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# Two Problems of Gerhard Ringel 

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# TWO PROBLEMS OF GERHARD RINGEL 

By

Adrián Pastine

## A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

In Mathematical Sciences

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This dissertation has been approved in partial fulfillment of the requirements for the Degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences.

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## Contents

List of Figures ..... ix
Preface ..... xi
Acknowledgments ..... xiii
Abstract ..... XV
Introduction ..... 1
1 On the Hamilton-Waterloo Problem with triangle factors and $C_{3 x^{-}}$ factors ..... 25
1.1 Introduction ..... 25
1.2 Constructions ..... 31
1.2.1 When $x$ is Odd ..... 33
1.2.2 When $x$ is Even ..... 36
1.2.3 A Weighting Construction ..... 41
1.3 Main Results ..... 45
1.3.1 When $x$ is Small ..... 52
1.4 Conclusions ..... 56
2 A Generalization of the Hamilton-Waterloo Problem on Complete
Equipartite Graphs ..... 59
2.1 Introduction ..... 59
2.2 Basic Definitions and Results ..... 63
2.3 Product and Decompositions ..... 67
2.4 Product of Cycles ..... 71
2.5 From $C_{(v: n)}$ to $K_{(v: m)}$ ..... 75
2.6 Hamilton-Waterloo Problem on Directed Complete Cyclic Multipartite Graphs ..... 79
2.7 Multivariable Functions ..... 97
2.8 Product ..... 115
2.9 Main Result ..... 123
2.10 Applications ..... 124
2.11 Appendix ..... 134
3 Finite Abelian Groups And Sequenceability ..... 137
3.1 Introduction ..... 137
3.2 First Stage of Proof ..... 141
3.3 The Gadget ..... 149
4 Conclusions and Future Work ..... 165
4.1 The Hamilton-Waterloo Problem ..... 166
4.1.1 Conclusions on the Hamilton-Waterloo Problem ..... 166
4.1.2 Future Work on the Hamilton-Waterloo Problem ..... 170
$4.2 \quad R$-Sequences ..... 170
4.2.1 Conclusions on the problem of $R$-Sequences ..... 170
4.2.2 Future Work on $R$-Sequences ..... 171
References ..... 173

## List of Figures

0.1 Representation of the graph in Example 1 ..... 3
0.2 Representation of the graph in Example 2 ..... 5
0.3 spikedmath.com/382.html ..... 5
0.4 A path with four vertices, $\Gamma \simeq P_{4}$, and a cycle with five vertices,
$H \simeq C_{5}$. ..... 8
0.5 A decomposition of the complete graph $\Gamma \simeq K_{6}$ ..... 9
0.6 The complete equipartite graph $\Gamma \simeq K_{(3: 4)}$ ..... 10
0.7 A Hamilton-Waterloo decomposition of $K_{10}$ into a 1-factor, $2 C_{10^{-}}$ factors and $2 C_{5}$-factors. ..... 13
0.8 A Hamilton-Waterloo decomposition of $K_{(5: 3)}$ into $3 C_{15}$-factors and 2
$C_{3}$-factors. ..... 14
0.9 The Cayley digraph $\overrightarrow{C a y}\left(\mathbb{Z}_{9} ;\{2,3\}\right)$ ..... 19
0.10 The Cayley digraph $\vec{K}\left(\mathbb{Z}_{6}\right)$ and its Cayley Factorization $\overrightarrow{\mathcal{F}}\left(\mathbb{Z}_{6} ;\{1,2,3,4,5\}\right)$ ..... 23
0.11 Orthogonal directed subgraphs ..... 24
(a) An orthogonal directed path for $\vec{K}\left(\mathbb{Z}_{10}\right)$. ..... 24
(b) An orthogonal directed cycle for $\vec{K}\left(\mathbb{Z}_{7}\right)$. ..... 24
2.1 A multipartite graph ..... 63
2.2 A multipartite graph labeled according to its partite sets ..... 63
2.3 Example of the partite product ..... 65
2.4 One directed cycle in $T_{5}(1)$, with $n=7$. ..... 80
$2.5 T_{3}(0), T_{3}(1), H_{3}(0,1), H_{3}(1,0)$. ..... 82
2.6 Triangle factors for $K_{(4: 3)}$. ..... 87
2.7 Base directed subgraphs for the decomposition of $\vec{C}_{(4: n)}$ into $\vec{C}_{n^{-}}$ factors and $\vec{C}_{2 n}$-factors ..... 88
2.8 A $\vec{C}_{7}$ factor of $\vec{C}_{(4: 7)}$ ..... 90
(a) $\Gamma_{0}$ when $n=7$ ..... 90
(b) $\Gamma_{0}$ when $n=7$ after identifying both $G_{6}$ columns ..... 90
$2.9 \quad T_{3}(1), T_{5}(2)$ and $T_{(3,5)}(1,2)$ ..... 99
$2.10 H_{3}(1,2), H_{5}(2,4)$ and $H_{(3,5)}(1,2)(2,4)$ ..... 100
2.11 Example of Theta-functions ..... 108
2.12 Example of Lemma 7.17 Case 1 ..... 116
2.13 Example of Lemma 7.17 Case 2 ..... 117
3.1 A path in $K_{10}$ satisfying the conditions of Lemma 3.2.6 ..... 145
3.2 A path in $K_{16}$ satisfying the conditions of Lemma 3.2.6 ..... 145

## Preface

Chapter 1. "On the Hamilton-Waterloo Problem with triangle factors and $C_{3 x^{-}}$ factors", was authored by J. Asplund, D. Kamin, M. Keranen, A. Pastine and S. Özkan, and published on Australasian Journal of Combinatorics, Volume 64 (part 3), pages 458-474. All the results in this work arose from the collective thinking and discussion of all the authors, and everyone deserves some credit for each of them. My main contributions to this work is coming up with the idea behind the constructions in Theorem 7 and Theorem 8. The resulting paper on this work was mainly written by M. Keranen and A. Pastine, and reviewed by J. Asplund.

Chapter 2, "A Generalization of the Hamilton-Waterloo Problem on Complete Equipartite Graphs", was authored by M. Keranen and A. Pastine, and submitted for publication to Journal of Combinatorial Designs and is currently in review. Every result in this work is due to the mutual cooperation of both authors, and they deserve equal credit for them. The resulting paper was mainly written by A. Pastine, and reviewed by M. Keranen.

Chapter 3. "Finite Abelian Groups And Sequenceability", was authored by B. Alspach, D.L. Kreher, and A. Pastine, and submitted for publication to Australasian Journal of Combinatorics and is currently in review. All results in this work are due
to the mutual cooperation of the three authors, and they deserve equal credit for them. The resulting paper was mainly written by B. Alspach, and reviewed by D.L. Kreher and A. Pastine.

Chapters 1, 2, and 3 are designed to be read independently. To achieve this some definitions and results occur in more than one chapter.

The Introduction to this dissertation was written using excerpts and ideas from each of the aforementioned papers.

Figure 0.3 is licensed under a Creative Commons Attribution-Noncommercial-Share Alike License.

## Acknowledgments

First, I would like to thank my co-advisors Don Kreher and Missy Keranen. Doing research with them was both a fun and gratifying experience. I feel like I have learned a lot from both of them, and their influence will be seen in all of my future work.

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I cannot finish this acknowledgments without thanking Daniel Jaume. He played an influential role in my academic career. He is the person that introduced me to Combinatorics and Graph Theory. He also pushed me to apply to scholarships and schools all over the world. Most of all he showed me how much fun research can be.

Finally, I would like to thank my family, who always showed me their unconditional support.


#### Abstract

Gerhard Ringel was an Austrian Mathematician, and is regarded as one of the most influential graph theorists of the twentieth century. This work deals with two problems that arose from Ringel's research: the Hamilton-Waterloo Problem, and the problem of $R$-Sequences.


The Hamilton-Waterloo Problem (HWP) in the case of $C_{m}$-factors and $C_{n}$-factors asks whether $K_{v}$, where $v$ is odd (or $K_{v}-F$, where $F$ is a 1 -factor and $v$ is even), can be decomposed into $r$ copies of a 2 -factor made entirely of $m$-cycles and $s$ copies of a 2 -factor made entirely of $n$-cycles. Chapter 1 gives some general constructions for such decompositions and apply them to the case where $m=3$ and $n=3 x$. This problem is settled for odd $v$, except for a finite number of $x$ values. When $v$ is even, significant progress is made on the problem, although open cases are left. In particular, the difficult case of $v$ even and $s=1$ is left open for many situations.

Chapter 2] generalizes the Hamilton-Waterloo Problem to complete equipartite graphs $K_{(n: m)}$ and shows that $K_{(x y z w: m)}$ can be decomposed into $s$ copies of a 2 -factor consisting of cycles of length $x z m$; and $r$ copies of a 2 -factor consisting of cycles of length $y z m$, whenever $m$ is odd, $s, r \neq 1, \operatorname{gcd}(x, z)=\operatorname{gcd}(y, z)=1$ and $x y z \neq 0(\bmod 4)$. Some more general constructions are given for the case when the cycles in a given
two factor may have different lengths. These constructions are used to find solutions to the Hamilton-Waterloo problem for complete graphs.

Chapter 3 completes the proof of the Friedlander, Gordon and Miller Conjecture that every finite abelian group whose Sylow 2-subgroup either is trivial or both nontrivial and non-cyclic is $R$-sequenceable. This settles a question of Ringel for abelian groups.

## Introduction

Gerhard Ringel was born in Kollnbrunn, Austria, in 1919. He received his PhD in 1951 at the Friedrich-Wilhelms Universität. He became a professor at Freie Universität Berlin in 1958, and from 1967 to 1970 he held the position of chairman of the mathematical institute. In 1970 he became a full professor at University of California, Santa Cruz, where he was chairman of the mathematics department from 1972 to 1984. He passed in Santa Cruz in June 24th 2008.

During his many years working in the field he collected many honors and awards, including two honorary doctorates. The Universität Karlsruhe awarded him a honorary doctorate in political science in 1983. In 1994 he received an honorary doctorate in mathematics from Freie Universität in Berlin.

Ringel is in the list of people that made graph theory an interesting and influential field of study. His conjectures are known by most graph theorists. Among them, we have the Ringel's conjecture, the Ringel-Kotzig conjecture, the Oberwolfach problem
and the Earth Moon problem. Asides from his conjectures, Ringel's most popular result is the Ringel-Youngs Theorem (also known as Heawood's Conjecture). This is one of the most important results in Graph Theory, where they proved that four colors are enough to create a political map on any surface, with no two adjacent countries sharing their color.

This manuscript is concerned with two problems first introduced by Ringel. Our focus will be on the Hamilton-Waterloo Problem, and $R$-sequenceability. The HamiltonWaterloo Problem is the natural continuation of the Oberwolfach problem, first posed by Ringel in 1967 at a conference in Oberwolfach. The problem of $R$-sequenceability was posed by Ringel in 1974, [36]. We now introduce the concepts necessary to understand what these problems ask, and what our contributions are.

Definition 0.0.1 A graph is an ordered pair $\Gamma=(V, E)$, where $V$ is a set of elements called vertices, and $E$ is a family of 2-element subsets of $V$ called edges.

If $\{x, y\} \in E$, then we say that $x$ and $y$ are adjacent. Sometimes the set of vertices and the family of edges of a graph $\Gamma$ are denoted $V(\Gamma)$, and $E(\Gamma)$, respectively. The degree of a vertex is the number of vertices that are adjacent to it.

Example 1: As an example we can take

$$
\begin{aligned}
& V=\{a, b, c, d, e, f, g, h\} \\
& \text { and } \\
& E=\{\{a, b\},\{a, c\},\{a, d\},\{b, d\},\{f, g\},\{f, h\},\{g, h\} .
\end{aligned}
$$

The graph $\Gamma=(V, E)$ has as vertices the elements $a, b, c, d, e, f, g, h$, where the vertex $a$ is adjacent to the vertices $b, c$, and $d$; the vertex $b$ is adjacent to the vertices $a$ and $d$, and so on. The degrees of the vertices are

| $x$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\operatorname{deg}(x)$ | 3 | 2 | 1 | 2 | 0 | 2 | 1 | 1 |

There are many ways to represent graphs. The most common way is by drawing a picture where each vertex is represented by a point or a circle, and each edge is represented by a line connecting the points (or circles) of its respective vertices. Figure 0.1 shows this kind of representation for the graph from Example 1.


Figure 0.1: Representation of the graph in Example 1

Related to graphs are directed graphs, where instead of having a family of subsets for the edges $E$, we have a set of ordered pairs $A$. Hence if $(x, y) \in A$ we say that there is an $\operatorname{arc}$ from $x$ to $y$. We sometimes write $A(\Gamma)$ to denote the set of $\operatorname{arcs}$ of $\Gamma$.

Example 2: As an example we can take

$$
\begin{aligned}
& V=\{a, b, c, d, e, f, g, h\} \\
& \text { and } \\
& A=\{(a, b),(a, c),(c, a),(d, a),(d, b),(f, g),(g, h),(h, f) .
\end{aligned}
$$

In this way the graph has as vertices the elements $a, b, c, d, e, f, g, h$, where there are arcs from vertex $a$ to the vertices $b, c$; there is an arc from vertex $c$ to vertex $a$, and so on.

Similar to the picture representation of a graph directed graphs can also be drawn, the only difference is that arcs are represented by arrows. In this way if $(x, y) \in A$, in the drawing there will be an arrow going pointing from $x$ to $y$. Figure 0.2 shows this representation for the graph described in Example 2,

Graphs are used in a wide range of fields to represent symmetric binary relations. A few examples of what graphs can represent follow:


Figure 0.2: Representation of the graph in Example 2

- Social networks, where each vertex represents a person, and two vertices are adjacent if the people are friends.
- Computer networks, where each vertex represents a computer, and two vertices are adjacent if the computers are connected in the network.
- Molecules, where each vertex represents an atom, and two vertices are adjacent if the atoms are bonded.

"MATHEMATICS IS THE ART OF GIVING THE SAME NAME TO DIFFERENT THINGS." JLLLES HENRI POINCARE (1854-1912)

Figure 0.3: spikedmath.com/382.html

Similarly directed graphs are used to represent asymmetric binary relations. For instance:

- Flight connections, where each vertex represents an airport, and an arrow $(x, y)$ represents a flight from airport $x$ to airport $y$.
- Websites, where each vertex represents a web page, and an arrow $(x, y)$ represents a link from the web page $x$ to the web page $y$.
- Ownership of companies, where each vertex represents a company, and an arrow $(x, y)$ means that the company $x$ owns the company $y$.

There are many different properties of graphs that can be studied. A particular property asks when two different graphs are technically the same. We say that two graphs $\Gamma_{1}=\left(V_{1}, E_{1}\right), \Gamma_{2}=\left(V_{2}, E_{2}\right)$ are isomorphic if there is a bijection $\varphi: V_{1} \rightarrow V_{2}$, such that $\{x, y\} \in E_{1}$ if and only if $\{\varphi(x), \varphi(y)\} \in E_{2}$. If this is the case, we say that $\varphi$ is an isomorphism between $\Gamma_{1}$ and $\Gamma_{2}$. If $\Gamma_{1}$ and $\Gamma_{2}$ are isomorphic we write $\Gamma_{1} \simeq \Gamma_{2}$.

Example 3: Let

$$
\begin{aligned}
& V_{1}=\{a, b, c, d, e, f, g, h\}, \\
& E_{1}=\{\{a, b\},\{a, c\},\{a, d\},\{b, d\},\{f, g\},\{f, h\},\{g, h\}, \\
& V_{2}=\{1,2,3,4,5,6,7,8\}, \\
& E_{2}=\{\{1,2\},\{1,3\},\{2,3\},\{4,5\},\{4,6\},\{4,7\},\{5,7\} .
\end{aligned}
$$

Then the bijection $\varphi$ defined by

$$
\begin{array}{ll}
\varphi(1)=f & \varphi(5)=b \\
\varphi(2)=g & \varphi(6)=c \\
\varphi(3)=h & \varphi(7)=d \\
\varphi(4)=a & \varphi(8)=e
\end{array}
$$



A path is a graph $\Gamma=(V, E)$, such that

$$
\begin{aligned}
V & =\left\{v_{1}, v_{2}, v_{3}, \ldots, v_{n}\right\} \\
E & =\left\{\left\{v_{1}, v_{2}\right\},\left\{v_{2}, v_{3}\right\}, \ldots,\left\{v_{i}, v_{i+1}\right\}, \ldots,\left\{v_{n-1}, v_{n}\right\}\right\}
\end{aligned}
$$

A path with $n$ vertices is usually denoted $P_{n}$. If we add the edge $\left\{v_{n}, v_{1}\right\}$, then the graph is called a cycle and is denoted $C_{n}$.

An example of a path with four vertices and a cycle with five vertices is given in Figure 0.4 .


Figure 0.4: A path with four vertices, $\Gamma \simeq P_{4}$, and a cycle with five vertices, $H \simeq C_{5}$.

A graph that has every pair of vertices adjacent to each other is called complete. The complete graph on $n$ vertices is usually denoted $K_{n}$.

A connected component of a graph is a maximal set of vertices $C$, such that for each pair of vertices $x, y \in C$, there is a path joining them.

Many interesting pure and applied problems can be described in the framework of graph decomposition, which asks whether the edges of a graph $\Gamma=(V, E)$ can be partitioned into subsets $E_{1}, \ldots, E_{k}$, such that the graphs $\Gamma_{1}=\left(V, E_{1}\right), \ldots, \Gamma_{k}=\left(V, E_{k}\right)$ are isomorphic to certain predetermined graphs called factors. Graph decomposition has applications in networking [26], block designs [14], and bioinformatics [38], among others.

Given two graphs $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=\left(V_{2}, E_{2}\right)$, we say that $\Gamma_{2}$ is a subgraph of $\Gamma_{1}$ if $V_{2} \subset V_{1}$ and $E_{2} \subset E_{1}$. A subgraph $F$ of $\Gamma$ is called a $n$-factor if every vertex of $F$ has degree $n$. Notice that in a 1-factor every connected component has exactly 2 vertices, and in a 2-factor every connected component is a cycle. A 2-factor were each
connected component has $x$ vertices is called $C_{x}$-factor. A $\left[n_{1}^{e_{1}}, n_{2}^{e_{2}}, \ldots, n_{p}^{e_{p}}\right]$-factor of $\Gamma$ is 2 -factor of $\Gamma$ with $e_{i}$ connected components of size $n_{i}, i=1,2, \ldots, p$. We allow $n_{i}=n_{j}$, which means that a $\left[3^{3}, 5^{2}\right]$-factor is also a $\left[3^{2}, 3^{1}, 5^{2}\right]$-factor.

Example 4: Figure 0.5 gives a decomposition of the complete graph $\Gamma \simeq K_{6}$. It is decomposed into 3 factors:

- $F_{1}$ is a 2-factor with one connected component. It is a $C_{6}$-factor, or a [6 $6^{1}$ ]-factor.
- $F_{2}$ is a 2-factor with two connected components. It is a $C_{3}$-factor, a $\left[3^{2}\right]$ factor, or a $\left[3^{1}, 3^{1}\right]$-factor.
- $F_{3}$ is a 1-factor, with three connected components.


Figure 0.5: A decomposition of the complete graph $\Gamma \simeq K_{6}$

If the vertices of a graph $\Gamma=(V, E)$ can be partitioned into sets $\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{m}$, such
that if $x, y \in \Gamma_{i}$ for some $i$, then $\{x, y\} \notin E$. The sets $\Gamma_{i}$ are called partite sets. If the number of parts is $m$, then the graph is also called $m$-partite. If all the partite sets have the same number of vertices, then the graph is called equipartite. If every pair of vertices belonging to different partite sets are adjacent, then the graph is called complete multipartite (equipartite). A complete equipartite graphs with $m$ partite sets of size $n$ is denoted $K_{(n: m)}$. Figure 0.6 shows a complete equipartite graph with 4 parts of size $3, K_{(3: 4)}$.


Figure 0.6: The complete equipartite graph $\Gamma \simeq K_{(3: 4)}$

In 1967, during a conference in Oberwolfach, Ringel asked whether it was possible to seat the $v$ conference attendees at $n$ round tables for dinner during $\frac{v-1}{2}$ nights, in such a way that every attendee sits next to every other attendee exactly once. This is equivalent to asking whether the complete graph $K_{v}$ can be decomposed into $\frac{v-1}{2}$ copies of a 2 -factor $F$ (in a 2-factor every component is a cycle, which represents a round table). To achieve this decomposition $v$ needs to be odd, because the vertices (attendees) need to have even degree. Later a version with $v$ even was introduced by Huang, Kotzig and Rosa [25]. In this case, the attendees will never sit next to their spouses (and we are assuming that every attendee has a spouse). This is equivalent to asking for a decomposition of $K_{v}$ into $\frac{v-2}{2}$ copies of a 2 -factor $F$, and one copy of a 1-factor (each attendee together with their spouse).

In [28] Liu first worked on the generalization of the Oberwolfach problem, where instead of avoiding their spouses, the attendees avoid all the other members of their delegation. The assumption was that all the delegations had the same number of people. Thus we are seeking to decompose the complete equipartite graph $K_{(m: n)}$ with $n$ partite sets (delegations) of size $m$ each (members of a delegations) into $\frac{(n-1) m}{2}$ copies of a 2 -factor $F$. Here $(n-1) m$ has to be even. In [22] Hoffman and Holliday worked on the equipartite generalization of the Oberwolfach problem when $(n-1) m$ is odd, decomposing into $\frac{(n-1) m-1}{2}$ copies of a 2 -factor $F$, and one copy of a 1 -factor.

The Hamilton-Waterloo problem, first mentioned in [17], is a generalization of the

Oberwolfach problem, in which the conference is being held at two different cities. Because the table arrangements are different, we have two 2-factors, $F_{1}$ and $F_{2}$. The Hamilton-Waterloo problem then asks whether the complete graph $K_{v}$ can be decomposed into $r$ copies of the 2-factor $F_{1}$ (tables at Hamilton) and $s$ copies of the 2-factor $F_{2}$ (tables at Waterloo), such that $s+r=\frac{v-1}{2}$, when $v$ is odd, and having $s+r=\frac{v-2}{2}$ and a 1 -factor when $v$ is even.

The uniform Oberwolfach problem (when all the tables have the same size, i.e. all the cycles of the 2-factor have the same size) has been completely solved by Alspach and Haagkvist [2] and Alspach, Schellenberg, Stinson and Wagner [3]. For the non-uniform case of the Oberwolfach problem, many results have been obtained. For a survey of results up to 2006 see [12]. The uniform Oberwolfach problem over equipartite graphs has been completely solved by Liu [29] and Hoffman and Holliday [22]. In the nonuniform case Bryant, Danziger and Pettersson [11] completely solved the case when the 2-factor is bipartite. For the Hamilton-Waterloo problem most of the results are uniform, see for example [5] or [13].

The Hamilton-Waterloo problem can be generalized for complete equipartite graphs in the same way as the Oberwolfach problem was generalized.

Figure 0.7 shows a Hamilton-Waterloo decomposition of $K_{10}$ into a 1-factor, $2 C_{10^{-}}$ factors, and $2 C_{5}$-factors. Figure 0.8 shows a Hamilton-Waterloo decomposition of $K_{(5: 3)}$ into $3 C_{15}$-factors and $2 C_{3}$-factors.


Figure 0.7: A Hamilton-Waterloo decomposition of $K_{10}$ into a 1-factor, 2 $C_{10}$-factors and $2 C_{5}$-factors.

Chapter 1 considers the Hamilton-Waterloo problem over the complete graph $K_{3 x y}$, looking for decompositions into $r C_{3}$-factors and $s C_{3 x}$-factors. In this chapter we show that such decompositions can be achieved except maybe when $s=1$, or when $x \in\{2,4,6,12\}$.


Figure 0.8: A Hamilton-Waterloo decomposition of $K_{(5: 3)}$ into $3 C_{15}$-factors and $2 C_{3}$-factors.

Chapter 2 is concerned with the Hamilton-Waterloo problem over the complete equipartite graph $K_{(v: m)}$, into $s F_{1}$-factors and $r F_{2}$-factors. In this chapter we show that such decompositions can be achieved if the following conditions are satisfied:

- $m=n m^{\prime}$ is odd, and $n$ divides the size of all the cycles;
- if there are cycles of size $x$ in one of the 2 -factors, then $x$ divides $v$, and there are $k \frac{v}{n x}$ such cycles.
- 4 does not divide the size of any cycle;
- if there are even cycles, then 4 divides $v$;
- $s, r \neq 1$;
except for certian lengths of cycles when $v$ is even.

We turn now to the problem of $R$-sequenceability. For this we first need to introduce the algebraic concept of group.

Definition 0.0.2 Given a set of elements $G$ and a binary operation $*: G \times G \rightarrow G$, we say that $(G, *)$ is a group if the following conditions are satisfied:

Closure: If $a, b \in G$, then $a * b \in G$.

Associativity: If $a, b, c \in G$, then $(a * b) * c=a *(b * c)$.

Identity: There exists an element $e \in G$, such that if $a \in G$, then $a * e=e * a=a$.

Inverse: If $a \in G$, then there is $b \in G$, such that $a * b=b * a=e$.

The element $e$ can be proven to be unique, and is known as the identity of the group. Given $a \in G$, the element $b \in G$ such that $a * b=e$ is also unique, it is called the inverse of $a$, and denoted $b=a^{-1}$. When the operation is known we sometimes refer as the group as $G$, instead of $(G, *)$. Because of the associativity property, it is common to write strings of products without parentheses. This means that instead of writing $(a * b) * c$, we would write $a * b * c$.

Example 5: The set of integers together with the usual addition, $(\mathbb{Z},+)$, is a group. We know that given two integers $a, b$, then $a+b$ is an integer, showing closure. Associativity is also one of the known properties of addition over the integers. The identity element is 0 , and the inverse of an element $a$ is $-a$.

Notice that we could not use the product as the operation, because we do not have inverses for our elements.

Some other examples of groups are:

- The integers, with addition as the operation.
- The rationals except for zero, with product as the operation.
- The set of non-singular matrices of a given size, with product as the operation.
- The permutations of a set, with composition as the operation.
- The set of vectors with $m$ coordinates and real numbers as entries, with vector addition as the operation.

We will mainly deal with finite groups. This is, groups containing only a finite number of elements.

Example 6: Usually the first finite groups introduced are the integers modulo $n$, $\left(\mathbb{Z}_{n},+_{n}\right)$. This group contains the numbers $\{0,1, \ldots, n-2, n-1\}$ as elements,
and the operation, called addition modulo $n$ is defined as:

$$
a+_{n} b=\left\{\begin{array}{ccc}
a+b & \text { if } & a+b \leq n-1 \\
a+b-n & \text { if } & a+b \geq n
\end{array} .\right.
$$

Because $0 \leq a+b \leq 2 n-2$, we get that $a+b \leq n-1$ or $0 \leq a+b-n \leq n-1$, hence the set is closed under $+_{n}$. Associativity is inherited from integer addition + . The identity element is 0 , and the inverse of an element $a \neq 0$ is $n-a$.

Some other examples of finite groups are:

- The $n$-th roots of unity, with multiplication as the operation.
- The symmetries of a regular polygon, with composition as the operation.
- The permutations of a finite set, with composition as the operation.
- The set of isomorphisms from a graph onto itself (automorphisms), with composition as the operation.
- The set of vectors with $m$ coordinates and integers modulo $n$ as entries, with vector addition modulo $n$ as the operation.

