TRUTHFUL IMPLEMENTATION AND PREFERENCE AGGREGATION IN RESTRICTED DOMAINS

JUAN CARLOS CARBAJAL ANDREW MCLENNAN RABEE TOURKY

Abstract. In a setting where agents have quasi-linear utilities over social alternatives and a transferable commodity, we consider three properties that a social choice function may possess: truthful implementation (in dominant strategies); monotonicity in differences; and lexicographic affine maximization. We introduce the notion of a *flexible* domain of preferences that allows *elevation of pairs* and study which of these conditions implies which others in such domain. We provide a generalization of the theorem of Roberts [36] in restricted valuation domains. Flexibility holds (and the theorem is not vacuous) if the domain of valuation profiles is restricted to the space of continuous functions defined on a compact metric space, or the space of piecewise linear functions defined on an affine space, or the space of smooth functions defined on a compact differentiable manifold. We provide applications of our results to public goods allocation settings, with finite and infinite alternative sets.

JEL Classification Numbers: C72, D70, D82.

Keywords: Roberts' theorem, truthful implementation, monotonicity in differences, lexicographic affine maximization, domain restrictions.

1. Introduction

The relationship between the implementability of a social choice function and other properties that this aggregation device may possess is a central theme in the mechanism design and social choice literatures. In reaction to the fundamental result of Gibbard [14] and Satterthwaite [40], a natural direction for investigation is to restrict the domain of profiles, aiming either at similar results with weaker hypotheses or positive results[1]. The most obvious restrictions on the domain of the

Date: September 2012.

School of Economics, The University of Queensland, St. Lucia, Queensland 4072, Australia. Phone: + 61 7 3365 6570. Email address: jc.carbajal@uq.edu.au (corresponding author), a.mclennan@economics.uq.edu.au, r.tourky@uq.edu.au.

We would like to thank the associate editor and a referee for their valuable comments. Thanks also to Ron Lavi, Claudio Mezzetti, Debasis Mishra, Ahuva Mu'alem, Rudolf Müller, Rakesh Vohra, audiences at the ACM–EC 2011 Workshop on Implementation Theory, the 2010 World Congress of the Econometric Society, the 2009 Workshop on Multidimensional Mechanism Design, HIM – Bonn University, the 2009 Conference on Economic Design, Maastricht University, and seminar participants at UNSW, Melbourne and Adelaide Universities, where earlier versions of this work was presented. Carbajal gratefully acknowledges support from the Hausdorff Research Institute for Mathematics – Bonn University. McLennan's work was funded in part by Australian Research Council grant DP0773324.

[1]The dictatorial conclusion of the Gibbard–Satterthwaite theorem holds even if the domain of preferences is restricted to the space of continuous functions defined on a metric space; cf. Barbera and Peleg [4]. In rich domains consisting of strict preferences, Dasgupta et al [10] show that a social choice function is truthfully implementable in dominant strategies if and only if it is monotonic, in the sense that for any two utility profiles, if alternative a is selected by the choice function under

social choice function are obtained by restricting each agent to have a preference drawn from some subset of the space of strict preference orderings[2], and our work considers only domain restrictions of this sort.

A particularly important possibility is that side payments may be feasible. Such environments occur naturally in connection with, for example, public good provision models with projects of variable size (Green and Laffont [15], [16], Laffont and Maskin [22]), cost sharing agreements (Moulin [31], [32]), allocation models of pollution permits and other common divisible resources (Dasgupta et al [11], Duggan and Roberts [13], Montero [30]), models of private good allocation with externalities (Jehiel et al [20], [19] and references therein), one-sided matching models with monetary transfers (Miyagawa [29], Schummer [41], Mishra and Roy [27], also Babaioff [3]), and so forth.

Roberts [36] considers an environment in which an outcome of the social choice process consists of an element of a finite set A of social alternatives and a vector of side payments. Roberts restricts agents to have preferences that are quasilinear: an agent's utility is the sum of a utility given by the societal alternative and the monetary transfer she receives. In the literature related to Roberts' work a *social choice function* (SCF) is a function from the domain of preference profiles to A. Such an SCF is said to be *truthfully implementable* (in dominant strategies) if it can be combined with a payment function, mapping preference profiles to vectors of side payments, to create a social choice function, in the more complete sense of Gibbard and Satterthwaite, for which truth telling is weakly dominant.

An SCF is an *affine maximizer* if it maximizes a social welfare function that is a weighted sum of the agents' utilities plus a function that may be thought of as representing societal values, such as externalities or the welfare of future generations, that are not captured by individual preferences. In this definition the weights on the individual utilities are required to be nonnegative, with at least one individual having a positive weight. Roberts [36] asserts that if $3 \leq |A| < \infty$ and the preference domain is unrestricted —that is, the agents can attach any values to the elements of A— then an SCF is truthfully implementable if and only if it is an affine maximizer, and that in turn these conditions hold if and only if a third condition called *positive association of differences* (PAD) holds. However, this is not quite correct because an affine maximizer can fail to be truthfully implementable[3]. The complication arises when some of the agents' preferences have zero weight in the affine combination. At each preference profile for the agents with positive weight, there is an induced SCF whose domain is the set of profiles for the agents with zero weight and whose range is the set of alternatives maximizing the affine functional. Without an additional restriction, these induced SCF's may not be truthfully implementable. Example 3.2 gives a concrete instance.

---

the first utility profile but a is not selected under the second profile, then there exist an alternative b and an agent in the society that weakly prefers a to b at the first situation but strictly prefers b to a at the second one. See also Maskin [26]. This monotonicity condition is sometimes referred to as strong positive association; cf. [33].

[2]Single peakedness (Black [8]) is an example of a well studied domain restriction that does not have this form.

[3]A referee informed us that this problem, and the resolution we describe, was independently observed before by Rozenshtrum [38].

If the given SCF is truthfully implementable, then so is each derived SCF, so it must be an affine maximizer when its image has three or more elements. Developing this condition recursively leads to the notion of a *lexicographic affine maximizer*, and the corrected result asserts that if $3 \leq |A| < \infty$ and the preference domain is unrestricted, then truthfully implementability, PAD, and lexicographic affine maximization are equivalent.

The assumption that A is finite is unnatural in many applications, e.g., typically the quantity of a public good is modeled as a real variable. Previous results characterizing implementable choice functions as affine maximizers[4] are not appropriate to handle such situations. When A is infinite the equivalence asserted by Roberts' theorem continues to hold in a formal sense, but one can easily construct profiles for which affine maximization is undefined, so the result is vacuous in this case. In order to have a meaningful generalization of Roberts' result in this direction, one must introduce restrictions on the preference domain. One of our major objectives is to extend Roberts' theorem to the case in which the set of alternatives is a compact metric space and the admissible preferences are continuous (or differentiable) functions on A. Our strategy is to impose conditions on the spaces of individual preferences that ensure that they are "rich enough" to support arguments leading to the conclusions of Roberts' theorem but are at the same time satisfied in a wide variety of applications.

The rest of this section describes the structure of the paper. Section 2 introduces the social choice setting and characterizes truthful implementation when the range of the SCF has one or two elements. Section 3 defines what it means for an SCF to be an affine maximizer, and a lexicographic affine maximizer. We show that a lexicographic affine maximizer is always truthfully implementable. In Section 3 we also introduce a condition called *monotonicity in differences*, and show that an affine maximizer satisfies this condition. For any two admissible profiles of preferences, monotonicity in differences requires that if alternative a is selected by the choice function under the first profile and alternative b is chosen under the second profile, then there exists at least one agent for whom the valuation difference between b and a is weakly greater in the second situation than in the first. Like PAD, monotonicity in differences is a collective condition insofar as it considers simultaneous changes of preferences for multiple agents. Unlike PAD, monotonicity in differences is not implied by the dominant strategy incentive constraints in every preference domain, nor does it necessarily imply truthful implementation or the affine maximization property in all domains.

In Section 4 we introduce two conditions on the preference domain, *elevation of pairs* and *flexibility*. These conditions are rather technical, but at this point we can say that they have the following character: given two (or in some cases three) alternatives and two (or in some cases three) individual preferences, there is another admissible preference that emphasizes the given alternatives because other alternatives become less desirable. If the domain of profiles is a cartesian product of spaces of admissible preferences for the various agents, and each agent's space allows elevation of pairs, then truthful implementability implies monotonicity in differences.

[4]In addition to Roberts [36], see Lavi et al [24], Dobzinski and Nisan [12], Mishra and Sen [28] and Vohra [42].

Our main result, Theorem 1, states that if the domain of preferences allows elevation of pairs and is flexible, and the image of the SCF has at least three elements, then the choice function is truthfully implementable if and only if it is a lexicographic affine maximizer. The main contribution of Robert's theorem is that truthful implementability implies affine maximization. The key technical result supporting our generalization of this part of the argument is that if the domain of profiles is flexible, then monotonicity in differences implies affine maximization. This result is proved in Section 8; the influence of the arguments of Lavi et al [24] will be evident. As we will demonstrate in Section 4, these results combine to imply Theorem 1.

In Section 5 we study some of the implications of our main result. To this end we introduce another condition called *comprehensiveness*, and show that comprehensiveness implies that the domain of preferences allows elevations of pairs and is flexible. Recall that a lattice of real valued functions is a vector space of functions that is closed under the pointwise minimum and pointwise maximum operators. A *resolving lattice* for A is a lattice with the additional property that for each $a \in A$, there is a function in the lattice whose value at a is different from its value at every other point in A. The key result of Section 5.2 shows that if a domain of valuations is a resolving lattice, then it is comprehensive. It is easy to see that the space of continuous functions on a compact metric space is a resolving lattice, as is the set of continuous piecewise affine functions on a convex subset of a Euclidean space. The space of C^r functions on a C^r manifold is not a lattice, so at least from the point of view of the proper development of our techniques, one should inquire as to whether Roberts' theorem also holds in this case. It turns out that it does, and we show this in Section 5.3.

Section 6 contains two applications of our results to public goods allocation problems. We first explore public goods models where the set of alternatives is a convex, compact subset of an Euclidean space and valuations for the public good are differentiable. Our motivation comes from the allocation of pollution permits considered by Montero [30]. Theorem 1 implies that in these environments an SCF is implementable if and only if it is a lexicographic affine maximizer. Our second application considers the selection of public projects of discrete size, when the participants have valuations that always include indifference between two or more public projects.[5] Incidentally, this will show how flexibility and elevation of pairs can hold in domains that are strict subsets of the unrestricted preference domain considered by Roberts [36], even when the set of alternatives is finite, so our result is a strict generalization in this case.

It is well known —e.g., Nisan [35]— that non affine maximizer SCFs can be implemented in single-dimensional domains; a prominent example is Myerson [34] optimal auction. Section 7 presents two counter examples intended to clarify which of the various implications that constitute Theorem 1 do not hold in domains that are not comprehensive. Example 7.1 illustrates the fact that, in certain domains, truthful implementation neither implies, nor is implied by, monotonicity in differences. This example also shows that a monotonic SCF is not necessarily an affine maximizer in every domain.[6] Example 7.2 shows that truthful implementation does

[5]As when the busy commuter is asked about his preference for the size of the sports arena.

[6]Our example draws inspiration from Bikhchandani et al [6].

not necessarily imply affine maximization in all domains, when the image of the SCF has three or more alternatives.[7] Thus these conditions are not equivalent in every domain.

Section 9 presents some final remarks.

2. The Social Choice Setting

There is a set A of social *alternatives*, which can be finite or infinite. There is a finite set $N = \{1, \ldots, n\}$ of *agents*. For each $i \in N$,- there is a space T_i of *types*. The space of *type profiles* is $T = T_1 \times \cdots \times T_n$.

A *difference* is a function $d\colon A \times A \to \mathbb{R}$ satisfying

$$d(a, b) \;+\; d(b, c) \;+\; d(c, a) \;=\; 0$$

for all $a, b, c \in A$. Note that

$$d(a, a) \;=\; 0 \quad \text{and} \quad d(b, a) \;=\; -\,d(a, b).$$

(Setting $a = b = c$ gives the first equation, after which setting $b = c$ gives the second.) Let $\mathcal{D}(A)$ be the space of differences. We assume that for each $t_i \in T_i$ there is an associated difference $d_{t_i} \in \mathcal{D}(A)$, signifying that for all $a, b \in A$ and $\tau_i \in \mathbb{R}$, agent i with type t_i is indifferent between having a implemented and receiving a transfer of τ_i and having b implemented with a transfer of $\tau_i + d_{t_i}(a, b)$. Thus agents have quasi-linear preferences. For $t \in T$, let $d_t = (d_{t_1}, \ldots, d_{t_n})$.

