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# On the Hamilton-Waterloo Problem for bipartite 2-factors *

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**Abstract**

Given two 2-regular graphs $F_1$ and $F_2$, both of order $n$, the Hamilton-Waterloo Problem for $F_1$ and $F_2$ asks for a factorisation of the complete graph $K_n$ into $\alpha_1$ copies of $F_1$, $\alpha_2$ copies of $F_2$, and a 1-factor if $n$ is even, for all non-negative integers $\alpha_1$ and $\alpha_2$ satisfying $\alpha_1 + \alpha_2 = \lfloor \frac{n-1}{2} \rfloor$. We settle the Hamilton-Waterloo problem for all bipartite 2-regular graphs $F_1$ and $F_2$ where $F_1$ can be obtained from $F_2$ by replacing each cycle with a bipartite 2-regular graph of the same order.

## 1 Introduction

For definitions of standard graph theoretic terminology used here see [48]. Given a 2-regular graph $F$ of order $n$, the well-known *Oberwolfach Problem* asks for a factorisation of the complete graph $K_n$ into copies of $F$ if $n$ is odd, or into copies of $F$ and a 1-factor if $n$ is even. More generally, given two 2-regular graphs $F_1$ and $F_2$, each of order $n$, and two non-negative integers $\alpha_1$ and $\alpha_2$ satisfying

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$\alpha_1 + \alpha_2 = \lfloor \frac{n-1}{2} \rfloor$, the *Hamilton-Waterloo Problem* asks for a factorisation of $K_n$ into $\alpha_1$ copies of $F_1$ and $\alpha_2$ copies of $F_2$ if $n$ is odd, or into $\alpha_1$ copies of $F_1$, $\alpha_2$ copies of $F_2$ and a 1-factor if $n$ is even.

For even $n$, the graph obtained from $K_n$ by removing the edges of a 1-factor is denoted by $K_n - I$. The 2-regular graph consisting of (vertex-disjoint) cycles of lengths $m_1, m_2, \ldots, m_t$ will be denoted by $[m_1, m_2, \ldots, m_t]$. We may also use $[m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t}]$ to denote the 2-regular graph consisting of $\alpha_i$ cycles of length $m_i$ $(i = 1, 2, \ldots, t)$. So, for example, the 2-regular graph order 14 consisting of two 4-cycles and a 6-cycle may be denoted by either $[4, 4, 6]$ or $[4^2, 6]$.

The only 2-regular graphs for which the Oberwolfach problem is known to have no solution are $[3^2]$, $[3^4]$, $[4, 5]$ and $[3^2, 5]\}$, and a solution exists for every other 2-regular graph of order at most 40 [18]. In [12], the Oberwolfach Problem is completely solved for infinitely many odd values of $n$ and for infinitely many even values of $n$. For any 2-regular graph $F$ with isomorphic connected components, except $F \cong [3^2]$ and $F \cong [3^4]$, the Oberwolfach Problem has a solution [4, 5, 29]. The Oberwolfach Problem is also known to have a solution whenever $F$ is bipartite [10, 27]. There are numerous other results on the Oberwolfach Problem, dealing with various special families of 2-regular graphs, see [9, 13, 14, 25, 28, 31, 33, 40, 42, 43, 45, 47] and see [11] for a survey of results up to 2006. Various generalisations of the Oberowlfach Problem have also been considered, for example see [15, 21, 26, 37, 38, 41, 44].

If a 2-regular graph $F_1$ can be obtained from a 2-regular graph $F_2$ by replacing each cycle of $F_2$ with a 2-regular graph on the same vertex set, then $F_1$ is said to be a *refinement* of $F_2$. For example, $[4, 8^3, 10^2, 12]$ is a refinement of $[4, 16, 18, 22]$, but $[4, 18^2, 20]$ is not. Of course, every 2-regular graph of order $n$ is a refinement of an $n$-cycle. In this paper we settle the Hamilton-Waterloo Problem in the case where the two given 2-regular graphs are bipartite and one is a refinement of the other, see Theorem 28. We obtain this result as a consequence of two more general results concerning factorisations of $K_n$ into specified numbers of copies of given bipartite 2-regular graphs and a 1-factor, see Theorems 26 and 27.

For non-isomorphic 2-regular graphs $F_1$ and $F_2$, both of order $n$, and non-negative integers $\alpha_1$ and $\alpha_2$ satisfying $\alpha_1 + \alpha_2 = \lfloor \frac{n-1}{2} \rfloor$, the problem of finding a factorisation of the complete graph $K_n$ into $\alpha_1$ copies of $F_1$ and $\alpha_2$ copies of $F_2$, or into $\alpha_1$ copies of $F_1$, $\alpha_2$ copies of $F_2$ and a 1-factor, is denoted by $\mathrm{HW}(F_1, F_2; \alpha_1, \alpha_2)$. If such a factorisation exists, then $\mathrm{HW}(F_1, F_2; \alpha_1, \alpha_2)$ is said to have a solution.

In addition to the four above-mentioned instances of the Oberwolfach Problem which have no solution, it is known that the following instances of the Hamilton-Waterloo Problem have no solution.

$$\mathrm{HW}([3,4],[7];2,1) \quad \mathrm{HW}([3,5],[4^2];2,1) \quad \mathrm{HW}([3,5],[4^2];1,2) \quad \mathrm{HW}([3^3],[4,5];2,2)$$

$$\mathrm{HW}([3^3],F;3,1) \text{ for } F \in \{[4,5],[3,6],[9]\} \quad \text{and}$$

$$\mathrm{HW}([3^5],F;6,1) \text{ for } F \in \{[3^2,4,5],[3,5,7],[5^3],[4^2,7],[7,8]\}.$$

Every other instance of the Hamilton-Waterloo Problem has a solution when $n \leq 17$ and odd [1, 22, 23], and when $n \leq 10$ and even [1, 6].

The Hamilton-Waterloo Problem has also been partially solved in the case of bipartite 2-regular graphs [10, 27]. In [27] it is shown that for bipartite 2-regular graphs $F_1$ and $F_2$ of order $n \equiv 2 \pmod 4$, $\mathrm{HW}(F_1, F_2; \alpha_1, \alpha_2)$ has a solution whenever $\alpha_1$ and $\alpha_2$ are both even. In [10] it is shown that for bipartite 2-regular graphs $F_1$ and $F_2$ or order $n \equiv 0 \pmod 4$, $\mathrm{HW}(F_1, F_2; \alpha_1, \alpha_2)$ has a solution except possibly when $\alpha_1 = 1$ or when $\alpha_2 = 1$. Our result finishes off these two partial results on the problem, but with the added restriction that $F_1$ is a refinement of $F_2$.

Apart from the above mentioned results, essentially all existing results on the Hamilton-Waterloo Problem concern special cases of the problem in which each 2-factor consists of isomorphic connected components. In [19, 20, 30], $\mathrm{HW}([3^{\frac{n}{3}}],[n];\alpha_1,\alpha_2)$ is shown to have a solution for all odd $n$ except that $\mathrm{HW}([3^3],[n];3,1)$ has no solution, and the existence of a solution is undecided when $\alpha_2 = 1$ and $n$ is any one of fourteen values in the range $93 \leq n \leq 249$. A partial solution to $\mathrm{HW}([3^{\frac{n}{3}}],[n];\alpha_1,\alpha_2)$ for $n$ even is given in [36]. In [16], $\mathrm{HW}([3^{\frac{n}{3}}],[4^{\frac{n}{4}}];\alpha_1,\alpha_2)$ is completely solved except for several cases when $n = 24$ and $n = 48$. The Hamilton-Waterloo Problem $\mathrm{HW}(F_1, F_2)$ has also been completely solved when

- $F_1 \cong [4,4,\ldots,4]$ and $F_2 \cong [2t,2t,\ldots,2t]$ for all $t \geq 3$ [24];
- $F_1 \cong [2t,2t,\ldots,2t]$ and $F_2 \cong [4t,4t,\ldots,4t]$ for all $t \geq 2$ [24];
- $F_1 \cong [4t,4t,\ldots,4t]$ and $F_2 \cong [n]$ for all $t \geq 1$ and all $n \equiv 0 \pmod{4t}$ [35].

Other results on the Hamilton-Waterloo Problem can be found in [2, 13, 32, 34], and a survey of results up to 2006 can be found in [11].

## 2 Notation, definitions and existing results

We now introduce some notation, definitions, and existing results that we will be using.

Let $\Gamma$ be a finite group and let $S$ be a subset of $\Gamma$ such that the identity $e \notin S$ and such that $S$ is inverse-closed, that is $S = -S$. The *Cayley graph* on $\Gamma$ with *connection set* $S$, denoted $\mathrm{Cay}(\Gamma, S)$, has the elements of $\Gamma$ as its vertices and there is an edge between vertices $g$ and $h$ if and only if $g = h + s$ for some $s \in S$.

We need the following two results on Hamilton cycle decompositions of Cayley graphs. The first was proved by Bermond et al [7], and the second by the third author of the current paper [17]. Both results address the open question of whether every connected Cayley graph on a finite abelian group has a Hamilton cycle decomposition [3].

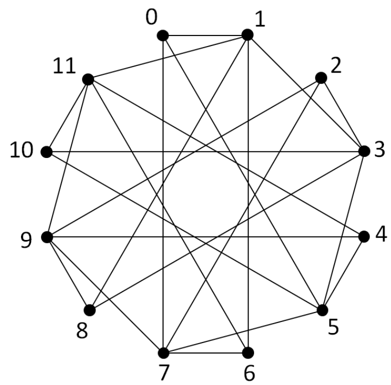
**Theorem 1** ([7]) *Every connected* 4*-regular Cayley graph on a finite abelian group has a Hamilton cycle decomposition.*

**Theorem 2** ([17]) *Every* 6*-regular Cayley graph on a cyclic group which has a generator of the group in its connection set has a Hamilton cycle decomposition.*

A Cayley graph on a cyclic group is called a *circulant graph* and we will be using these, and certain subgraphs of them, frequently. Thus, we introduce the following notation.

The length of an edge $\{x, y\}$ in a graph with vertex set $\mathbb{Z}_m$ is defined to be either $x - y$ or $y - x$, whichever is in $\{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$ (calculations in $\mathbb{Z}_m$). For even $m$ and $s \in \{1, 2, \ldots, \frac{m}{2}\}$, we call $\{\{x, x+s\} : x = 0, 2, \ldots, m-2\}$ the even edges of length $s$ and we call $\{\{x, x+s\} : x = 1, 3, \ldots, m-1\}$ the odd edges of length $s$. Note that half the edges of length $s$ are even and half are odd, except when $m \equiv 2 \pmod 4$ and $s = \frac{m}{2}$, and in this case each edge of length $s$ is both even and odd.