A pair of elements $a$ and $b$ in a group commute if $a * b=b * a$. Notice that in particular the identity of a group commutes with every element of the group. If every pair of
elements of a group commute, then the group is said to be abelian. If we look back at our examples, the first two are abelian, the last two are not.

If the group $G$ is abelian, usually,+ 0 , and $-a$ are used to represent the operation, identity, and inverse of $a \in G$. For groups in general the operation is usually represented by $*$ or by $\cdot ;$ the identity element is represented by $1, I$ or $e$; and the inverse of $a \in G$ by $a^{-1}$. When the operation is represented by $\cdot$, it is common to write $a b$ instead of $a \cdot b$.

A nice connection between Graph Theory and Group Theory are Cayley Digraphs. Given a group $G$, and $S$ a subset of elements of $G$, such that the identity element is not in $S$, the Cayley Digraph $\Gamma=\overrightarrow{C a y}(G ; S)$ is the directed graph that has:

$$
\begin{aligned}
& V(\Gamma)=\{g \mid g \in G\}, \\
& A(\Gamma)=\{(g, g s) \mid g \in G, s \in S\} .
\end{aligned}
$$

We will say that arcs of the form $(g, g s)$ are generated by the element $s$.

Example 7: The set of integers modulo $9, \mathbb{Z}_{9}$, is a group, with elements $0,1,2,3,4,5,6,7,8$ and addition modulo 9 as the operation, +. The Cayley digraph $\overrightarrow{C a y}\left(\mathbb{Z}_{9} ;\{2,3\}\right)$ is shown in Figure 0.9 , where the blue arcs are of the form $(g, g+2)$, and the red arcs are of the form $(g, g+3)$.


Figure 0.9: The Cayley digraph $\overrightarrow{C a y}\left(\mathbb{Z}_{9} ;\{2,3\}\right)$

In 1961 B. Gordon [19] defined a group $G$ to be sequenceable when there exists a permutation

$$
g_{0}, g_{1}, g_{2}, \ldots, g_{n-1}
$$

of its elements so that the sequence of partial products

$$
g_{0}, g_{0} g_{1}, g_{0} g_{1} g_{2}, \ldots, g_{0} g_{1} g_{2} \cdots g_{n-1}
$$

are distinct. In that same paper he proved the following theorem.

Theorem $1 A$ finite abelian group $G$ is sequenceable if and only if $G$ contains $a$ unique non-identity element $a$, such that $a=-a$.

In 1974 G. Ringel [36] asked when there exists a permutation

$$
g_{1}, g_{2}, \ldots, g_{n-1}
$$

of the non-identity elements of a group such that the sequence

$$
g_{2} g_{1}^{-1}, g_{3} g_{2}^{-1}, \ldots, g_{n-1} g_{n-2}^{-1}, g_{1} g_{n-1}^{-1}
$$

also is a permutation of the non-identity elements. A group $G$ that admits such a permutation is called $R$-sequenceable. As a matter of fact, L. Paige [33] used this concept in 1951, but it was Ringel's problem that motivated the most important paper on this topic.

One can investigate these problems using Cayley digraphs. In particular the Cayley digraph with $S=G \backslash\{e\}$, that is, the set $S$ has everything in it other than the identity element. We use the special notation $\vec{K}(G)$ for this Cayley digraph.

It is easy to see that a fixed element $s \in S$ generates a subdigraph consisting of directed cycles of length $|s|$, the order of $s$ in $G$. Thus, we obtain a decomposition of $\overrightarrow{C a y}(G ; S)$ into $|S|$ directed 2-factors. We call this decomposition the Cayley factorization of $\overrightarrow{C a y}(G ; S)$ and denote it by $\overrightarrow{\mathcal{F}}(G ; S)$. Figure 0.10 shows the Cayley factorization $\overrightarrow{\mathcal{F}}\left(\mathbb{Z}_{6} ;\{1,2,3,4,5\}\right)$ of the Cayley digraph $\vec{K}\left(\mathbb{Z}_{6}\right)$.

If $\vec{D}$ is a subdigraph of $\overrightarrow{C a y}(G ; S)$ with $|S|$ arcs, and $\vec{D}$ has exactly one arc from each directed 2-factor in $\overrightarrow{\mathcal{F}}(G ; S)$, then we say that $\vec{D}$ is orthogonal to $\overrightarrow{\mathcal{F}}(G ; S)$. In this language, the group $G$ is sequenceable when $\vec{K}(G)$ has an orthogonal directed path of length $|G|$, and $G$ is $R$-sequenceable when $\vec{K}(G)$ has an orthogonal directed
cycle of length $|G|-1$.

Example 8: Figure $0.11(\mathrm{a})$ shows an orthogonal directed path of length 10 for $\mathbb{Z}_{10}$. The vertices used by the path are:

$$
0,1,9,2,8,3,7,4,6,5 .
$$

The arcs used are generated by the elements:

$$
1,8,3,6,5,4,7,2,9
$$

Notice that each non-identity element from $G$ generates exactly one of these arcs, thus the directed path is orthogonal.

Figure 0.11(b) shows an orthogonal directed cycle of length 6 for $\mathbb{Z}_{6}$. The vertices used by the cycle are:

$$
1,4,6,5,2,3,1
$$

The arcs used are generated by the elements:

$$
3,2,6,4,1,5
$$

Notice that each non-identity element from $G$ generates exactly one of these
arcs, thus the directed cycle is orthogonal.

Chapter 3 studies the $R$-sequenceability of abelian groups. In this chapter we show that every abelian group is either sequenceable or $R$-sequenceable. This completely solves the problem of Ringel for abelian groups.


Figure 0.10: The Cayley digraph $\vec{K}\left(\mathbb{Z}_{6}\right)$ and its Cayley Factorization $\vec{F}\left(\mathbb{Z}_{6} ;\{1,2,3,4,5\}\right)$


Figure 0.11: Orthogonal directed subgraphs

## Chapter 1

## On the Hamilton-Waterloo

## Problem with triangle factors and $C_{3 x}$-factors ${ }^{1 /}$

### 1.1 Introduction

The Oberwolfach problem was first proposed by Ringel in 1967, and involves seating $v$ conference attendees at $t$ round tables over $\frac{v-1}{2}$ nights such that each attendee sits next to each other attendee exactly once. It is mathematically equivalent to

[^0]decomposing $K_{v}$ into 2-factors where $K_{v}$ is the complete graph on $v$ vertices and each 2-factor is isomorphic to a given 2-factor $Q$. In the original statement of the problem, we have that $v$ must be odd. It was later extended to the spouse-avoiding Oberwolfach problem, allowing for even $v$ by decomposing $K_{v}-F$, where $F$ is a 1-factor.

The Hamilton-Waterloo Problem (HWP) is an extension of the Oberwolfach Problem. Instead of seating $v$ attendees at the same $t$ tables each night, the Hamilton-Waterloo problem asks how the $v$ attendees can be seated if they split their nights between two different venues. The attendees will all spend the same $r$ nights in Hamilton, which has round tables of size $m_{1}, m_{2}, \ldots, m_{k}$, and $s$ nights in Waterloo, which has round tables of size $n_{1}, n_{2}, \ldots, n_{p}$ where $\sum_{i=1}^{k} m_{i}=\sum_{i=1}^{p} n_{i}=v$. The case when $m_{1}=m_{2}=\cdots=m_{k}=m$ and $n_{1}=n_{2}=\cdots=n_{p}=n$ is called the HamiltonWaterloo Problem with uniform cycle sizes, and this variant of the problem gets most of the attention. Graph theoretically, this problem is equivalent to decomposing $K_{v}$ (or $K_{v}-F$ when $v$ is even) into 2-factors where each 2-factor consists entirely of $m$-cycles (a $C_{m}$-factor) or entirely of $n$-cycles (a $C_{n}$-factor). Throughout this paper, the word factor is assumed to be a 2 -factor unless otherwise stated. We frequently refer to a $C_{3}$-factor as a triangle factor and a Hamilton cycle as a Hamilton factor.

A decomposition of a graph $G$ is a partition of the edge set of $G$. A decomposition of $K_{v}$ into $C_{m}$-factors is called a $C_{m}$-factorization. We will refer to a solution to
the Hamilton-Waterloo Problem with $r$ factors of $m$-cycles, $s$ factors of $n$-cycles, and $v$ points as a resolvable $\left(C_{m}, C_{n}\right)$-decomposition of $K_{v}$ into $r C_{m}$-factors and $s C_{n^{-}}$ factors, and we will let $(m, n)-\operatorname{HWP}(v ; r, s)$ denote such a decomposition. In order for an $(m, n)-\operatorname{HWP}(v ; r, s)$ to exist, it is clear that $r+s=\frac{v-1}{2}$ (or $r+s=\frac{v-2}{2}$, for even $v$ ), and both $m$ and $n$ must divide $v$. These conditions are summarized in the following theorem.

Theorem 2 [1] The necessary conditions for the existence of an $(m, n)-\operatorname{HWP}(v ; r, s)$ are

1. If $v$ is odd, $r+s=\frac{v-1}{2}$,
2. If $v$ is even, $r+s=\frac{v-2}{2}$,
3. If $r>0, m \mid v$,
4. If $s>0, n \mid v$.

Recall that the Oberwolfach Problem involves seating $v$ conference attendees at $t$ round tables such that each attendee sits next to each other attendee exactly once. The Oberwolfach Problem for constant cycle lengths was solved in [2, 3, 23, 34]. This is equivalent to the Hamilton-Waterloo Problem with $r=0$ or $s=0$.

Theorem 3 [2, 3, 23, 34] There exists a resolvable m-cycle decomposition of $K_{v}$ (or $K_{v}-F$ when $v$ is even) if and only if $v \equiv 0(\bmod m),(v, m) \neq(6,3)$ and
$(v, m) \neq(12,3)$.

An equipartite graph is a graph whose vertex set can be partitioned into $u$ subsets of size $h$ such that no two vertices from the same subset are connected by an edge. The complete equipartite graph with $u$ subsets of size $h$ is denoted $K_{(h: u)}$, and it contains every edge between vertices of different subsets. Another key result solves the Oberwolfach Problem for constant cycle lengths over complete equipartite graphs (as opposed to $K_{v}$ ). That is to say, with finitely many exceptions, $K_{(h: u)}$ has a resolvable $C_{m}$-factorization.

Theorem 4 [28] For $m \geq 3$ and $u \geq 2, K_{(h: u)}$ has a resolvable $C_{m}$-factorization if and only if hu is divisible by $m, h(u-1)$ is even, $m$ is even if $u=2$, and $(h, u, m) \notin$ $\{(2,3,3),(6,3,3),(2,6,3)$, $(6,2,6)\}$.

Much of the attention to the HWP has been dedicated to the case of triangle factors and Hamilton factors. The results for this case have been summarized in the following theorem.

Theorem 5 [15, 16, 24, 27] There exists a $(3, v)-\operatorname{HWP}(v ; r, s)$ with

- $2 \leq s \leq \frac{v-1}{2}$ and $v \equiv 3(\bmod 6)$ except possibly when:

$$
v \equiv 15 \quad(\bmod 18) \text { and } 2 \leq s \leq \frac{v-3}{6} \text { or } s=\frac{v+3}{6}+1
$$

- $s=1$ and $v \equiv 3(\bmod 6)$ except when $v=9$ and possibly when:

$$
v \in\{93,111,123,129,141,153,159,177,183,201,207,213,237,249\}
$$

- $2 \leq s \leq(v-2) / 2$ and $v \equiv 0(\bmod 6)$ except possibly when $(v, s) \in$ $\{(36,2),(36,4)\}$ or when $v \equiv 12(\bmod 18)$ and $2 \leq s \leq(v / 6)-1$; and
- $s=1$ and $v \equiv 0(\bmod 6)$ except possibly when $v=18, v \equiv 12(\bmod 18)$ or $v \equiv 6(\bmod 36)$.

When considering the HWP for triangle factors and Hamilton factors, the focus is on a specific case of the problem. This paper considers a more general family of decompositions, namely, triangle factors and $3 x$-factors of $K_{v}$ for any $v$ that is divisible by both 3 and $3 x$. In this instance of the problem, $v$ is of the form $3 x y$. When $x=1$, the problem of finding a $(3,3 x)-\operatorname{HWP}(v ; r, s)$ is simply that of finding a resolvable $C_{3^{-}}$ factorization of $K_{v}$, which is also known as a Kirkman triple system $(K T S(v))$. It was shown in 1971 by Ray-Chadhuri and Wilson [34] and independently by Lu (see [30]) that a $K T S(v)$ exists if and only if $v \equiv 3(\bmod 6)$. When $y=1$, then the problem asks for a decomposition of $K_{v}$ into triangle factors and Hamilton cycles. This case
is addressed in [15], [16], and [24], and the results were presented in Theorem 5 . Therefore, we focus on the cases where $x \geq 2$ and $y \geq 2$. It is a different type of decomposition than what was considered in [15, 16, 24], because in our case, we let both $x$ and $y$ vary . However, as expected, the results given in Theorem 5 can be used in the decompositions we are interested in.

The Hamilton-Waterloo Problem was studied in 2002 by Adams, et. al. [1]. The paper provides solutions to all Hamilton-Waterloo decompositions on less than 18 vertices. Some notable results involving $v=6$ and $v=12$ will be relevant to this paper.

Theorem 6 [1] There exists a $(3,6)-\operatorname{HWP}(12 ; r, s)$ if and only if $r+s=5$ except $(r, s)=(5,0)$. There exists a $(3,12)-\operatorname{HWP}(12 ; r, s)$ if and only if $r+s=5$ except $(r, s)=(5,0)$. There exists a $(3,6)-\operatorname{HWP}(6 ; r, s)$ if and only if $r+s=2$ except $(r, s)=(2,0)$.

The authors in [1] also developed a tripartite construction that could be used when considering $m=3$ and $n=3 x$. However, it leaves many open cases, because it relies on the existence of a $(3, v)-\operatorname{HWP}(v ; r, s)$ for all $(r, s)$ and for all $v \equiv 3(\bmod 6)$. According to Theorem 5, there are some gaps in the existence of these. The problem is that the construction given in [1] uses a uniform decomposition of $K_{(x: 3)}$. Therefore, we proceed in this paper by developing a new construction that is a bit more general,
and in particular, depends on the decomposition of $K_{(x: 3)}$ into $r_{p} C_{m}$-factors and $s_{p}$ $C_{n}$-factors. The flexibility in this construction allows us to settle all but 14 cases of the existence of a $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ for all possible $(r, s)$ whenever both $x \geq 3$ and $y \geq 3$ are odd. We also introduce a modified construction that is used in the cases where at least one of $x$ or $y$ is even. We give almost complete results for these cases as well. In Section 1.3.1 we handle the cases when $x \in\{2,4\}$ and collect all of the results into a summarizing theorem in Section 1.4 .

### 1.2 Constructions

In this section, we develop constructions that will later be used to prove our main results about the Hamilton-Waterloo Problem in the case of triangle factors and $C_{3 x^{-}}$ factors.

Recall that $K_{(x: 3)}$ is the complete multipartite graph with 3 parts of size $x$. Let the parts be $G_{0}, G_{1}$ and $G_{2}$ and the vertices be $(a, b)$ with $0 \leq a \leq 2,0 \leq b \leq x-1$. Consider the edge $\left\{\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)\right\}$ which has one vertex from $G_{a_{1}}$ and one vertex from $G_{a_{2}}$. With computations being done in $\mathbb{Z}_{x}$, we say this edge has difference $b_{2}-b_{1}$. Let $T_{x}(i)$ for $0 \leq i \leq x-1$ be the subgraph of $K_{(x: 3)}$ obtained by taking all edges of difference: $2 i$ between vertices of $G_{0}$ and vertices of $G_{1},-i$ between $G_{1}$ and $G_{2}$, and $-i$ between $G_{2}$ and $G_{0}$.

Lemma 1.2.1 $T_{x}(i)$ is a triangle factor of $K_{(x: 3)}$ for any $i$.

Proof: It is easy to see that the triangles are of the form $\{(0, k),(1, k+2 i),(2, k+i)\}$ for every $0 \leq k \leq x-1$.

Let $H_{x}(i, j)$ be the subgraph of $K_{(x: 3)}$ obtained by taking all edges of difference: $2 i$ between $G_{0}$ and $G_{1},-i$ between $G_{1}$ and $G_{2}$, and $-j$ between $G_{2}$ and $G_{0}$.

Lemma 1.2.2 If $\operatorname{gcd}(x, i-j)=1$ then $H_{x}(i, j)$ is a Hamiltonian cycle of $K_{(x: 3)}$.

Proof: Since the edges are given by differences it is clear that all vertices have degree 2. We need to show that all the vertices are connected. We will first show that there is a path between any 2 vertices of $G_{0}$. Without loss of generality, we will show that $(0,0)$ is connected to $(0, k)$ for any $k$. Starting at $(0,0)$, we may traverse the path: $(0,0),(1,2 i),(2, i),(0, i-j)$. Thus the next time that we reach $G_{0}$ it is via the vertex $i-j$. Since $g c d(x, i-j)=1$, the order of $i-j$ in the cyclic group $\mathbb{Z}_{x}$ is $x$. Therefore, any $k$ modulo $x$ can be written as $k^{\prime}(i-j)$, which means that we reach the vertex $(0, k)$ after visiting the part $G_{0} k^{\prime}$ times. Hence $(0,0)$ is connected to all the vertices of $G_{0}$ via a path.

Because we are taking every edge of a particular difference, it follows that every vertex in $G_{1}$ is connected to a vertex in $G_{0}$, and the same is true for vertices in $G_{2}$. Hence all the vertices are connected, and the cycle is Hamiltonian, as we wanted to prove.

### 1.2.1 When $x$ is Odd

We can think of a decomposition of a graph $G$ as a partition of the edge set or as a union of edge disjoint subgraphs. This means that a decomposition of $G$ can be given by $E(G)=\cup E\left(F_{i}\right)$ or by $G=\oplus F_{i}$, where each $F_{i}$ is an edge disjoint subgraph of $G$. The next lemma shows that $K_{(x: 3)}$ can be decomposed entirely into triangle factors or Hamilton cycles when $x$ is odd.

Lemma 1.2.3 Let $x$ be an odd integer, and let $\phi$ be a bijection of the set $\{0,1, \ldots, x-$ 1\} into itself. Then

$$
K_{(x: 3)}=\bigoplus_{i=0}^{x-1} T_{x}(i)=\bigoplus_{i=0}^{x-1} H_{x}(i, \phi(i))
$$

Proof: To prove the first equality,

$$
K_{(x: 3)}=\bigoplus_{i=0}^{x-1} T_{x}(i)
$$

we need to show that between each pair of parts in $K_{(x: 3)}$, each difference is covered by the edges in one of the triangle factors exactly once. It is clear that edges of difference $i$ between $G_{1}$ and $G_{2}$ and between $G_{2}$ and $G_{0}$ are covered in $T_{x}(i)$. Now consider groups
$G_{0}$ and $G_{1}$. Each factor $T_{x}(i)$ uses the difference $2 i$. Because $\operatorname{gcd}(x, 2)=1$, the order of 2 in the cyclic group $\mathbb{Z}_{x}$ is $x$. So it follows that any $i$ modulo $x$ can be written as $2 i^{\prime}$, and thus the difference $i$ between $G_{0}$ and $G_{1}$ is covered in $T_{x}\left(i^{\prime}\right)$. Notice that we cover the edges of exactly one difference between any two parts per subgraph, and we only have $x$ subgraphs. This together with the fact that we are covering all the differences imply that we cover each difference exactly once. Thus it is equivalent to decomposing $K_{(x: 3)}$.

The second equality

$$
\bigoplus_{i=0}^{x-1} T_{x}(i)=\bigoplus_{i=0}^{x-1} H_{x}(i, \phi(i))
$$

is true because we again cover each difference between any pair of parts exactly once by the edges in the factors.

Notice that the subgraph $H_{x}(i, i)$ is the same as $T_{x}(i)$. Therefore, decomposing $K_{(x: 3)}$ into $s$ Hamilton cycles and $x-s$ triangle factors is equivalent to finding a bijection $\phi$ such that $\operatorname{gcd}(x, i-\phi(i))=1$ for $s$ elements of $\{0,1, \ldots, x-1\}$ and $\phi(i)=i$ for the rest.

Theorem 7 Let $x$ be odd and let $s \in\{0,2,3, \ldots, x\}$. Then:

- there exists a bijection $\phi$ on the set $\{0,1, \ldots, x-1\}$ with $\operatorname{gcd}(x, i-\phi(i))=1$ for $s$ elements and $r=x-s$ fixed points; and
- $K_{(x: 3)}$ can be decomposed into $s$ Hamiltonian cycles and $r=x-s$ triangle factors.

Proof: If $s=0$ we just use the identity mapping. Let $2 \leq s \leq x$, and let $e$ be the smallest integer such that $s \leq 2^{e}+1$. We have

$$
2^{e-1}+1<s \leq \min \left\{2^{e}+1, x\right\}=t
$$

Let $r=t-s$ and define $\phi$ as follows:

$$
\phi(i)= \begin{cases}0 & \text { for } \quad i=1 \\ i+2 & \text { for } \quad i \equiv 0 \quad(\bmod 2), 0 \leq i \leq s-3 \\ i-2 & \text { for } \quad i \equiv 1 \quad(\bmod 2), 3 \leq i \leq s-1 \\ s-2 & \text { for } \quad i \equiv 0 \quad(\bmod 2), i=s-1 \\ s-1 & \text { for } \quad i \equiv 0 \quad(\bmod 2), i=s-2 \\ i & \text { for } \quad s \leq i \leq x-1\end{cases}
$$

It is an easy exercise to check that $\phi$ is a bijection with $r=x-s$ fixed points. Furthermore, for any non-fixed point we have $(i-\phi(i)) \in\{ \pm 1, \pm 2\}$ and, because $x$ is odd, $\operatorname{gcd}(x, i-\phi(i))=1$. Hence by Lemma 1.2.3.

$$
K_{(x: 3)}=\bigoplus_{i=0}^{x-1} H_{x}(i, \phi(i))
$$

is a decomposition of $K_{(x: 3)}$ into $s$ Hamiltonian cycles and $r=x-s$ triangle factors.

Unfortunately this construction only works when $x$ is odd. For the cases when $x$ is even we can get a similar result, although only when $x=2 \bar{x}$, with $\bar{x}$ odd.

### 1.2.2 When $x$ is Even

In this subsection, we develop a construction similar to what is described in Section 1.2.1. It relies on the following decomposition of $K_{(4: 3)}$ into triangle factors. Define $\Gamma(i)$ for $i \in\{0,1,2,3\}$ as follows.


Note that the edges that join $G_{0}$ to $G_{2}$ are dashed since they will need to be distinguished from the other two edges in each $C_{3}$. It is easy to see that $\bigoplus_{i=0}^{3} \Gamma(i)$ is a $C_{3}$-factorization of $K_{(4: 3)}$.

Lemma 1.2.4 There exist a decomposition of $K_{(4: 3)}$ into $s C_{6}$-factors and $4-s C_{3^{-}}$factors for any $s \in\{0,2,3,4\}$.

Proof: Consider the $C_{3}$ factorization of $K_{(4: 3)}, \bigoplus_{i=0}^{3} \Gamma(i)$. Let $\Lambda(\alpha, \beta)$ be the graph that has edges between $G_{0}$ (the first column) and $G_{1}$ (the second column) from $\Gamma(\alpha)$, has edges between $G_{1}$ and $G_{2}$ from $\Gamma(\alpha)$, and has dashed edges from $\Gamma(\beta)$. Notice that if $\alpha \neq \beta$ then $\Lambda(\alpha, \beta)$ is a union of cycles of size 6 .

This way we can get $2 C_{6}$-factors by using $\Lambda(0,1)$ and $\Lambda(1,0)$ instead of $\Gamma(0)$ and $\Gamma(1)$. We can get $3 C_{6}$-factors by using edges $\Lambda(0,1), \Lambda(1,2)$ and $\Lambda(2,0)$ instead of $\Gamma(0), \Gamma(1)$ and $\Gamma(2)$. And finally we can get $4 C_{6}$-factors by using $\Lambda(0,1), \Lambda(1,2)$, $\Lambda(2,3)$ and $\Lambda(3,0)$. This construction gives the desired decompositions.

For $\bar{x}=1$, Lemma 1.2 .4 gives a decomposition of $K_{(4 \bar{x}: 3)}$ into triangle factors and $C_{6 \bar{x}}$-factors. We will extend this result to work on any $K_{(4 \bar{x}: 3)}$ where $\bar{x}>1$ and odd. The construction works by giving weight $\bar{x}$ to each vertex in $K_{(4: 3)}$.

Replace each vertex in $K_{(4: 3)}$ by a set of $\bar{x}$ vertices. Thus for $a=0,1,2$, we have $G_{a}=\{(a, b, c): b=1,2,3,4 ; c=1,2, \ldots, \bar{x}\}$ is a set of $4 \bar{x}$ vertices.

For $\alpha=0,1,2,3$, we construct the triangle factor $T_{2 \bar{x}}(\alpha, i)$ of $K_{(4 \bar{x}: 3)}$ as follows. For each triangle $\left\{\left(0, b_{0}\right),\left(1, b_{1}\right),\left(2, b_{2}\right)\right\}$ in $\bigoplus_{\alpha=0}^{3} \Gamma(\alpha)$, construct the complete equipartite graph $K_{(\bar{x}: 3)}$ on the set of vertices $\left\{\left(0, b_{0}, c\right),\left(1, b_{1}, c\right),\left(2, b_{2}, c\right): c=1,2, \ldots, \bar{x}\right\}$.

To visualize this weighting construction with $\bar{x}=3$ we show a picture of a triangle from $K_{(4: 3)}$. After giving weight 3 to each vertex, the triangle becomes $K_{(3: 3)}$.


Decompose each $K_{(\bar{x}: 3)}$ into triangle factors $T_{\bar{x}}(i)$ for $i=0,1, \ldots, \bar{x}-1$ by Lemma 1.2 .3 . Thus we have a decomposition of $K_{(4 \bar{x}: 3)}$ into triangle factors.

Define $H_{2 \bar{x}}(\alpha, i)(\beta, j)$ as the graph obtained by taking $T_{2 \bar{x}}(\alpha, i)$ and replacing the edges between $G_{0}$ and $G_{2}$ with the same edges from $T_{2 \bar{x}}(\beta, j)$. Then we have that

$$
H_{2 \bar{x}}(\alpha, i)(\beta, j) \oplus H_{2 \bar{x}}(\beta, j)(\alpha, i)=T_{2 \bar{x}}(\alpha, i) \oplus T_{2 \bar{x}}(\beta, j)
$$

If $\alpha \neq \beta$ and $\operatorname{gcd}(i-j, \bar{x})=1$ then we claim that $H_{2 \bar{x}}(\alpha, i)(\beta, j)$ is a $C_{6 \bar{x}}$-factor. Suppose $(a, b, c) \in H_{2 \bar{x}}(\alpha, i)(\beta, j)$. Then $(a, b)$ is a vertex of $K_{(4: 3)}$. Because $\alpha \neq \beta$, $\Lambda(\alpha, \beta)$ is a $C_{6}$-factor on $K_{(4: 3)}$, as shown in the proof of Lemma 1.2 .3 . Because $\operatorname{gcd}(i-j, x)=1$, it follows from Lemma 1.2 .2 that $K_{(\bar{x}: 3)}$ is a $3 \bar{x}$-cycle. Thus $(a, b, c)$ is contained in a cycle of length $\operatorname{lcm}(6,3 \bar{x})=6 \bar{x}$. Hence $H_{2 \bar{x}}(\alpha, i)(\beta, j)$ is a $C_{6 \bar{x}}$-factor.

