminus \{(r, K) \mid (r, K) \in \mathscr{F}\} \to \mathscr{B}' : (q, L) \mapsto (p, (L \wedge K) \vee p).$$

Clearly, the map f_K is a smooth retraction onto $\mathscr{B}'$. This implies that $\mathscr{B}'$ is a submanifold of $\mathscr{F}'_K := \mathscr{F}' \setminus \{(r,K)(r,K) \in \mathscr{F}\}$, see, e.g., [1], (2.5). Since $\mathscr{F}'_K$ is an open subset of $\mathscr{F}'$, we infer that $\mathscr{B}'$ is also a submanifold of $\mathscr{F}'$. Applying the diffeomorphism $\pi_{\mathscr{L}|\mathscr{B}'}$ and noting that $\pi_{\mathscr{L}}(\mathscr{B}') = \mathscr{L}'_p$ holds, this proves (v).

The following easy lemma plays the key role in the proof that a smooth subplane of a smooth projective plane induces a closed subplane of the tangent plane $\mathscr{A}_o$ of some inner point o. Recall that any two lines of a smooth projective plane always meet transversally, see [6].

Lemma 2.3. *Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a smooth projective plane and let $\mathscr{P}' = (P', \mathscr{L}', \mathscr{F}')$ be a smooth subplane of $\mathscr{P}$. Then every outer point row L meets P' transversally.*

Proof. The point set P' is a submanifold of P by hypothesis. Let p' be some point of $\mathscr{P}'$ and let $L \in \mathscr{L}_{p'}$ be an outer line. The point rows of $\mathscr{P}'$ through p' cover the point space P' and every tangent vector $v \in \mathrm{T}_{p'}P'$ is contained in a (unique) subspace $\mathrm{T}_{p'}L'$ for some line $L' \in \mathscr{L}'$ through p'. Choose a tangent vector $v \in \mathrm{T}_{p'}P$ belonging to both T_pL and T_pP'. Then there is a line $L' \in \mathscr{L}'$ such that $v \in \mathrm{T}_{p'}L'$. Since L and L' are different, they are transversal by Breitsprecher [6], see also [1], (1.13). This shows that $v = 0$ and hence P' is transversal to L.

The next result involves the concept of a Baer subplane. For Baer subplanes in the topological context the reader is referred to [24], 21.7, 41.11, and § 55.

Proposition 2.4. *Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a smooth projective plane with $\mathscr{P}' = (P', \mathscr{L}', \mathscr{F}')$ as a smooth subplane of $\mathscr{P}$. For every inner point o of $\mathscr{P}$ the incidence structure $\mathscr{B}_o = (\mathrm{T}_oP', \mathscr{S}'_o)$ defines a closed subplane of the tangent plane $\mathscr{A}_o = (\mathrm{T}_oP, \mathscr{S}_o)$, where $\mathscr{S}'_o = \{\mathrm{T}_oK \mid K \in \mathscr{L}'_o\}$. The subplane $\mathscr{B}_o$ is a Baer subplane of $\mathscr{A}_o$ if and only if $\mathscr{P}'$ is a Baer subplane of $\mathscr{P}$.*

Proof. We have to check that $\{\mathrm{T}_oK \cap \mathrm{T}_oP' \mid K \in \mathscr{L}'_o\}$ is a spread on T_oP'. Since the tangent spaces T_oK intersect pairwise transversally, it remains to verify that T_oP' is covered by $\bigcup_{K \in \mathscr{L}'_o} \mathrm{T}_oK$. Let $v \in \mathrm{T}_oP' \setminus \{0\}$. Then there is a unique tangent space T_oM of some line $M \in \mathscr{L}_o$ with $v \in \mathrm{T}_oM$ (remember that $\mathscr{S}_o = \{\mathrm{T}_oK \mid K \in \mathscr{L}_o\}$ is a spread of T_oP). Thus we have $v \in \mathrm{T}_oM \cap \mathrm{T}_oP'$ and so $v \in \bigcup_{K \in \mathscr{L}'_o} \mathrm{T}_oK$, because outer lines are transversal to P' according to Lemma (2.3). In order to show that $\mathscr{B}_o$ is a subplane of $\mathscr{A}_o$ we have to prove that $\mathrm{T}_oK \in \mathscr{S}'_o$ if and only if $\mathrm{T}_oK \cap \mathrm{T}_oP' \neq \{0\}$. But this follows again from the fact that exactly the outer lines of $\mathscr{P}$ through o are transversal to P'. Since P' is a closed submanifold of P and because o is an inner point, the tangent subspace T_oP' is closed in T_oP. Hence we have shown that $\mathscr{B}_o$ is a subplane of $\mathscr{A}_o$ which is closed by Lemma (2.1). A (closed) subplane of a topological plane of dimension $2l$ is a Baer subplane if and only if the dimension of the subplane is l, see e.g., Salzmann, [21], 1.4 and [24], 41.11. Thus, the last part of the proposition follows from $\dim \mathrm{T}_oP' = \dim P'$ and $\dim \mathrm{T}_oP = \dim P$.