For any $m \geq 2$ and any $S \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$, we denote by $\langle S \rangle_m$ the graph with vertex set $\mathbb{Z}_m$ and edge set consisting of the edges of length $s$ for each $s \in S$ (that is, $\langle S \rangle_m = \mathrm{Cay}(\mathbb{Z}_m, S \cup -S)$). For $m$ even, if we wish to include only the even edges of length $s$ then we give $s$ the superscript "e". Similarly, if we wish to include only the odd edges of length $s$ then we give $s$ the superscript "o". For example, the graph $\langle \{1^{\mathrm{e}}, 2^{\mathrm{o}}, 5\} \rangle_{12}$ is shown below.




The graph $\langle\{1^{\mathrm{e}}, 2^{\mathrm{o}}, 5\}\rangle_{12}$

The *wreath product* $G \wr H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and edge set given by joining $(g_1, h_1)$ to $(g_2, h_2)$ precisely when $g_1$ is joined to $g_2$ in $G$ or $g_1 = g_2$ and $h_1$ is joined to $h_2$ in $H$. For each even $m \geq 6$, we shall use $G_{2m}$ to denote the graph $\langle\{1, 3^{\mathrm{e}}\}\rangle_m \wr \langle\{1\}\rangle_2$, and use $G_{2m} - I$ to denote the graph $\langle\{1, 3^{\mathrm{e}}\}\rangle_m \wr \langle\emptyset\rangle_2$. Thus, $G_{2m}$ is 7-regular of order $2m \geq 12$, and $G_{2m} - I$ is 6-regular of order $2m \geq 12$.

We will be dealing frequently with the wreath product of a graph $K$ and the empty graph with vertex set $\mathbb{Z}_2$, so we introduce the following special notation for this graph. The graph $K^{\square}$ is defined by $V(K^{\square}) = V(K) \times \mathbb{Z}_2$ and $E(K^{\square}) = \{\{(x, a), (y, b)\} : \{x, y\} \in E(K),\ a, b \in \mathbb{Z}_2\}$. Thus, $G_{2m} - I = \langle\{1, 3^e\}\rangle_m^{\square}$. If $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ is a set of graphs then we define $\mathcal{F}^{\square} = \{F_1^{\square}, F_2^{\square}, \ldots, F_t^{\square}\}$. Observe that if $\mathcal{F}$ is a factorisation of $K$, then $\mathcal{F}^{\square}$ is a factorisation of $K^{\square}$.

We need the following four results. Lemma 3 is a very useful result proved by Häggkvist [27], Lemma 4 was proven independently in [8] and [46], Theorem 5 is a special case of the main result in [8], and Theorem 6 was proved in [10].

**Lemma 3** ([27]) *For each 2-regular bipartite graph $F$ of order $2m$, there is a 2-factorisation of $C_m^{\square}$ into two copies of $F$.*

**Lemma 4** ([8],[46]) *For each $m \geq 5$ and every 2-regular graph $F$ of order $m$, there is a factorisation of $\langle\{1, 2\}\rangle_m$ into a Hamilton cycle and copy of $F$.*

**Theorem 5** ([8]) *For each $m \geq 3$ and each 2-regular graph $F$ of order $m$, there is a factorisation of $K_m$ into $\frac{m-3}{2}$ Hamilton cycles and copy of $F$ when $m$ is odd, and there is a factorisation of $K_m$ into $\frac{m-4}{2}$ Hamilton cycles, a copy of $F$, and a 1-factor when $m$ is even.*

**Theorem 6** ([10]) *If $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order $n$ and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that $\alpha_1+\alpha_2+\cdots+\alpha_t = \frac{n-2}{2}$, $\alpha_1 \geq 3$ is odd, and $\alpha_i$ is even for $i = 2, 3, \ldots, t$, then there exists a 2-factorisation of $K_n - I$ in which there are exactly $\alpha_i$ copies of $F_i$ for $i = 1, 2, \ldots, t$.*

In [10], the following complete solution to the Oberwolfach Problem for bipartite 2-regular graphs is established as an easy consequence of Häggkvist's result (Lemma 3) and Theorem 6.

**Theorem 7** ([10, 27]) *If $F$ is a bipartite 2-regular graph of order $n$ then there is a factorisation of $K_n$ into $\frac{n-2}{2}$ 2-factors which are isomorphic to $F$ and a 1-factor.*

# 3 Preliminary results

**Lemma 8** *For each even $m \geq 6$ and each 2-regular graph $F$ of order $m$, there is a factorisation of $K_m$ into $\frac{m-6}{2}$ Hamilton cycles, a copy of $F$, and a copy of $\langle\{1, 3^{\mathrm{e}}\}\rangle_m$; except that there is no such factorisation when $m = 6$ and $F = [6]$ nor when $m = 8$ and $F = [3, 5]$.*

**Proof** For $m = 6$, $\langle\{1, 3^{\mathrm{e}}\}\rangle_m$ is $K_{3,3}$ and the graph that remains when the edges of a copy of $K_{3,3}$ are removed from $K_6$ is $[3, 3]$. This proves the result for $m = 6$. For $m = 8$, $\langle\{1, 3^{\mathrm{e}}\}\rangle_m$ is the 3-cube and the graph that remains when a 3-cube is removed from $K_8$ consists of a pair of vertex-disjoint copies of $K_4$ joined by a perfect matching. It is straightforward to decompose this graph into two Hamilton cycles, or into a Hamilton cycle and a pair of vertex-disjoint 4-cycles. It is also easy to see that it does not contain the 2-factor $[3, 5]$. This proves the result for $m = 8$.

We now deal with the case $m = 10$. The permutation $(0)(1\,5\,7\,3)(2\,6\,8\,4)(9)$ is an isomorphism from $\langle\{1, 3^{\mathrm{e}}\}\rangle_{10}$ to $\langle\{1, 5\}\rangle_{10}$. Thus, it is sufficient to show that $\langle\{2, 3, 4\}\rangle_{10}$ can be factorised into two Hamilton cycles and a copy of $F$ for each 2-regular graph $F$ of order 10. There are five such graphs: $[10], [3, 7], [4, 6], [5, 5]$ and $[3, 3, 4]$. For $F \cong [10]$ we can use Theorem 2. For the remaining four graphs we have the decompositions given below.

| $F \cong [3,3,4]$ | $F \cong [3,7]$ | $F \cong [4,6]$ | $F \cong [5,5]$ |
|---|---|---|---|
| (0,3,6) ∪ (2,5,9) ∪ (1,7,4,8) | (0,3,6) ∪ (2,8,4,7,1,5,9) | (0,3,9,6) ∪ (1,5,2,8,4,7) | (0,2,4,6,8) ∪ (1,3,5,7,9) |
| (0,2,4,6,8,5,3,1,9,7) | (0,7,9,1,3,5,2,4,6,8) | (0,7,5,3,1,9,2,4,6,8) | (0,3,9,5,1,7,4,8,2,6) |
| (0,4,1,5,7,3,9,6,2,8) | (0,2,6,9,3,7,5,8,1,4) | (0,2,6,3,7,9,5,8,1,4) | (0,4,1,8,5,2,9,6,3,7) |

We now deal with $m \geq 12$. By Lemma 4, the result follows if there is a factorisation of $K_m$ into $\frac{m-8}{2}$ Hamilton cycles, a copy of $\langle\{1,2\}\rangle_m$, and a copy of $\langle\{1,3^{\rm e}\}\rangle_m$. We now show that such a factorisation exists, by dealing separately with the cases $m \equiv 0 \pmod 4$ ($m \geq 12$) and $m \equiv 2 \pmod 4$ ($m \geq 14$).

For $m \equiv 2 \pmod 4$ observe that the mapping

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & m-3 & m-2 & m-1 \\ 0 & \frac{m}{2} & \frac{m}{2}+1 & 1 & 2 & \frac{m}{2}+2 & \frac{m}{2}+3 & 3 & 4 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-1 & m-1 \end{pmatrix}$$

is an isomorphism from $\langle\{1,3^{\rm e}\}\rangle_m$ to $\langle\{1,\frac{m}{2}\}\rangle_m$, and that $\langle\{1,2\}\rangle_m$ is isomorphic to $\langle 4,\frac{m}{2}-2\rangle_m$. So in the case $m \equiv 2 \pmod 4$ it is sufficient to show that $\langle\{1,2,\ldots,\frac{m}{2}\}\setminus\{1,4,\frac{m}{2}-2,\frac{m}{2}\}\rangle_m$ has a decomposition into Hamilton cycles. This is straightforward as $\{\langle\{2,3\}\rangle_m, \langle\{5,6\}\rangle_m, \langle\{7,8\}\rangle_m, \ldots, \langle\{\frac{m}{2}-6,\frac{m}{2}-5\}\rangle_m, \langle\{\frac{m}{2}-4,\frac{m}{2}-3,\frac{m}{2}-1\}\rangle_m\}$ is a factorisation of $\langle\{1,2,\ldots,\frac{m}{2}\}\setminus\{1,4,\frac{m}{2}-2,\frac{m}{2}\}\rangle_m$ in which each 4-factor has a Hamilton cycle decomposition by Theorem 1, and the 6-factor has a Hamilton cycle decomposition by Theorem 2 (since $\gcd(\frac{m}{2}-4,m)=1$ when $m \equiv 2 \pmod 4$).