Let $\psi$ be a bijection on $\{(\alpha, i) \mid 0 \leq \alpha \leq 3,0 \leq i \leq \bar{x}-1\}$. The previous discussion
leads us to the following result.

Lemma 1.2.5 Let $\bar{x}$ be odd. Let $s$ and $r$ be non-negative integers such that $s+r=4 \bar{x}$. If $\psi$ satisfies the following:

- $\psi(\alpha, i)=(\alpha, i)$ for $r$ pairs $(\alpha, i)$; and
- $\psi(\alpha, i)=(\beta, j)$ with $\alpha \neq \beta$ and $\operatorname{gcd}(i-j, \bar{x})=1$ for the s remaining pairs;
then $K_{(4 \bar{x}: 3)}=\bigoplus H_{2 \bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a decomposition of $K_{(4 \bar{x}: 3)}$ intor triangle factors and s $C_{6 \bar{x}}$-factors.

Proof: Notice that $H_{2 \bar{x}}(\alpha, i)(\alpha, i)=T_{2 \bar{x}}(\alpha, i)$, so if $\psi(\alpha, i)=(\alpha, i), H_{2 \bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a triangle factor. When $\psi(\alpha, i)=(\beta, j)$ with $\alpha \neq \beta$ and $\operatorname{gcd}(i-j, \bar{x})=1$, by the discussion preceding the lemma, $H_{2 \bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a $C_{6 \bar{x}}$-factor. Therefore $K_{(4 \bar{x}: 3)}=\bigoplus H_{2 \bar{x}}(\alpha, i)(\psi(\alpha, i))$ is a decomposition of $K_{(4 \bar{x}: 3)}$ into $r$ triangle factors and $s C_{6 \bar{x}}$-factors.

Thanks to Lemma 1.2 .5 we only need to show that for any $r \in\{0,1, \ldots, 4 \bar{x}-2,4 \bar{x}\}$ we have a bijection $\psi$ satisfying the conditions of the lemma and with $r$ fixed points.

Theorem 8 Let $\bar{x}$ be odd and $s \in\{0,2,3, \ldots, 4 \bar{x}-1,4 \bar{x}\}$, then:

- There exists a bijection $\psi$ satisfying the conditions of Lemma 1.2 .5 with $r=$ $4 \bar{x}-s$ fixed points.
- $K_{(4 \bar{x}: 3)}$ can be decomposed into s $C_{6 \bar{x}}$-factors and $r$ triangle factors.

Proof: If $s=0$ we just use the identity mapping.

If $2 \leq s \leq 4 \bar{x}$ we let $s_{0}, s_{1}, s_{2}, s_{3} \in\{0,2,3 \ldots, \bar{x}-1\}$ be such that $s=s_{0}+s_{1}+s_{2}+s_{3}$.
We define $\psi$ as follows, where $m \in\{0,1,2,3\}$ and $i+m$ is taken $(\bmod 4)$ :

$$
\psi(i+m, i)= \begin{cases}(m, 0) & \text { for } \quad i=1 \\ (i+m+2, i+2) & \text { for } \quad i \equiv 0 \quad(\bmod 2), 0 \leq i \leq s_{m}-3 \\ (i+m-2, i-2) & \text { for } \quad i \equiv 1 \quad(\bmod 2), 3 \leq i \leq s_{m}-1 \\ \left(s_{m}+m-2, s_{m}-2\right) & \text { for } \quad i \equiv 0 \quad(\bmod 2), i=s_{m}-1 \\ \left(s_{m}+m-1, s_{m}-1\right) & \text { for } \quad i \equiv 0 \quad(\bmod 2), i=s_{m}-2 \\ (i+m, i) & \text { for } \quad s_{m} \leq i \leq \bar{x}-1\end{cases}
$$

It is an easy exercise to check that $\psi$ is a bijection with $4 \bar{x}-\left(s_{0}+s_{1}+s_{2}+s_{3}\right)=r$ fixed points. Notice that $\psi(\alpha, i)-(\alpha, i) \in\{(0,0),( \pm 1, \pm 1),( \pm 2, \pm 2)\}$. This gives that if $\psi(\alpha, i)=(\beta, j)$ is not a fixed point of $\psi, \alpha \neq \beta$ and $\operatorname{gcd}(i-j, \bar{x})=1$.

Hence by Lemma 1.2.5

$$
K_{(4 \bar{x}: 3)}=\bigoplus H_{2 \bar{x}}(\alpha, i)(\psi(\alpha, i))
$$

is a decomposition of $K_{(4 \bar{x}: 3)}$ into $s C_{6 \bar{x}}$-factors and $4 \bar{x}-s$ triangle factors.

### 1.2.3 A Weighting Construction

A group divisible design $(k, \lambda)-\operatorname{GDD}\left(h^{u}\right)$ is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a finite set of size $v=h u, \mathcal{G}$ is a partition of $\mathcal{V}$ into $u$ groups each containing $h$ elements, and $\mathcal{B}$ is a collection of $k$ element subsets of $\mathcal{V}$ called blocks which satisfy the following properties.

- If $B \in \mathcal{B}$, then $|B|=k$.
- If a pair of elements from $\mathcal{V}$ appear in the same group, then the pair cannot be in any block.
- Two points that are not in the same group, called a transverse pair, appear in exactly $\lambda$ blocks.
- $|\mathcal{G}|>1$.

Here we use the term group to indicate an element of $\mathcal{G}$. In this context, group simply means a set of elements without any algebraic structure. A resolvable GDD
(RGDD) has the additional condition that the blocks can be partitioned into parallel classes such that for each element of $\mathcal{V}$ there is exactly one block in each parallel class containing it. If $\lambda=1$, we refer to the $\operatorname{RGDD}$ as a $k-\operatorname{RGDD}\left(h^{u}\right)$. In this paper, we will only talk about RGDDs with $\lambda=1$. Necessary and sufficient conditions for the existence of $3-\operatorname{RGDD}\left(h^{u}\right)$ s have been established except in a finite number of cases.

Theorem 9 [35] $A(3, \lambda)-\operatorname{RGDD}\left(h^{u}\right)$ exists if and only if $u \geq 3, \lambda h(u-1)$ is even, $h u \equiv 0(\bmod 3)$, and $(\lambda, h, u) \notin\{(1,2,6),(1,6,3)\} \bigcup\{(2 j+1,2,3),(4 j+2,1,6): j \geq$ $0\}$.

In particular, we have that a $3-\operatorname{RGDD}\left(3^{u}\right)$ exists for all odd $u \geq 3$ and a $3-\operatorname{RGDD}\left(6^{u}\right)$ exists for all $u \geq 4$. Note that when a $3-\operatorname{RGDD}\left(h^{u}\right)$ exists, then $\mathcal{B}$ can be partitioned in $\frac{h(u-1)}{2}$ parallel classes.

Lemma 1.2.6 Let $m \geq 3, n \geq 3$ and $x$ be positive integers such that both $m$ and $n$ divide $3 x$. Suppose the following conditions are satisfied:

- There exists a 3-RGDD $\left(h^{u}\right)$,
- there exists a decomposition of $K_{(x: 3)}$ into $r_{p} C_{m}$-factors and $s_{p} C_{n}$-factors, for $p \in\left\{1,2, \ldots, \frac{h(u-1)}{2}\right\}$,
- there exists an $(m, n)-\operatorname{HWP}\left(h x ; r_{\beta}, s_{\beta}\right)$.

Let

$$
r_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}} r_{p} \text { and } s_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}} s_{p} .
$$

Then there exists a $(m, n)-\operatorname{HWP}\left(h u x ; r_{\alpha}+r_{\beta}, s_{\alpha}+s_{\beta}\right)$.

Proof: For $a=1,2, \ldots, u$, let the groups of the $3-\operatorname{RGDD}\left(h^{u}\right)$ be denoted by $G_{a}=\{(a, b): b=1,2, \ldots, h\}$. Let $\left\{\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{\frac{h(u-1)}{2}}\right\}$ denote the parallel classes of the $3-\operatorname{RGDD}\left(h^{u}\right)$, and for $a=1,2, \ldots, u$, define $G_{a}^{*}=\{(a, b, c): b=$ $1,2, \ldots, h ; c=1,2, \ldots, x\}$ to be a set of $h x$ vertices. Consider each parallel class $\mathcal{P}_{p}$ with $p \in\left\{1,2, \ldots, \frac{h(u-1)}{2}\right\}$. For each block $\left\{\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right),\left(a_{3}, b_{3}\right)\right\} \in \mathcal{P}_{p}$, construct a decomposition of $K_{(x: 3)}$ into $r_{p} C_{m}$-factors and $s_{p} C_{n}$-factors on the set of vertices $\left\{\left(a_{1}, b_{1}, c\right),\left(a_{2}, b_{2}, c\right),\left(a_{3}, b_{3}, c\right): c=1,2, \ldots, x\right\}$. Thus we have a decomposition of $K_{(h x: u)}$ into $r_{\alpha} C_{m}$-factors and $s_{\alpha} C_{n}$-factors where

$$
r_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}} r_{p} \text { and } s_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}} s_{p}
$$

Now each part of $K_{(h x: u)}$ can be decomposed into $r_{\beta} C_{m}$-factors and $s_{\beta} C_{n}$-factors. Thus there exists an $(m, n)-\operatorname{HWP}(h u x ; r, s)$ where $r=r_{\alpha}+r_{\beta}$ and $s=s_{\alpha}+s_{\beta}$.

Lemma 1.2.7 Let $m \geq 3, n \geq 3$ and $x$ be positive integers such that both $m$ and $n$ divide $3 x$. Suppose the following conditions are satisfied:

- There exists a 3-RGDD $\left(h^{u}\right)$,
- there exists an $(m, n)-\operatorname{HWP}\left(3 x ; r_{\beta}, s_{\beta}\right)$,
- there exists a decomposition of $K_{(x: h)}$ into $r_{\gamma} C_{m}$-factors and $s_{\gamma} C_{n}$-factors,
- there exists a decomposition of $K_{(x: 3)}$ into $r_{p} C_{m}$-factors and $s_{p} C_{n}$-factors, for $p \in\left\{1,2, \ldots, \frac{h(u-1)}{2}\right\}$.

Let

$$
r_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}-1} r_{p} \text { and } s_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}-1} s_{p} .
$$

Then there exists a $(m, n)-\operatorname{HWP}\left(h u x ; r_{\alpha}+r_{\beta}+r_{\gamma}, s_{\alpha}+s_{\beta}+s_{\gamma}\right)$.

Proof: Let $\left\{\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{\frac{h(u-1)}{2}}\right\}$ denote the parallel classes of the $3-\operatorname{RGDD}\left(h^{u}\right)$, and let $W=\{1,2, \ldots, x\}$. Consider each parallel class $\mathcal{P}_{p}$ with $p \in\left\{1,2, \ldots, \frac{h(u-1)}{2}-1\right\}$. For each block $\left\{a_{1}, a_{2}, a_{3}\right\} \in \mathcal{P}_{p}$, construct a decomposition of $K_{(x: 3)}$ into $r_{p} C_{m^{-}}$ factors and $s_{p} C_{n}$-factors with parts $\left\{a_{i}\right\} \times W, i=1,2,3$. For each block $\left\{a_{1}, a_{2}, a_{3}\right\}$ in parallel class $\mathcal{P}_{\beta}$ where $\beta=\frac{h(u-1)}{2}$, construct an $(m, n)-\operatorname{HWP}\left(3 x ; r_{\beta}, s_{\beta}\right)$ on $\left\{a_{1} \times W, a_{2} \times W, a_{3} \times W\right\}$. Take a decomposition of $K_{(x: h)}$ into $r_{\gamma} C_{m}$-factors and $s_{\gamma} C_{n}$-factors simultaneously on each group of the $3-\operatorname{RGDD}\left(h^{u}\right)$. This makes an $(m, n)-\operatorname{HWP}(h u x ; r, s)$ where $r=r_{\alpha}+r_{\beta}+r_{\gamma}$ and $s=s_{\alpha}+s_{\beta}+s_{\gamma}$.

### 1.3 Main Results

In this section, we use the constructions given in Section 1.2 to obtain results on the existence of a $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$. We consider four different cases depending on the parity of $x$ and $y$.

Lemma 1.3.1 Suppose $x$ is even. If there exists a decomposition of $K_{3 x}-F$ into $r_{\delta} C_{3}$-factors and $s_{\delta}$ Hamilton cycles, then there exists a decomposition of $K_{6 x}-F^{\prime}$ into $r_{\delta} C_{3}$-factors and $s_{\delta}+\frac{3 x}{2} C_{3 x}$-factors, where $F$ is a 1 -factor of $K_{3 x}$ and $F^{\prime}$ is a 1-factor of $K_{6 x}$.

Proof: Let $G_{1}$ and $G_{2}$ be a partition of the $6 x$ points into two subsets of size $3 x$. Decompose $K_{G_{1}}-F_{1}$ (the complete graph on $G_{1}$ minus a 1-factor $F_{1}$ ) into $r_{\delta} C_{3^{-}}$ factors and $s_{\delta}$ Hamilton cycles. In the same manner, decompose $K_{G_{2}}-F_{2}$ (the complete graph on $G_{2}$ minus a 1-factor) into $r_{\delta} C_{3}$-factors and $s_{\delta}$ Hamilton cycles. Then there is a decomposition of $\left(K_{G_{1}} \cup K_{G_{2}}\right)-\left(F_{1} \cup F_{2}\right)$ into $r_{\delta} C_{3}$-factors and $s_{\delta}$ $C_{3 x}$-factors. Notice that $F^{\prime}=F_{1} \cup F_{2}$ is a 1-factor of $K_{6 x}$. By Theorem 4, there exists a decomposition of $K_{(3 x: 2)}$ into $\frac{3 x}{2} C_{3 x}$-factors. The union of these edges is $K_{6 x}$.

Therefore there is a decomposition of $K_{6 x}-F^{\prime}$ into $r_{\delta} C_{3}$-factors and $s_{\delta}+\frac{3 x}{2} C_{3 x}$ factors.

Theorem 10 For each pair of odd integers $x \geq 3$ and $y \geq 3$, there exists $a$ $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ if and only if $r+s=\frac{v-1}{2}$ except when $s=1$ and $x=3$, and possibly when $s=1$ and
$x \in\{31,37,41,43,47,51,53,59,61,67,69,71,79,83\}$.

Proof: By Theorem 9 there exists a $3-\operatorname{RGDD}\left(3^{y}\right)$ for all odd $y \geq 3$. There exists a decomposition of $K_{(x: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{3 x}$-factors for $\left(r_{p}, s_{p}\right) \in\{(x, 0),(x-$ $2,2),(x-3,3), \ldots,(0, x)\}$ by Theorem 7. There exists a $(3,3 x)-\operatorname{HWP}\left(3 x ; r_{\beta}, s_{\beta}\right)$ whenever $\left(r_{\beta}, s_{\beta}\right) \in\left\{\left(\frac{3 x-1}{2}, 0\right),\left(\frac{3 x-3}{2}, 1\right),\left(0, \frac{3 x-1}{2}\right)\right\}$ by Theorems 3 and 5 (excluding the exception and possible exceptions listed in the statements of these theorems). So apply Lemma 1.2 .6 with $m=3$ and $n=3 x$. We must now show that for each $s \in\left\{0,1, \ldots, \frac{3 x y-1}{2}\right\}$, there exists a $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$. It is easy to see that if $s_{\alpha} \in$ $\left\{0,2,3, \ldots, \frac{3 x y-3 x}{2}\right\}$, then we can write $s_{\alpha}=\sum_{i=1}^{(3 y-3) / 2} s_{p}$ where $s_{p} \in\{0,2,3, \ldots, x\}$. Thus if $s \in\left\{0,2,3, \ldots, \frac{3 x y-3 x}{2}\right\}$, then we may write $s=s_{\alpha}+s_{\beta}$ by choosing $s_{\alpha}=s$ and $s_{\beta}=0$. If $s=1$, then choose $s_{\alpha}=0$ and $s_{\beta}=1$. If $s=\frac{3 x y-3 x}{2}+1$, choose $s_{\alpha}=\frac{3 x y-3 x}{2}$ and $s_{\beta}=1$. Finally, let $i=2,3, \ldots, \frac{3 x-1}{2}$, and consider $s=\frac{3 x y-3 x}{2}+i$. We may choose $s_{\alpha}=s-\left(\frac{3 x-1}{2}\right)$ and $s_{\beta}=\frac{3 x-1}{2}$ because

$$
2 \leq s-\frac{3 x-1}{2} \leq \frac{3 x y-3 x}{2}
$$

Theorem 11 For each odd integer $x \geq 3$ and each even integer $y \geq 8$, there exists $a$ $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ if and only if $r+s=\frac{3 x y-1}{2}$ except possibly when $s=1$.

Proof: By Theorem 9, there exists a $3-\operatorname{RGDD}\left(6^{y / 2}\right)$ for all even $y \geq 8$. By Theorem 7, for each $p \in\left\{1,2, \ldots, \frac{6(y / 2-1)}{2}\right\}, K_{(x: 3)}$ can be decomposed into $r_{p} C_{3}$-factors and $s_{p}$ $C_{3 x}$-factors where $\left(r_{p}, s_{p}\right) \in\{(x, 0),(x-2,2),(x-3,3), \ldots,(0, x)\}$, so that $r_{\alpha}=$ $\sum_{p=1}^{3(y / 2-1)} r_{p}$ and $s_{\alpha}=\sum_{p=1}^{3(y / 2-1)} s_{p}$. By Theorem 3, $K_{6 x}$ can be decomposed into $r_{\beta}$ $C_{3}$-factors, $s_{\beta} C_{3 x}$-factors, and a 1-factor where $\left(r_{\beta}, s_{\beta}\right) \in\{((6 x-2) / 2,0),(0,(6 x-$ $2) / 2)\}$. We must show that for each $s \in\{0,2,3, \ldots,(3 x y-2) / 2\}$ there exists a $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$. It is easy to see that such a decomposition exists when $s \in$ $\{0,2,3, \ldots,(3 x y-6 x) / 2\}$ by choosing $s_{\alpha}=s$ and $s_{\beta}=0$. For each $i \in\{1,2, \ldots,(6 x-$ $2) / 2\}$, when $s=(3 x y-6 x) / 2+i$, choose $s_{\alpha}=s-(6 x-2) / 2$ and $s_{\beta}=(6 x-2) / 2$. Notice that

$$
2 \leq s_{\alpha}=\frac{3 x y-6 x}{2}+i-\left(\frac{6 x-2}{2}\right) \leq \frac{3 x y-6 x}{2}+\left(\frac{6 x-2}{2}\right)-\left(\frac{6 x-2}{2}\right) \leq \frac{3 x y-6 x}{2} .
$$

Therefore by Lemma 1.2 .6 , the proposed $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ exists for all specified pairs $(r, s)$.

Note that when $x$ is even we cannot apply Theorem 7 to decompose $K_{(x: 3)}$. Instead we can apply Theorem 4 to get a decomposition of $K_{(x: 3)}$ into $x C_{3}$-factors or a decomposition of $K_{(x: 3)}$ into $x C_{3 x}$-factors. In this way we can use Theorem 4 to
decompose $K_{(x: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{3 x}$-factors, where $\left(r_{p}, s_{p}\right) \in\{(x, 0),(0, x)\}$.

Theorem 12 For each even integer $x \geq 8$ and each odd integer $y \geq 3$, there exists $a$ $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ if and only if $r+s=\frac{3 x y-2}{2}$ except possibly when:

- $(s, x) \in\{(2,12),(4,12)\}$,
- $1 \leq s \leq \frac{x}{2}-1$ and $x \equiv 4(\bmod 6)$,
- $s=1$ and $x \equiv 2(\bmod 12)$.

Proof: Suppose $x \geq 8$ is even. By Theorem 9, there exists a $3-\operatorname{RGDD}\left(3^{y}\right)$ for all odd integers $y \geq 3$. By Theorem 4, for each $p \in\left\{1,2, \ldots, \frac{3(y-1)}{2}\right\}, K_{(x: 3)}$ can be decomposed into $r_{p} C_{3}$-factors and $s_{p} C_{3 x}$-factors, where $\left(r_{p}, s_{p}\right) \in\{(x, 0),(0, x)\}$. By Theorem 5, there exists a decomposition of $K_{3 x}$ into $r_{\beta} C_{3}$-factors and $s_{\beta} C_{3 x}$-factors and a 1 -factor for $\left(r_{\beta}, s_{\beta}\right) \in\left\{\left(\frac{3 x-2}{2}, 0\right),\left(\frac{3 x-4}{2}, 1\right), \ldots,\left(0, \frac{3 x-2}{2}\right)\right\}$, except possibly when $\left(s_{\beta}, x\right) \in\{(2,12),(4,12)\} ; 1 \leq s_{\beta} \leq \frac{x}{2}-1$ and $x \equiv 4(\bmod 6) ;$ or $s_{\beta}=1$ and $x \equiv 2$ $(\bmod 12)$. We apply Lemma 1.2 .6 to obtain a $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ with $r=r_{\alpha}+r_{\beta}$ and $s=s_{\alpha}+s_{\beta}$ for all $s \in\left\{0,1, \ldots, \frac{3 x y-2}{2}\right\}$ (with the exceptions listed in the statement of this theorem) as follows. We may write $s_{\alpha}=\sum_{p=1}^{\frac{3(y-1)}{2}} s_{p}$ where $s_{p} \in\{0, x\}$, so that $s_{\alpha} \in\left\{0, x, 2 x, \ldots, x \cdot \frac{3 y-3}{2}\right\}$. Write $s=t \cdot x+i$, where $t \in\left\{0,1, \ldots, \frac{3 y-3}{2}\right\}$ and $i \in\left\{0,1, \ldots, \frac{3 x-2}{2}\right\}$. We may choose $s_{\alpha}=s-i$ and $s_{\beta}=i$.

Note that the cases of $x=2,4$ are not considered in the previous theorem. They will be handled in Section 1.3.1. We leave open the case of $x=6$ and $y$ odd.

Theorem 13 For each even integer $x \geq 8$ and each even integer $y \geq 8$, there exists $a$
$(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ if and only if $r+s=\frac{3 x y-2}{2}$ except possibly when:

- $(s, x) \in\{(2,12),(4,12)\}$,
- $2 \leq s \leq \frac{x}{2}-1$ and $x \equiv 4$ or $10(\bmod 12)$,
- $s=1$ and $x \equiv 2,4$, or $10(\bmod 12)$.

Proof: There exists a $3-\operatorname{RGDD}\left(6^{y / 2}\right)$ for all even $y \geq 8$ by Theorem 9. There exists a decomposition of $K_{(x: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{3 x}$-factors for $\left(r_{p}, s_{p}\right) \in$ $\{(0, x),(x, 0)\}$ by Theorem 4. By the same result, we also get a decomposition of $K_{(x: 6)}$ into $r_{\gamma} C_{3}$-factors and $s_{\gamma} C_{3 x}$-factors for $\left(r_{\gamma}, s_{\gamma}\right) \in\left\{\left(0, \frac{5 x}{2}\right),\left(\frac{5 x}{2}, 0\right)\right\}$. By Theorem 5, there exists a decomposition of $K_{3 x}$ into $r_{\beta} C_{3}$-factors, $s_{\beta} C_{3 x}$-factors, and a 1 -factor for $\left.\left(r_{\beta}, s_{\beta}\right) \in\left\{\frac{3 x-2}{2}, 0\right),\left(\frac{3 x-4}{2}, 1\right), \ldots,\left(0, \frac{3 x-2}{2}\right)\right\}$, except possibly when $\left(s_{\beta}, x\right) \in\{(2,12),(4,12)\} ; 1 \leq s_{\beta} \leq \frac{x}{2}-1$ and $x \equiv 4(\bmod 6) ;$ or $s_{\beta}=1$ and $x \equiv 2(\bmod 12)$. Write $s_{\alpha}=\sum_{p=1}^{\frac{3 y}{2}-4} s_{p}$ so $s_{\alpha} \in\left\{0, x, 2 x, \ldots, x\left(\frac{3 y}{2}-4\right)\right\} . \quad$ By Lemma 1.2 .7 , we obtain a $(3,12)-\operatorname{HWP}(3 x y ; r, s)$ for all $s \in\left\{0,1, \ldots, \frac{3 x y-2}{2}\right\}$ as follows. If $s \in\left\{0,1, \ldots, \frac{3 x y}{2}-\frac{5 x}{2}-1\right\}$, it is easy to see that we can let $s_{\gamma}=0$ and write $s$ as $s=s_{\alpha}+s_{\beta}$. If $s=\frac{3 x y}{2}-\frac{5 x}{2}+i$, for $i=0,1, \ldots, \frac{3 x}{2}-1$ choose $s_{\alpha}=\left(\frac{3 y}{2}-5\right) x$,
$s_{\beta}=i$, and $s_{\gamma}=\frac{5 x}{2}$. If $s=\frac{3 x y}{2}-x+i$ for $i=0,1, \ldots, x-1$, choose $s_{\alpha}=\left(\frac{3 y}{2}-4\right) x$, $s_{\beta}=\frac{x}{2}+i$ and $s_{\gamma}=\frac{5 x}{2}$.

We can fill in some of the gaps that we have left by using Theorem 8.

Theorem 14 For each odd integer $\bar{x} \geq 3$ and each even integer $y \geq 6$, there exists $a$ $(3,6 \bar{x})-\operatorname{HWP}(6 \bar{x} y ; r, s)$ if and only if $r+s=\frac{6 \bar{x} y-2}{2}$ except possibly when $s=1$.

Proof: Assume that $y \equiv 2(\bmod 4)$ and $y \geq 6$. For all such $y$, there exists a $3-\operatorname{RGDD}\left(3^{\frac{y}{2}}\right)$ by Theorem 9. There exists a $(3,6 \bar{x})-\operatorname{HWP}\left(12 \bar{x} ; r_{\beta}, s_{\beta}\right)$ for all $\left(r_{\beta}, s_{\beta}\right) \in\left\{\left(0, \frac{12 \bar{x}-2}{2}\right),\left(\frac{12 \bar{x}-2}{2}, 0\right)\right\}$ by Theorem 3. By Theorem 8, we have that $K_{(4 \bar{x}: 3)}$ can be decomposed into $r_{p} C_{3}$-factors and $s_{p} C_{6 \bar{x}}$-factors for $\left(r_{p}, s_{p}\right) \in\{(0,4 \bar{x}),(1,4 \bar{x}-$ $1), \ldots,(4 \bar{x}-2,2),(4 \bar{x}, 0)\}$. Apply Lemma 1.2 .6 with $m=3, n=6 \bar{x}$, and $x=4 \bar{x}$. Let $s_{\alpha}=\sum_{p=1}^{3\left(\frac{y}{2}-1\right) / 2} s_{p}$, then it is easy to see that $s_{\alpha} \in\{0,2,3, \ldots, 3 \bar{x} y-6 \bar{x}\}$. Write $s=s_{\alpha}+s_{\beta}$ where $s_{\alpha} \in\{0,2,3, \ldots, 3 \bar{x} y-6 \bar{x}\}$ and $s_{\beta} \in\{0,6 \bar{x}-1\}$. Then we can write $s$ as $s_{\alpha}+s_{\beta}$ for every $s \in\left\{0,2,3, \ldots, \frac{6 \bar{x} y-2}{2}\right\}$ in this way. Thus we can construct a $(3,6 \bar{x})-\operatorname{HWP}(6 \bar{x} y ; r, s)$ for all $s \in\left\{0,1, \ldots, \frac{6 \bar{x} y-2}{2}\right\}$.

