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TWO PROBLEMS OF GERHARD RINGEL

By

Adrián Pastine

A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

In Mathematical Sciences

MICHIGAN TECHNOLOGICAL UNIVERSITY

2016

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This dissertation has been approved in partial fulfillment of the requirements for the Degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences.

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Preface

Chapter 1, "On the Hamilton-Waterloo Problem with triangle factors and C_{3x} -factors", was authored by J. Asplund, D. Kamin, M. Keranen, A. Pastine and S. Özkan, and published on Australasian Journal of Combinatorics, Volume 64 (part 3), pages 458-474. All the results in this work arose from the collective thinking and discussion of all the authors, and everyone deserves some credit for each of them. My main contributions to this work is coming up with the idea behind the constructions in Theorem 7 and Theorem 8. The resulting paper on this work was mainly written by M. Keranen and A. Pastine, and reviewed by J. Asplund.

Chapter 2, "A Generalization of the Hamilton-Waterloo Problem on Complete Equipartite Graphs", was authored by M. Keranen and A. Pastine, and submitted for publication to Journal of Combinatorial Designs and is currently in review. Every result in this work is due to the mutual cooperation of both authors, and they deserve equal credit for them. The resulting paper was mainly written by A. Pastine, and reviewed by M. Keranen.

Chapter 3, "Finite Abelian Groups And Sequenceability", was authored by B. Alspach, D.L. Kreher, and A. Pastine, and submitted for publication to Australasian Journal of Combinatorics and is currently in review. All results in this work are due to the mutual cooperation of the three authors, and they deserve equal credit for them. The resulting paper was mainly written by B. Alspach, and reviewed by D.L. Kreher and A. Pastine.

Chapters 1, 2, and 3 are designed to be read independently. To achieve this some definitions and results occur in more than one chapter.

The Introduction to this dissertation was written using excerpts and ideas from each of the aforementioned papers.

Figure 0.3 is licensed under a Creative Commons Attribution-Noncommercial-Share Alike License.

Acknowledgments

First, I would like to thank my co-advisors Don Kreher and Missy Keranen. Doing research with them was both a fun and gratifying experience. I feel like I have learned a lot from both of them, and their influence will be seen in all of my future work.

I also would like to thank all the friends that I made at Michigan Tech, both in the Math Department and elsewhere. They made Houghton a better place to live. Specially among them, I have to thank Ezequiel Medici. He was the first person I met in the Upper Peninsula, and has accompanied me throughout the whole time I was there. He was always willing to give me a hand any time I needed help, and his unconditional friendship in all this years has granted him this special mention.

I cannot finish this acknowledgments without thanking Daniel Jaume. He played an influential role in my academic career. He is the person that introduced me to Combinatorics and Graph Theory. He also pushed me to apply to scholarships and schools all over the world. Most of all he showed me how much fun research can be.

Finally, I would like to thank my family, who always showed me their unconditional support.

Abstract

Gerhard Ringel was an Austrian Mathematician, and is regarded as one of the most influential graph theorists of the twentieth century. This work deals with two problems that arose from Ringel's research: the Hamilton-Waterloo Problem, and the problem of R-Sequences.

The Hamilton-Waterloo Problem (HWP) in the case of C_m -factors and C_n -factors asks whether K_v , where v is odd (or $K_v - F$, where F is a 1-factor and v is even), can be decomposed into r copies of a 2-factor made entirely of m-cycles and s copies of a 2-factor made entirely of n-cycles. Chapter 1 gives some general constructions for such decompositions and apply them to the case where m = 3 and n = 3x. This problem is settled for odd v, except for a finite number of x values. When v is even, significant progress is made on the problem, although open cases are left. In particular, the difficult case of v even and s = 1 is left open for many situations.

Chapter 2 generalizes the Hamilton-Waterloo Problem to complete equipartite graphs $K_{(n:m)}$ and shows that $K_{(xyzw:m)}$ can be decomposed into s copies of a 2-factor consisting of cycles of length xzm; and r copies of a 2-factor consisting of cycles of length yzm, whenever m is odd, $s, r \neq 1$, gcd(x, z) = gcd(y, z) = 1 and $xyz \neq 0 \pmod{4}$. Some more general constructions are given for the case when the cycles in a given two factor may have different lengths. These constructions are used to find solutions to the Hamilton-Waterloo problem for complete graphs.

Chapter 3 completes the proof of the Friedlander, Gordon and Miller Conjecture that every finite abelian group whose Sylow 2-subgroup either is trivial or both nontrivial and non-cyclic is R-sequenceable. This settles a question of Ringel for abelian groups.

Introduction

Gerhard Ringel was born in Kollnbrunn, Austria, in 1919. He received his PhD in 1951 at the Friedrich-Wilhelms Universität. He became a professor at Freie Universität Berlin in 1958, and from 1967 to 1970 he held the position of chairman of the mathematical institute. In 1970 he became a full professor at University of California, Santa Cruz, where he was chairman of the mathematics department from 1972 to 1984. He passed in Santa Cruz in June 24th 2008.

During his many years working in the field he collected many honors and awards, including two honorary doctorates. The Universität Karlsruhe awarded him a honorary doctorate in political science in 1983. In 1994 he received an honorary doctorate in mathematics from Freie Universität in Berlin.

Ringel is in the list of people that made graph theory an interesting and influential field of study. His conjectures are known by most graph theorists. Among them, we have the Ringel's conjecture, the Ringel-Kotzig conjecture, the Oberwolfach problem and the Earth Moon problem. Asides from his conjectures, Ringel's most popular result is the Ringel-Youngs Theorem (also known as Heawood's Conjecture). This is one of the most important results in Graph Theory, where they proved that four colors are enough to create a political map on any surface, with no two adjacent countries sharing their color.

This manuscript is concerned with two problems first introduced by Ringel. Our focus will be on the Hamilton-Waterloo Problem, and R-sequenceability. The Hamilton-Waterloo Problem is the natural continuation of the Oberwolfach problem, first posed by Ringel in 1967 at a conference in Oberwolfach. The problem of R-sequenceability was posed by Ringel in 1974, [36]. We now introduce the concepts necessary to understand what these problems ask, and what our contributions are.

Definition 0.0.1 A graph is an ordered pair $\Gamma = (V, E)$, where V is a set of elements called vertices, and E is a family of 2-element subsets of V called edges.

If $\{x, y\} \in E$, then we say that x and y are adjacent. Sometimes the set of vertices and the family of edges of a graph Γ are denoted $V(\Gamma)$, and $E(\Gamma)$, respectively. The *degree* of a vertex is the number of vertices that are adjacent to it. **Example 1:** As an example we can take

$$\begin{split} V &= \left\{ a, b, c, d, e, f, g, h \right\} \\ & \text{and} \\ E &= \left\{ \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{f, g\}, \{f, h\}, \{g, h\} \right\}. \end{split}$$

The graph $\Gamma = (V, E)$ has as vertices the elements a, b, c, d, e, f, g, h, where the vertex a is adjacent to the vertices b, c, and d; the vertex b is adjacent to the vertices a and d, and so on. The degrees of the vertices are

x		b	С	d	e	f	g	h
$\deg(x)$	3	2	1	2	0	2	1	1

There are many ways to represent graphs. The most common way is by drawing a picture where each vertex is represented by a point or a circle, and each edge is represented by a line connecting the points (or circles) of its respective vertices. Figure 0.1 shows this kind of representation for the graph from Example 1.



Figure 0.1: Representation of the graph in Example 1

Related to graphs are *directed graphs*, where instead of having a family of subsets for the edges E, we have a set of ordered pairs A. Hence if $(x, y) \in A$ we say that there is an arc from x to y. We sometimes write $A(\Gamma)$ to denote the set of arcs of Γ .

Example 2: As an example we can take

$$V = \{a, b, c, d, e, f, g, h\}$$

and

$$A = \{(a, b), (a, c), (c, a), (d, a), (d, b), (f, g), (g, h), (h, f).$$

In this way the graph has as vertices the elements a, b, c, d, e, f, g, h, where there are arcs from vertex a to the vertices b, c; there is an arc from vertex c to vertex a, and so on.

Similar to the picture representation of a graph directed graphs can also be drawn, the only difference is that arcs are represented by arrows. In this way if $(x, y) \in A$, in the drawing there will be an arrow going pointing from x to y. Figure 0.2 shows this representation for the graph described in Example 2.

Graphs are used in a wide range of fields to represent symmetric binary relations. A few examples of what graphs can represent follow:



Figure 0.2: Representation of the graph in Example 2

- Social networks, where each vertex represents a person, and two vertices are adjacent if the people are friends.
- Computer networks, where each vertex represents a computer, and two vertices are adjacent if the computers are connected in the network.
- Molecules, where each vertex represents an atom, and two vertices are adjacent if the atoms are bonded.



Figure 0.3: spikedmath.com/382.html

Similarly directed graphs are used to represent asymmetric binary relations. For instance:

- Flight connections, where each vertex represents an airport, and an arrow (x, y) represents a flight from airport x to airport y.
- Websites, where each vertex represents a web page, and an arrow (x, y) represents a link from the web page x to the web page y.
- Ownership of companies, where each vertex represents a company, and an arrow (x, y) means that the company x owns the company y.

There are many different properties of graphs that can be studied. A particular property asks when two different graphs are technically the same. We say that two graphs $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $\varphi : V_1 \to V_2$, such that $\{x, y\} \in E_1$ if and only if $\{\varphi(x), \varphi(y)\} \in E_2$. If this is the case, we say that φ is an *isomorphism* between Γ_1 and Γ_2 . If Γ_1 and Γ_2 are isomorphic we write $\Gamma_1 \simeq \Gamma_2$.

Example 3: Let

$$V_{1} = \{a, b, c, d, e, f, g, h\},$$

$$E_{1} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{f, g\}, \{f, h\}, \{g, h\},$$

$$V_{2} = \{1, 2, 3, 4, 5, 6, 7, 8\},$$

$$E_{2} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 7\}.$$

Then the bijection φ defined by

$$\varphi(1) = f \qquad \qquad \varphi(5) = b$$

$$\varphi(2) = g \qquad \qquad \varphi(6) = c$$

$$\varphi(3) = h \qquad \qquad \varphi(7) = d$$

$$\varphi(4) = a \qquad \qquad \varphi(8) = e$$

is an isomorphism between $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$.

A path is a graph $\Gamma = (V, E)$, such that

$$V = \{v_1, v_2, v_3, \dots, v_n\},\$$
$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_i, v_{i+1}\}, \dots, \{v_{n-1}, v_n\}\}.$$

A path with n vertices is usually denoted P_n . If we add the edge $\{v_n, v_1\}$, then the graph is called a *cycle* and is denoted C_n .

An example of a path with four vertices and a cycle with five vertices is given in Figure 0.4.



Figure 0.4: A path with four vertices, $\Gamma \simeq P_4$, and a cycle with five vertices, $H \simeq C_5$.

A graph that has every pair of vertices adjacent to each other is called *complete*. The complete graph on n vertices is usually denoted K_n .

A connected component of a graph is a maximal set of vertices C, such that for each pair of vertices $x, y \in C$, there is a path joining them.

Many interesting pure and applied problems can be described in the framework of graph decomposition, which asks whether the edges of a graph $\Gamma = (V, E)$ can be partitioned into subsets E_1, \ldots, E_k , such that the graphs $\Gamma_1 = (V, E_1), \ldots, \Gamma_k = (V, E_k)$ are isomorphic to certain predetermined graphs called *factors*. Graph decomposition has applications in networking [26], block designs [14], and bioinformatics [38], among others.

Given two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, we say that Γ_2 is a subgraph of Γ_1 if $V_2 \subset V_1$ and $E_2 \subset E_1$. A subgraph F of Γ is called a *n*-factor if every vertex of F has degree n. Notice that in a 1-factor every connected component has exactly 2 vertices, and in a 2-factor every connected component is a cycle. A 2-factor were each

connected component has x vertices is called C_x -factor. A $[n_1^{e_1}, n_2^{e_2}, \ldots, n_p^{e_p}]$ -factor of Γ is 2-factor of Γ with e_i connected components of size n_i , $i = 1, 2, \ldots, p$. We allow $n_i = n_j$, which means that a $[3^3, 5^2]$ -factor is also a $[3^2, 3^1, 5^2]$ -factor.

Example 4: Figure 0.5 gives a decomposition of the complete graph $\Gamma \simeq K_6$. It is decomposed into 3 factors:

- F_1 is a 2-factor with one connected component. It is a C_6 -factor, or a $[6^1]$ -factor.
- F_2 is a 2-factor with two connected components. It is a C_3 -factor, a $[3^2]$ -factor, or a $[3^1, 3^1]$ -factor.
- F_3 is a 1-factor, with three connected components.



Figure 0.5: A decomposition of the complete graph $\Gamma \simeq K_6$

If the vertices of a graph $\Gamma = (V, E)$ can be partitioned into sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, such

that if $x, y \in \Gamma_i$ for some *i*, then $\{x, y\} \notin E$. The sets Γ_i are called *partite sets*. If the number of parts is *m*, then the graph is also called *m*-partite. If all the partite sets have the same number of vertices, then the graph is called *equipartite*. If every pair of vertices belonging to different partite sets are adjacent, then the graph is called *complete multipartite (equipartite)*. A complete equipartite graphs with *m* partite sets of size *n* is denoted $K_{(n:m)}$. Figure 0.6 shows a complete equipartite graph with 4 parts of size 3, $K_{(3:4)}$.



Figure 0.6: The complete equipartite graph $\Gamma\simeq K_{(3:4)}$

In 1967, during a conference in Oberwolfach, Ringel asked whether it was possible to seat the v conference attendees at n round tables for dinner during $\frac{v-1}{2}$ nights, in such a way that every attendee sits next to every other attendee exactly once. This is equivalent to asking whether the complete graph K_v can be decomposed into $\frac{v-1}{2}$ copies of a 2-factor F (in a 2-factor every component is a cycle, which represents a round table). To achieve this decomposition v needs to be odd, because the vertices (attendees) need to have even degree. Later a version with v even was introduced by Huang, Kotzig and Rosa [25]. In this case, the attendees will never sit next to their spouses (and we are assuming that every attendee has a spouse). This is equivalent to asking for a decomposition of K_v into $\frac{v-2}{2}$ copies of a 2-factor F, and one copy of a 1-factor (each attendee together with their spouse).

In [28] Liu first worked on the generalization of the Oberwolfach problem, where instead of avoiding their spouses, the attendees avoid all the other members of their delegation. The assumption was that all the delegations had the same number of people. Thus we are seeking to decompose the complete equipartite graph $K_{(m:n)}$ with n partite sets (delegations) of size m each (members of a delegations) into $\frac{(n-1)m}{2}$ copies of a 2-factor F. Here (n-1)m has to be even. In [22] Hoffman and Holliday worked on the equipartite generalization of the Oberwolfach problem when (n-1)mis odd, decomposing into $\frac{(n-1)m-1}{2}$ copies of a 2-factor F, and one copy of a 1-factor.

The Hamilton-Waterloo problem, first mentioned in [17], is a generalization of the

Oberwolfach problem, in which the conference is being held at two different cities. Because the table arrangements are different, we have two 2-factors, F_1 and F_2 . The Hamilton-Waterloo problem then asks whether the complete graph K_v can be decomposed into r copies of the 2-factor F_1 (tables at Hamilton) and s copies of the 2-factor F_2 (tables at Waterloo), such that $s + r = \frac{v-1}{2}$, when v is odd, and having $s + r = \frac{v-2}{2}$ and a 1-factor when v is even.

The uniform Oberwolfach problem (when all the tables have the same size, i.e. all the cycles of the 2-factor have the same size) has been completely solved by Alspach and Haagkvist [2] and Alspach, Schellenberg, Stinson and Wagner [3]. For the non-uniform case of the Oberwolfach problem, many results have been obtained. For a survey of results up to 2006 see [12]. The uniform Oberwolfach problem over equipartite graphs has been completely solved by Liu [29] and Hoffman and Holliday [22]. In the non-uniform case Bryant, Danziger and Pettersson [11] completely solved the case when the 2-factor is bipartite. For the Hamilton-Waterloo problem most of the results are uniform, see for example [5] or [13].

The Hamilton-Waterloo problem can be generalized for complete equipartite graphs in the same way as the Oberwolfach problem was generalized.

Figure 0.7 shows a Hamilton-Waterloo decomposition of K_{10} into a 1-factor, 2 C_{10} factors, and 2 C_5 -factors. Figure 0.8 shows a Hamilton-Waterloo decomposition of $K_{(5:3)}$ into 3 C_{15} -factors and 2 C_3 -factors.



Figure 0.7: A Hamilton-Waterloo decomposition of K_{10} into a 1-factor, 2 C_{10} -factors and 2 C_5 -factors.

Chapter 1 considers the Hamilton-Waterloo problem over the complete graph K_{3xy} , looking for decompositions into $r \ C_3$ -factors and $s \ C_{3x}$ -factors. In this chapter we show that such decompositions can be achieved except maybe when s = 1, or when $x \in \{2, 4, 6, 12\}$.



Figure 0.8: A Hamilton-Waterloo decomposition of $K_{(5:3)}$ into 3 C_{15} -factors and 2 C_3 -factors.

Chapter 2 is concerned with the Hamilton-Waterloo problem over the complete equipartite graph $K_{(v:m)}$, into s F_1 -factors and r F_2 -factors. In this chapter we show that such decompositions can be achieved if the following conditions are satisfied:

- m = nm' is odd, and n divides the size of all the cycles;
- if there are cycles of size x in one of the 2-factors, then x divides v, and there are $k\frac{v}{nx}$ such cycles.
- 4 does not divide the size of any cycle;
- if there are even cycles, then 4 divides v;

• $s, r \neq 1;$

except for certian lengths of cycles when v is even.

We turn now to the problem of R-sequenceability. For this we first need to introduce the algebraic concept of group.

Definition 0.0.2 Given a set of elements G and a binary operation $*: G \times G \to G$, we say that (G, *) is a group if the following conditions are satisfied:

Closure: If $a, b \in G$, then $a * b \in G$.

Associativity: If $a, b, c \in G$, then (a * b) * c = a * (b * c).

Identity: There exists an element $e \in G$, such that if $a \in G$, then a * e = e * a = a.

Inverse: If $a \in G$, then there is $b \in G$, such that a * b = b * a = e.

The element e can be proven to be unique, and is known as the *identity* of the group. Given $a \in G$, the element $b \in G$ such that a * b = e is also unique, it is called the *inverse* of a, and denoted $b = a^{-1}$. When the operation is known we sometimes refer as the group as G, instead of (G, *). Because of the associativity property, it is common to write strings of products without parentheses. This means that instead of writing (a * b) * c, we would write a * b * c. **Example 5:** The set of integers together with the usual addition, $(\mathbb{Z}, +)$, is a group. We know that given two integers a, b, then a + b is an integer, showing closure. Associativity is also one of the known properties of addition over the integers. The identity element is 0, and the inverse of an element a is -a.

Notice that we could not use the product as the operation, because we do not have inverses for our elements.

Some other examples of groups are:

- The integers, with addition as the operation.
- The rationals except for zero, with product as the operation.
- The set of non-singular matrices of a given size, with product as the operation.
- The permutations of a set, with composition as the operation.
- The set of vectors with *m* coordinates and real numbers as entries, with vector addition as the operation.

We will mainly deal with *finite groups*. This is, groups containing only a finite number of elements.

Example 6: Usually the first finite groups introduced are the integers modulo n, $(\mathbb{Z}_n, +_n)$. This group contains the numbers $\{0, 1, \ldots, n-2, n-1\}$ as elements,

and the operation, called addition modulo n is defined as:

$$a +_n b = \begin{cases} a+b & \text{if } a+b \le n-1 \\ a+b-n & \text{if } a+b \ge n \end{cases}$$

•

Because $0 \le a + b \le 2n - 2$, we get that $a + b \le n - 1$ or $0 \le a + b - n \le n - 1$, hence the set is closed under $+_n$. Associativity is inherited from integer addition +. The identity element is 0, and the inverse of an element $a \ne 0$ is n - a.

Some other examples of finite groups are:

- The *n*-th roots of unity, with multiplication as the operation.
- The symmetries of a regular polygon, with composition as the operation.
- The permutations of a finite set, with composition as the operation.
- The set of isomorphisms from a graph onto itself (automorphisms), with composition as the operation.
- The set of vectors with m coordinates and integers modulo n as entries, with vector addition modulo n as the operation.

A pair of elements a and b in a group *commute* if a * b = b * a. Notice that in particular the identity of a group commutes with every element of the group. If every pair of elements of a group commute, then the group is said to be *abelian*. If we look back at our examples, the first two are abelian, the last two are not.

If the group G is abelian, usually +, 0, and -a are used to represent the operation, identity, and inverse of $a \in G$. For groups in general the operation is usually represented by * or by \cdot ; the identity element is represented by 1, I or e; and the inverse of $a \in G$ by a^{-1} . When the operation is represented by \cdot , it is common to write abinstead of $a \cdot b$.

A nice connection between Graph Theory and Group Theory are Cayley Digraphs. Given a group G, and S a subset of elements of G, such that the identity element is not in S, the Cayley Digraph $\Gamma = \overrightarrow{Cay}(G; S)$ is the directed graph that has:

> $V(\Gamma) = \{g \,|\, g \in G\},\$ $A(\Gamma) = \{(g, gs) \,|\, g \in G, \, s \in S\}.$

We will say that arcs of the form (g, gs) are generated by the element s.

Example 7: The set of integers modulo 9, \mathbb{Z}_9 , is a group, with elements 0, 1, 2, 3, 4, 5, 6, 7, 8 and addition modulo 9 as the operation, +. The Cayley digraph $\overrightarrow{Cay}(\mathbb{Z}_9; \{2, 3\})$ is shown in Figure 0.9, where the blue arcs are of the form (g, g + 2), and the red arcs are of the form (g, g + 3).



Figure 0.9: The Cayley digraph $\overrightarrow{Cay}(\mathbb{Z}_9; \{2, 3\})$

In 1961 B. Gordon [19] defined a group G to be *sequenceable* when there exists a permutation

$$g_0, g_1, g_2, \ldots, g_{n-1}$$

of its elements so that the sequence of partial products

 $g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1g_2\cdots g_{n-1}$

are distinct. In that same paper he proved the following theorem.

Theorem 1 A finite abelian group G is sequenceable if and only if G contains a unique non-identity element a, such that a = -a.

In 1974 G. Ringel [36] asked when there exists a permutation

$$g_1, g_2, \ldots, g_{n-1}$$
of the non-identity elements of a group such that the sequence

$$g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_{n-1}g_{n-2}^{-1}, g_1g_{n-1}^{-1}$$

also is a permutation of the non-identity elements. A group G that admits such a permutation is called *R-sequenceable*. As a matter of fact, L. Paige [33] used this concept in 1951, but it was Ringel's problem that motivated the most important paper on this topic.

One can investigate these problems using Cayley digraphs. In particular the Cayley digraph with $S = G \setminus \{e\}$, that is, the set S has everything in it other than the identity element. We use the special notation $\overrightarrow{K}(G)$ for this Cayley digraph.

It is easy to see that a fixed element $s \in S$ generates a subdigraph consisting of directed cycles of length |s|, the order of s in G. Thus, we obtain a decomposition of $\overrightarrow{Cay}(G; S)$ into |S| directed 2-factors. We call this decomposition the *Cayley factorization* of $\overrightarrow{Cay}(G; S)$ and denote it by $\overrightarrow{\mathcal{F}}(G; S)$. Figure 0.10 shows the Cayley factorization $\overrightarrow{\mathcal{F}}(\mathbb{Z}_6; \{1, 2, 3, 4, 5\})$ of the Cayley digraph $\overrightarrow{K}(\mathbb{Z}_6)$.

If \overrightarrow{D} is a subdigraph of $\overrightarrow{Cay}(G; S)$ with |S| arcs, and \overrightarrow{D} has exactly one arc from each directed 2-factor in $\overrightarrow{\mathcal{F}}(G; S)$, then we say that \overrightarrow{D} is *orthogonal* to $\overrightarrow{\mathcal{F}}(G; S)$. In this language, the group G is sequenceable when $\overrightarrow{K}(G)$ has an orthogonal directed path of length |G|, and G is R-sequenceable when $\overrightarrow{K}(G)$ has an orthogonal directed cycle of length |G| - 1.

Example 8: Figure 0.11(a) shows an orthogonal directed path of length 10 for \mathbb{Z}_{10} . The vertices used by the path are:

The arcs used are generated by the elements:

Notice that each non-identity element from G generates exactly one of these arcs, thus the directed path is orthogonal.

Figure 0.11(b) shows an orthogonal directed cycle of length 6 for \mathbb{Z}_6 . The vertices used by the cycle are:

The arcs used are generated by the elements:

Notice that each non-identity element from G generates exactly one of these

arcs, thus the directed cycle is orthogonal.

Chapter 3 studies the R-sequenceability of abelian groups. In this chapter we show that every abelian group is either sequenceable or R-sequenceable. This completely solves the problem of Ringel for abelian groups.





Figure 0.10: The Cayley digraph $\overrightarrow{K}(\mathbb{Z}_6)$ and its Cayley Factorization $\overrightarrow{\mathcal{F}}(\mathbb{Z}_6; \{1, 2, 3, 4, 5\})$





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Figure 0.11: Orthogonal directed subgraphs

Chapter 1

On the Hamilton-Waterloo Problem with triangle factors and C_{3x} -factors¹

1.1 Introduction

The Oberwolfach problem was first proposed by Ringel in 1967, and involves seating v conference attendees at t round tables over $\frac{v-1}{2}$ nights such that each attendee sits next to each other attendee exactly once. It is mathematically equivalent to

¹The material in this paper was previously published by Australasian Journal of Combinatorics

decomposing K_v into 2-factors where K_v is the complete graph on v vertices and each 2-factor is isomorphic to a given 2-factor Q. In the original statement of the problem, we have that v must be odd. It was later extended to the spouse-avoiding Oberwolfach problem, allowing for even v by decomposing $K_v - F$, where F is a 1-factor.

The Hamilton-Waterloo Problem (HWP) is an extension of the Oberwolfach Problem. Instead of seating v attendees at the same t tables each night, the Hamilton-Waterloo problem asks how the v attendees can be seated if they split their nights between two different venues. The attendees will all spend the same r nights in Hamilton, which has round tables of size m_1, m_2, \ldots, m_k , and s nights in Waterloo, which has round tables of size n_1, n_2, \ldots, n_p where $\sum_{i=1}^k m_i = \sum_{i=1}^p n_i = v$. The case when $m_1 = m_2 = \cdots = m_k = m$ and $n_1 = n_2 = \cdots = n_p = n$ is called the Hamilton-Waterloo Problem with uniform cycle sizes, and this variant of the problem gets most of the attention. Graph theoretically, this problem is equivalent to decomposing K_v (or $K_v - F$ when v is even) into 2-factors where each 2-factor consists entirely of m-cycles (a C_m -factor) or entirely of n-cycles (a C_n -factor). Throughout this paper, the word factor is assumed to be a 2-factor unless otherwise stated. We frequently refer to a C_3 -factor as a triangle factor and a Hamilton cycle as a Hamilton factor.

A decomposition of a graph G is a partition of the edge set of G. A decomposition of K_v into C_m -factors is called a C_m -factorization. We will refer to a solution to the Hamilton-Waterloo Problem with r factors of m-cycles, s factors of n-cycles, and v points as a resolvable (C_m, C_n) -decomposition of K_v into r C_m -factors and s C_n -factors, and we will let (m, n)-HWP(v; r, s) denote such a decomposition. In order for an (m, n)-HWP(v; r, s) to exist, it is clear that $r + s = \frac{v-1}{2}$ (or $r + s = \frac{v-2}{2}$, for even v), and both m and n must divide v. These conditions are summarized in the following theorem.

Theorem 2 [1] The necessary conditions for the existence of an (m, n)-HWP(v; r, s) are

- 1. If v is odd, $r + s = \frac{v-1}{2}$,
- 2. If v is even, $r + s = \frac{v-2}{2}$,
- 3. If r > 0, m|v,
- 4. If s > 0, n|v.

Recall that the Oberwolfach Problem involves seating v conference attendees at tround tables such that each attendee sits next to each other attendee exactly once. The Oberwolfach Problem for constant cycle lengths was solved in [2, 3, 23, 34]. This is equivalent to the Hamilton-Waterloo Problem with r = 0 or s = 0.

Theorem 3 [2, 3, 23, 34] There exists a resolvable *m*-cycle decomposition of K_v (or $K_v - F$ when v is even) if and only if $v \equiv 0 \pmod{m}$, $(v,m) \neq (6,3)$ and $(v,m) \neq (12,3).$

An equipartite graph is a graph whose vertex set can be partitioned into u subsets of size h such that no two vertices from the same subset are connected by an edge. The complete equipartite graph with u subsets of size h is denoted $K_{(h:u)}$, and it contains every edge between vertices of different subsets. Another key result solves the Oberwolfach Problem for constant cycle lengths over complete equipartite graphs (as opposed to K_v). That is to say, with finitely many exceptions, $K_{(h:u)}$ has a resolvable C_m -factorization.

Theorem 4 [28] For $m \ge 3$ and $u \ge 2$, $K_{(h:u)}$ has a resolvable C_m -factorization if and only if hu is divisible by m, h(u-1) is even, m is even if u = 2, and $(h, u, m) \notin$ $\{(2,3,3), (6,3,3), (2,6,3),$ $(6,2,6)\}.$

Much of the attention to the HWP has been dedicated to the case of triangle factors and Hamilton factors. The results for this case have been summarized in the following theorem.

Theorem 5 [15, 16, 24, 27] There exists a (3, v)-HWP(v; r, s) with

• $2 \le s \le \frac{v-1}{2}$ and $v \equiv 3 \pmod{6}$ except possibly when:

$$v \equiv 15 \pmod{18}$$
 and $2 \le s \le \frac{v-3}{6}$ or $s = \frac{v+3}{6} + 1$,

• s = 1 and $v \equiv 3 \pmod{6}$ except when v = 9 and possibly when:

 $v \in \{93, 111, 123, 129, 141, 153, 159, 177, 183, 201, 207, 213, 237, 249\}.$

- $2 \le s \le (v-2)/2$ and $v \equiv 0 \pmod{6}$ except possibly when $(v,s) \in \{(36,2), (36,4)\}$ or when $v \equiv 12 \pmod{18}$ and $2 \le s \le (v/6) 1$; and
- $s \equiv 1$ and $v \equiv 0 \pmod{6}$ except possibly when $v \equiv 18$, $v \equiv 12 \pmod{18}$ or $v \equiv 6 \pmod{36}$.