We now deal with the case $m \equiv 0 \pmod 4$. First observe that for $m \equiv 0 \pmod 4$, $\langle\{1,2\}\rangle_m$ is isomorphic to $\langle\{2,\frac{m}{2}-1\}\rangle_m$, and that $\{\langle\{4,5\}\rangle_m, \langle\{6,7\}\rangle_m, \ldots, \langle\{\frac{m}{2}-4,\frac{m}{2}-3\}\rangle_m\}$ is a 4-factorisation of $\langle\{4,5,\ldots,\frac{m}{2}-3\}\rangle_m$ in which each 4-factor has a Hamilton cycle decomposition by Theorem 1. Thus it is sufficient to show that $\langle\{3^{\rm o},\frac{m}{2}-2,\frac{m}{2}\}\rangle_m$ has a Hamilton cycle decomposition. But it is easy to see that $\langle\{3^{\rm o},\frac{m}{2}-2,\frac{m}{2}\}\rangle_m \cong \mathrm{Cay}(\mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_2, \{(\frac{m}{4}-1,0),(\frac{m}{4},0),(0,1)\})$ and so the result follows by Theorem 1. □

**Lemma 9** *If $m \geq 3$ is odd and $F$ is any 2-regular graph of order $m$, then there is a factorisation of $K_{2m}$ into $\frac{m-3}{2}$ copies of $C_m^{\square}$ and a copy of $F \wr K_2$.*

**Proof** Let $\mathcal{F}$ be a factorisation of $K_m$ into $\frac{m-3}{2}$ copies of $C_m$ and a copy of $F$, which exists by Theorem 5. Then $\mathcal{F}^{\square}$ is a factorisation of $K_{2m}-I$ into $\frac{m-3}{2}$ copies of $C_m^{\square}$ and a copy of $F^{\square}$. If we add the edges of the removed 1-factor to the copy of $F^{\square}$, then we obtain $F \wr K_2$ and hence the required factorisation of $K_{2m}$. □

**Lemma 10** *If $m \geq 6$ is even and $F \notin \{[6],[3,5]\}$ is a 2-regular graph of order $m$, then there is a factorisation of $K_{2m}$ into $\frac{m-6}{2}$ copies of $C_m^{\square}$, a copy of $F \wr K_2$, and copy of $G_{2m}-I$, and there is a factorisation of $K_{2m}$ into $\frac{m-6}{2}$ copies of $C_m^{\square}$, a copy of $F^{\square}$, and copy of $G_{2m}$.*

**Proof** Let $\mathcal{F}$ be a factorisation of $K_m$ into $\frac{m-6}{2}$ Hamilton cycles, a copy of $F$, and a copy of $\langle\{1,3^{\rm e}\}\rangle_m$, which exists by Lemma 8 (since $F \notin \{[6],[3,5]\}$). Then $\mathcal{F}^{\square}$ is a factorisation of $K_{2m}-I$ into $\frac{m-6}{2}$ copies of $C_m^{\square}$, a copy of $F^{\square}$, and copy of $G_{2m}-I$. We obtain the first required factorisation of $K_{2m}$ by adding the edges of the removed 1-factor to the copy of $F^{\Box}$, and the second by adding the edges of the removed 1-factor to the copy of $G_{2m} - I$. $\Box$

**Lemma 11** *If $m \geq 3$ and $F$ is any bipartite 2-regular graph of order $2m$, then there exists a factorisation of $C_m \wr K_2$ into a copy of $F$, a Hamilton cycle, and a 1-factor.*

**Proof** Define three graphs $F_1$, $F_2$ and $F_3$, each with vertex set $\mathbb{Z}_m \times \mathbb{Z}_2$, by

- $E(F_1) = \{\{(x,i),(x+1,i)\} : x \in \mathbb{Z}_m, i \in \mathbb{Z}_2\}$;
- $E(F_2) = \{\{(x,i),(x+1,i+1)\} : x \in \mathbb{Z}_m, i \in \mathbb{Z}_2\}$; and
- $E(F_3) = \{\{(x,0),(x,1))\} : x \in \mathbb{Z}_m\}$.

It is clear that $\{F_1, F_2, F_3\}$ is a factorisation of $C_m \wr K_2$ in which $F_1$ and $F_2$ are 2-factors and $F_3$ is a 1-factor. We obtain the required factorisation by making alterations to this factorisation. If $F = [2m_1, 2m_2, \ldots, 2m_t]$, then define $S_1$ and $S_2$ by

$$\begin{aligned} S_1 = \{ \quad & \{(m_1+m_2+\ldots+m_i,0),(m_1+m_2+\ldots+m_i+1,0)\}, \\ & \{(m_1+m_2+\ldots+m_i-1,1),(m_1+m_2+\ldots+m_i,1)\} : i = 1,2,\ldots,t\} \end{aligned}$$

and

$$\begin{aligned} S_2 = \{ \quad & \{(m_1+m_2+\ldots+m_i-1,1),(m_1+m_2+\ldots+m_i,0)\}, \\ & \{(m_1+m_2+\ldots+m_i,1),(m_1+m_2+\ldots+m_i+1,0)\} : i = 1,2,\ldots,t\} \end{aligned}$$

Define new 2-factors $F_1'$ and $F_2'$ by $E(F_1') = (E(F_1) \setminus S_1) \cup S_2$ and $E(F_2') = (E(F_2) \setminus S_2) \cup S_1$. Then $\{F_1', F_2'\}$ is a 2-factorisation of $C_m^{\Box}$ in which each of $F_1'$ and $F_2'$ is isomorphic to $F$ (this is the construction used in [27] to prove Lemma 3). If we let $\{I_1, I_2\}$ be any 1-factorisation of $F_1'$, then $\{I_1 \cup F_3, F_2', I_2\}$ is the required factorisation of $C_m \wr K_2$ with $I_1 \cup F_3$ being a Hamilton cycle, $F_2'$ being a 2-factor isomorphic to $F$, and $I_2$ being a 1-factor. $\Box$

**Lemma 12** *Let $F_2$ be any bipartite 2-regular graph of order $2m \geq 6$, say $F_2 \cong [4^r, 2m_1, 2m_2, \ldots, 2m_t]$ with $3 \leq m_1 \leq m_2 \leq \cdots \leq m_t$. If $F_1$ is any bipartite refinement of $F_2$, and $F$ is the 2-regular graph of order $m$ given by*

- *$F \cong [2r]$ if $F_2$ consists entirely of 4-cycles;*
- *$F \cong [2r + m_1, m_2, \ldots, m_t]$ otherwise,*

*then there is a factorisation of $F \wr K_2$ consisting of a 1-factor, a 2-factor isomorphic to $F_1$ and a 2-factor isomorphic to $F_2$.*

**Proof** If $F_2$ consists of 4-cycles only, then so does $F_1$ and the result follows immediately by applying Lemma 3 with $F \cong C_{2r}$. Thus, we can assume $F_2 \cong [4^r, 2m_1, 2m_2, \dots, 2m_t]$ where $t \geq 1$, $r \geq 0$, $m_i \geq 3$ for $i = 1, 2, \dots, t$ and $2r + m_1 + m_2 + \dots + m_t = m$. Let $F \cong [2r + m_1, m_2, \dots, m_t]$ so that $F \wr K_2$ consists of $t$ components: $[2r + m_1] \wr K_2$ and $[m_i] \wr K_2$ for $i = 2, 3, \dots, t$.

Now, $F_1$ consists of $t$ vertex-disjoint 2-regular graphs $G_1, G_2, \dots, G_t$ where $G_1$ is a bipartite refinement of $[4^r, 2m_1]$ and $G_i$ is a refinement of $[2m_i]$ for $i = 2, 3, \dots, t$. By Lemma 11, there is a factorisation of $[m_i] \wr K_2$ consisting of a 1-factor, a $2m_i$-cycle, and a 2-factor isomorphic to $G_i$ for $i = 2, 3, \dots, t$. Thus, the result follows if there is a factorisation of $[2r + m_1] \wr K_2$ into a 1-factor, a 2-factor isomorphic to $[4^r, 2m_1]$, and a 2-factor isomorphic to $G_1$. We now show that such a factorisation exists.

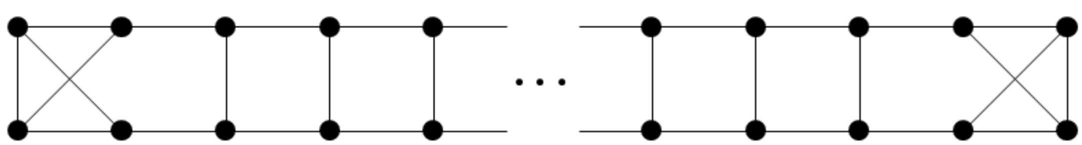
Let $s = 2r + m_1$ and let $K \cong [s] \wr K_2$ be the graph with vertex set $\mathbb{Z}_s \times \mathbb{Z}_2$ and edge set

$$\{ \{(x,0),(x,1)\} : x \in \mathbb{Z}_s \} \cup \{ \{(x,i),(x+1,j)\} : x \in \mathbb{Z}_s, i \in \mathbb{Z}_2, j \in \mathbb{Z}_2 \}.$$

Let $H \cong [4^r, 2m_1]$ be the 2-factor of $K$ consisting of the 4-cycle $((x,0),(x+1,0),(x,1),(x+1,1))$ for $x = 1, 3, \dots, 2r-1$ and the $2m_1$-cycle with edge set

$$\{ \{(0,0),(0,1)\}, \{(2r+1,0),(2r+1,1)\} \} \cup$$
$$\{ \{(x,0),(x+1,1)\}\{(x,1),(x+1,0)\} : x = 2r+1, 2r+2, \dots, 2r+m_1-1 \}.$$

Let $G$ be the graph obtained from $K$ by removing the edges of $H$. Then $G$ is a 3-regular graph consisting of $r-1$ copies of $K_4$ and one copy of the graph of order $2m_1 + 4$ shown below.

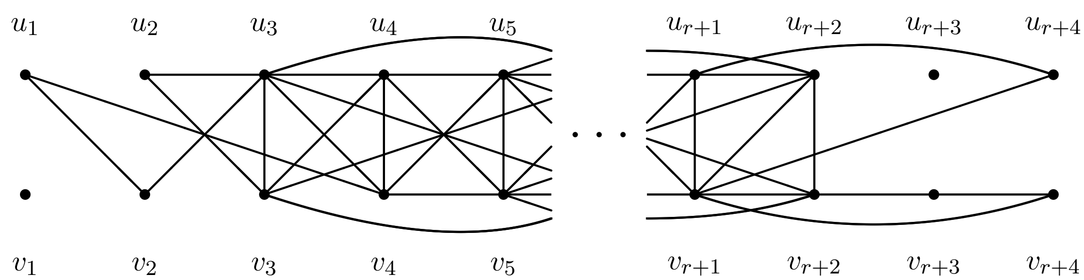


It is easy to see that this graph contains every bipartite 2-regular graph of order $2m_1 + 4$ and it follows that there is a factorisation of $G$ into a 1-factor $I$ and a 2-regular graph $H'$ that is isomorphic to $G_1$. Thus, $\{H, H', I\}$ is the required factorisation of $K \cong [s] \wr K_2$. □

# 4 Factorisations of $G_n - I$

The purpose of this section is to prove Lemma 16 below, which gives factorisations of $G_n - I$ into three copies of almost any bipartite 2-regular graph of order $n$. To achieve this we introduce classes of subgraphs of $G_n$. For each even $r$ we define $J_{2r}$ (see the figure below) to be the graph with vertex set $V(J_{2r}) = \{u_1, u_2, \ldots, u_{r+4}\} \cup \{v_1, v_2, \ldots, v_{r+4}\}$ and edge set

$$\begin{aligned} E(J_{2r}) = \; & \{\{u_i, v_i\} : i = 3, 4, \ldots, r+2\} \cup \\ & \{\{u_{i+1}, u_{i+2}\}, \{v_{i+3}, v_{i+4}\}, \{u_i, v_{i+1}\}, \{v_{i+1}, u_{i+2}\} : i = 1, 2, \ldots, r\} \cup \\ & \{\{u_{i+2}, u_{i+5}\}, \{v_{i+2}, v_{i+5}\}, \{u_i, v_{i+3}\}, \{v_{i+2}, u_{i+5}\} : i = 1, 3, 5, \ldots, r-1\}. \end{aligned}$$




We define $J_{2r} - I$ to be the graph obtained from $J_{2r}$ by removing the edges $\{\{u_i, v_i\} : i = 3, 4, \ldots, r+2\}$. Notice that for $r \geq 6$, if we take the graph $J_{2r}$ or $J_{2r} - I$ and identify $u_i$ with $u_{r+i}$ and $v_i$ with $v_{r+i}$ for each $i \in \{1, 2, 3, 4\}$, then the resulting graph is isomorphic to $G_{2r}$ or $G_{2r} - I$ respectively.