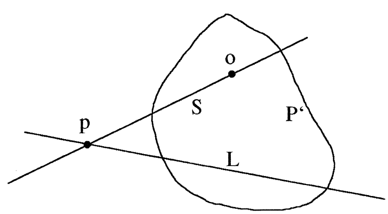
3. Smooth Hughes planes. The first examples of Hughes planes are due to D. R. Hughes. He constructed them from finite nearfields (see the book of Hughes-Piper, [12], IX.6). Ostrom [16] characterized these planes by the property that there is a Desarguesian Baer subplane $\mathscr{D}$ such that every axial collineation of $\mathscr{D}$ can be extended to a central collineation of the whole plane. Dembowski [7] generalized Hughes' construction in order to obtain infinite planes. In 1981, Salzmann [22] introduced analogues of Hughes planes in the category of topological planes. For compact planes of dimension greater then 4 there are no Hughes planes in the strict sense, since in these cases homologies of $\mathscr{D}$ cannot be extended to homologies of the whole plane. On the other hand, there are non-classical planes $\mathscr{P}$ having a Desarguesian Baer subplane $\mathscr{D}$ such that every collineation of $\mathscr{D}$ can be extended to $\mathscr{P}$. This gives rise to the following definition.

Definition 3.1. A compact projective plane $\mathscr{P}$ is called a *Hughes plane* if it contains a closed Desarguesian Baer subplane $\mathscr{D}$ such that every collineation of $\mathscr{D}$ can be extended to a collineation of the whole plane $\mathscr{P}$.

Clearly, a 2-dimensional projective plane cannot be a Hughes plane since such a plane does not contain any closed proper subplane. The classical planes $\mathscr{P}_2\mathbb{F}$ over one of the domains $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ are examples of Hughes planes. If $\mathscr{D} \cong \mathscr{P}_2\mathbb{R}$, Salzmann has proved that $\mathscr{P} \cong \mathscr{P}_2\mathbb{C}$ (see [20]), so the complex projective plane is the only 4-dimensional Hughes plane. Thus it remains to study 8- and 16-dimensional Hughes planes. The eight-dimensional Hughes planes have been classified also by Salzmann, [22], § 4, while the 16-dimensional case is due to Hähl, [11]. For both dimensions, there is a single one-parameter family of (pairwise non-isomorphic) Hughes planes $\mathscr{P}_\alpha$. For a unified treatment of 8- and 16-dimensional Hughes planes see [24], § 86.

In this section, we will show that for smooth Hughes planes the situation is exactly the same as for 4-dimensional *topological* Hughes planes: they are always classical. In our proof we will heavily make use of the collineation groups of Hughes planes. It turns out that we only need to consider the stabilizer of a suitable flag (p, L) in order to exclude non-classical Hughes planes.

Let $\mathscr{P}_\mathbb{F} = (P, \mathscr{L}, \mathscr{F})$ be a smooth Hughes plane with a closed Γ-invariant Desarguesian Baer subplane $\mathscr{D}_\mathbb{F} = (P', \mathscr{L}', \mathscr{F}') \cong \mathscr{P}_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. Fix an outer point p and an outer line S through p. Let L be the unique inner line through p and dually let o be the unique inner point on S.



We start by investigating the tangent translation planes $\mathscr{A}_q$ of $\mathscr{D}_\mathbb{F}$. We show that these planes are always classical affine planes.