When considering the HWP for triangle factors and Hamilton factors, the focus is on a specific case of the problem. This paper considers a more general family of decompositions, namely, triangle factors and 3x-factors of K_v for any v that is divisible by both 3 and 3x. In this instance of the problem, v is of the form 3xy. When x = 1, the problem of finding a (3, 3x)-HWP(v; r, s) is simply that of finding a resolvable C_3 factorization of K_v , which is also known as a Kirkman triple system (KTS(v)). It was shown in 1971 by Ray-Chadhuri and Wilson [34] and independently by Lu (see [30]) that a KTS(v) exists if and only if $v \equiv 3 \pmod{6}$. When y = 1, then the problem asks for a decomposition of K_v into triangle factors and Hamilton cycles. This case is addressed in [15], [16], and [24], and the results were presented in Theorem 5. Therefore, we focus on the cases where $x \ge 2$ and $y \ge 2$. It is a different type of decomposition than what was considered in [15, 16, 24], because in our case, we let both x and y vary. However, as expected, the results given in Theorem 5 can be used in the decompositions we are interested in.

The Hamilton-Waterloo Problem was studied in 2002 by Adams, et. al. [1]. The paper provides solutions to all Hamilton-Waterloo decompositions on less than 18 vertices. Some notable results involving v = 6 and v = 12 will be relevant to this paper.

Theorem 6 [1] There exists a (3,6)-HWP(12;r,s) if and only if r + s = 5 except (r,s) = (5,0). There exists a (3,12)-HWP(12;r,s) if and only if r + s = 5 except (r,s) = (5,0). There exists a (3,6)-HWP(6;r,s) if and only if r + s = 2 except (r,s) = (2,0).

The authors in [1] also developed a tripartite construction that could be used when considering m = 3 and n = 3x. However, it leaves many open cases, because it relies on the existence of a (3, v)-HWP(v; r, s) for all (r, s) and for all $v \equiv 3 \pmod{6}$. According to Theorem 5, there are some gaps in the existence of these. The problem is that the construction given in [1] uses a uniform decomposition of $K_{(x:3)}$. Therefore, we proceed in this paper by developing a new construction that is a bit more general, and in particular, depends on the decomposition of $K_{(x:3)}$ into $r_p \ C_m$ -factors and $s_p \ C_n$ -factors. The flexibility in this construction allows us to settle all but 14 cases of the existence of a (3,3x)-HWP(3xy;r,s) for all possible (r,s) whenever both $x \ge 3$ and $y \ge 3$ are odd. We also introduce a modified construction that is used in the cases where at least one of x or y is even. We give almost complete results for these cases as well. In Section 1.3.1 we handle the cases when $x \in \{2,4\}$ and collect all of the results into a summarizing theorem in Section 1.4.

1.2 Constructions

In this section, we develop constructions that will later be used to prove our main results about the Hamilton-Waterloo Problem in the case of triangle factors and C_{3x} factors.

Recall that $K_{(x:3)}$ is the complete multipartite graph with 3 parts of size x. Let the parts be G_0 , G_1 and G_2 and the vertices be (a, b) with $0 \le a \le 2, 0 \le b \le x - 1$. Consider the edge $\{(a_1, b_1), (a_2, b_2)\}$ which has one vertex from G_{a_1} and one vertex from G_{a_2} . With computations being done in \mathbb{Z}_x , we say this edge has difference $b_2 - b_1$. Let $T_x(i)$ for $0 \le i \le x - 1$ be the subgraph of $K_{(x:3)}$ obtained by taking all edges of difference: 2i between vertices of G_0 and vertices of G_1 , -i between G_1 and G_2 , and -i between G_2 and G_0 . **Lemma 1.2.1** $T_x(i)$ is a triangle factor of $K_{(x:3)}$ for any *i*.

Proof: It is easy to see that the triangles are of the form $\{(0,k), (1, k+2i), (2, k+i)\}$ for every $0 \le k \le x - 1$.

Let $H_x(i, j)$ be the subgraph of $K_{(x:3)}$ obtained by taking all edges of difference: 2i between G_0 and G_1 , -i between G_1 and G_2 , and -j between G_2 and G_0 .

Lemma 1.2.2 If gcd(x, i - j) = 1 then $H_x(i, j)$ is a Hamiltonian cycle of $K_{(x:3)}$.

Proof: Since the edges are given by differences it is clear that all vertices have degree 2. We need to show that all the vertices are connected. We will first show that there is a path between any 2 vertices of G_0 . Without loss of generality, we will show that (0,0) is connected to (0,k) for any k. Starting at (0,0), we may traverse the path: (0,0), (1,2i), (2,i), (0,i-j). Thus the next time that we reach G_0 it is via the vertex i-j. Since gcd(x,i-j) = 1, the order of i-j in the cyclic group \mathbb{Z}_x is x. Therefore, any k modulo x can be written as k'(i-j), which means that we reach the vertex (0,k) after visiting the part G_0 k' times. Hence (0,0) is connected to all the vertices of G_0 via a path.

Because we are taking every edge of a particular difference, it follows that every vertex in G_1 is connected to a vertex in G_0 , and the same is true for vertices in G_2 . Hence all the vertices are connected, and the cycle is Hamiltonian, as we wanted to prove.

1.2.1 When x is Odd

We can think of a decomposition of a graph G as a partition of the edge set or as a union of edge disjoint subgraphs. This means that a decomposition of G can be given by $E(G) = \bigcup E(F_i)$ or by $G = \bigoplus F_i$, where each F_i is an edge disjoint subgraph of G. The next lemma shows that $K_{(x:3)}$ can be decomposed entirely into triangle factors or Hamilton cycles when x is odd.

Lemma 1.2.3 Let x be an odd integer, and let ϕ be a bijection of the set $\{0, 1, \dots, x-1\}$ into itself. Then

$$K_{(x:3)} = \bigoplus_{i=0}^{x-1} T_x(i) = \bigoplus_{i=0}^{x-1} H_x(i,\phi(i))$$

Proof: To prove the first equality,

$$K_{(x:3)} = \bigoplus_{i=0}^{x-1} T_x(i)$$

we need to show that between each pair of parts in $K_{(x:3)}$, each difference is covered by the edges in one of the triangle factors exactly once. It is clear that edges of difference *i* between G_1 and G_2 and between G_2 and G_0 are covered in $T_x(i)$. Now consider groups G_0 and G_1 . Each factor $T_x(i)$ uses the difference 2i. Because gcd(x, 2) = 1, the order of 2 in the cyclic group \mathbb{Z}_x is x. So it follows that any i modulo x can be written as 2i', and thus the difference i between G_0 and G_1 is covered in $T_x(i')$. Notice that we cover the edges of exactly one difference between any two parts per subgraph, and we only have x subgraphs. This together with the fact that we are covering all the differences imply that we cover each difference exactly once. Thus it is equivalent to decomposing $K_{(x:3)}$.

The second equality

$$\bigoplus_{i=0}^{x-1} T_x(i) = \bigoplus_{i=0}^{x-1} H_x(i,\phi(i))$$

is true because we again cover each difference between any pair of parts exactly once by the edges in the factors.

Notice that the subgraph $H_x(i, i)$ is the same as $T_x(i)$. Therefore, decomposing $K_{(x:3)}$ into s Hamilton cycles and x - s triangle factors is equivalent to finding a bijection ϕ such that $gcd(x, i - \phi(i)) = 1$ for s elements of $\{0, 1, \ldots, x - 1\}$ and $\phi(i) = i$ for the rest.

Theorem 7 Let x be odd and let $s \in \{0, 2, 3, \dots, x\}$. Then:

• there exists a bijection ϕ on the set $\{0, 1, \dots, x - 1\}$ with $gcd(x, i - \phi(i)) = 1$ for s elements and r = x - s fixed points; and • $K_{(x:3)}$ can be decomposed into s Hamiltonian cycles and r = x - s triangle factors.

Proof: If s = 0 we just use the identity mapping. Let $2 \le s \le x$, and let e be the smallest integer such that $s \le 2^e + 1$. We have

$$2^{e-1} + 1 < s \le \min\{2^e + 1, x\} = t.$$

Let r = t - s and define ϕ as follows:

$$\phi(i) = \begin{cases} 0 & \text{for} \quad i = 1 \\ i+2 & \text{for} \quad i \equiv 0 \pmod{2}, 0 \le i \le s-3 \\ i-2 & \text{for} \quad i \equiv 1 \pmod{2}, 3 \le i \le s-1 \\ s-2 & \text{for} \quad i \equiv 0 \pmod{2}, i = s-1 \\ s-1 & \text{for} \quad i \equiv 0 \pmod{2}, i = s-2 \\ i & \text{for} \quad s \le i \le x-1 \end{cases}$$

It is an easy exercise to check that ϕ is a bijection with r = x - s fixed points. Furthermore, for any non-fixed point we have $(i - \phi(i)) \in \{\pm 1, \pm 2\}$ and, because x is odd, $gcd(x, i - \phi(i)) = 1$. Hence by Lemma 1.2.3,

$$K_{(x:3)} = \bigoplus_{i=0}^{x-1} H_x(i,\phi(i))$$

is a decomposition of $K_{(x:3)}$ into s Hamiltonian cycles and r = x - s triangle factors.

Unfortunately this construction only works when x is odd. For the cases when x is even we can get a similar result, although only when $x = 2\bar{x}$, with \bar{x} odd.

1.2.2 When x is Even

In this subsection, we develop a construction similar to what is described in Section 1.2.1. It relies on the following decomposition of $K_{(4:3)}$ into triangle factors. Define $\Gamma(i)$ for $i \in \{0, 1, 2, 3\}$ as follows.



Note that the edges that join G_0 to G_2 are dashed since they will need to be distinguished from the other two edges in each C_3 . It is easy to see that $\bigoplus_{i=0}^{3} \Gamma(i)$ is a C_3 -factorization of $K_{(4:3)}$.

Lemma 1.2.4 There exist a decomposition of $K_{(4:3)}$ into $s \ C_6$ -factors and $4 - s \ C_3$ -factors for any $s \in \{0, 2, 3, 4\}$.

Proof: Consider the C_3 factorization of $K_{(4:3)}$, $\bigoplus_{i=0}^3 \Gamma(i)$. Let $\Lambda(\alpha, \beta)$ be the graph that has edges between G_0 (the first column) and G_1 (the second column) from $\Gamma(\alpha)$, has edges between G_1 and G_2 from $\Gamma(\alpha)$, and has dashed edges from $\Gamma(\beta)$. Notice that if $\alpha \neq \beta$ then $\Lambda(\alpha, \beta)$ is a union of cycles of size 6.

This way we can get 2 C_6 -factors by using $\Lambda(0,1)$ and $\Lambda(1,0)$ instead of $\Gamma(0)$ and $\Gamma(1)$. We can get 3 C_6 -factors by using edges $\Lambda(0,1)$, $\Lambda(1,2)$ and $\Lambda(2,0)$ instead of $\Gamma(0)$, $\Gamma(1)$ and $\Gamma(2)$. And finally we can get 4 C_6 -factors by using $\Lambda(0,1)$, $\Lambda(1,2)$, $\Lambda(2,3)$ and $\Lambda(3,0)$. This construction gives the desired decompositions.

For $\bar{x} = 1$, Lemma 1.2.4 gives a decomposition of $K_{(4\bar{x}:3)}$ into triangle factors and $C_{6\bar{x}}$ -factors. We will extend this result to work on any $K_{(4\bar{x}:3)}$ where $\bar{x} > 1$ and odd. The construction works by giving weight \bar{x} to each vertex in $K_{(4:3)}$.

Replace each vertex in $K_{(4:3)}$ by a set of \bar{x} vertices. Thus for a = 0, 1, 2, we have $G_a = \{(a, b, c) : b = 1, 2, 3, 4; c = 1, 2, ..., \bar{x}\}$ is a set of $4\bar{x}$ vertices.

For $\alpha = 0, 1, 2, 3$, we construct the triangle factor $T_{2\bar{x}}(\alpha, i)$ of $K_{(4\bar{x}:3)}$ as follows. For each triangle $\{(0, b_0), (1, b_1), (2, b_2)\}$ in $\bigoplus_{\alpha=0}^3 \Gamma(\alpha)$, construct the complete equipartite graph $K_{(\bar{x}:3)}$ on the set of vertices $\{(0, b_0, c), (1, b_1, c), (2, b_2, c) : c = 1, 2, \dots, \bar{x}\}$. To visualize this weighting construction with $\bar{x} = 3$ we show a picture of a triangle from $K_{(4:3)}$. After giving weight 3 to each vertex, the triangle becomes $K_{(3:3)}$.



Decompose each $K_{(\bar{x}:3)}$ into triangle factors $T_{\bar{x}}(i)$ for $i = 0, 1, ..., \bar{x}-1$ by Lemma 1.2.3. Thus we have a decomposition of $K_{(4\bar{x}:3)}$ into triangle factors.

Define $H_{2\bar{x}}(\alpha, i)(\beta, j)$ as the graph obtained by taking $T_{2\bar{x}}(\alpha, i)$ and replacing the edges between G_0 and G_2 with the same edges from $T_{2\bar{x}}(\beta, j)$. Then we have that

$$H_{2\bar{x}}(\alpha,i)(\beta,j) \oplus H_{2\bar{x}}(\beta,j)(\alpha,i) = T_{2\bar{x}}(\alpha,i) \oplus T_{2\bar{x}}(\beta,j)$$

If $\alpha \neq \beta$ and $gcd(i - j, \bar{x}) = 1$ then we claim that $H_{2\bar{x}}(\alpha, i)(\beta, j)$ is a $C_{6\bar{x}}$ -factor. Suppose $(a, b, c) \in H_{2\bar{x}}(\alpha, i)(\beta, j)$. Then (a, b) is a vertex of $K_{(4:3)}$. Because $\alpha \neq \beta$, $\Lambda(\alpha, \beta)$ is a C_6 -factor on $K_{(4:3)}$, as shown in the proof of Lemma 1.2.3. Because gcd(i - j, x) = 1, it follows from Lemma 1.2.2 that $K_{(\bar{x}:3)}$ is a $3\bar{x}$ -cycle. Thus (a, b, c)is contained in a cycle of length $lcm(6, 3\bar{x}) = 6\bar{x}$. Hence $H_{2\bar{x}}(\alpha, i)(\beta, j)$ is a $C_{6\bar{x}}$ -factor.

Let ψ be a bijection on $\{(\alpha, i) | 0 \le \alpha \le 3, 0 \le i \le \overline{x} - 1\}$. The previous discussion

leads us to the following result.

Lemma 1.2.5 Let \bar{x} be odd. Let s and r be non-negative integers such that $s+r = 4\bar{x}$. If ψ satisfies the following:

- $\psi(\alpha, i) = (\alpha, i)$ for r pairs (α, i) ; and
- $\psi(\alpha, i) = (\beta, j)$ with $\alpha \neq \beta$ and $gcd(i j, \bar{x}) = 1$ for the s remaining pairs;

then $K_{(4\bar{x}:3)} = \bigoplus H_{2\bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a decomposition of $K_{(4\bar{x}:3)}$ into r triangle factors and $s \ C_{6\bar{x}}$ -factors.

Proof: Notice that $H_{2\bar{x}}(\alpha, i)(\alpha, i) = T_{2\bar{x}}(\alpha, i)$, so if $\psi(\alpha, i) = (\alpha, i)$, $H_{2\bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a triangle factor. When $\psi(\alpha, i) = (\beta, j)$ with $\alpha \neq \beta$ and $gcd(i - j, \bar{x}) = 1$, by the discussion preceding the lemma, $H_{2\bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a $C_{6\bar{x}}$ -factor. Therefore $K_{(4\bar{x}:3)} = \bigoplus H_{2\bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a decomposition of $K_{(4\bar{x}:3)}$ into r triangle factors and $s C_{6\bar{x}}$ -factors.

Thanks to Lemma 1.2.5 we only need to show that for any $r \in \{0, 1, \dots, 4\bar{x} - 2, 4\bar{x}\}$ we have a bijection ψ satisfying the conditions of the lemma and with r fixed points.

Theorem 8 Let \bar{x} be odd and $s \in \{0, 2, 3, ..., 4\bar{x} - 1, 4\bar{x}\}$, then:

- There exists a bijection ψ satisfying the conditions of Lemma 1.2.5 with $r = 4\bar{x} s$ fixed points.
- $K_{(4\bar{x}:3)}$ can be decomposed into s $C_{6\bar{x}}$ -factors and r triangle factors.

Proof: If s = 0 we just use the identity mapping.

If $2 \le s \le 4\bar{x}$ we let $s_0, s_1, s_2, s_3 \in \{0, 2, 3..., \bar{x} - 1\}$ be such that $s = s_0 + s_1 + s_2 + s_3$. We define ψ as follows, where $m \in \{0, 1, 2, 3\}$ and i + m is taken (mod 4):

$$\psi(i+m,i) = \begin{cases} (m,0) & \text{for} \quad i=1 \\ (i+m+2,i+2) & \text{for} \quad i \equiv 0 \pmod{2}, 0 \le i \le s_m - 3 \\ (i+m-2,i-2) & \text{for} \quad i \equiv 1 \pmod{2}, 3 \le i \le s_m - 1 \\ (s_m+m-2,s_m-2) & \text{for} \quad i \equiv 0 \pmod{2}, i = s_m - 1 \\ (s_m+m-1,s_m-1) & \text{for} \quad i \equiv 0 \pmod{2}, i = s_m - 2 \\ (i+m,i) & \text{for} \quad s_m \le i \le \bar{x} - 1 \end{cases}$$

It is an easy exercise to check that ψ is a bijection with $4\bar{x} - (s_0 + s_1 + s_2 + s_3) = r$ fixed points. Notice that $\psi(\alpha, i) - (\alpha, i) \in \{(0, 0), (\pm 1, \pm 1), (\pm 2, \pm 2)\}$. This gives that if $\psi(\alpha, i) = (\beta, j)$ is not a fixed point of $\psi, \alpha \neq \beta$ and $gcd(i - j, \bar{x}) = 1$. Hence by Lemma 1.2.5

$$K_{(4\bar{x}:3)} = \bigoplus H_{2\bar{x}}(\alpha, i)(\psi(\alpha, i))$$

is a decomposition of $K_{(4\bar{x}:3)}$ into $s C_{6\bar{x}}$ -factors and $4\bar{x} - s$ triangle factors.

1.2.3 A Weighting Construction

A group divisible design (k, λ) -GDD (h^u) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where \mathcal{V} is a finite set of size v = hu, \mathcal{G} is a partition of \mathcal{V} into u groups each containing h elements, and \mathcal{B} is a collection of k element subsets of \mathcal{V} called *blocks* which satisfy the following properties.

- If $B \in \mathcal{B}$, then |B| = k.
- If a pair of elements from \mathcal{V} appear in the same group, then the pair cannot be in any block.
- Two points that are not in the same group, called a *transverse pair*, appear in exactly λ blocks.
- $|\mathcal{G}| > 1.$

Here we use the term group to indicate an element of \mathcal{G} . In this context, group simply means a set of elements without any algebraic structure. A *resolvable* GDD

(RGDD) has the additional condition that the blocks can be partitioned into parallel classes such that for each element of \mathcal{V} there is exactly one block in each parallel class containing it. If $\lambda = 1$, we refer to the RGDD as a k-RGDD (h^u) . In this paper, we will only talk about RGDDs with $\lambda = 1$. Necessary and sufficient conditions for the existence of 3-RGDD (h^u) s have been established except in a finite number of cases.

Theorem 9 [35] $A(3,\lambda)$ -RGDD (h^u) exists if and only if $u \ge 3, \lambda h(u-1)$ is even, $hu \equiv 0 \pmod{3}$, and $(\lambda, h, u) \notin \{(1, 2, 6), (1, 6, 3)\} \bigcup \{(2j+1, 2, 3), (4j+2, 1, 6) : j \ge 0\}.$

In particular, we have that a 3–RGDD(3^{*u*}) exists for all odd $u \ge 3$ and a 3–RGDD(6^{*u*}) exists for all $u \ge 4$. Note that when a 3–RGDD(h^u) exists, then \mathcal{B} can be partitioned in $\frac{h(u-1)}{2}$ parallel classes.

Lemma 1.2.6 Let $m \ge 3$, $n \ge 3$ and x be positive integers such that both m and n divide 3x. Suppose the following conditions are satisfied:

- There exists a 3-RGDD (h^u) ,
- there exists a decomposition of $K_{(x:3)}$ into $r_p \ C_m$ -factors and $s_p \ C_n$ -factors, for $p \in \{1, 2, \dots, \frac{h(u-1)}{2}\},$
- there exists an (m, n)-HWP $(hx; r_{\beta}, s_{\beta})$.

Let

$$r_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}} r_p \text{ and } s_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}} s_p.$$

Then there exists a (m, n)-HWP $(hux; r_{\alpha} + r_{\beta}, s_{\alpha} + s_{\beta})$.

Proof: For a = 1, 2, ..., u, let the groups of the 3-RGDD (h^u) be denoted by $G_a = \{(a, b) : b = 1, 2, ..., h\}$. Let $\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_{\frac{h(u-1)}{2}}\}$ denote the parallel classes of the 3-RGDD (h^u) , and for a = 1, 2, ..., u, define $G_a^* = \{(a, b, c) : b = 1, 2, ..., h; c = 1, 2, ..., x\}$ to be a set of hx vertices. Consider each parallel class \mathcal{P}_p with $p \in \{1, 2, ..., \frac{h(u-1)}{2}\}$. For each block $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} \in \mathcal{P}_p$, construct a decomposition of $K_{(x:3)}$ into $r_p \ C_m$ -factors and $s_p \ C_n$ -factors on the set of vertices $\{(a_1, b_1, c), (a_2, b_2, c), (a_3, b_3, c) : c = 1, 2, ..., x\}$. Thus we have a decomposition of $K_{(hx:u)}$ into $r_\alpha \ C_m$ -factors and $s_\alpha \ C_n$ -factors where

$$r_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}} r_p \text{ and } s_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}} s_p.$$

Now each part of $K_{(hx:u)}$ can be decomposed into $r_{\beta} C_m$ -factors and $s_{\beta} C_n$ -factors. Thus there exists an (m, n)-HWP(hux; r, s) where $r = r_{\alpha} + r_{\beta}$ and $s = s_{\alpha} + s_{\beta}$.

Lemma 1.2.7 Let $m \ge 3$, $n \ge 3$ and x be positive integers such that both m and n divide 3x. Suppose the following conditions are satisfied:

- There exists a 3-RGDD (h^u) ,
- there exists an (m, n)-HWP $(3x; r_{\beta}, s_{\beta})$,
- there exists a decomposition of $K_{(x:h)}$ into $r_{\gamma} C_m$ -factors and $s_{\gamma} C_n$ -factors,
- there exists a decomposition of $K_{(x:3)}$ into $r_p \ C_m$ -factors and $s_p \ C_n$ -factors, for

 $p \in \{1, 2, \dots, \frac{h(u-1)}{2}\}.$

Let

$$r_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}-1} r_p \text{ and } s_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}-1} s_p.$$

 $Then \ there \ exists \ a \ (m,n) - \mathsf{HWP}(hux;r_{\alpha} + r_{\beta} + r_{\gamma},s_{\alpha} + s_{\beta} + s_{\gamma}).$

Proof: Let $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{\frac{h(u-1)}{2}}\}$ denote the parallel classes of the 3-RGDD (h^u) , and let $W = \{1, 2, \ldots, x\}$. Consider each parallel class \mathcal{P}_p with $p \in \{1, 2, \ldots, \frac{h(u-1)}{2} - 1\}$. For each block $\{a_1, a_2, a_3\} \in \mathcal{P}_p$, construct a decomposition of $K_{(x:3)}$ into r_p C_m factors and s_p C_n -factors with parts $\{a_i\} \times W$, i = 1, 2, 3. For each block $\{a_1, a_2, a_3\}$ in parallel class \mathcal{P}_β where $\beta = \frac{h(u-1)}{2}$, construct an (m, n)-HWP $(3x; r_\beta, s_\beta)$ on $\{a_1 \times W, a_2 \times W, a_3 \times W\}$. Take a decomposition of $K_{(x:h)}$ into r_γ C_m -factors and s_γ C_n -factors simultaneously on each group of the 3-RGDD (h^u) . This makes an (m, n)-HWP(hux; r, s) where $r = r_\alpha + r_\beta + r_\gamma$ and $s = s_\alpha + s_\beta + s_\gamma$.

1.3 Main Results

In this section, we use the constructions given in Section 1.2 to obtain results on the existence of a (3, 3x)-HWP(3xy; r, s). We consider four different cases depending on the parity of x and y.

Lemma 1.3.1 Suppose x is even. If there exists a decomposition of $K_{3x} - F$ into $r_{\delta} C_3$ -factors and s_{δ} Hamilton cycles, then there exists a decomposition of $K_{6x} - F'$ into $r_{\delta} C_3$ -factors and $s_{\delta} + \frac{3x}{2} C_{3x}$ -factors, where F is a 1-factor of K_{3x} and F' is a 1-factor of K_{6x} .

Proof: Let G_1 and G_2 be a partition of the 6x points into two subsets of size 3x. Decompose $K_{G_1} - F_1$ (the complete graph on G_1 minus a 1-factor F_1) into r_{δ} C_3 -factors and s_{δ} Hamilton cycles. In the same manner, decompose $K_{G_2} - F_2$ (the complete graph on G_2 minus a 1-factor) into r_{δ} C_3 -factors and s_{δ} Hamilton cycles. Then there is a decomposition of $(K_{G_1} \cup K_{G_2}) - (F_1 \cup F_2)$ into r_{δ} C_3 -factors and s_{δ} C_{3x} -factors. Notice that $F' = F_1 \cup F_2$ is a 1-factor of K_{6x} . By Theorem 4, there exists a decomposition of $K_{(3x:2)}$ into $\frac{3x}{2}$ C_{3x} -factors. The union of these edges is K_{6x} .

Therefore there is a decomposition of $K_{6x} - F'$ into $r_{\delta} C_3$ -factors and $s_{\delta} + \frac{3x}{2} C_{3x}$ -factors.

Theorem 10 For each pair of odd integers $x \ge 3$ and $y \ge 3$, there exists a (3, 3x)-HWP(3xy; r, s) if and only if $r + s = \frac{v-1}{2}$ except when s = 1 and x = 3, and possibly when s = 1 and

 $x \in \{31, 37, 41, 43, 47, 51, 53, 59, 61, 67, 69, 71, 79, 83\}.$

Proof: By Theorem 9 there exists a $3 - \mathsf{RGDD}(3^y)$ for all odd $y \ge 3$. There exists a decomposition of $K_{(x:3)}$ into $r_p C_3$ -factors and $s_p C_{3x}$ -factors for $(r_p, s_p) \in \{(x, 0), (x - 2, 2), (x - 3, 3), \dots, (0, x)\}$ by Theorem 7. There exists a (3, 3x)-HWP $(3x; r_\beta, s_\beta)$ whenever $(r_\beta, s_\beta) \in \{(\frac{3x-1}{2}, 0), (\frac{3x-3}{2}, 1), (0, \frac{3x-1}{2})\}$ by Theorems 3 and 5 (excluding the exception and possible exceptions listed in the statements of these theorems). So apply Lemma 1.2.6 with m = 3 and n = 3x. We must now show that for each $s \in \{0, 1, \dots, \frac{3xy-1}{2}\}$, there exists a (3, 3x)-HWP(3xy; r, s). It is easy to see that if $s_\alpha \in \{0, 2, 3, \dots, \frac{3xy-3x}{2}\}$, then we can write $s_\alpha = \sum_{i=1}^{(3y-3)/2} s_p$ where $s_p \in \{0, 2, 3, \dots, x\}$. Thus if $s \in \{0, 2, 3, \dots, \frac{3xy-3x}{2}\}$, then we may write $s = s_\alpha + s_\beta$ by choosing $s_\alpha = s$ and $s_\beta = 0$. If s = 1, then choose $s_\alpha = 0$ and $s_\beta = 1$. If $s = \frac{3xy-3x}{2} + 1$, choose $s_\alpha = \frac{3xy-3x}{2}$ and $s_\beta = 1$. Finally, let $i = 2, 3, \dots, \frac{3x-1}{2}$ because

$$2 \le s - \frac{3x - 1}{2} \le \frac{3xy - 3x}{2}.$$

Theorem 11 For each odd integer $x \ge 3$ and each even integer $y \ge 8$, there exists a (3, 3x)-HWP(3xy; r, s) if and only if $r + s = \frac{3xy-1}{2}$ except possibly when s = 1.

Proof: By Theorem 9, there exists a 3–RGDD($6^{y/2}$) for all even $y \ge 8$. By Theorem 7, for each $p \in \{1, 2, \ldots, \frac{6(y/2-1)}{2}\}$, $K_{(x:3)}$ can be decomposed into $r_p \ C_3$ -factors and $s_p \ C_{3x}$ -factors where $(r_p, s_p) \in \{(x, 0), (x - 2, 2), (x - 3, 3), \ldots, (0, x)\}$, so that $r_\alpha = \sum_{p=1}^{3(y/2-1)} r_p$ and $s_\alpha = \sum_{p=1}^{3(y/2-1)} s_p$. By Theorem 3, K_{6x} can be decomposed into $r_\beta \ C_3$ -factors, $s_\beta \ C_{3x}$ -factors, and a 1-factor where $(r_\beta, s_\beta) \in \{((6x - 2)/2, 0), (0, (6x - 2)/2)\}$. We must show that for each $s \in \{0, 2, 3, \ldots, (3xy - 2)/2\}$ there exists a (3, 3x)–HWP(3xy; r, s). It is easy to see that such a decomposition exists when $s \in \{0, 2, 3, \ldots, (3xy - 6x)/2\}$ by choosing $s_\alpha = s$ and $s_\beta = 0$. For each $i \in \{1, 2, \ldots, (6x - 2)/2\}$, when s = (3xy - 6x)/2 + i, choose $s_\alpha = s - (6x - 2)/2$ and $s_\beta = (6x - 2)/2$. Notice that

$$2 \le s_{\alpha} = \frac{3xy - 6x}{2} + i - \left(\frac{6x - 2}{2}\right) \le \frac{3xy - 6x}{2} + \left(\frac{6x - 2}{2}\right) - \left(\frac{6x - 2}{2}\right) \le \frac{3xy - 6x}{2}.$$

Therefore by Lemma 1.2.6, the proposed (3, 3x)-HWP(3xy; r, s) exists for all specified pairs (r, s).