Since agents' types are private information, the scope for incentive compatible social choice functions that discriminate between two types with the same difference is quite limited. Nevertheless, in the corrected version of Robert's theorem the additional generality of general type spaces is significant. Specifically, given an affine maximizer, the derived mechanisms for agents with zero weight in the affine functional can depend in an arbitrary way on the additional information, beyond the differences, reported by the agents with positive weight.

A *valuation* is a function $\nu\colon A \to \mathbb{R}$. One may derive a difference from a valuation ν by setting $d_\nu(a, b) = \nu(a) - \nu(b)$, so a valuation may be thought of as a pair consisting of a difference and a shift parameter that does not affect incentives. In this sense valuations are intermediate between differences and fully general types. Most of the literature works with valuations, but at least in principle working with differences, as we do in this paper, should be psychologically valuable, disciplining the analysis by systematically excluding certain extraneous information, which tends to result in greater simplicity and clarity. However, we will see some assumptions that are most naturally expressed in terms of valuations, and in those contexts we will work with them.

A *social choice function*[8] (SCF) is a function

$$f\colon T \to A.$$

[7]This has also been shown by Mishra and Sen [28], who characterize *neutral* social choice functions via weighted welfare maximizers in open interval domains, still assuming that the alternative set A is finite. The neutrality of a choice function may be a reasonable assumption for certain environments, but is violated whenever the choice function discriminates a priori among alternatives, as occurs for instance when the social objective function includes a function embodying societal values. On the other hand, as Mishra and Sen [28] point out, neutrality is an essential component for restricting the domain to open bounded intervals.

[8]It is customary to require that f is surjective. This assumption is known as *non imposition* (e.g. Roberts [36]) and *citizen sovereignty* (e.g. Dasgupta et al [10]). While surjectivity may be

We fix such an f. An SCF will be combined with a *payment scheme*, which is an n-tuple

$$p = (p_1, \dots, p_n)\colon T \to \mathbb{R}^n$$

of functions specifying taxes imposed on the agents. We follow the customary notational conventions: for $i \in N$, $T_{-i} = \times_{j \neq i} T_j$, and thus $t = (t_i, t_{-i})$ denotes the type profile with components $t_i \in T_i$ and $t_{-i} \in T_{-i}$.

Definition 2.1. We say that p *truthfully implements* f if

$$d_{t_i}(f(t), f(t'_i, t_{-i})) \geq p_i(t) - p_i(t'_i, t_{-i})$$

for all $i \in N$, $t \in T$, and $t'_i \in T_i$. We say that f is *truthfully implementable (in dominant strategies)* if there is a payment scheme that truthfully implements it.

Roberts [36] is concerned with characterizing truthfully implementable SCFs with $|f(T)| \geq 3$, but we cannot restrict the discussion to such SCFs. Note that any constant choice function (i.e., $|f(T)| = 1$) is truthfully implementable. We say that f is a *binary implementable SCF* if it is truthfully implementable in dominant strategies and $|f(T)| = 2$. Such choice functions are characterized by cutoff differences.

Proposition 1. *If $f(T) = \{x, y\} \subset A$, $x \neq y$, then f is truthfully implementable if and only if for all $i \in N$ and all $t_{-i} \in T_{-i}$, there exists $\delta^*_i(t_{-i}) \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that the following conditions hold:*

(a) *$t_i \in T_i$ and $d_{t_i}(x, y) > \delta^*_i(t_{-i})$ imply $f(t) = x$;*
(b) *$t_i \in T_i$ and $d_{t_i}(x, y) < \delta^*_i(t_{-i})$ imply $f(t) = y$.*

Similar results are available in the literature; for a somewhat more general version, with a proof, see Theorem 9.36 of Nisan [35]. In more recent work, Marchant and Mishra [25] show that under reasonably weak conditions related to the agents decisiveness, the cutoff characterization of a binary implementable SCF can be replaced by a condition formulated in terms of maximization of *generalized utility functions*.

The following construction shows that the class of binary implementable SCFs is —in comparison with the affine maximizers we will see below— quite rich. Suppose that the function $G\colon \mathbb{R}^n \to \mathbb{R}$ is weakly increasing, in the sense that $G(\delta') \geq G(\delta)$ whenever $\delta'_i \geq \delta_i$ for all $i \in N$. We say that G is *strictly increasing at zero* if, for any $\delta \in \mathbb{R}^n$ with $G(\delta) = 0$, one has $G(\delta_i - \epsilon, \delta_{-i}) < 0 < G(\delta_i + \epsilon, \delta_{-i})$ for each i and every $\epsilon > 0$. Suppose that this is the case, that $f(T) = \{x, y\}$, and that $f(t) = x$ for all t such that $G(d_t(x, y)) > 0$ and $f(t) = y$ for all t such that $G(d_t(x, y)) < 0$. For each i and $t_{-i} \in T_{-i}$ let $\delta^*_i(t_{-i})$ be the infimum of the set of δ_i such that $G(\delta_i, d_{t_{-i}}(x, y)) > 0$ if this set is nonempty, and otherwise set $\delta^*_i(t_{-i}) = \infty$. The result above implies that f is binary implementable.

There are also binary implementable choice functions that are not derived from this construction. Consider the following example, in which $\delta^*_2(t_1)$ does not depend solely on the difference d_{t_1}.

---

philosophical significant in some settings, in formal analysis it can be restrictive, for example because we will have derived SCF's with smaller images.

Example 2.2. Let $N = \{1, 2\}$, $T_1 = \mathbb{R} \cup \{t_1'\}$ where $t_1' \notin \mathbb{R}$, and $T_2 = \mathbb{R}$. Suppose that $d_{t_1}(x, y) = t_1$ if $t_1 \neq t_1'$, $d_{t_1'}(x, y) = 0$, and $d_{t_2}(x, y) = t_2$. If $t_1 \neq t_1'$ we set

$$f(t_1, t_2) \;=\; \begin{cases} x, & \text{if } d_{t_1}(x, y) \geq 0 \text{ and } d_{t_2}(x, y) \geq 0; \\ y, & \text{otherwise.} \end{cases}$$

Let $\delta_1^*(t_2) = 0$ if $t_2 \geq 0$, and otherwise set $\delta_1^*(t_2) = \infty$. If $t_1 \neq t_1'$ let $\delta_2^*(t_1) = 0$ if $t_1 \geq 0$, and otherwise set $\delta_2^*(t_1) = \infty$. It is easy to see that in the absence of t_1' the hypotheses of Proposition 1 are satisfied. The definition of $f \colon T \to A$ is completed by setting

$$f(t_1', t_2) \;=\; \begin{cases} x, & \text{if } d_{t_2}(x, y) > 1; \\ y, & \text{if } d_{t_2}(x, y) \leq 1. \end{cases}$$

Let $\delta_2^*(t_1') = 1$. It is never the case that $d_{t_1'}(x, y) > \delta_1^*(t_2)$, and when $d_{t_1'}(x, y) < \delta_1^*(t_2)$, we have $\delta_1^*(t_2) = \infty$ and $f(t_1', t_2) = y$, so the hypotheses of Proposition 1 continue to hold.

3. Affine Maximization

Let $\Delta^{n-1} = \{\, \sigma \in \mathbb{R}^n_+ : \sigma_1 + \cdots + \sigma_n = 1 \,\}$ be the $(n-1)$-dimensional simplex.

Definition 3.1. We say that the SCF f is an *affine maximizer* if there exist $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ such that

$$\sum_{i=1}^{n} \sigma_i d_{t_i}\big(f(t), a\big) \;+\; q\big(f(t), a\big) \;\geq\; 0, \tag{1}$$

for all $t \in T$ and $a \in f(T)$.

An affine maximizer need not be truthfully implementable.

Example 3.2. Suppose that $n = 2$ and A is finite. Suppose that f always chooses one of agent 1's favorite alternatives, and if the set of favorites has more than one element, then f chooses one of agent 2's least favorite elements of this set. Clearly f is an affine maximizer because it maximizes agent 1's utility, but it is not truthfully implementable because, for any payment scheme, agent 2 is motivated to misreport some of her types.

As this example suggests, in order to obtain an exact characterization of truthful implementability we need to recursively impose conditions when the affine functional has more than one maximizer.

Definition 3.3. Suppose that f is an affine maximizer, with q and σ as in Definition 3.1, and let $S = \{\, i \in N : \sigma_i = 0 \,\}$, $T_S = \times_{i \in S} T_i$, and $T_{N \setminus S} = \times_{i \in N \setminus S} T_i$. We say that f is a *lexicographic affine maximizer* if, for each $t_{N \setminus S} \in T_{N \setminus S}$, one has:

(a) if $|f(t_{N \setminus S}, T_S)| = 2$, then the function $t_S \mapsto f(t_{N \setminus S}, t_S)$ is a binary implementable SCF;
(b) if $|f(t_{N \setminus S}, T_S)| \geq 3$, then the function $t_S \mapsto f(t_{N \setminus S}, t_S)$ is a lexicographic affine maximizer.

Of course the appearance of circularity is resolved by working inductively, assuming that lexicographic affine maximization when there are fewer than n agents has already been defined. Definition 3.3 identifies a "correct" recursive tie-breaking procedure to preserve truthful implementability of an affine maximizer SCF. In that sense, it is analogous to the *sequential decision process* of Larsson and Svensson [23] in the context of Gibbard–Satterthwaite theorem, and to the *scoring function* of Young [43] in the context of symmetric and consistent social choice functions in environments without side payments.[9]

Proposition 2. *If f is a lexicographic affine maximizer, then it is truthfully implementable.*

Proof. We construct generalized VCG payment functions to implement f. Let $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ be as in Definition 3.1. As above, let $S = \{\, i \in N : \sigma_i = 0 \,\}$. For each $i \in N \setminus S$, choose some $t_i^* \in T_i$ and let $p_i(t_i^*; \cdot)\colon T \to \mathbb{R}$ be the payment function given by[10]

$$p_i(t_i^*; t) \;=\; \frac{1}{\sigma_i}\Bigg[\sum_{j \neq i} \sigma_j d_{t_j}(f(t_i^*, t_{-i}), f(t)) \;+\; q(f(t_i^*, t_{-i}), f(t))\Bigg].$$

This construction guarantees that

$$d_{t_i}(f(t), f(t_i', t_{-i})) \;\geq\; p_i(t_i^*; t) - p_i(t_i^*; t_i', t_{-i})$$

for all $t \in T$, $t_i' \in T_i$, and $i \in N \setminus S$,

For each type profile $t_{N\setminus S} \in T_{N\setminus S}$, the derived SCF $f(t_{N\setminus S}, \cdot)\colon T_S \to A$ is trivially truthfully implementable (by constant payments) if it is a constant function. If it is instead a binary implementable SCF, i.e., if $f(t_{N\setminus S}, T_S) = \{x, y\}$, $x \neq y$, then for each $i \in S$ we can use the cutoffs differences $T_{S\setminus\{i\}} \ni t_{S\setminus\{i\}} \mapsto \delta_i^*(t_{N\setminus S}, t_{S\setminus\{i\}})$ of Proposition 1 to implement the derived SCF:

$$p_i(t_{N\setminus S}, t_S) \;=\; \begin{cases} \delta_i^*(t_{N\setminus S}, t_{S\setminus\{i\}}), & \text{if } f(t_{N\setminus S}, t_S) = x; \\ 0, & \text{if } f(t_{N\setminus S}, t_S) = y. \end{cases}$$

Otherwise $f(t_{N\setminus S}, \cdot)$ is a lexicographic affine maximizer. Therefore this argument can be repeated inductively to construct a payment scheme such that truth-telling is incentive compatible for all $i \in N$. $\square$

We now introduce another condition that an SCF may satisfy. For all $\delta, \delta' \in \mathbb{R}^n$, write $\delta \gg \delta'$ to indicate that $\delta_i > \delta_i'$ for all $i \in N$, and $\delta \ll \delta'$ to indicate $\delta_i < \delta_i'$ for all $i \in N$.