We need the following facts from Hähl [11], (3.9) and from [24], § 86. There is a closed subgroup $\Sigma \cong \mathrm{SL}_3\mathbb{F}$ of $\mathrm{Aut}(\mathscr{P}_\mathbb{F})$ which induces the group $\mathrm{PSL}_3\mathbb{F}$ on the Baer subplane $\mathscr{D}_\mathbb{F}$. The stabilizer $\Delta := \Sigma_{o,L}$ induces on $\mathscr{D}_\mathbb{F}$ a group isomorphic to $\mathrm{SL}_2\mathbb{F} \cdot \mathbb{F}^\times$. The subgroup $\Pi := \Sigma_{p,S}$ fixes the unique inner line L through p and the unique inner point o. Note that Π also fixes o and L, since it leaves the Baer subplane $\mathscr{D}_\mathbb{F}$ invariant, and o and L are uniquely determined by $\mathscr{D}_{\mathbb{F}}$, p and S. By Salzmann, [22], p. 355, [19], (7.26), Hähl, [11], 3.11 and 3.14, and [24], 86.34, the stabilizer Π induces on $\mathscr{D}_{\mathbb{F}}$ a group $\mathrm{SU}_2\mathbb{F}\cdot \mathrm{P}$, where P is a closed non-compact one-parameter subgroup of $\mathbb{F}^{\times}$. In fact, we have $\mathrm{P} = \{e^{(1+i\alpha)t} t \in \mathbb{R}\} \subset \mathbb{C} \subset \mathbb{H}$ for some real nonnegative constant α, cp. [11], Lemma (2.6). Thus we have the situation:

$$\begin{array}{c} \Sigma \\ | \\ \Delta = \Sigma_{o,L} \\ | \\ \Pi = \Delta_{p,S} = \Sigma_{p,S} \\ | \\ \mathrm{P} \\ | \\ \mathbb{1} \end{array}$$

Consider the derivation map $\mathrm{D}_o : \Sigma_o \to \mathrm{Aut}(\mathscr{A}_o)_0$. Since (o, L) is an antiflag, the restriction of D_p to the group Δ is an injection by [2], (1.10). Let $\mathscr{S}_o$ be the spread that defines the tangent translation plane $\mathscr{A}_o$. The actions of Δ on $\mathscr{L}_o$ and of $\mathrm{D}_p\Delta$ on $\mathscr{S}_o$ are equivalent via the equivariant homeomorphism $\mathscr{L}_o \to \mathscr{S}_o : M \mapsto \mathrm{T}_o M$, see [1], (3.14).

Theorem 3.2. *Let $\mathscr{P}_{\mathbb{F}} = (P, \mathscr{L}, \mathscr{F})$ be a smooth Hughes plane of dimension $n \geqq 8$ and let $q \in P$ be some inner point. Then the affine tangent translation plane $\mathscr{A}_q$ is isomorphic to $\mathscr{A}_2\mathbb{H}$ (if $\mathbb{F} = \mathbb{C}$) or to $\mathscr{A}_2\mathbb{O}$ (if $\mathbb{F} = \mathbb{H}$), respectively.*

Proof. The stabilizer $\Delta = \Sigma_{o,L}$ of an inner antiflag (o, L) contains a closed subgroup Λ which is isomorphic to $\mathrm{SL}_2\mathbb{F}$. Since $\mathrm{D}_o : \Lambda \to \mathrm{Aut}(\mathscr{A}_o)$ is a closed embedding (this is true, because Λ is semi-simple, see [8] or [9]), the plane $\mathscr{A}_o$ is a so-called $\mathrm{SL}_2\mathbb{F}$-plane, see Hähl [10] and Löwe [14]. Such a plane is either the classical affine plane over $\mathbb{H}$ (over $\mathbb{O}$) or its collineation group has dimension at most 16 (at most 35). By [24], 86.35, the automorphism group Γ of the Hughes plane $\mathscr{P}_{\mathbb{F}}$ contains a closed subgroup B of dimension 17 (if $\mathbb{F} = \mathbb{C}$) or 36 (if $\mathbb{F} = \mathbb{H}$). This subgroup B induces the full automorphism group on the Baer subplane $\mathscr{D}_{\mathbb{F}} \cong \mathscr{P}_2\mathbb{F}$. In particular, we infer that B acts transitively on the set of antiflags of $\mathscr{D}_{\mathbb{F}}$, and thus we get

$$\dim \Gamma_{o,L} \geqq \dim \mathrm{B}_{o,L} = \dim \mathrm{B} - n = \begin{cases} 17 - 8 = 9 & \text{if } n = 8 \\ 36 - 16 = 20 & \text{if } n = 16. \end{cases}$$