Let $F$ be a 2-regular graph of order $2r$. We write $J_{2r} - I \mapsto F$ if there is a decomposition $\{F_1, F_2, F_3\}$ of $J_{2r} - I$ such that $F_1 \cong F_2 \cong F_3 \cong F$ and the following conditions (1), (2) and (3) hold.

(1) $V(F_1) = \{u_5, u_6, \ldots, u_{r+1}\} \cup \{u_2, u_3, u_{r+4}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$.

(2) $V(F_2) = \{u_3, u_4, \ldots, u_r\} \cup \{u_1, u_{r+2}\} \cup \{v_2, v_3, \ldots, v_{r+1}\}$.

(3) $V(F_3) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_5, v_6, \ldots, v_{r+4}\}$.

It is easy to see that the next result follows immediately from the discussion in the preceding two paragraphs, as conditions (1)-(3) ensure that the subgraphs $F_1$, $F_2$ and $F_3$ become 2-factors upon the above-described identification of vertices of $J_{2r} - I$ to form $G_{2r} - I$.

**Lemma 13** *If $J_{|V(F)|} - I \mapsto F$, then $G_{|V(F)|} - I$ factorises into three copies of $F$.*

For each integer $k \geq 0$ define the mapping $\phi_k$ on $\{u_1, u_2, \ldots\} \cup \{v_1, v_2, \ldots\}$ by

$$\phi_k(u_i) = u_{i+k} \text{ and } \phi_k(v_i) = v_{i+k}$$

and for any subgraph $H$ of $J_{2r}$ define $\phi_k(H)$ to be the graph with vertex set $\{\phi_k(x) : x \in V(H)\}$ and edge set $\{\phi_k(\{x, y\}) : \{x, y\} \in E(H)\}$. Thus, $J_{2r+2s} = J_{2r} \cup \phi_r(J_{2s})$. Moreover, if $F$ is the union of vertex disjoint 2-regular graphs $F'$ and $F''$, $\{F'_1, F'_2, F'_3\}$ is a decomposition $J_{2r} - I \mapsto F'$ and $\{F''_1, F''_2, F''_3\}$ is a decomposition $J_{2r} - I \mapsto F''$ (where $F'_i$ and $F''_i$ satisfy condition $(i)$ above for $i = 1, 2, 3$), then it is clear that $\{F'_1 \cup \phi_r(F''_1), F'_2 \cup \phi_r(F''_2), F'_3 \cup \phi_r(F''_3)\}$ is a decomposition $J_{2r+2s} - I \mapsto F$. Hence we have the following result.

**Lemma 14** *If $F$ is the union of vertex disjoint 2-regular graphs $F'$ and $F''$, $J_{|V(F')|} - I \mapsto F'$, and $J_{|V(F'')|} - I \mapsto F''$, then $J_{|V(F)|} - I \mapsto F$.*

**Lemma 15** *For each graph $F$ in the following list we have $J_{|V(F)|} - I \mapsto F$.*

*(i)* $[k]$ *for each* $k \in \{8, 12, 16, \ldots\}$

*(ii)* $[k, k']$ *for each* $k \in \{6, 10, 14, \ldots\}$ *and each* $k' \in \{10, 14, \ldots\}$

*(iii)* $[4, k]$ *for each* $k \in \{4, 8, 12, \ldots\}$

*(iv)* $[4, k, k']$ *for each* $k \in \{6, 10, 14, \ldots\}$ *and each* $k' \in \{10, 14, \ldots\}$

*(v)* $[4, 4, 4]$

*(vi)* $[6, 6, k]$ *for each* $k \in \{8, 12, 16, \ldots\}$

*(vii)* $[6, 6, k, k']$ *for each* $k \in \{6, 10, 14, \ldots\}$ *and each* $k' \in \{6, 10, \ldots\}$

*(viii)* $[4, 6, 6, k]$ *for each* $k \in \{4, 8, 12, \ldots\}$

*(ix)* $[4, 6, 6, k, k']$ *for each* $k \in \{6, 10, 14, \ldots\}$ *and each* $k' \in \{6, 10, \ldots\}$

*(x)* $[4, 4, 4, 6, 6]$

**Proof** We introduce the twelve graph decompositions shown in Figure 1 which we call *pieces*. Each piece has three subgraphs indexed by the subscripts 1, 2 and 3. In each piece the subgraph indexed by subscript 1 is shown with thin solid lines, the subgraph indexed by subscript 2 is shown with dotted lines, and the subgraph indexed by subscript 3 is shown with thick solid lines.

If $\mathcal{X} = \{X_1, X_2, X_3\}$ and $\mathcal{Y} = \{Y_1, Y_2, Y_3\}$ are two pieces, we define the *concatenation* of piece $\mathcal{X}$ with piece $\mathcal{Y}$, denoted by $\mathcal{X} \oplus \mathcal{Y}$, to be the decomposition $\{X_1 \cup Y_1, X_2 \cup Y_2, X_3 \cup Y_3\}$ of the graph obtained by identifying each of the four right-most vertices, say $u_{z-1}, v_{z-1}, u_z, v_z$ of $\mathcal{X}$, with the corresponding left-most vertex, say $u_1, v_1, u_2, v_2$ respectively, of $\mathcal{Y}$. For example, Figure 2 shows the concatenation $\mathcal{L}_1 \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$. Notice that this particular concatenation is a decomposition $J_{16} - I \mapsto [16]$. Generally speaking, a left piece

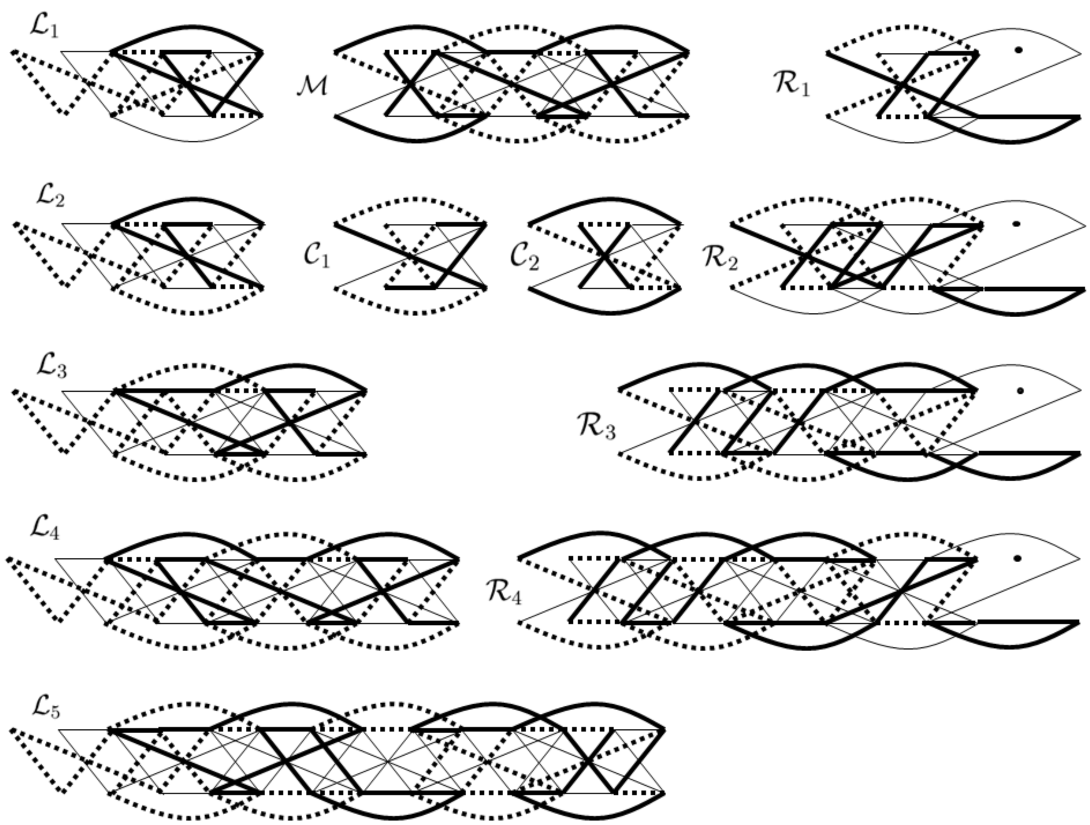



Figure 1: Twelve pieces

$(\mathcal{L}_i)$, perhaps some middle pieces $(\mathcal{C}_1, \mathcal{C}_2$ or $\mathcal{M})$, and then a right piece $(\mathcal{R}_j)$ will be concatenated to yield a decomposition $J_{V(F)} - I \mapsto F$ for each required 2-regular graph $F$. We are now ready to construct each of the decompositions $J_{|V(F)|} - I \mapsto F$ listed in (i)-(x) as required to prove the lemma.