Definition 3.4. We say that f is *monotonic in differences* if for any $t^x, t^y \in T$ with $f(t^x) = x$ and $f(t^y) = y$, there is some $i \in N$ such that

$$d_{t_i^x}(x, y) \;\geq\; d_{t_i^y}(x, y).$$

Equivalently, for all $t^x, t \in T$ such that $f(t^x) = x$ and all $a \in A \setminus \{x\}$, if $d_t(x, a) \gg d_{t^x}(x, a)$, then $f(t) \neq a$. The latter formulation of the condition is called *negative unanimity*.

[9] We are grateful to an anonymous referee for drawing our attention to this related literature.

[10] In this construction, the role of t_i^* is akin the role of the auxiliary function $t_{-i} \mapsto h_i(t_{-i})$ in the standard definition of VCG payments in terms of valuations (rather than differences).

Proposition 3. *If f is an affine maximizer, then it is monotonic in differences.*

Proof. Let $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ be such that (1) holds for all $t \in T$ and $a \in f(T)$. To obtain a contradiction, suppose that negative unanimity fails: there is a pair of type profiles $t^x, t^y \in T$ such that

$$d_{t^x}(x,y) \ \ll \ d_{t^y}(x,y)$$

where $x = f(t^x)$ and $y = f(t^y)$. Since $\sigma_i > 0$ for some i this implies that

$$\sum_{i=1}^n \sigma_i d_{t_i^x}(x,y) \ < \ \sum_{i=1}^n \sigma_i d_{t_i^y}(x,y).$$

Inequality (1) with $a = y$ and $t = t^x$ is:

$$0 \ \geq \ -\sum \sigma_i d_{t_i^x}(x,y) \ - \ q(x,y).$$

Combining these two equations and using the fact that $d(b,a) = -d(a,b)$ for any difference gives

$$0 \ > \ \sum_{i=1}^n \sigma_i d_{t_i^y}(y,x) \ + \ q(y,x)$$

which contradicts (1) with t replaced by t^y and $a = x$. □

Thus, a lexicographic affine maximizer is truthfully implementable and monotonic in differences in every domain T. However, these three conditions are not equivalent in all domains. We provide examples in Section 7.

4. The Main Result

Our generalization of the key implication of Robert's theorem —that truthful revelation implies lexicographic affine maximization— depends on the domains being in certain senses "rich." The two assumptions of this sort introduced in this section assert that, in comparison with a given pair or triple of alternatives, and given pairs or triples of preferences, it is possible to find preferences or profiles that further enhance the attractiveness of the distinguished alternatives.

Definition 4.1. We say that T_i *allows elevation of pairs* if, for all $t_i^x, t_i^y \in T_i$ and $x, y \in A$ such that $d_{t_i^y}(x,y) > d_{t_i^x}(x,y)$, there is a $t_i \in T_i$ such that

$$\begin{aligned} d_{t_i}(x,a) \ &> \ d_{t_i^x}(x,a), \qquad \text{for all } a \neq x, \\ d_{t_i}(y,a) \ &> \ d_{t_i^y}(y,a), \qquad \text{for all } a \neq y. \end{aligned}$$

We say that T *allows elevation of pairs* if each T_i allows elevation of pairs.

For example, suppose that $d_{t_i^y}(x,y) > d_{t_i^x}(x,y)$, where $d_{t_i^x}$ and $d_{t_i^y}$ are derived from i's valuations v_i^x and v_i^y, respectively. After adding a suitable constant, we may assume that $v_i^y(x) > v_i^x(x)$ and $v_i^x(y) > v_i^y(y)$. Elevation of pairs requires that there be a t_i such that d_{t_i} is derived from an admissible valuation v_i with $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, and $v_i(a) < \min\{v_i^x(a), v_i^y(a)\}$ for all $a \neq x, y$. Note that if A is a metric space and v_i^x and v_i^y are continuous, then such a continuous v_i exists (see Figure 1).

Proposition 4. *If T allows elevation of pairs and f is truthfully implementable, then f is monotonic in differences.*

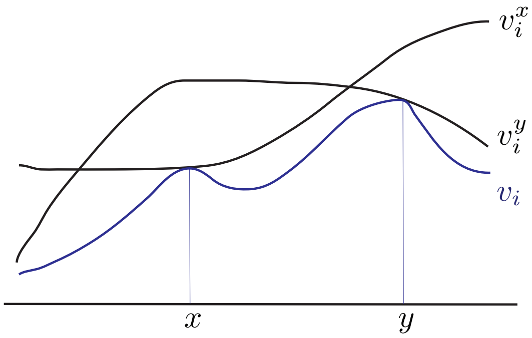

FIGURE 1. Elevation of pairs for continuous valuations

Proof. By hypothesis there is a payment scheme $p\colon T \to \mathbb{R}^n$ that truthfully implements f. Aiming at a contradiction, suppose that f is not monotonic in differences, so there is a failure of negative unanimity: for some $t^x, t^y \in T$ we have

$$d_{t^y}(x,y) \gg d_{t^x}(x,y).$$

where $x = f(t^x)$ and $y = f(t^y)$ respectively. Since each T_i allows elevation of pairs, there is a type profile $t \in T$ such that

$$d_t(x,a) \gg d_{t^x}(x,a), \qquad \text{for all } a \neq x, \tag{2}$$

$$d_t(y,a) \gg d_{t^y}(y,a), \qquad \text{for all } a \neq y. \tag{3}$$

Let $t^0 = t^x$, and define $t^1, \ldots, t^n$ inductively by setting $t^i = (t_i, t^{i-1}_{-i})$. Truthful implementation implies that for each $i \in N$ we have both

$$d_{t_i^i}(f(t^i), f(t^{i-1})) \geq p_i(t^i) - p_i(t^{i-1})$$

and

$$d_{t_i^{i-1}}(f(t^{i-1}), f(t^i)) \geq p_i(t^{i-1}) - p_i(t^i).$$

Combining these, and recognizing that $t_i^{i-1} = t_i^x$ and $t_i^i = t_i$, gives

$$d_{t_i^x}(f(t^{i-1}), f(t^i)) \geq d_{t_i}(f(t^{i-1}), f(t^i)).$$

Since $t^0 = t^x$ we have $f(t^0) = x$, and if j is the first index such that $f(t^j) \neq x$, this inequality contradicts (2). Therefore we have

$$x = f(t^0) = \cdots = f(t^n) = f(t).$$

The same argument with t^y in place of t^x gives $y = f(t)$, but $x = y$ is impossible. $\square$

Allowing elevation of pairs is a condition that is imposed on each T_i individually. In the following condition, on the other hand, the restriction is coordinated across the different T_i.

Definition 4.2. We say that T is *flexible* if:

(F1) For any distinct $x, y \in A$ there are disjoint sets $B_x, B_y \subset A$ with $x \in B_x$ and $y \in B_y$ such that for all $i \in N$, $t_i^x, t_i^y \in T_i$, and $\delta_{xyi} \in \mathbb{R}$, there is a $t_i \in T_i$ satisfying

$$d_{t_i}(x,y) = \delta_{xyi},$$

and further:

(a) $d_{t_i}(a,x) < d_{t_i^x}(a,x)$, for all $a \in A \setminus (\{x\} \cup B_y)$;

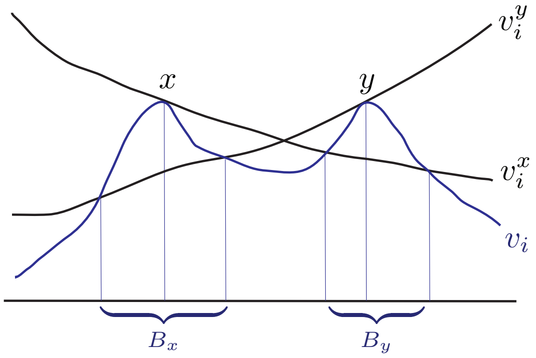

FIGURE 2. Flexibility for continuous valuations and $\delta_{xyi} = 0$

(b) $d_{t_i}(a, y) < d_{t_i^y}(a, y)$, for all $a \in A \setminus (\{y\} \cup B_x)$.

(F2) For any distinct $x, y, z \in A$ there are pairwise disjoint sets $B'_x, B'_y, B'_z \subset A$ with $x \in B'_x$, $y \in B'_y$, and $z \in B'_z$, such that for all $i \in N$, $t_i^x, t_i^y, t_i^z \in T_i$, and $\delta_{xyi}, \delta_{yzi} \in \mathbb{R}$, there is a $t_i \in T_i$ such that

$$d_{t_i}(x, y) \;=\; \delta_{xyi} \quad \text{and} \quad d_{t_i}(y, z) \;=\; \delta_{yzi}\,,$$

and further:
(a) $d_{t_i}(a, x) < d_{t_i^x}(a, x)$, for all $a \in A \setminus \big(\{x\} \cup B'_y \cup B'_z\big)$;
(b) $d_{t_i}(a, y) < d_{t_i^y}(a, y)$, for all $a \in A \setminus \big(\{y\} \cup B'_x \cup B'_z\big)$;
(c) $d_{t_i}(a, z) < d_{t_i^z}(a, z)$, for all $a \in A \setminus \big(\{z\} \cup B'_x \cup B'_y\big)$.

These properties abstract the key features of a topological setting that we will employ in Section 5. To develop intuition for them it may help to imagine that each difference profile d_t as arising from continuous valuations: $d_{t_i}(a, b) = v_i(a) - v_i(b)$ for each $i \in N$. What flexibility requires is that we can associate, to any pair (or triple) of alternatives x, y (or x, y, z), two (or three) neighborhoods around them such that given any two valuation profiles $v^x = (v_1^x, \ldots, v_n^x)$ and $v^y = (v_1^y, \ldots, v_n^y)$, there exists a valuation profile $v = (v_1, \ldots, v_n)$ that enhances the value of x relative to v^x everywhere except perhaps in the neighborhood around y, and similarly it enhances the value of y relative to v^y everywhere except perhaps in the neighborhood around x. See Figure 2 for an illustration.

The following result will be proved in Section 8.

Proposition 5. *If T is flexible, f is monotonic in differences, and $|f(T)| \geq 3$, then f is an affine maximizer.*

Our main result is the following characterization theorem.

Theorem 1. *If T allows elevation of pairs and is flexible, then f is truthfully implementable if and only if one of the following hold:*
(a) *f is a constant function;*
(b) *f is a binary implementable SCF;*
(c) *f is a lexicographic affine maximizer.*

Proof of Theorem 1. It is obvious that f is truthfully implementable if (a) holds, truthful implementability is part of the definition of (b), and Proposition 2 states that (c) implies truthful implementability.

Suppose that f is truthfully implementable. Evidently there is nothing to prove unless $|f(T)| \geq 3$, in which case Propositions 4 and 5 imply that f is an affine maximizer, so there are q and σ such that the conditions of Definition 3.1 are satisfied. Let $S = \{ i : \sigma_i = 0 \}$, $T_S = \times_{i \in S} T_i$, and $T_{N \setminus S} = \times_{i \in N \setminus S} T_i$, and consider $t_{N \setminus S} \in T_{N \setminus S}$. Of course $f(t_{N \setminus S}, \cdot) \colon T_S \to A$ is truthfully implementable. By induction we may assume that the result has already been established for the derived SCFs with fewer than n agents, so $f(t_{N \setminus S}, \cdot)$ is constant, a binary implementable SCF, or a lexicographic affine maximizer. Since $t_{N \setminus S}$ was arbitrary, the proof is complete. □

Fix a lexicographic affine maximizer f with weights $\sigma \in \Delta^{n-1}$, and suppose that agent i is *decisive*[11] in the sense that for every profile $t_{-i} \in T_{-i}$ and every alternative $a \in A$, there exists a type $t_i \in T_i$ such that $f(t_i, t_{-i}) = a$. Clearly, $\sigma_i > 0$ in this case. Thus, when $|A| \geq 3$ and every agent is decisive, if T satisfies elevation of pairs and is flexible, then we have that an SCF f is truthfully implementable if and only if it is an affine maximizer.

5. Extensions

Here we explore some extensions that the additional generality of our result opens up. As we explained in the introduction, quasi-linear preferences are natural and commonly assumed in connection with public goods, public bads such as pollution, cost sharing, and various other allocation problems. In such models it is common to treat the infinite set of alternatives A as a compact metric or topological space, and the domain of admissible valuation profiles as containing only continuous, piecewise linear, or even differentiable functions. We show in this section how Theorem 1 can be employed in these cases. In particular, if the space of individual valuations is the space of continuous functions on a compact metric space A, then it will follow from Corollary 5.10 that any truthfully implementable SCF must be a lexicographic affine maximizer. We are able to show, with some additional work, that this result holds even if the space of individual valuations is the space of smooth valuations on a smooth compact manifold.