Assume $y \equiv 0(\bmod 4)$, and $y \geq 12$. Then there exists a $3-\operatorname{RGDD}\left(6^{\frac{y}{4}}\right)$ by Theorem 9 . There exists a decomposition of $K_{(4 \bar{x}: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{6 \bar{x}}$-factors for $s_{p} \in$ $\{0,2,3, \ldots, 4 \bar{x}\}$ by Theorem 8. By Theorem 4, there exists a $\left(C_{3}, C_{6 \bar{x}}\right)$-factorization of $K_{(4 \bar{x}: 6)}$ for $\left(r_{\gamma}, s_{\gamma}\right) \in\{(0,10 \bar{x}),(10 \bar{x}, 0)\}$. There exists a $(3,6 \bar{x})-\operatorname{HWP}\left(12 \bar{x} ; r_{\beta}, s_{\beta}\right)$ for $s_{\beta} \in\left\{0, \frac{12 \bar{x}-2}{2}\right\}$ by Theorem 3. Now we can easily write $s=s_{\alpha}+s_{\beta}+s_{\gamma}$ for
$s \in\{0,2,3, \ldots, 3 \bar{x} y-1\}$ and apply Lemma 1.2 .7 .

By writing $x=2 \bar{x}$ Theorem 14 covers the cases when $s \neq 1$ and $x=6$ and also some of the cases when $s \neq 1$ and $x \equiv 4(\bmod 6)($ namely the ones where $x \equiv 10$ $(\bmod 12))$. When $x \geq 6$ is even and $y \geq 8$ is even, the cases that are not covered by Theorems 13 and 14 are as follows:

- $(s, x) \in\{(2,12),(4,12)\}$,
- $2 \leq s \leq \frac{x}{2}-1$ and $x \equiv 4(\bmod 12)$,
- $s=1$ and $x \equiv 2,4,10(\bmod 12)$.

Because there is no $3-\operatorname{RGDD}\left(6^{u}\right)$ for $u \leq 3$, Lemmas 1.2 .6 and 1.2 .7 are not useful when $y \in\{2,4,6\}$. However, we still have some results. When $y=2$ and $x$ is even we may apply Lemma 1.3.1 to find a $(3,3 x)-\operatorname{HWP}(6 x ; r, s)$ for $s=s_{1}+\frac{3 x}{2}, r=r_{1}$, where $\left(s_{1}, r_{1}\right)$ is a solution of the Hamilton-Waterloo Problem with triangles and Hamilton cycles for $K_{3 x}$.

When $y=4$ and $x \geq 2$ is even, consider $K_{12 x}$. We can partition the vertices into four parts of size $3 x$. In the four copies of $K_{3 x}$ we have some solutions for the HamiltonWaterloo Problem with triangles and Hamilton cycles. The remaining edges give us $K_{(3 x: 4)}$, which can be decomposed into all $C_{3 x}$-factors or into all triangle factors. In this way we can get either all triangle factors, or $s=s_{1}+e_{1} \frac{9 x}{2}, r=r_{1}+e_{2} \frac{9 x}{2}$, where
$\left(s_{1}, r_{1}\right)$ is a solution of the Hamilton-Waterloo problem with triangles and Hamilton cycles for $K_{3 x}$ and $e_{1}+e_{2}=1, e_{1}, e_{2} \geq 0$. If $y=6$ and $x$ is even, consider $K_{18 x}$. By following the same method, we can get either all triangle factors, or $s=s_{1}+e_{1} \frac{15 x}{2}$, $r=r_{1}+e_{2} \frac{15 x}{2}$, where $\left(s_{1}, r_{1}\right)$ is a solution of the Hamilton-Waterloo Problem with triangles and Hamilton cycles for $K_{3 x}$ and $e_{1}+e_{2}=1, e_{1}, e_{2} \geq 0$.

### 1.3.1 When $x$ is Small

In this subsection, we consider the small values of $x$ for which the general constructions used in Section 1.3 cannot be readily applied. By applying the methods described at the end of Section 1.3 , it is easy to see that the following decompositions exist when $x=2: \mathrm{a}(3,6)-\operatorname{HWP}(24 ; r, s)$ for $s \in\{0,1,2,7,8,9,10,11\}$, and a $(3,6)-\operatorname{HWP}(48 ; r, s)$ for $s \in\{0,1,2,3,4,5,12,13,14,19,20,21,22,23\}$. The following three results gives solutions to the Hamilton-Waterloo Problem, $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$, for all other values of $y$ when $x=2$.

Theorem 15 There exists a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ for all $y \equiv 2(\bmod 4)$ if and only if $r+s=\frac{6 y-2}{2}$, except when $y=2$ and $s=0$.

Proof: If $y=2$, then there exists a $(3,6)-\operatorname{HWP}(12 ; r, s)$ for all possible $r$ and $s$ except when $s=0$ by Theorem 6. We now assume that $y \equiv 2(\bmod 4)$ and
$y \geq 6$. For all such $y$, there exists a $3-\operatorname{RGDD}\left(3^{\frac{y}{2}}\right)$ by Theorem 9 . There exists a $(3,6)-\operatorname{HWP}\left(12 ; r_{\beta}, s_{\beta}\right)$ for all $\left(r_{\beta}, s_{\beta}\right) \in\{(0,5),(1,4),(2,3),(3,2),(4,1)\}$ by Theorem 6. By Lemma 1.2.4, we have that $K_{(4: 3)}$ can be decomposed into $r_{p} C_{3}$-factors and $s_{p} C_{6}$-factors for $\left(r_{p}, s_{p}\right) \in\{(0,4),(1,3),(2,2),(4,0)\}$. Apply Lemma 1.2 .6 with $m=3, n=6$, and $x=4$. Let $s_{\alpha}=\sum_{p=1}^{3\left(\frac{y}{2}-1\right) / 2} s_{p}$, then it is easy to see that $s_{\alpha} \in\{0,2,3, \ldots, 3 y-6\}$. Write $s=s_{\alpha}+s_{\beta}$ where $s_{\alpha} \in\{0,2,3, \ldots, 3 y-6\}$ and $s_{\beta} \in\{1,2,3,4,5\}$. Then we can write $s$ as $s_{\alpha}+s_{\beta}$ for every $s \in\left\{1,2, \ldots, \frac{6 y-2}{2}\right\}$ in this way. If $s=0$, then there exists a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ by Theorem 3. Thus we can construct a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ for all $s \in\left\{0,1, \ldots, \frac{6 y-2}{2}\right\}$.

Theorem 16 There exists a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ for all $y \equiv 0(\bmod 4)$ if and only if $r+s=\frac{6 y-2}{2}$, except possibly when $y=4$ or $y=8$.

Proof: Assume $y \equiv 0(\bmod 4)$, and $y \geq 12$. Then there exists a $3-\operatorname{RGDD}\left(6^{\frac{y}{4}}\right)$ by Theorem 9. There exists a decomposition of $K_{(4: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{6}$-factors for $s_{p} \in\{0,2,3,4\}$ by Lemma 1.2.4. By Theorem 4 , there exists a $\left(C_{3}, C_{6}\right)$-factorization of $K_{(4: 6)}$ for $\left(r_{\gamma}, s_{\gamma}\right) \in\{(0,10),(10,0)\}$. There exists a $(3,6)-\operatorname{HWP}\left(12 ; r_{\beta}, s_{\beta}\right)$ for $s_{\beta} \in\{1,2,3,4,5\}$ by Theorem 6. Now we can easily write $s=s_{\alpha}+s_{\beta}+s_{\gamma}$ for $s \in\{0,1, \ldots, 3 y-1\}$ and apply Lemma 1.2.7.

Theorem 17 There exists a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ when $y$ is odd and $s \in\left\{1,2, \frac{3(y-1)}{2}+1, \frac{3(y-1)}{2}+2, \ldots, 3 y-1\right\}$.

Proof: If $y=1$, then exists a $(3,6)-\operatorname{HWP}(6 ; r, s)$ for all possible $r$ and $s$ except for $(r, s)=(2,0)$ by Theorem 6. Assume $y \geq 3$ is odd, then there exists a $3-\operatorname{RGDD}\left(3^{y}\right)$ by Theorem 9. There exists a $(3,6)-\operatorname{HWP}\left(6 ; r_{\beta}, s_{\beta}\right)$ for $\left(r_{\beta}, s_{\beta}\right) \in\{(1,1),(0,2)\}$ by Theorem 6. It is easy to see that $K_{(2: 3)}$ can be decomposed into a $C_{3}$-factor and a $C_{6}$-factor or two $C_{6}$-factors. Apply Lemma 1.2 .6 with $m=3, n=6$ and $x=2$. Let $s_{\alpha}=\sum_{p=1}^{3(y-1) / 2} s_{p}$ with $s_{p} \in\{1,2\}$ and notice that $s_{\alpha} \in\left\{\frac{3(y-1)}{2}, \frac{3(y-1)}{2}+1, \ldots, 3(y-1)\right\}$. Then we can write $s$ as $s_{\alpha}+s_{\beta}$ for every $s \in\left\{\frac{3(y-1)}{2}+1, \frac{3(y-1)}{2}+2, \ldots, 3 y-1\right\}$. Thus we obtain a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ for all such $s$. We can also obtain a $(3,6)-\operatorname{HWP}(6 y ; r, s)$ for $s=1$ and $s=2$ as follows. There exists a $3-\operatorname{RGDD}\left(6^{y}\right)$ by Theorem 9; it has $3(y-1)$ parallel classes. There exists a $(3,6)-\operatorname{HWP}\left(6 ; r_{\beta}, s_{\beta}\right)$ for $s_{\beta} \in\{1,2\}$. Apply Lemma 1.2 .6 with $m=3, n=6$ and $x=1$, and write $s=s_{\alpha}+s_{\beta}$ with $s_{\alpha}=0$ and $s_{\beta}=1$ or $s_{\beta}=2$.

Recall from Theorem 6 that there exists a $(3,12)-\operatorname{HWP}\left(12 ; r_{\delta}, s_{\delta}\right)$ if and only if $s_{\delta} \in\{1,2,3,4,5\}$. For each possible decomposition of $K_{12}$, let $s_{\beta}=s_{\delta}+6$, and apply Lemma 1.3.1 to obtain a $(3,12)-\operatorname{HWP}(24 ; r, s)$ for all $s \in\{7,8,9,10,11\}$. If $s=0$, then simply apply Theorem 3. Similarly, apply Theorem 3 to obtain a $(3,12)-\operatorname{HWP}(48 ; r, s)$ for $s=0$. Consider the equipartite graph $K_{(12: 4)}$. It has a $C_{12}$-factorization and a $C_{3}$-factorization by Theorem 4. On each part, construct a $(3,12)-\operatorname{HWP}(12 ; r, s)$ for $s \in\{1,2,3,4,5\}$. Thus we have a $(3,12)-\operatorname{HWP}(48 ; r, s)$ for $s \in\{0,1,2,3,4,5,19,20,21,22,23\}$. The next theorem settles the Hamilton-Waterloo Problem, $(3,3 x)-\operatorname{HWP}(3 x y ; r, s)$ when $x=4$ for the remaining values of $y$.

Theorem 18 For $y=3$ and all $y \geq 5$, there exists a $(3,12)-\operatorname{HWP}(12 y ; r, s)$ if and only if $r+s=\frac{v-2}{2}$.

Proof: Let $y \geq 6$ be even. There exists a $3-\operatorname{RGDD}\left(6^{y / 2}\right)$ by Theorem 9. There exists a decomposition of $K_{(4: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{12}$-factors for $\left(r_{p}, s_{p}\right) \in$ $\{(0,4),(4,0)\}$ by Lemma 4. By the same result, we also get a decomposition of $K_{(4: 6)}$ into $r_{\gamma} C_{3}$-factors and $s_{\gamma} C_{12}$-factors for $\left(r_{\gamma}, s_{\gamma}\right) \in\{(0,10),(10,0)\}$. Recall that there exists a $(3,12)-\operatorname{HWP}\left(12 ; r_{\beta}, s_{\beta}\right)$ for $\left(r_{\beta}, s_{\beta}\right) \in\{(0,5),(1,4),(2,3),(3,2),(4,1)\}$ by Theorem 6. Write $s_{\alpha}=\sum_{p=1}^{\frac{3 y}{2}-4} s_{p}$ so $s_{\alpha} \in\{0,4,8, \ldots, 6 y-16\}$. By Lemma 1.2.7. we obtain a $(3,12)-\operatorname{HWP}(3 x y ; r, s)$ for all $s \in\{0,1, \ldots, 6 y-1\}$ as follows. If $s=0$, apply Theorem 3. If $s \in\{1,2, \ldots, 6 y-11\}$, it is easy to see that we can let $s_{\gamma}=0$ and write $s$ as $s=s_{\alpha}+s_{\beta}$. If $s=6 y-10$, choose $s_{\alpha}=6 y-24, s_{\beta}=4$, and $s_{\gamma}=10$. If $s=6 y-i$ for $i=9,8,7,6$, choose $s_{\alpha}=6 y-20, s_{\beta}=10-i$ and $s_{\gamma}=10$. If $s=6 y-i$ for $i=5,4,3,2,1$, choose $s_{\alpha}=6 y-16, s_{\beta}=6-i$ and $s_{\gamma}=10$.

If $y \geq 3$ is odd, there exists a $3-\operatorname{RGDD}\left(3^{y}\right)$ by Lemma 9 . There exists a decomposition of $K_{(4: 3)}$ into $r_{p} C_{3}$-factors and $s_{p} C_{12}$-factors for $\left(r_{p}, s_{p}\right) \in\{(0,4),(4,0)\}$ by Theorem.4. Write $s_{\alpha}=\sum_{p=1}^{\frac{3(y-1)}{2}} s_{p}$, so $s_{\alpha} \in\{0,4,8, \ldots, 6(y-1)\}$. Recall the existence of a $(3,12)-\operatorname{HWP}\left(12 ; r_{\beta}, s_{\beta}\right)$ for $s_{\beta} \in\{1,2,3,4,5\}$. Then it is easy to see that we can write $s$ as $s_{\alpha}+s_{\beta}$ for all $s \in\{0,1,2, \ldots, 6 y-1\}$. Thus we may apply Lemma 1.2.6 for the result.

### 1.4 Conclusions

The following theorem combines the results from Theorems 10, 11, 12, 13 , 14. 15, 16, 17, and 18 (note that we did not include all of the small partially complete results such as those at the end of Section 1.3):

Theorem 19 Let $x \geq 2, y \geq 2$, and $r, s \geq 0$ such that $r+s=\left\lfloor\frac{3 x y-1}{2}\right\rfloor$. Then there exist a $(3,3 x)-H W P(3 x y ; r, s)$ except possibly when:

- $s=1, y \geq 3$, and $x \in\{3,31,37,41,43,47,51,53,59,61,67,69,71,79,83\}$.
- $s=1, x$ is odd and $y$ is even.
- $s=1, x \geq 6, x \equiv 2(\bmod 12)$.
- $s=1, y \geq 8$ is even and $x \equiv 10(\bmod 12)$.
- $s=1, x \geq 3$ is odd and $y$ is even.
- $1 \leq s \leq \frac{x}{2}-1, x \geq 16, x \equiv 4(\bmod 12), y$ is even.
- $1 \leq s \leq \frac{x}{2}-1, x \geq 10, x \equiv 4(\bmod 6), y$ is odd.
- $(s, x) \in\{(2,12),(4,12)\}$.
- $s=0, x=2, y=2$.
- $x=2$ and $y \in\{4,8\}$.
- $s \in\left\{3,4, \ldots \frac{3(y-1)}{2}\right\}, x=2$ and $y \geq 3$ is odd.
- $x \notin\{2,4\}$ and $y \in\{2,4,6\}$.
- $x=4$ and $y \in\{2,4\}$.
- $x=6$ and $y$ odd.


## Chapter 2

## A Generalization of the

## Hamilton-Waterloo Problem on <br> Complete Equipartite Graphs ${ }^{1}$

### 2.1 Introduction

The Oberwolfach Problem was first posed by Ringel in 1967 during a conference in Oberwolfach. The question was whether it was possible to seat the $v$ conference attendees at $n$ round tables for dinner during $\frac{v-1}{2}$ nights, in such a way that every

[^1]attendee sits next to every other attendee exactly once. This is equivalent to asking whether the complete graph $K_{v}$ can be decomposed into $\frac{v-1}{2}$ copies of a 2-factor $F$ (in a 2-factor every component is a cycle, which represents a round table). To achieve this decomposition $v$ needs to be odd, because the vertices (attendees) need to have even degree. Later a version with $v$ even was studied. In this case, the attendees will never sit next to their spouses (and we are assuming that every attendee has a spouse). This is equivalent to asking for a decomposition of $K_{v}$ into $\frac{v-2}{2}$ copies of a 2-factor $F$, and one copy of a 1 -factor (each attendee together with their spouse).

In [28] Liu first worked on the generalization of the Oberwolfach problem, where instead of avoiding their spouses, the attendees avoid all the other members of their delegation. The assumption was that all the delegations had the same number of people. Thus we are seeking to decompose the complete equipartite graph $K_{(m: n)}$ with $n$ partite sets (delegations) of size $m$ each (members of a delegations) into $\frac{(n-1) m}{2}$ copies of a 2-factor $F$. Here $(n-1) m$ has to be even. In 22] Hoffman and Holliday worked on the equipartite generaliztion of the Oberwolfach problem when $(n-1) m$ is odd, decomposing into $\frac{(n-1) m-1}{2}$ copies of a 2 -factor $F$, and one copy of a 1 -factor.

The Hamilton-Waterloo problem is a generalization of the Oberwolfach problem, in which the conference is being held at two different cities. Because the table arrangements are different, we have two 2-factors, $F_{1}$ and $F_{2}$. The Hamilton-Waterloo problem then asks whether the complete graph $K_{v}$ can be decomposed into $r$ copies
of the 2 -factor $F_{1}$ (tables at Hamilton) and $s$ copies of the 2-factor $F_{2}$ (tables at Waterloo), such that $s+r=\frac{v-1}{2}$, when $v$ is odd, and having $s+r=\frac{v-2}{2}$ and a 1-factor when $v$ is even.

The uniform Oberwolfach problem (when all the tables have the same size, i.e. all the cycles of the 2-factor have the same size) has been completely solved by Alspach and Haagkvist [2] and Alspach, Schellenberg, Stinson and Wagner [3]. For the non-uniform case of the Oberwolfach problem, many results have been obtained. For a survey of results up to 2006 see [12. The uniform Oberwolfach problem over equipartite graphs has been completely solved by Liu [29] and Hoffman and Holliday [22]. In the nonuniform case Bryant, Danziger and Pettersson [11] completely solved the case when the 2 -factor is bipartite. For the Hamilton-Waterloo problem most of the results are uniform, see for example [5] or [13]. In particular, Burgess, Danziger and Traetta [13] proved the following theorem.

Theorem 20 [13] If $m$ and $n$ are odd integers with $n \geq m \geq 3$ and $t>1$, then there is a decomposition of $K_{m n t}$ into $s C_{m}$-factors and $r C_{n}$-factors if and only if $t$ is odd, $s, r \geq 0$ and $s+r=(m n t-1) / 2$, except possibly when $r=1$ or 3 , or $(m, n, r)=(5,9,5),(5,9,7),(7,9,5),(7,9,7),(3,13,5)$.

Theorem 20 covers most of the odd ordered uniform cases. The authors in [13] point out that it is possible to have solutions where the number of vertices is not a multimple
of $m n$. Thus if $l=\operatorname{lcm}(m, n)$ and the number of vertices is a multiple of $l$, but not divisible by $m n$, then Theorem 20 cannot be applied. The constructions given in this paper can be applied to cover some of these cases.

There are some results in the non-uniform case, some examples are Bryant, Danzinger [9], Bryant, Danzinger, Dean [10] and Haggkvist [20].

The Hamilton-Waterloo problem can be generalized for complete equipartite graphs in the same way as the Oberwolfach problem was generalized, but not much work has been done in this direction. Asplund, Kamin, Keranen, Pastine and Özkan [5] gave some constructions for complete equipartite graphs with 3 parts. Burgess, Danziger and Traetta [13] studied the case when the graph consists of $m$ partite sets of size $n$, and the cycle sizes are $m$ and $n$. In both papers the constructions were done in order to get a result on the Hamilton-Waterloo problem for complete graphs. The focus of this paper is to give a generalization of the Hamilton-Waterloo problem for complete equipartite graphs with an odd number of partite sets. We obtain results both in the uniform and non-uniform cases.


Figure 2.1: A multipartite graph


Figure 2.2: A multipartite graph labeled according to its partite sets

### 2.2 Basic Definitions and Results

Let $G$ be a multipartite graph with $k$ partite sets, $G_{0}, G_{1}, \ldots, G_{k-1}$. We identify each vertex $g$ of $G$ as an ordered pair $(g, i)$, where $g \in G_{i}$.

Example 9: Consider the graph in Figure 2.1. If we consider each column as a partite set, we have 3 partite sets; $G_{0}$, with vertices $a, d$ and $f ; G_{1}$ with vertices $b$ and $e$; and $G_{2}$ with $c$ as its only vertex. Then the vertices are $(a, 0),(b, 1)$, $(c, 2),(d, 0),(e, 1),(f, 0)$.

When it is convenient, we will just denote the vertices $(0,0),(0,1),(0,2),(1,0)$, $(1,1),(2,0)$, as in Figure 2.2 .

Definition 2.2.1 Let $G$ and $H$ be multipartite graphs. Then we define the partite product of $G$ and $H, G \otimes H$ as follows:

- $V(G \otimes H)=\{(g, h, i) \mid(g, i) \in V(G)$ and $(h, i) \in V(H)\}$.
- $E(G \otimes H)=\left\{\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \mid\left\{\left(g_{1}, i\right),\left(g_{2}, j\right)\right\} \quad \in \quad E(G)\right.$ and $\left.\left\{\left(h_{1}, i\right),\left(h_{2}, j\right)\right\} \in E(H)\right\}$.

Notice that this definition is quite similar to that of the direct product. The main difference is that we are doing this product "just in 1 coordinate". To see that they are different it suffices to count the number of vertices in the product. If $G=H=K_{3,3,3}$ then $|V(G \times H)|=81$ but $|V(G \otimes H)|=27$.

Indeed, if the $k$ partite sets of $G$ and $H$ have sizes $g_{0}, g_{1}, \ldots, g_{k-1}$ and $h_{0}, h_{1}, \ldots, h_{k-1}$, respectively, then $|V(G \otimes H)|=g_{0} h_{0}+g_{1} h_{1}+\cdots+g_{k-1} h_{k-1}$, whereas $|V(G \times H)|=$ $\left(\sum g_{i}\right)\left(\sum h_{i}\right)$.

Remark 2.2.2 The partite product depends on the multipartite representation chosen for a graph. For example, the graphs $G$ and $H$ in Figure 2.3 are isomorphic, but they behave differently in the product (where we understand that each column is a part of the multipartite graph).

The next result follows directly from Definition 2.2.1


Figure 2.3: Example of the partite product

Lemma 2.2.3 The product is commutative, that is, $G \otimes H=H \otimes G$.

Most of our results will be concerning complete multipartite graphs. We will denote by $K_{(n: m)}$ the complete multipartite graph with $m$ parts, each of size $n$.

Lemma 2.2.4 If $G$ is $k$-partite, then $G \otimes K_{(1: k)}$ is isomorphic to $G$.

Proof: Here each part of $K_{(1: k)}$ has just 1 vertex and so $|V(G)|=\left|V\left(G \otimes K_{(1: k)}\right)\right|$. Because all the vertices of $K_{(1: k)}$ are neighbors, two vertices $\left(g_{1}, k_{1}, i\right),\left(g_{2}, k_{2}, j\right)$ of $G \otimes K_{(1: k)}$ are neighbors if and only if $\left(g_{1}, i\right)$ and $\left(g_{2}, j\right)$ are neighbors. Therefore $G \otimes K_{(1: k)}$ is isomorphic to $G$.

Definition 2.2.5 The complete cyclic multipartite graph $C_{(x: k)}$ is the graph with $k$ parts of size $x$, where two vertices $(g, i)$ and $(h, j)$ are neighbors if and only if $i-j=$ $\pm 1(\bmod k)$, with this subtraction being done modulo $k$. The directed complete cyclic multipartite graph $\vec{C}_{(x: k)}$ is the graph with $k$ parts of size $x$, with arcs of the form $((g, i),(h, i+1))$ for every $0 \leq g, h \leq x-1,0 \leq i \leq k-1$.

It should be noted that any decomposition of $\vec{C}_{(x: k)}$ gives a decomposition of $C_{(x: k)}$. Notice that $C_{(1: k)}$ is the cycle with $k$ vertices and $C_{(x: 3)}$ is isomorphic to $K_{(x: 3)}$. The next three results are easy to see, so the proofs are left to the reader.

Lemma 2.2.6 Let $G$ and $H$ be $k$-partite graphs. If each part of $G$ has $\frac{|V(G)|}{k}$ vertices and each part of $H$ has $\frac{|V(H)|}{k}$ vertices, then:

- Each part of $G \otimes H$ has $\frac{|V(G)| \times|V(H)|}{k^{2}}$ vertices.
- $|V(G \otimes H)|=\frac{|V(G)| \times|V(H)|}{k}$.

Lemma 2.2.7 • $K_{(x: k)} \otimes K_{(y: k)}$ is isomorphic to $K_{(x y: k)}$.

- $C_{(x: k)} \otimes C_{(y: k)}$ is isomorphic to $C_{(x y: k)}$.
- $\vec{C}_{(x: k)} \otimes \vec{C}_{(y: k)}$ is isomorphic to $\vec{C}_{(x y: k)}$.

Lemma 2.2.8 The complete cyclic multipartite graph is the product of the complete multipartite graph by the cycle. This is: $K_{(x: k)} \otimes C_{(1: k)}=C_{(x: k)}$.

### 2.3 Product and Decompositions

We can consider a decomposition of a graph as a partition of the edge set or as a union of edge disjoint subgraphs. This means that a decomposition of $G$ into $H_{1}, \ldots, H_{s}$ is given by $E(G)=\cup E\left(H_{i}\right)$ or by $G=\oplus H_{i}$. We will think of $\oplus$ as a boolean sum, which means that $H_{i} \oplus H_{i}=\emptyset$.

We have the following easy result.

Theorem 21 (Distribution) Let $G=\oplus_{i} G_{i}$ and $H=\oplus_{j} H_{j}$ be $k$-partite graphs. Then $G \otimes H=\left(\oplus_{i} G_{i}\right) \otimes\left(\oplus_{j} H_{j}\right)$. Furthermore, the following distributive property holds:

$$
\left(\oplus_{i} G_{i}\right) \otimes\left(\oplus_{j} H_{j}\right)=\oplus_{i}\left(G_{i} \otimes \oplus_{j} H_{j}\right)=\oplus_{i} \oplus_{j}\left(G_{i} \otimes H_{j}\right)
$$

Proof: It is enough to prove that

$$
G \otimes\left(H_{1} \oplus H_{2}\right)=\left(G \otimes H_{1}\right) \oplus\left(G \otimes H_{2}\right)
$$

where $E\left(H_{1}\right) \cap E\left(H_{2}\right)=\emptyset$.