Note that when x is even we cannot apply Theorem 7 to decompose $K_{(x:3)}$. Instead we can apply Theorem 4 to get a decomposition of $K_{(x:3)}$ into x C_3 -factors or a decomposition of $K_{(x:3)}$ into x C_{3x} -factors. In this way we can use Theorem 4 to decompose $K_{(x:3)}$ into $r_p C_3$ -factors and $s_p C_{3x}$ -factors, where $(r_p, s_p) \in \{(x, 0), (0, x)\}$.

Theorem 12 For each even integer $x \ge 8$ and each odd integer $y \ge 3$, there exists a (3, 3x)-HWP(3xy; r, s) if and only if $r + s = \frac{3xy-2}{2}$ except possibly when:

- $(s, x) \in \{(2, 12), (4, 12)\},\$
- $1 \le s \le \frac{x}{2} 1$ and $x \equiv 4 \pmod{6}$,
- s = 1 and $x \equiv 2 \pmod{12}$.

Proof: Suppose $x \ge 8$ is even. By Theorem 9, there exists a 3-RGDD(3^y) for all odd integers $y \ge 3$. By Theorem 4, for each $p \in \{1, 2, \ldots, \frac{3(y-1)}{2}\}$, $K_{(x:3)}$ can be decomposed into $r_p C_3$ -factors and $s_p C_{3x}$ -factors, where $(r_p, s_p) \in \{(x, 0), (0, x)\}$. By Theorem 5, there exists a decomposition of K_{3x} into $r_\beta C_3$ -factors and $s_\beta C_{3x}$ -factors and a 1-factor for $(r_\beta, s_\beta) \in \{(\frac{3x-2}{2}, 0), (\frac{3x-4}{2}, 1), \ldots, (0, \frac{3x-2}{2})\}$, except possibly when $(s_\beta, x) \in \{(2, 12), (4, 12)\}; 1 \le s_\beta \le \frac{x}{2} - 1$ and $x \equiv 4 \pmod{6}$; or $s_\beta = 1$ and $x \equiv 2 \pmod{12}$. We apply Lemma 1.2.6 to obtain a (3, 3x)-HWP(3xy; r, s) with $r = r_\alpha + r_\beta$ and $s = s_\alpha + s_\beta$ for all $s \in \{0, 1, \ldots, \frac{3xy-2}{2}\}$ (with the exceptions listed in the statement of this theorem) as follows. We may write $s_\alpha = \sum_{p=1}^{\frac{3(y-1)}{2}} s_p$ where $s_p \in \{0, x\}$, so that $s_\alpha \in \{0, x, 2x, \ldots, x \cdot \frac{3y-3}{2}\}$. Write $s = t \cdot x + i$, where $t \in \{0, 1, \ldots, \frac{3y-3}{2}\}$ and $i \in \{0, 1, \ldots, \frac{3x-2}{2}\}$. We may choose $s_\alpha = s - i$ and $s_\beta = i$.

Note that the cases of x = 2, 4 are not considered in the previous theorem. They will be handled in Section 1.3.1. We leave open the case of x = 6 and y odd.

Theorem 13 For each even integer $x \ge 8$ and each even integer $y \ge 8$, there exists a

(3, 3x)-HWP(3xy; r, s) if and only if $r + s = \frac{3xy-2}{2}$ except possibly when:

- $(s, x) \in \{(2, 12), (4, 12)\},\$
- $2 \le s \le \frac{x}{2} 1$ and $x \equiv 4$ or 10 (mod 12),
- s = 1 and $x \equiv 2, 4, \text{ or } 10 \pmod{12}$.

Proof: There exists a 3-RGDD($6^{y/2}$) for all even $y \ge 8$ by Theorem 9. There exists a decomposition of $K_{(x:3)}$ into $r_p \ C_3$ -factors and $s_p \ C_{3x}$ -factors for $(r_p, s_p) \in \{(0, x), (x, 0)\}$ by Theorem 4. By the same result, we also get a decomposition of $K_{(x:6)}$ into $r_{\gamma} \ C_3$ -factors and $s_{\gamma} \ C_{3x}$ -factors for $(r_{\gamma}, s_{\gamma}) \in \{(0, \frac{5x}{2}), (\frac{5x}{2}, 0)\}$. By Theorem 5, there exists a decomposition of K_{3x} into $r_{\beta} \ C_3$ -factors, $s_{\beta} \ C_{3x}$ -factors, and a 1-factor for $(r_{\beta}, s_{\beta}) \in \{\frac{3x-2}{2}, 0\}, (\frac{3x-4}{2}, 1), \dots, (0, \frac{3x-2}{2})\}$, except possibly when $(s_{\beta}, x) \in \{(2, 12), (4, 12)\}; 1 \le s_{\beta} \le \frac{x}{2} - 1$ and $x \equiv 4 \pmod{6}$; or $s_{\beta} = 1$ and $x \equiv 2 \pmod{12}$. Write $s_{\alpha} = \sum_{p=1}^{\frac{3y}{2}-4} s_p$ so $s_{\alpha} \in \{0, x, 2x, \dots, x(\frac{3y}{2} - 4)\}$. By Lemma 1.2.7, we obtain a (3, 12)-HWP(3xy; r, s) for all $s \in \{0, 1, \dots, \frac{3xy-2}{2}\}$ as follows. If $s \in \{0, 1, \dots, \frac{3xy}{2} - \frac{5x}{2} - 1\}$, it is easy to see that we can let $s_{\gamma} = 0$ and write s as $s = s_{\alpha} + s_{\beta}$. If $s = \frac{3xy}{2} - \frac{5x}{2} + i$, for $i = 0, 1, \dots, \frac{3x}{2} - 1$ choose $s_{\alpha} = (\frac{3y}{2} - 5)x$,

$$s_{\beta} = i$$
, and $s_{\gamma} = \frac{5x}{2}$. If $s = \frac{3xy}{2} - x + i$ for $i = 0, 1, \dots, x - 1$, choose $s_{\alpha} = (\frac{3y}{2} - 4)x$,
 $s_{\beta} = \frac{x}{2} + i$ and $s_{\gamma} = \frac{5x}{2}$.

We can fill in some of the gaps that we have left by using Theorem 8.

Theorem 14 For each odd integer $\bar{x} \ge 3$ and each even integer $y \ge 6$, there exists a $(3, 6\bar{x})$ -HWP $(6\bar{x}y; r, s)$ if and only if $r + s = \frac{6\bar{x}y-2}{2}$ except possibly when s = 1.

Proof: Assume that $y \equiv 2 \pmod{4}$ and $y \geq 6$. For all such y, there exists a 3-RGDD $(3^{\frac{y}{2}})$ by Theorem 9. There exists a $(3, 6\bar{x})$ -HWP $(12\bar{x}; r_{\beta}, s_{\beta})$ for all $(r_{\beta}, s_{\beta}) \in \{(0, \frac{12\bar{x}-2}{2}), (\frac{12\bar{x}-2}{2}, 0)\}$ by Theorem 3. By Theorem 8, we have that $K_{(4\bar{x}:3)}$ can be decomposed into $r_p C_3$ -factors and $s_p C_{6\bar{x}}$ -factors for $(r_p, s_p) \in \{(0, 4\bar{x}), (1, 4\bar{x} - 1), \ldots, (4\bar{x} - 2, 2), (4\bar{x}, 0)\}$. Apply Lemma 1.2.6 with $m = 3, n = 6\bar{x}$, and $x = 4\bar{x}$. Let $s_{\alpha} = \sum_{p=1}^{3(\frac{y}{2}-1)/2} s_p$, then it is easy to see that $s_{\alpha} \in \{0, 2, 3, \ldots, 3\bar{x}y - 6\bar{x}\}$. Write $s = s_{\alpha} + s_{\beta}$ where $s_{\alpha} \in \{0, 2, 3, \ldots, 3\bar{x}y - 6\bar{x}\}$ and $s_{\beta} \in \{0, 6\bar{x} - 1\}$. Then we can write s as $s_{\alpha} + s_{\beta}$ for every $s \in \{0, 2, 3, \ldots, \frac{6\bar{x}y-2}{2}\}$ in this way. Thus we can construct a $(3, 6\bar{x})$ -HWP $(6\bar{x}y; r, s)$ for all $s \in \{0, 1, \ldots, \frac{6\bar{x}y-2}{2}\}$.

Assume $y \equiv 0 \pmod{4}$, and $y \ge 12$. Then there exists a 3-RGDD($6^{\frac{y}{4}}$) by Theorem 9. There exists a decomposition of $K_{(4\bar{x}:3)}$ into $r_p \ C_3$ -factors and $s_p \ C_{6\bar{x}}$ -factors for $s_p \in \{0, 2, 3, \ldots, 4\bar{x}\}$ by Theorem 8. By Theorem 4, there exists a $(C_3, C_{6\bar{x}})$ -factorization of $K_{(4\bar{x}:6)}$ for $(r_{\gamma}, s_{\gamma}) \in \{(0, 10\bar{x}), (10\bar{x}, 0)\}$. There exists a $(3, 6\bar{x})$ -HWP($12\bar{x}; r_{\beta}, s_{\beta}$) for $s_{\beta} \in \{0, \frac{12\bar{x}-2}{2}\}$ by Theorem 3. Now we can easily write $s = s_{\alpha} + s_{\beta} + s_{\gamma}$ for By writing $x = 2\bar{x}$ Theorem 14 covers the cases when $s \neq 1$ and x = 6 and also some of the cases when $s \neq 1$ and $x \equiv 4 \pmod{6}$ (namely the ones where $x \equiv 10 \pmod{12}$). When $x \ge 6$ is even and $y \ge 8$ is even, the cases that are not covered by Theorems 13 and 14 are as follows:

- $(s, x) \in \{(2, 12), (4, 12)\},\$
- $2 \le s \le \frac{x}{2} 1$ and $x \equiv 4 \pmod{12}$,
- s = 1 and $x \equiv 2, 4, 10 \pmod{12}$.

Because there is no 3–RGDD(6^u) for $u \leq 3$, Lemmas 1.2.6 and 1.2.7 are not useful when $y \in \{2, 4, 6\}$. However, we still have some results. When y = 2 and x is even we may apply Lemma 1.3.1 to find a (3, 3x)–HWP(6x; r, s) for $s = s_1 + \frac{3x}{2}, r = r_1$, where (s_1, r_1) is a solution of the Hamilton-Waterloo Problem with triangles and Hamilton cycles for K_{3x} .

When y = 4 and $x \ge 2$ is even, consider K_{12x} . We can partition the vertices into four parts of size 3x. In the four copies of K_{3x} we have some solutions for the Hamilton-Waterloo Problem with triangles and Hamilton cycles. The remaining edges give us $K_{(3x:4)}$, which can be decomposed into all C_{3x} -factors or into all triangle factors. In this way we can get either all triangle factors, or $s = s_1 + e_1 \frac{9x}{2}$, $r = r_1 + e_2 \frac{9x}{2}$, where (s_1, r_1) is a solution of the Hamilton-Waterloo problem with triangles and Hamilton cycles for K_{3x} and $e_1 + e_2 = 1$, $e_1, e_2 \ge 0$. If y = 6 and x is even, consider K_{18x} . By following the same method, we can get either all triangle factors, or $s = s_1 + e_1 \frac{15x}{2}$, $r = r_1 + e_2 \frac{15x}{2}$, where (s_1, r_1) is a solution of the Hamilton-Waterloo Problem with triangles and Hamilton cycles for K_{3x} and $e_1 + e_2 = 1$, $e_1, e_2 \ge 0$.

1.3.1 When x is Small

In this subsection, we consider the small values of x for which the general constructions used in Section 1.3 cannot be readily applied. By applying the methods described at the end of Section 1.3, it is easy to see that the following decompositions exist when x = 2: a (3,6)-HWP(24; r, s) for $s \in \{0, 1, 2, 7, 8, 9, 10, 11\}$, and a (3,6)-HWP(48; r, s) for $s \in \{0, 1, 2, 3, 4, 5, 12, 13, 14, 19, 20, 21, 22, 23\}$. The following three results gives solutions to the Hamilton-Waterloo Problem, (3, 3x)-HWP(3xy; r, s), for all other values of y when x = 2.

Theorem 15 There exists a (3,6)-HWP(6y;r,s) for all $y \equiv 2 \pmod{4}$ if and only if $r + s = \frac{6y-2}{2}$, except when y = 2 and s = 0.

Proof: If y = 2, then there exists a (3, 6)-HWP(12; r, s) for all possible r and s except when s = 0 by Theorem 6. We now assume that $y \equiv 2 \pmod{4}$ and

 $y \ge 6$. For all such y, there exists a 3-RGDD $(3^{\frac{y}{2}})$ by Theorem 9. There exists a (3,6)-HWP $(12; r_{\beta}, s_{\beta})$ for all $(r_{\beta}, s_{\beta}) \in \{(0,5), (1,4), (2,3), (3,2), (4,1)\}$ by Theorem 6. By Lemma 1.2.4, we have that $K_{(4:3)}$ can be decomposed into r_p C_3 -factors and s_p C_6 -factors for $(r_p, s_p) \in \{(0, 4), (1, 3), (2, 2), (4, 0)\}$. Apply Lemma 1.2.6 with m = 3, n = 6, and x = 4. Let $s_{\alpha} = \sum_{p=1}^{3(\frac{y}{2}-1)/2} s_p$, then it is easy to see that $s_{\alpha} \in \{0, 2, 3, \dots, 3y - 6\}$. Write $s = s_{\alpha} + s_{\beta}$ where $s_{\alpha} \in \{0, 2, 3, \dots, 3y - 6\}$ and $s_{\beta} \in \{1, 2, 3, 4, 5\}$. Then we can write s as $s_{\alpha} + s_{\beta}$ for every $s \in \{1, 2, \dots, \frac{6y-2}{2}\}$ in this way. If s = 0, then there exists a (3, 6)-HWP(6y; r, s) by Theorem 3. Thus we can construct a (3, 6)-HWP(6y; r, s) for all $s \in \{0, 1, \dots, \frac{6y-2}{2}\}$.

Theorem 16 There exists a (3,6)-HWP(6y;r,s) for all $y \equiv 0 \pmod{4}$ if and only if $r + s = \frac{6y-2}{2}$, except possibly when y = 4 or y = 8.

Proof: Assume $y \equiv 0 \pmod{4}$, and $y \geq 12$. Then there exists a 3-RGDD $(6^{\frac{y}{4}})$ by Theorem 9. There exists a decomposition of $K_{(4:3)}$ into $r_p C_3$ -factors and $s_p C_6$ -factors for $s_p \in \{0, 2, 3, 4\}$ by Lemma 1.2.4. By Theorem 4, there exists a (C_3, C_6) -factorization of $K_{(4:6)}$ for $(r_{\gamma}, s_{\gamma}) \in \{(0, 10), (10, 0)\}$. There exists a (3, 6)-HWP $(12; r_{\beta}, s_{\beta})$ for $s_{\beta} \in \{1, 2, 3, 4, 5\}$ by Theorem 6. Now we can easily write $s = s_{\alpha} + s_{\beta} + s_{\gamma}$ for $s \in \{0, 1, \ldots, 3y - 1\}$ and apply Lemma 1.2.7.

Theorem 17 There exists a (3, 6)-HWP(6y; r, s) when y is odd and $s \in \{1, 2, \frac{3(y-1)}{2} + 1, \frac{3(y-1)}{2} + 2, \dots, 3y - 1\}.$

Proof: If y = 1, then exists a (3, 6)-HWP(6; r, s) for all possible r and s except for (r, s) = (2, 0) by Theorem 6. Assume $y \ge 3$ is odd, then there exists a 3-RGDD (3^y) by Theorem 9. There exists a (3, 6)-HWP $(6; r_\beta, s_\beta)$ for $(r_\beta, s_\beta) \in \{(1, 1), (0, 2)\}$ by Theorem 6. It is easy to see that $K_{(2:3)}$ can be decomposed into a C_3 -factor and a C_6 -factor or two C_6 -factors. Apply Lemma 1.2.6 with m = 3, n = 6 and x = 2. Let $s_\alpha = \sum_{p=1}^{3(y-1)/2} s_p$ with $s_p \in \{1, 2\}$ and notice that $s_\alpha \in \{\frac{3(y-1)}{2}, \frac{3(y-1)}{2} + 1, \dots, 3(y-1)\}$. Then we can write s as $s_\alpha + s_\beta$ for every $s \in \{\frac{3(y-1)}{2} + 1, \frac{3(y-1)}{2} + 2, \dots, 3y-1\}$. Thus we obtain a (3, 6)-HWP(6y; r, s) for all such s. We can also obtain a (3, 6)-HWP(6y; r, s) for s = 1 and s = 2 as follows. There exists a 3-RGDD (6^y) by Theorem 9; it has 3(y - 1) parallel classes. There exists a (3, 6)-HWP $(6; r_\beta, s_\beta)$ for $s_\beta \in \{1, 2\}$. Apply Lemma 1.2.6 with m = 3, n = 6 and x = 1, and write $s = s_\alpha + s_\beta$ with $s_\alpha = 0$ and $s_\beta = 1$ or $s_\beta = 2$.

Recall from Theorem 6 that there exists a (3, 12)-HWP $(12; r_{\delta}, s_{\delta})$ if and only if $s_{\delta} \in \{1, 2, 3, 4, 5\}$. For each possible decomposition of K_{12} , let $s_{\beta} = s_{\delta} + 6$, and apply Lemma 1.3.1 to obtain a (3, 12)-HWP(24; r, s) for all $s \in \{7, 8, 9, 10, 11\}$. If s = 0, then simply apply Theorem 3. Similarly, apply Theorem 3 to obtain a (3, 12)-HWP(48; r, s) for s = 0. Consider the equipartite graph $K_{(12:4)}$. It has a C_{12} -factorization and a C_3 -factorization by Theorem 4. On each part, construct a (3, 12)-HWP(12; r, s) for $s \in \{1, 2, 3, 4, 5\}$. Thus we have a (3, 12)-HWP(48; r, s) for $s \in \{0, 1, 2, 3, 4, 5, 19, 20, 21, 22, 23\}$. The next theorem settles the Hamilton-Waterloo Problem, (3, 3x)-HWP(3xy; r, s) when x = 4 for the remaining values of y.

Theorem 18 For y = 3 and all $y \ge 5$, there exists a (3, 12)-HWP(12y; r, s) if and only if $r + s = \frac{v-2}{2}$.

Proof: Let $y \ge 6$ be even. There exists a 3-RGDD($6^{y/2}$) by Theorem 9. There exists a decomposition of $K_{(4:3)}$ into $r_p \ C_3$ -factors and $s_p \ C_{12}$ -factors for $(r_p, s_p) \in \{(0, 4), (4, 0)\}$ by Lemma 4. By the same result, we also get a decomposition of $K_{(4:6)}$ into $r_\gamma \ C_3$ -factors and $s_\gamma \ C_{12}$ -factors for $(r_\gamma, s_\gamma) \in \{(0, 10), (10, 0)\}$. Recall that there exists a (3, 12)-HWP($12; r_\beta, s_\beta$) for $(r_\beta, s_\beta) \in \{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1)\}$ by Theorem 6. Write $s_\alpha = \sum_{p=1}^{\frac{3y}{2}-4} s_p$ so $s_\alpha \in \{0, 4, 8, \dots, 6y - 16\}$. By Lemma 1.2.7, we obtain a (3, 12)-HWP(3xy; r, s) for all $s \in \{0, 1, \dots, 6y - 1\}$ as follows. If s = 0, apply Theorem 3. If $s \in \{1, 2, \dots, 6y - 11\}$, it is easy to see that we can let $s_\gamma = 0$ and write s as $s = s_\alpha + s_\beta$. If s = 6y - 10, choose $s_\alpha = 6y - 24$, $s_\beta = 4$, and $s_\gamma = 10$. If s = 6y - i for i = 9, 8, 7, 6, choose $s_\alpha = 6y - 20$, $s_\beta = 10 - i$ and $s_\gamma = 10$.

If $y \ge 3$ is odd, there exists a 3-RGDD(3^y) by Lemma 9. There exists a decomposition of $K_{(4:3)}$ into $r_p \ C_3$ -factors and $s_p \ C_{12}$ -factors for $(r_p, s_p) \in \{(0, 4), (4, 0)\}$ by Theorem 4. Write $s_{\alpha} = \sum_{p=1}^{\frac{3(y-1)}{2}} s_p$, so $s_{\alpha} \in \{0, 4, 8, \dots, 6(y-1)\}$. Recall the existence of a (3, 12)-HWP($12; r_{\beta}, s_{\beta}$) for $s_{\beta} \in \{1, 2, 3, 4, 5\}$. Then it is easy to see that we can write s as $s_{\alpha} + s_{\beta}$ for all $s \in \{0, 1, 2, \dots, 6y - 1\}$. Thus we may apply Lemma 1.2.6 for the result.
1.4 Conclusions

The following theorem combines the results from Theorems 10, 11, 12, 13, 14, 15, 16, 17, and 18 (note that we did not include all of the small partially complete results such as those at the end of Section 1.3):

Theorem 19 Let $x \ge 2$, $y \ge 2$, and $r, s \ge 0$ such that $r + s = \lfloor \frac{3xy-1}{2} \rfloor$. Then there exist a (3, 3x)-HWP(3xy; r, s) except possibly when:

- $s = 1, y \ge 3$, and $x \in \{3, 31, 37, 41, 43, 47, 51, 53, 59, 61, 67, 69, 71, 79, 83\}.$
- s = 1, x is odd and y is even.
- $s = 1, x \ge 6, x \equiv 2 \pmod{12}$.
- $s = 1, y \ge 8$ is even and $x \equiv 10 \pmod{12}$.
- $s = 1, x \ge 3$ is odd and y is even.
- $1 \le s \le \frac{x}{2} 1$, $x \ge 16$, $x \equiv 4 \pmod{12}$, y is even.
- $1 \le s \le \frac{x}{2} 1$, $x \ge 10$, $x \equiv 4 \pmod{6}$, y is odd.
- $(s, x) \in \{(2, 12), (4, 12)\}.$
- s = 0, x = 2, y = 2.
- x = 2 and $y \in \{4, 8\}$.
- $s \in \{3, 4, \dots, \frac{3(y-1)}{2}\}, x = 2 \text{ and } y \ge 3 \text{ is odd.}$
- $x \notin \{2,4\}$ and $y \in \{2,4,6\}$.

- x = 4 and $y \in \{2, 4\}$.
- x = 6 and y odd.

Chapter 2

A Generalization of the Hamilton-Waterloo Problem on Complete Equipartite Graphs¹

2.1 Introduction

The Oberwolfach Problem was first posed by Ringel in 1967 during a conference in Oberwolfach. The question was whether it was possible to seat the v conference attendees at n round tables for dinner during $\frac{v-1}{2}$ nights, in such a way that every

¹The material in this chapter has been submitted to Journal of Combinatorial Designs

attendee sits next to every other attendee exactly once. This is equivalent to asking whether the complete graph K_v can be decomposed into $\frac{v-1}{2}$ copies of a 2-factor F(in a 2-factor every component is a cycle, which represents a round table). To achieve this decomposition v needs to be odd, because the vertices (attendees) need to have even degree. Later a version with v even was studied. In this case, the attendees will never sit next to their spouses (and we are assuming that every attendee has a spouse). This is equivalent to asking for a decomposition of K_v into $\frac{v-2}{2}$ copies of a 2-factor F, and one copy of a 1-factor (each attendee together with their spouse).

In [28] Liu first worked on the generalization of the Oberwolfach problem, where instead of avoiding their spouses, the attendees avoid all the other members of their delegation. The assumption was that all the delegations had the same number of people. Thus we are seeking to decompose the complete equipartite graph $K_{(m:n)}$ with n partite sets (delegations) of size m each (members of a delegations) into $\frac{(n-1)m}{2}$ copies of a 2-factor F. Here (n-1)m has to be even. In [22] Hoffman and Holliday worked on the equipartite generalization of the Oberwolfach problem when (n-1)mis odd, decomposing into $\frac{(n-1)m-1}{2}$ copies of a 2-factor F, and one copy of a 1-factor.

The Hamilton-Waterloo problem is a generalization of the Oberwolfach problem, in which the conference is being held at two different cities. Because the table arrangements are different, we have two 2-factors, F_1 and F_2 . The Hamilton-Waterloo problem then asks whether the complete graph K_v can be decomposed into r copies of the 2-factor F_1 (tables at Hamilton) and s copies of the 2-factor F_2 (tables at Waterloo), such that $s + r = \frac{v-1}{2}$, when v is odd, and having $s + r = \frac{v-2}{2}$ and a 1-factor when v is even.

The uniform Oberwolfach problem (when all the tables have the same size, i.e. all the cycles of the 2-factor have the same size) has been completely solved by Alspach and Haagkvist [2] and Alspach, Schellenberg, Stinson and Wagner [3]. For the non-uniform case of the Oberwolfach problem, many results have been obtained. For a survey of results up to 2006 see [12]. The uniform Oberwolfach problem over equipartite graphs has been completely solved by Liu [29] and Hoffman and Holliday [22]. In the non-uniform case Bryant, Danziger and Pettersson [11] completely solved the case when the 2-factor is bipartite. For the Hamilton-Waterloo problem most of the results are uniform, see for example [5] or [13]. In particular, Burgess, Danziger and Traetta [13] proved the following theorem.

Theorem 20 [13] If m and n are odd integers with $n \ge m \ge 3$ and t > 1, then there is a decomposition of K_{mnt} into $s \ C_m$ -factors and $r \ C_n$ -factors if and only if t is odd, $s, r \ge 0$ and s + r = (mnt - 1)/2, except possibly when r = 1 or 3, or (m, n, r) = (5, 9, 5), (5, 9, 7), (7, 9, 5), (7, 9, 7), (3, 13, 5).

Theorem 20 covers most of the odd ordered uniform cases. The authors in [13] point out that it is possible to have solutions where the number of vertices is not a multimple of mn. Thus if l = lcm(m, n) and the number of vertices is a multiple of l, but not divisible by mn, then Theorem 20 cannot be applied. The constructions given in this paper can be applied to cover some of these cases.

There are some results in the non-uniform case, some examples are Bryant, Danzinger [9], Bryant, Danzinger, Dean [10] and Haggkvist [20].

The Hamilton-Waterloo problem can be generalized for complete equipartite graphs in the same way as the Oberwolfach problem was generalized, but not much work has been done in this direction. Asplund, Kamin, Keranen, Pastine and Özkan [5] gave some constructions for complete equipartite graphs with 3 parts. Burgess, Danziger and Traetta [13] studied the case when the graph consists of m partite sets of size n, and the cycle sizes are m and n. In both papers the constructions were done in order to get a result on the Hamilton-Waterloo problem for complete graphs. The focus of this paper is to give a generalization of the Hamilton-Waterloo problem for complete equipartite graphs with an odd number of partite sets. We obtain results both in the uniform and non-uniform cases.



Figure 2.1: A multipartite graph



Figure 2.2: A multipartite graph labeled according to its partite sets

2.2 Basic Definitions and Results

Let G be a multipartite graph with k partite sets, $G_0, G_1, \ldots, G_{k-1}$. We identify each vertex g of G as an ordered pair (g, i), where $g \in G_i$.

Example 9: Consider the graph in Figure 2.1. If we consider each column as a partite set, we have 3 partite sets; G_0 , with vertices a, d and f; G_1 with vertices b and e; and G_2 with c as its only vertex. Then the vertices are (a, 0), (b, 1), (c, 2), (d, 0), (e, 1), (f, 0).

When it is convenient, we will just denote the vertices (0,0), (0,1), (0,2), (1,0), (1,1), (2,0), as in Figure 2.2.

Definition 2.2.1 Let G and H be multipartite graphs. Then we define the partite product of G and H, $G \otimes H$ as follows:

- $V(G \otimes H) = \{(g, h, i) | (g, i) \in V(G) \text{ and } (h, i) \in V(H) \}.$
- $E(G \otimes H) = \{\{(g_1, h_1, i), (g_2, h_2, j)\} | \{(g_1, i), (g_2, j)\} \in E(G) \text{ and} \{(h_1, i), (h_2, j)\} \in E(H)\}.$

Notice that this definition is quite similar to that of the direct product. The main difference is that we are doing this product "just in 1 coordinate". To see that they are different it suffices to count the number of vertices in the product. If $G = H = K_{3,3,3}$ then $|V(G \times H)| = 81$ but $|V(G \otimes H)| = 27$.

Indeed, if the k partite sets of G and H have sizes $g_0, g_1, \ldots, g_{k-1}$ and $h_0, h_1, \ldots, h_{k-1}$, respectively, then $|V(G \otimes H)| = g_0 h_0 + g_1 h_1 + \cdots + g_{k-1} h_{k-1}$, whereas $|V(G \times H)| = (\sum g_i) (\sum h_i)$.

Remark 2.2.2 The partite product depends on the multipartite representation chosen for a graph. For example, the graphs G and H in Figure 2.3 are isomorphic, but they behave differently in the product (where we understand that each column is a part of the multipartite graph).

The next result follows directly from Definition 2.2.1







Figure 2.3: Example of the partite product

Lemma 2.2.3 The product is commutative, that is, $G \otimes H = H \otimes G$.

Most of our results will be concerning complete multipartite graphs. We will denote by $K_{(n:m)}$ the complete multipartite graph with m parts, each of size n. **Lemma 2.2.4** If G is k-partite, then $G \otimes K_{(1:k)}$ is isomorphic to G.

Proof: Here each part of $K_{(1:k)}$ has just 1 vertex and so $|V(G)| = |V(G \otimes K_{(1:k)})|$. Because all the vertices of $K_{(1:k)}$ are neighbors, two vertices $(g_1, k_1, i), (g_2, k_2, j)$ of $G \otimes K_{(1:k)}$ are neighbors if and only if (g_1, i) and (g_2, j) are neighbors. Therefore $G \otimes K_{(1:k)}$ is isomorphic to G.

Definition 2.2.5 The complete cyclic multipartite graph $C_{(x:k)}$ is the graph with k parts of size x, where two vertices (g, i) and (h, j) are neighbors if and only if $i - j = \pm 1 \pmod{k}$, with this subtraction being done modulo k. The directed complete cyclic multipartite graph $\overrightarrow{C}_{(x:k)}$ is the graph with k parts of size x, with arcs of the form ((g,i), (h, i + 1)) for every $0 \le g, h \le x - 1, 0 \le i \le k - 1$.

It should be noted that any decomposition of $\overrightarrow{C}_{(x:k)}$ gives a decomposition of $C_{(x:k)}$. Notice that $C_{(1:k)}$ is the cycle with k vertices and $C_{(x:3)}$ is isomorphic to $K_{(x:3)}$. The next three results are easy to see, so the proofs are left to the reader.