5.1. Comprehensive Domains

Insofar as the valuation $\nu \colon A \to \mathbb{R}$ induces a difference $d_\nu \in \mathcal{D}(A)$ given by $d_\nu(a, b) = \nu(a) - \nu(b)$, we may regard a valuation as a particular sort of type. Let $\mathcal{V}(A)$ be the space of all valuations. We assume now that each agent $i \in N$ is endowed with a preference domain $V_i \subseteq \mathcal{V}(A)$. Let

$$V \;=\; V_1 \times \cdots \times V_n \;\subseteq\; \mathcal{V}(A)^n.$$

An element $v = (v_1, \ldots, v_n)$ of V is called an admissible valuation profile. Our goal is to develop conditions on V that imply that it is flexible and allows elevation of pairs.

Of course $\mathcal{V}(A)$ is a vector space when endowed with the pointwise addition and scalar multiplication operations, and it is partially ordered by the pointwise ordering. For $\nu, \nu' \in \mathcal{V}(A)$ the pointwise max and pointwise min functions, $\nu \vee \nu'$ and $\nu \wedge \nu'$ respectively, are:

$$\nu \vee \nu'(a) \;=\; \max\{\nu(a), \nu'(a)\} \quad \text{and} \quad \nu \wedge \nu'(a) \;=\; \min\{\nu(a), \nu'(a)\}.$$

[11] See Lavi et al [24].

These operations extend component-wise to vectors of valuations.

Definition 5.1. For $i \in N$ we say that V_i is *combinative* if:

(C1) For all distinct $x, y \in A$ and $v_i^x, v_i^y \in V_i$ there is a $v_i \in V_i$ such that $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, and $v_i \leq v_i^x \vee v_i^y$.

(C2) For all distinct $x, y, z \in A$ and $v_i^x, v_i^y, v_i^z \in V_i$ there is a $v_i \in V_i$ such that $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, $v_i(z) = v_i^z(z)$, and $v_i \leq v_i^x \vee v_i^y \vee v_i^z$.

(C3) For all $x, y \in A$ and $v_i^x, v_i^y \in V_i$ such that $v_i^x(x) \neq v_i^x(y)$ and $v_i^y(x) \neq v_i^y(y)$, there is a $v_i \in V_i$ such that $v_i(x) = v_i^x \wedge v_i^y(x)$, $v_i(y) = v_i^x \wedge v_i^y(y)$, and $v_i \leq v_i^x \wedge v_i^y$.

(C4) For any finite set $B \subset A$ and collection of valuations $\{v_i^x\}_{x \in B}$ in V_i, there is a $v_i \in V_i$ such that $v_i \leq \bigwedge_{x \in B} v_i^x$.

We say that V is *combinative* if each V_i is combinative.

We now introduce a collection of functions that will be used to perturb valuations. The *support* of a function $\mu : A \to \mathbb{R}$ is the set

$$\operatorname{supp}(\mu) \;=\; \{\, a \in A : \mu(a) \neq 0 \,\}.$$

Definition 5.2. A set $\mathcal{U}$ of functions from A to $[0, 1]$ is called a *separating family* if, for any distinct $x, y, z \in A$ there are $\mu^x, \mu^y, \mu^z \in \mathcal{U}$ satisfying $\mu^x(x) = \mu^y(y) = \mu^z(z) = 1$ whose supports are pairwise disjoint:

$$\operatorname{supp}(\mu^x) \cap \operatorname{supp}(\mu^y) = \operatorname{supp}(\mu^x) \cap \operatorname{supp}(\mu^z) = \operatorname{supp}(\mu^y) \cap \operatorname{supp}(\mu^z) = \emptyset.$$

Recall that a topological space is *normal* if any two disjoint closed sets are contained in disjoint open sets. Urysohn's lemma asserts that in a normal space, for any two disjoint closed sets there is a continuous function from the space to $[0, 1]$ that is identically zero on the first set and identically one on the second. If, in addition, the space is T_1 (singletons are closed) and consequently Hausdorff, then the set of continuous real valued functions on the space contains a separating family.

Fix a separating family $\mathcal{U}$.

Definition 5.3. For $i \in N$ we say that V_i is *$\mathcal{U}$-perturbative* if for all $x \in A$, $v_i^x \in V_i$, $\mu \in \mathcal{U}$, and $\delta_i \in \mathbb{R}_+$, there is a valuation $v_i \in V_i$ such that

$$v_i(a) \;\leq\; v_i^x(a) \,+\, \delta_i \mu(a)$$

for all $a \in A$ with strict inequality if and only if $a \neq x$. We say that V is *$\mathcal{U}$-perturbative* if each V_i is $\mathcal{U}$-perturbative.

Definition 5.4. For $i \in N$ we say that V_i is *$\mathcal{U}$-comprehensive* if it is combinative and $\mathcal{U}$-perturbative. We say that V is *$\mathcal{U}$-comprehensive* if each V_i is $\mathcal{U}$-comprehensive.

Lemma 5.5. *Suppose that V is $\mathcal{U}$-comprehensive and $x, y \in A$ are distinct. Then for any $v^x, v^y \in V$ there are disjoint sets $B_x, B_y \subset A$, with $x \in B_x$ and $y \in B_y$, and an admissible profile $v \in V$ such that $v(x) = v^x(x)$, $v(y) = v^y(y)$, and*

$$\begin{aligned} v(a) &\ll v^x(a), && \textit{for all } a \in B_x \setminus \{x\}, \\ v(a) &\ll v^y(a), && \textit{for all } a \in B_y \setminus \{y\}, \\ v(a) &\ll v^x \wedge v^y(a), && \textit{for all } a \in A \setminus (B_x \cup B_y). \end{aligned}$$

Proof. Fix functions $\mu^x, \mu^y : A \to [0,1]$ in $\mathcal{U}$ with disjoint supports B_x and B_y and $\mu^x(x) = \mu^y(y) = 1$. Fix a particular $i \in N$.

Applying (C4), choose $v_{i1} \in V_i$ such that

$$v_{i1} \;\le\; v_i^x \wedge v_i^y.$$

Choose $\delta_i > 0$ large enough that

$$v_i^x(x) < v_{i1}(x) + \delta_i \quad \text{and} \quad v_i^y(y) < v_{i1}(y) + \delta_i.$$

Since V_i is $\mathcal{U}$-perturbative there is a $v_{i2}^x \in V_i$ such that

$$v_{i2}^x \le v_{i1} + \delta_i \mu^x$$

with equality at x and strict inequality at all other $a \in A$. Applying (C3) to v_i^x and v_{i2}^x gives a $v_{i3}^x \in V_i$ such that

$$v_{i3}^x \le v_{i2}^x \wedge v_i^x$$

with equality at x. Applying the assumption that V is $\mathcal{U}$-perturbative (with $\delta_i = 0$), there is a $v_{i4}^x \in V_i$ such that

$$v_{i4}^x \le v_{i3}^x$$

with equality at x and strict inequality at all other $a \in A$.

Similarly, there are $v_{i2}^y, v_{i3}^y, v_{i4}^y \in V_i$ such that

$$v_{i2}^y \le v_{i1} + \delta_i \mu^y, \quad v_{i3}^y \le v_{i2}^y \wedge v_i^y, \quad v_{i4}^y \le v_{i3}^y$$

with equality at y and strict inequality at all other $a \in A$, in all three cases. Finally, applying (C1) to v_{i4}^x and v_{i4}^y gives $v_i \in V_i$ such that

$$v_i(x) \;=\; v_{i4}^x(x), \quad v_i(y) \;=\; v_{i4}^y(y), \quad \text{and} \quad v_i(a) \;\le\; v_{i4}^x \vee v_{i4}^y.$$

We have

$$v_i(x) = v_{i4}^x(x) = v_{i3}^x(x) = v_{i2}^x \wedge v_i^x(x) = (v_{i1} + \delta_i) \wedge v_i^x(x) = v_i^x(x)$$

and $v_i(y) = v_i^y(y)$ by symmetry. For $a \in B_x \setminus \{x\}$ we have $\mu^y(a) = 0$ and thus

$$v_{i3}^y(a) \;\le\; v_{i2}^y(a) \;<\; v_{i1}(a) \;\le\; v_i^x \wedge v_i^y(a) \le v_i^x(a).$$

Since $v_{i3}^x(a) \le v_i^x(a)$, we conclude that

$$v_i(a) \;\le\; v_{i4}^x \vee v_{i4}^y(a) < v_{i3}^x \vee v_{i3}^y(a) \le v_i^x(a).$$

Symmetrically, $v_i(a) < v_i^y(a)$ for all $a \in B_y \setminus \{y\}$. For $a \in A \setminus (B_x \cup B_y)$ we have $v_{i3}^x(a) < v_{i1}(a)$ and $v_{i3}^y(a) < v_{i1}(a)$, so

$$v_i(a) \;\le\; v_{i4}^x \vee v_{i4}^y(a) \;\le\; v_{i3}^x \vee v_{i3}^y(a) \;<\; v_{i1}(a) \le v_i^x \wedge v_i^y(a).$$

This gives us the desired result. □

Proposition 6. *If V is $\mathcal{U}$-comprehensive, then it allows elevation of pairs.*

Proof. Consider $v^x, v^y \in V$ and $x, y \in A$ such that $d_{v^y}(x,y) \gg d_{v^x}(x,y)$. Assume without loss of generality that $v^y(x) \gg v^x(x)$ and $v^y(y) \ll v^x(y)$. We can apply (C3) to v^x and v^y, at x and y respectively, to obtain a profile v_1 in V with $v_1 \le v^x \wedge v^y$, $v_1(x) = v^x(x)$, and $v_1(y) = v^y(y)$. Since V is $\mathcal{U}$-perturbative there are $v_2^x \in V$ such that $v_2^x(x) = v_1(x) = v^x(x)$ and $v_2^x(a) \ll v_1(a)$ for all $a \ne x$, and $v_2^y \in V$ such that $v_2^y(y) = v_1(y) = v^y(y)$ and $v_2^y(a) \ll v_1(a)$ for all $a \ne y$. By Lemma 5.5, there exists an admissible profile $v \in V$ such that $v(x) = v_2^x(x) = v^x(x)$, $v(y) = v_2^y(y) = v^y(y)$, and $v(a) \ll v_2^x \vee v_2^y(a) \le v^x \wedge v^y(a)$, for each $a \ne x, y$. □

We omit the proof of the following because it is slightly more complex, but otherwise similar to the proof of Lemma 5.5 above.

Lemma 5.6. *Suppose that V is $\mathcal{U}$-comprehensive and $x, y, z \in A$ are distinct. Then for any $v^x, v^y, v^z \in V$ there are pairwise disjoint sets $B_x, B_y, B_z \subset A$, with $w \in B_w$ for each $w = x, y, z$, and an admissible profile $v \in V$ such that*

$$v(a) \ll v^x \wedge v^y \wedge v^z(a), \qquad \text{for all } a \in A \setminus (B_x \cup B_y \cup B_z),$$

and further, for each $w = x, y, z$, $v(w) = v^w(w)$ and $v(a) \ll v^w(a)$ for all $a \in B_w \setminus \{w\}$.

Proposition 7. *If V is $\mathcal{U}$-comprehensive, then it is flexible.*

Proof. We will only prove (F1); the proof of (F2) follows the same pattern. Fix distinct $x, y \in A$. Choose $\mu^x, \mu^y \in \mathcal{U}$ such that $\mu^x(x) = \mu^y(y) = 1$ and $B_x = \operatorname{supp}(\mu^x)$ and $B_y = \operatorname{supp}(\mu^y)$ are disjoint. Fix $v^x, v^y \in V$ and $\delta_{xy} \in \mathbb{R}^n$. It suffices to prove that there is a profile $v \in V$ such that $d_v(x, y) = \delta_{xy}$, $d_v(a, x) \ll d_{v^x}(a, x)$ for all $a \in A \setminus (\{x\} \cup B_y)$, and $d_v(a, y) \ll d_{v^y}(a, y)$ for all $a \in A \setminus (\{y\} \cup B_x)$.