Let

$$
\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \in E\left(G \otimes\left(H_{1} \oplus H_{2}\right)\right)
$$

This means that

$$
\left\{\left(h_{1}, i\right),\left(h_{2}, j\right)\right\} \in E\left(H_{1}\right) \cup E\left(H_{2}\right) .
$$

But since $E\left(H_{1}\right) \cap E\left(H_{2}\right)=\emptyset$, without loss of generality we may assume

$$
\left\{\left(h_{1}, i\right),\left(h_{2}, j\right)\right\} \in E\left(H_{1}\right),
$$

and so

$$
\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \in E\left(G \otimes H_{1}\right) \subset E\left(\left(G \otimes H_{1}\right) \oplus\left(G \otimes H_{2}\right)\right) .
$$

Hence

$$
E\left(G \otimes\left(H_{1} \oplus H_{2}\right)\right) \subset E\left(\left(G \otimes H_{1}\right) \oplus\left(G \otimes H_{2}\right)\right)
$$

Let now

$$
\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \in E\left(\left(G \otimes H_{1}\right) \oplus\left(G \otimes H_{2}\right)\right) .
$$

This means that

$$
\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \in E\left(\left(G \otimes H_{1}\right)\right) \cup E\left(\left(G \otimes H_{2}\right)\right)
$$

Without loss of generality we may assume

$$
\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \in E\left(\left(G \otimes H_{1}\right)\right),
$$

which gives

$$
\begin{gathered}
\left\{\left(h_{1}, i\right),\left(h_{2}, j\right)\right\} \in E\left(H_{1}\right) \subset E\left(H_{1} \oplus H_{2}\right), \text { and } \\
\left\{\left(g_{1}, h_{1}, i\right),\left(g_{2}, h_{2}, j\right)\right\} \in E\left(G \otimes\left(H_{1} \oplus H_{2}\right)\right) .
\end{gathered}
$$

Hence

$$
E\left(\left(G \otimes H_{1}\right) \oplus\left(G \otimes H_{2}\right)\right) \subset E\left(G \otimes\left(H_{1} \oplus H_{2}\right)\right)
$$

Therefore

$$
G \otimes\left(H_{1} \oplus H_{2}\right)=\left(G \otimes H_{1}\right) \oplus\left(G \otimes H_{2}\right),
$$

and by induction we get that the product and additions in

$$
G \otimes H=\left(\oplus_{i} G_{i}\right) \otimes\left(\oplus_{j} H_{j}\right)
$$

are distributive.

Corollary 2.3.1 Let $G$ and $H$ be multipartite graphs with $k$ partite sets.

- If $G$ can be decomposed into isomorphic copies of $\Gamma$ and $H$ can be decomposed into isomorphic copies of $K_{(1: k)}=K_{k}$, then $G \otimes H$ can be decomposed into isomorphic copies of $\Gamma$.
- If $G$ can be factored into isomorphic copies of $\Gamma$ and $H$ can be factored into unions of copies of $K_{(1: k)}=K_{k}$, then $G \otimes H$ can be factored into unions of copies of $\Gamma$.

Proof: If $G$ is decomposed into copies of $\Gamma$, it means that $G=\oplus G_{i}$, where each $G_{i}$ is isomorphic to $\Gamma$. If $H$ is decomposed into copies of $K_{(1: k)}$ (or union of them), it means that $H=\oplus H_{i}$, where each $H_{i}$ is isomorphic to $K_{(1: k)}$ (or union of them).

By the Distribution Theorem we only need to show that $G_{i} \otimes H_{i}$ is isomorphic to $\Gamma$ or to $G_{i}$. But by Lemma 2.2.4 we know this is true.

It is interesting to notice that the set of $k$-partite graphs, with $\oplus$ as a sum and $\otimes$ as
a product form a commutative ring. The empty graph is the 0 element, and $K_{(1: k)}$ is the 1 element. All of the elements are additive involutions, Theorem 21 gives us the distribution laws, and Lemma 2.2 .3 shows that the product is commutative.

### 2.4 Product of Cycles

In this section we will concern ourselves with the product of two or more cycles. Since our product depends on what kind of partition we are using, we need to ask something more from our cycles in order to get results.

Definition 2.4.1 Given a graph $G$ we will say that $C$ is a $C_{n}$-factor of $G$ if $C$ is a 2-factor of $G$ where each connected component is of size $n$. This means that $C$ is a spanning subgraph of $G$ and $C$ is a union of disjoint cycles of size $n$. When it is understood that the graph is $G$, then we will just call $C$ a $C_{n}$-factor (instead of a $C_{n}$ factor of $G$ ). Similarly given a directed graph $\vec{G}$ we will say that $\vec{C}$ is a $\vec{C}_{n}$-factor of $G$ if $\vec{C}$ is a 2-factor of $\vec{G}$ where each connected component is a directed cycle of size n. When it is understood that the graph is $\vec{G}$, the we will just call $\vec{C}$ a $\vec{C}_{n}$-factor.

The following lemmas give us an idea of how directed cycles work under the product. They also illustrate why we introduce $\vec{C}_{(x: k)}$ instead of just working with $C_{(x: k)}$.

Lemma 2.4.2 Let $\vec{C}$ be a directed cycle of length $n$ of $\vec{C}_{(x: k)}$, and let $\vec{C}^{\prime}$ be a directed cycle of length $m$ of $\vec{C}_{(y: k)}$. Then $\vec{C} \otimes \vec{C}^{\prime}$ is a set of $\frac{\operatorname{gcd}(n, m)}{k}$ disjoint directed cycles of $\vec{C}_{(x y: k)}$ of length $l=\frac{n m}{\operatorname{gcd}(n, m)}$ and $x y k-\frac{n m}{k}$ isolated vertices.

Proof: Notice that $\vec{C}$ has $x k-n$ isolated vertices, because it is a subgraph of $\vec{C}_{(x: k)}$. Likewise, $\vec{C}^{\prime}$ has $y k-m$ isolated vertices.

Let $\left(x_{0}, y_{0}, i\right)$ be a vertex in $\vec{C} \otimes \vec{C}^{\prime}$. If either $\left(x_{0}, i\right)$ or $\left(y_{0}, i\right)$ are isolated vertices, then $\left(x_{0}, y_{0}, i\right)$ is isolated. If neither $\left(x_{0}, i\right)$ and $\left(y_{0}, i\right)$ are isolated, then they respectively have an arrow coming from $\left(x_{1}, i-1\right)$ and $\left(y_{1}, i-1\right)$; and an arrow going to $\left(x_{2}, i+1\right)$ and ( $y_{2}, i+1$ ), for some $0 \leq x_{1}, x_{2} \leq x-1,0 \leq y_{1}, y_{2} \leq y-1$. Hence $\left(x_{0}, y_{0}, i\right)$ has an arrow coming from $\left(x_{1}, y_{1}, i-1\right)$ and an arrow going to $\left(x_{2}, y_{2}, i+1\right)$. So $\vec{C} \otimes \vec{C}^{\prime}$ is composed of directed cycles and isolated vertices.

Assume without loss of generality $n \leq m$. Let $i_{0}, i_{1}, \ldots, i_{n-1}$ be the non-isolated vertices of $\vec{C}$, in the order they appear, with $i_{0} \in \vec{C}_{0}$; and let $j_{0}, j_{1}, \ldots, j_{m-1}$ be the non-isolated vertices of $\vec{C}^{\prime}$, in the order they appear with $j_{0} \in \vec{C}_{0}^{\prime}$. Then the directed cycle starting at $\left(i_{0}, j_{0}\right)$ in $\vec{C} \otimes \vec{C}^{\prime}$ consists of the vertices:

$$
\left(i_{0}, j_{0}\right),\left(i_{1}, j_{1}\right), \ldots,\left(i_{n-1}, j_{n-1}\right),\left(i_{0}, j_{n}\right), \ldots\left(i_{n-1}, i_{m-1}\right)
$$

This directed cycle has length $l=\frac{n m}{g c d(n, m)}$. Notice that $\vec{C} \otimes \vec{C} \vec{C}^{\prime}$ has $\frac{n m}{k}$ non-isolated vertices, which means that the number of directed cycles is $\frac{n m}{k} \frac{\operatorname{gcd}(n, m)}{n m}=\frac{\operatorname{gcd}(n, m)}{k}$. Thus
$\vec{C} \otimes \vec{C}^{\prime}$ is a set of $\frac{\operatorname{gcd}(n, m)}{k}$ disjoint directed cycles of $\vec{C}_{(x y: k)}$ of length $l=\frac{n m}{\operatorname{gcd}(n, m)}$ and $x y k-\frac{n m}{k}$ isolated vertices.

Lemma 2.4.3 Let $\vec{G}$ and $\vec{H}$ be a $\vec{C}_{n}$-factor and a $\vec{C}_{m}$-factor of $\vec{C}_{(x: k)}$ and $\vec{C}_{(y: k)}$, respectively. Then $\vec{G} \otimes \vec{H}$ is a $\vec{C}_{l}$-factor of $\vec{C}_{(x y: k)}$, where $l=\frac{n m}{g c d(n, m)}$.

Proof: Notice that neither $\vec{G}$ nor $\vec{H}$ have isolated vertices. Let $\left(x_{0}, y_{0}, i\right)$ be a vertex in $\vec{G} \otimes \vec{H}$. We know that $\left(x_{0}, i\right)$ has an arrow coming from $\left(x_{1}, i-1\right)$ and an arrow going to $\left(x_{2}, i+1\right)$, for exactly one pair $0 \leq x_{1}, x_{2} \leq x-1$, because $\vec{G}$ is a $\vec{C}_{n^{-}}$ factor. Likewise $\left(y_{0}, i\right)$ has has an arrow coming from $\left(y_{1}, i-1\right)$ and an arrow going to $\left(y_{2}, i+1\right)$, for exactly one pair $0 \leq y_{1}, y_{2} \leq y-1$. Hence $\left(x_{0}, y_{0}, i\right)$ has an arrow coming from $\left(x_{1}, y_{1}, i-1\right)$ and an arrow going to $\left(x_{2}, y_{2}, i+1\right)$, so each vertex in $\vec{G} \otimes \vec{H}$ is in exactly one directed cycle.

Let $\vec{G}=\oplus_{i} \vec{C}(i)$ and $\vec{H}=\oplus_{j} \vec{C}^{\prime}(j)$, where each $\vec{C}(i)$ is a directed cycle of length $n$, and each $\vec{C}^{\prime}(j)$ is a directed cycle of length $m$.

Then by Theorem 21 we get:

$$
\begin{aligned}
\vec{G} \otimes \vec{H} & =\left(\oplus_{i} \vec{C}(i)\right) \otimes\left(\oplus_{j} \vec{C}^{\prime}(j)\right) \\
& =\oplus_{i} \oplus_{j}\left(\vec{C}(i) \otimes \vec{C}^{\prime}(j)\right)
\end{aligned}
$$

But by Lemma 2.4.2 we know that $\vec{C}(i) \otimes \vec{C}^{\prime}(j)$ is composed of $\frac{\operatorname{gcd}(n, m)}{k}$ directed cycles of length $l=\frac{n m}{g c d(n, m)}$. Hence each directed cycle in $\vec{G} \otimes \vec{H}$ has size $l=\frac{n m}{g c d(n, m)}$ and $\vec{G} \otimes \vec{H}$ is a $\vec{C}_{l}$-factor.

Definition 2.4.4 Given a graph $G$ we will say that $F$ is a $\left[n_{1}^{e_{1}}, n_{2}^{e_{2}}, \ldots, n_{p}^{e_{p}}\right]$-factor of $G$ if $F$ is a 2-factor of $G$ with $e_{i}$ connected components of size $n_{i}, i=1,2, \ldots, p$.

If $e_{i}$ is not listed, we will just assume that it is 1 . Also, we allow $n_{i}=n_{j}$, so that the number of cycles of a certain size is just the sum of the exponents of that number in the expression $\left[n_{1}^{e_{1}}, \ldots, n_{p}^{e_{p}}\right]$.

Example 10: A $\left[3^{2}, 3^{3}, 5^{2}, 11,13\right]$-factor is a subgraph of a graph on 59 vertices, consisting of 5 cycles of size 3,2 cycles of size 5,1 cycle of size 11 and 1 cycle of size 13. This subgraph can also be written as a $[3,3,3,3,3,5,5,11,13]$-factor or as a $\left[3^{5}, 5^{2}, 11,13\right]$-factor.

Notice that a $C_{n}$-factor of a graph on $m$ vertices is a $\left[n^{\frac{m}{n}}\right]$-factor.

### 2.5 From $C_{(v: n)}$ to $K_{(v: m)}$

It is advantageous to find solutions to the the Hamilton-Waterloo problem on complete multipartite graphs because they can then be used to obtain solutions to the Hamilton-Waterloo problem on complete graphs:

Lemma 2.5.1 Let $m, n, x, y$ and $v$ be positive integers. Suppose the following conditions are satisfied:

- There exists a decomposition of $K_{v}$ into $s_{\alpha} C_{x n}$-factors and $r_{\alpha} C_{y n}$-factors.
- There exists a decomposition of $K_{(v: m)}$ into $s_{\beta} C_{x n}$-factors and $r_{\beta} C_{y n}$-factors.

Then there exists a decomposition of $K_{v m}$ into $s=s_{\alpha}+s_{\beta} C_{x n}$-factors and $r=r_{\alpha}+r_{\beta}$ $C_{y n}$-factors.

Proof: Partition the vertices of $K_{v m}$ into $m$ sets $A_{1}, \ldots, A_{m}$ of size $v$ each. The graph that contains the edges between vertices belonging to a same partite set is the union of $m$ disjoint copies of $K_{v}$. We can decompose each copy of $K_{v}$ into $s_{\alpha} C_{x n}$-factors and $r_{\alpha} C_{y n}$-factors. The graph that contains the edges between vertices belonging to different parts is isomorphic to $K_{(v: m)}$. We can decompose this graph into $s_{\beta}$ $C_{x n}$-factors and $r_{\beta} C_{y n}$-factors.

Thus we have a decomposition of $K_{v m}$ into $s=s_{\alpha}+s_{\beta} C_{x n}$-factors and $r=r_{\alpha}+r_{\beta}$ $C_{y n}$-factors.

One could write a version of the lemma for non-uniform solutions as follows, where by $m K_{v}$ we understand the graph consisting of $m$ disconnected copies of $K_{v}$.

Lemma 2.5.2 Let $m$, and $v$ be positive integers. Let $F_{1}$ and $F_{2}$ be two 2-factors on $v m$ vertices. Suppose the following conditions are satisfied:

- There exists a decomposition of $m K_{v}$ into $s_{\alpha}$ copies of $F_{1}$ and $r_{\alpha}$ copies of $F_{2}$.
- There exists a decomposition of $K_{(v: m)}$ into $s_{\beta}$ copies of $F_{1}$ and $r_{\beta}$ copies of $F_{2}$.

Then there exists a decomposition of $K_{v m}$ into $s=s_{\alpha}+s_{\beta}$ copies of $F_{1}$ and $r=r_{\alpha}+r_{\beta}$ copies of $F_{2}$.

In order to use these lemmas we need two types of ingredients. The first one is the decomposition of the complete graph $K_{v}$. In [2, 3] uniform decompositions were given:

Theorem 22 [2, (3] There exists a decomposition of $K_{v}$ into $C_{n}$-factors if and only if $v \equiv 0(\bmod n),(v, n) \neq(6,3)$ and $(v, n) \neq(12,3)$.

The other ingredient is the decomposition of the complete multipartite graph $K_{(v: m)}$. In [13] the authors used decompositions of $C_{(v: m)}$ to obtain decompositions of $K_{(v: m)}$. We describe this type of construction in a formal fashion in the following lemma.

Lemma 2.5.3 Let $m, x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{p}$, and $v$ be positive integers. Let $s_{1}, \ldots, s_{\frac{m-1}{2}}$ be non-negative integers. Suppose the following conditions are satisfied:

- There exists a decomposition of $K_{m}$ into $\left[n_{1}, \ldots, n_{p}\right]$-factors.
- For every $1 \leq i \leq p$, and for every $1 \leq t \leq \frac{m-1}{2}$ there exists a decomposition of $C_{\left(v: n_{i}\right)}$ into $s_{t} C_{x_{i} n_{i}}$-factors and $r_{t} C_{y_{i} n_{i}}$-factors.

Let

$$
s=\sum_{t=1}^{\frac{(m-1)}{2}} s_{t} \quad \text { and } \quad r=\sum_{t=1}^{\frac{(m-1)}{2}} r_{t}
$$

Then there exists a decomposition of $K_{(v: m)}$ into $s\left[\left(x_{1} n_{1}\right)^{\frac{v}{x_{1}}}, \ldots,\left(x_{p} n_{p}\right)^{\frac{v}{x_{p}}}\right]$-factors and $r\left[\left(y_{1} n_{1}\right)^{\frac{v}{y_{1}}}, \ldots,\left(y_{p} n_{p}\right)^{\frac{v}{y_{p}}}\right]$-factors.

Proof: Using the decomposition of $K_{m}$ into $\left[n_{1}, n_{2}, \ldots, n_{p}\right]$-factors, we have:

$$
K_{(1: m)}=K_{m}=\bigoplus_{t=1}^{\frac{m-1}{2}} N_{t}
$$

where each $N_{t}$ is a $\left[n_{1}, n_{2}, \ldots, n_{p}\right]$-factor.

We know that $K_{(v: m)}=K_{(v: m)} \otimes K_{(1: m)}$. This gives:

$$
\begin{aligned}
K_{(v: m)} & =K_{(v: m)} \otimes K_{(1: m)} \\
& =K_{(v: m)} \otimes\left(\bigoplus_{t=1}^{\frac{m-1}{2}} N_{t}\right) \\
& =\bigoplus_{t=1}^{\frac{m-1}{2}}\left(K_{(v: m)} \otimes N_{t}\right)
\end{aligned}
$$

Notice that $K_{(v: m)} \otimes N_{t}$ is a spanning subgraph of $K_{(v: m)}$. Since $N_{t}$ is a $\left[n_{1}, n_{2}, \ldots, n_{p}\right]$ factor, we have $N_{t}=\bigoplus_{i=1}^{p} C\left(w_{i}\right)$, where $C\left(w_{i}\right)$ is a cycle of size $n_{i}$. Therefore,

$$
\begin{aligned}
K_{(v: m)} \otimes N_{t} & =K_{(v: m)} \otimes\left(\bigoplus_{i=1}^{p} C\left(w_{i}\right)\right) \\
& =\bigoplus_{i=1}^{p}\left(K_{(v: m)} \otimes C\left(w_{i}\right)\right)
\end{aligned}
$$

But $K(v: m) \otimes C\left(w_{i}\right)$ is isomorphic to $C_{\left(v: n_{i}\right)}$ because $C\left(w_{i}\right)$ is isomorphic to $C_{\left(1: n_{i}\right)}$. So for each $w_{i}$ we can decompose $K(v: m) \otimes C\left(w_{i}\right)$ into $s_{t} C_{x_{i} n_{i}}$-factors and $r_{t} C_{y_{i} n_{i}}$ factors. Since $C_{\left(v: n_{i}\right)}$ has $v n_{i}$ vertices, a $C_{x_{i} n_{i}}$-factor has $\frac{v}{x_{i}}$ cycles, hence it is a $\left[\left(x_{i} n_{i}\right)^{\frac{v}{x_{i}}}\right]$-factor. Likewise a $C_{y_{i} n_{i}}$-factor is a $\left[\left(y_{i} n_{i}\right)^{\frac{v}{y_{i}}}\right]$-factor.

Taking the union of one $C_{x_{i} n_{i}}$-factor for each $i$ gives a $\left[\left(x_{1} n_{1}\right)^{\frac{v}{x_{1}}}, \ldots,\left(x_{p} n_{p}\right)^{\frac{v}{x_{p}}}\right]$-factor. Thus we get a decomposition of $K(v: m) \otimes N_{t}$ into $s_{t}\left[\left(x_{1} n_{1}\right)^{\frac{v}{x_{1}}}, \ldots,\left(x_{p} n_{p}\right)^{\frac{v}{x_{p}}}\right]$-factors and $r_{t}\left[\left(y_{1} n_{1}\right)^{\frac{v}{y_{1}}}, \ldots,\left(y_{p} n_{p}\right)^{\frac{v}{y_{p}}}\right]$-factors.

Doing this for every $1 \leq t \leq \frac{m-1}{2}$, we end up with a decomposition of $K_{(v: m)}$ into $s$ $\left[\left(x_{1} n_{1}\right)^{\frac{v}{x_{1}}}, \ldots,\left(x_{p} n_{p}\right)^{\frac{v}{x_{p}}}\right]$-factors and $r\left[\left(y_{1} n_{1}\right)^{\frac{v}{y_{1}}}, \ldots,\left(y_{p} n_{p}\right)^{\frac{v}{y_{p}}}\right]$-factors.

It is important to note that Theorem 22 can be used to obtain decompositions of $K_{m}$ into $C_{n}$-factors. Thus the focus of the next three sections is to find decompositions of $C_{(v: n)}$. Any decomposition of $\vec{C}_{(v: n)}$ is equivalent to a decomposition of $C_{(v: n)}$, by simply removing the direction of each edge. We will work with the partite product on directed graphs to find decompositions of $\vec{C}_{(v: n)}$, and thus obtain decompositions of $C_{(v: n)}$.

### 2.6 Hamilton-Waterloo Problem on Directed Complete Cyclic Multipartite Graphs

For the entirety of this section, we assume $x$ is odd. In this section we will decompose $\vec{C}_{(x: n)}$ into $\vec{C}_{n^{-}}$factors and $\vec{C}_{x n}$-factors (Hamilton Cycles), and $\vec{C}_{(4 x: n)}$ into $\vec{C}_{n^{-}}$ factors and $\vec{C}_{2 x n}$-factors.

Suppose $G_{\alpha}$ and $G_{\beta}$ are two parts of size $x$ in an equipartite directed graph $G$. We say an arc in $G$ has difference $d$ if $\left(\left(g_{1}, \alpha\right),\left(g_{2}, \beta\right)\right) \in E(G)$ and $g_{2}-g_{1} \equiv d(\bmod x)$. If $\left\{\left(\left(g_{1}, \alpha\right),\left(g_{2}, \beta\right)\right): g_{2}-g_{1} \equiv d(\bmod x)\right\} \subset E(G)$, then we say that difference $d$ between parts $\alpha$ and $\beta$ is covered by $G$.

Let the partite sets of $\vec{C}_{(x: n)}$ be $G_{0}, G_{1}, \ldots, G_{n-1}$. Write $n-1=2^{e_{1}}+2^{e_{2}}+\ldots+2^{e_{k}}$ with $e_{\alpha}>e_{\beta}$ if $\alpha<\beta$. Notice that $k \leq \frac{n}{2}$. Working modulo $x$, let $T_{x}(i)$ be the


Figure 2.4: One directed cycle in $T_{5}(1)$, with $n=7$.
directed subgraph of $\vec{C}_{(x: n)}$ obtained by taking differences:

- $2^{e_{j}} i$ between $G_{j-1}$ and $G_{j}$ for $1 \leq j \leq k$.
- $-2 i$ between $G_{j-1}$ and $G_{j}$ for $k+1 \leq j \leq 2 k-1$.
- $-i$ between $G_{j-1}$ and $G_{j}$ for $2 k \leq j \leq n-1$.
- $-i$ between $G_{n-1}$ and $G_{0}$.

Example 11: We construct $T_{5}(1)$ with $n=7$. We have $n-1=6=2^{2}+2^{1}$, and $k=2$.

This means that from the first column to the second one we add $2^{2}$, from the second to the third we add 2 ; since $k=2$, from the third to the forth we subtract 2, and for the rest of the arcs we just subtract 1. Figure 2.4 shows one directed cycle of this construction.

The rest of the arcs are obtained by developing this base directed cycle modulo $x$,
i.e. if $((\alpha, j),(\beta, j+1)) \in E\left(T_{x}(i)\right)$ then $((\alpha+h, j),(\beta+h, j+1)) \in E\left(T_{x}(i)\right)$ for all $h \in \mathbb{Z}_{x}$.

Lemma 2.6.1 $T_{x}(i)$ is a $\vec{C}_{n}$-factor for any $i$.

Proof: It suffices to show that the construction gives a base directed cycle of length $n$. The directed cycle containing the vertex $(0,0)$ can be tracked by considering the first coordinate of each vertex that is visited while passing from $G_{0}$ to $G_{1}$ to $G_{2} \ldots$ to $G_{0}$. If we add the respective differences of the edges between $G_{0}$ and $G_{1}, G_{1}$ and $G_{2}, \ldots$, $G_{n-1}$ and $G_{0}$, we must show that this total sum is 0 . Because $2^{e_{1}}+2^{e_{2}}+\ldots+2^{e_{k}}=n-1$, we have for the sum:

$$
i(n-1)-2 i(k-1)-i(n-2 k+1)=i(n-1)-i(n-1)=0 .
$$

Let $F_{h}(G)$ be the directed subgraph of the directed graph $G$ that contains only the arcs between parts $h-1$ and $h$. That is

$$
E\left(F_{h}(G)\right)=\left\{\left(\left(g_{1}, h-1\right)\left(g_{2}, h\right)\right) \mid\left\{\left(g_{1}, h-1\right)\left(g_{2}, h\right)\right\} \in E(G)\right\}
$$

In particular $F_{n}(G)$ contains the arcs between $G_{n-1}$ and $G_{0}$.


Figure 2.5: $T_{3}(0), T_{3}(1), H_{3}(0,1), H_{3}(1,0)$.

Let $H_{x}(i, s)=T_{x}(i) \oplus F_{n}\left(T_{x}(i)\right) \oplus F_{n}\left(T_{x}(s)\right)$. This means that $H_{x}(i, s)$ is the directed subgraph of $\vec{C}_{(x: n)}$ obtained by taking the same arcs as $T_{x}(i)$ between $G_{j}$ and $G_{j+1}$ for $0 \leq j \leq n-2$ and $T_{x}(s)$ between $G_{n-1}$ and $G_{0}$.

Example 12: Figure 2.5 ilustrates $T_{3}(0), T_{3}(1), H_{3}(0,1)$ and $H_{3}(1,0)$ with $n=3$.

Lemma 2.6.2 If $\operatorname{gcd}(x, i-s)=1$ then $H_{x}(i, s)$ is a directed Hamiltonian cycle.

Proof: Because the arcs are given by differences it is clear that each vertex has indegree and out-degree both equal to 1 . We need to show that all of the vertices are connected. We will first show that there is a directed path between any 2 vertices of $G_{0}$. Without loss of generality, we will show that $(0,0)$ is connected to $(\alpha, 0)$ for any $\alpha$. Because the arcs between groups $G_{j}$ and $G_{j+1}$ are the same as the $\operatorname{arcs}$ in $T_{x}(i)$ for $j=0, \ldots, n-2$ it is easy to see that there is a path from $(0,0)$ to $(i, n-1)$. Now the arc leaving from $(i, n-1)$ has its other end as $(i-s, 0)$. So $(0,0)$ is connected
to $(i-s, 0)$. If we continue on this path, every time we arrive back in $G_{0}$, we arrive at the vertex $\left(\alpha^{\prime}(i-s), 0\right)$. Because $\operatorname{gcd}(x, i-s)=1$, the order of $i-s$ in the cyclic group $\mathbb{Z}_{x}$ is $x$. Thus any $\alpha$ modulo $x$ can be written as $\alpha^{\prime}(i-s)$. Hence $(0,0)$ is connected to all the vertices of $G_{0}$.

Because we are defining arcs by differences, every vertex in $G_{1}$ is connected to a vertex in $G_{0}$, every vertex in $G_{2}$ to a vertex in $G_{1}$, and so on. Therefore all the vertices are connected, and the directed cycle is Hamiltonian as we wanted to prove.