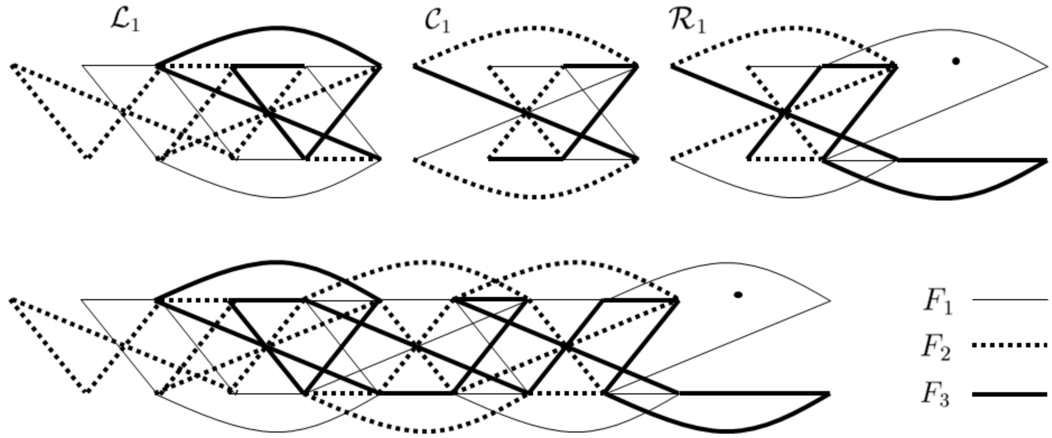



Figure 2: The concatenation $\mathcal{L}_1 \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ yields $J_{16} - I \mapsto [16]$

**(i)** The small case $J_8 - I \mapsto [8]$ is shown in Figure 3. For $k \geq 12$, the concatenation $\mathcal{L}_1 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ with $\frac{k-12}{4}$ occurrences of $\mathcal{C}_1$ gives a decomposition. The case $k = 16$ is shown in Figure 2.

**(ii)** If $k = 6$, then a decomposition is given by $\mathcal{L}_3 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ with $\frac{k'-10}{4}$ occurrences of $\mathcal{C}_1$. If $k = 10$, a decomposition is given by $\mathcal{L}_4 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ with $\frac{k'-10}{4}$ occurrences of $\mathcal{C}_1$. For $k \geq 14$, a decomposition is given by $\mathcal{L}_2 \oplus \mathcal{C}_2 \oplus \ldots \oplus \mathcal{C}_2 \oplus \mathcal{M} \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ with $\frac{k-14}{4}$ occurrences of $\mathcal{C}_2$ and $\frac{k'-10}{4}$ occurrences of $\mathcal{C}_1$.

**(iii)** Decompositions $J_8 \mapsto [4,4]$ and $J_{12} \mapsto [4,8]$ are given in Figure 3. For $k \geq 12$, the concatenation $\mathcal{L}_1 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_2$ with $\frac{k-12}{4}$ occurrences of $\mathcal{C}_1$ gives the result.

**(iv)** A decomposition $J_{4+k+k'} - I \mapsto [4, k, k']$ is found by the same method as for $J_{k+k'} - I \mapsto [k, k']$ in case (ii) above, except that the piece $\mathcal{R}_2$ is required instead of $\mathcal{R}_1$.

**(v)** A decomposition $J_{12} - I \mapsto [4,4,4]$ is given in Figure 3.

**(vi)** A decomposition $J_{20} - I \mapsto [6,6,8]$ is given in Figure 3. For $k \geq 12$, the concatenation $\mathcal{L}_3 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_3$ with $\frac{k-12}{4}$ occurrences of $\mathcal{C}_1$ gives the decomposition.

**(vii)** A decomposition $J_{24} - I \mapsto [6,6,6,6]$ is given in Figure 3. For $k = 6$ and $k' \geq 10$, the concatenation $\mathcal{L}_5 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ with $\frac{k'-10}{4}$ occurrences of $\mathcal{C}_1$ gives a decomposition. Similarly, for $k' = 6$ and $k \geq 10$, the concatenation $\mathcal{L}_5 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_1$ with $\frac{k-10}{4}$ occurrences of $\mathcal{C}_1$ suffices. If $k \geq 10$ and $k' \geq 10$, then we can write $[6,6,k,k']$ as the vertex disjoint union of $[6,k]$ and $[6,k']$ and use (ii) and Lemma 14 to obtain

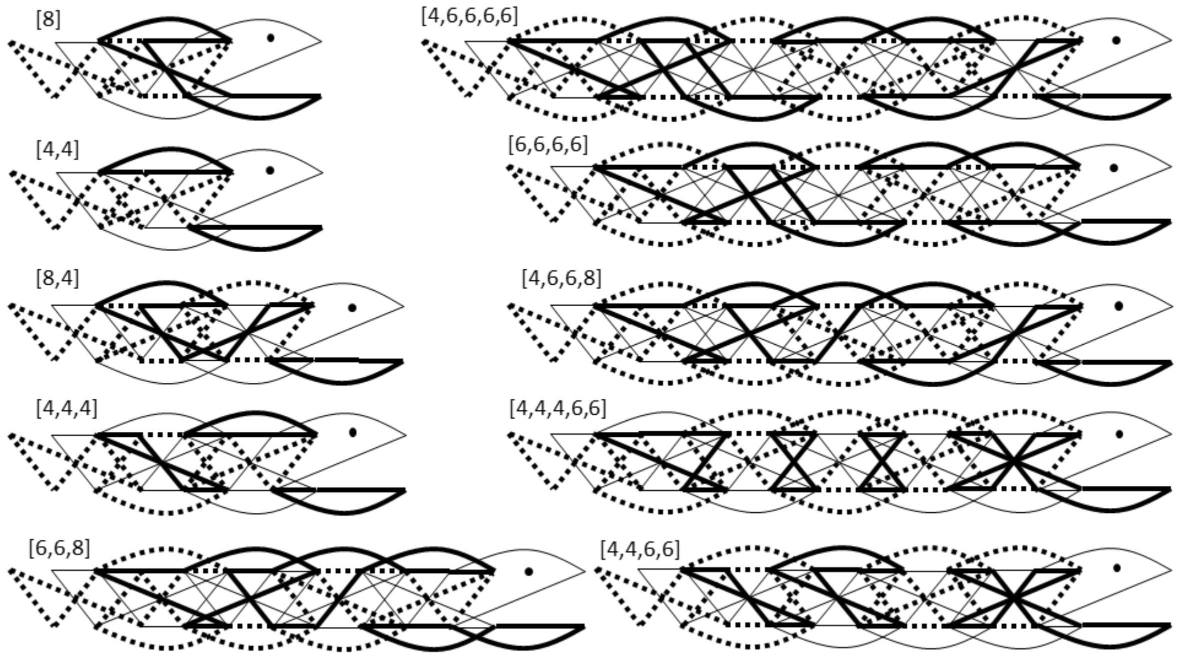



Figure 3: $J_{2r} - I \mapsto F$ for some small $F$

a required decomposition.

**(viii)** Decompositions $J_{20} - I \mapsto [4, 6, 6, 4]$ and $J_{24} - I \mapsto [4, 6, 6, 8]$ are given in Figure 3. For $k \geq 12$, the concatenation $\mathcal{L}_3 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_4$ with $\frac{k-12}{4}$ occurrences of $\mathcal{C}_1$ gives a required decomposition.

**(ix)** A decomposition $J_{28} - I \mapsto [4, 6, 6, 6, 6]$ is given in Figure 3. For $k = 6$ and $k' \geq 10$, the concatenation $\mathcal{L}_5 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_2$ with $\frac{k'-10}{4}$ occurrences of $\mathcal{C}_1$ suffices. Similarly, for $k' = 6$ and $k \geq 10$, the concatenation $\mathcal{L}_5 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_1 \oplus \mathcal{R}_2$ with $\frac{k-10}{4}$ occurrences of $\mathcal{C}_1$ gives a decomposition. If $k \geq 10$ and $k' \geq 10$, then we can write $[4, 6, 6, k, k']$ as the vertex disjoint union of $[4, 6, k]$ and $[6, k']$ and use (ii), (iv) and Lemma 14 to obtain a required decomposition.

**(x)** A decomposition $J_{24} - I \mapsto [4, 4, 4, 6, 6]$ is given in Figure 3. □

We are now ready to prove the main result of this section.

**Lemma 16** *Let $n \equiv 0 \pmod 4$ with $n \geq 12$. For each bipartite 2-regular graph $F$ of order $n$, there is a factorisation of $G_n - I$ into three copies of $F$; except possibly when $F \in \{[6^r], [4, 6^r] : r \equiv 2 \pmod 4\}$.*

**Proof** Let $F$ be a 2-regular graph of order $2m$ such that $F$ is neither $[6^r]$ with $r \equiv 2 \pmod 4$ nor $[4, 6^r]$ with $r \equiv 2 \pmod 4$. We will show that $F$ can be written as the vertex-disjoint union of 2-regular subgraphs $H_1, H_2, \ldots, H_w$ where each $H_i$ is covered by Lemma 15. The result then follows by application of Lemmas

13 and 14. Note that since $m$ is even, $|V(F)| \equiv 0 \pmod 4$ and so the number of cycles in $F$ of length congruent to $2 \pmod 4$ is also even, we will use this fact often in the remainder of the proof.

If $F$ is any graph satisfying the conditions of the Lemma and containing at least four 6-cycles, then either $F \cong [6,6,6,6]$, $F \cong [4,6,6,6,6]$, or the graph obtained from $F$ by removing four 6-cycles satisfies the conditions of the Lemma. Thus, since $[6,6,6,6]$ and $[4,6,6,6,6]$ are covered by Lemma 15, we can assume that $F$ contains at most three 6-cycles.

If $F$ contains at most one 6-cycle, then it is clear that $F$ can be written as a union of copies of graphs covered by (i)-(v) of Lemma 15. Noting that (vi)-(x) of Lemma 15 cover $[6,6] \cup H$ for each $H$ that is covered by (i)-(v) of Lemma 15, it is clear that we can deal similarly with the case where $F$ contains either two or three 6-cycles. Note that $F \cong [6,6]$ and $F \cong [4,6,6]$ are excluded by the conditions of the Lemma. □

# 5 Factorisations of $G_n$

The purpose of this section is to prove Lemma 20 below, which gives factorisations of $G_n$ into two copies of $F'$, a copy of $F$, and a 1-factor, for each 2-regular graph $F$ that has a refinement $F' \in \{[4,6^r],[6^r] : r \equiv 2 \pmod 4\}$. The need for these factorisations arises because of the listed possible exceptions in Lemma 16.