Choose $\delta_x, \delta_y \in \mathbb{R}^n_+$ such that

$$v^x(x) - v^y(y) + \delta_x - \delta_y \;=\; \delta_{xy}.$$

Since V is $\mathcal{U}$-perturbative there are $v_1^x, v_2^y \in V$ such that $v_1^x(x) = v^x(x) + \delta_x$, $v_2^y(y) = v^y(y) + \delta_y$, $v_1^x(a) < v^x(a) + \mu^x(a)\delta_x$ for all $a \in A \setminus \{x\}$, and $v_2^y(a) < v^y(a) + \mu^y(a)\delta_y$ for all $a \in A \setminus \{y\}$. Lemma 5.5 now gives a $v \in V$ such that

$$v(x) = v_1^x(x) = v^x(x) + \delta_x, \qquad v(y) = v_2^y(y) = v^y(y) + \delta_y,$$

and:

$$\begin{aligned} v(a) &\ll v_1^x(a) && \text{for all } a \in B_x \setminus \{x\}; \\ v(a) &\ll v_2^y(a) && \text{for all } a \in B_y \setminus \{y\}; \\ v(a) &\ll v_1^x \wedge v_2^y(a) && \text{for all } a \in A \setminus (B_x \cup B_y). \end{aligned}$$

If $a \in A \setminus (B_x \cup B_y)$, then

$$v(a) \ll v_1^x \wedge v_2^y(a) \ll v^x \wedge v^y(a).$$

If $a \in B_x \setminus \{x\}$, then

$$v(a) \ll v_1^x(a) \ll v^x(a) + \mu^x(a)\delta_x.$$

Of course $v(x) = v^x(x) + \delta_x$, so in either case one has

$$d_v(a, x) \ll d_{v^x}(a, x).$$

Reversing x and y in this argument, we find that $d_v(a, y) \ll d_{v^y}(a, y)$ for all $a \in A \setminus (\{y\} \cup B_x)$, as desired. ☐

5.2. Resolving Lattices

One of our major concerns is the possibility that A be a compact metric space and each V_i the space of all continuous real valued functions on A. In this subsection we explain the details of the verification of $\mathcal{U}$-comprehensiveness for this case in a way that allows somewhat more generality (in the next subsection we show directly that the space of smooth valuations is $\mathcal{U}$-comprehensive).

Definition 5.7. A function $\nu^x \in \mathcal{V}(A)$ is a *resolving function* for $x \in A$ if $\nu^x(x) \neq \nu^x(a)$ for all $a \in A \setminus \{x\}$. A subset $\mathcal{R}$ of $\mathcal{V}(A)$ is a *resolving set* if it contains a resolving function for each $x \in A$.

For example, the distance from a point in a metric space to x is a continuous resolving function for x. Recall that $\mathcal{L} \subset \mathcal{V}(A)$ is a *lattice* if $\nu \vee \nu', \nu \wedge \nu' \in \mathcal{L}$ for all $\nu, \nu' \in \mathcal{L}$.

Definition 5.8. A subset $\mathcal{L}$ of $\mathcal{V}(A)$ is called a *resolving lattice* if:

(a) $\mathcal{L}$ is a linear subspace of $\mathcal{V}(A)$;
(b) $\mathcal{L}$ contains all constant functions;
(c) $\mathcal{L}$ is a lattice;
(d) $\mathcal{L}$ is a resolving set.

Henceforth $\mathcal{L}$ will denote a given resolving lattice. Note that for any subset B of A, the space of restricted functions $\{\nu|_B : \nu \in \mathcal{L}\}$ is also a resolving lattice. We assume that $\mathcal{L}$ contains a separating family $\mathcal{U}$. As we mentioned earlier, this is certainly the case when $\mathcal{L}$ is the set of all continuous functions on a T_1 normal space. In the next two results we consider a fixed $i \in N$.

Proposition 8. *If $V_i = \mathcal{L}$, then V_i is combinative.*

Proof. To prove (C1) suppose that $v_i^x, v_i^y \in V_i$ and that $x, y \in A$ are distinct. Since $\mathcal{U}$ is a separating family there are $\mu^x, \mu^y \in \mathcal{L}$ taking values in $[0,1]$ with disjoint supports and $\mu^x(x) = \mu^y(y) = 1$. If we let $\delta_i \geq v_i^x(x) - v_i^y(x), v_i^y(y) - v_i^x(y)$, then

$$v_i \;=\; [v_i^x + \delta_i \mu^y] \wedge [v_i^y + \delta_i \mu^x]$$

belongs to V_i and has the desired properties.

To prove (C2) suppose that $v_i^x, v_i^y, v_i^z \in V_i$ and that $x, y, z \in A$ are distinct. Since $\mathcal{U}$ is a separating family there are $\mu^x, \mu^y, \mu^z \in \mathcal{L}$ taking values in $[0,1]$ with pairwise disjoint supports and $\mu^x(x) = \mu^y(y) = \mu^z(z) = 1$. If $\delta_i \geq 0$ is sufficiently large, then

$$v_i \;=\; [v_i^x + \delta_i(\mu^y + \mu^z)] \wedge [v_i^y + \delta_i(\mu^x + \mu^z)] \wedge [v_i^z + \delta_i(\mu^x + \mu^y)]$$

belongs to V_i and has the desired properties.

Note that (C3) and (C4) holds simply because V_i is a lattice. □

In the following result and below we abuse notation, letting a scalar also denote the constant function with that particular value. We also let ν^+ and $|\nu|$ denote $\nu \vee 0$ and $\nu^+ + (-\nu)^+$, respectively.

Proposition 9. *If $V_i = \mathcal{L}$, then V_i is $\mathcal{U}$-perturbative.*

Proof. Consider $x \in A$, $v_i^x \in V_i = \mathcal{L}$, $\mu \in \mathcal{L}$ mapping A to $[0,1]$, and $\delta_i \in \mathbb{R}_+$. We know there is a resolving function $\nu^x \in \mathcal{L}$ such that $\nu^x(x) \neq \nu^x(a)$ for all $a \in A$, $a \neq x$. Let $\nu = |\nu^x - \nu^x(x)|$. Then i's valuation

$$v_i \;=\; v_i^x + \delta_i \mu - \nu$$

is in V_i and has the desired properties. □

Corollary 5.9. *If $V_i = \mathcal{L}$ for all $i \in N$, then V is flexible and allows elevation of pairs.*

If A is a topological space, $C(A)$ is the space of all continuous functions on A.

Corollary 5.10. *If A is a topological space such that there exists a continuous resolving function ν^x for each $x \in A$ and $C(A)$ contains a separating family, then $V = C(A)^n$ is $C(A)$-comprehensive.*

If A is a metric space with metric ρ, functions of the form

$$\nu^x_\alpha : a \mapsto \max\{0, 1 - \alpha\rho(x, a)\}$$

show that $C(A)$ is a resolving set, and they can also obviously be used to construct a separating family. The advantage of this indirect approach is that one can also look at proper subsets of $C(A)$.

Example 5.11. Let $\mathcal{R}$ be a resolving set. Let $\mathcal{L}$ be the space of all functions of the form

$$\max\nolimits_{j=1}^{k} \min\nolimits_{j'=1}^{k'} \alpha_{jj'} + \nu_{jj'},$$

where k and k' are two natural numbers, each $\alpha_{jj'}$ is a real number, and each $\nu_{jj'}$ is a linear combination of functions in $\mathcal{R}$. Then $\mathcal{L}$ is a resolving lattice.

For a concrete example, suppose that A is a convex subset of a finite dimensional Euclidean space $\mathbb{R}^m$. A function $\nu\colon A \to \mathbb{R}$ is said to be piecewise affine if there is a finite number $\nu^1, \ldots, \nu^k$ of affine functions from A to $\mathbb{R}$ such that for each $a \in A$ we have $\nu(a) = \nu^j(a)$ for some $j = 1, \ldots, k$. Let $\mathcal{P}(A)$ be the space of continuous piecewise affine functions on A.

Corollary 5.12. *If A is a convex subset of $\mathbb{R}^m$, for some positive integer m, then $V = \mathcal{P}(A)^n$ is $\mathcal{P}(A)$-comprehensive.*

It turns out that resolving lattices of valuations are uninteresting when A is finite.

Lemma 5.13. *If A is finite, then $\mathcal{L}$ is all of $\mathbb{R}^A$.*

Proof. Fix $x \in A$ and let $\nu^x \in \mathcal{L}$ be a resolving function for x. There are numbers c_1 and c_2 such that $c_1 < \nu^x(x) < c_2$ and for all $a \in A \setminus \{x\}$, either $\nu^x(y) < c_1$ or $c_2 < \nu^x(y)$. The function $(\nu^x - c_1)^+ \wedge (c_2 - \nu^x)^+$ is in $\mathcal{L}$, because $\mathcal{L}$ contains all constant functions and is a lattice, and it nonzero at x and nowhere else. Since x was arbitrary, $\mathcal{L}$ contains a basis of $\mathbb{R}^A$. $\square$

5.3. Smooth Valuations

Fix an order of differentiability $1 \le r \le \infty$. We now assume that A is a compact C^r manifold, and as usual we let $C^r(A)$ denote the space of all C^r real valued functions on A. We will often describe elements of $C^r(A)$ as *smooth* functions. The closure of $C^r(A)$ in the topology of uniform convergence is all of $C(A)$, so there is little economic content in assuming that valuations are smooth, but the ability to use such functions might be technically convenient in some contexts. We point out that this is an example in which the set of valuations is *not* a lattice, and it illustrates the additional generality obtained by not collapsing (C3) and (C4) into a single condition asserting that for any $x, y \in A$ and $v_i^x, v_i^y \in V_i$ such that there is a $v_i \in V_i$ such that $v_i \le v_i^x \wedge v_i^y$ with $v_i(x) = v_i^x \wedge v_i^y(x)$ and $v_i(y) = v_i^x \wedge v_i^y(y)$.

For any $x \in A$ and neighborhood B of x, standard constructions (cf. Section 2.2 of Hirsch [17]) give an element of $C^r(A)$ whose range is $[0,1]$ that attains the value 1 uniquely at x and whose support is contained in B. Let $\mathcal{U}$ be a subset of $C^r(A)$ whose elements map into $[0,1]$, and which contains such a function for each x and U. Of course $\mathcal{U}$ is a separating family.

Lemma 5.14. *If $V_i = C^r(A)$, then V_i is $\mathcal{U}$-perturbative.*

Proof. Consider $x \in A$, $v_i^x \in V_i$, $\mu \in \mathcal{U}$, and $\delta_i \in \mathbb{R}$. Let ν be an element of $C^r(A)$ with $\nu(x) = 0$ and $\nu(a) < 0$ for all $a \neq x$. It suffices to set $v_i = v_i^x + \delta_i \mu + \nu$. □

Lemma 5.15. *If $V_i = C^r(A)$, then V is combinative.*

Proof. (C1) Fix $v_i^x, v_i^y \in V$ and distinct alternatives $x, y \in A$. Let B_x, B_y be neighborhoods of x and y, respectively, whose closures are disjoint. Let c_i be a constant such that $c_i \leq v_i^x \wedge v_i^y$ for all $a \in A$ (since A is compact and all functions are continuous, such a c_i exists) and let $\nu\colon A \to [0,1]$ be a smooth function such that $\nu(a) = 1$ if $a = x, y, z$, $\nu(a) < 1$ otherwise, and $\nu(a) = 0$ for all $a \notin (B_x \cup B_y \cup B_z)$. Let v_i be defined on A by

$$v_i(a) \;=\; \begin{cases} (1-\nu(a))c_i + \nu(a)v_i^x(a), & \text{if } a \in B_x, \\ (1-\nu(a))c_i + \nu(a)v_i^y(a), & \text{if } a \in B_y, \\ c_i, & \text{otherwise.} \end{cases}$$

Then v_i agrees with $(1-\nu(a))c_i + \nu(a)v_i^x(a)$ on $A \setminus \overline{B}_y$, so it is smooth on this set, and it is also smooth on $A \setminus \overline{B}_x$, so it is smooth on all of A. Obviously we have $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, and $v_i \leq v_i^x \vee v_i^y$.

(C2) With obvious modifications, the proof is the same as for (C1).