Next we show how to decompose $\vec{C}_{(x: n)}$ by using the $H_{x}(i, j)$ graphs. First we will decompose $\vec{C}_{(x: n)}$ into $\vec{C}_{n}$-factors using the $T_{x}(i)$ graphs. Then we will show how to switch some edges in the $T_{x}(i)$ graphs to obtain $H_{x}(i, j)$ graphs. It is important to notice that $H_{x}(i, i)=T_{x}(i)$.

Lemma 2.6.3 Let $x$ be odd, and let $\phi$ be a bijection on $\{0, \ldots, x-1\}$. Then

$$
\vec{C}_{(x: n)}=\bigoplus_{i=0}^{x-1} T_{x}(i)=\bigoplus_{i=0}^{x-1} H_{x}(i, \phi(i))
$$

Proof: To prove the first equality,

$$
\vec{C}_{(x: n)}=\bigoplus_{i=0}^{x-1} T_{x}(i)
$$

we need to show that any difference between consecutive parts is covered by one of the $T_{x}(i)$ graphs. Notice that all the differences in the $T_{x}$ graphs are given by a power of 2 times $i$, or $-2 i$ or $-i$. It is clear that between parts which use difference $-i$, we cover all the differences, with difference $\delta$ being covered in $T_{x}(x-\delta)$. Because $x$ is odd, $\operatorname{gcd}\left(x, 2^{e}\right)=1$, so the order of $2^{e}$ in the cyclic group $C_{x}$ is $x$ for any $1 \leq e \leq x-1$. This means that any $\delta \equiv 2^{e_{j}} i(\bmod x)$ can be written as $2^{e} \delta^{\prime}$. Therefore, the difference $\delta$ between the remaining pairs of consecutive groups is covered in some $T_{x}\left(\delta^{\prime}\right)$. A similar calculation can be used for differences of the form $-2 i$, because $\operatorname{gcd}(x, x-2)=1$. Here we write $\delta$ as $-2 \delta^{\prime}$, and the difference is covered in $T_{x}\left(\delta^{\prime}\right)$.

For the second equality we have

$$
\begin{aligned}
\bigoplus_{i=0}^{x-1} H_{x}(i, \phi(i)) & =\bigoplus_{i=0}^{x-1}\left(T_{x}(i) \oplus F_{n}\left(T_{x}(i)\right) \oplus F_{n}\left(T_{x}(\phi(i))\right)\right) \\
& =\bigoplus_{i=0}^{x-1} T_{x}(i) \bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(i)\right) \bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(\phi(i))\right)
\end{aligned}
$$

and

$$
\bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(\phi(i))\right)=\bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(i)\right)
$$

because $\phi$ is a bijection. So:

$$
\begin{aligned}
\bigoplus_{i=0}^{x-1} T_{x}(i) \bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(i)\right) \bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(\phi(i))\right) & =\bigoplus_{i=0}^{x-1} T_{x}(i) \bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(i)\right) \bigoplus_{i=0}^{x-1} F_{n}\left(T_{x}(i)\right) \\
& =\bigoplus_{i=0}^{x-1} T_{x}(i)
\end{aligned}
$$

Hence:

$$
\bigoplus_{i=0}^{x-1} T_{x}(i)=\bigoplus_{i=0}^{x-1} H_{x}(i, \phi(i))
$$

In some cases we have $H_{x}(i, i)$, notice that this is the same as $T_{x}(i)$. Decomposing $\vec{C}_{(x: n)}$ into $s$ directed Hamilton cycles and $x-s \vec{C}_{n}$-factors is now equivalent to finding a bijection $\phi$ with $\operatorname{gcd}(x, i-\phi(i))=1$ for $s$ elements of $\{0, \ldots, x-1\}$ and $\phi(i)=i$ for the rest. We will use these functions extensively throughout the paper, so we will refer to them as "the phi-functions": Let $2 \leq s \leq x$. Define $\phi_{s}: \mathbb{Z}_{x} \rightarrow \mathbb{Z}_{x}$ as follows:

## The phi-functions

$$
\phi_{s}(i)= \begin{cases}i+1 & \text { for } \quad i \leq(s-3), i \text { even } \\ i-1 & \text { for } \\ i \leq(s-3), i \text { odd } \\ i+1 & \text { for } \\ i-2 & \text { for } \\ i=s-2 \\ i-1 & \text { for } \\ i=s-1, s \text { even } \\ i & \text { for } \quad s \leq i \leq x-1\end{cases}
$$

For example, if $s=7$ and $x=11$, then

$$
\phi_{7}=(01)(23)(456)(7)(8)(9)(10)
$$

Notice that $\phi_{s}$ has $x-s$ fixed points. For any non-fixed point we have $i-\phi(i) \in$ $\{ \pm 1,2\}$, and so $\operatorname{gcd}(x, i-\phi(i))=1$ if $x$ is odd.

Theorem 23 Let $x$ be odd. Let $s \in\{0, \ldots, x\}, s \neq 1$. Then $\vec{C}_{(x: n)}$ can be decomposed into $s \vec{C}_{x n}$-factors and $x-s \vec{C}_{n}$-factors.

Proof: If $s=0$ we just use the identity mapping. Otherwise we use the phi-function


Figure 2.6: Triangle factors for $K_{(4: 3)}$.
$\phi_{s}$. Hence the discussion that precedes this theorem shows that

$$
\vec{C}_{(x: n)}=\bigoplus_{i=0}^{x-1} H_{x}\left(i, \phi_{s}(i)\right)
$$

is a decomposition of $\vec{C}_{(x: n)}$ into $s \vec{C}_{x n}$-factors and $x-s \vec{C}_{n}$-factors.

Next we turn to the case of $x$ even. We begin by considering $\vec{C}_{(4: n)}$. In [5] the graphs in Figure 2.6 were used to decompose $K_{(4: 3)}$ into triangle factors and $C_{6}$-factors. We extend the ideas used in [5] to decompose $\vec{C}_{(4: n)}$ into $\vec{C}_{n}$-factors and $\vec{C}_{2 n}$-factors. We will define directed subgraphs $\gamma_{i, j}$ and build the directed graphs $\Gamma(j)$ as the sum of some of these directed subgraphs. In each of these directed subgraphs, the vertices in the top row will be said to have height 0 , in the second row height 1 , and so on. We begin with the base directed subgraphs from Figure 2.7. In each subgraph, the rows are indexed by their height. The following result is easy to verify by inspection of the graphs $\gamma_{i, j}$.


Figure 2.7: Base directed subgraphs for the decomposition of $\vec{C}_{(4: n)}$ into $\vec{C}_{n}$-factors and $\vec{C}_{2 n}$-factors

Lemma 2.6.4 For any $i, j, h$, the directed path beginning at height $h$ in the first column of $\gamma_{i, j}$ finishes at height $h$ in the last column of $\gamma_{i, j}$.

Let $n=3 b+a$ with $0 \leq a<3, b \geq 2$. For $0 \leq t \leq b-2$, we define $\gamma_{0, j}(t)$ as the directed graph $\gamma_{0, j}$ on the parts $G_{3 t-1}, G_{3 t}, G_{3 t+1}, G_{3 t+2}$, with calculations done in $\mathbb{Z}_{n}$. We define $\gamma_{a, j}(n)$ as the directed graph $\gamma_{a, j}$ on the parts $G_{3 b-4}, G_{3 b-3}, \ldots, G_{3 b+a-1}$. Let $\Gamma(j)=\bigoplus_{t=0}^{b-2} \gamma_{0, j}(t) \oplus \gamma_{a, j}(n)$. Notice that $\gamma_{0, j}(0)$ is on the parts $G_{-1}, G_{0}, G_{1}$ and $G_{2}$. This means that $F_{n}(\Gamma(j))$ is the matching between the first and second columns in $\gamma_{0, j}(0)$.

Example 13: Let $n=7$. We will construct $\Gamma(0)$. Since $n=6+1$, we have $b=2$ and $a=1$. This means that $\Gamma(0)=\gamma_{0,0}(0) \oplus \gamma_{1,0}(7)$, where $\gamma_{0,0}(0)$ is on the parts $G_{-1}=G_{7-1}=G_{6}, G_{0}, G_{1}$ and $G_{2} ;$ and $\gamma_{1,0}(7)$ is on the parts $G_{2}, G_{3}, G_{4}, G_{5}$ and $G_{6}$. So we get the picture in Figure $2.8(\mathrm{a})$, where the $\operatorname{arcs}$ from $\gamma_{0,0}(0)$ are dashed.

Notice that the first and last columns are both $G_{6}$. Because the directed graph $\vec{C}_{(4: n)}$ has $G_{0}$ as the first column, we connect the vertices from $G_{0}$ to the last column instead, obtaining the picture in Figure 2.8(b) where the dashed arcs still belong to $\gamma_{0,0}(0)$.

Using Lemma 2.6.4 we have the following result.

(a) $\Gamma_{0}$ when $n=7$

(b) $\Gamma_{0}$ when $n=7$ after identifying both $G_{6}$ columns

Figure 2.8: A $\vec{C}_{7}$ factor of $\vec{C}_{(4: 7)}$

## Lemma 2.6.5

$$
\vec{C}_{(4: n)}=\Gamma(0) \oplus \Gamma(1) \oplus \Gamma(2) \oplus \Gamma(3)=\bigoplus_{j=0}^{3} \Gamma(j)
$$

is a $\vec{C}_{n}$-factorization of $\vec{C}_{(4: n)}$.

Proof: It is easy to verify from the pictures that for any given $0 \leq i \leq 2$, the directed graphs $\gamma_{i, 0}, \gamma_{i, 1}, \gamma_{i, 2}$ and $\gamma_{i, 3}$ are arc disjoint. By Lemma 2.6.4, in each $\gamma_{i, j}$ the directed paths start and end at the same height. Thus when we connect all the directed paths in each factor $\Gamma(j)$, we obtain four directed cycles of length $n$. Therefore $\bigoplus_{j=0}^{3} \Gamma(j)$
is a $\vec{C}_{n}$-factorization of $\vec{C}_{(4: n)}$.

To construct directed cycles of size $2 n$, we perform switches on the edges between columns. Define $\lambda_{i, j}=\gamma_{0, i}(0) \oplus F_{n}\left(\gamma_{0, i}(0)\right) \oplus F_{n}\left(\gamma_{0, j}(0)\right)$. Keep in mind that $\gamma_{0, i}(0)$ is on the parts $G_{n-1}, G_{0}, G_{1}, G_{2}$, and so $F_{n}\left(\gamma_{0, i}(0)\right)$ only consists of the edges between parts $G_{n-1}$ and $G_{0}$.

Lemma 2.6.6 If the directed path that starts at height $h_{1}$ in part $G_{n-1}$ in $\lambda_{i, j}$ ends at height $h_{2}$ in $G_{2}$, then the directed path that starts at height $h_{2}$ in $G_{n-1}$ ends at height $h_{1}$ in $G_{2}$. Even more, if $i \neq j$ then no directed path starts and ends at the same height.

Proof: We build tables that show for each possible combination of $i$ and $j$, the starting and ending heights of the directed paths $\lambda_{i, j}$. We have one table for each $i$, with the rows indexed by the options for $j$, and the columns indexed by the options for the starting height of each directed path. The entry in the table gives the finishing height.

$i=0$|  | height 0 | height 1 | height 2 | height 3 |
| :--- | :---: | :---: | :---: | :---: |
| $j=0$ | 0 | 1 | 2 | 3 |
| $j=1$ | 3 | 2 | 1 | 0 |
| $j=2$ | 1 | 0 | 3 | 2 |
| $j=3$ | 2 | 3 | 0 | 1 |


$i=1$|  | height 0 | height 1 | height 2 | height 3 |
| :--- | :---: | :---: | :---: | :---: |
| $j=0$ | 3 | 2 | 1 | 0 |
| $j=1$ | 0 | 1 | 2 | 3 |
| $j=2$ | 2 | 3 | 0 | 1 |
| $j=3$ | 1 | 0 | 3 | 2 |


$i=2$|  | height 0 | height 1 | height 2 | height 3 |
| :--- | :---: | :---: | :---: | :---: |
| $j=0$ | 1 | 0 | 3 | 2 |
| $j=1$ | 2 | 3 | 0 | 1 |
| $j=2$ | 0 | 1 | 2 | 3 |
| $j=3$ | 3 | 2 | 1 | 0 |


$i=3$|  | height 0 | height 1 | height 2 | height 3 |
| :---: | :---: | :---: | :---: | :---: |
| $j=0$ | 2 | 3 | 0 | 1 |
| $j=1$ | 1 | 0 | 3 | 2 |
| $j=2$ | 3 | 2 | 1 | 0 |
| $j=3$ | 0 | 1 | 2 | 3 |

Notice that whenever $i=j$ we have $\lambda_{i, i}=\gamma_{0, i}(0)$, in which case we already know that the starting and ending heights of each directed path are the same. When $i \neq j$ the starting and ending heights are never the same, but if the starting height in $\lambda_{i, j}$ is $h_{1}$ and the ending height is $h_{2}$, then the directed path with starting height $h_{2}$ has
ending height $h_{1}$. Therefore the result is proven.

Let $\Lambda(i, j)=\Gamma(i) \oplus F_{n}(\Gamma(i)) \oplus F_{n}(\Gamma(j))$.

Lemma 2.6.7 If $i \neq j, \Lambda(i, j)$ is a $\vec{C}_{2 n}$-factor.

Proof: Notice that

$$
\begin{aligned}
\Lambda(i, j) & =\Gamma(i) \oplus F_{n}(\Gamma(i)) \oplus F_{n}(\Gamma(j)) \\
& =\left(\bigoplus_{t=0}^{b-2} \gamma_{0, i}(t)\right) \oplus \gamma_{a, i}(n) \oplus F_{n}(\Gamma(i)) \oplus F_{n}(\Gamma(j))
\end{aligned}
$$

Because $F_{n}(\Gamma(i))$ is the matching between the first and second columns in $\gamma_{0, i}(0)$, we have $F_{n}(\Gamma(i))=F_{n}\left(\gamma_{0, i}(0)\right)$. Therefore,

$$
\begin{aligned}
\Lambda(i, j) & =\left(\bigoplus_{t=0}^{b-2} \gamma_{0, i}(t)\right) \oplus \gamma_{a, i}(n) \oplus F_{n}(\Gamma(i)) \oplus F_{n}(\Gamma(j)) \\
& =\left(\bigoplus_{t=0}^{b-2} \gamma_{0, i}(t)\right) \oplus \gamma_{a, i}(n) \oplus F_{n}\left(\gamma_{0, i}(0)\right) \oplus F_{n}\left(\gamma_{0, j}(0)\right) \\
& =\left(\bigoplus_{t=1}^{b-2} \gamma_{0, i}(t)\right) \oplus \gamma_{a, i}(n) \oplus\left(\gamma_{0, i}(0) \oplus F_{n}\left(\gamma_{0, i}(0)\right) \oplus F_{n}\left(\gamma_{0, j}(0)\right)\right) \\
& =\left(\bigoplus_{t=1}^{b-2} \gamma_{0, i}(t)\right) \oplus \gamma_{a, i}(n) \oplus \lambda_{i, j}
\end{aligned}
$$

Consider the directed cycle that contains the vertex at height $h_{1}$ in $G_{n-1}$. From Lemma 2.6.6 we know that in $\lambda_{i, j}$ the directed path that starts at height $h_{1}$ in $G_{n-1}$ finishes at height $h_{2}$ in $G_{2}$. By Lemma 2.6.4. the directed paths through all the $\gamma_{i, j}(l)$, with $l \in\{1,2, \ldots, b-1, n\}$ start and end at the same heights. So when we reach $G_{n-1}$ again, it is at at height $h_{2}$. We leave $G_{2}$ at height $h_{1}$ this time, and as we move through all the $\gamma_{i, j}(l)$, with $l \in\{1,2, \ldots, b-1, n\}$, the heights never change. Therefore, we reach $G_{n-1}$ again at height $h_{1}$, closing the directed cycle. This produces one directed cycle of size $2 n$. By repeating the process with the directed cycle starting at one of the vertices that we have not used yet, we get the second directed cycle. Therefore $\Lambda(i, j)$ consists of two directed cycles of length $2 n$.

Notice that if $i=j$, then $\Lambda(i, j)=\Gamma(i)$ consists of 4 directed cycles of length $n$.

Theorem 24 If $s \in\{0,2,3,4\}$, then $\vec{C}_{(4: n)}$ can be decomposed into $s \vec{C}_{2 n}$-factors and $4-s \vec{C}_{n}$-factors.

Proof: Let $\pi$ be a permutation of the set $\{0,1,2,3\}$ with exactly $4-s$ fixed points. Then

$$
\vec{C}_{(4: n)}=\bigoplus_{j=0}^{3} \Gamma(j)=\bigoplus_{j=0}^{3}\left(\Gamma(j) \oplus F_{n}(\Gamma(j)) \oplus F_{n}(\Gamma(\pi(j)))=\bigoplus_{j=0}^{3} \Lambda(j, \pi(j))\right.
$$

Since $\Lambda(j, \pi(j))$ is a $\vec{C}_{2 n}$-factor if $j \neq \pi(j)$ and a $\vec{C}_{n}$-factor otherwise, the theorem
is proven.

Remark 2.6.8 Notice that if $n=5$, we have $b=1$ and $a=2$. This means we have $\Gamma(j)=\gamma_{a, j}(5)=\gamma_{2, j}(5)$, which is on the parts $G_{-1}=G_{4}, G_{0}, G_{1}, G_{2}, G_{3}, G_{4}$. This will actually close the directed cycle. The results given in Lemmas 2.6.6 and 2.6.7 only apply to $b \geq 2$, but it can be shown that the same results are true with $b=1$ by applying similar techniques on $\gamma_{a, j}$ instead of $\gamma_{0, j}$.

There is one more basic decomposition that we will use, based on the resolvable gregarious decomposition of $K_{(w: n)}$ from [6]. We make use of the constructions given in Lemma 3.1 and Corollary 3.2 of [6], and apply them to $\vec{C}_{(w: n)}$ instead of $K_{(w: n)}$.

Definition 2.6.9 A quasigroup $(Q, *)$ is a set $Q$ with a binary operation $*$ such that for each $a$ and $b$ in $Q$, there exist unique elements $x$ and $y$ in $Q$ such that:

- $a * x=b$;
- $y * a=b$.

Definition 2.6.10 Two quasigroups on the same set $(Q, *),(Q, \circ)$ are said to be orthogonal if $i * j \neq i \circ j$ for every $i, j$ in $Q$.

The reader may be familiar with Latin Squares, which are the multiplication tables of quasigroups, and mutually orthogonal Latin Squares, which are the multiplication tables of orthogonal quasigroups. In [7], [8] it was shown that if $|Q| \notin\{1,2,6\}$ then there are at least 2 orthogonal quasigroups on $Q$. Again, the decomposition in the following theorem is obtained by modifying the construction from Lemma 3.2 in [6] to work with $\vec{C}_{(w: n)}$ instead of $K_{(w: n)}$ :

Theorem 25 Let $w \notin\{2,6\}$ and $n$ odd. Then there is a decomposition of $\vec{C}_{(w: n)}$ into $\vec{C}_{n}$-factors.

Proof: Since $w \notin\{2,6\}$ there exist two orthogonal quasigroups (Latin Squares) $(Q, \circ)$ and $(Q, *)$ of order $w$, with $Q=\{0,1,2, \ldots, w-1\}$. We take directed cycles of the form:
$(i, 0)(j, 1)(i, 2)(j, 3) \ldots(i, n-3)(j, n-2)(k, n-1), \quad$ where $\quad 0 \leq i, j \leq w-1, k=i \circ j$.

This produces a decomposition of $\vec{C}_{(w: n)}$ into $w^{2}$ directed cycles of size $n$. To form a $\vec{C}_{n}$-factor, given $l \in Q$ we take all cycles arising from the pairs $i, j$ with $i * j=l$ in the second quasigroup $(Q, *)$. Thus we have a decomposition of $\vec{C}_{(w: n)}$ into $w$ $\vec{C}_{n}$-factors.

### 2.7 Multivariable Functions

Definition 2.7.1 Let $x$ and $y$ be odd. We define $T_{(x y)}(i, \alpha)$ to be the directed subgraph of $\vec{C}_{(x y: n)}$ obtained by taking $T_{(x y)}(i, \alpha)=T_{x}(i) \otimes T_{y}(\alpha)$. We also define

$$
H_{(x y)}(i, \alpha)(j, \beta)=T_{(x y)}(i, \alpha) \oplus F_{n}\left(T_{(x y)}(i, \alpha)\right) \oplus F_{n}\left(T_{(x y)}(j, \beta)\right)
$$

This means that $H_{(x y)}(i, \alpha)(j, \beta)$ is the directed graph obtained by taking the arcs of $T_{(x y)}(i, \alpha)$ between parts $t$ and $t+1$ for $0 \leq t \leq n-2$, and the arcs between parts $n-1$ and 0 from $T_{(x y)}(j, \beta)$.

Example 14: Figure 2.9 illustrates the first part of Definition 2.7.1 by showing $T_{x}(i)$, $T_{y}(\alpha)$ and $T_{(x y)}(i, \alpha)$, for $x=3, y=5, i=1$ and $\alpha=2$, with 3 partite sets.

Figure 2.10 illustrates the second part of Definition 2.7.1 by showing $H_{(x y)}(i, \alpha)(j, \beta)$, for $x=3, y=5, i=1, \alpha=2, j=2, \beta=4$, with 3 partite sets. Figure 2.10 also shows $H_{x}(i, j)$ and $H_{y}(\alpha, \beta)$, to illustrate Lemma 2.7.2.

Notice that in both figures instead of giving all the coordinates in each vertex, we give the first two coordinates of all the vertices in each row (the third coordinate would specify which partite set the vertex belongs to).

Lemma 2.7.2 Let $x, y$ and $n$ be odd. Then:

$$
H_{(x y)}(i, \alpha)(j, \beta)=H_{x}(i, j) \otimes H_{y}(\alpha, \beta)
$$

Proof: Notice that

$$
F_{n}\left(T_{(x y)}(i, \alpha)\right)=F_{n}\left(T_{x}(i) \otimes T_{y}(\alpha)\right)=F_{n}\left(T_{x}(i)\right) \otimes F_{n}\left(T_{y}(\alpha)\right)
$$

Notice also that

$$
F_{n}\left(T_{x}(i) \otimes T_{y}(\alpha)\right)=F_{n}\left(T_{x}(i)\right) \otimes T_{y}(\alpha)=T_{x}(i) \otimes F_{n}\left(T_{y}(\alpha)\right)
$$

Then we have

$$
\begin{aligned}
H_{x}(i, j) \otimes H_{y}(\alpha, \beta)= & \left(T_{x}(i) \oplus F_{n}\left(T_{x}(i)\right) \oplus F_{n}\left(T_{x}(j)\right)\right) \\
& \otimes\left(T_{y}(\alpha) \oplus F_{n}\left(T_{y}(\alpha)\right) \oplus F_{n}\left(T_{y}(\beta)\right)\right) \\
= & T_{x}(i) \otimes T_{y}(\alpha) \oplus F_{n}\left(T_{x}(i)\right) \\
& \otimes F_{n}\left(T_{y}(\alpha)\right) \oplus F_{n}\left(T_{x}(j)\right) \otimes F_{n}\left(T_{y}(\beta)\right) \\
= & T_{(x y)}(i, \alpha) \oplus F_{n}\left(T_{(x y)}(i, \alpha)\right) \oplus F_{n}\left(T_{(x y)}(j, \beta)\right) \\
= & H_{(x y)}(i, \alpha)(j, \beta)
\end{aligned}
$$



Figure 2.9: $T_{3}(1), T_{5}(2)$ and $T_{(3,5)}(1,2)$


Figure 2.10: $H_{3}(1,2), H_{5}(2,4)$ and $H_{(3,5)}(1,2)(2,4)$

Lemma 2.7.3 Let $\psi$ be a bijection on the set $\{(i, \alpha) \mid 0 \leq i \leq x-1,0 \leq \alpha \leq y-1\}$.
Then

$$
\vec{C}_{(x y: n)}=\bigoplus_{(i, \alpha)} H_{(x y)}(i, \alpha) \psi(i, \alpha)
$$

Proof: We know that $\vec{C}_{(x y: n)}=\vec{C}_{(x: n)} \otimes \vec{C}_{(y: n)}=\left(\bigoplus_{i} T_{x}(i)\right) \otimes\left(\bigoplus_{\alpha} T_{y}(\alpha)\right)$. By definition of $T_{(x y)}(i, \alpha)$ we get

$$
\vec{C}_{(x y: n)}=\bigoplus_{(i, \alpha)} T_{(x, y)}(i, \alpha)
$$

We also have

$$
\bigoplus_{(i, \alpha)} T_{(x y)}(i, \alpha)=\bigoplus_{(i, \alpha)} H_{(x y)}(i, \alpha) \psi(i, \alpha)
$$

Combining both we get:

$$
\vec{C}_{(x y: n)}=\bigoplus_{(i, \alpha)} H_{(x y)}(i, \alpha) \psi(i, \alpha)
$$

as we wanted to prove.

If $\psi(i, \alpha)=(j, \beta)$ we will denote $\psi_{1}(i, \alpha)=j$ and $\psi_{2}(i, \alpha)=\beta$. If $\operatorname{gcd}(x, i-j)=1$ and $\alpha=\beta$, then $H_{(x y)}(i, \alpha)(j, \beta)$ is a $\vec{C}_{x n}$-factor. This is because

$$
H_{(x y)}(i, \alpha)(j, \beta)=H_{x}(i, j) \otimes H_{y}(\alpha, \alpha)=H_{x}(i, j) \otimes T_{y}(\alpha)
$$

By Lemma 2.6.2 $H_{x}(i, j)$ is a $\vec{C}_{x n}$-factor. By Lemma 2.6.1 $T_{y}(\alpha)$ is a $\vec{C}_{n}$-factor. Then by Lemma 2.4.3 $H_{x}(i, j) \otimes T_{y}(\alpha)$ is a $\vec{C}_{x n}$-factor. Thus to obtain a decomposition of $\vec{C}_{(x y: n)}$ into $\vec{C}_{x n}$-factors and $\vec{C}_{y n}$-factors we need a bijection $\psi$ that satisfies the following set of conditions

Conditions 2.7.4 a) For all $(i, \alpha), \operatorname{gcd}\left(x, i-\psi_{1}(i, \alpha)\right)=1$ and $\psi_{2}(i, \alpha)=\alpha$, or
b) $\operatorname{gcd}\left(y, \alpha-\psi_{2}(i, \alpha)\right)=1$ and $\psi_{1}(i, \alpha)=i$.

Lemma 2.7.5 Let $x, y$, and $n$ be odd. Let $s_{p} \neq 1, x y-1$. Then there is a decomposition of $\vec{C}_{(x y: n)}$ into $s_{p} \vec{C}_{x n}$-factors and $r_{p}=x y-s_{p} \vec{C}_{y n}$-factors.

Proof: We will describe a bijection $\psi$ that satisfies conditions 2.7.4 with $r_{p}$ pairs $(i, \alpha)$ that satisfy $i=\psi_{1}(i, \alpha)$.

Let $r_{\alpha}, 0 \leq \alpha \leq y-1$ be such that:

- $\sum_{\alpha} r_{\alpha}=r_{p}$,
- $r_{i} \geq r_{j}$ if $i \leq j$,
- $r_{0}=r_{1}$,
- $0 \leq r_{\alpha} \leq x, r_{\alpha} \neq x-1$.