Since the restriction of D_o to $\Gamma_{o,L}$ is an injection, we obtain

$$\dim \mathrm{Aut}(\mathscr{A}_o) \geqq \dim \mathrm{D}_o\Gamma_{o,L} + \dim \mathrm{Aut}(\mathscr{A}_o)_{[L_\infty, L_\infty]} \geqq \begin{cases} 9 + 8 = 17 & \text{if } n = 8 \\ 20 + 16 = 36 & \text{if } n = 16. \end{cases}$$

According to [10], (2.5) or [13], this implies that $\mathscr{A}_o \cong \mathscr{A}_2\mathbb{H}$ in the case $n = 8$. If $n = 16$, we infer that $\mathscr{A}_o \cong \mathscr{A}_2\mathbb{O}$ by a recent result of Löwe [14]. Since the collineation group of $\mathscr{P}_{\mathbb{F}}$ acts transitively on the set P' of inner points, we get $\mathscr{A}_q \cong \mathscr{A}_o$ for every $q \in P'$. This finishes the proof.

Proposition 3.3. *The subplane $\mathscr{D}_{\mathbb{F}}$ is a smooth subplane of $\mathscr{P}_{\mathbb{F}}$.*

Proof. Since $\mathscr{P}_{\mathbb{F}}$ is a Hughes plane, the stabilizer Δ acts transitively on the set $L' \approx \mathbb{S}_{l/2}$ of inner points of L. This is true even for a maximal compact subgroup K of Δ: either use the fact that $\mathscr{D}_{\mathbb{F}}$ is Desarguesian or utilize a well-known theorem on transformation groups (see [24], 96.19): if G is a Lie group acting transitively on a simply connected locally compact space X, then every maximal compact subgroup of G also acts transitively on X. Thus the set of inner points L' of L is the orbit of the compact Lie group K, which implies that L' is a submanifold of L. See, e.g., Onishchik [15], § 1, Th. 1. According to Lemma 2.2 this shows that the point set P' of $\mathscr{D}_{\mathbb{F}}$ is a submanifold of P and hence $\mathscr{D}_{\mathbb{F}}$ is a smooth subplane of $\mathscr{P}_{\mathbb{F}}$.

Set $A' := P' \setminus L'$ and let $\mathscr{D}_{\mathbb{F}}^L$ be the affine plane $(A', \mathscr{L}' \setminus \{L\})$. Since the subplane $\mathscr{D}_{\mathbb{F}}$ is Desarguesian, we can represent the point space A' of $\mathscr{D}_{\mathbb{F}}^L$ by the translation group $\mathrm{T} = \mathrm{Aut}(\mathscr{D}_{\mathbb{F}})_{[L,L]} \cong \mathbb{R}^l$, cp. Lemma (3.4) of [4]. The lines through the origin $o \in \mathrm{T}$ are then linear subspaces that build up the ordinary complex or quaternion spread $\mathscr{S}_o'$ on T. Using this representation as a chart h for the point space of the affine plane $\mathscr{D}_{\mathbb{F}}^L$, we get an isomorphism between the smooth affine planes $\mathscr{D}_{\mathbb{F}}^L$ and $\mathscr{A}_o'$ that is equivariant with respect to the actions of $\mathrm{Aut}(\mathscr{D}_{\mathbb{F}}^L)_o$ on T and of $\mathrm{Aut}(\mathscr{A}_o')_0$ on $\mathrm{T}_o P' \cong \mathbb{R}^l$. The stabilizer Π consists of linear mappings on T, whence the restriction $\mathrm{D}_o' : \Pi \to \mathrm{Aut}(\mathscr{A}_o')$ of the derivation map D_o to the submanifold A' is just the identity and the action of Π_o on $A' = \mathrm{T}$ is equivalent to the action of $\mathrm{D}_o'\Pi$ on $\mathrm{T}_o P' = \mathrm{T}^h$. With respect to a suitable basis of T we can write

$$\mathrm{D}_o'\Pi = \{e^{(1+i\alpha)t} M \mid M \in \mathrm{SU}_2\mathbb{F},\ t \in \mathbb{R}\}.$$

The different types of Hughes planes can be distinguished by the way how $\mathrm{D}_o'\Pi$ acts on the translation group T. For different values of α, these actions are inequivalent, and $\mathscr{P}_{\mathbb{F}}^{\alpha}$ is a Moufang plane if and only if $\alpha = 0$ (Hähl [11], 4.3, 4.6, and [24], 86.36, 86.37).