(C3) Let $v_i^x, v_i^y \in V$ and $x, y \in A$ be such that $v_i^x(x) \neq v_i^y(x)$ and $v_i^x(y) \neq v_i^y(y)$. Let B_x and B_y be disjoint neighborhoods of x and y, respectively, such that the sign of $v_i^x(a) - v_i^y(a)$ coincides with the sign of $v_i^x(x) - v_i^y(x)$ for all a in the closure of B_x, and similarly the sign of $v_i^x(a) - v_i^y(a)$ coincides with that of $v_i^x(y) - v_i^y(y)$ for all a in the closure of B_y. Let δ_i be a constant such that $\delta_i \leq v_i^x \wedge v_i^y$. There is a smooth function $\nu\colon A \to [0,1]$ such that $\nu(x) = \nu(y) = 1$, $\nu(a) < 1$ if $a \neq x, y$, and $\nu(a) = 0$ if $a \notin B_x \cup B_y$. (One can take a sum of two appropriate elements of $\mathcal{U}$.) Let v_i be the function

$$v_i(a) \;=\; (1-\nu(a))\delta_i \,+\, \nu(a)\min\{v_i^x(a), v_i^y(a)\}.$$

Then $v_i \leq v_i^x \wedge v_i^y$ with $v_i(x) = v_i^x \wedge v_i^y(x)$ and $v_i(y) = v_i^x \wedge v_i^y(y)$. This function agrees with either $(1-\nu(a))\delta_i - v_i^x(a)$ or $(1-\nu(a))\delta_i - v_i^y(a)$ on a neighborhood of the closure of B_x, and also on a neighborhood of the closure of B_y, and it is constant on $A \setminus (B_x \cup B_y)$, so it is smooth.

(C4) For any finite collection $\{v_i^x\}_{x\in S}$ there is some constant function δ_i such that $\delta_i \leq \bigwedge_{x\in S} v_i^x$. □

Combining the last two results for all i gives:

Proposition 10. *If, for every agent $i \in N$, $V_i = C^r(A)$, then V is $\mathcal{U}$-comprehensive.*

6. Applications

In this section, we illustrate our results with some economic applications.

6.1. Providing a Divisible Public Good (Bad)

Our results can be applied to various collective decision-making problems: public good provision models with projects of variable size, as in Green and Laffont [16], and Laffont and Maskin [22]; cost sharing agreements, as in Moulin [31], [32]; allocation models of pollution permits and other public divisible resources, as Dasgupta et al [11], Duggan and Roberts [13], and more recently Montero [30]; etc. It is common in these kinds of models to treat the set of alternatives A as a topological or metric space. If the domain of individual valuations is the space of continuous functions on a metric space, or the space of piecewise affine functions or piecewise concave functions on a finite-dimensional vector space, or the space of smooth functions on a compact topological space, then it follows from our results in Section 5 that in all these cases any truthfully implementable SCF must be a lexicographic affine maximizer.

A concrete illustration is provided by a generalization of the model of Montero [30], who studies an efficient auction mechanism to allocate a profile of pollution rights $x = (x_1, \dots, x_m)$ among m firms. Here x_i is firm i's pollution level; it is assumed that $0 \leq x_i \leq x_i^0$, for all $i = 1, \dots, m$. Montero also assumes that each firm has a privately known differentiable inverse demand function $P_i(x_i)$, with $P_i'(x_i) < 0$ and $P_i(x_i^0) = 0$, so that its cost function of emission reduction is $C_i(x_i) = \int_{x_i}^{x_i^0} P_i(z)dz$. The objective function of the regulator includes, in addition to the aggregate clean-up costs, a differentiable social cost function $D(\overline{x})$, where $\overline{x} = \sum_{i=1}^m x_i$. It is possible, though, that firms have non-monotone, non-convex cost functions due to indivisibilities or complementarities in their production technologies. As long as firms or, more generally, different stakeholders in the society have continuous or smooth valuations for $x = (x_1, \dots, x_m)$ —it may matter for some agents if pollution levels are concentrated in a certain region or industry— any SCF that truthfully implements emission rights must be a lexicographic affine maximizer. Moreover, truthful implementation can always be accomplished by means of generalized VCG payments.

6.2. Discrete Public Projects with Indifference Sets

Consider an alternate model of public good provision, where there is a finite number of projects that can be adopted, but each project must be built in fixed increments; e.g., a bridge or highway connecting two locations can have two, four or six lanes, an airport (hospital) can have one, two or three terminals (units), a sports arena can sit 50, 60 or 70 thousand spectators, etc. An alternative a here describes a chosen project and its associated size. We consider a situation where the selection of certain projects generates the same valuation for an agent, regardless of the size of the chosen project. For instance, the daily commuter may assign different values to the bridge or the highway, depending on the number of lanes specified for each project, but he is indifferent with respect to the capacity of the football stadium.[12]

To capture an scenario where every agent has a non-empty indifference set of alternatives, we assume that a type $t_i = (I_{t_i}, \delta_{t_i}, (v_{t_i}(a))_{a \in A \setminus I_{t_i}})$ for agent $i \in N$ specifies an indifference set $I_{t_i} \subset A$ with two or more alternatives, a number $\delta_{t_i} \geq 0$ such that if $x \in I_{t_i}$ then $v_{t_i}(x) = \delta_{t_i}$, and a vector of nonnegative valuations $(v_{t_i}(a))_{a \in A \setminus I_{t_i}}$ for alternatives outside the indifference set. Let $\mathcal{T}^I$ be the space of all such types. Note that since every type contains a indifference set, the space of valuations $\left\{(v_{t_i}(a))_{a \in A} \colon t_i \in \mathcal{T}^I\right\}$ is a strict subset of $\mathbb{R}_+^{|A|}$ when $|A| \geq 5$.[13]

---

[12] Jaramillo and Manjunath [18] study the consequences of accommodating indifference sets in the assignment problem of indivisible objects without money.

[13] Here we use the fact that the indifference set of every type contains two or more alternatives. Otherwise the domain is unrestricted, as in [36].

Lemma 6.1. *Let A be a finite set of alternatives, with $|A| \geq 5$. If for all $i \in N$, $T_i = \mathcal{T}^I$, then $T = \times_i^n T_i$ allows elevation of pairs and is flexible.*

Proof. To show that T satisfies elevation of pairs, fix $i \in N$, $x, y \in A$, and $t_i^x, t_i^y \in T_i$ such that $d_{t_i^y}(x,y) = v_{t_i^y}(x) - v_{t_i^y}(y) > v_{t_i^x}(x) - v_{t_i^x}(y) = d_{t_i^x}(x,y)$. Notice that by assumption it is impossible to have $\{x,y\} \subset I_{t_i^x}$ and $\{x,y\} \subset I_{t_i^y}$ holding simultaneously. Choose $\epsilon_i, \delta_i > 0$ to satisfy $v_{t_i^y}(y) - v_{t_i^x}(y) < \epsilon_i - \delta_i < v_{t_i^y}(x) - v_{t_i^x}(x)$. Let $t_i \in T_i = \mathcal{T}^I$ be such that $I_{t_i} = A \setminus \{x,y\}$, $v_{t_i}(a) = \min_{b \in A}\{v_{t_i^x}(b), v_{t_i^y}(b)\}$ for all $a \in I_{t_i}$, and $v_{t_i}(x) = v_{t_i^x}(x) + \epsilon_i$, $v_{t_i}(y) = v_{t_i^y}(y) + \delta_i$. It is immediate to verify that $d_{t_i}(x,a) > d_{t_i^x}(x,a)$ for all $a \neq x$ and $d_{t_i}(y,a) > d_{t_i^y}(y,a)$ for all $a \neq y$.

To show that T satisfies (F2), fix $x, y, z \in A$, $i \in N$, $t_i^x, t_i^y, t_i^z \in T_i = \mathcal{T}^I$ and $\delta_{xyi}, \delta_{yzi} \in \mathbb{R}$. Consider a type $t_i \in T_i$ for which $I_{t_i} = A \setminus \{x,y,z\}$, $v_{t_i}(a) = 0$ for all $a \in I_{t_i}$, and further $v_{t_i}(x) > v_{t_i^x}(x) \geq 0$, $v_{t_i}(y) > v_{t_i^y}(y) \geq 0$, $v_{t_i}(z) > v_{t_i^z}(z) \geq 0$, and $d_{t_i}(x,y) = \delta_{xyi}, d_{t_i}(y,z) = \delta_{yzi}$. Clearly, type t_i satisfies (a) to (c) of condition (F2) with $B'_x = \{x\}, B'_y = \{y\}, B'_z = \{z\}$. Condition (F1) is similarly shown. Thus T is flexible. □

If the type domain T_i of each agent equals $\mathcal{T}^I$, then from Lemma 6.1, it follows that in these public good provision environments with indifference sets an SCF is implementable if and only if it is a lexicographic affine maximizer.

7. Counterexamples

We mentioned in Section 1 that Myerson [34] auction model is a prominent example of a domain sufficiently restrictive to include non affine maximizers as truthfully implementable SCFs. More generally, one-dimensional preference domains admit implementation via non affine maximiers; cf. Nisan [35] and references therein. Recent work by Mishra and Roy [27] provides an interesting example of a multidimensional domain where Robert's theorem does not hold. They study allocation problems with finitely many alternatives and *dichotomous domains*: a type domain T_i is dichotomous if each $t_i = (v_{t_i}, A_{t_i}) \in T_i$ specifies a positive number v_{t_i} and a non-empty set of "acceptable" alternatives $A_{t_i} \subset A$, such that the value associated with every alternative $a \in A_{t_i}$ is v_{t_i} and the value associated with every alternative $b \in A \setminus A_{t_i}$ is zero. A prominent example of a dichotomous domain is one-sided matching with transfers, for which Mishra and Roy [27] obtain the revenue maximizing SCF. A dichotomous domain is smaller than the domain we consider in Section 6.2.

Here we present two simple examples of multidimensional domains that, we hope, help clarify the extent to which the various implications that constitute Theorem 1 do not hold in domains in the absence of comprehensiveness. Example 7.1, inspired by Example S3 in Bikhchandani et al [6], considers an auction-like environment without externalities to show that truthful implementation does not imply monotonicity in differences in every domain V. It also illustrates that the affine maximization property of an SCF does not necessarily follow from monotonicity in differences without flexibility. Example 7.2 shows an environment in which the affine maximization property does not follow from truthful implementation, even though the preference domain in partially ordered. This example is somewhat similar to the one in Mishra and Sen [28], which they attribute to Meyer-ter-Vehn and Moldovanu[14].

Example 7.1. There are three units of a good to be allocated among agents with unit demands. We use $o_i = \mathsf{y}$ to indicate that $i \in N = \{1,2,3\}$ receives one unit of the good, and $o_i = \mathsf{n}$ to indicate otherwise. Thus, an alternative is $a = o_1o_2o_3$ and $A = \{o_1o_2o_3 : o_i = \mathsf{y}, \mathsf{n},\ i \in N\}$. Agent i's domain V_i contains all vectors $v_i \in \mathbb{R}^{|A|}$ satisfying:

$$v_i(o_1o_2o_3) \;=\; \begin{cases} \alpha_i, & \text{if } o_i = \mathsf{y}, \\ 0, & \text{if } o_i = \mathsf{n}; \end{cases}$$

where $\alpha_i \in \mathbb{R}$ for each $i = 1,2,3$. Note v_i is not a split function (cf. Section 4).

We first give an SCF that is truthfully implementable but not monotonic in differences, which shows that Proposition 4 does not hold without the assumption of elevation of pairs. Define $f_1 \colon V \to A$ by specifying that $f_1(v) = o_1o_2o_3$, with $o_i = \mathsf{y}$ if and only if $\alpha_i > \alpha_j + \alpha_k - 10$, where j and k denote the two other agents. Immediately, f_1 is truthfully implemented using the payment rule $p \colon V \to A$ defined by $p_i(v) = \alpha_j + \alpha_k - 10$ if $f_1(v)$ satisfies $o_i = \mathsf{y}$ and $p_i(v) = 0$ otherwise. However, f_1 is not monotonic in differences. To see this, let $v, v' \in V$ be two profiles such that for each i, $\alpha_i = 9$ and $\alpha'_i = 11$. Clearly, $f_1(v) = \mathsf{yyy}$ whereas $f_1(v') = \mathsf{nnn}$, which implies $d_{v_i}(f_1(v), f_1(v')) = 9 < 11 = d_{v'_i}(f_1(v), f_1(v'))$ for all $i \in N$.