Define the function $\phi_{\alpha}(i)=\phi_{s}(i)$, with $\phi_{s}(i)$ as the phi-function over the set $\{0,1, \ldots, x-1\}$ with $s=x-r_{\alpha}$. Let $\pi(i)=\left|\left\{\alpha \mid \phi_{\alpha}(i)=i\right\}\right|$. Let $\sigma_{i}(\alpha)=\phi_{s}(\alpha)$, with $\phi_{s}(\alpha)$ as the phi-function over the set $\{0,1, \ldots, y-1\}$ with $s=\pi(i)$. Notice that

$$
\psi(i, \alpha)=\left(\phi_{\alpha}(i), \sigma_{i}(\alpha)\right)
$$

is a function satisfying conditions 2.7 .4 because if $\alpha \leq \pi(i)$, then $\psi_{\alpha}(i)=i$ and $\operatorname{gcd}\left(y, \alpha-\sigma_{i}(\alpha)\right)=1$. If, on the other hand $\alpha \geq \pi(i)$, then we have $\sigma_{i}(\alpha)=\alpha$, and $\operatorname{gcd}\left(x, i-\psi_{\alpha}(i)\right)=1$. Finally, notice that there are $r_{p}$ pairs $(i, \alpha)$ that satisfy $i=\varphi_{1}(i, \alpha)$. Therefore there is a decomposition of $\vec{C}_{(x y: n)}$ into $s_{p} \vec{C}_{x n}$-factors and $r_{p}=x y-s_{p} \vec{C}_{y n}$-factors.

We can work with $\Gamma(i)$ and $\Lambda(i)$ in a similar fashion as to what we did with $T_{x}(i)$.

Definition 2.7.6 Let $x$ be odd. We define $T_{(2 x)}(i, \alpha)$ to be the directed subgraph of $\vec{C}_{(4 x: n)}$ obtained by taking $T_{(x y)}(i, \alpha)=T_{x}(i) \otimes \Gamma(\alpha)$. We also define $H_{(2 x)}(i, \alpha)(j, \beta)=$ $T_{(2 x)}(i, \alpha) \oplus F_{n}\left(T_{(2 x)}(i, \alpha)\right) \oplus F_{n}\left(T_{(2 x)}(j, \beta)\right)$. This is the directed graph obtained by taking the arcs of $T_{(2 x)}(i, \alpha)$ between parts $t$ and $t+1$ for $0 \leq t \leq n-2$, and the arcs between parts $n-1$ and 0 from $T_{(2 x)}(j, \beta)$.

Now we can apply the same techniques that we did to $T_{(x y)}$ and $H_{(x y)}$.

Lemma 2.7.7 Let $x$ and $n$ be odd. Then:

$$
H_{(2 x)}(i, \alpha)(j, \beta)=H_{x}(i, j) \otimes \Lambda(\alpha, \beta)
$$

Proof: Notice that

$$
F_{n}\left(T_{(2 x)}(i, \alpha)\right)=F_{n}\left(T_{x}(i) \otimes \Gamma(\alpha)\right)=F_{n}\left(T_{x}(i)\right) \otimes F_{n}(\Gamma(\alpha))
$$

and

$$
F_{n}\left(T_{x}(i) \otimes \Gamma(\alpha)\right)=F_{n}\left(T_{x}(i)\right) \otimes \Gamma(\alpha)=T_{x}(i) \otimes F_{n}(\Gamma(\alpha))
$$

Using this the result is trivial.

Lemma 2.7.8 Let $\varphi$ by a bijection on the set $\{(i, \alpha) \mid 0 \leq i \leq x-1,0 \leq \alpha \leq 3\}$.
Then

$$
\vec{C}_{(4 x: n)}=\bigoplus_{(i, \alpha)} H_{(2 x)}(i, \alpha) \varphi(i, \alpha)
$$

Proof: We know that $\vec{C}_{(4 x: n)}=\vec{C}_{(x: n)} \otimes \vec{C}_{(4: n)}=\left(\bigoplus_{i} T_{x}(i)\right) \otimes\left(\bigoplus_{\alpha} \Gamma(\alpha)\right)$. By definition of $T_{(2 x)}(i, \alpha)$ we get

$$
\vec{C}_{(4 x: n)}=\bigoplus_{(i, \alpha)} T_{(2 x)}(i, \alpha)
$$

We also have

$$
\bigoplus_{(i, \alpha)} T_{(2 x)}(i, \alpha)=\bigoplus_{(i, \alpha)} H_{(2 x)}(i, \alpha) \varphi(i, \alpha)
$$

Combining both we get:

$$
\vec{C}_{(4 x: n)}=\bigoplus_{(i, \alpha)} H_{(2 x)}(i, \alpha) \varphi(i, \alpha)
$$

as we wanted to prove.

Next we develop the conditions needed for our decompositions. Recall that if $\varphi(i, \alpha)=$ $(j, \beta)$ we will denote $\varphi_{1}(i, \alpha)=j$ and $\varphi_{2}(i, \alpha)=\beta$.

- If $\alpha \neq \beta$ and $\operatorname{gcd}(x, i-j)=1$ then

$$
H_{(2 x)}(i, \alpha)(j, \beta)=H_{x}(i, j) \otimes \Lambda(\alpha, \beta)
$$

is a $\vec{C}_{2 x n}$-factor by Lemmas 2.6.2, 2.6.7, and 2.4.3.

- If $i=j$ and $\alpha \neq \beta$, then $H_{(2 x)}(i, \alpha)(j, \beta)$ is a $\vec{C}_{2 n}$-factor by Lemmas 2.6.1, 2.6.7. and 2.4.3.
- If $\alpha=\beta$ and $\operatorname{gcd}(x, i-j)=1$, then $H_{(2 x)}(i, \alpha)(j, \beta)$ is a $\vec{C}_{x n}$-factor by Lemmas 2.6.2, 2.6.5, and 2.4.3.
- If $i=j$ and $\alpha=\beta$, then $H_{(x y)}(i, \alpha)(j, \beta)$ is a $\vec{C}_{n}$-factor by Lemmas 2.6.1, 2.6.5.

So for a decomposition of $\vec{C}_{(4 x: n)}$ into $\vec{C}_{2 x n}$-factors and $\vec{C}_{n}$-factors we need a bijection $\varphi$ that satisfies:

Conditions 2.7.9 For all $(i, \alpha)$ such that $\varphi(i, \alpha) \neq(i, \alpha), \alpha \neq \varphi_{2}(i, \alpha)$ and $\operatorname{gcd}(x, i-$ $\left.\varphi_{1}(i, \alpha)\right)=1$.

For a decomposition of $\vec{C}_{(4 x: n)}$ into $\vec{C}_{2 n}$-factors and $\vec{C}_{x n}$-factors we need a bijection $\varphi$ that satisfies:

Conditions 2.7.10 a) For all $(i, \alpha)$ either $\alpha \neq \varphi_{2}(i, \alpha)$ and $i=j$, or
b) $\alpha=\varphi_{2}(i, \alpha)$ and $\operatorname{gcd}\left(x, i-\varphi_{1}(i, \alpha)\right)=1$.

We define a new family of functions $\theta_{s}: \mathbb{Z}_{x} \times \mathbb{Z}_{4} \rightarrow \mathbb{Z}_{x} \times \mathbb{Z}_{4}$. These functions will be referred as theta-functions. Let $s \in\{0,1, \ldots, 4 x\}, s \notin\{1,4 x-1\}$, and write $s=4 k+2 a+3 b$, with $a, b \in\{0,1\}, k \leq x$. We define:

## Theta-functions:

$$
\theta_{s}(i, \alpha):=\left\{\begin{array}{ccc}
(i+1, \alpha+1) & \text { if } & 0 \leq i \leq k-1, \alpha=0,2 \\
(i-1, \alpha-1) & \text { if } & 1 \leq i \leq k, \alpha=1,3 \\
(i+1, \alpha+1) & \text { if } & i=k, a=1, \alpha=0 \\
(i-1, \alpha-1) & \text { if } & i=k+1, a=1, \alpha=1 \\
(i+1, \alpha+1) & \text { if } & i=k, b=1, \alpha=2 \\
(i+1, \alpha-2) & \text { if } & i=k+1, b=1, \alpha=3 \\
(i-2, \alpha+1) & \text { if } & i=k+2, b=1, \alpha=1 \\
(i, \alpha) & & \text { otherwise }
\end{array}\right.
$$

If $s=4 x-1=4(x-1)+3$, we define $\theta_{s}$ in a similar way, with a small change:

$$
\theta_{4 x-1}(i, \alpha):=\left\{\begin{array}{ccc}
(i+1, \alpha+1) & \text { if } & 0 \leq i \leq x-2, \alpha=0,2 \\
(i-1, \alpha-1) & \text { if } & 1 \leq i \leq x-2, \alpha=1,3 \\
(x-2,2) & \text { if } & i=x-1, \alpha=1 \\
(0,1) & \text { if } & i=x-1, \alpha=3 \\
(x-1,0) & \text { if } & i=0, \alpha=1 \\
(0,3) & \text { if } & i=x-1, \alpha=0 \\
(x-2,0) & \text { if } & i=0, \alpha=3 \\
(i, \alpha) & & \text { otherwise }
\end{array}\right.
$$



Figure 2.11: Example of Theta-functions

We give a visual example of $\theta_{9}$ and $\theta_{19}$, for $x=5$ in Figure 2.11.

The following lemma is a generalization of a result given in [5].

Lemma 2.7.11 Let $s_{p} \in\{0,2, \ldots, 4 x-1,4 x\}, x$ odd. Then there exists a decomposition of $C_{(4 x: n)}$ into $s_{p} C_{2 x n}$-factors and $r_{p}=4 x-s_{p} C_{n}$-factors.

Proof: The bijection $\psi=\theta_{s_{p}}$ satisfies Conditions 7.9. In particular if $\psi(i, \alpha) \neq(i, \alpha)$, then $\alpha \neq \psi_{2}(i, \alpha)$ and $i-\psi_{1}(i, \alpha) \in\{ \pm 1, \pm 2\}$; and as $x$ is odd, $\operatorname{gcd}\left(x, i-\psi_{1}(i, \alpha)\right)=1$. Furthermore, $\psi$ has $s_{p}$ non-fixed points. Therefore there exists a decomposition of $C_{(4 x: n)}$ into $s_{p} C_{2 x n}$-factors and $4 x-s_{p} C_{n}$-factors.

Lemma 2.7.12 Let $s_{p} \in\{0,2,3, \ldots, 4 x-3,4 x-2,4 x\}$. Then there exists a decomposition of $\vec{C}_{(4 x: n)}$ into $s_{p} \vec{C}_{x n}$-factors and $r_{p}=4 x-s_{p} \vec{C}_{2 n}$-factors.

Proof: We provide a bijection $\varphi$ that satisfies Conditions 2.7.10 with $r_{p}$ pairs $(i, \alpha)$ that satisfy $i=\varphi_{1}(i, \alpha)$.

Let $r_{\alpha}, 0 \leq \alpha \leq 3$ be such that:

- $\sum_{\alpha} r_{\alpha}=r_{p}$,
- $r_{i} \geq r_{j}$ if $i \leq j$,
- $r_{0}=r_{1}$,
- $r_{\alpha} \leq x, r_{\alpha} \neq x-1$.

The only case where such a choice of $r_{\alpha}$ cannot be made is when $x=3, s=7$. This case is covered in Lemma 2.11.2 in the Appendix.

Define the function $\psi_{\alpha}(i)=\phi_{s}(i)$, with $\phi_{s}(i)$ as the phi-functions over the set $\{0,1, \ldots, x-1\}$ with $s=x-r_{\alpha}$. Let $\pi(i)=\left|\left\{\alpha \mid \psi_{\alpha}(i)=i\right\}\right|$. Let $\sigma_{i}(\alpha)$ be the permutation on the set $\{0,1,2,3\}$ that cyclically permutes the first $\pi(i)$ elements and fixes the rest. Notice that

$$
\varphi(i, \alpha)=\left(\psi_{\alpha}(i), \sigma_{i}(\alpha)\right)
$$

is a function satisfying conditions 2.7.10 because if $\alpha \leq \pi(i)$, then $\psi_{\alpha}(i)=i$ and $\sigma_{i}(\alpha) \neq \alpha$. If, on the other hand $\alpha \geq \pi(i)$, then we have $\sigma_{i}(\alpha)=\alpha$, and $\operatorname{gcd}(x, i-$ $\left.\psi_{\alpha}(i)\right)=1$. Finally, notice that there are $r_{p}$ pairs $(i, \alpha)$ that satisfy $i=\varphi_{1}(i, \alpha)$. Therefore there exists a decomposition of $\vec{C}_{(4 x: n)}$ into $s_{p} \vec{C}_{x n}$-factors and $r_{p}=4 x-s_{p}$ $\vec{C}_{2 n}$-factors.

We are interested in one more type of decomposition, into $\vec{C}_{2 x n}$ and $\vec{C}_{y n}$ factors. To do this we introduce the following:

Definition 2.7.13 Let $x$ and $y$ be odd. Define $T_{(2 x y)}(i, \alpha, \gamma)$ to be the directed subgraph of $\vec{C}_{(4 x y: n)}$ obtained by taking $T_{(2 x y)}(i, \alpha, \gamma)=T_{(2 x)}(i, \alpha) \otimes T_{y}(\gamma)$. We also define

$$
H_{(2 x y)}(i, \alpha, \gamma)(j, \beta, \delta)=T_{(2 x y)}(i, \alpha, \gamma) \oplus F_{n}\left(T_{(2 x y)}(i, \alpha, \gamma)\right) \oplus F_{n}\left(T_{(2 x y)}(j, \beta, \delta)\right.
$$

This means that $H_{(2 x y)}(i, \alpha, \gamma)(j, \beta, \delta)$ is the directed graph obtained by taking the arcs of $T_{(2 x y)}(i, \alpha, \gamma)$ between parts $t$ and $t+1$ for $0 \leq t \leq n-2$, and the arcs between parts $n-1$ and 0 from $T_{(2 x y)}(j, \beta, \delta)$.

Now we have all the usual results:

Lemma 2.7.14 Let $x$, $y$ and $n$ be odd. Then:

$$
H_{(2 x y)}(i, \alpha, \gamma)(j, \beta, \delta)=H_{(2 x)}(i, \alpha)(j, \beta) \otimes H_{y}(\gamma, \delta)
$$

Proof: Notice that

$$
F_{n}\left(T_{(2 x y)}(i, \alpha, \gamma)\right)=F_{n}\left(T_{(2 x)}(i, \alpha) \otimes T_{y}(\gamma)\right)=F_{n}\left(T_{(2 x)}(i, \alpha)\right) \otimes F_{n}\left(T_{y}(\gamma)\right)
$$

Using this the result is trivial.

Lemma 2.7.15 Let $\varphi$ be a bijection on the set $\{(i, \alpha, \gamma) \mid 0 \leq i \leq x-1,0 \leq \alpha \leq$ $3,0 \leq \gamma \leq y-1\}$. Then

$$
\vec{C}_{(4 x y: n)}=\bigoplus_{(i, \alpha, \gamma)} H_{(2 x y)}(i, \alpha, \gamma) \varphi(i, \alpha, \gamma)
$$

Proof: We know that $\vec{C}_{(4 x y: n)}=\vec{C}_{(4 x: n)} \otimes \vec{C}_{(y: n)}=\left(\bigoplus_{(i, \alpha)} T_{(2 x)}(i, \alpha)\right) \otimes\left(\bigoplus_{\gamma} T_{y}(\gamma)\right)$. By the definition of $T_{(2 x y)}(i, \alpha, \gamma)$ we get

$$
\vec{C}_{(4 x y: n)}=\bigoplus_{(i, \alpha, \gamma)} T_{(2 x y)}(i, \alpha)
$$

We also have

$$
\bigoplus_{(i, \alpha, \gamma)} T_{(2 x y)}(i, \alpha, \gamma)=\bigoplus_{(i, \alpha, \gamma)} H_{(2 x y)}(i, \alpha, \gamma) \varphi(i, \alpha, \gamma)
$$

Combining both we get:

$$
\vec{C}_{(4 x y: n)}=\bigoplus_{(i, \alpha, \gamma)} H_{(2 x y)}(i, \alpha, \gamma) \varphi(i, \alpha, \gamma)
$$

as we wanted to prove.

We have the following properties:

- If $\alpha \neq \beta, \gamma=\delta$, and $\operatorname{gcd}(x, i-j)=1$, then

$$
\begin{aligned}
H_{(2 x y)}(i, \alpha, \gamma)(j, \beta, \delta) & =H_{(2 x)}(i, \alpha)(j, \beta) \otimes H_{y}(\gamma, \gamma) \\
& =H_{x}(i, j) \otimes \Lambda(\alpha, \beta) \otimes T_{y}(\gamma)
\end{aligned}
$$

is a $\vec{C}_{2 x n}$-factor by Lemmas 2.6.2, 2.6.7, 2.6.1 and 2.4.3.

- If $i=j, \alpha=\beta$, and $\operatorname{gcd}(y, \gamma-\delta)=1$ then

$$
\begin{aligned}
H_{(2 x y)}(i, \alpha, \gamma)(j, \beta, \delta) & =H_{(2 x)}(i, \alpha)(i, \alpha) \otimes H_{y}(\gamma, \delta) \\
& =T_{x}(i) \otimes \Gamma(\alpha) \otimes H_{y}(\gamma, \delta)
\end{aligned}
$$

is a $\vec{C}_{y n}$-factor by Lemmas 2.6.1, 2.6.5, 2.6.2, and 2.4.3.

To get a decomposition of $\vec{C}_{(4 x y: n)}$ into $\vec{C}_{2 x n}$-factors and $\vec{C}_{y n}$-factors we need a bijection $\varphi$ that satisfies:

Conditions 2.7.16 a) For all $(i, \alpha, \gamma), \operatorname{gcd}\left(y, \gamma-\varphi_{3}(i, \alpha, \gamma)\right)=1$ or $\gamma=\varphi_{3}(i, \alpha, \gamma)$.
b) If $\gamma=\varphi_{3}(i, \alpha, \gamma)$, then $\operatorname{gcd}\left(x, i-\varphi_{1}(i, \alpha, \gamma)\right)=1$ and $\alpha \neq \varphi_{2}(i, \alpha, \gamma)$.
c) If $\operatorname{gcd}\left(y, \gamma-\varphi_{3}(i, \alpha, \gamma)\right)=1$, then $i=\varphi_{1}(i, \alpha, \gamma)$ and $\alpha=\varphi_{2}(i, \alpha, \gamma)$.

Now we can write our lemma:

Lemma 2.7.17 Let $x, y$, and $n$ be odd. Let $s_{p} \neq 1,4 x y-1$. Then there is a decomposition of $C_{(4 x y: n)}$ into $s_{p} C_{2 x n}$-factors and $r_{p}=4 x y-s_{p} C_{y n}$-factors.

Proof: We give a bijection $\varphi$ that satisfies Conditions 7.16 with $r_{p}$ elements ( $i, \alpha, \gamma$ ) that satisfy $i=\varphi_{1}(i, \alpha, \gamma)$. Let $s_{p}=4 x k+q$, with $0 \leq q \leq 4 x-1$. We have two cases, $k \leq y-3$, and $k \geq y-2$.

Case 1 If $k \leq y-3$, let $s_{p}=4 x k+a-\epsilon$, with $2 \leq a \leq 4 x-1,0 \leq \epsilon \leq 2$. For $y-k+1 \leq \gamma \leq y-1$, let $\psi_{\gamma}(i, \alpha)=\theta_{4 x}(i, \alpha)$, the theta-function with $s=4 x$. Let $\psi_{y-k}(i, \alpha)=\theta_{4 x-\epsilon}(i, \alpha)$, the theta-function with $s=4 x-\epsilon$. Let
$\psi_{y-k-1}(i, \alpha)=\theta_{a}(i, \alpha)$, the theta function with $s=a$. For $0 \leq \gamma \leq y-k-2$ let $\psi_{\gamma}(i, \alpha)=(i, \alpha)$, the identity function.

Case 2 If $k \geq y-2$, let $s_{p}=4 x k^{\prime}+2 a-\epsilon$, with $2 \leq a \leq 4 x-\epsilon, 0 \leq \epsilon \leq 4$, where $k^{\prime} \in\{y-2, y-3\}$ because $2 a$ may be greater than $4 x$. For $y-k+1 \leq \gamma \leq y-1$, let $\psi_{\gamma}(i, \alpha)=\theta_{4 x}(i, \alpha)$, the theta-function with $s=4 x$. Let $\psi_{y-k}(i, \alpha)=\theta_{4 x-\epsilon}(i, \alpha)$, the theta-function with $s=4 x-\epsilon$. Let $\psi_{y-k-1}(i, \alpha)=\psi_{y-k-2}(i, \alpha)=\theta_{a}(i, \alpha)$, the theta function with $s=a$. For $0 \leq \gamma \leq y-k-3$ let $\psi_{\gamma}(i, \alpha)=(i, \alpha)$, the identity function.

Notice that the fixed point of $\theta_{4 x-1}$ is $(x-1,2)$, the fixed points of $\theta_{4 x-2}$ are $\{(x-$ $1,2),(0,3)\}$, the fixed points of $\theta_{4 x-3}$ are $\{(x-1,2),(0,3),(x-1,0)\}$, and the fixed points of $\theta_{4 x-4}$ are $\{(x-1,2),(0,3),(x-1,0),(0,1)\}$. This means that if $0 \leq \epsilon \leq 4$ and $a \leq 4 x-\epsilon$, the fixed points of $\theta_{4 x-\epsilon}$ are a subset of the fixed points of $\theta_{a}$. Hence if $\psi_{\delta}(i, \alpha)=(i, \alpha)$, then $\psi_{\gamma}(i, \alpha)=(i, \alpha)$ for all $\gamma \leq \delta$. Notice also that $\psi_{0}(i, \alpha)=\psi_{1}(i, \alpha)$, hence $\max \left\{\delta \in\{0, \ldots, y-1\} \mid \psi_{\delta}(i, \alpha)=(i, \alpha)\right\} \neq 1$. Therefore we can define $\sigma_{i, \alpha}(\gamma)=\phi_{s}(\gamma)$, the phi-function over the set $\{0,1, \ldots, y-1\}$ with $s=\max \left\{\delta \in\{0, \ldots, y-1\} \mid \psi_{\delta}(i, \alpha)=(i, \alpha)\right\}$. Then:

$$
\rho(i, \alpha, \gamma)=\left(\psi_{\gamma}(i, \alpha), \sigma_{i, \alpha}(\gamma)\right)
$$

is a function satisfying conditions 7.16.

Example 15: We provide two visual examples, with $x=3$ and $y=5$. To make the picture easier to understand, the points satisfying $i=\varphi_{1}(i, \alpha, \gamma)$ have been boxed and underlined, and rearranged with their images at the right side.

In Figure 2.12 we have $s_{p}=25=2 \cdot 12+1=2 \cdot 12+2-1$, giving us $r_{p}=60-25=35, k=2, a=2, \epsilon=1$.

In Figure 2.13 we have $s_{p}=37=3 \cdot 12+1=3 \cdot 12+2 \cdot 2-3$, giving us $r_{p}=60-37=23, k^{\prime}=3, a=2, \epsilon=3$.

### 2.8 Product

In this section the partite product will be applied to the decompositions obtained in the previous section, to obtain decompositions of larger graphs.

Lemma 2.8.1 Let $m=z w$ with $z$ odd and $w \notin\{2,6\}$. Then there is a decomposition of $\vec{C}_{(m: n)}$ into $\vec{C}_{z n}$-factors and a decomposition of $\vec{C}_{(4 m: n)}$ into $\vec{C}_{2 z n}$-factors.

Proof: We know that $\vec{C}_{(m: n)}=\vec{C}_{(z: n)} \otimes \vec{C}_{(w: n)}$. By Theorem 23 we can decompose $\vec{C}_{(z: n)}=\bigoplus_{i} H_{z}(i, \phi(i))$ into $\vec{C}_{z n}$-factors, and by Lemma 2.6.1 $\vec{C}_{(w: n)}=\bigoplus_{j} T_{w}(j, j)$ into $\vec{C}_{n}$-factors.

$$
\begin{aligned}
& \gamma=4\{(2,0,4) \\
& s=12 \begin{cases}(1,0,4) \\
(0,0,4) & (0,2,4)\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
& x=3, y=5, s_{p}=25, r_{p}=35, k=2, a=2, \epsilon=1
\end{aligned}
$$

Figure 2.12: Example of Lemma 7.17 Case 1

$$
\begin{aligned}
& \begin{array}{l}
\gamma=4 \\
s=12 \\
(2,0,4)(2,1,4) \\
(2,2,4)(2,4) \\
(0,0,4)^{(0,1,4)} \\
(0,2,4)
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
& (\underset{(1,0,1)}{(1,0,0)})\left(\begin{array}{c}
\left(\frac{(2,1,1)}{(2,1,0)}\right. \\
\left(\frac{(1,2,1)}{(1,2,0)}\right.
\end{array}\left(\underset{\left(\frac{(2,3,1)}{(2,3)}\right.}{( }\right)\right. \\
& \left(\begin{array}{l}
(0,2,1) \\
(0,2,0) \\
\left(\frac{(1,3,1)}{(1,3,0)}\right. \\
\hline
\end{array}\right. \\
& x=3, y=5, s_{p}=37, r_{p}=23, k=4, a=2, \epsilon=3, k^{\prime}=3
\end{aligned}
$$

Figure 2.13: Example of Lemma 7.17 Case 2

Then

$$
\begin{aligned}
\vec{C}_{(m: n)} & =\vec{C}_{(z: n)} \otimes \vec{C}_{(w: n)} \\
& =\left(\bigoplus_{i} H_{z}(i, \phi(i))\right) \otimes\left(\bigoplus_{j} H_{w}(j, j)\right) \\
& =\bigoplus_{i} \bigoplus_{j} H_{z}(i, \phi(i)) \otimes H_{w}(j, j)
\end{aligned}
$$

By Lemma 2.4.3 $H_{z}(i, \phi(i)) \otimes H_{w}(j, j)$ is a $\vec{C}_{z n}$-factor, and so $\vec{C}_{(m: n)}$ can be decomposed into $\vec{C}_{z n}$-factors.

For the result on $\vec{C}_{(4 m: n)}$ we just multiply by $\vec{C}_{(4: n)}$ :

$$
\vec{C}_{(4 m: n)}=\vec{C}_{(m: n)} \otimes \vec{C}_{(4: n)}
$$

We can decompose $\vec{C}_{(4: n)}$ into $\vec{C}_{2 n}$-factors by Theorem 24 and so when multiplying by $\vec{C}_{(m: n)}$ we obtain a decomposition of $\vec{C}_{(4 m: n)}$ into $\vec{C}_{2 z n-}$ factors.

We may now use Lemma 2.8.1 and the decompositions obtained in Section 7 to get a decomposition of $\vec{C}_{(x y z w: n)}$ into $\vec{C}_{x z n}$-factors and $\vec{C}_{y z n}$-factors:

Lemma 2.8.2 Suppose $\vec{C}_{(m: n)}$ can be decomposed into $s_{p} \vec{C}_{m_{1} n}$-factors and $r_{p} \vec{C}_{m_{2} n}$ factors, with $s_{p}, r_{p} \neq 1, r_{p}+s_{p}=m$. Let $z$ be odd with $\operatorname{gcd}\left(m_{1}, z\right)=\operatorname{gcd}\left(m_{2}, z\right)=1$
and $w \notin\{2,6\}$. Then there is a decomposition of $\vec{C}_{(m z w: n)}$ into $s \vec{C}_{m_{1} z n}$-factors and $r \vec{C}_{m_{2} z n}$-factors for any $s, r \neq 1, s+r=m z w$. Furthermore, if $m_{1}$ and $m_{2}$ are odd, there is a decomposition of $\vec{C}_{(4 m z w: n)}$ into $s^{\prime} \vec{C}_{2 m_{1} z n}$-factors and $r^{\prime} \vec{C}_{2 m_{2} z n}$-factors for any $s^{\prime}, r^{\prime} \neq 1, s^{\prime}+r^{\prime}=4 m z w$.