Lemma 2.2.6 Let G and H be k-partite graphs. If each part of G has $\frac{|V(G)|}{k}$ vertices and each part of H has $\frac{|V(H)|}{k}$ vertices, then:

- Each part of $G \otimes H$ has $\frac{|V(G)| \times |V(H)|}{k^2}$ vertices.
- $|V(G \otimes H)| = \frac{|V(G)| \times |V(H)|}{k}$.

Lemma 2.2.7 • $K_{(x:k)} \otimes K_{(y:k)}$ is isomorphic to $K_{(xy:k)}$.

- $C_{(x:k)} \otimes C_{(y:k)}$ is isomorphic to $C_{(xy:k)}$.
- $\overrightarrow{C}_{(x:k)} \otimes \overrightarrow{C}_{(y:k)}$ is isomorphic to $\overrightarrow{C}_{(xy:k)}$.

Lemma 2.2.8 The complete cyclic multipartite graph is the product of the complete multipartite graph by the cycle. This is: $K_{(x:k)} \otimes C_{(1:k)} = C_{(x:k)}$.

2.3 **Product and Decompositions**

We can consider a decomposition of a graph as a partition of the edge set or as a union of edge disjoint subgraphs. This means that a decomposition of G into H_1, \ldots, H_s is given by $E(G) = \bigcup E(H_i)$ or by $G = \bigoplus H_i$. We will think of \bigoplus as a boolean sum, which means that $H_i \oplus H_i = \emptyset$.

We have the following easy result.

Theorem 21 (Distribution) Let $G = \bigoplus_i G_i$ and $H = \bigoplus_j H_j$ be k-partite graphs. Then $G \otimes H = (\bigoplus_i G_i) \otimes (\bigoplus_j H_j)$. Furthermore, the following distributive property holds:

$$(\oplus_i G_i) \otimes (\oplus_j H_j) = \oplus_i (G_i \otimes \oplus_j H_j) = \oplus_i \oplus_j (G_i \otimes H_j)$$

Proof: It is enough to prove that

$$G \otimes (H_1 \oplus H_2) = (G \otimes H_1) \oplus (G \otimes H_2),$$

where $E(H_1) \cap E(H_2) = \emptyset$.

Let

$$\{(g_1, h_1, i), (g_2, h_2, j)\} \in E(G \otimes (H_1 \oplus H_2)).$$

This means that

$$\{(h_1, i), (h_2, j)\} \in E(H_1) \cup E(H_2).$$

But since $E(H_1) \cap E(H_2) = \emptyset$, without loss of generality we may assume

$$\{(h_1, i), (h_2, j)\} \in E(H_1),\$$

and so

$$\{(g_1, h_1, i), (g_2, h_2, j)\} \in E(G \otimes H_1) \subset E((G \otimes H_1) \oplus (G \otimes H_2)).$$

Hence

$$E(G \otimes (H_1 \oplus H_2)) \subset E((G \otimes H_1) \oplus (G \otimes H_2)).$$

Let now

$$\{(g_1, h_1, i), (g_2, h_2, j)\} \in E((G \otimes H_1) \oplus (G \otimes H_2)).$$

This means that

$$\{(g_1, h_1, i), (g_2, h_2, j)\} \in E((G \otimes H_1)) \cup E((G \otimes H_2)).$$

Without loss of generality we may assume

$$\{(g_1, h_1, i), (g_2, h_2, j)\} \in E((G \otimes H_1)),\$$

which gives

$$\{(h_1, i), (h_2, j)\} \in E(H_1) \subset E(H_1 \oplus H_2), \text{ and}$$

 $\{(g_1, h_1, i), (g_2, h_2, j)\} \in E(G \otimes (H_1 \oplus H_2)).$

Hence

$$E\left(\left(G\otimes H_1\right)\oplus \left(G\otimes H_2\right)\right)\subset E\left(G\otimes \left(H_1\oplus H_2\right)\right).$$

Therefore

$$G \otimes (H_1 \oplus H_2) = (G \otimes H_1) \oplus (G \otimes H_2),$$

and by induction we get that the product and additions in

$$G \otimes H = (\oplus_i G_i) \otimes (\oplus_j H_j)$$

are distributive.

Corollary 2.3.1 Let G and H be multipartite graphs with k partite sets.

- If G can be decomposed into isomorphic copies of Γ and H can be decomposed into isomorphic copies of K_(1:k) = K_k, then G ⊗ H can be decomposed into isomorphic copies of Γ.
- If G can be factored into isomorphic copies of Γ and H can be factored into unions of copies of K_(1:k) = K_k, then G ⊗ H can be factored into unions of copies of Γ.

Proof: If G is decomposed into copies of Γ , it means that $G = \bigoplus G_i$, where each G_i is isomorphic to Γ . If H is decomposed into copies of $K_{(1:k)}$ (or union of them), it means that $H = \bigoplus H_i$, where each H_i is isomorphic to $K_{(1:k)}$ (or union of them).

By the Distribution Theorem we only need to show that $G_i \otimes H_i$ is isomorphic to Γ or to G_i . But by Lemma 2.2.4 we know this is true.

It is interesting to notice that the set of k-partite graphs, with \oplus as a sum and \otimes as

a product form a commutative ring. The empty graph is the 0 element, and $K_{(1:k)}$ is the 1 element. All of the elements are additive involutions, Theorem 21 gives us the distribution laws, and Lemma 2.2.3 shows that the product is commutative.

2.4 Product of Cycles

In this section we will concern ourselves with the product of two or more cycles. Since our product depends on what kind of partition we are using, we need to ask something more from our cycles in order to get results.

Definition 2.4.1 Given a graph G we will say that C is a C_n -factor of G if C is a 2-factor of G where each connected component is of size n. This means that C is a spanning subgraph of G and C is a union of disjoint cycles of size n. When it is understood that the graph is G, then we will just call C a C_n -factor (instead of a C_n factor of G). Similarly given a directed graph \overrightarrow{G} we will say that \overrightarrow{C} is a \overrightarrow{C}_n -factor of G if \overrightarrow{C} is a 2-factor of \overrightarrow{G} where each connected component is a directed cycle of size n. When it is understood that the graph is \overrightarrow{G} , the we will just call \overrightarrow{C} a \overrightarrow{C}_n -factor.

The following lemmas give us an idea of how directed cycles work under the product. They also illustrate why we introduce $\overrightarrow{C}_{(x:k)}$ instead of just working with $C_{(x:k)}$. **Lemma 2.4.2** Let \overrightarrow{C} be a directed cycle of length n of $\overrightarrow{C}_{(x:k)}$, and let \overrightarrow{C}' be a directed cycle of length m of $\overrightarrow{C}_{(y:k)}$. Then $\overrightarrow{C} \otimes \overrightarrow{C}'$ is a set of $\frac{\operatorname{gcd}(n,m)}{k}$ disjoint directed cycles of $\overrightarrow{C}_{(xy:k)}$ of length $l = \frac{nm}{\operatorname{gcd}(n,m)}$ and $xyk - \frac{nm}{k}$ isolated vertices.

Proof: Notice that \overrightarrow{C} has xk - n isolated vertices, because it is a subgraph of $\overrightarrow{C}_{(x:k)}$. Likewise, \overrightarrow{C}' has yk - m isolated vertices.

Let (x_0, y_0, i) be a vertex in $\overrightarrow{C} \otimes \overrightarrow{C'}$. If either (x_0, i) or (y_0, i) are isolated vertices, then (x_0, y_0, i) is isolated. If neither (x_0, i) and (y_0, i) are isolated, then they respectively have an arrow coming from $(x_1, i-1)$ and $(y_1, i-1)$; and an arrow going to $(x_2, i+1)$ and $(y_2, i+1)$, for some $0 \le x_1, x_2 \le x - 1, 0 \le y_1, y_2 \le y - 1$. Hence (x_0, y_0, i) has an arrow coming from $(x_1, y_1, i-1)$ and an arrow going to $(x_2, y_2, i+1)$. So $\overrightarrow{C} \otimes \overrightarrow{C'}$ is composed of directed cycles and isolated vertices.

Assume without loss of generality $n \leq m$. Let $i_0, i_1, \ldots, i_{n-1}$ be the non-isolated vertices of \overrightarrow{C} , in the order they appear, with $i_0 \in \overrightarrow{C}_0$; and let $j_0, j_1, \ldots, j_{m-1}$ be the non-isolated vertices of \overrightarrow{C}' , in the order they appear with $j_0 \in \overrightarrow{C}'_0$. Then the directed cycle starting at (i_0, j_0) in $\overrightarrow{C} \otimes \overrightarrow{C}'$ consists of the vertices:

$$(i_0, j_0), (i_1, j_1), \dots, (i_{n-1}, j_{n-1}), (i_0, j_n), \dots, (i_{n-1}, i_{m-1})$$

This directed cycle has length $l = \frac{nm}{gcd(n,m)}$. Notice that $\overrightarrow{C} \otimes \overrightarrow{C}'$ has $\frac{nm}{k}$ non-isolated vertices, which means that the number of directed cycles is $\frac{nm}{k} \frac{gcd(n,m)}{nm} = \frac{gcd(n,m)}{k}$. Thus

 $\overrightarrow{C} \otimes \overrightarrow{C}'$ is a set of $\frac{\gcd(n,m)}{k}$ disjoint directed cycles of $\overrightarrow{C}_{(xy:k)}$ of length $l = \frac{nm}{\gcd(n,m)}$ and $xyk - \frac{nm}{k}$ isolated vertices.

Lemma 2.4.3 Let \overrightarrow{G} and \overrightarrow{H} be a \overrightarrow{C}_n -factor and a \overrightarrow{C}_m -factor of $\overrightarrow{C}_{(x:k)}$ and $\overrightarrow{C}_{(y:k)}$, respectively. Then $\overrightarrow{G} \otimes \overrightarrow{H}$ is a \overrightarrow{C}_l -factor of $\overrightarrow{C}_{(xy:k)}$, where $l = \frac{nm}{gcd(n,m)}$.

Proof: Notice that neither \overrightarrow{G} nor \overrightarrow{H} have isolated vertices. Let (x_0, y_0, i) be a vertex in $\overrightarrow{G} \otimes \overrightarrow{H}$. We know that (x_0, i) has an arrow coming from $(x_1, i - 1)$ and an arrow going to $(x_2, i + 1)$, for exactly one pair $0 \leq x_1, x_2 \leq x - 1$, because \overrightarrow{G} is a \overrightarrow{C}_n factor. Likewise (y_0, i) has has an arrow coming from $(y_1, i - 1)$ and an arrow going to $(y_2, i + 1)$, for exactly one pair $0 \leq y_1, y_2 \leq y - 1$. Hence (x_0, y_0, i) has an arrow coming from $(x_1, y_1, i - 1)$ and an arrow going to $(x_2, y_2, i + 1)$, so each vertex in $\overrightarrow{G} \otimes \overrightarrow{H}$ is in exactly one directed cycle.

Let $\overrightarrow{G} = \bigoplus_i \overrightarrow{C}(i)$ and $\overrightarrow{H} = \bigoplus_j \overrightarrow{C}'(j)$, where each $\overrightarrow{C}(i)$ is a directed cycle of length n, and each $\overrightarrow{C}'(j)$ is a directed cycle of length m.

Then by Theorem 21 we get:

$$\overrightarrow{G} \otimes \overrightarrow{H} = \left(\oplus_i \overrightarrow{C}(i) \right) \otimes \left(\oplus_j \overrightarrow{C}'(j) \right)$$
$$= \oplus_i \oplus_j \left(\overrightarrow{C}(i) \otimes \overrightarrow{C}'(j) \right)$$

But by Lemma 2.4.2 we know that $\overrightarrow{C}(i) \otimes \overrightarrow{C}'(j)$ is composed of $\frac{\gcd(n,m)}{k}$ directed cycles of length $l = \frac{nm}{gcd(n,m)}$. Hence each directed cycle in $\overrightarrow{G} \otimes \overrightarrow{H}$ has size $l = \frac{nm}{gcd(n,m)}$ and $\overrightarrow{G} \otimes \overrightarrow{H}$ is a \overrightarrow{C}_l -factor.

Definition 2.4.4 Given a graph G we will say that F is a $[n_1^{e_1}, n_2^{e_2}, \ldots, n_p^{e_p}]$ -factor of G if F is a 2-factor of G with e_i connected components of size n_i , $i = 1, 2, \ldots, p$.

If e_i is not listed, we will just assume that it is 1. Also, we allow $n_i = n_j$, so that the number of cycles of a certain size is just the sum of the exponents of that number in the expression $[n_1^{e_1}, \ldots, n_p^{e_p}]$.

Example 10: A [3², 3³, 5², 11, 13]-factor is a subgraph of a graph on 59 vertices, consisting of 5 cycles of size 3, 2 cycles of size 5, 1 cycle of size 11 and 1 cycle of size 13. This subgraph can also be written as a [3, 3, 3, 3, 3, 5, 5, 11, 13]-factor or as a [3⁵, 5², 11, 13]-factor.

Notice that a C_n -factor of a graph on m vertices is a $[n^{\frac{m}{n}}]$ -factor.

2.5 From $C_{(v:n)}$ to $K_{(v:m)}$

It is advantageous to find solutions to the Hamilton-Waterloo problem on complete multipartite graphs because they can then be used to obtain solutions to the Hamilton-Waterloo problem on complete graphs:

Lemma 2.5.1 Let m, n, x, y and v be positive integers. Suppose the following conditions are satisfied:

- There exists a decomposition of K_v into $s_{\alpha} C_{xn}$ -factors and $r_{\alpha} C_{yn}$ -factors.
- There exists a decomposition of $K_{(v:m)}$ into $s_{\beta} C_{xn}$ -factors and $r_{\beta} C_{yn}$ -factors.

Then there exists a decomposition of K_{vm} into $s = s_{\alpha} + s_{\beta} C_{xn}$ -factors and $r = r_{\alpha} + r_{\beta}$ C_{yn} -factors.

Proof: Partition the vertices of K_{vm} into m sets A_1, \ldots, A_m of size v each. The graph that contains the edges between vertices belonging to a same partite set is the union of m disjoint copies of K_v . We can decompose each copy of K_v into $s_{\alpha} C_{xn}$ -factors and $r_{\alpha} C_{yn}$ -factors. The graph that contains the edges between vertices belonging to different parts is isomorphic to $K_{(v:m)}$. We can decompose this graph into s_{β} C_{xn} -factors and $r_{\beta} C_{yn}$ -factors. Thus we have a decomposition of K_{vm} into $s = s_{\alpha} + s_{\beta} C_{xn}$ -factors and $r = r_{\alpha} + r_{\beta}$ C_{yn} -factors.

One could write a version of the lemma for non-uniform solutions as follows, where by mK_v we understand the graph consisting of m disconnected copies of K_v .

Lemma 2.5.2 Let m, and v be positive integers. Let F_1 and F_2 be two 2-factors on vm vertices. Suppose the following conditions are satisfied:

- There exists a decomposition of mK_v into s_α copies of F_1 and r_α copies of F_2 .
- There exists a decomposition of $K_{(v:m)}$ into s_{β} copies of F_1 and r_{β} copies of F_2 .

Then there exists a decomposition of K_{vm} into $s = s_{\alpha} + s_{\beta}$ copies of F_1 and $r = r_{\alpha} + r_{\beta}$ copies of F_2 .

In order to use these lemmas we need two types of ingredients. The first one is the decomposition of the complete graph K_v . In [2, 3] uniform decompositions were given:

Theorem 22 [2, 3] There exists a decomposition of K_v into C_n -factors if and only if $v \equiv 0 \pmod{n}$, $(v, n) \neq (6, 3)$ and $(v, n) \neq (12, 3)$.

The other ingredient is the decomposition of the complete multipartite graph $K_{(v:m)}$. In [13] the authors used decompositions of $C_{(v:m)}$ to obtain decompositions of $K_{(v:m)}$. We describe this type of construction in a formal fashion in the following lemma.

Lemma 2.5.3 Let $m, x_1, \ldots, x_p, y_1, \ldots, y_p$, and v be positive integers. Let $s_1, \ldots, s_{\frac{m-1}{2}}$ be non-negative integers. Suppose the following conditions are satisfied:

- There exists a decomposition of K_m into $[n_1, \ldots, n_p]$ -factors.
- For every $1 \le i \le p$, and for every $1 \le t \le \frac{m-1}{2}$ there exists a decomposition of $C_{(v:n_i)}$ into $s_t \ C_{x_in_i}$ -factors and $r_t \ C_{y_in_i}$ -factors.

Let

$$s = \sum_{t=1}^{\frac{(m-1)}{2}} s_t$$
 and $r = \sum_{t=1}^{\frac{(m-1)}{2}} r_t$

Then there exists a decomposition of $K_{(v:m)}$ into $s [(x_1n_1)^{\frac{v}{x_1}}, \ldots, (x_pn_p)^{\frac{v}{x_p}}]$ -factors and $r [(y_1n_1)^{\frac{v}{y_1}}, \ldots, (y_pn_p)^{\frac{v}{y_p}}]$ -factors.

Proof: Using the decomposition of K_m into $[n_1, n_2, \ldots, n_p]$ -factors, we have:

$$K_{(1:m)} = K_m = \bigoplus_{t=1}^{\frac{m-1}{2}} N_t$$

where each N_t is a $[n_1, n_2, \ldots, n_p]$ -factor.

We know that $K_{(v:m)} = K_{(v:m)} \otimes K_{(1:m)}$. This gives:

$$K_{(v:m)} = K_{(v:m)} \otimes K_{(1:m)}$$
$$= K_{(v:m)} \otimes \left(\bigoplus_{t=1}^{\frac{m-1}{2}} N_t\right)$$
$$= \bigoplus_{t=1}^{\frac{m-1}{2}} \left(K_{(v:m)} \otimes N_t\right)$$

Notice that $K_{(v:m)} \otimes N_t$ is a spanning subgraph of $K_{(v:m)}$. Since N_t is a $[n_1, n_2, \ldots, n_p]$ factor, we have $N_t = \bigoplus_{i=1}^p C(w_i)$, where $C(w_i)$ is a cycle of size n_i . Therefore,

$$K_{(v:m)} \otimes N_t = K_{(v:m)} \otimes \left(\bigoplus_{i=1}^p C(w_i)\right)$$
$$= \bigoplus_{i=1}^p \left(K_{(v:m)} \otimes C(w_i)\right)$$

But $K(v:m) \otimes C(w_i)$ is isomorphic to $C_{(v:n_i)}$ because $C(w_i)$ is isomorphic to $C_{(1:n_i)}$. So for each w_i we can decompose $K(v:m) \otimes C(w_i)$ into $s_t C_{x_in_i}$ -factors and $r_t C_{y_in_i}$ -factors. Since $C_{(v:n_i)}$ has vn_i vertices, a $C_{x_in_i}$ -factor has $\frac{v}{x_i}$ cycles, hence it is a $[(x_in_i)^{\frac{v}{x_i}}]$ -factor. Likewise a $C_{y_in_i}$ -factor is a $[(y_in_i)^{\frac{v}{y_i}}]$ -factor.

Taking the union of one $C_{x_in_i}$ -factor for each i gives a $[(x_1n_1)^{\frac{v}{x_1}}, \ldots, (x_pn_p)^{\frac{v}{x_p}}]$ -factor. Thus we get a decomposition of $K(v:m) \otimes N_t$ into $s_t [(x_1n_1)^{\frac{v}{x_1}}, \ldots, (x_pn_p)^{\frac{v}{x_p}}]$ -factors and $r_t [(y_1n_1)^{\frac{v}{y_1}}, \ldots, (y_pn_p)^{\frac{v}{y_p}}]$ -factors.

Doing this for every $1 \le t \le \frac{m-1}{2}$, we end up with a decomposition of $K_{(v:m)}$ into $s = [(x_1n_1)^{\frac{v}{x_1}}, \dots, (x_pn_p)^{\frac{v}{x_p}}]$ -factors and $r [(y_1n_1)^{\frac{v}{y_1}}, \dots, (y_pn_p)^{\frac{v}{y_p}}]$ -factors.

It is important to note that Theorem 22 can be used to obtain decompositions of K_m into C_n -factors. Thus the focus of the next three sections is to find decompositions of $C_{(v:n)}$. Any decomposition of $\overrightarrow{C}_{(v:n)}$ is equivalent to a decomposition of $C_{(v:n)}$, by simply removing the direction of each edge. We will work with the partite product on directed graphs to find decompositions of $\overrightarrow{C}_{(v:n)}$, and thus obtain decompositions of $C_{(v:n)}$.

2.6 Hamilton-Waterloo Problem on Directed Complete Cyclic Multipartite Graphs

For the entirety of this section, we assume x is odd. In this section we will decompose $\overrightarrow{C}_{(x:n)}$ into \overrightarrow{C}_n -factors and \overrightarrow{C}_{xn} -factors (Hamilton Cycles), and $\overrightarrow{C}_{(4x:n)}$ into \overrightarrow{C}_n -factors and \overrightarrow{C}_{2xn} -factors.

Suppose G_{α} and G_{β} are two parts of size x in an equipartite directed graph G. We say an arc in G has difference d if $((g_1, \alpha), (g_2, \beta)) \in E(G)$ and $g_2 - g_1 \equiv d \pmod{x}$. If $\{((g_1, \alpha), (g_2, \beta)) : g_2 - g_1 \equiv d \pmod{x}\} \subset E(G)$, then we say that difference dbetween parts α and β is covered by G.

Let the partite sets of $\overrightarrow{C}_{(x:n)}$ be $G_0, G_1, \ldots, G_{n-1}$. Write $n-1 = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_k}$ with $e_{\alpha} > e_{\beta}$ if $\alpha < \beta$. Notice that $k \leq \frac{n}{2}$. Working modulo x, let $T_x(i)$ be the



Figure 2.4: One directed cycle in $T_5(1)$, with n = 7.

directed subgraph of $\overrightarrow{C}_{(x:n)}$ obtained by taking differences:

- $2^{e_j}i$ between G_{j-1} and G_j for $1 \le j \le k$.
- -2i between G_{j-1} and G_j for $k+1 \le j \le 2k-1$.
- -i between G_{j-1} and G_j for $2k \le j \le n-1$.
- -i between G_{n-1} and G_0 .

Example 11: We construct $T_5(1)$ with n = 7. We have $n - 1 = 6 = 2^2 + 2^1$, and k = 2.

This means that from the first column to the second one we add 2^2 , from the second to the third we add 2; since k = 2, from the third to the forth we subtract 2, and for the rest of the arcs we just subtract 1. Figure 2.4 shows one directed cycle of this construction.

The rest of the arcs are obtained by developing this base directed cycle modulo x,

i.e. if $((\alpha, j), (\beta, j+1)) \in E(T_x(i))$ then $((\alpha + h, j), (\beta + h, j+1)) \in E(T_x(i))$ for all $h \in \mathbb{Z}_x$.

Lemma 2.6.1 $T_x(i)$ is a \overrightarrow{C}_n -factor for any *i*.

Proof: It suffices to show that the construction gives a base directed cycle of length n. The directed cycle containing the vertex (0,0) can be tracked by considering the first coordinate of each vertex that is visited while passing from G_0 to G_1 to G_2 ... to G_0 . If we add the respective differences of the edges between G_0 and G_1 , G_1 and G_2 ,..., G_{n-1} and G_0 , we must show that this total sum is 0. Because $2^{e_1}+2^{e_2}+\ldots+2^{e_k}=n-1$, we have for the sum:

$$i(n-1) - 2i(k-1) - i(n-2k+1) = i(n-1) - i(n-1) = 0.$$

Let $F_h(G)$ be the directed subgraph of the directed graph G that contains only the arcs between parts h-1 and h. That is

$$E(F_h(G)) = \{ ((g_1, h-1)(g_2, h)) | \{ (g_1, h-1)(g_2, h) \} \in E(G) \}.$$

In particular $F_n(G)$ contains the arcs between G_{n-1} and G_0 .



Figure 2.5: $T_3(0), T_3(1), H_3(0,1), H_3(1,0).$

Let $H_x(i,s) = T_x(i) \oplus F_n(T_x(i)) \oplus F_n(T_x(s))$. This means that $H_x(i,s)$ is the directed subgraph of $\overrightarrow{C}_{(x:n)}$ obtained by taking the same arcs as $T_x(i)$ between G_j and G_{j+1} for $0 \le j \le n-2$ and $T_x(s)$ between G_{n-1} and G_0 .

Example 12: Figure 2.5 illustrates $T_3(0)$, $T_3(1)$, $H_3(0,1)$ and $H_3(1,0)$ with n = 3.

Lemma 2.6.2 If gcd(x, i - s) = 1 then $H_x(i, s)$ is a directed Hamiltonian cycle.

Proof: Because the arcs are given by differences it is clear that each vertex has indegree and out-degree both equal to 1. We need to show that all of the vertices are connected. We will first show that there is a directed path between any 2 vertices of G_0 . Without loss of generality, we will show that (0,0) is connected to $(\alpha,0)$ for any α . Because the arcs between groups G_j and G_{j+1} are the same as the arcs in $T_x(i)$ for $j = 0, \ldots, n-2$ it is easy to see that there is a path from (0,0) to (i, n-1). Now the arc leaving from (i, n - 1) has its other end as (i - s, 0). So (0, 0) is connected to (i - s, 0). If we continue on this path, every time we arrive back in G_0 , we arrive at the vertex $(\alpha'(i - s), 0)$. Because gcd(x, i - s) = 1, the order of i - s in the cyclic group \mathbb{Z}_x is x. Thus any α modulo x can be written as $\alpha'(i - s)$. Hence (0, 0) is connected to all the vertices of G_0 .

Because we are defining arcs by differences, every vertex in G_1 is connected to a vertex in G_0 , every vertex in G_2 to a vertex in G_1 , and so on. Therefore all the vertices are connected, and the directed cycle is Hamiltonian as we wanted to prove.

Next we show how to decompose $\overrightarrow{C}_{(x:n)}$ by using the $H_x(i, j)$ graphs. First we will decompose $\overrightarrow{C}_{(x:n)}$ into \overrightarrow{C}_n -factors using the $T_x(i)$ graphs. Then we will show how to switch some edges in the $T_x(i)$ graphs to obtain $H_x(i, j)$ graphs. It is important to notice that $H_x(i, i) = T_x(i)$.

Lemma 2.6.3 Let x be odd, and let ϕ be a bijection on $\{0, ..., x - 1\}$. Then

$$\overrightarrow{C}_{(x:n)} = \bigoplus_{i=0}^{x-1} T_x(i) = \bigoplus_{i=0}^{x-1} H_x(i,\phi(i))$$

Proof: To prove the first equality,

$$\overrightarrow{C}_{(x:n)} = \bigoplus_{i=0}^{x-1} T_x(i),$$

we need to show that any difference between consecutive parts is covered by one of the $T_x(i)$ graphs. Notice that all the differences in the T_x graphs are given by a power of 2 times i, or -2i or -i. It is clear that between parts which use difference -i, we cover all the differences, with difference δ being covered in $T_x(x-\delta)$. Because x is odd, $gcd(x, 2^e) = 1$, so the order of 2^e in the cyclic group C_x is x for any $1 \le e \le x-1$. This means that any $\delta \equiv 2^{e_j}i \pmod{x}$ can be written as $2^e\delta'$. Therefore, the difference δ between the remaining pairs of consecutive groups is covered in some $T_x(\delta')$. A similar calculation can be used for differences of the form -2i, because gcd(x, x - 2) = 1. Here we write δ as $-2\delta'$, and the difference is covered in $T_x(\delta')$.

For the second equality we have

$$\bigoplus_{i=0}^{x-1} H_x(i,\phi(i)) = \bigoplus_{i=0}^{x-1} (T_x(i) \oplus F_n(T_x(i)) \oplus F_n(T_x(\phi(i))))$$
$$= \bigoplus_{i=0}^{x-1} T_x(i) \bigoplus_{i=0}^{x-1} F_n(T_x(i)) \bigoplus_{i=0}^{x-1} F_n(T_x(\phi(i)))$$

and

$$\bigoplus_{i=0}^{x-1} F_n(T_x(\phi(i))) = \bigoplus_{i=0}^{x-1} F_n(T_x(i))$$

because ϕ is a bijection. So:

$$\bigoplus_{i=0}^{x-1} T_x(i) \bigoplus_{i=0}^{x-1} F_n(T_x(i)) \bigoplus_{i=0}^{x-1} F_n(T_x(\phi(i))) = \bigoplus_{i=0}^{x-1} T_x(i) \bigoplus_{i=0}^{x-1} F_n(T_x(i)) \bigoplus_{i=0}^{x-1} F_n(T_x(i))$$

$$= \bigoplus_{i=0}^{x-1} T_x(i)$$

Hence:

$$\bigoplus_{i=0}^{x-1} T_x(i) = \bigoplus_{i=0}^{x-1} H_x(i,\phi(i))$$

In some cases we have $H_x(i, i)$, notice that this is the same as $T_x(i)$. Decomposing $\overrightarrow{C}_{(x:n)}$ into s directed Hamilton cycles and $x - s \overrightarrow{C}_n$ -factors is now equivalent to finding a bijection ϕ with $gcd(x, i - \phi(i)) = 1$ for s elements of $\{0, ..., x - 1\}$ and $\phi(i) = i$ for the rest. We will use these functions extensively throughout the paper, so we will refer to them as "the phi-functions": Let $2 \leq s \leq x$. Define $\phi_s : \mathbb{Z}_x \to \mathbb{Z}_x$ as follows:

The phi-functions

 $\phi_{s}(i) = \begin{cases} i+1 & \text{for} \quad i \leq (s-3), i \text{ even} \\ i-1 & \text{for} \quad i \leq (s-3), i \text{ odd} \\ i+1 & \text{for} \quad i = s-2 \\ i-2 & \text{for} \quad i = s-1, s \text{ odd} \\ i-1 & \text{for} \quad i = s-1, s \text{ even} \\ i & \text{for} \quad s \leq i \leq x-1 \end{cases}$

For example, if s = 7 and x = 11, then

$$\phi_7 = (01)(23)(456)(7)(8)(9)(10)$$

Notice that ϕ_s has x - s fixed points. For any non-fixed point we have $i - \phi(i) \in \{\pm 1, 2\}$, and so $gcd(x, i - \phi(i)) = 1$ if x is odd.

Theorem 23 Let x be odd. Let $s \in \{0, ..., x\}$, $s \neq 1$. Then $\overrightarrow{C}_{(x:n)}$ can be decomposed into $s \overrightarrow{C}_{xn}$ -factors and $x - s \overrightarrow{C}_n$ -factors.

Proof: If s = 0 we just use the identity mapping. Otherwise we use the phi-function



Figure 2.6: Triangle factors for $K_{(4:3)}$.