Let $F$ be a bipartite 2-regular graph of order $2r \equiv 0$ or $4 \pmod 6$ such that $[6^{\frac{2r}{6}}]$ is a refinement of $F$ when $2r \equiv 0 \pmod 6$ and such that $[4,6^{\frac{2r}{6}}]$ is a refinement of $F$ when $2r \equiv 4 \pmod 6$. We write $J_{2r} \searrow F$ if there is a decomposition $\{F_1,F_2,F_3,F_4\}$ of $J_{2r}$ such that $F_3 \cong F$ and

(1) $V(F_1) = \{u_5,u_6,\ldots,u_{r+1}\} \cup \{u_2,u_3,u_{r+4}\} \cup \{v_3,v_4,\ldots,v_{r+2}\}$

(2) $V(F_2) = \{u_3,u_4,\ldots,u_r\} \cup \{u_1,u_{r+2}\} \cup \{v_2,v_3,\ldots,v_{r+1}\}$

(3) $V(F_3) = \{u_3,u_4,\ldots,u_{r+2}\} \cup \{v_5,v_6,\ldots,v_{r+4}\}$

(4) $F_1 \cong F_2 \cong [6^{\frac{2r}{6}}]$ if $2r \equiv 0 \pmod 6$

(5) $F_1 \cong F_2 \cong [4,6^{\frac{2r}{6}}]$ if $2r \equiv 4 \pmod 6$

(6) $F_4$ is 1-regular with vertex set $\{u_3,u_4,\ldots,u_{r+2}\} \cup \{v_3,v_4,\ldots,v_{r+2}\}$

Note that conditions (1), (2) and (3) in the definition of $J_{2r} \searrow F$ are the same as conditions (1), (2) and (3) in the definition of $J_{2r} - I \mapsto F$. It is clear that we also have the following two results which are analogues of Lemmas 13 and 14.

**Lemma 17** *If $J_{|V(F)|} \searrow F$, then $G_{|V(F)|}$ factorises into two copies of $H$, one copy of $F$, and a 1-factor where $H \cong [6^{\frac{|V(F)|}{6}}]$ if $|V(F)| \equiv 0 \pmod 6$ and $H \cong [4, 6^{\frac{|V(F)-4|}{6}}]$ if $|V(F)| \equiv 4 \pmod 6$.*

**Lemma 18** *If $F$ is the union of vertex disjoint 2-regular graphs $F'$ and $F''$ where at most one of $F'$ and $F''$ has order congruent to $4 \pmod 6$, $J_{|V(F')|} \searrow F'$, and $J_{|V(F'')|} \searrow F''$, then $J_{|V(F)|} \searrow F$.*

**Lemma 19** *For each graph $F$ in the following list, $J_{|V(F)|} \searrow F$.*

*(i)* $[12k]$ *for each* $k \geq 1$

*(ii)* $[12j + 6, 12k + 6]$ *for each* $j \geq 0$ *and each* $k \geq 1$

*(iii)* $[6, 6, 12k]$ *for each* $k \geq 1$

*(iv)* $[6, 6, 12j + 6, 12k + 6]$ *for each* $j \geq 0$ *and each* $k \geq 0$

*(v)* $[4, 12k]$ *for each* $k \geq 1$

*(vi)* $[4, 12j + 6, 12k + 6]$ *for each* $j \geq 0$ *and each* $k \geq 0$

*(vii)* $[12k + 4]$ *for each* $k \geq 1$

*(viii)* $[12j + 10, 12k + 6]$ *for each* $j \geq 0$ *and each* $k \geq 0$

*(ix)* $[6, 6, 12k + 4]$ *for each* $k \geq 0$

*(x)* $[6, 6, 12j + 6, 12k + 10]$ *for each* $j \geq 0$ *and for each* $k \geq 0$

**Proof** Suppose $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ is a decomposition $J_{2r} \searrow F$ for some $F$ with the property that for some $x$, the edges $\{u_x, u_{x+1}\}$ and $\{v_x, v_{x+1}\}$ are in $F_3$ and the edges $\{u_x, v_x\}$ and $\{u_{x+1}, v_{x+1}\}$ are in $F_4$. If we define $F_3'$ to be the graph obtained from $F_3$ by replacing the edges $\{u_x, u_{x+1}\}$ and $\{v_x, v_{x+1}\}$ with $\{u_x, v_x\}$ and $\{u_{x+1}, v_{x+1}\}$, and define $F_4'$ to be the graph obtained from $F_4$ by replacing the edges $\{u_x, v_x\}$ and $\{u_{x+1}, v_{x+1}\}$ with $\{u_x, u_{x+1}\}$ and $\{v_x, v_{x+1}\}$, then $\{F_1, F_2, F_3', F_4'\}$ is a decomposition $J_{2r} \searrow F'$ (where $F' \cong F_3'$). We shall call this process *performing a 4-edge swap at $u_x$*, and denote the new decomposition by $\mathcal{F}(u_x)$.

We shall be performing 4-edge swaps at $u_x$ when the edges $\{u_x, u_{x+1}\}$ and $\{v_x, v_{x+1}\}$ are in distinct cycles of $F_3$. Thus, when we obtain a new decomposition $J_{2r} \searrow F'$ from $J_{2r} \searrow F$ by performing a 4-edge swap, $F'$ will be isomorphic to a graph obtained from $F$ by replacing an $a$-cycle and a $b$-cycle with a single $(a + b)$-cycle.

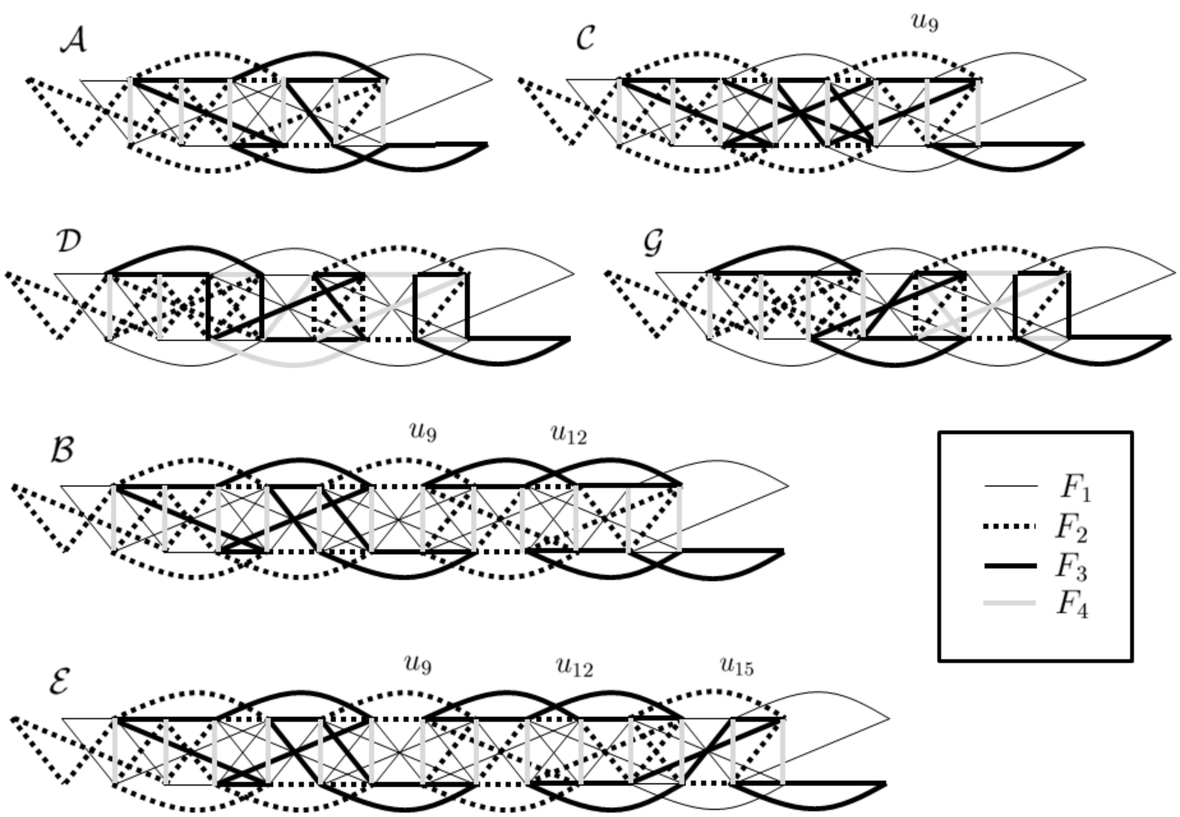



Figure 4: Some small decompositions $J_{2r} \searrow F$

Decompositions $J_{12} \searrow [12]$, $J_{24} \searrow [6,6,6,6]$, $J_{16} \searrow [12,4]$, $J_{16} \searrow [10,6]$, $J_{28} \searrow [6,6,6,6,4]$, and $J_{16} \searrow [4,6,6]$ are given by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{G}$ respectively in Figure 4.

Moreover, it is straightforward to check that

$$\begin{array}{lll} \mathcal{C}(u_9) & \text{yields} & J_{16} \searrow [16] \\ \mathcal{B}(u_{12}) & \text{yields} & J_{24} \searrow [6,6,12] \\ \mathcal{B}(u_9,u_{12}) & \text{yields} & J_{24} \searrow [6,18] \\ \mathcal{E}(u_9) & \text{yields} & J_{28} \searrow [6,12,6,4] \\ \mathcal{E}(u_9,u_{12}) & \text{yields} & J_{28} \searrow [6,18,4] \\ \mathcal{E}(u_{15}) & \text{yields} & J_{28} \searrow [6,6,6,10] \\ \mathcal{E}(u_{12},u_{15}) & \text{yields} & J_{28} \searrow [6,6,16]. \end{array}$$

Suppose $\mathcal{F}'$ and $\mathcal{F}''$ are any two of the above thirteen decompositions, that $\mathcal{F}'$ is a decomposition $J_{2r} \searrow F'$, that $\mathcal{F}''$ is a decomposition $J_{2s} \searrow F''$, and that $\mathcal{F}$ is the decomposition given by applying Lemma 18 to $\mathcal{F}'$ and $\mathcal{F}''$. We can then perform a 4-edge swap at $u_{r+3}$ in $\mathcal{F}$ to obtain the new decomposition $\mathcal{F}(u_{r+3})$. We denote any such new decomposition obtained in this manner by $\mathcal{F}' * \mathcal{F}''$. We are now ready to construct the decompositions needed to prove the lemma.

(i) A decomposition $J_{12k} \searrow [12k]$ for $k \geq 1$ is given by $\mathcal{A} * \mathcal{A} * \ldots * \mathcal{A}$ with $k$ occurrences of $\mathcal{A}$.