Our second SCF is monotonic in differences but not an affine maximizer. Since its image has more than two elements, this shows that Proposition 5 requires the assumption of flexibility. Define $f_2 \colon V \to A$ by $f_2(v) = o_1o_2o_3$, where $o_i = \mathsf{y}$ if and only if $\alpha_i < \alpha_j + \alpha_k - 10$, for $i = 1,2$, and $o_3 = \mathsf{y}$ if and only if $\alpha_3 > 0$. It is easy to see that f_2 is not an affine maximizer. Monotonicity in differences is however satisfied. Choose $v, v' \in V$ arbitrarily, and let $f_2(v) = o_1o_2o_3$ and $f_2(v') = o'_1o'_2o'_3$. If $o_3 = o'_3$, then immediately one has

$$d_{v_3}(f_2(v), f_2(v')) \;=\; 0 \;=\; d_{v'_3}(f_2(v), f_2(v')).$$

If, on the other hand, $o_3 = \mathsf{y}$ and $o'_3 = \mathsf{n}$, so that $\alpha_3 > 0 \geq \alpha'_3$, then

$$d_{v_3}(f_2(v), f_2(v')) \;=\; \alpha_3 \;>\; \alpha'_3 \;=\; d_{v'_3}(f_2(v), f_2(v')).$$

Example 7.2. The alternative set is $A = \{w, x, y, z\}$. There are two agents, and the domain of valuation profiles is $V = V_1 \times V_2$ where:

$$\begin{aligned} V_1 \;&=\; \big\{v_1 \in \mathbb{R}^4 : 0 < v_1(w) \leq v_1(y), v_1(z) \leq v_1(x)\big\}, \\ V_2 \;&=\; \big\{v_2 \in \mathbb{R}^4 : 0 < v_2(w) \leq v_2(x) \leq v_2(y), v_2(z)\big\}. \end{aligned}$$

Thus V_1 is the set of positive valuations consistent with the partial order $x \succeq_1 y, z \succeq_1 w$, while V_2 is the set of positive valuations consistent with $y, z \succeq_2 x \succeq_2 w$. Notice that V is open, convex (hence connected) and unbounded from above, but it is not comprehensive.

Consider now the following subsets of V_1 and V_2:

$$\begin{aligned} V_1^\circ \;&=\; \big\{v_1 \in V_1 : 0 < v_1(w) < \tfrac{1}{2}, v_1(y) = v_1(z) = v_1(w), v_1(x) = v_1(w) + \tfrac{1}{2}\big\}, \\ V_2^\circ \;&=\; \big\{v_2 \in V_2 : 0 < v_2(w) < \tfrac{1}{2}, v_2(x) = v_2(w), v_2(y) = v_2(z) = v_2(w) + \tfrac{1}{2}\big\}. \end{aligned}$$

[14]Private communication.

Define the SCF $f\colon V \to A$ by $f(v) = w$ if and only if $v_1 \in V_1^\circ$ and $v_2 \in V_2^\circ$, and $f(v) = \arg\max\{v_1(a) + v_2(a) : a \in A, a \neq w\}$ otherwise. Clearly f is not an affine maximizer. It is however truthfully implementable. Indeed, using the Taxation Principle, we write the payment function p_i in terms of v_j and chosen alternative $a \in A$. Consider the payment scheme given by:

$$p_1(a, v_2) \ = \ \begin{cases} v_2(w), & \text{for } v_2 \in V_2^\circ, a = w, y, z, \\ v_2(w) + \frac{1}{2}, & \text{for } v_2 \in V_2^\circ, a = x, \\ -\,v_2(a), & \text{for } v_2 \in V_2 \setminus V_2^\circ, a \in A; \end{cases}$$

and similarly

$$p_2(a, v_1) \ = \ \begin{cases} v_1(w), & \text{for } v_1 \in V_1^\circ, a = w, x, \\ v_1(w) + \frac{1}{2}, & \text{for } v_1 \in V_1^\circ, a = y, z, \\ -\,v_1(a), & \text{for } v_1 \in V_1 \setminus V_1^\circ, a \in A. \end{cases}$$

The reader can verify without difficulty that this payment scheme truthfully implements f in dominant strategies.

8. Proof of Proposition 5

We now prove Proposition 5, which is the most challenging aspect of Robert's theorem. The argument is modelled on one of the proofs given in Lavi et al [24]. Throughout the section we assume that T is flexible, that $|f(T)| \geq 3$, and that f satisfies monotonicity in differences. We begin with some elementary consequences of flexibility.

Lemma 8.1. *For any distinct $x, y \in f(T)$ and any $\delta_{xy} \in \mathbb{R}^n$, there exists an profile $t \in T$ such that $d_t(x, y) = \delta_{xy}$ and $f(t) \in \{x, y\}$.*

Proof. Choose $t^x, t^y \in T$ such that $f(t^x) = x$ and $f(t^y) = y$. Then (F1) provides disjoint sets $B_x, B_y \subset A$ containing x and y, respectively, and a type profile $t \in T$ such that $d_t(x, y) = \delta_{xy}$ and:

(a) $d_t(a, x) \ll d_{t^x}(a, x)$, for all $a \in A \setminus (\{x\} \cup B_y)$;
(b) $d_t(a, y) \ll d_{t^y}(a, y)$, for all $a \in A \setminus (\{y\} \cup B_x)$.

If $f(t) \notin \{x, y\}$, then either $f(t) \in A \setminus (\{x\} \cup B_y)$ or $f(t) \in A \setminus (\{y\} \cup B_x)$, so that either (a) or (b) gives a failure of negative unanimity. □

Lemma 8.2. *For any distinct $x, y, z \in f(T)$ and any $\delta_{xy}, \delta_{yz} \in \mathbb{R}^n$ there exists a type profile $t \in T$ such that $d_t(x, y) = \delta_{xy}$, $d_t(y, z) = \delta_{yz}$, and $f(t) \in \{x, y, z\}$.*

Proof. The argument follows the same pattern as the proof of Lemma 8.1, with (F2) in place of (F1), so it is left to the reader. □

For distinct $x, y \in f(T)$, let

$$Q(x, y) \ = \ \{\, \alpha \in \mathbb{R}^n : d_t(x, y) \ll \alpha \text{ for some } t \in T \text{ with } f(t) = x \,\}.$$

Lemma 8.3. *For any distinct $x, y \in f(T)$, $Q(x, y) \cap -Q(y, x) \ = \ \emptyset$.*

Proof. Suppose that $\alpha \in Q(x, y)$ and $-\alpha \in Q(y, x)$. Choose $t^x, t^y \in T$ such that $f(t^x) = x$, $f(t^y) = y$, $d_{t^x}(x, y) \ll \alpha$ and $d_{t^y}(y, x) \ll -\alpha$. Since $d_{t^y}(y, x) = -\,d_{t^y}(x, y)$, we have

$$d_{t^y}(x, y) \ \gg \ \alpha \ \gg \ d_{t^x}(x, y),$$

which is a violation of negative unanimity. □

The next two results develop the pertinent consequences of flexibility.

Lemma 8.4. *For any distinct $x, y \in f(T)$ and $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \gg \beta$, either $\alpha \in Q(x, y)$ or $\beta \in -Q(y, x)$.*

Proof. Lemma 8.1 gives a $t \in T$ such that $d_t(x, y) = \alpha$ and $f(t) \in \{x, y\}$. Therefore either $f(t) = x$ and thus $\alpha \in Q(x, y)$, or $f(t) = y$, in which case $-\beta \in Q(y, x)$ because $d_t(y, x) + (\alpha - \beta) = -\beta$ and $\alpha - \beta \gg 0$. □

Lemma 8.5. *For any distinct $x, y, z \in f(T)$, $Q(x, y) + Q(y, z) \subseteq Q(x, z)$.*

Proof. Consider $\alpha \in Q(x, y)$ and $\beta \in Q(y, z)$. Choose t^x and t^y such that $f(t^x) = x$, $f(t^y) = y$, $d_{t^x}(x, y) \ll \alpha$, $d_{t^y}(y, z) \ll \beta$. Lemma 8.2 gives a profile $t \in T$ such that $d_t(x, y) \ll d_{t^x}(x, y)$, $d_t(y, z) \ll d_{t^y}(y, z)$, and $f(t) \in \{x, y, z\}$. Then negative unanimity implies that $f(t) \neq y$ and $f(t) \neq z$, so $f(t) = x$; since $d_t(x, z) = d_t(x, y) + d_t(y, z) \ll \alpha + \beta$, the result is proven. □

Let $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^n$. For each distinct $x, y \in f(T)$, let

$$q(x, y) \ = \ \sup\{\, s \in \mathbb{R} : -\, s\mathbf{e} \in Q(x, y)\,\}.$$

Lemma 8.6. *For any distinct $x, y \in f(T)$, $q(x, y) = -\,q(y, x)$ and*

$$-\,q(x, y)\mathbf{e} \notin Q(x, y) \cup -Q(y, x).$$

Proof. In view of Lemma 8.3, $-q(x, y)\mathbf{e} \ll q(y, x)\mathbf{e}$ is impossible, so $-q(x, y) \geq q(y, x)$. The inequality cannot be strict because then $(-\frac{1}{3}q(x, y) + \frac{2}{3}q(y, x))\mathbf{e}$ and $(-\frac{2}{3}q(x, y) + \frac{1}{3}q(y, x))\mathbf{e}$ would be in neither $Q(x, y)$ nor $-Q(y, x)$, which would contradict Lemma 8.4. In view of the definition of $Q(x, y)$ it is obvious that $-q(x, y)\mathbf{e} \notin Q(x, y)$ and $-q(y, x)\mathbf{e} \notin Q(y, x)$. □

Lemma 8.7. $q \in \mathcal{D}(f(T))$.

Proof. Since $Q(x, y) + Q(y, z) \subseteq Q(x, z)$ and $Q(x, z) + Q(z, y) \subseteq Q(x, y)$, it must be the case that $q(x, y) + q(y, z) \leq q(x, z)$ and $q(x, z) + q(z, y) \leq q(x, y)$. But $q(z, y) = -\,q(y, z)$. Thus, $q(x, y) + q(y, z) = q(x, z)$, as desired. □

For distinct $x, y \in f(T)$, let $Q^*(x, y) = Q(x, y) + q(x, y)\mathbf{e}$.

Lemma 8.8. *For all distinct x, y, z in $f(T)$ we have:*

(a) *$Q^*(x, y) + Q^*(y, z) \subseteq Q^*(x, z)$.*
(b) *$Q^*(x, y) \cap -Q^*(y, x) = \emptyset$.*
(c) *If $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \gg \beta$, then $\alpha \in Q^*(x, y)$ or $\beta \in -Q^*(y, x)$.*
(d) *$Q^*(x, y) = Q^*(x, z) = Q^*(y, z)$.*

Proof. We have:

(a) This follows from $Q(x, y) + Q(y, z) \subseteq Q(x, z)$ and $q(x, y) + q(y, z) = q(x, z)$.
(b) This follows from $Q(x, y) \cap -Q(y, x) = \emptyset$ and $q(x, y) = -\,q(y, x)$.
(c) Since $q(x, y) = -\,q(y, x)$, Lemma 8.4 implies that $\alpha - q(x, y)\mathbf{e} \in Q(x, y)$ or $\beta + q(y, x)\mathbf{e} \in -Q(y, x)$.
(d) Since $q(y, z)$ is in the boundary of $Q(y, z)$, the origin is in the boundary of $Q^*(y, z)$, and $Q^*(x, y) + Q^*(y, z) \subseteq Q^*(x, z)$, so $Q^*(x, y) \subseteq \overline{Q^*(x, z)}$. If $\alpha \in Q^*(x, y)$, there is $\beta \in Q^*(x, y)$ with $\beta \ll \alpha$ and $\gamma \in Q^*(x, z)$ arbitrarily close to β. Therefore there is $\gamma \in Q^*(x, z)$ with $\gamma \ll \alpha$, which implies that

$\alpha \in Q^*(x,z)$. This implies the first equality, and the same argument with the roles of $Q^*(x,y)$ and $Q^*(y,z)$ reversed gives the second.

□

Proof of Proposition 5. If $w, x, y, z \in f(T)$ with $x \neq y$ and $w \neq z$, then (d) implies that $Q^*(x,y) = Q^*(w,z)$ except when the four alternatives are distinct, and in that case applying (d) twice gives $Q^*(x,y) = Q^*(x,z) = Q^*(w,z)$. Therefore there is a single set Q^* such that $Q^*(x,y) = Q^*$ for all distinct x, y.