 ϕ_s . Hence the discussion that precedes this theorem shows that

$$\overrightarrow{C}_{(x:n)} = \bigoplus_{i=0}^{x-1} H_x(i, \phi_s(i))$$

is a decomposition of $\overrightarrow{C}_{(x:n)}$ into $s \overrightarrow{C}_{xn}$ -factors and $x - s \overrightarrow{C}_n$ -factors.

Next we turn to the case of x even. We begin by considering $\overrightarrow{C}_{(4:n)}$. In [5] the graphs in Figure 2.6 were used to decompose $K_{(4:3)}$ into triangle factors and C_6 -factors. We extend the ideas used in [5] to decompose $\overrightarrow{C}_{(4:n)}$ into \overrightarrow{C}_n -factors and \overrightarrow{C}_{2n} -factors. We will define directed subgraphs $\gamma_{i,j}$ and build the directed graphs $\Gamma(j)$ as the sum of some of these directed subgraphs. In each of these directed subgraphs, the vertices in the top row will be said to have height 0, in the second row height 1, and so on. We begin with the base directed subgraphs from Figure 2.7 . In each subgraph, the rows are indexed by their height. The following result is easy to verify by inspection of the graphs $\gamma_{i,j}$.



Figure 2.7: Base directed subgraphs for the decomposition of $\overrightarrow{C}_{(4:n)}$ into \overrightarrow{C}_n -factors and \overrightarrow{C}_{2n} -factors

Lemma 2.6.4 For any i, j, h, the directed path beginning at height h in the first column of $\gamma_{i,j}$ finishes at height h in the last column of $\gamma_{i,j}$.

Let n = 3b + a with $0 \le a < 3$, $b \ge 2$. For $0 \le t \le b - 2$, we define $\gamma_{0,j}(t)$ as the directed graph $\gamma_{0,j}$ on the parts $G_{3t-1}, G_{3t}, G_{3t+1}, G_{3t+2}$, with calculations done in \mathbb{Z}_n . We define $\gamma_{a,j}(n)$ as the directed graph $\gamma_{a,j}$ on the parts $G_{3b-4}, G_{3b-3}, \ldots, G_{3b+a-1}$. Let $\Gamma(j) = \bigoplus_{t=0}^{b-2} \gamma_{0,j}(t) \oplus \gamma_{a,j}(n)$. Notice that $\gamma_{0,j}(0)$ is on the parts G_{-1}, G_0, G_1 and G_2 . This means that $F_n(\Gamma(j))$ is the matching between the first and second columns in $\gamma_{0,j}(0)$.

Example 13: Let n = 7. We will construct $\Gamma(0)$. Since n = 6+1, we have b = 2 and a = 1. This means that $\Gamma(0) = \gamma_{0,0}(0) \oplus \gamma_{1,0}(7)$, where $\gamma_{0,0}(0)$ is on the parts $G_{-1} = G_{7-1} = G_6, G_0, G_1$ and G_2 ; and $\gamma_{1,0}(7)$ is on the parts G_2, G_3, G_4, G_5 and G_6 . So we get the picture in Figure 2.8(a), where the arcs from $\gamma_{0,0}(0)$ are dashed.

Notice that the first and last columns are both G_6 . Because the directed graph $\overrightarrow{C}_{(4:n)}$ has G_0 as the first column, we connect the vertices from G_0 to the last column instead, obtaining the picture in Figure 2.8(b); where the dashed arcs still belong to $\gamma_{0,0}(0)$.

Using Lemma 2.6.4 we have the following result.





(b) Γ_0 when n = 7 after identifying both G_6 columns

Figure 2.8: A \overrightarrow{C}_7 factor of $\overrightarrow{C}_{(4:7)}$

Lemma 2.6.5

$$\overrightarrow{C}_{(4:n)} = \Gamma(0) \oplus \Gamma(1) \oplus \Gamma(2) \oplus \Gamma(3) = \bigoplus_{j=0}^{3} \Gamma(j)$$

is a \overrightarrow{C}_n -factorization of $\overrightarrow{C}_{(4:n)}$.

Proof: It is easy to verify from the pictures that for any given $0 \le i \le 2$, the directed graphs $\gamma_{i,0}, \gamma_{i,1}, \gamma_{i,2}$ and $\gamma_{i,3}$ are arc disjoint. By Lemma 2.6.4, in each $\gamma_{i,j}$ the directed paths start and end at the same height. Thus when we connect all the directed paths in each factor $\Gamma(j)$, we obtain four directed cycles of length n. Therefore $\bigoplus_{j=0}^{3} \Gamma(j)$ To construct directed cycles of size 2n, we perform switches on the edges between columns. Define $\lambda_{i,j} = \gamma_{0,i}(0) \oplus F_n(\gamma_{0,i}(0)) \oplus F_n(\gamma_{0,j}(0))$. Keep in mind that $\gamma_{0,i}(0)$ is on the parts G_{n-1}, G_0, G_1, G_2 , and so $F_n(\gamma_{0,i}(0))$ only consists of the edges between parts G_{n-1} and G_0 .

Lemma 2.6.6 If the directed path that starts at height h_1 in part G_{n-1} in $\lambda_{i,j}$ ends at height h_2 in G_2 , then the directed path that starts at height h_2 in G_{n-1} ends at height h_1 in G_2 . Even more, if $i \neq j$ then no directed path starts and ends at the same height.

Proof: We build tables that show for each possible combination of i and j, the starting and ending heights of the directed paths $\lambda_{i,j}$. We have one table for each i, with the rows indexed by the options for j, and the columns indexed by the options for the starting height of each directed path. The entry in the table gives the finishing height.

		height 0	height 1	height 2	height 3
i = 0	j = 0	0	1	2	3
	j = 1	3	2	1	0
	j = 2	1	0	3	2
	j = 3	2	3	0	1
i = 1		height 0	height 1	height 2	height 3
-------	-------	----------	----------	----------	----------
	j = 0	3	2	1	0
	j = 1	0	1	2	3
	j = 2	2	3	0	1
	j = 3	1	0	3	2

i = 2		height 0	height 1	height 2	height 3
	j = 0	1	0	3	2
	j = 1	2	3	0	1
	j = 2	0	1	2	3
	j = 3	3	2	1	0

		height 0	height 1	height 2	height 3
i = 3	j = 0	2	3	0	1
	j = 1	1	0	3	2
	j = 2	3	2	1	0
	j = 3	0	1	2	3

Notice that whenever i = j we have $\lambda_{i,i} = \gamma_{0,i}(0)$, in which case we already know that the starting and ending heights of each directed path are the same. When $i \neq j$ the starting and ending heights are never the same, but if the starting height in $\lambda_{i,j}$ is h_1 and the ending height is h_2 , then the directed path with starting height h_2 has

Let
$$\Lambda(i,j) = \Gamma(i) \oplus F_n(\Gamma(i)) \oplus F_n(\Gamma(j)).$$

Lemma 2.6.7 If $i \neq j$, $\Lambda(i, j)$ is a \overrightarrow{C}_{2n} -factor.

Proof: Notice that

$$\Lambda(i,j) = \Gamma(i) \oplus F_n(\Gamma(i)) \oplus F_n(\Gamma(j))$$
$$= \left(\bigoplus_{t=0}^{b-2} \gamma_{0,i}(t)\right) \oplus \gamma_{a,i}(n) \oplus F_n(\Gamma(i)) \oplus F_n(\Gamma(j))$$

Because $F_n(\Gamma(i))$ is the matching between the first and second columns in $\gamma_{0,i}(0)$, we have $F_n(\Gamma(i)) = F_n(\gamma_{0,i}(0))$. Therefore,

$$\Lambda(i,j) = \left(\bigoplus_{t=0}^{b-2} \gamma_{0,i}(t)\right) \oplus \gamma_{a,i}(n) \oplus F_n(\Gamma(i)) \oplus F_n(\Gamma(j))$$
$$= \left(\bigoplus_{t=0}^{b-2} \gamma_{0,i}(t)\right) \oplus \gamma_{a,i}(n) \oplus F_n(\gamma_{0,i}(0)) \oplus F_n(\gamma_{0,j}(0))$$
$$= \left(\bigoplus_{t=1}^{b-2} \gamma_{0,i}(t)\right) \oplus \gamma_{a,i}(n) \oplus (\gamma_{0,i}(0) \oplus F_n(\gamma_{0,i}(0)) \oplus F_n(\gamma_{0,j}(0)))$$
$$= \left(\bigoplus_{t=1}^{b-2} \gamma_{0,i}(t)\right) \oplus \gamma_{a,i}(n) \oplus \lambda_{i,j}$$

Consider the directed cycle that contains the vertex at height h_1 in G_{n-1} . From Lemma 2.6.6 we know that in $\lambda_{i,j}$ the directed path that starts at height h_1 in G_{n-1} finishes at height h_2 in G_2 . By Lemma 2.6.4, the directed paths through all the $\gamma_{i,j}(l)$, with $l \in \{1, 2, \ldots, b - 1, n\}$ start and end at the same heights. So when we reach G_{n-1} again, it is at at height h_2 . We leave G_2 at height h_1 this time, and as we move through all the $\gamma_{i,j}(l)$, with $l \in \{1, 2, \ldots, b - 1, n\}$, the heights never change. Therefore, we reach G_{n-1} again at height h_1 , closing the directed cycle. This produces one directed cycle of size 2n. By repeating the process with the directed cycle starting at one of the vertices that we have not used yet, we get the second directed cycle. Therefore $\Lambda(i, j)$ consists of two directed cycles of length 2n.

Notice that if i = j, then $\Lambda(i, j) = \Gamma(i)$ consists of 4 directed cycles of length n.

Theorem 24 If $s \in \{0, 2, 3, 4\}$, then $\overrightarrow{C}_{(4:n)}$ can be decomposed into $s \overrightarrow{C}_{2n}$ -factors and $4 - s \overrightarrow{C}_n$ -factors.

Proof: Let π be a permutation of the set $\{0, 1, 2, 3\}$ with exactly 4 - s fixed points. Then

$$\overrightarrow{C}_{(4:n)} = \bigoplus_{j=0}^{3} \Gamma(j) = \bigoplus_{j=0}^{3} (\Gamma(j) \oplus F_n(\Gamma(j)) \oplus F_n(\Gamma(\pi(j))) = \bigoplus_{j=0}^{3} \Lambda(j,\pi(j))$$

Since $\Lambda(j, \pi(j))$ is a \overrightarrow{C}_{2n} -factor if $j \neq \pi(j)$ and a \overrightarrow{C}_n -factor otherwise, the theorem

is proven.

Remark 2.6.8 Notice that if n = 5, we have b = 1 and a = 2. This means we have $\Gamma(j) = \gamma_{a,j}(5) = \gamma_{2,j}(5)$, which is on the parts $G_{-1} = G_4, G_0, G_1, G_2, G_3, G_4$. This will actually close the directed cycle. The results given in Lemmas 2.6.6 and 2.6.7 only apply to $b \ge 2$, but it can be shown that the same results are true with b = 1 by applying similar techniques on $\gamma_{a,j}$ instead of $\gamma_{0,j}$.

There is one more basic decomposition that we will use, based on the resolvable gregarious decomposition of $K_{(w:n)}$ from [6]. We make use of the constructions given in Lemma 3.1 and Corollary 3.2 of [6], and apply them to $\overrightarrow{C}_{(w:n)}$ instead of $K_{(w:n)}$.

Definition 2.6.9 A quasigroup (Q, *) is a set Q with a binary operation * such that for each a and b in Q, there exist unique elements x and y in Q such that:

- a * x = b;
- y * a = b.

Definition 2.6.10 Two quasigroups on the same set (Q, *), (Q, \circ) are said to be orthogonal if $i * j \neq i \circ j$ for every i, j in Q.

The reader may be familiar with Latin Squares, which are the multiplication tables of quasigroups, and mutually orthogonal Latin Squares, which are the multiplication tables of orthogonal quasigroups. In [7], [8] it was shown that if $|Q| \notin \{1, 2, 6\}$ then there are at least 2 orthogonal quasigroups on Q. Again, the decomposition in the following theorem is obtained by modifying the construction from Lemma 3.2 in [6] to work with $\overrightarrow{C}_{(w:n)}$ instead of $K_{(w:n)}$:

Theorem 25 Let $w \notin \{2, 6\}$ and n odd. Then there is a decomposition of $\overrightarrow{C}_{(w:n)}$ into \overrightarrow{C}_n -factors.

Proof: Since $w \notin \{2, 6\}$ there exist two orthogonal quasigroups (Latin Squares) (Q, \circ) and (Q, *) of order w, with $Q = \{0, 1, 2, ..., w - 1\}$. We take directed cycles of the form:

 $(i,0)(j,1)(i,2)(j,3)\dots(i,n-3)(j,n-2)(k,n-1),$ where $0 \le i,j \le w-1, k = i \circ j.$

This produces a decomposition of $\overrightarrow{C}_{(w:n)}$ into w^2 directed cycles of size n. To form a \overrightarrow{C}_n -factor, given $l \in Q$ we take all cycles arising from the pairs i, j with i * j = lin the second quasigroup (Q, *). Thus we have a decomposition of $\overrightarrow{C}_{(w:n)}$ into w \overrightarrow{C}_n -factors.

2.7 Multivariable Functions

Definition 2.7.1 Let x and y be odd. We define $T_{(xy)}(i, \alpha)$ to be the directed subgraph of $\overrightarrow{C}_{(xy:n)}$ obtained by taking $T_{(xy)}(i, \alpha) = T_x(i) \otimes T_y(\alpha)$. We also define

$$H_{(xy)}(i,\alpha)(j,\beta) = T_{(xy)}(i,\alpha) \oplus F_n(T_{(xy)}(i,\alpha)) \oplus F_n(T_{(xy)}(j,\beta))$$

This means that $H_{(xy)}(i, \alpha)(j, \beta)$ is the directed graph obtained by taking the arcs of $T_{(xy)}(i, \alpha)$ between parts t and t + 1 for $0 \le t \le n - 2$, and the arcs between parts n - 1 and 0 from $T_{(xy)}(j, \beta)$.

Example 14: Figure 2.9 illustrates the first part of Definition 2.7.1 by showing $T_x(i)$, $T_y(\alpha)$ and $T_{(xy)}(i, \alpha)$, for x = 3, y = 5, i = 1 and $\alpha = 2$, with 3 partite sets.

Figure 2.10 illustrates the second part of Definition 2.7.1 by showing $H_{(xy)}(i,\alpha)(j,\beta)$, for x = 3, y = 5, i = 1, $\alpha = 2$, j = 2, $\beta = 4$, with 3 partice sets. Figure 2.10 also shows $H_x(i,j)$ and $H_y(\alpha,\beta)$, to illustrate Lemma 2.7.2.

Notice that in both figures instead of giving all the coordinates in each vertex, we give the first two coordinates of all the vertices in each row (the third coordinate would specify which partite set the vertex belongs to).

Lemma 2.7.2 Let x, y and n be odd. Then:

$$H_{(xy)}(i,\alpha)(j,\beta) = H_x(i,j) \otimes H_y(\alpha,\beta)$$

Proof: Notice that

$$F_n(T_{(xy)}(i,\alpha)) = F_n(T_x(i) \otimes T_y(\alpha)) = F_n(T_x(i)) \otimes F_n(T_y(\alpha))$$

Notice also that

$$F_n(T_x(i) \otimes T_y(\alpha)) = F_n(T_x(i)) \otimes T_y(\alpha) = T_x(i) \otimes F_n(T_y(\alpha))$$

Then we have

$$H_x(i,j) \otimes H_y(\alpha,\beta) = (T_x(i) \oplus F_n(T_x(i)) \oplus F_n(T_x(j)))$$
$$\otimes (T_y(\alpha) \oplus F_n(T_y(\alpha)) \oplus F_n(T_y(\beta)))$$
$$= T_x(i) \otimes T_y(\alpha) \oplus F_n(T_x(i))$$
$$\otimes F_n(T_y(\alpha)) \oplus F_n(T_x(j)) \otimes F_n(T_y(\beta))$$
$$= T_{(xy)}(i,\alpha) \oplus F_n(T_{(xy)}(i,\alpha)) \oplus F_n(T_{(xy)}(j,\beta))$$
$$= H_{(xy)}(i,\alpha)(j,\beta)$$



Figure 2.9: $T_3(1)$, $T_5(2)$ and $T_{(3,5)}(1,2)$

 $H_{(3,5)}(1,2)(2,4)$ $H_3(1,2)$ (0, 0)(0, 1)0 (0, 2)1 (0,3)2(0, 4) $H_5(2,4)$ (1, 0)0 (1, 1)1 (1, 2)2(1, 3)3 (1, 4)4(2, 0)(2, 1)(2, 2)(2, 3)(2, 4)

Figure 2.10: $H_3(1,2), H_5(2,4)$ and $H_{(3,5)}(1,2)(2,4)$

Lemma 2.7.3 Let ψ be a bijection on the set $\{(i, \alpha) | 0 \le i \le x - 1, 0 \le \alpha \le y - 1\}$. Then

$$\overrightarrow{C}_{(xy:n)} = \bigoplus_{(i,\alpha)} H_{(xy)}(i,\alpha)\psi(i,\alpha).$$

Proof: We know that $\overrightarrow{C}_{(xy:n)} = \overrightarrow{C}_{(x:n)} \otimes \overrightarrow{C}_{(y:n)} = (\bigoplus_i T_x(i)) \otimes (\bigoplus_{\alpha} T_y(\alpha))$. By definition of $T_{(xy)}(i, \alpha)$ we get

$$\overrightarrow{C}_{(xy:n)} = \bigoplus_{(i,\alpha)} T_{(x,y)}(i,\alpha)$$

We also have

$$\bigoplus_{(i,\alpha)} T_{(xy)}(i,\alpha) = \bigoplus_{(i,\alpha)} H_{(xy)}(i,\alpha)\psi(i,\alpha)$$

Combining both we get:

$$\overrightarrow{C}_{(xy:n)} = \bigoplus_{(i,\alpha)} H_{(xy)}(i,\alpha)\psi(i,\alpha)$$

as we wanted to prove.

If $\psi(i, \alpha) = (j, \beta)$ we will denote $\psi_1(i, \alpha) = j$ and $\psi_2(i, \alpha) = \beta$. If gcd(x, i - j) = 1and $\alpha = \beta$, then $H_{(xy)}(i, \alpha)(j, \beta)$ is a \overrightarrow{C}_{xn} -factor. This is because

$$H_{(xy)}(i,\alpha)(j,\beta) = H_x(i,j) \otimes H_y(\alpha,\alpha) = H_x(i,j) \otimes T_y(\alpha)$$

By Lemma 2.6.2 $H_x(i, j)$ is a \overrightarrow{C}_{xn} -factor. By Lemma 2.6.1 $T_y(\alpha)$ is a \overrightarrow{C}_n -factor. Then by Lemma 2.4.3 $H_x(i, j) \otimes T_y(\alpha)$ is a \overrightarrow{C}_{xn} -factor. Thus to obtain a decomposition of $\overrightarrow{C}_{(xy:n)}$ into \overrightarrow{C}_{xn} -factors and \overrightarrow{C}_{yn} -factors we need a bijection ψ that satisfies the following set of conditions

Conditions 2.7.4 a) For all (i, α) , $gcd(x, i - \psi_1(i, \alpha)) = 1$ and $\psi_2(i, \alpha) = \alpha$, or

b) $gcd(y, \alpha - \psi_2(i, \alpha)) = 1$ and $\psi_1(i, \alpha) = i$.

Lemma 2.7.5 Let x, y, and n be odd. Let $s_p \neq 1, xy - 1$. Then there is a decomposition of $\overrightarrow{C}_{(xy:n)}$ into $s_p \overrightarrow{C}_{xn}$ -factors and $r_p = xy - s_p \overrightarrow{C}_{yn}$ -factors.

Proof: We will describe a bijection ψ that satisfies conditions 2.7.4 with r_p pairs (i, α) that satisfy $i = \psi_1(i, \alpha)$.

Let r_{α} , $0 \leq \alpha \leq y - 1$ be such that:

- $\sum_{\alpha} r_{\alpha} = r_p,$
- $r_i \ge r_j$ if $i \le j$,
- $r_0 = r_1$,
- $0 \le r_{\alpha} \le x, r_{\alpha} \ne x 1.$

Define the function $\phi_{\alpha}(i) = \phi_s(i)$, with $\phi_s(i)$ as the phi-function over the set $\{0, 1, \ldots, x-1\}$ with $s = x - r_{\alpha}$. Let $\pi(i) = |\{\alpha | \phi_{\alpha}(i) = i\}|$. Let $\sigma_i(\alpha) = \phi_s(\alpha)$, with $\phi_s(\alpha)$ as the phi-function over the set $\{0, 1, \ldots, y-1\}$ with $s = \pi(i)$. Notice that

$$\psi(i,\alpha) = (\phi_{\alpha}(i), \sigma_i(\alpha))$$

is a function satisfying conditions 2.7.4 because if $\alpha \leq \pi(i)$, then $\psi_{\alpha}(i) = i$ and $gcd(y, \alpha - \sigma_i(\alpha)) = 1$. If, on the other hand $\alpha \geq \pi(i)$, then we have $\sigma_i(\alpha) = \alpha$, and $gcd(x, i - \psi_{\alpha}(i)) = 1$. Finally, notice that there are r_p pairs (i, α) that satisfy $i = \varphi_1(i, \alpha)$. Therefore there is a decomposition of $\overrightarrow{C}_{(xy:n)}$ into $s_p \ \overrightarrow{C}_{xn}$ -factors and $r_p = xy - s_p \ \overrightarrow{C}_{yn}$ -factors.

We can work with $\Gamma(i)$ and $\Lambda(i)$ in a similar fashion as to what we did with $T_x(i)$.

Definition 2.7.6 Let x be odd. We define $T_{(2x)}(i, \alpha)$ to be the directed subgraph of $\overrightarrow{C}_{(4x:n)}$ obtained by taking $T_{(xy)}(i, \alpha) = T_x(i) \otimes \Gamma(\alpha)$. We also define $H_{(2x)}(i, \alpha)(j, \beta) = T_{(2x)}(i, \alpha) \oplus F_n(T_{(2x)}(i, \alpha)) \oplus F_n(T_{(2x)}(j, \beta))$. This is the directed graph obtained by taking the arcs of $T_{(2x)}(i, \alpha)$ between parts t and t + 1 for $0 \le t \le n - 2$, and the arcs between parts n - 1 and 0 from $T_{(2x)}(j, \beta)$.

Now we can apply the same techniques that we did to $T_{(xy)}$ and $H_{(xy)}$.

Lemma 2.7.7 Let x and n be odd. Then:

$$H_{(2x)}(i,\alpha)(j,\beta) = H_x(i,j) \otimes \Lambda(\alpha,\beta)$$

Proof: Notice that

$$F_n(T_{(2x)}(i,\alpha)) = F_n(T_x(i) \otimes \Gamma(\alpha)) = F_n(T_x(i)) \otimes F_n(\Gamma(\alpha))$$

and

$$F_n(T_x(i) \otimes \Gamma(\alpha)) = F_n(T_x(i)) \otimes \Gamma(\alpha) = T_x(i) \otimes F_n(\Gamma(\alpha))$$

Using this the result is trivial.

Lemma 2.7.8 Let φ by a bijection on the set $\{(i, \alpha) | 0 \le i \le x - 1, 0 \le \alpha \le 3\}$.

Then

$$\overrightarrow{C}_{(4x:n)} = \bigoplus_{(i,\alpha)} H_{(2x)}(i,\alpha)\varphi(i,\alpha).$$

Proof: We know that $\overrightarrow{C}_{(4x:n)} = \overrightarrow{C}_{(x:n)} \otimes \overrightarrow{C}_{(4:n)} = (\bigoplus_i T_x(i)) \otimes (\bigoplus_{\alpha} \Gamma(\alpha))$. By definition of $T_{(2x)}(i, \alpha)$ we get

$$\overrightarrow{C}_{(4x:n)} = \bigoplus_{(i,\alpha)} T_{(2x)}(i,\alpha)$$

We also have

$$\bigoplus_{(i,\alpha)} T_{(2x)}(i,\alpha) = \bigoplus_{(i,\alpha)} H_{(2x)}(i,\alpha)\varphi(i,\alpha)$$

Combining both we get:

$$\overrightarrow{C}_{(4x:n)} = \bigoplus_{(i,\alpha)} H_{(2x)}(i,\alpha)\varphi(i,\alpha)$$

as we wanted to prove.

Next we develop the conditions needed for our decompositions. Recall that if $\varphi(i, \alpha) = (j, \beta)$ we will denote $\varphi_1(i, \alpha) = j$ and $\varphi_2(i, \alpha) = \beta$.

• If $\alpha \neq \beta$ and gcd(x, i - j) = 1 then

$$H_{(2x)}(i,\alpha)(j,\beta) = H_x(i,j) \otimes \Lambda(\alpha,\beta)$$

is a \overrightarrow{C}_{2xn} -factor by Lemmas 2.6.2, 2.6.7, and 2.4.3.

- If i = j and $\alpha \neq \beta$, then $H_{(2x)}(i, \alpha)(j, \beta)$ is a \overrightarrow{C}_{2n} -factor by Lemmas 2.6.1, 2.6.7, and 2.4.3.
- If $\alpha = \beta$ and gcd(x, i j) = 1, then $H_{(2x)}(i, \alpha)(j, \beta)$ is a \overrightarrow{C}_{xn} -factor by Lemmas 2.6.2, 2.6.5, and 2.4.3.
- If i = j and $\alpha = \beta$, then $H_{(xy)}(i, \alpha)(j, \beta)$ is a \overrightarrow{C}_n -factor by Lemmas 2.6.1, 2.6.5,

and 2.4.3.

So for a decomposition of $\overrightarrow{C}_{(4x:n)}$ into \overrightarrow{C}_{2xn} -factors and \overrightarrow{C}_n -factors we need a bijection φ that satisfies:

Conditions 2.7.9 For all (i, α) such that $\varphi(i, \alpha) \neq (i, \alpha)$, $\alpha \neq \varphi_2(i, \alpha)$ and $gcd(x, i - \varphi_1(i, \alpha)) = 1$.

For a decomposition of $\overrightarrow{C}_{(4x:n)}$ into \overrightarrow{C}_{2n} -factors and \overrightarrow{C}_{xn} -factors we need a bijection φ that satisfies:

Conditions 2.7.10 a) For all (i, α) either $\alpha \neq \varphi_2(i, \alpha)$ and i = j, or

b) $\alpha = \varphi_2(i, \alpha)$ and $gcd(x, i - \varphi_1(i, \alpha)) = 1$.

We define a new family of functions $\theta_s : \mathbb{Z}_x \times \mathbb{Z}_4 \to \mathbb{Z}_x \times \mathbb{Z}_4$. These functions will be referred as *theta-functions*. Let $s \in \{0, 1, \dots, 4x\}$, $s \notin \{1, 4x - 1\}$, and write s = 4k + 2a + 3b, with $a, b \in \{0, 1\}$, $k \leq x$. We define:

Theta-functions:

$$\theta_{s}(i,\alpha) \coloneqq \left\{ \begin{array}{ll} (i+1,\alpha+1) & \text{if} & 0 \leq i \leq k-1, \alpha=0,2\\ (i-1,\alpha-1) & \text{if} & 1 \leq i \leq k, \alpha=1,3\\ (i+1,\alpha+1) & \text{if} & i=k, \ a=1, \ \alpha=0\\ (i-1,\alpha-1) & \text{if} & i=k+1, \ a=1, \ \alpha=1\\ (i+1,\alpha+1) & \text{if} & i=k, \ b=1, \ \alpha=2\\ (i+1,\alpha-2) & \text{if} & i=k+1, \ b=1, \ \alpha=3\\ (i-2,\alpha+1) & \text{if} & i=k+2, \ b=1, \ \alpha=1\\ (i,\alpha) & \text{otherwise} \end{array} \right.$$

If s = 4x - 1 = 4(x - 1) + 3, we define θ_s in a similar way, with a small change:

$$\theta_{4x-1}(i,\alpha) := \begin{cases} (i+1,\alpha+1) & \text{if} \quad 0 \le i \le x-2, \alpha = 0, 2\\ (i-1,\alpha-1) & \text{if} \quad 1 \le i \le x-2, \alpha = 1, 3\\ (x-2,2) & \text{if} \quad i = x-1, \alpha = 1\\ (0,1) & \text{if} \quad i = x-1, \alpha = 3\\ (x-1,0) & \text{if} \quad i = 0, \alpha = 1\\ (0,3) & \text{if} \quad i = x-1, \alpha = 0\\ (x-2,0) & \text{if} \quad i = 0, \alpha = 3\\ (i,\alpha) & \text{otherwise} \end{cases}$$



Figure 2.11: Example of Theta-functions

We give a visual example of θ_9 and θ_{19} , for x = 5 in Figure 2.11.

The following lemma is a generalization of a result given in [5].

Lemma 2.7.11 Let $s_p \in \{0, 2, ..., 4x - 1, 4x\}$, x odd. Then there exists a decomposition of $C_{(4x:n)}$ into $s_p \ C_{2xn}$ -factors and $r_p = 4x - s_p \ C_n$ -factors.

Proof: The bijection $\psi = \theta_{s_p}$ satisfies Conditions 7.9. In particular if $\psi(i, \alpha) \neq (i, \alpha)$, then $\alpha \neq \psi_2(i, \alpha)$ and $i - \psi_1(i, \alpha) \in \{\pm 1, \pm 2\}$; and as x is odd, $gcd(x, i - \psi_1(i, \alpha)) = 1$. Furthermore, ψ has s_p non-fixed points. Therefore there exists a decomposition of $C_{(4x:n)}$ into $s_p C_{2xn}$ -factors and $4x - s_p C_n$ -factors. **Lemma 2.7.12** Let $s_p \in \{0, 2, 3, ..., 4x - 3, 4x - 2, 4x\}$. Then there exists a decomposition of $\overrightarrow{C}_{(4x:n)}$ into $s_p \overrightarrow{C}_{xn}$ -factors and $r_p = 4x - s_p \overrightarrow{C}_{2n}$ -factors.

Proof: We provide a bijection φ that satisfies Conditions 2.7.10 with r_p pairs (i, α) that satisfy $i = \varphi_1(i, \alpha)$.

Let r_{α} , $0 \leq \alpha \leq 3$ be such that:

- $\sum_{\alpha} r_{\alpha} = r_p,$
- $r_i \ge r_j$ if $i \le j$,
- $r_0 = r_1$,
- $r_{\alpha} \leq x, r_{\alpha} \neq x 1.$

The only case where such a choice of r_{α} cannot be made is when x = 3, s = 7. This case is covered in Lemma 2.11.2 in the Appendix.