(ii) A decomposition $J_{12j+12k+12} \searrow [12j+6, 12k+6]$ for $j \geq 0$ and $k \geq 1$ is given by $\mathcal{A} * \ldots * \mathcal{A} * \mathcal{B}(u_9,u_{12}) * \mathcal{A} * \ldots * \mathcal{A}$ with $j$ occurrences of $\mathcal{A}$ to the left of $\mathcal{B}(u_9,u_{12})$ and $k-1$ occurrences of $\mathcal{A}$ to the right.

(iii) A decomposition $J_{12k+12} \searrow [6,6,12k]$ for $k \geq 1$ is given by $\mathcal{B}(u_{12}) * \mathcal{A} * \ldots * \mathcal{A}$ with $k-1$ occurrences of $\mathcal{A}$.

(iv) For all $j \geq 0$, a decomposition $J_{12j+24} \searrow [6,6,12j+6,6]$ is given by $\mathcal{B} * \mathcal{A} * \ldots * \mathcal{A}$ with $j$ occurrences of $\mathcal{A}$. Thus, we may assume $j, k \geq 1$. In this case, we can write $[6,6,12j+6,12k+6]$ as the vertex-disjoint union of $[6,12j+6]$ and $[6,12k+6]$ and use (ii) and Lemma 18 to obtain the required decomposition.

(v) A decomposition $J_{12k+4} \searrow [4,12k]$ for $k \geq 1$ is given by $\mathcal{A} * \ldots * \mathcal{A} * \mathcal{C}$ with $k-1$ occurrences of $\mathcal{A}$.

(vi) A decomposition $J_{16} \searrow [4,6,6]$ is given by $\mathcal{G}$ in Figure 4. A decomposition $J_{28} \searrow [4,6,18]$ is given by $\mathcal{E}(u_9,u_{12})$ and a decomposition $J_{40} \searrow [4,18,18]$ is given by $\mathcal{A} * \mathcal{E}(u_9,u_{12})$. This covers the cases where $j$ and $k$ are both in $\{0,1\}$. Hence (by the symmetry between $j$ and $k$) we can assume $k \geq 2$. The decomposition $\mathcal{A} * \ldots * \mathcal{A} * \mathcal{B}(u_9,u_{12}) * \mathcal{A} * \ldots * \mathcal{A} * \mathcal{C}$ with $j$ occurrences of $\mathcal{A}$ to the left of $\mathcal{B}(u_9,u_{12})$, and $k-2$ occurrences to the right is the required decomposition $J_{12j+12k+16} \searrow [4,12j+6,12k+6]$ for $j \geq 0$ and $k \geq 2$.

(vii) A decomposition $J_{12k+4} \searrow [12k+4]$ for $k \geq 1$ is given by $\mathcal{A} * \ldots * \mathcal{A} * \mathcal{C}(u_9)$ with $k-1$ occurrences of $\mathcal{A}$.

(viii) A decomposition $J_{12j+12k+16} \searrow [12j+10, 12k+6]$ for $j \geq 0$ and $k \geq 0$ is given by $\mathcal{A} * \ldots * \mathcal{A} * \mathcal{D} * \mathcal{A} * \ldots * \mathcal{A}$ with $j$ occurrences of $\mathcal{A}$ to the left of $\mathcal{D}$ and $k$ occurrences to the right.

(ix) We have already noted the existence of a decomposition $J_{16} \searrow [4,6,6]$ in (vi), and $J_{12k+16} \searrow [6,6,12k+4]$ for $k \geq 1$ is given by $\mathcal{E}(u_{12}, u_{15}) * \mathcal{A} * \ldots * \mathcal{A}$ with $k-1$ occurrences of $\mathcal{A}$.

(x) A decomposition $J_{12j+12k+28} \searrow [6,6,12j+10,12k+6]$ for $j \geq 0$ and $k \geq 0$ is given by $\mathcal{A} * \ldots * \mathcal{A} * \mathcal{E}(u_{15}) * \mathcal{A} * \ldots * \mathcal{A}$ with $j$ occurrences of $\mathcal{A}$ to the right of $\mathcal{E}(u_{15})$ and $k$ occurrences to the left. □

We are now ready to prove the main result of this section.

**Lemma 20** *Let $n \equiv 0 \pmod 4$ with $n \geq 12$. If $F$ is a 2-regular graph of order $n$ having a refinement $F' \in \{[4,6^r],[6^r] : r \equiv 2 \pmod 4\}$, then there is a factorisation of $G_n$ into two copies of $F'$, a copy of $F$, and a 1-factor.*

**Proof** Let $F_3$ be a 2-regular graph of order $2m$, which has a refinement isomorphic to $[6^r]$ or $[6^r,4]$ where $r \equiv 2 \pmod 4$. Note that since $r$ is even, $F_3$ contains an even number of cycles of length congruent to $6 \pmod{12}$. We will be using this fact implicitly in the remainder of the proof.

We deal first with the special case where $F_3 \cong [6^r]$. It is easy to see that $G_{12} \cong \langle\{1,3,5,6\}\rangle_{12}$ and that the orbit of the 2-factor consisting of the two 6-cycles $(0,1,7,10,11,6)$ and $(2,5,8,3,4,9)$ under the permutation $x \mapsto x+4 \pmod{12}$ is a factorisation of $\langle\{1,3,5,6\}\rangle_{12}$ into three copies of $[6^2]$ and a 1-factor. Also, it was shown in [10] that there is a factorisation of $G_n$ into three copies of $[6^r]$ and a 1-factor for all $r \equiv 2 \pmod 4$ with $r \geq 6$. Hence, the result holds when $F_3 \cong [6^r]$ so we may assume that $F_3 \ncong [6^r]$.

For $F_3 \ncong [6^r]$, we will show that $F_3$ can be written as a vertex-disjoint union of 2-regular subgraphs $H_1, H_2, \ldots H_w$ where each subgraph $H_i$ is listed in Lemma 19 (and where at most one of the subgraphs has order congruent to $4 \pmod 6$)). It then follows by Lemma 18 that $J_{2m} \searrow F_3$, and consequently by Lemma 17 that $G_{2m}$ has the required decomposition.

Clearly, we can write $F_3$ as a union of vertex disjoint graphs isomorphic to $[6,6,6,6]$ and a graph satifying the conditions of the lemma and having at most three 6-cycles. Hence, since $[6,6,6,6]$ is covered by Lemma 19, we can assume that $F_3$ contains at most three 6-cycles (and $F_3 \ncong [6,6]$).

The proof now splits into the following five cases which we deal with one at a time.

(1) $[6^r]$ is a refinement of $F_3$ and $F_3$ contains zero or one 6-cycles.

(2) $[6^r]$ is a refinement of $F_3$ and $F_3$ contains two or three 6-cycles.

(3) $[4,6^r]$ is a refinement of $F_3$ and $F_3$ contains a 4-cycle.

(4) $[4, 6^r]$ is a refinement of $F_3$, $F_3$ does not contain a 4-cycle, and $F_3$ contains zero or one 6-cycles.

(5) $[4, 6^r]$ is a refinement of $F_3$, $F_3$ does not contain a 4-cycle, and $F_3$ contains two or three 6-cycles.

**(1)** It is easy to see that if $[6^r]$ is a refinement of $F_3$ and $F_3$ contains zero or one 6-cycles, then $F_3$ can be written as the union of copies of graphs covered by (i) and (ii) of Lemma 19. We will refer back to this observation.

**(2)** If $[6^r]$ is a refinement of $F_3$ where $F_3$ contains two or three 6-cycles, then $F_3$ can be written as the union of a graph covered by either (iii) or (iv) of Lemma 19, and a graph which falls into case (1).

**(3)** If $[4, 6^r]$ is a refinement of $F_3$ and $F_3$ contains a 4-cycle, then $F_3$ can be written as the union of a graph covered by either (v) or (vi) of Lemma 19, and a graph which falls into case (1).

**(4)** If $[4, 6^r]$ is a refinement of $F_3$, $F_3$ does not contains a 4-cycle and $F_3$ contains zero or one 6-cycles, then $F_3$ can be written as the union of a graph covered by either (vii) or (viii) of Lemma 19, and a graph which falls into case (1).

**(5)** If $[4, 6^r]$ is a refinement of $F_3$, $F_3$ does not contains a 4-cycle and $F_3$ contains two or three 6-cycles, then $F_3$ can be written as the union of a graph covered by either (ix) or (x) of Lemma 19, and a graph which falls into case (1). □

# 6 Factorisations of $K_{12}$ and $K_{16}$

In this section we give some additional factorisations of $K_{12}$ and $K_{16}$ which we will need because our general approach does not work completely in these small cases. The following result is Theorem 4.1 in [2].

**Theorem 21** ([2]) *If $F_1$ and $F_2$ are non-isomorphic bipartite 2-regular graphs of order $n \leq 16$, each consisting of isomorphic connected components, then the Hamilton Waterloo Problem* $\mathrm{HW}(F_1, F_2; \alpha_1, \alpha_2)$ *has a solution for all $\alpha_1, \alpha_2$ satisfying $\alpha_1 + \alpha_2 = \frac{n-2}{2}$.*

**Lemma 22** *For each*

$$(F_1, F_2) \in \{([4,4,4],[4,8])\ ,\ ([4,4,4],[12])\ ,\ ([4,8],[12])\ ,\ ([6,6],[12])\}$$

*there is a factorisation of $K_{12}$ into a copy of $F_1$, four copies of $F_2$, and a 1-factor.*

**Proof** For $(F_1, F_2) = ([4,4,4],[4,8])$ we use a factorisation of $K_6$ into two Hamilton cycles and a 1-factor to obtain a factorisation of $K_{12}$ into two copies of $C_6^{\square}$, a copy of $[4,4,4]$ and a 1-factor. By Lemma 3, each copy of $C_6^\square$ can then be factored into two copies $[4,8]$ to obtain the required factorisation of $K_{12}$. In all other cases, the result follows by Theorem 5. $\square$

**Lemma 23** *For each*

$$(F_1,F_2) \in \{([4,4,4],[4,8])\ ,\ ([4,4,4],[12])\ ,\ ([4,8],[12])\ ,\ ([6,6],[12])\}$$

*there is a factorisation of $K_{12}$ into a four copies of $F_1$, a copy of $F_2$, and a 1-factor.*

**Proof** The cases $(F_1,F_2) \in \{([4,4,4],[12]),([6,6],[12])\}$ are covered by Theorem 21. We now deal with $(F_1,F_2) = ([4,4,4],[4,8])$. Let $\mathcal{F}$ be a factorisation of $K_6$ into a Hamilton cycle and three 1-factors. Then $\mathcal{F}^\square$ is a factorisation of $K_{12}-I$ into into a copy of $C_6^\square$ and three copies of $[4,4,4]$. If we add the edges of the removed 1-factor to the copy of $C_6^\square$, then we obtain a factorisation of $K_{12}$ into a copy of $C_6 \wr K_2$ and three copies of $[4,4,4]$. A factorisation of $C_6 \wr K_2$ into a copy of $[4,4,4]$, a copy of $[4,8]$ and a 1-factor exists by Lemma 12. This yields the required factorisation of $K_{12}$ when $(F_1,F_2) = ([4,4,4],[4,8])$.