We are going to introduce several groups of collineations acting on the projective tangent translation plane $\mathscr{P}_o$ at the point o which correspond to the groups Σ, Δ, Π, and P of $\mathscr{P}_{\mathbb{F}}$. To do so, we will simply add an asterisk to these groups. Let $\Gamma^* = \mathrm{Aut}(\mathscr{P}_o)_{\mathscr{P}_o'}$ be the subgroup of $\mathrm{Aut}(\mathscr{P}_o)$ which leaves the Baer subplane $\mathscr{P}_o'$ of $\mathscr{P}_o$ invariant. Since $\mathscr{P}_o$ is classical by Theorem 3.2, there exists a unique subgroup Σ^* of Γ^* isomorphic to $\mathrm{SL}_3\mathbb{F}$ which leaves the Baer subplane $\mathscr{P}_o'$ invariant. Again, let 0 denote the origin of $\mathrm{T}_p P$. Then $\mathrm{D}_o : \Sigma_{o,L} \to (\mathrm{Aut}\mathscr{A}_o)_0$ maps $\Sigma_{o,L}$ injectively into $\Sigma_0^* \cap (\mathrm{Aut}\mathscr{A}_o)_0 = \Sigma_{0,L_\infty}^*$. Both groups Σ and Σ^* act transitively on the set of antiflags of $\mathscr{D}_{\mathbb{F}}$ and of $\mathscr{P}_o'$, respectively. Hence, we infer that $\dim \Sigma_{o,L} = \dim \Sigma_{0,L_\infty}^*$ und thus we get $\mathrm{D}_o \Sigma_{o,L} = \Sigma_{0,L_\infty}^*$. Putting $\Pi^* := \Sigma_{\mathrm{T}_o S \wedge L_\infty, L_\infty}^*$, we get $\mathrm{D}_o \Pi \subseteqq \Pi^*$ and, using the same argument as before, end up with $\mathrm{D}_o \Pi = \Pi^*$.

Before we finish our argument, let us summarize what we have done so far. On the one hand we have started with the group $\Sigma \cong \mathrm{SL}_3\mathbb{F}$ acting on $\mathscr{P}$, considered the stabilizers Δ and Π of Σ, and ended up with an action of Π on the translation group T. On the other hand, we took the group $\Sigma^* \cong \mathrm{SL}_3\mathbb{F}$ acting on the tangent translation plane $\mathscr{P}_o$, defined corresponding stabilizers Δ^* and Π^*, and finally obtained in just the same manner an action of Π^* on T^h. Since the tangent plane $\mathscr{P}_o$ is classical by Theorem 3.2, we can represent Π^* on T^h as

$$\Pi^* = \{e^t N \mid N \in \mathrm{SU}_2\mathbb{F},\ t \in \mathbb{R}\}$$

with respect to a suitable basis of T^h. Since the actions of Π on T and of $\mathrm{D}_o'\Pi = \Pi^*$ on T^h are equivalent with respect to the equivariant homeomorphism $h : \mathrm{T} \to \mathrm{T}^h$, we conclude that $\alpha = 0$, and so $\mathscr{P}_{\mathbb{F}}$ is a Moufang plane. Because the smooth structure of a smooth plane is uniquely determined ([5]), we have proved Theorem A.

Let $\mathscr{P}$ be an 8-dimensional compact projective plane with automorphism group Σ. If $\dim \Sigma \geqq 17$, a theorem of H. Salzmann, [23] says that either $\mathscr{P}$ or its dual is a translation plane, or $\mathscr{P}$ is a Hughes plane. According to J. Otte, [17], smooth translation planes are classical. Thus we have derived Theorem B from Theorem A.

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