Define the function $\psi_{\alpha}(i) = \phi_s(i)$, with $\phi_s(i)$ as the phi-functions over the set $\{0, 1, \ldots, x - 1\}$ with $s = x - r_{\alpha}$. Let $\pi(i) = |\{\alpha | \psi_{\alpha}(i) = i\}|$. Let $\sigma_i(\alpha)$ be the permutation on the set $\{0, 1, 2, 3\}$ that cyclically permutes the first $\pi(i)$ elements and fixes the rest. Notice that

$$\varphi(i,\alpha) = (\psi_{\alpha}(i), \sigma_i(\alpha))$$

is a function satisfying conditions 2.7.10 because if $\alpha \leq \pi(i)$, then $\psi_{\alpha}(i) = i$ and $\sigma_i(\alpha) \neq \alpha$. If, on the other hand $\alpha \geq \pi(i)$, then we have $\sigma_i(\alpha) = \alpha$, and $\gcd(x, i - \psi_{\alpha}(i)) = 1$. Finally, notice that there are r_p pairs (i, α) that satisfy $i = \varphi_1(i, \alpha)$. Therefore there exists a decomposition of $\overrightarrow{C}_{(4x:n)}$ into $s_p \overrightarrow{C}_{xn}$ -factors and $r_p = 4x - s_p$ \overrightarrow{C}_{2n} -factors.

We are interested in one more type of decomposition, into \overrightarrow{C}_{2xn} and \overrightarrow{C}_{yn} factors. To do this we introduce the following:

Definition 2.7.13 Let x and y be odd. Define $T_{(2xy)}(i, \alpha, \gamma)$ to be the directed subgraph of $\overrightarrow{C}_{(4xy:n)}$ obtained by taking $T_{(2xy)}(i, \alpha, \gamma) = T_{(2x)}(i, \alpha) \otimes T_y(\gamma)$. We also define

$$H_{(2xy)}(i,\alpha,\gamma)(j,\beta,\delta) = T_{(2xy)}(i,\alpha,\gamma) \oplus F_n(T_{(2xy)}(i,\alpha,\gamma)) \oplus F_n(T_{(2xy)}(j,\beta,\delta))$$

This means that $H_{(2xy)}(i, \alpha, \gamma)(j, \beta, \delta)$ is the directed graph obtained by taking the arcs of $T_{(2xy)}(i, \alpha, \gamma)$ between parts t and t + 1 for $0 \le t \le n - 2$, and the arcs between parts n - 1 and 0 from $T_{(2xy)}(j, \beta, \delta)$.

Now we have all the usual results:

Lemma 2.7.14 Let x, y and n be odd. Then:

$$H_{(2xy)}(i,\alpha,\gamma)(j,\beta,\delta) = H_{(2x)}(i,\alpha)(j,\beta) \otimes H_y(\gamma,\delta)$$

Proof: Notice that

$$F_n(T_{(2xy)}(i,\alpha,\gamma)) = F_n(T_{(2x)}(i,\alpha) \otimes T_y(\gamma)) = F_n(T_{(2x)}(i,\alpha)) \otimes F_n(T_y(\gamma))$$

Using this the result is trivial.

Lemma 2.7.15 Let φ be a bijection on the set $\{(i, \alpha, \gamma) | 0 \le i \le x - 1, 0 \le \alpha \le 3, 0 \le \gamma \le y - 1\}$. Then

$$\overrightarrow{C}_{(4xy:n)} = \bigoplus_{(i,\alpha,\gamma)} H_{(2xy)}(i,\alpha,\gamma)\varphi(i,\alpha,\gamma).$$

Proof: We know that $\overrightarrow{C}_{(4xy:n)} = \overrightarrow{C}_{(4x:n)} \otimes \overrightarrow{C}_{(y:n)} = \left(\bigoplus_{(i,\alpha)} T_{(2x)}(i,\alpha)\right) \otimes \left(\bigoplus_{\gamma} T_y(\gamma)\right)$. By the definition of $T_{(2xy)}(i,\alpha,\gamma)$ we get

$$\overrightarrow{C}_{(4xy:n)} = \bigoplus_{(i,\alpha,\gamma)} T_{(2xy)}(i,\alpha)$$

We also have

$$\bigoplus_{(i,\alpha,\gamma)} T_{(2xy)}(i,\alpha,\gamma) = \bigoplus_{(i,\alpha,\gamma)} H_{(2xy)}(i,\alpha,\gamma)\varphi(i,\alpha,\gamma)$$

Combining both we get:

$$\overrightarrow{C}_{(4xy:n)} = \bigoplus_{(i,\alpha,\gamma)} H_{(2xy)}(i,\alpha,\gamma)\varphi(i,\alpha,\gamma)$$

as we wanted to prove.

We have the following properties:

• If $\alpha \neq \beta$, $\gamma = \delta$, and gcd(x, i - j) = 1, then

$$H_{(2xy)}(i,\alpha,\gamma)(j,\beta,\delta) = H_{(2x)}(i,\alpha)(j,\beta) \otimes H_y(\gamma,\gamma)$$
$$= H_x(i,j) \otimes \Lambda(\alpha,\beta) \otimes T_y(\gamma)$$

is a \overrightarrow{C}_{2xn} -factor by Lemmas 2.6.2, 2.6.7, 2.6.1 and 2.4.3.

• If i = j, $\alpha = \beta$, and $gcd(y, \gamma - \delta) = 1$ then

$$H_{(2xy)}(i,\alpha,\gamma)(j,\beta,\delta) = H_{(2x)}(i,\alpha)(i,\alpha) \otimes H_y(\gamma,\delta)$$
$$= T_x(i) \otimes \Gamma(\alpha) \otimes H_y(\gamma,\delta)$$

is a \overrightarrow{C}_{yn} -factor by Lemmas 2.6.1, 2.6.5, 2.6.2, and 2.4.3.

To get a decomposition of $\overrightarrow{C}_{(4xy:n)}$ into \overrightarrow{C}_{2xn} -factors and \overrightarrow{C}_{yn} -factors we need a bijection φ that satisfies:

Conditions 2.7.16 a) For all (i, α, γ) , $gcd(y, \gamma - \varphi_3(i, \alpha, \gamma)) = 1$ or $\gamma = \varphi_3(i, \alpha, \gamma)$.

b) If $\gamma = \varphi_3(i, \alpha, \gamma)$, then $gcd(x, i - \varphi_1(i, \alpha, \gamma)) = 1$ and $\alpha \neq \varphi_2(i, \alpha, \gamma)$.

c) If
$$gcd(y, \gamma - \varphi_3(i, \alpha, \gamma)) = 1$$
, then $i = \varphi_1(i, \alpha, \gamma)$ and $\alpha = \varphi_2(i, \alpha, \gamma)$.

Now we can write our lemma:

Lemma 2.7.17 Let x, y, and n be odd. Let $s_p \neq 1, 4xy - 1$. Then there is a decomposition of $C_{(4xy:n)}$ into $s_p \ C_{2xn}$ -factors and $r_p = 4xy - s_p \ C_{yn}$ -factors.

Proof: We give a bijection φ that satisfies Conditions 7.16 with r_p elements (i, α, γ) that satisfy $i = \varphi_1(i, \alpha, \gamma)$. Let $s_p = 4xk + q$, with $0 \le q \le 4x - 1$. We have two cases, $k \le y - 3$, and $k \ge y - 2$.

Case 1 If $k \leq y - 3$, let $s_p = 4xk + a - \epsilon$, with $2 \leq a \leq 4x - 1$, $0 \leq \epsilon \leq 2$. For $y - k + 1 \leq \gamma \leq y - 1$, let $\psi_{\gamma}(i, \alpha) = \theta_{4x}(i, \alpha)$, the theta-function with s = 4x. Let $\psi_{y-k}(i, \alpha) = \theta_{4x-\epsilon}(i, \alpha)$, the theta-function with $s = 4x - \epsilon$. Let $\psi_{y-k-1}(i,\alpha) = \theta_a(i,\alpha)$, the theta function with s = a. For $0 \le \gamma \le y - k - 2$ let $\psi_{\gamma}(i,\alpha) = (i,\alpha)$, the identity function.

Case 2 If $k \ge y - 2$, let $s_p = 4xk' + 2a - \epsilon$, with $2 \le a \le 4x - \epsilon$, $0 \le \epsilon \le 4$, where $k' \in \{y-2, y-3\}$ because 2a may be greater than 4x. For $y-k+1 \le \gamma \le y-1$, let $\psi_{\gamma}(i, \alpha) = \theta_{4x}(i, \alpha)$, the theta-function with s = 4x. Let $\psi_{y-k}(i, \alpha) = \theta_{4x-\epsilon}(i, \alpha)$, the theta-function with $s = 4x - \epsilon$. Let $\psi_{y-k-1}(i, \alpha) = \psi_{y-k-2}(i, \alpha) = \theta_a(i, \alpha)$, the theta function with s = a. For $0 \le \gamma \le y - k - 3$ let $\psi_{\gamma}(i, \alpha) = (i, \alpha)$, the identity function.

Notice that the fixed point of θ_{4x-1} is (x-1,2), the fixed points of θ_{4x-2} are $\{(x-1,2), (0,3)\}$, the fixed points of θ_{4x-3} are $\{(x-1,2), (0,3), (x-1,0)\}$, and the fixed points of θ_{4x-4} are $\{(x-1,2), (0,3), (x-1,0), (0,1)\}$. This means that if $0 \le \epsilon \le 4$ and $a \le 4x - \epsilon$, the fixed points of $\theta_{4x-\epsilon}$ are a subset of the fixed points of θ_a . Hence if $\psi_{\delta}(i,\alpha) = (i,\alpha)$, then $\psi_{\gamma}(i,\alpha) = (i,\alpha)$ for all $\gamma \le \delta$. Notice also that $\psi_0(i,\alpha) = \psi_1(i,\alpha)$, hence $\max\{\delta \in \{0,\ldots,y-1\} | \psi_{\delta}(i,\alpha) = (i,\alpha)\} \ne 1$. Therefore we can define $\sigma_{i,\alpha}(\gamma) = \phi_s(\gamma)$, the phi-function over the set $\{0,1,\ldots,y-1\}$ with $s = \max\{\delta \in \{0,\ldots,y-1\} | \psi_{\delta}(i,\alpha) = (i,\alpha)\}$. Then:

$$\rho(i, \alpha, \gamma) = (\psi_{\gamma}(i, \alpha), \sigma_{i, \alpha}(\gamma))$$

is a function satisfying conditions 7.16.

Example 15: We provide two visual examples, with x = 3 and y = 5. To make the picture easier to understand, the points satisfying $i = \varphi_1(i, \alpha, \gamma)$ have been boxed and underlined, and rearranged with their images at the right side.

In Figure 2.12 we have $s_p = 25 = 2 \cdot 12 + 1 = 2 \cdot 12 + 2 - 1$, giving us $r_p = 60 - 25 = 35, k = 2, a = 2, \epsilon = 1.$

In Figure 2.13 we have $s_p = 37 = 3 \cdot 12 + 1 = 3 \cdot 12 + 2 \cdot 2 - 3$, giving us $r_p = 60 - 37 = 23, \ k' = 3, \ a = 2, \ \epsilon = 3.$

2.8 Product

In this section the partite product will be applied to the decompositions obtained in the previous section, to obtain decompositions of larger graphs.

Lemma 2.8.1 Let m = zw with z odd and $w \notin \{2, 6\}$. Then there is a decomposition of $\overrightarrow{C}_{(m:n)}$ into \overrightarrow{C}_{zn} -factors and a decomposition of $\overrightarrow{C}_{(4m:n)}$ into \overrightarrow{C}_{2zn} -factors.

Proof: We know that $\overrightarrow{C}_{(m:n)} = \overrightarrow{C}_{(z:n)} \otimes \overrightarrow{C}_{(w:n)}$. By Theorem 23 we can decompose $\overrightarrow{C}_{(z:n)} = \bigoplus_i H_z(i, \phi(i))$ into \overrightarrow{C}_{zn} -factors, and by Lemma 2.6.1 $\overrightarrow{C}_{(w:n)} = \bigoplus_j T_w(j, j)$ into \overrightarrow{C}_n -factors.



 $x = 3, y = 5, s_p = 25, r_p = 35, k = 2, a = 2, \epsilon = 1$

Figure 2.12: Example of Lemma 7.17 Case 1

$$\begin{split} \gamma &= 4 \\ s &= 12 \\ \begin{pmatrix} (0,1,4) & (0,3,4) \\ (2,0,4) & (2,1,4) & (2,2,4) & (2,3,4) \\ (1,0,4) & (1,1,4) & (1,2,4) & (1,3,4) \\ (0,0,4) & (0,2,4) \\ \end{pmatrix} \\ \gamma &= 3 \\ s &= 12 \\ \begin{pmatrix} (0,1,3) & (0,3,3) \\ (2,0,3) & (2,1,3) & (2,2,3) & (2,3,3) \\ (1,0,3) & (1,1,3) & (1,2,3) & (1,3,3) \\ (0,0,3) & (0,2,3) \\ \end{pmatrix} \\ \gamma &= 2 \\ s &= 9 \\ \begin{pmatrix} (0,1,2) & (0,1,2) & (0,3,2) \\ (1,0,2) & (1,1,2) & (1,2,2) & (1,3,2) \\ (1,0,2) & (1,1,2) & (1,2,2) & (1,3,2) \\ (0,0,2) & (0,2,2) \\ \end{pmatrix} \\ \gamma &= 1 \\ s &= 2 \\ \begin{pmatrix} (0,1,2) & (0,1,2) & (0,3,1) \\ (1,0,1) & (1,2,1) & (1,2,2) & (1,3,2) \\ (0,0,2) & (0,2,2) \\ \end{pmatrix} \\ \gamma &= 1 \\ s &= 2 \\ \begin{pmatrix} (0,1,1) & (0,1,1) & (0,1,2) & (1,2,2) \\ (1,0,2) & (1,1,2) & (1,2,2) & (1,3,2) \\ (0,0,2) & (0,2,2) \\ \end{pmatrix} \\ \gamma &= 1 \\ s &= 2 \\ \begin{pmatrix} (0,1,1) & (0,1,1) & (0,2,1) & (0,3,1) \\ (1,0,1) & (0,1,1) & (1,2,1) & (1,3,1) \\ (0,0,1) & (1,1,1) & (1,2,1) & (1,3,1) \\ (0,0,1) & (1,1,1) & (1,2,1) & (1,3,1) \\ (0,0,1) & (1,1,0) & (2,2,0) & (2,3,0) \\ s &= 2 \\ \begin{pmatrix} (0,1,1) & (0,1,1) & (0,2,1) & (0,3,1) \\ (1,0,0) & (1,2,0) & (1,3,0) \\ (1,0,0) & (1,0,0) & (1,2,0) & (1,3,0) \\ (0,2,0) & (0,2,0) \\ \end{pmatrix} \\ \gamma &= 0 \\ s &= 2 \\ \begin{pmatrix} (0,2,1) & (0,1,2) & (0,2,1) & (0,3,0) \\ (1,0,0) & (0,2,0) & (0,3,0) \\ (1,0,0) & (0,2,0) & (1,3,0) \\ (0,2,0) & (0,2,0) \\ \end{pmatrix} \\ \end{array}$$

 $x = 3, y = 5, s_p = 37, r_p = 23, k = 4, a = 2, \epsilon = 3, k' = 3$

Figure 2.13: Example of Lemma 7.17 Case 2

Then

$$\overrightarrow{C}_{(m:n)} = \overrightarrow{C}_{(z:n)} \otimes \overrightarrow{C}_{(w:n)}$$
$$= \left(\bigoplus_{i} H_{z}(i,\phi(i))\right) \otimes \left(\bigoplus_{j} H_{w}(j,j)\right)$$
$$= \bigoplus_{i} \bigoplus_{j} H_{z}(i,\phi(i)) \otimes H_{w}(j,j)$$

By Lemma 2.4.3 $H_z(i, \phi(i)) \otimes H_w(j, j)$ is a \overrightarrow{C}_{zn} -factor, and so $\overrightarrow{C}_{(m:n)}$ can be decomposed into \overrightarrow{C}_{zn} -factors.

For the result on $\overrightarrow{C}_{(4m:n)}$ we just multiply by $\overrightarrow{C}_{(4:n)}$:

$$\overrightarrow{C}_{(4m:n)} = \overrightarrow{C}_{(m:n)} \otimes \overrightarrow{C}_{(4:n)}$$

We can decompose $\overrightarrow{C}_{(4:n)}$ into \overrightarrow{C}_{2n} -factors by Theorem 24 and so when multiplying by $\overrightarrow{C}_{(m:n)}$ we obtain a decomposition of $\overrightarrow{C}_{(4m:n)}$ into \overrightarrow{C}_{2zn} -factors.

We may now use Lemma 2.8.1 and the decompositions obtained in Section 7 to get a decomposition of $\overrightarrow{C}_{(xyzw:n)}$ into \overrightarrow{C}_{xzn} -factors and \overrightarrow{C}_{yzn} -factors:

Lemma 2.8.2 Suppose $\overrightarrow{C}_{(m:n)}$ can be decomposed into $s_p \overrightarrow{C}_{m_1n}$ -factors and $r_p \overrightarrow{C}_{m_2n}$ factors, with $s_p, r_p \neq 1$, $r_p + s_p = m$. Let z be odd with $gcd(m_1, z) = gcd(m_2, z) = 1$ and $w \notin \{2,6\}$. Then there is a decomposition of $\overrightarrow{C}_{(mzw:n)}$ into $s \overrightarrow{C}_{m_1zn}$ -factors and $r \overrightarrow{C}_{m_2zn}$ -factors for any $s, r \neq 1, s + r = mzw$. Furthermore, if m_1 and m_2 are odd, there is a decomposition of $\overrightarrow{C}_{(4mzw:n)}$ into $s' \overrightarrow{C}_{2m_1zn}$ -factors and $r' \overrightarrow{C}_{2m_2zn}$ -factors for any $s', r' \neq 1, s' + r' = 4mzw$.

Proof: We start with the product $\overrightarrow{C}_{(mzw:n)} = \overrightarrow{C}_{(m:n)} \otimes \overrightarrow{C}_{(zw:n)}$. By Lemma 2.8.1 we can decompose $\overrightarrow{C}_{(zw:n)} = \bigoplus_{i=1}^{zw} Z_i$, where each Z_i is a \overrightarrow{C}_{zn} -factor.

Let s = mt + u, with $0 \le u \le m$. If $u \ne 1, m - 1$ we decompose as follows:

$$\vec{C}_{(mzw:n)} = \vec{C}_{(m:n)} \otimes \vec{C}_{(zw:n)}$$

$$= \vec{C}_{(m:n)} \otimes (\bigoplus_{i=1}^{zw} Z_i)$$

$$= \bigoplus_{i=1}^{zw} \vec{C}_{(m:n)} \otimes Z_i$$

$$= \left(\bigoplus_{i=1}^t \vec{C}_{(m:n)} \otimes Z_i \right) \oplus \vec{C}_{(m:n)} \otimes Z_{t+1} \oplus \left(\bigoplus_{i=t+2}^{zw} \vec{C}_{(m:n)} \otimes Z_i \right)$$

From the theorem hypothesis on $\overrightarrow{C}_{(m:n)}$, we have the following decompositions:

- We decompose $\overrightarrow{C}_{(m:n)}$ into $m \overrightarrow{C}_{m_1n}$ -factors for the product $\bigoplus_{i=1}^t \overrightarrow{C}_{(m:n)} \otimes Z_i$.
- We decompose $\overrightarrow{C}_{(m:n)}$ into $m \overrightarrow{C}_{m_2n}$ -factors for the product $\bigoplus_{i=t+2}^{zw} C_{(m:n)} \otimes Z_i$.

• We decompose $\overrightarrow{C}_{(m:n)}$ into $u \overrightarrow{C}_{m_1n}$ -factors and $m - u \overrightarrow{C}_{m_2n}$ -factors for the product $\overrightarrow{C}_{(m:n)} \otimes Z_{t+1}$.

Because m_1 and m_2 are coprime with z, by Lemma 2.4.3 there is a decomposition into $mt + u = s \overrightarrow{C}_{m_1 z n}$ -factors and $r \overrightarrow{C}_{m_2 z n}$ -factors.

If u = 1, we decompose as follows:

$$\overrightarrow{C}_{(mzw:n)} = \left(\bigoplus_{i=1}^{t-1} \overrightarrow{C}_{(m:n)} \otimes Z_i \right) \oplus \left(\overrightarrow{C}_{(m:n)} \otimes Z_t \right) \oplus \left(\overrightarrow{C}_{(m:n)} \otimes Z_{t+1} \right) \oplus \left(\bigoplus_{i=t+2}^{zw} \overrightarrow{C}_{(m:n)} \otimes Z_i \right)$$

We also have the following decompositions:

- We decompose $\overrightarrow{C}_{(m:n)}$ into $m \overrightarrow{C}_{m_1n}$ -factors for the product $\bigoplus_{i=1}^{t-1} \overrightarrow{C}_{(m:n)} \otimes Z_i$.
- We decompose $\overrightarrow{C}_{(m:n)}$ into $m \overrightarrow{C}_{m_2n}$ -factors for the product $\bigoplus_{i=t+2}^{zw} \overrightarrow{C}_{(m:n)} \otimes Z_i$.
- We decompose $\overrightarrow{C}_{(m:n)}$ into m-2 $\overrightarrow{C}_{m_1n}$ -factors and 2 $\overrightarrow{C}_{m_2n}$ -factors for the product $\overrightarrow{C}_{(m:n)} \otimes Z_t$.
- We decompose $\overrightarrow{C}_{(m:n)}$ into 3 $\overrightarrow{C}_{m_1n}$ -factors and m-3 $\overrightarrow{C}_{m_2n}$ -factors for the product $\overrightarrow{C}_{(m:n)} \otimes Z_{t+1}$.

Because m_1 and m_2 are coprime with z, this gives a decomposition into $m(t-1) + m_1$

 $m-2+3 = mt+1 = s \ \overrightarrow{C}_{m_1 zn}$ -factors and $r \ \overrightarrow{C}_{m_2 zn}$ -factors. If u = m-1 we just change the roles of m_1 and m_2 and take the decomposition for u = 1. Therefore there is a decomposition of $\overrightarrow{C}_{(m z w:n)}$ into $s \ \overrightarrow{C}_{m_1 zn}$ factors and $r \ \overrightarrow{C}_{m_2 zn}$ -factors for any $s, r \neq 1, r + s = m z w$. The decomposition of $\overrightarrow{C}_{(4m z w:n)}$ into $s' \ \overrightarrow{C}_{2m_1 zn}$ -factors and $r' \ \overrightarrow{C}_{2m_2 zn}$ -factors works in the same way.

We can now combine this result with our decompositions from Section 7 to obtain the following result, which we write using non-directed graphs, as we are getting ready to apply the results from Section 5.

Theorem 26 Let x, y, z, n be odd numbers with gcd(x, z) = gcd(y, z) = 1 and $w \notin \{2, 6\}$. Then we have the following decompositions:

- a) $C_{(xyzw:n)}$ can be decomposed into $s C_{xzn}$ -factors and $r C_{yzn}$ -factors for any $s, r \neq 1$, s + r = xyzw.
- b) $C_{(4xzw:n)}$ can be decomposed into $s C_{2xzn}$ -factors and $r C_{2zn}$ -factors for any $s, r \neq 1$, s + r = 4xzw.
- c) $C_{(4xzw:n)}$ can be decomposed into $s C_{2xzn}$ -factors and $r C_{zn}$ -factors for any $s, r \neq 1$, s + r = 4xzw.
- d) $C_{(4xzw:n)}$ can be decomposed into s C_{xzn} -factors and r C_{2zn} -factors for any $s, r \neq 1$,

$$s + r = 4xzw.$$

- e) $C_{(4xyzw:n)}$ can be decomposed into $s C_{2xzn}$ -factors and $r C_{yzn}$ -factors for any $s, r \neq 1$, s + r = 4xyzw.
- f) $C_{(4xyzw:n)}$ can be decomposed into $s C_{2xzn}$ -factors and $r C_{2yzn}$ -factors for any $s, r \neq 1$, s + r = 4xyzw.

Proof:

- a) $\overrightarrow{C}_{(xy:n)}$ can be decomposed into $s_p \overrightarrow{C}_{xn}$ -factors and $r_p \overrightarrow{C}_{yn}$ -factors by Lemma 2.7.5. So by Lemma 2.8.2; $\overrightarrow{C}_{(xywz:n)}$ can be decomposed into $s \overrightarrow{C}_{xzn}$ -factors and $r \overrightarrow{C}_{yzn}$ -factors.
- b) $\overrightarrow{C}_{(x:n)}$ can be decomposed into $s_p \overrightarrow{C}_{xn}$ -factors and $r_p \overrightarrow{C}_n$ -factors by Lemma 23. Now apply Lemma 2.8.2.
- c) $\overrightarrow{C}_{(4x:n)}$ can be decomposed into $s_p \overrightarrow{C}_{2xn}$ -factors and $r_p \overrightarrow{C}_n$ -factors by Lemma 2.7.11. Now apply Lemma 2.8.2.
- d) $\overrightarrow{C}_{(4x:n)}$ can be decomposed into $s_p \ \overrightarrow{C}_{xn}$ -factors and $r_p \ \overrightarrow{C}_{2n}$ -factors by Lemma 2.7.12. Now apply Lemma 2.8.2.
- e) $\overrightarrow{C}_{(4xy:n)}$ can be decomposed into $s_p \ \overrightarrow{C}_{2xn}$ -factors and $r_p \ \overrightarrow{C}_{yn}$ -factors by Lemma 2.7.17. Now apply Lemma 2.8.2.

f) $\overrightarrow{C}_{(xy:n)}$ can be decomposed into $s_p \ \overrightarrow{C}_{xn}$ -factors and $r_p \ \overrightarrow{C}_{yn}$ -factors by Lemma 2.7.5. Now apply Lemma 2.8.2.

2.9 Main Result

We now to use the decompositions that we obtained for $C_{(v:n)}$ to obtain decompositions of $K_{(v:m)}$ via Lemmas 2.5.3 and 2.5.1.

Theorem 27 Let m and n be odd, such that $m \equiv 0 \pmod{n}$. Let s and r be such that $s, r \neq 1$ and $s+r = v \frac{m-1}{2}$. Let $x_1, \ldots, x_{m/n}, y_1, \ldots, y_{m/n}, z_1, \ldots, z_{m/n}$ and $w_1, \ldots, w_{m/n}$ be such that:

- $gcd(x_i, z_i) = gcd(y_i, z_i) = 1;$
- $w_i \notin \{2, 6\};$
- 2 divides at most one of x_i, y_i and z_i ;
- $v = x_i y_i z_i w_i$ if 2 divides none of x_i, y_i, z_i ; and
- $v = 2x_iy_iz_iw_i$ if 2 divides one of x_i, y_i, z_i .

Furthermore, let F_1 be a $[(x_1z_1n)^{\frac{v}{x_1z_1}}, \ldots, (x_{m/n}z_{m/n}n)^{\frac{v}{x_{m/n}z_{m/n}}}]$ -factor, and let F_2 be a $[(y_1z_1n)^{\frac{v}{y_1z_1}}, \ldots, (y_{m/n}z_{m/n}n)^{\frac{v}{y_{m/n}z_{m/n}}}]$ -factor. Then there is a decomposition of $K_{(v:m)}$ into s copies of F_1 and r copies of F_2 .

Proof: By Theorem 22 there is a decomposition of K_m into C_n -factors.

Pick $s_1, \ldots, s_{\frac{m-1}{2}}$ such that $s = \bigoplus_{i=1}^{v^{\frac{m-1}{2}}} s_i$, with $0 \le s_i \le v$ and $s_i \notin \{1, v-1\}$.

By Theorem 26 there is a decomposition of $C_{(v:n)}$ into $s_i C_{x_i z_i n}$ -factors and $r_i = v - s_i$ $C_{y_i z_i n}$ -factors.

Therefore by Theorem 2.5.3 there is a decomposition of $K_{(v:m)}$ into s copies of the 2-factor F_1 and r copies of the 2-factor F_2 .

2.10 Applications

We can use our results to solve many cases of the Hamilton-Waterloo Problem for complete graphs. For some of them we will need the notion of resolvable group divisible design.

A resolvable group divisible design (k, λ) -RGDD (h^u) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where \mathcal{V} is a finite set of size v = hu, \mathcal{G} is a partition of \mathcal{V} into u groups each containing h elements, and \mathcal{B} is a collection of k element subsets of \mathcal{V} called *blocks* which satisfy the following properties.

- If $B \in \mathcal{B}$, then |B| = k.
- If a pair of elements from \mathcal{V} appear in the same group, then the pair cannot be in any block.
- Two points that are not in the same group, called a *transverse pair*, appear in exactly λ blocks.
- $|\mathcal{G}| > 1.$
- The blocks can be partitioned into parallel classes such that for each element of \mathcal{V} there is exactly one block in each parallel class containing it.

Here we use the term group to indicate an element of \mathcal{G} . In this context, group simply means a set of elements without any algebraic structure. If $\lambda = 1$, we refer to the RGDD as a k-RGDD (h^u) .

In [35] the following characterization theorem was proven:

Theorem 28 [35] $A(3,\lambda)$ -RGDD (h^u) exists if and only if $u \ge 3, \lambda h(u-1)$ is even, $hu \equiv 0 \pmod{3}$, and $(\lambda, h, u) \notin \{(1, 2, 6), (1, 6, 3)\} \bigcup \{(2j + 1, 2, 3), (4j + 2, 1, 6) : j \ge 0\}.$

In [5] the authors used resolvable group divisible designs together with Theorem 28

to decompose complete graphs into C_3 -factors and C_{3x} -factors.

Lemma 2.10.1 [5] Let $x \ge 3$, $y \ge 3$ and m be positive integers such that both x and y divide 3m. Suppose the following conditions are satisfied:

- There exists a 3-RGDD (h^u) ,
- there exists a decomposition of $K_{(m:3)}$ into $s_p \ C_x$ -factors and $r_p \ C_y$ -factors, for $p \in \{1, 2, \dots, \frac{h(u-1)}{2}\},$
- there exists a decomposition of K_{hm} into $s_{\beta} C_x$ -factors and $r_{\beta} C_y$ -factors.

Let

$$s_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}} s_p \text{ and } r_{\alpha} = \sum_{p=1}^{\frac{h(u-1)}{2}} r_p.$$

Then there exists a decomposition of K_{hum} into $s_{\alpha} + s_{\beta} C_x$ -factors and $r_{\alpha} + r_{\beta} C_y$ -factors.