We now deal with $(F_1,F_2) = ([4,8],[12])$. Let $\mathcal{F}$ be a factorisation of $K_6$ into a Hamilton cycle and a copy of $\langle\{2,3\}\rangle_6$. Then $\mathcal{F}^\square$ is a factorisation of $K_{12}-I$ into into a copy of $C_6^\square$ and a copy of $\langle\{2,3\}\rangle_6^\square$. Since $C_6^\square$ can be factored into two copies of $[4,8]$, the required factorisation of $K_{12}$ can be obtained if there is a factorisation of $\langle\{2,3\}\rangle_6^\square$ into two copies of $[4,8]$ and one copy of $[12]$. This factorisation is given below where $(a,b)$ is denoted by $a_b$.

$$\begin{aligned} \{\quad & (0_0,4_0,1_0,4_1)\cup(2_0,5_1,3_1,1_1,3_0,5_0,2_1,0_1),\\ & (0_0,3_0,0_1,3_1)\cup(2_0,4_0,2_1,4_1,1_1,5_1,1_0,5_0),\\ & (0_0,2_0,4_1,0_1,4_0,1_1,5_0,3_1,1_0,3_0,5_1,2_1) \quad \} \end{aligned}$$

$\square$

**Lemma 24** *There is a factorisation of $K_{16}$ into a copy of $[4,6,6]$, six copies of $[6,10]$, and a 1-factor.*

**Proof** Applying Lemma 10 with $F \cong [8]$, we obtain a factorisation of $K_{16}$ into two copies of $C_8^\square$ and a copy of $G_{16}$. We can factorise each copy of $C_8^\square$ into two copies of $[6,10]$ using Lemma 3, and this leaves us needing a factorisation of $G_{16}$ into two copies of $[6,10]$, a copy of $[4,6,6]$, and a 1-factor. This factorisation is given below with $(a,b)$ denoted by $a_b$.

$$\begin{aligned} \{\quad & (0_0,1_1,0_1,1_0)\ \cup\ (2_0,7_0,6_1,5_0,6_0,7_1)\ \cup\ (2_1,3_0,4_1,5_1,4_0,3_1),\\ & (1_0,2_1,1_1,4_1,5_0,4_0)\ \cup\ (0_0,5_1,0_1,7_1,6_1,3_0,2_0,3_1,6_0,7_0),\\ & (0_0,5_0,0_1,7_0,2_1,7_1)\ \cup\ (1_0,2_0,1_1,4_0,3_0,6_0,5_1,6_1,3_1,4_1) \quad \} \end{aligned}$$

$\square$

**Lemma 25** *There is a factorisation of $K_{16}$ into six copies of $[4, 6, 6]$, a copy of $[6, 10]$, and a 1-factor.*

**Proof** In the proof of Lemma 24, we noted the existence of a factorisation of $K_{16}$ into two copies of $C_8^{\square}$ and a copy of $G_{16}$. We can factorise each copy of $C_8^{\square}$ into two copies of $[4, 6, 6]$ using Lemma 3, and this leaves us needing a factorisation of $G_{16}$ into two copies of $[4, 6, 6]$, a copy of $[6, 10]$, and a 1-factor. This factorisation exists by Lemma 20. □

# 7 Proofs of main results

**Theorem 26** *If $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order $n \equiv 2 \pmod 4$, $F_1$ is a refinement of $F_2$, and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$ with $\alpha_i$ even for $i = 3, 4 \ldots, t$, then $K_n$ has a factorisation into $\alpha_i$ copies of $F_i$ for $i = 1, 2, \ldots, t$ and a 1-factor.*

**Proof** Let $m = \frac{n}{2}$. It follows from $n \equiv 2 \pmod 4$ that $\alpha_1$ and $\alpha_2$ are either both even or both odd. If they are both even, then we let $F \cong C_m$, and if they are both odd, then we let $F$ be a 2-regular graph of order $m$, which exists by Lemma 12, such that $F \wr K_2$ has a factorisation into a copy of $F_1$, a copy of $F_2$ and a 1-factor. By Lemma 9, there is a factorisation of $K_{2m}$ into $\frac{m-3}{2}$ copies of $C_m^{\square}$ and a copy of $F \wr K_2$. The required factorisation of $K_n$ can thus be obtained by applying Lemma 3 (when $\alpha_1$ and $\alpha_2$ are both even we first factorise $F \wr K_2$ into a copy of $C_m^{\square}$ and a 1-factor). □

**Theorem 27** *If $t \geq 3$, $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order $n$, and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are positive integers such that*

- *$F_1$ is a refinement of $F_2$;*
- *$\alpha_1$, $\alpha_2$, $\alpha_3$ are odd with $\alpha_3 \geq 3$;*
- *$\alpha_i$ is even for $i = 4, 5, \ldots, t$;*
- *$\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$;*
- *$F_2 \notin \{[4, 4, 4], [4, 8], [12], [4, 6, 6], [6, 10]\}$; and*
- *$F_3 \notin \{[6^r], [4, 6^r] : r \equiv 2 \pmod 4\}$;*

*then $K_n$ has a factorisation into $\alpha_i$ copies of $F_i$ for $i = 1, 2, \ldots, t$ and a 1-factor.*

**Proof** The conditions imply $n \equiv 0 \pmod 4$ and $n \geq 12$. Let $F$ be the 2-regular graph of order $m = \frac{n}{2}$, given by Lemma 12, such that $F \wr K_2$ has a factorisation into a copy of $F_1$, and copy of $F_2$, and a 1-factor. Since $F_2 \notin \{[4,4,4],[4,8],[12],[4,6,6],[6,10]\}$, we have $F \notin \{[6],[3,5]\}$. Hence, by Lemma 10 there is a factorisation of $K_{2m}$ into $\frac{m-6}{2}$ copies of $C_m^{\square}$, a copy of $F \wr K_2$, and copy of $G_n - I$. We can thus obtain the required factorisation of $K_{2m}$ by using the above-mentioned factorisation of $F \wr K_2$, using Lemma 16 to factorise $G_n - I$ into three copies of $F_3$ (as $F_3 \ncong [6^r]$ and $F_3 \ncong [4, 6^r]$), and using Lemma 3 to factorise each copy of $C_m^{\square}$. □

Theorems 26 and 27 together with the factorisations of $K_{12}$ and $K_{16}$ given in Section 6 allow us to prove our main result on the Hamilton-Waterloo Problem.

**Theorem 28** *If $F_2$ is a bipartite 2-regular graph of order $n$ and $F_1$ is a bipartite refinement of $F_2$, then for all non-negative $\alpha_1, \alpha_2$ satisfying $\alpha_1 + \alpha_2 = \frac{n-2}{2}$ there is a factorisation of $K_n$ into $\alpha_1$ copies of $F_1$, $\alpha_2$ copies of $F_2$, and a 1-factor.*

**Proof** If $n \equiv 2 \pmod 4$, then we can obtain the required factorisation by applying Theorem 26 with $t = 2$. The case $n = 4$ is trivial and the required factorisations for $n = 8$ are known to exist, see [11]. Thus, we can assume $n \equiv 0 \pmod 4$ and $n \geq 12$. For $n \equiv 0 \pmod 4$, exactly one of $\alpha_1$ and $\alpha_2$ is odd. If $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$, then we can obtain the required factorisation by applying Theorem 6 with $t = 2$. This leaves only cases $\alpha_1 = 1$ and $\alpha_2 = 1$.

If $\alpha_1 = 1$, then we can obtain the required factorisation by applying Theorem 27 with $t = 3$ and $F_3 \cong F_2$, except if $F_2 \in \{[4,4,4],[4,8],[12],[4,6,6],[6,10]\} \cup \{[6^r],[4,6^r] : r \equiv 2 \pmod 4\}$. In most of these exceptional cases the only bipartite refinement of $F_2$ is $F_2$ itself, which means that $F_1 \cong F_2$ and the result follows by Theorem 7. The exceptional cases where $F_2$ has a non-isomorphic bipartite refinement are precisely those covered by Lemmas 22 and 24.

If $\alpha_2 = 1$, then we can obtain the required factorisation by applying Theorem 27 with $t = 3$ and $F_3 \cong F_1$, except if $F_2 \in \{[4,4,4],[4,8],[12],[4,6,6],[6,10]\}$ or $F_1 \in \{[6^r],[4,6^r] : r \equiv 2 \pmod 4\}$. For $F_2 \in \{[4,4,4],[4,6,6]\}$, the only bipartite refinement of $F_2$ is $F_2$ itself, which means that $F_1 \cong F_2$ and the result follows by Theorem 7. For $F_2 \in \{[4,8],[12],[6,10]\}$, the required factorisations are precisely those given in Lemmas 23 and 25. For $F_1 = [6,6]$ we have either $F_2 = [6,6]$ or $F_2 = [12]$. In the former case, the result follows by Theorem 7, and in the latter case the required factorisation is given by Lemma 23 (as we have already noted in dealing with the case $F_2 = [12]$).

Finally, for $\alpha_2 = 1$ and $F_1 \in \{[6^r],[4,6^r] : r \equiv 2 \pmod 4\} \setminus \{[6,6]\}$, we first apply Lemma 10 with $F \cong C_m$ (and $m = \frac{n}{2}$) to obtain a factorisation of $K_n$ into $\frac{n-8}{4}$ copies of $C_{\frac{n}{2}}^{\square}$ and a copy of $G_n$. Using Lemma

3 we can then factorise each copy of $C^{\square}_{\frac{n}{2}}$ into two copies of $F_1$, and using Lemma 20 we can factorise $G_n$ into two copies of $F_1$, one copy of $F_2$, and a 1-factor. This yields the required factorisation of $K_n$. □

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