We claim that Q^* is convex. It suffices to show that $\frac{1}{2}\alpha + \frac{1}{2}\beta \in Q^*$ whenever $\alpha, \beta \in Q^*$ because for any $0 \leq s' \leq 1$ we can find s arbitrarily close to 1 such that

$$(1-s')\alpha + s'\beta = (1-r)\alpha + r(\alpha + s(\beta - \alpha))$$

where r is a integer multiple of 2^{-k} for some positive integer k, and $\alpha + s(\beta - \alpha) \in Q^*$ when s is sufficiently close to 1 because Q^* is open. In turn, since $Q^* + Q^* \subset Q^*$ it suffices to show that $\frac{1}{2}\alpha \in Q^*$ whenever $\alpha \in Q^*$. But if $\frac{1}{2}\alpha \notin Q^*$, then for $\beta \ll \frac{1}{2}\alpha$ we have $\beta \in -Q^*$ and $2\beta \in -Q^*$, which implies that α is in the closure of $-Q^*$, and this is impossible because Q^* is open.

Now the separating hyperplane theorem gives a nonzero $\sigma \in \mathbb{R}^n$ such that $\sigma \cdot \alpha > 0$ for all $\alpha \in Q^*$. Since $Q^* + \mathbb{R}^n_{++} \subset Q^*$, all the components of σ are nonnegative, so we can take $\sigma \in \Delta^{n-1}$. It must be the case that $\alpha \in Q^*$ whenever $\sigma \cdot \alpha > 0$ because otherwise $\beta \in -Q^*$ whenever $\beta \ll \alpha$, and for β close to α we would have $\sigma \cdot \beta > 0$, which is impossible. Thus $Q^* = \{\, \alpha \in \mathbb{R}^n : \sigma \cdot \alpha > 0 \,\}$.

Suppose that $t \in T$ and $a \in f(T) \setminus \{f(t)\}$. Then $d_t(f(t), a)$ is in the closure of $Q(f(t), a)$, so $d_t(f(t), a) + q(f(t), a)\mathbf{e}$ is in the closure of Q^*, and consequently

$$0 \leq \sigma \cdot \big(d_t(f(t), a) + q(f(t), a)\mathbf{e}\big) = \sigma \cdot d_t(f(t), a) + q(f(t), a).$$

Thus f is an affine maximizer. □

9. Concluding Remarks

We have provided a generalization of Roberts theorem: if the space of type profiles T allows elevation of pairs and is flexible, then an SCF f is truthfully implementable if and only if it is a constant function, binary implementable, or a lexicographic affine maximizer. A flexible domain that allows elevation of pairs can be smaller than the unrestricted quasi-linear domain, even with finite alternative sets. Indeed, we show that the related notion of comprehensiveness is satisfied if the set of alternatives A is a topological space and the domain of valuations is the space of continuous functions; or if A is a convex subset of a finite dimensional Euclidean space and the domain of valuations is the space of continuous piecewise affine functions; or if A is a compact differentiable manifold and the domain of valuations is the space of smooth functions. Applications of our results include standard models of public good provision.

To what extent do our results allow economically significant generality beyond the case of all continuous valuations on a metric space? If $C(A)$ contains a resolving set, then A must be a T_1 space, because continuity of the function ν^x implies that $\{x\}$ is closed. The metrization theorem (e.g., Kelley [21], p. 127) states that a topological space is metrizable if and only if it is T_1 and regular (that is, every neighborhood of a point contains a closed neighborhood) and it has a base that is a countable union of collections of open sets that are locally finite. Thus there is some room for other sorts of topological spaces, which must be in some sense exotic, and at present there are no obvious applications in view.

Unfortunately we have not been able to fully assess the extent to which our results allow one to go beyond the space of all continuous valuations, in an economically interesting manner, by further restricting the agents' preferences. In particular, we do not know of a proper subset of the space of continuous valuations that is closed in the topology of uniform convergence, contains all constant functions, and is $\mathcal{U}$-comprehensive, nor have we been able to prove that no such subspace exists. More generally, we do not know whether there are proper subspaces of the space of continuous differences that are closed in the topology of uniform convergence, flexible, and allow elevation of pairs.

Other characterization of truthful implementability available in the literature are based on cyclic monotonicity a la Rochet [37] or one of its weaker forms. If the set of alternatives is finite, then weak monotonicity characterizes truthful implementation in order-based, auction-like domains (Bikhchandani et al [7]), convex domains (Saks and Yu [39]), and monotone domains (Ashlagi et al [2]). With infinite allocation sets and quasi-linear preferences parameterized by multi-dimensional types, weak monotonicity (in addition to an integral path-independence condition) is sufficient for truthful implementation when valuations are linear in types (Archer and Kleinberg [1]), while integral weak monotonicity (plus the path-integrability condition) suffices when valuations are convex or differentiable in types (Berger et al [5]), or Lipschitz continuous in types (Carbajal and Ely [9]).

These results do not provide a description, in terms of a functional form, of implementable choice functions, which is given by monotonicity in differences in comprehensive domains. It is known that non affine maximizer choice functions can be truthfully implemented in settings with single-dimensional type spaces (cf. Nisan [35] and references therein; see also Section 2). Clearly, single dimensional type domains are not comprehensive. However, the relationship between monotonicity in differences and weak person-by-person monotonicity in multi-dimensional parametric, quasi-linear utility settings remains to be studied.

References

[1] Aaron Archer and Robert Kleinberg. Truthful germs are contagious: a local-to-global characterization of truthfulness. In *10th ACM Conference on Electronic Commerce*, 2008.

[2] Itai Ashlagi, Mark Braverman, Avinatan Hassidim, and Dov Monderer. Monotonicity and implementability. *Econometrica*, 78(5):1749–1772, 2010.

[3] Moshe Babaioff, Ron Lavi, and Elan Pavlov. Mechanism design for single-value domains. In *Proceedings of the 20th National Conference on Artificial Intelligence (AAAI-05)*, pages 241–247, 2005.

[4] Salvador Barberà and Bezalel Peleg. Strategy-proof voting schemes with continuous preferences. *Soc. Choice Welfare*, 7(1):31–38, 1990.

[5] André Berger, Rudolf Müller, and Seyed Hossein Naeemi. Path-monotonicity and truthful implementation. Technical report, Maastricht University, 2010.

[6] Sushil Bikhchandani, Shurojit Chatterji, Ron Lavi, Ahuva Mu'alem, Noam Nisan, and Arunava Sen. Supplement to weak monotonicity characterizes deterministic dominant-strategy implementation. *Econometrica* Supplement Material, 2006.

[7] Sushil Bikhchandani, Shurojit Chatterji, Ron Lavi, Ahuva Mu'alem, Noam Nisan, and Arunava Sen. Weak monotonicity characterizes deterministic dominant-strategy implementation. *Econometrica*, 74(4):1109–1132, 2006.

[8] Duncan Black. *The Theory of Committees and Elections*. Cambridge University Press, Cambridge, England, 1958.

[9] Juan Carlos Carbajal and Jeffrey C. Ely. Mechanism design without revenue equivalence. Forthcoming: *J. Econ. Theory*, 2012.

[10] Partha Dasgupta, Peter Hammond, and Eric Maskin. The implementation of social choice rules: some general results on incentive compatibility. *Rev. Econ. Stud.*, 46(2):185–216, 1979.

[11] Partha Dasgupta, Peter Hammond, and Eric Maskin. On imperfect information and optimal pollution control. *Rev. Econ. Stud.*, 47(5):857–860, 1980.

[12] Shahar Dobzinski and Noam Nisan. A modular approach to Roberts' theorem. In Marios Mavronicolas and Vicky G. Papadopoulou, editors, *Algorithmic Game Theory: SAGT 2009 Proceedings*, volume 5814 of *Lecture Notes in Computer Science*, pages 14–23, Berlin, 2009. Springer.

[13] John Duggan and Joanne Roberts. Implementing the efficient allocation of pollution. *Am. Econ. Rev.*, 92(4):1070–1078, 2002.

[14] Allan Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, 41(4):587–601, 1973.

[15] Jerry R. Green and Jean-Jacques Laffont. Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica*, 45(2):427–438, 1977.

[16] Jerry R. Green and Jean-Jacques Laffont. *Incentives in Public Decision-Making*. Studies in Public Economics. North-Holland, New York, NY, 1979.

[17] Morris W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1976.

[18] Paula Jaramillo and Vikram Manjunath. The difference indifference makes in strategy-proof allocation of objects. *J. Econ. Theory*, 147(5):1913–1946, 2012.

[19] Philippe Jehiel and Benny Moldovanu. Allocative and informational externalities in auctions and related mechanisms. In Richard Blundell, Whitney K. Newey, and Torsten Persson, editors, *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, chapter 3. Cambridge University Press, Cambridge, UK, 2006.

[20] Philippe Jehiel, Benny Moldovanu, and Ennio Stacchetti. Multidimentional mechanism design for auctions with externalities. *J. Econ. Theory*, 85(2):258–294, 1999.

[21] John L. Kelley. *General Topology*. Van Nostrand, New York, 1955.

[22] Jean-Jacques Laffont and Eric Maskin. A differential approach to dominant strategy mechanisms. *Econometrica*, 48(6):1507–1520, 1980.

[23] Bo Larsson and Lars-Gunnar Svensson. Strategy-proof voting on the full preference domain. *Math. Soc. Sci.*, 52:272–287, 2006.

[24] Ron Lavi, Ahuva Mu'alem, and Noam Nisan. Two simplified proofs for Roberts' theorem. *Soc. Choice Welfare*, 32(3):407–423, 2009.

[25] Thierry Marchant and Debasis Mishra. Mechanism design with two alternatives in quasi-linear environments. Technical report, Indian Statistical Institute, 2012.

[26] Eric Maskin. The theory of implementation in nash equilibrium: a survey. In Leonid Hurwicz, David Schmeidler, and Hugo Sonnenschein, editors, *Social Goals and Social Organization*, chapter 6. Cambridge University Press, Cambridge, UK, 1985.

[27] Debasis Mishra and Souvik Roy. Implementation in multidimensional dichotomous domains. Forthcoming: *Theor. Econ.*, 2012.

[28] Debasis Mishra and Arunava Sen. Roberts' theorem with neutrality: a social welfare ordering approach. *Games Econ. Behav.*, 75(1):283–298, 2012.

[29] Eiichi Miyagawa. House allocation with transfers. *J. Econ. Theory*, 100(2):329–355, 2001.

[30] Juan-Pablo Montero. A simple auction mechanism for the optimal allocation of the commons. *Amer. Econ. Rev.*, 98(1):496–518, 2008.

[31] Hervé Moulin. *Axioms of Cooperative Decision Making.* Econometric Society Monographs. Cambridge University Press, Cambridge, UK, 1988.

[32] Hervé Moulin. An efficient and almost budget balanced cost sharing method. *Games Econ. Behav.*, 70(1):107–131, 2010.

[33] Eitan Muller and Mark A. Satterthwaite. The equivalence of strong positive association and strategy-proofness. *J. Econ. Theory*, 14(2):412–418, 1977.

[34] Roger B. Myerson. Optimal auction design. *Math. Oper. Res.*, 6(1):58–73, 1981.

[35] Noam Nisan. Introduction to mechanism design (for computer scientists). In Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V. Vazirani, editors, *Algorithmic Game Theory*, chapter 9. Cambridge University Press, Cambridge, UK, 2007.

[36] Kevin Roberts. The characterization of implementable choice rules. In Jean-Jacques Laffont, editor, *Aggregation and Revelation of Preferences*, chapter 18. North-Holland, Amsterdam, The Netherlands, 1979.

[37] Jean-Charles Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. *J. Math. Econ.*, 16(2):191–200, 1987.

[38] Irit Rozenshtrum. Dominant strategy implementation with quasi-linear preferences. Master's thesis, Department of Economics, The Hebrew University of Jerusalem, 1999.

[39] Michael Saks and Lan Yu. Weak monotonicity suffices for truthfulness on convex domains. In *7th ACM Conference on Electronic Commerce*, 2005.

[40] Mark Allen Satterthwaite. Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. *J. Econ. Theory*, 10(1):187–217, 1975.

[41] James Schummer. Eliciting preferences to assign positions and compensation. *Games Econ. Behav.*, 30(2):293–318, 2000.

[42] Rakesh V. Vohra. *Mechanism Design: a Linear Programming Approach.* Econometric Society Monographs. Cambridge University Press, 2011.

[43] H. Peyton Young. Social choice scoring functions. *SIAM J. Appl. Math.*, 28(4):824–838, 1975.