We can now apply our decompositions to extend the result from [5]. We will be concerned with prime numbers whose greatest power that divides x is the same as their greatest power that divides y. Thus we give the following definition:

Definition 2.10.2 Let x and y be natural numbers, with $x = p_1^{a_1} \dots p_n^{a_n}$ and $y = p_1^{b_1} \dots p_n^{b_n}$ their prime factorization. Then we define the special greatest common divisor of x and y, s(x, y) as the smallest number such that $p_i^{a_i}$ divides s(x, y) if and

only if $a_i = b_i$.

Example 16: For example if $x = 2^3 3^2 5^2 7^2$ and $y = 3^2 5^3 7^2 11^4$, then $s(x, y) = 3^2 7^2$.

Corollary 2.10.3 Let x, y, n be integers such that

- $\frac{xy}{s(x,y)}$ divides n,
- $x, y \not\equiv 0 \pmod{4}$, and 4 divides n if 2 divides xy;
- $3n = \frac{huxyw}{s(x,y)}$, with $h \equiv 0 \pmod{3}$, $u \geq 3$, h(u-1) even, and $(h,u) \notin \{(2,6), (6,3)\}$.

Then there exists a decomposition of K_{3n} into $s \ C_{3x}$ -factors and $r \ C_{3y}$ -factors for every pair (s,r) such that $s + r = \lfloor \frac{3n-1}{2} \rfloor$, $s, r \neq 1$.

Proof: Let (s, r) be such that $s + r = \lfloor \frac{3n-1}{2} \rfloor$ and $s, r \neq 1$. If $s \geq r$ let $s_0 = \lfloor \frac{3n-1}{2} \rfloor$ and $r_0 = 0$. Otherwise let $s_0 = 0$ and $r_0 = \lfloor \frac{3n-1}{2} \rfloor$. Let $s_1, \ldots, s_{\frac{h(u-1)}{2}}$ and $r_1, \ldots, r_{\frac{h(u-1)}{2}}$ be such that $s_i + r_i = \frac{xyw}{s(x,y)}$, $s_i, r_i \neq 1$ for all i and

$$r_{\alpha} = \sum_{i=0}^{\frac{h(u-1)}{2}} r_i \text{ and } s_{\alpha} = \sum_{i=0}^{\frac{h(u-1)}{2}} s_i.$$
From Theorem 28 we know that there is a 3-RGDD (h^u) . Let z = s(x, y), $x_1 = \frac{x}{z}$ and $y_1 = \frac{y}{z}$. Then

$$\frac{xyw}{z} = \frac{x_1 z y_1 z w}{z} = x_1 y_1 z w$$

So we may apply Theorem 27 to obtain a decomposition of $K_{\left(\frac{xyw}{z}:3\right)}$ into $s_i \ C_{3x_1z}$ -factors and $r_i \ C_{3y_1z}$ -factors for each *i*.

Because $hxyw = hx_1zyw = hxy_1zw$ we have that $3x_1z|(hxyw)$ and $3y_1z|(hxyw)$. From Theorem 22 there is a decomposition of $K_{\frac{hxyw}{z}}$ into C_{3x_1z} -factors, and there is also a decomposition of $K_{\frac{hxyw}{z}}$ into C_{3y_1z} -factors.

Thus we may apply Lemma 2.10.1 to obtain a decomposition of $K_{\frac{huxyw}{z}} = K_{3n}$ into s C_{3x} -factors and $r C_{3y}$ -factors.

We can also use Lemma 2.5.1 to obtain decompositions of complete graphs into C_x -factors and C_y -factors:

Corollary 2.10.4 Let m, x, and y be integers such that:

•
$$z = s\left(x, y\right), w = \frac{\gcd(x, y)}{z} \ge 2,$$

- $\frac{xy}{z}$ divides m,
- 4 does not divide x nor y.

• Neither x nor y is 3 if $\frac{m}{w} \in \{6, 12\}$.

Then there exists a decomposition of K_m into s C_x -factors and r C_y -factors for every $s, r \neq 1$.

Proof: Let s, r be such that $s + r = \lfloor \frac{m-1}{2} \rfloor$ and $s, r \neq 1$.

Let

$$k = \frac{mz}{xy} \quad x' = \frac{x}{zw} \quad y' = \frac{y}{zw} \quad m' = \frac{m}{w} = \frac{xyk}{zw} = x'y'zwk$$

Let $s_{\alpha}, s_{\beta}, r_{\alpha}, r_{\beta}$ be such that $s_{\beta}, r_{\beta} \neq 1$, $\{s_{\alpha}, r_{\alpha}\} = \{0, \lfloor \frac{m'-1}{2} \rfloor\}$, $s = s_{\alpha} + s_{\beta}$ and $r = r_{\alpha} + r_{\beta}$. By Theorem 27 there is a decomposition of $K_{(m':w)}$ into $s_{\beta} C_{x'zw}$ -factors and $r_{\beta} C_{y'zw}$ -factors. This is a decomposition of $K_{(m':w)}$ into $s_{\beta} C_x$ -factors and $r_{\beta} C_y$ -factors. Because $\frac{xy}{z}$ divides m, it follows that both x and y divide $m' = \frac{m}{w} = m \frac{z}{\gcd(x,y)}$. Thus by Theorem 22 there is decompositon of $K_{m'}$ into $s_{\alpha} C_x$ -factors and $r_{\alpha} C_y$ -factors (keep in mind that one of s_{α} and r_{α} is 0). Then by Lemma 2.5.1 there is a decomposition of $K_{m'w}$ into $s C_x$ -factors and $r C_y$ -factors.

Notice that by asking $\frac{xy}{z}$ to divide *m* we cover some of the cases left open in [13] for the odd order case.

Example 17: Let $m = 3^{3}5^{3}$, $x = 3^{1}5^{2}$ and $y = 3^{2}5^{2}$. We have:

$$z = s(x, y) = 5^2,$$
 $w = 3^1,$ $k = 5^1$

And $\frac{xy}{z} = 3^3 5^2$ divides m. So we can decompose K_m into $s C_x$ -factors and $r C_y$ -factors for any $s, r \neq 1$. Note that $l = \text{lcm}(x, y) = 3^2 5^2$. The number of vertices, m, is a multiple of l, however xy / m. Thus, Theorem 20 cannot be applied here.

Example 18: Let $m = 4^{1}3^{4}5^{3}7^{1}$, $x = 2^{1}3^{1}5^{2}7^{1}$ and $y = 3^{3}5^{2}$. We have:

$$z = s(x, y) = 5^2,$$
 $w = 3^1,$ $k = 2^1 5^1$

And $\frac{xy}{z} = 2^1 3^4 5^2 7^1$ divides m. So we can decompose K_m into $s C_x$ -factors and $r C_y$ -factors for any $s, r \neq 1$. Note that because x is even Theorem 20 cannot be applied here.

We can also use our Lemma 2.5.2 to obtain non-uniform decompositions of complete graphs:

Corollary 2.10.5 Let $v, m, n, x_1, \ldots, x_k, y_1, \ldots, y_k$ be integers such that:

• $m \ge 3$ is odd,

- n divides m, x_i , and y_i for every i,
- $k = \frac{m}{n}$,

•
$$z_i = s\left(x_i, y_i\right),$$

- $\frac{x_i y_i}{z_i n}$ divides v for each i,
- x_i divides v for each i,
- 4 does not divide x_i nor y_i for any i,
- $3 \notin \{x_1, \dots, x_k, y_1, \dots, y_k\}$ if $k \in \{6, 12\}$.

Then there exists a decomposition of K_{vm} into $s [x_1^{vn/x_1}, \ldots, x_k^{vn/x_k}]$ -factors and $r [y_1^{vn/y_1}, \ldots, y_k^{vn/y_k}]$ -factors for every $s, r \neq 1$.

Proof: Let s, r be such that $s + r = \lfloor \frac{vm-1}{2} \rfloor$ and $s, r \neq 1$.

Let

$$x'_i = \frac{x_i}{z_i n} \quad y'_i = \frac{y_i}{z_i n} \quad k_i = \frac{v z_i n}{x_i y_i}$$

Then

$$v = \frac{x_i y_i k_i}{z_i n} = \frac{x'_i z_i n y'_i z_i n k_i}{z_i n} = x'_i y'_i z_i n k_i$$

Let $s_{\alpha}, s_{\beta}, r_{\alpha}, r_{\beta}$ be such that $s_{\beta}, r_{\beta} \neq 1$, $\{s_{\alpha}, r_{\alpha}\} = \{0, \lfloor \frac{v-1}{2} \rfloor\}$, $s = s_{\alpha} + s_{\beta}$ and $r = s_{\alpha} + s_{\beta}$

 $r_{\alpha} + r_{\beta}$. By Theorem 27 there is a decomposition of $K_{(v:m)}$ into $s_{\beta} [x_1^{vn/x_1}, \ldots, x_k^{vn/x_k}]$ factors and $r_{\beta} [y_1^{vn/y_1}, \ldots, y_k^{vn/y_k}]$ -factors. We partition mK_v into k copies of nK_v ,
labeled $\kappa_1, \ldots \kappa_k$. Because $\frac{x_i y_i}{z_i n}$ and x_i divide v, we get that both x_i and y_i divide v,
by Theorem 22 there is a decomposition of K_v into $s_{\alpha} C_{x_i}$ -factors and $r_{\alpha} C_{y_i}$ -factors
(keep in mind that one of s_{α} and r_{α} is 0). This means that κ_i can be decomposed into $s_{\alpha} [x_i^{vn/x_i}]$ -factors and $r_{\alpha} [y_i^{vn/y_i}]$ -factors. Combining these decompositions we get a
decomposition of mK_v into $s_{\alpha} [x_1^{vn/x_1}, \ldots, x_k^{vn/x_k}]$ -factors and $r_{\alpha} [y_1^{vn/y_i}, \ldots, y_k^{vn/y_k}]$ factors.

Then by Lemma 2.5.2 there is a decomposition K_{vm} into $s [x_1^{vn/x_1}, \ldots, x_k^{vn/x_k}]$ -factors and $r [y_1^{vn/y_1}, \ldots, y_k^{vn/y_k}]$ -factors for every $s, r \neq 1$.

Example 19: Let $v = 5^3 7^2 11^3 13^4$, $m = 3^1 5^1$, n = 5, k = 3, $x_1 = 5^2 7^2$, $y_1 = 5^1 7^2 11^1 13^1$, $x_2 = 5^1 11^1 13^1$, $y_2 = 5^1 7^2 13^3$, $x_3 = 5^2 7^1 11^1 13^4$, and $y_3 = 5^1$. We have:

$$z_{1} = s (x_{1}, y_{1}) = 7^{2}, \qquad \frac{x_{1}y_{1}}{z_{1}n} = 5^{2}7^{2}$$
$$z_{2} = s (x_{2}, y_{2}) = 5^{1}, \qquad \frac{x_{2}y_{2}}{z_{2}n} = 7^{2}11^{1}13^{4}$$
$$z_{3} = s (x_{3}, y_{3}) = 1, \qquad \frac{x_{3}y_{3}}{z_{3}n} = 5^{2}7^{1}11^{1}13^{4}$$

We can see that $\frac{x_i y_i}{z_i n}$ and x_i divide v for each i.

Let F_1 be the 2-factor consisting of:

• $\frac{vn}{x_1} = 5^2 11^3 13^4$ cycles of size $x_1 = 5^2 7^2$,

•
$$\frac{vn}{x_2} = 5^3 7^2 11^2 13^3$$
 cycles of size $x_2 = 5^1 11^1 13^1$,

• and
$$\frac{vn}{x_3} = 5^2 6^1 11^2$$
 cycles of size $x_3 = 5^2 7^1 11^1 13^4$.

Let F_2 be the 2-factor consisting of:

•
$$\frac{vn}{y_1} = 5^3 11^2 13^3$$
 cycles of size $y_1 = 5^1 7^2 11^1 13^1$,

•
$$\frac{vn}{y_2} = 5^3 11^3 13^1$$
 cycles of size $y_2 = 5^1 7^2 13^3$,

• and $\frac{vn}{y_3} = 5^3 7^2 11^3 13^4$ cycles of size $y_3 = 5^1$.

Then we can decompose K_{vm} into s copies of F_1 and r copies of F_2 for any $s, r \neq 1$.

2.11 Appendix

Lemma 2.11.1 There is a decomposition of $C_{(12:3)}$ into 7 C_{12} -factors and 5 C_{6} -factors.

Proof: Let G_0 , G_1 , and G_2 be the partite sets. We will construct each factors by taking a difference between each pair of partite sets. Notice that if the sum of the three differences is congruent to 6 modulo 12, this gives a C_6 -factor. Likewise, if the sum of the differences is congruent to 4 or 8 modulo 12 we get a C_{12} -factor.

The	C_6	factors	are	obtained	by	taking	differences:
					•/	()	

G_0 to G_1	G_1 to G_2	G_2 to G_3	total mod 12
0	3	3	6
1	1	4	6
2	2	2	6
3	9	6	6
4	4	10	6

G_0 to G_1	G_1 to G_2	G_2 to G_3	total mod 12
5	10	5	8
6	6	8	8
7	0	9	4
8	5	7	8
9	7	0	4
10	11	11	8
11	8	1	8

The C_{12} -factors are obtained by taking differences:

Because all possible difference between each pair of partite sets has been taken once, this provides the desired decomposition.

Lemma 2.11.2 There is a decomposition of $C_{(12:n)}$ into 7 C_{3n} -factors and 5 C_{2n} -factors, for any $n \ge 5$.

Proof: We have to pick differences between n pairs of partite sets. For the first n-3 differences sets choose differences that add up to 0. This can be achieved by taking the difference one would take to decompose $C_{(12:n-3)}$ into C_{n-3} -factors. For the last three differences, choose in the same way as we did in the previous lemma.

Chapter 3

Finite Abelian Groups And Sequenceability¹

3.1 Introduction

In 1961 B. Gordon [19] defined a group G to be *sequenceable* when there exists a permutation

 $g_0, g_1, g_2, \ldots, g_{n-1}$

¹The material in this chapter has been submitted to Australasian Journal of Combinatorics

of its elements so that the sequence of partial products

$$g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1g_2\cdots g_{n-1}$$

are distinct. In that same paper he proved the following theorem.

Theorem 3.1.1 A finite abelian group G is sequenceable if and only if its Sylow 2-subgroup is non-trivial and cyclic.

In 1974 G. Ringel [36] asked when there exists a permutation

$$g_1, g_2, \ldots, g_{n-1}$$

of the non-identity elements of a group such that the sequence

$$g_2g_1^{-1}, g_3g_2^{-1}, \dots, g_{n-1}g_{n-2}^{-1}, g_1g_{n-1}^{-1}$$

also is a permutation of the non-identity elements. A group G that admits such a permutation is called *R*-sequenceable. As a matter of fact, L. Paige [33] used this concept in 1951, but it was Ringel's problem that motivated the most important paper on this topic (discussed below).

We now provide a context which establishes the close connection between the two

concepts. Given a group G and a subset S of G such that S does not contain the identity element of G, we define the Cayley digraph $\overrightarrow{Cay}(G; S)$ by letting its vertices be the elements of G and having an arc (g_1, g_2) if and only if $g_2 = g_1 s$ for some $s \in S$. One special such Cayley digraph in which we are particularly interested is when $S = G - \{1\}$, that is, the set S has everything in it other than the identity element. We use the special notation $\overrightarrow{K}(G)$ for this Cayley digraph.

It is easy to see that a fixed element $s \in S$ generates a subdigraph consisting of directed cycles whose lengths are all |s|, where |s| denotes the order of s. Thus, we obtain a factorization of $\overrightarrow{Cay}(G; S)$ into |S| directed 2-factors. We call this factorization the *Cayley factorization* of $\overrightarrow{Cay}(G; S)$ and denote it by $\overrightarrow{\mathcal{F}}(G; S)$.

If \overrightarrow{D} is a subdigraph of $\overrightarrow{Cay}(G; S)$ with |S| arcs, and \overrightarrow{D} has exactly one arc from each directed 2-factor in $\overrightarrow{\mathcal{F}}(G; S)$, then we say that \overrightarrow{D} is *orthogonal* to $\overrightarrow{\mathcal{F}}(G; S)$. In this language, the group G is sequenceable when $\overrightarrow{K}(G)$ admits an orthogonal Hamilton directed path, and G is R-sequenceable when $\overrightarrow{K}(G)$ admits an orthogonal directed cycle of length |G| - 1.

In spite of the similarity between these two concepts, they arose from quite different settings. Gordon was interested in complete Latin squares, whereas, Ringel was considering embeddings of complete graphs into orientable surfaces of positive genus.

We now say a few words about some notational conventions in this paper. We use

(x, y) to denote an arc from x to y in a digraph, and xy to denote an edge joining x and y in a graph. Continuing in this vein, $(x_1, x_2, x_3, \ldots, x_n)$ denotes a directed path of length n - 1, $(x_1, x_2, \ldots, x_n, x_1)$ denotes a directed cycle of length n, xy denotes an edge joining x and y in a graph, $x_1x_2 \ldots x_n$ denotes a path of length n - 1 in a graph and $x_1x_2 \ldots x_nx_1$ denotes a cycle of length n in a graph. We use cyclic notation for permutations and in order to distinguish permutations from directed paths, we are careful with the exposition. Thus, as a permutation, (1, 2, 3, 4) is the cyclic permutation mapping 4 to 1, and i to i + 1 for i = 1, 2, 3.

For the rest of this paper, we consider only finite abelian groups and use additive notation with one exception. For the direct sum of a copies of the cyclic group Z_n , we write Z_n^a rather than aZ_n .

As mentioned above, R. Friedlander, B. Gordon and M. Miller [18] wrote the most significant paper on Ringel's problem. They conjectured that if G is a finite abelian group whose Sylow 2-subgroup is either trivial or both non-trivial and non-cyclic, then G is R-sequenceable. (In other words, the conjecture is saying that if G is not covered by Theorem 3.1.1, then it is R-sequenceable.) They established that the conjecture holds in many cases and introduced the following important strengthening of R-sequenceability. If $\overrightarrow{C} = (g_1, g_2, \ldots, g_{n-1}, g_1)$ is a directed cycle of length n-1 that is orthogonal to $\overrightarrow{K}(G)$, where G is an abelian group of order n, with the additional properties that 0 is the vertex missed by \overrightarrow{C} , and there exist three successive elements g_i, g_{i+1}, g_{i+2} on \overrightarrow{C} such that $g_i + g_{i+2} = g_{i+1}$, then we say that G is R^* -sequenceable. We sometimes say that $g_1, g_2, \ldots, g_{n-1}$ is an R^* -sequence.

Friedlander, Gordon and Miller made considerable progress on the conjecture in [18], but did not solve it completely. Nevertheless, several of their results are important tools for the general conjecture. Some of the missing cases were settled in [21, 31, 37]. The proof of the conjecture is completed in this paper. We express the completion in the form of the following theorem that includes all finite abelian groups.

Theorem 3.1.2 If G is a finite abelian group, then the following hold:

(1) G is sequenceable if the Sylow 2-subgroup is cyclic and non-trivial; and

(2) G is R-sequenceable if the Sylow 2-subgroup either is trivial, or the Sylow 2subgroup is non-trivial and non-cyclic.

3.2 First Stage of Proof

Part (1) of Theorem 3.1.2 is covered by Theorem 3.1.1. So we move to part (2) which has a natural partition into two subcases. The first subcase is that G has even order with its Sylow 2-subgroup non-trivial and non-cyclic. The second subcase is that Ghas odd order, that is, the Sylow 2-subgroup is trivial. We consider the first subcase next beginning with some useful results from [18]. **Lemma 3.2.1** The cyclic group Z_n is R^* -sequenceable for all odd n > 5.

Lemma 3.2.2 Let G be an R^* -sequenceable abelian group and Z_n , n > 1, an odd order cyclic group. Then the following hold:

(1) If G has even order, then $G \oplus Z_n$ is R^* -sequenceable; and

(2) If G has odd order, then $G \oplus Z_n$ is R^* -sequenceable whenever 3 does not divide n.

Lemma 3.2.3 Elementary abelian groups are *R*-sequenceable.

The next two results are from [21, 32], respectively.

Lemma 3.2.4 If G is an even order abelian group and its Sylow 2-subgroup is neither Z_2^3 nor $Z_2 \oplus Z_4$, then G is R-sequenceable.

Lemma 3.2.5 If G is R^* -sequenceable, then $Z_2^3 \oplus G$ is R^* -sequenceable.

We now establish a method for handling the missing even order abelian groups. This is inspired by Häggkvist's Lemma in [20]. Consider the cycle $u_0u_1u_2...u_ru_0$. The edge u_iu_j divides the cycle into two subpaths with common end vertices u_i and u_j . The *length* of the edge u_iu_j is the length of the shorter of the two paths unless both subpaths have the same length in which case the length of the edge is (r + 1)/2. **Lemma 3.2.6** If we label the vertices of K_n cyclically as $u_0, u_1, u_2, \ldots, u_{n-1}$, where n = 2m > 4, then there is a Hamilton path whose first edge has length m and every other edge length is used twice.

Proof: When m is odd, start a path with the edge u_0u_m which has length m. Continue with the edge u_mu_1 and then zig zag back and forth decreasing the length by one with each edge until finishing with the edge $u_{(m-1)/2}u_{(m+1)/2}$. We refer to this kind of path as a *zig-zag path*. At this point we have used one edge of each of the lengths $1, 2, 3, \ldots, m$.

Next we add the edge $u_{(m+1)/2}u_{(3m-1)/2}$ which has length m-1. The unused vertices are u_{m+1}, u_{m+2} through $u_{(3m-3)/2}$, of which there are (m-3)/2 such vertices, and $u_{(3m+1)/2}, u_{(3m+3)/2}$ through u_{2m-1} , of which there are (m-1)/2 such vertices. We now continue with an increasing zig-zag path starting with the edge $u_{(3m-1)/2}u_{(3m+1)/2}$ and finishing with the edge $u_{m+1}u_{2m-1}$ of length m-2. The resulting path satisfies the conclusions of the lemma. Figure 3.1 shows the path for m = 5.

The solution when m is even is different in that we describe an iterative procedure for which we show that it results in a path with the desired properties. We require some notation. We denote the current path by P and say the *terminal vertex* of P is the end vertex distinct from u_0 . The interval $I[u_i, u_j]$, $i \leq j$, denotes the set of vertices $\{u_i, u_{i+1}, \ldots, u_j\}$. Suppose P misses the α vertices $I[u_{2m-\alpha}, u_{2m-1}]$. If, in addition, the remaining vertices missed by P are $I[u_1, u_{\alpha-1}]$ and $u_{\alpha+1}$, the terminal vertex of P is u_{α} , and the edge lengths not used twice by P are $2, 3, \ldots, 2\alpha + 1$, then we say the P is R-*sided*. Note that the interval notation makes no sense when $\alpha = 1$. In this case, we treat the interval $[u_1, u_0]$ as empty so that P terminates at u_1 and the vertex u_2 is not on P.

The other possibility is that the remaining vertices missed by P are $I[u_1, u_{\alpha+2}]$ and $u_{2m-\alpha-2}$, the terminal vertex of P is $u_{2m-\alpha-1}$, and the edge lengths not used twice by P are $2, 3, \ldots, 2\alpha + 4$. In this case we say the P is L-sided. The interval notation makes no sense here for $\alpha = 0$. So we treat the interval $[u_{2m}, u_{2m-1}]$ as empty and maintain the remaining conditions.

If P is R-sided with $\alpha \geq 3$, then extend P by adding the 3-path $u_{\alpha}u_{2m-\alpha}u_{\alpha+1}$ $u_{2m-\alpha+2}$. These new edges have lengths $2\alpha - 1, 2\alpha, 2\alpha + 1$ and the terminal vertex of the updated P is now $u_{2m-\alpha+1}$. Thus, P is now L-sided and α has decreased by 3.

On the other hand, if P is L-sided with $\alpha \geq 1$, then extend P by adding the 3-path $u_{2m-\alpha-1}u_{\alpha+2}u_{2m-\alpha-2}u_{\alpha}$. These new edges have lengths $2\alpha + 1$, $2\alpha + 2$, $2\alpha + 3$ and the terminal vertex of the updated P is now u_{α} . Thus, P is now R-sided and α has not changed.



Figure 3.1: A path in K_{10} satisfying the conditions of Lemma 3.2.6



Figure 3.2: A path in K_{16} satisfying the conditions of Lemma 3.2.6

Construct the initial path P by starting with the edge u_0u_m . Then add an increasing zig-zag path starting with the 2-path $u_m u_{m-1} u_{m+1}$ and continue until finishing with the edge from $u_{(3m-2)/2}$ to $u_{m/2}$ of length m-1. Then add the 3-path $u_{m/2}u_{(3m+2)/2}u_{3m/2}u_{(m-4)/2}$ to complete the initial P. Note that P is an R-sided path with $\alpha = (m-4)/2$.

We now begin iterations of the procedure described above and may continue until

we reach a path P that is L-sided with $\alpha = 0$, or R-sided with $\alpha \in \{1, 2\}$. If P is L-sided with $\alpha = 0$, then P terminates at u_{2m-1} , is missing the vertices u_1, u_2, u_{2m-2} and requires edges of lengths 2, 3, and 4. The completion $u_{2m-1}u_1u_{2m-2}u_2$ does the job.

If P is R-sided with $\alpha = 1$, then the terminal vertex is u_1 , the missing vertices are u_{2m-1}, u_2 , and the unused lengths are 2 and 3. The completion $u_1u_{2m-1}u_2$ works.

If P is R-sided with $\alpha = 2$, then the terminal vertex is u_2 , the missing vertices are $u_{2m-2}, u_{2m-1}, u_1, u_3$, and the unused lengths are 2, 3, 4 and 5. There is no completion for this case. If this is the initial P, then m = 8 and Figure 3.2 gives a solution for m = 8. If this is not the initial P, then before the last iteration P was L-sided with $\alpha = 2$. So the vertices missed by P are $u_{2m-4}, u_{2m-2}, u_{2m-1}, I[u_1, u_4]$, the terminal vertex is u_{2m-3} , and the missing lengths are 2 through 8. The completion that works is

$$u_{2m-3}u_3u_{2m-4}u_4u_{2m-1}u_1u_{2m-2}u_2.$$

This completes the proof.

Lemma 3.2.6 allows us to complete the even order case. Suppose the Sylow 2-subgroup of G is $Z_2 \oplus Z_4$. If this is the entire group G, then

$$(0,2), (1,3), (0,3), (1,1), (1,0), (1,2), (0,1)$$

is an *R*-sequence.

Write G as the direct sum of its Sylow subgroups. From the preceding paragraph we may assume that there is a summand of the form Z_q , where q is an odd prime power. So G is a direct sum of $Z_2 \oplus Z_4 \oplus Z_q \cong Z_2 \oplus Z_{4q}$ and an odd order abelian group.

Lemma 3.2.6 tells us that there is a path P (undirected) of length 4q - 1 in K_{4q} , where we are thinking of this as a Cayley graph on Z_{4q} , such that an initial edge of P has length 2q (that is, joins 0 and 2q) and all remaining edge lengths occur twice in P. Display the vertices of $Z_{4q} \oplus Z_2$ as a $2 \times 4q$ array with the obvious coordinate system from Z_2 and Z_{4q} .

Build an undirected cycle C of length 8q - 1 as follows. Join (0, 1) to both (2q, 0)and (2q, 1). Given two edges g_1g_2 and g_3g_4 of the same length in P, join $(g_1, 0)$ to $(g_2, 0)$ and $(g_1, 1)$ to $(g_2, 1)$, and join $(g_3, 0)$ to $(g_4, 1)$ and $(g_3, 1)$ to $(g_4, 0)$. Finally, if g is the terminal vertex of P distinct from 0, join (g, 0) to (g, 1).

The preceding construction yields a cycle C (undirected) of length 8q - 1. Note that the vertex (0,0) is not included in C. Also note that three successive vertices are (2q,0), (0,1), (2q,1) and (2q,1) + (2q,0) = (0,1). Hence, if we direct C in either direction to obtain a directed cycle, both directed cycles provide an R^* -sequence for $Z_2 \oplus Z_{4q}$. As the remaining summands in the direct sum of G have odd order, we may apply part (1) of Lemma 3.2.2 as many times as required to obtain that G is If the Sylow 2-subgroup of G is Z_2^3 , then Lemma 3.2.3 takes care of the case that $G \cong Z_2^3$, and Lemmas 3.2.1 and 3.2.5 takes care of the case that there is a cyclic group of odd order bigger than 5 in the direct sum of Sylow p-subgroups. Also, if both Z_3 and Z_5 appear in the Sylow subgroups of G, then Lemma 3.2.1 tells us that Z_{15} is R^* -sequenceable. Lemma 3.2.5 then takes care of this situation.

So we are left with groups of the form $Z_2^3 \oplus Z_3^a$ and $Z_2^3 \oplus Z_5^b$, where a, b > 0. Following are R^* -sequences for $Z_2^3 \oplus Z_3 \cong Z_2^2 \oplus Z_6$ and $Z_2^3 \oplus Z_5$, respectively:

$$(0, 0, 1), (0, 1, 1), (0, 1, 0), (0, 0, 5), (1, 0, 0), (1, 0, 1), (0, 0, 4), (1, 1, 0), (1, 1, 4), (1, 0, 5), (1, 1, 2), (1, 1, 5), (0, 1, 5), (1, 0, 2), (0, 1, 3), (1, 1, 1), (1, 0, 3), (0, 1, 2), (0, 1, 4), (0, 0, 2), (1, 0, 4), (0, 0, 3), (1, 1, 3)$$

and

(0, 0, 1), (1, 1, 6), (1, 1, 5), (0, 1, 9), (0, 1, 5), (0, 0, 3), (1, 1, 3), (0, 1, 2),(1, 1, 4), (0, 1, 0), (1, 0, 3), (0, 1, 7), (0, 0, 6), (0, 0, 9), (1, 0, 9), (1, 0, 1),(0, 1, 3), (1, 1, 8), (1, 0, 4), (1, 0, 8), (0, 1, 6), (1, 1, 9), (1, 1, 7), (1, 0, 2),(0, 1, 1), (0, 0, 2), (0, 1, 4), (1, 1, 1), (0, 0, 7), (0, 0, 8), (0, 1, 8), (0, 0, 5),(1, 1, 2), (1, 0, 6), (0, 0, 4), (1, 0, 5), (1, 0, 0), (1, 0, 7), (1, 1, 0).

We then use part (1) of Lemma 3.2.2 to obtain that G is R^* -sequenceable for both forms. This completes the proof of Theorem 3.1.2 when G has even order.

3.3 The Gadget

To complete the proof of Theorem 3.1.2 for groups of odd order, we first state the following corollary which is an easy consequence of Lemma 3.2.1 and Lemma 3.2.2.

Corollary 3.3.1 If G is an odd order abelian group whose Sylow 3-subgroup either is trivial, or non-trivial and cyclic, or R^* -sequenceable, then G itself is R^* -sequenceable

unless $G \cong Z_3$ or $G \cong Z_5$ both of which are R-sequenceable.

The preceding corollary means that we need only show that abelian groups whose Sylow 3-subgroups are non-trivial and non-cyclic are *R*-sequenceable. The method we employ works, in fact, for all odd order groups and there is no gain in efficiency by restricting ourelves to those groups satisfying the preceding condition on the Sylow 3-subgroups. Thus, we present the general method.

We work with direct sums. Given the direct sum $G \oplus H$, we shall display the vertices as an $|H| \times |G|$ array, where the columns correspond to the elements of G and the rows correspond to elements of H. We develop some lemmas which prove to be very useful, but we need a definition first.

Definition 3.3.2 Let f be a permutation of H and let $g_1, g_2 \in G$. We define the f-lift of the arc (g_1, g_2) onto $\overrightarrow{K}(G \oplus H)$ to be the set of arcs $\{((g_1, h), (g_2, f(h))) : h \in H\}$. We denote this set of arcs by $\pi_f(g_1, g_2)$.

In spite of the fact we use functional notation for permutations, we compose permutations from left to right because we move through the arrays from left to right. This gives us the composition rule (fg)(x) = g(f(x)).

Lemma 3.3.1 Let G and H be abelian groups. If $(g_1, g_2, \ldots, g_{r+1})$ is a directed path

in $\overrightarrow{K}(G)$ of length r, and f_1, f_2, \ldots, f_r are permutations of H, then the set of arcs

$$\pi_{f_1}(g_1, g_2) \cup \pi_{f_2}(g_2, g_3) \cup \cdots \cup \pi_{f_r}(g_r, g_{r+1})$$

forms n = |H| vertex-disjoint directed paths of length r in $\overrightarrow{K}(G \oplus H)$, where the last vertex of the directed path with initial vertex (g_1, h) is $(g_{r+1}, f_1 f_2 \cdots f_r(h))$.

If $(g_1, g_2, \ldots, g_r, g_1)$ is a directed cycle in $\overrightarrow{K}(G)$ of length r, and f_1, f_2, \ldots, f_r are permutations of Z_n , then the set of arcs

$$\pi_{f_1}(g_1, g_2) \cup \pi_{f_2}(g_2, g_3) \cup \cdots \cup \pi_{f_r}(g_r, g_1)$$

forms vertex-disjoint directed cycles. The number of directed cycles equals the number of cycles in the disjoint cycle decomposition of $f_1 f_2 \cdots f_r$.

Proof: It is easy to see that $\pi_f(g_1, g_2)$ for any permutation f of H generates an orientation of a perfect matching between vertices whose first coordinate is g_1 and vertices whose first coordinate is g_2 so that every arc is oriented from g_1 to g_2 . It then follows directly that we obtain n vertex-disjoint directed paths as claimed.

If we consider the directed path starting at (g_1, h) , it is straightforward to see that its terminal vertex is $(g_{r+1}, f_1 f_2 f_3 \cdots f_r(h))$. The argument for a directed cycle in $\overrightarrow{K}(G)$ is essentially the same except that π_{f_r} generates an arc from vertices in $G \oplus H$ whose first coordinate is g_r to vertices whose first coordinate is g_1 . It is then easy to see that a cycle of length t in the disjoint cycle decomposition of $f_1 f_2 f_3 \cdots f_r$ generates a directed cycle of length rt in $G \oplus H$. The rest of the lemma now follows.

Lemma 3.3.1 gives us a way of controlling arcs in $\overrightarrow{K}(G \oplus H)$. But we really would like the arcs in the projection of an arc of $\overrightarrow{K}(G)$ to be generated by distinct elements of $G \oplus H$. This leads naturally to a known type of permutation. A permutation $f: H \to H$ is an *orthomorphism* if the function g(x) = f(x) - x also is a permutation. The next lemma tells us that orthomorphisms are precisely what we need.

Lemma 3.3.2 Let G and H be abelian groups. If f is an orthomorphism of H, then the arcs of $\pi_f(g_i, g_j)$ in $\overrightarrow{K}(G \oplus H)$ are generated by the group elements $(g_j - g_i, h)$ as h runs through H.

Proof: This follows immediately from the definition of an orthomorphism.

There are some special orthomorphisms we use. Let |H| be odd and define the permutation T_0 on H by $T_0(h) = -h$ for $h \in H$. It is easy to see that T_0 is an orthomorphism because H contains no involutions. We extend this particular orthomorphism to T_a , $a \in H$, by defining $T_a(h) = 2a - h$. It is straightforward to check that T_a also is an orthomorphism. An important feature of these particular orthomorphisms is the following. When $H \cong Z_n, n$ odd, then the composition

$$T_0T_1 = h + 2 = (0, 2, \dots, n - 1, 1, 3, \dots, n - 2),$$
(3.1)

that is, the product is an n-cycle.

If G is an R^* -sequenceable abelian group of order m, then we have a directed cycle of length m-1 that misses the vertex 0 and has three successive vertices a, b, c for which a + c = b. Label the vertices of the directed cycle in succession as $g_1, g_2, \ldots, g_{m-1}$ so that $a = g_1, b = g_2, c = g_3$. The *canonical labelling* of the group $G \oplus H$ has the columns labelled so that the leftmost column is labelled g_1 , the next column is labelled 0, and the remaining columns are labelled g_2 through g_{m-1} from left to right in that order.

We want to prove that $G \oplus H$ is R^* -sequenceable whenever possible. It is natural to work with lifts of arcs of the directed cycle in $\overrightarrow{K}(G)$, but this directed cycle misses the vertex 0 so that we need to get the vertices of the column labelled 0 involved. We now describe how to do so.

Definition 3.3.3 Suppose that G is an abelian group with non-zero elements g_1, g_2, g_3 satisfying $g_1 + g_3 = g_2$. Consider $G \oplus H$ with H abelian of odd order $n \geq 3$. The lifts $\pi_{T_0}(g_1, g_2) \cup \pi_{T_0}(g_2, g_3)$ consist of n vertex-disjoint directed paths of length 2 using all the vertices of columns g_1, g_2, g_3 , and whose arcs are generated by $(g_2 - g_1, h)$ and $(g_3 - g_2, h)$ as h runs through H.

Now for each pair h, -h of additive inverses, replace the pair of directed 2-paths

$$((g_1, h), (g_2, -h), (g_3, h))$$
 and $((g_1, -h), (g_2, h), (g_3, -h)), h \neq 0,$

by the directed 3-paths

$$((g_1, h), (0, -h), (0, h)(g_3, -h))$$
 and $((g_1, -h), (g_2, h), (g_2, -h)(g_3, h).$

The directed 2-path $((g_1, 0), (g_2, 0), (g_3, 0))$ is left unaltered. The new collection of directed paths is called the *gadget on columns* $g_1, 0, g_2, g_3$.

Lemma 3.3.3 The arcs of the gadget on columns $g_1, 0, g_2, g_3$ are generated by the elements $(g_2 - g_1, h), (g_3 - g_2, h), (0, h')$ for all $h \in H$ and all $h' \neq 0$ in H. Moreover, the terminal vertex of the directed path whose initial vertex is (g_1, h) is $(g_3, -h)$.

Proof: The new arc $((g_1, h), (0, -h))$ of the gadget is generated by the group element $(g_3 - g_2, -2h)$ because $g_1 + g_3 = g_2$. Similarly, the arc $((0, h), (g_3, -h))$ is generated by the group element $(g_2 - g_1, -2h)$. The two vertical arcs ((0, -h), (0, h)) and $((g_2, h), (g_2, -h))$ are generated by the group elements (0, 2h) and (0, -2h). Finally, the arc $((g_1, -h), (g_2, h))$ is generated by $(g_2 - g_1, 2h)$, and the arc $((g_2, -h), (g_3, h))$ is generated by $(g_3 - g_2, 2h)$. Hence, the claims about which group elements generate the arcs of the gadget follow.

It is easy to see that the directed path beginning at (g_1, h) terminates at $(g_3, -h)$ for all $h \in \mathbb{Z}_n$.

The next lemma is the basis for establishing Theorem 3.1.2 when G has odd order.

Lemma 3.3.4 Let G be an R^* -sequenceable abelian group of order m. If H is an odd order abelian group for which there are orthomorphisms f_1, f_2, \ldots, f_t of H such that $T_0f_1f_2\cdots f_t$ is an |H|-cycle and $m-t-3 \ge 0$ is even, then $G \oplus H$ is R^* -sequenceable.

Proof: We use the canonical labelling of $G \oplus H$. The first four columns of the array correspond to the group elements $g_1, 0, g_2, g_3$ in that order, where $g_1 + g_3 = g_2$. Employ the gadget on these first four columns. Because of Lemma 3.3.3, it follows that if for each remaining (g_i, g_{i+1}) and (g_{r-1}, g_1) , we employ a lift arising from an orthomorphism of H, the arcs will have been generated by all elements of $G \oplus H$ other than (0,0). Moreover, the vertex (0,0) is isolated and the vertices $(g_1,0), (g_2,0), (g_3,0)$ occur in succession. Because $(g_1,0)+(g_3,0)=(g_2,0)$, if the arcs form a single directed cycle, then $G \oplus H$ is R^* -sequenceable.

From Lemma 3.3.3, the permutation from column g_1 to column g_3 is T_0 . We then successively employ the orthomorphisms f_1, f_2, \ldots, f_t for the following lifts. By hypothesis, the product $T_0 f_1 f_2 \cdots f_t$ is a cycle of length |H|.

There are m - (t + 3) further lifts to be employed. If m - (t + 3) = 0, we already have an |H|-cycle and we are done. If m - (t + 3) > 0, then it is even and we use T_0 for each subsequent lift. The product of an even number of T_0 permutations is the identity as T_0 is an involution. Thus, the final product is a cycle of length |H|completing the proof.

This method of lifts brings to the fore why the prime 3 is a nagging problem. For $a \in Z_n$ satisfying gcd(n, a) = 1, let M_a denote the permutation of Z_n defined by $M_a(x) = ax$. When 3 does not divide n, it is straightforward to check that both M_2 and $M_{(n-1)/2}$ are orthomorphisms. Note that $M_2M_{(n-1)/2} = T_0$. Then $T_0M_2M_{(n-1)/2}T_0T_1 = T_0T_1$ is an n-cycle and Lemma 3.2.2 applies for $m \ge 7$. When 3 divides n, unfortunately, $M_{(n-1)/2}$ is not an orthomorphism forcing us to find special arguments for the prime 3. This is what we now examine.

Corollary 3.3.4 If G is an R^* -sequenceable abelian group of odd order, then $G \oplus Z_{3^e}$

Proof: It is easy to verify that the permutations $f_0 = T_0, f_1 = M_2$, and $f_2 = (0,1)(2,6,3,5,8,4)(7)$ satisfy $f_0f_1f_2 = (0,1,7,2,8,6,3,5,4)$ for e = 3. This means that $G \oplus Z_9$ is R^* -sequenceable when G is R^* - sequenceable according to Lemma 3.3.4.

For e = 3, let $f_0 = T_0$. Let f_1 and f_2 be the following permutations, respectively:

(0, 26, 3, 8, 19, 7, 10, 16, 5, 24, 17, 12, 20, 14, 4, 22, 23, 25, 11, 18, 1, 13, 9, 6, 15, 2)(21)

and

(0, 22, 21, 13, 11)(1, 6, 7)(2, 8, 15, 5, 23, 10, 19, 4, 24, 20, 3, 16, 18, 26, 14, 25)(9, 12)(17).

Again it is easy to verify that the permutation $f_0f_1f_2$ is a 27-cycle as required. Lemma 3.3.4 then implies that $G \oplus Z_{27}$ is R^* -sequenceable when G is R^* -sequenceable.

We now want to show that $G \oplus Z_{3^e}$ is R^* -sequenceable, when G is R^* -sequenceable, for all $e \ge 2$ and we proceed by induction on e having established the result for e = 2, 3. Consider $e \ge 4$. Let N be the subgroup of Z_{3^e} of order 3^{e-2} so that Z_{3^e}/N is isomorphic to Z_9 . Use $0, 1, \ldots, 8$ as the coset representatives and let \overline{x} correspond to the element N + x in the quotient group of order 9.

From above we know there are three orthomorphisms $\overline{f_0}, \overline{f_1}, \overline{f_2}$ of Z_{3^e}/N so that $\overline{f_0f_1f_2} = (\overline{0}, \overline{1}, \overline{7}, \overline{2}, \overline{8}, \overline{6}, \overline{3}, \overline{5}, \overline{4})$, and $\overline{f_0}(\overline{0}) = \overline{f_1}(\overline{0}) = \overline{0}$ and $\overline{f_2}(\overline{0}) = \overline{1}$. Suppose that $\overline{f_i}(\overline{x}) = \overline{y}$. Then let α be any orthomorphism of N. Define the α -lift action of f_i on N + x by letting $f_i(n + x) = \alpha(n) + y$, $n \in N$. It is easy to see that f_i acting on the coset N + x picks up all elements of the form N + (y - x) via $f_i(n + x) - (n + x)$. Thus, f_i is an orthomorphism of Z_{3^e} if the action on each coset is defined via the lift of an orthomorphism of N as just described.

We now define f_0 , f_1 , f_2 to ensure that $f_0f_1f_2$ is a cycle of length 3^e . Let α_0 , α_1 , α_2 be orthomorphisms of N such that $\alpha_0\alpha_1\alpha_2$ is a cycle of length 3^{e-2} on N by induction. We have that $\overline{f_0}$ maps $\overline{0}$ to itself. We use the lift of the orthomorphism α_0 on N to define f_0 on N. Continuing, we know that $\overline{f_1}$ also maps $\overline{0}$ to $\overline{0}$. We use the lift of α_1 to define f_1 acting on N. Finally, to get the action of f_2 on N, use the lift of α_2 to define the action of f_2 mapping N to N + 1. For all other lifts, use T_0 on N.

We claim that $f_0f_1f_2$ is a cycle of length 3^e . To see this, first note that $f_0f_1f_2$ acts as $[\overline{0}, \overline{1}, \overline{7}, \overline{2}, \overline{8}, \overline{6}, \overline{3}, \overline{5}, \overline{4}]$ on the cosets. Because $\alpha_0\alpha_1\alpha_2$ is a cycle of length 3^{e-2} on N and we use the lifts of these three orthomorphisms to give the action of f_0, f_1, f_2 on N, we see that if $\alpha_0\alpha_1\alpha_2(n_1) = n_2$, then $f_0f_1f_2(n_1) = n_2 + 1$. All remaining lifts use T_0 and there are an even number of them so that $f_0f_1f_2$ is a full cycle of length $9 \cdot 3^{e-2} = 3^e$ as required.

If every summand in the Sylow 3-subgroup has order at least 9, then any summand is R^* -sequenceable by Lemma 3.2.1. Repeated applications of Corollary 3.3.4 yield that the Sylow 3-subgroup is R^* -sequenceable. Corollary 3.3.1 then implies that G is R^* -sequenceable.

When exactly one summand in the Sylow 3-subgroup is Z_3 , we require a lemma. Two useful items for the proof are given first.

The following are R^* -sequences for $Z_3 \oplus Z_9$ and $Z_3 \oplus Z_{27}$, respectively:

(2,0), (2,3), (0,3), (0,5), (1,2), (2,4), (1,1), (1,8)(0,1), (1,4), (1,0), (2,1), (2,2), (2,8), (1,6), (0,2), (1,7), (0,8), (1,3), (0,7), (1,5), (0,4), (2,7), (0,6), (2,6), (2,5)

and

(0, 1), (0, 26), (0, 25), (1, 24), (2, 10), (1, 11), (1, 25), (0, 11), (2, 16), (0, 8), (2, 26), (1, 15), (0, 14), (2, 4), (1, 23), (0, 23), (2, 20), (1, 8), (2, 15), (1, 0), (0, 10), (0, 17), (1, 19), (2, 14), (0, 19), (1, 20), (1, 13), (1, 7), (1, 18), (1, 3), (2, 13), (2, 17), (2, 7), (0, 22), (2, 25), (1, 6), (0, 20), (2, 0), (2, 8), (2, 5), (0, 2), (1, 10), (2, 1), (2, 3), (0, 7), (2, 18), (2, 9), (0, 18), (1, 14), (0, 12), (1, 26), (1, 2), (0, 6), (2, 12), (2, 22), (1, 4), (2, 2), (2, 21), (0, 21), (2, 23), (0, 16), (1, 22), (2, 11), (2, 24), (2, 6), (1, 1), (0, 24), (1, 9), (0, 3), (0, 4), (0, 9), (0, 15), (1, 5), (1, 21), (1, 17), (1, 12), (0, 5), (1, 16), (2, 19), (0, 13).

Lemma 3.3.5 The group $G = Z_3 \oplus Z_{3^e}$, $e \ge 2$, is R^* -sequenceable.

Proof: The statement is true for e = 2, 3 because R^* -sequences are given above. We proceed by induction on e and let e > 3. Let N be the cyclic subgroup of order 3^{e-2} . The quotient group G/N is isomorphic to $Z_3 \oplus Z_9$. Let the coset representatives be $\{(i, j) : 0 \le i \le 2, 0 \le j \le 8\}$ and let $\overline{(i, j)}$ denote the element N + (i, j) of the quotient group. Display the elements of G as a $3^{e-2} \times 9$ array where the columns are cosets of the cyclic subgroup of order 3^{e-2} and they are written left to right in the order of the R^* -sequence for $Z_3 \oplus Z_9$ given above, where column $\overline{(0,0)}$ is inserted between $\overline{(2,0)}$ and $\overline{(2,3)}$.

Even though the columns now correspond to cosets of $Z_{3^{e-2}}$ rather than the group itself—as they did earlier when we defined the lift of an arc onto the array for a direct sum—it should be clear how we define a lift now. Namely, if there is an arc from $\overline{(i,j)}$ to $\overline{(i',j')}$ in $\overrightarrow{K}(G/N)$ and f is a permutation of N, then for each $(i,j) + n \in \overline{(i,j)}$, we have an arc to (i',j') + f(n). We then use the same notation π_f for the lift.

We then use π_{T_0} as the lift for the arcs from (2,0) to (2,3), and from (2,3) to (0,3). Note that one of the directed paths is (2,0), (2,3)(0,3) and this sequence of three vertices satisfies (2,0) + (0,3) = (2,3). So if we end up with a directed cycle of length $3^{e+1} - 1$, we have that $Z_3 \oplus Z_{3^e}$ is R^* -sequenceable.

It is now clear that if we carry out the obvious gadget operation, we end up with directed paths of length 3, except for the unaltered directed path, whose initial and terminal vertices behave like π_{T_0} from column $\overline{(2,0)}$ to column $\overline{(0,3)}$. In the proof of Corollary 3.3.4, we show that for all e > 1 there are two orthomorphisms f_1, f_2 such that $T_0f_1f_2$ is a cycle of length of length 3^e . So we use these two orthomorphisms for the next two lifts of arcs along the R^* -sequence for $Z_3 \oplus Z_9$. We then use T_0 for all subsequent lifts and this leads to a directed cycle of length $3^{e+1} - 1$ as required.

We continue now with the subcase that the Sylow 3-subgroup has exactly one Z_3 term in the direct sum. The Sylow 3-subgroup is not cyclic so that Lemma 3.3.5 and repeated applications of Corollary 3.3.4 imply that the Sylow 3-subgroup is R^* sequenceable. Lemma 3.3.1 then implies that G is R^* -sequenceable.

If there are two or more Z_3 terms in the direct sum for the Sylow 3-subgroup, there is a useful fact we exploit. Let

$$f_1 = \left((0,0), (2,0), (0,2), (1,2), (1,0), (0,1)\right) \left((1,1), (2,2)\right) \left((2,1)\right)$$

and

$$f_2 = \left((0,0), (1,0), (1,1), (0,2), (2,2), (0,1)\right) \left((1,2), (2,0)\right) \left((2,1)\right)$$

be two permutations of $Z_3 \oplus Z_3$. It is easy to check that both are orthomorphisms and that $T_0f_1f_2$ is a 9-cycle.

We then conclude that $Z_3 \oplus Z_3 \oplus G$ is R^* -sequenceable when G is R^* -sequenceable and has odd order from Lemma 3.3.4. So consider the Sylow 3-subgroup H itself. If H has a summand Z whose order is at least 9, then both Z and $Z_3 \oplus Z$ are R^* -sequenceable by Lemma 3.2.1 or Lemma 3.3.5. Then H is R^* -sequenceable by starting with Z if there are an even number of Z_3 terms in the direct sum, or starting with $Z_3 \oplus Z$ if there are an odd number, and using the preceding fact. Therefore, H is R^* -sequenceable and Lemma 3.3.1 implies that G is R^* -sequenceable.

The preceding paragraph means we are left with the subcase that the Sylow 3subgroup is Z_3^a for some $a \ge 2$. If this is all of G, then G is R-sequenceable by Lemma 3.2.3. So we may assume that there is a non-trivial Sylow p-subgroup for some prime p > 3. If p > 5, then we may repeatedly apply Lemmas 3.2.1, 3.2.2, and the above fact to obtain that G is R^* -sequenceable.

The same process works for p = 5 except $Z_3 \oplus Z_3 \oplus Z_5$. Following is an R^* -sequence for this group which completes the proof of Theorem 3.1.2.

(0, 0, 1), (0, 2, 2), (0, 2, 1), (1, 1, 0), (0, 2, 3), (0, 1, 1), (0, 1, 2), (1, 2, 3), (0, 0, 2),(2, 1, 2), (0, 0, 4), (0, 1, 0), (1, 0, 3), (2, 0, 0), (2, 1, 3), (2, 0, 3), (0, 1, 3), (1, 2, 1),(2, 2, 1), (1, 1, 2), (2, 1, 0), (1, 0, 2), (1, 0, 0), (2, 0, 4), (1, 1, 1), (2, 2, 0), (2, 2, 2),(2, 0, 1), (2, 2, 3), (0, 1, 4), (2, 1, 1), (1, 2, 2), (0, 2, 0), (2, 1, 4), (1, 1, 4), (1, 2, 4),(1, 1, 3), (0, 0, 3), (1, 0, 4), (2, 2, 4), (1, 2, 0), (2, 0, 2), (1, 0, 1), (0, 2, 4).
Chapter 4

Conclusions and Future Work

In this dissertation research related to two problems introduced by the Austrian mathematician Gerhard Ringel is presented. The first was the Hamilton-Waterloo problem. This asks whether complete graphs can be decomposed into combinations of two 2-factors. The second asks whether the non-identity elements of a group can be written in a certain order $g_1, g_2, \ldots, g_{n-1}$ such that each non-identity element appears exactly once in the sequence $g_2g_1^{-1}, g_3g_2^{-1}, \ldots, g_{n-1}g_{n-2}^{-1}, g_1g_{n-1}^{-1}$.

4.1 The Hamilton-Waterloo Problem

4.1.1 Conclusions on the Hamilton-Waterloo Problem

For the Hamilton-Waterloo problem, in Chapter 1 we considered the Hamilton-Waterloo problem over the complete graph K_v , were each connected component of F_1 is a cycle of size 3 and each connected component of F_2 is a cycle of size 3x. The results obtained are summarized in the following theorem:

Theorem 29 Let $x \ge 2$, $y \ge 2$, and $r, s \ge 0$ such that $r + s = \lfloor \frac{3xy-1}{2} \rfloor$. Then there is a decomposition of K_{3xy} into $r \ C_3$ -factors and $s \ C_{3x}$ -factors, except possibly when:

- $s = 1, y \ge 3, and x \in \{3, 31, 37, 41, 43, 47, 51, 53, 59, 61, 67, 69, 71, 79, 83\}.$
- s = 1, x is odd and y is even.
- $s = 1, x \ge 6, x \equiv 2 \pmod{12}$.
- $s = 1, y \ge 8$ is even and $x \equiv 10 \pmod{12}$.
- $s = 1, x \ge 3$ is odd and y is even.
- $1 \le s \le \frac{x}{2} 1$, $x \ge 16$, $x \equiv 4 \pmod{12}$, y is even.
- $1 \le s \le \frac{x}{2} 1$, $x \ge 10$, $x \equiv 4 \pmod{6}$, y is odd.
- $(s, x) \in \{(2, 12), (4, 12)\}.$

- s = 0, x = 2, y = 2.
- x = 2 and $y \in \{4, 8\}$.
- $s \in \{3, 4, \dots, \frac{3(y-1)}{2}\}, x = 2 \text{ and } y \ge 3 \text{ is odd.}$
- $x \notin \{2,4\}$ and $y \in \{2,4,6\}$.
- x = 4 and $y \in \{2, 4\}$.
- x = 6 and y odd.

The methods used in Chapter 1 relied on Hamilton-Waterloo decompositions of the complete equipartite graphs $K_{(x:3)}$ and $K_{(4:3)}$. This inspired the work done in Chapter 2.

For the generalization of the Hamilton-Waterloo problem over complete equipartite graphs, in Chapter 2 decompositions of $K_{(v:m)}$ into $s \ F_1$ -factors and $r \ F_2$ -factors were found. The main result presented was:

Theorem 30 Let m and n be odd, such that $m \equiv 0 \pmod{n}$. Let s and r be such that $s, r \neq 1$ and $s+r = v \frac{m-1}{2}$. Let $x_1, \ldots, x_{m/n}, y_1, \ldots, y_{m/n}, z_1, \ldots, z_{m/n}$ and $w_1, \ldots, w_{m/n}$ be such that:

- $gcd(x_i, z_i) = gcd(y_i, z_i) = 1;$
- $w_i \notin \{2, 6\};$
- 2 divides at most one of x_i, y_i and z_i ;

- $v = x_i y_i z_i w_i$ if 2 divides none of x_i, y_i, z_i ; and
- $v = 2x_iy_iz_iw_i$ if 2 divides one of x_i, y_i, z_i .

Furthermore, let F_1 be a $[(x_1z_1n)^{\frac{v}{x_1z_1}}, \ldots, (x_{m/n}z_{m/n}n)^{\frac{v}{x_{m/n}z_{m/n}}}]$ -factor, and let F_2 be a $[(y_1z_1n)^{\frac{v}{y_1z_1}}, \ldots, (y_{m/n}z_{m/n}n)^{\frac{v}{y_{m/n}z_{m/n}}}]$ -factor. Then there is a decomposition of $K_{(v:m)}$ into s copies of F_1 and r copies of F_2 .

These results were applied to obtain decompositions of complete graphs, obtaining the following corollaries:

Corollary 4.1.1 Let m, x, and y be integers such that:

- $z = s\left(x, y\right), w = \frac{\gcd(x, y)}{z} \ge 2,$
- $\frac{xy}{z}$ divides m,
- 4 does not divide x nor y.
- Neither x nor y is 3 if $\frac{m}{w} \in \{6, 12\}.$

Then there exists a decomposition of K_m into s C_x -factors and r C_y -factors for every $s, r \neq 1$.

Corollary 4.1.2 Let $v, m, n, x_1, \ldots, x_k, y_1, \ldots, y_k$ be integers such that:

- $m \geq 3$ is odd,
- n divides m, x_i , and y_i for every i,
- $k = \frac{m}{n}$,
- $z_i = s\left(x_i, y_i\right),$
- $\frac{x_i y_i}{z_i n}$ divides v for each i,
- x_i divides v for each i,
- 4 does not divide x_i nor y_i for any i,
- $3 \notin \{x_1, \dots, x_k, y_1, \dots, y_k\}$ if $k \in \{6, 12\}$.

Then there exists a decomposition of K_{vm} into $s [x_1^{vn/x_1}, \ldots, x_k^{vn/x_k}]$ -factors and $r [y_1^{vn/y_1}, \ldots, y_k^{vn/y_k}]$ -factors for every $s, r \neq 1$.

The importance of Chapter 2 goes beyond the results obtained. The equipartite product we introduced will change the way people study problems on equipartite graphs, and will become one of the main tools to attack such problems. The special greatest common divisor is an extremely interesting concept that can be studied in depth and may yield some extremely useful results.

4.1.2 Future Work on the Hamilton-Waterloo Problem

On the Hamilton-Waterloo problem the first goal is to show that K_v can be decomposed into C_x -factors and C_y -factors if $\frac{xy}{\gcd(x,y)}$ divides v for all x, y odd. To do this I plan to study decompositions of $C_{(x:n)}$ into C_{x_1n} -factors and C_{x_2n} -factors, when x_1x_2 is odd and $\frac{x_1x_2}{s(x_1,x_2)}$ does not divide x.

Some further research problems on this topic are:

- Find decompositions into cycle sizes divisible by 4.
- Study the Hamilton-Waterloo problem over $K_{(v:m)}$, when m is even.
- Find decompositions of $K_{(v:m)}$ into 1 F_1 -factor and $\left\lfloor \frac{(m-1)v}{2} \right\rfloor F_2$ -factors.

4.2 *R*-Sequences

4.2.1 Conclusions on the problem of *R*-Sequences

On the problem of R-sequences, in Chapter 3 the problem over abelian groups was completely solved, as shown in the following theorem: **Theorem 4.2.1** If G is a finite abelian group, then the following hold:

(1) G is sequenceable if the Sylow 2-subgroup is cyclic and non-trivial; and

(2) G is R-sequenceable if the Sylow 2-subgroup either is trivial, or the Sylow 2subgroup is non-trivial and non-cyclic.

The problem was studied as two separate cases, abelian groups of even order, and abelian groups of odd order. The even case was proved by finding some special kind of cycles of \mathbb{Z}_{4n} , and using them to find R^* -sequences of $\mathbb{Z}_2 \oplus \mathbb{Z}_{4n}$. The odd case was proved by using orthomorphisms of \mathbb{Z}_{3^e} to find R^* -sequences of $\mathbb{Z}_n \oplus \mathbb{Z}_{3^e}$.

4.2.2 Future Work on *R*-Sequences

In the case of R-Sequences, a natural research problem is to study non-abelian groups. One possible starting point is to use the even order construction from Chapter 3 to study when dihedral groups are R-sequenceable. Another is to develop a tool using normal subgroups and quotient groups instead of direct products, to aide in the study of R-sequenceable non-abelian solvable groups.

I also plan to study the concept of orthogonalizable groups. A group G is called *orthogonalizable* if for every identity free subset $S \subset G$, the Cayley digraph $\overrightarrow{Cay}(G; S)$ admits either an orthogonal path or an orthogonal cycle. So far the only results in this problem are over abelian groups of small size [4]. The results obtained in Chapter 3 are a first step towards finding a solution to this problem. The next step would be to work on cyclic groups. I will start by studying groups of prime order.

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