Proof: We start with the product $\vec{C}_{(m z w: n)}=\vec{C}_{(m: n)} \otimes \vec{C}_{(z w: n)}$. By Lemma 2.8.1 we can decompose $\vec{C}_{(z w: n)}=\oplus_{i=1}^{z w} Z_{i}$, where each $Z_{i}$ is a $\vec{C}_{z n}$-factor.

Let $s=m t+u$, with $0 \leq u \leq m$. If $u \neq 1, m-1$ we decompose as follows:

$$
\begin{aligned}
\vec{C}_{(m z w: n)} & =\vec{C}_{(m: n)} \otimes \vec{C}_{(z w: n)} \\
& =\vec{C}_{(m: n)} \otimes\left(\oplus_{i=1}^{z w} Z_{i}\right) \\
& =\oplus_{i=1}^{z w} \vec{C}_{(m: n)} \otimes Z_{i} \\
& =\left(\oplus_{i=1}^{t} \vec{C}_{(m: n)} \otimes Z_{i}\right) \oplus \vec{C}_{(m: n)} \otimes Z_{t+1} \oplus\left(\oplus_{i=t+2}^{z w} \vec{C}_{(m: n)} \otimes Z_{i}\right)
\end{aligned}
$$

From the theorem hypothesis on $\vec{C}_{(m: n)}$, we have the following decompositions:

- We decompose $\vec{C}_{(m: n)}$ into $m \vec{C}_{m_{1} n}$-factors for the product $\oplus_{i=1}^{t} \vec{C}_{(m: n)} \otimes Z_{i}$.

- We decompose $\vec{C}_{(m: n)}$ into $u \vec{C}_{m_{1} n}$-factors and $m-u \vec{C}_{m_{2} n}$-factors for the product $\vec{C}_{(m: n)} \otimes Z_{t+1}$.

Because $m_{1}$ and $m_{2}$ are coprime with $z$, by Lemma 2.4.3 there is a decomposition into $m t+u=s \vec{C}_{m_{1} z n}$-factors and $r \vec{C}_{m_{2} z n}$-factors.

If $u=1$, we decompose as follows:

$$
\begin{aligned}
\vec{C}_{(m z w: n)}= & \left(\oplus_{i=1}^{t-1} \vec{C}_{(m: n)} \otimes Z_{i}\right) \oplus\left(\vec{C}_{(m: n)} \otimes Z_{t}\right) \oplus \\
& \left(\vec{C}_{(m: n)} \otimes Z_{t+1}\right) \oplus\left(\oplus_{i=t+2}^{z w} \vec{C}_{(m: n)} \otimes Z_{i}\right)
\end{aligned}
$$

We also have the following decompositions:

- We decompose $\vec{C}_{(m: n)}$ into $m \vec{C}_{m_{1} n}$-factors for the product $\oplus_{i=1}^{t-1} \vec{C}_{(m: n)} \otimes Z_{i}$.
- We decompose $\vec{C}_{(m: n)}$ into $m \vec{C}_{m_{2} n}$-factors for the product $\oplus_{i=t+2}^{z w} \vec{C}_{(m: n)} \otimes Z_{i}$.
- We decompose $\vec{C}_{(m: n)}$ into $m-2 \vec{C}_{m_{1} n}$-factors and $2 \vec{C}_{m_{2} n}$-factors for the product $\vec{C}_{(m: n)} \otimes Z_{t}$.
- We decompose $\vec{C}_{(m: n)}$ into $3 \vec{C}_{m_{1} n}$-factors and $m-3 \vec{C}_{m_{2} n}$-factors for the product $\vec{C}_{(m: n)} \otimes Z_{t+1}$.

Because $m_{1}$ and $m_{2}$ are coprime with $z$, this gives a decomposition into $m(t-1)+$
$m-2+3=m t+1=s \vec{C}_{m_{1} z n}$-factors and $r \vec{C}_{m_{2} z n}$-factors. If $u=m-1$ we just change the roles of $m_{1}$ and $m_{2}$ and take the decomposition for $u=1$. Therefore there is a decomposition of $\vec{C}_{(m z w: n)}$ into $s \vec{C}_{m_{1} z n}$ factors and $r \vec{C}_{m_{2} z n}$-factors for any $s, r \neq 1, r+s=m z w$. The decomposition of $\vec{C}_{(4 m z w: n)}$ into $s^{\prime} \vec{C}_{2 m_{1} z n}$-factors and $r^{\prime} \vec{C}_{2 m_{2} z n}$-factors works in the same way.

We can now combine this result with our decompositions from Section 7 to obtain the following result, which we write using non-directed graphs, as we are getting ready to apply the results from Section 5.

Theorem 26 Let $x, y, z, n$ be odd numbers with $\operatorname{gcd}(x, z)=\operatorname{gcd}(y, z)=1$ and $w \notin$ $\{2,6\}$. Then we have the following decompositions:
a) $C_{(x y z w: n)}$ can be decomposed into s $C_{x z n}$-factors and $r C_{y z n}$-factors for any $s, r \neq 1$, $s+r=x y z w$.
b) $C_{(4 x z w: n)}$ can be decomposed into s $C_{2 x z n}$-factors and $r C_{2 z n}$-factors for any $s, r \neq 1$, $s+r=4 x z w$.
c) $C_{(4 x z w: n)}$ can be decomposed into s $C_{2 x z n}$-factors and $r C_{z n}$-factors for any $s, r \neq 1$, $s+r=4 x z w$.
d) $C_{(4 x z w: n)}$ can be decomposed into s $C_{x z n}$-factors and $r C_{2 z n}$-factors for any $s, r \neq 1$,
$s+r=4 x z w$.
e) $C_{(4 x y z w: n)}$ can be decomposed into $s C_{2 x z n}$-factors and $r C_{y z n}$-factors for any $s, r \neq$ $1, s+r=4 x y z w$.
f) $C_{(4 x y z w: n)}$ can be decomposed into s $C_{2 x z n}$-factors and $r C_{2 y z n}$-factors for any $s, r \neq$ $1, s+r=4 x y z w$.

Proof:
a) $\vec{C}_{(x y: n)}$ can be decomposed into $s_{p} \vec{C}_{x n}$-factors and $r_{p} \vec{C}_{y n}$-factors by Lemma 2.7.5. So by Lemma 2.8.2, $\vec{C}_{(x y w z: n)}$ can be decomposed into $s \vec{C}_{x z n}$-factors and $r \vec{C}_{y z n}$-factors.
b) $\vec{C}_{(x: n)}$ can be decomposed into $s_{p} \vec{C}_{x n}$-factors and $r_{p} \vec{C}_{n}$-factors by Lemma 23 . Now apply Lemma 2.8.2.
c) $\vec{C}_{(4 x: n)}$ can be decomposed into $s_{p} \vec{C}_{2 x n}$-factors and $r_{p} \vec{C}_{n}$-factors by Lemma 2.7.11. Now apply Lemma 2.8.2.
d) $\vec{C}_{(4 x: n)}$ can be decomposed into $s_{p} \vec{C}_{x n}$-factors and $r_{p} \vec{C}_{2 n}$-factors by Lemma 2.7.12. Now apply Lemma 2.8.2.
e) $\vec{C}_{(4 x y: n)}$ can be decomposed into $s_{p} \vec{C}_{2 x n}$-factors and $r_{p} \vec{C}_{y n}$-factors by Lemma 2.7.17. Now apply Lemma 2.8.2.
f) $\vec{C}_{(x y: n)}$ can be decomposed into $s_{p} \vec{C}_{x n}$-factors and $r_{p} \vec{C}_{y n}$-factors by Lemma 2.7.5. Now apply Lemma 2.8.2.

### 2.9 Main Result

We now to use the decompositions that we obtained for $C_{(v: n)}$ to obtain decompositions of $K_{(v: m)}$ via Lemmas 2.5.3 and 2.5.1.

Theorem 27 Let $m$ and $n$ be odd, such that $m \equiv 0(\bmod n)$. Let $s$ and $r$ be such that $s, r \neq 1$ and $s+r=v \frac{m-1}{2}$. Let $x_{1}, \ldots x_{m / n}, y_{1}, \ldots y_{m / n}, z_{1}, \ldots, z_{m / n}$ and $w_{1}, \ldots, w_{m / n}$ be such that:

- $\operatorname{gcd}\left(x_{i}, z_{i}\right)=\operatorname{gcd}\left(y_{i}, z_{i}\right)=1$;
- $w_{i} \notin\{2,6\} ;$
- 2 divides at most one of $x_{i}, y_{i}$ and $z_{i}$;
- $v=x_{i} y_{i} z_{i} w_{i}$ if 2 divides none of $x_{i}, y_{i}, z_{i}$; and
- $v=2 x_{i} y_{i} z_{i} w_{i}$ if 2 divides one of $x_{i}, y_{i}, z_{i}$.

Furthermore, let $F_{1}$ be $a\left[\left(x_{1} z_{1} n\right)^{\frac{v}{x_{1} z_{1}}}, \ldots,\left(x_{m / n} z_{m / n} n\right)^{\frac{v}{x_{m / n} z_{m / n}}}\right]$-factor, and let $F_{2}$ be $a$ $\left[\left(y_{1} z_{1} n\right)^{\frac{v}{y_{1} z_{1}}}, \ldots,\left(y_{m / n} z_{m / n} n\right)^{\frac{v}{y_{m / n} z_{m / n}}}\right]$-factor. Then there is a decomposition of $K_{(v: m)}$ into $s$ copies of $F_{1}$ and $r$ copies of $F_{2}$.

Proof: By Theorem 22 there is a decomposition of $K_{m}$ into $C_{n}$-factors.

Pick $s_{1}, \ldots, s_{\frac{m-1}{2}}$ such that $s=\bigoplus_{i=1}^{v \frac{m-1}{2}} s_{i}$, with $0 \leq s_{i} \leq v$ and $s_{i} \notin\{1, v-1\}$.

By Theorem 26 there is a decomposition of $C_{(v: n)}$ into $s_{i} C_{x_{i} z_{i} n}$-factors and $r_{i}=v-s_{i}$ $C_{y_{i} z_{i} n}$-factors.

Therefore by Theorem 2.5.3 there is a decomposition of $K_{(v: m)}$ into $s$ copies of the 2-factor $F_{1}$ and $r$ copies of the 2-factor $F_{2}$.

### 2.10 Applications

We can use our results to solve many cases of the Hamilton-Waterloo Problem for complete graphs. For some of them we will need the notion of resolvable group divisible design.

A resolvable group divisible design $(k, \lambda)-\operatorname{RGDD}\left(h^{u}\right)$ is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a finite set of size $v=h u, \mathcal{G}$ is a partition of $\mathcal{V}$ into $u$ groups each containing $h$
elements, and $\mathcal{B}$ is a collection of $k$ element subsets of $\mathcal{V}$ called blocks which satisfy the following properties.

- If $B \in \mathcal{B}$, then $|B|=k$.
- If a pair of elements from $\mathcal{V}$ appear in the same group, then the pair cannot be in any block.
- Two points that are not in the same group, called a transverse pair, appear in exactly $\lambda$ blocks.
- $|\mathcal{G}|>1$.
- The blocks can be partitioned into parallel classes such that for each element of $\mathcal{V}$ there is exactly one block in each parallel class containing it.

Here we use the term group to indicate an element of $\mathcal{G}$. In this context, group simply means a set of elements without any algebraic structure. If $\lambda=1$, we refer to the RGDD as a $k-\operatorname{RGDD}\left(h^{u}\right)$.

In [35] the following characterization theorem was proven:

Theorem 28 [35] $A(3, \lambda)-\operatorname{RGDD}\left(h^{u}\right)$ exists if and only if $u \geq 3, \lambda h(u-1)$ is even, $h u \equiv 0(\bmod 3)$, and $(\lambda, h, u) \notin\{(1,2,6),(1,6,3)\} \bigcup\{(2 j+1,2,3),(4 j+2,1,6): j \geq$ $0\}$.

In [5] the authors used resolvable group divisible designs together with Theorem 28
to decompose complete graphs into $C_{3}$-factors and $C_{3 x}$-factors.

Lemma 2.10.1 [5] Let $x \geq 3, y \geq 3$ and $m$ be positive integers such that both $x$ and $y$ divide $3 m$. Suppose the following conditions are satisfied:

- There exists a 3-RGDD $\left(h^{u}\right)$,
- there exists a decomposition of $K_{(m: 3)}$ into $s_{p} C_{x}$-factors and $r_{p} C_{y}$-factors, for $p \in\left\{1,2, \ldots, \frac{h(u-1)}{2}\right\}$,
- there exists a decomposition of $K_{h m}$ into $s_{\beta} C_{x}$-factors and $r_{\beta} C_{y}$-factors.

Let

$$
s_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}} s_{p} \text { and } r_{\alpha}=\sum_{p=1}^{\frac{h(u-1)}{2}} r_{p} .
$$

Then there exists a decomposition of $K_{\text {hum }}$ into $s_{\alpha}+s_{\beta} C_{x}$-factors and $r_{\alpha}+r_{\beta} C_{y}$ factors.

We can now apply our decompositions to extend the result from [5]. We will be concerned with prime numbers whose greatest power that divides $x$ is the same as their greatest power that divides $y$. Thus we give the following definition:

Definition 2.10.2 Let $x$ and $y$ be natural numbers, with $x=p_{1}^{a_{1}} \ldots p_{n}^{a_{n}}$ and $y=$ $p_{1}^{b_{1}} \ldots p_{n}^{b_{n}}$ their prime factorization. Then we define the special greatest common divisor of $x$ and $y, s(x, y)$ as the smallest number such that $p_{i}^{a_{i}}$ divides $s(x, y)$ if and
only if $a_{i}=b_{i}$.

Example 16: For example if $x=2^{3} 3^{2} 5^{2} 7^{2}$ and $y=3^{2} 5^{3} 7^{2} 11^{4}$, then $s(x, y)=3^{2} 7^{2}$.

Corollary 2.10.3 Let $x, y, n$ be integers such that

- $\frac{x y}{s(x, y)}$ divides $n$,
- $x, y \not \equiv 0(\bmod 4)$, and 4 divides $n$ if 2 divides $x y$;
- $3 n=\frac{\text { huxyw }}{s(x, y)}$, with $h \equiv 0(\bmod 3), u \geq 3, h(u-1)$ even, and $(h, u) \notin$ $\{(2,6),(6,3)\}$.

Then there exists a decomposition of $K_{3 n}$ into $s C_{3 x}$-factors and r $C_{3 y}$-factors for every pair $(s, r)$ such that $s+r=\left\lfloor\frac{3 n-1}{2}\right\rfloor, s, r \neq 1$.

Proof: Let $(s, r)$ be such that $s+r=\left\lfloor\frac{3 n-1}{2}\right\rfloor$ and $s, r \neq 1$. If $s \geq r$ let $s_{0}=\left\lfloor\frac{3 n-1}{2}\right\rfloor$ and $r_{0}=0$. Otherwise let $s_{0}=0$ and $r_{0}=\left\lfloor\frac{3 n-1}{2}\right\rfloor$. Let $s_{1}, \ldots, s_{\frac{h(u-1)}{2}}$ and $r_{1}, \ldots, r_{\frac{h(u-1)}{2}}$ be such that $s_{i}+r_{i}=\frac{x y w}{s(x, y)}, s_{i}, r_{i} \neq 1$ for all $i$ and

$$
r_{\alpha}=\sum_{i=0}^{\frac{h(u-1)}{2}} r_{i} \text { and } s_{\alpha}=\sum_{i=0}^{\frac{h(u-1)}{2}} s_{i}
$$

From Theorem 28 we know that there is a $3-\operatorname{RGDD}\left(h^{u}\right)$. Let $z=s(x, y), x_{1}=\frac{x}{z}$ and $y_{1}=\frac{y}{z}$. Then

$$
\frac{x y w}{z}=\frac{x_{1} z y_{1} z w}{z}=x_{1} y_{1} z w .
$$

So we may apply Theorem 27 to obtain a decomposition of $K_{\left(\frac{x y w}{z}: 3\right)}$ into $s_{i} C_{3 x_{1} z^{-}}$ factors and $r_{i} C_{3 y_{1} z}$-factors for each $i$.

Because $h x y w=h x_{1} z y w=h x y_{1} z w$ we have that $3 x_{1} z \mid(h x y w)$ and $3 y_{1} z \mid(h x y w)$. From Theorem 22 there is a decomposition of $K_{\frac{h x y w}{z}}$ into $C_{3 x_{1} z}$-factors, and there is also a decomposition of $K_{\frac{h x y w}{z}}$ into $C_{3 y_{1} z}$-factors.

Thus we may apply Lemma 2.10 .1 to obtain a decomposition of $K_{\frac{\text { huxyw }}{z}}=K_{3 n}$ into $s$ $C_{3 x}$-factors and $r C_{3 y}$-factors.

We can also use Lemma 2.5.1 to obtain decompositions of complete graphs into $C_{x^{-}}$ factors and $C_{y}$-factors:

Corollary 2.10.4 Let $m, x$, and $y$ be integers such that:

- $z=s(x, y), w=\frac{\operatorname{gcd}(x, y)}{z} \geq 2$,
- $\frac{x y}{z}$ divides $m$,
- 4 does not divide $x$ nor $y$.
- Neither $x$ nor $y$ is 3 if $\frac{m}{w} \in\{6,12\}$.

Then there exists a decomposition of $K_{m}$ into $s C_{x}$-factors and $r C_{y}$-factors for every $s, r \neq 1$.

Proof: Let $s, r$ be such that $s+r=\left\lfloor\frac{m-1}{2}\right\rfloor$ and $s, r \neq 1$.

Let

$$
k=\frac{m z}{x y} \quad x^{\prime}=\frac{x}{z w} \quad y^{\prime}=\frac{y}{z w} \quad m^{\prime}=\frac{m}{w}=\frac{x y k}{z w}=x^{\prime} y^{\prime} z w k
$$

Let $s_{\alpha}, s_{\beta}, r_{\alpha}, r_{\beta}$ be such that $s_{\beta}, r_{\beta} \neq 1,\left\{s_{\alpha}, r_{\alpha}\right\}=\left\{0,\left\lfloor\frac{m^{\prime}-1}{2}\right\rfloor\right\}, s=s_{\alpha}+s_{\beta}$ and $r=r_{\alpha}+r_{\beta}$. By Theorem 27 there is a decomposition of $K_{\left(m^{\prime}: w\right)}$ into $s_{\beta} C_{x^{\prime} z w}$-factors and $r_{\beta} C_{y^{\prime} z w^{-}}$-factors. This is a decomposition of $K_{\left(m^{\prime}: w\right)}$ into $s_{\beta} C_{x^{-}}$-factors and $r_{\beta} C_{y^{-}}$ factors. Because $\frac{x y}{z}$ divides $m$, it follows that both $x$ and $y$ divide $m^{\prime}=\frac{m}{w}=m \frac{z}{\operatorname{gcd}(x, y)}$. Thus by Theorem 22 there is decompositon of $K_{m^{\prime}}$ into $s_{\alpha} C_{x}$-factors and $r_{\alpha} C_{y^{-}}$ factors (keep in mind that one of $s_{\alpha}$ and $r_{\alpha}$ is 0 ). Then by Lemma 2.5.1 there is a decomposition of $K_{m^{\prime} w}$ into $s C_{x}$-factors and $r C_{y}$-factors. Therefore there is a decompostion of $K_{m}$ into $s C_{x}$-factors and $r C_{y}$-factors.

Notice that by asking $\frac{x y}{z}$ to divide $m$ we cover some of the cases left open in [13] for the odd order case.

Example 17: Let $m=3^{3} 5^{3}, x=3^{1} 5^{2}$ and $y=3^{2} 5^{2}$. We have:

$$
z=s(x, y)=5^{2}, \quad w=3^{1}, \quad k=5^{1}
$$

And $\frac{x y}{z}=3^{3} 5^{2}$ divides $m$. So we can decompose $K_{m}$ into $s C_{x}$-factors and $r$ $C_{y}$-factors for any $s, r \neq 1$. Note that $l=\operatorname{lcm}(x, y)=3^{2} 5^{2}$. The number of vertices, $m$, is a multiple of $l$, however $x y \nmid m$. Thus, Theorem 20 cannot be applied here.

Example 18: Let $m=4^{1} 3^{4} 5^{3} 7^{1}, x=2^{1} 3^{1} 5^{2} 7^{1}$ and $y=3^{3} 5^{2}$. We have:

$$
z=s(x, y)=5^{2}, \quad w=3^{1}, \quad k=2^{1} 5^{1}
$$

And $\frac{x y}{z}=2^{1} 3^{4} 5^{2} 7^{1}$ divides $m$. So we can decompose $K_{m}$ into $s C_{x}$-factors and $r C_{y}$-factors for any $s, r \neq 1$. Note that because $x$ is even Theorem 20 cannot be applied here.

We can also use our Lemma 2.5.2 to obtain non-uniform decompositions of complete graphs:

Corollary 2.10.5 Let $v, m, n, x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{k}$ be integers such that:

- $m \geq 3$ is odd,
- $n$ divides $m, x_{i}$, and $y_{i}$ for every $i$,
- $k=\frac{m}{n}$,
- $z_{i}=s\left(x_{i}, y_{i}\right)$,
- $\frac{x_{i} y_{i}}{z_{i} n}$ divides $v$ for each $i$,
- $x_{i}$ divides $v$ for each $i$,
- 4 does not divide $x_{i}$ nor $y_{i}$ for any $i$,
- $3 \notin\left\{x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{k}\right\}$ if $k \in\{6,12\}$.

Then there exists a decomposition of $K_{v m}$ into $s\left[x_{1}^{v n / x_{1}}, \ldots, x_{k}^{v n / x_{k}}\right]$-factors and $r$ $\left[y_{1}^{v n / y_{1}}, \ldots, y_{k}^{v n / y_{k}}\right]$-factors for every $s, r \neq 1$.

Proof: Let $s, r$ be such that $s+r=\left\lfloor\frac{v m-1}{2}\right\rfloor$ and $s, r \neq 1$.

Let

$$
x_{i}^{\prime}=\frac{x_{i}}{z_{i} n} \quad y_{i}^{\prime}=\frac{y_{i}}{z_{i} n} \quad k_{i}=\frac{v z_{i} n}{x_{i} y_{i}}
$$

Then

$$
v=\frac{x_{i} y_{i} k_{i}}{z_{i} n}=\frac{x_{i}^{\prime} z_{i} n y_{i}^{\prime} z_{i} n k_{i}}{z_{i} n}=x_{i}^{\prime} y_{i}^{\prime} z_{i} n k_{i}
$$

Let $s_{\alpha}, s_{\beta}, r_{\alpha}, r_{\beta}$ be such that $s_{\beta}, r_{\beta} \neq 1,\left\{s_{\alpha}, r_{\alpha}\right\}=\left\{0,\left\lfloor\frac{v-1}{2}\right\rfloor\right\}, s=s_{\alpha}+s_{\beta}$ and $r=$
$r_{\alpha}+r_{\beta}$. By Theorem 27 there is a decomposition of $K_{(v: m)}$ into $s_{\beta}\left[x_{1}^{v n / x_{1}}, \ldots, x_{k}^{v n / x_{k}}\right]$ factors and $r_{\beta}\left[y_{1}^{v n / y_{1}}, \ldots, y_{k}^{v n / y_{k}}\right]$-factors. We partition $m K_{v}$ into $k$ copies of $n K_{v}$, labeled $\kappa_{1}, \ldots \kappa_{k}$. Because $\frac{x_{i} y_{i}}{z_{i} n}$ and $x_{i}$ divide $v$, we get that both $x_{i}$ and $y_{i}$ divide $v$, by Theorem 22 there is a decompostion of $K_{v}$ into $s_{\alpha} C_{x_{i}}$-factors and $r_{\alpha} C_{y_{i}}$-factors (keep in mind that one of $s_{\alpha}$ and $r_{\alpha}$ is 0 ). This means that $\kappa_{i}$ can be decomposed into $s_{\alpha}\left[x_{i}^{v n / x_{i}}\right]$-factors and $r_{\alpha}\left[y_{i}^{v n / y_{i}}\right]$-factors. Combining these decompositions we get a decomposition of $m K_{v}$ into $s_{\alpha}\left[x_{1}^{v n / x_{1}}, \ldots, x_{k}^{v n / x_{k}}\right]$-factors and $r_{\alpha}\left[y_{1}^{v n / y_{1}}, \ldots, y_{k}^{v n / y_{k}}\right]$ factors.

Then by Lemma 2.5 .2 there is a decomposition $K_{v m}$ into $s\left[x_{1}^{v n / x_{1}}, \ldots, x_{k}^{v n / x_{k}}\right]$-factors and $r\left[y_{1}^{v n / y_{1}}, \ldots, y_{k}^{v n / y_{k}}\right]$-factors for every $s, r \neq 1$.

Example 19: Let $v=5^{3} 7^{2} 11^{3} 13^{4}, m=3^{1} 5^{1}, n=5, k=3, x_{1}=5^{2} 7^{2}, y_{1}=$ $5^{1} 7^{2} 11^{1} 13^{1}, x_{2}=5^{1} 11^{1} 13^{1}, y_{2}=5^{1} 7^{2} 13^{3}, x_{3}=5^{2} 7^{1} 11^{1} 13^{4}$, and $y_{3}=5^{1} . \mathrm{We}$ have:

$$
\begin{aligned}
z_{1} & =s\left(x_{1}, y_{1}\right)=7^{2}, & \frac{x_{1} y_{1}}{z_{1} n}=5^{2} 7^{2} \\
z_{2} & =s\left(x_{2}, y_{2}\right)=5^{1}, & \frac{x_{2} y_{2}}{z_{2} n}=7^{2} 11^{1} 13^{4} \\
z_{3} & =s\left(x_{3}, y_{3}\right)=1, & \frac{x_{3} y_{3}}{z_{3} n}=5^{2} 7^{1} 11^{1} 13^{4}
\end{aligned}
$$

We can see that $\frac{x_{i} y_{i}}{z_{i} n}$ and $x_{i}$ divide $v$ for each $i$.

Let $F_{1}$ be the 2-factor consisting of:

- $\frac{v n}{x_{1}}=5^{2} 11^{3} 13^{4}$ cycles of size $x_{1}=5^{2} 7^{2}$,
- $\frac{v n}{x_{2}}=5^{3} 7^{2} 11^{2} 13^{3}$ cycles of size $x_{2}=5^{1} 11^{1} 13^{1}$,
- and $\frac{v n}{x_{3}}=5^{2} 6^{1} 11^{2}$ cycles of size $x_{3}=5^{2} 7^{1} 11^{1} 13^{4}$.

Let $F_{2}$ be the 2-factor consisting of:

- $\frac{v n}{y_{1}}=5^{3} 11^{2} 13^{3}$ cycles of size $y_{1}=5^{1} 7^{2} 11^{1} 13^{1}$,
- $\frac{v n}{y_{2}}=5^{3} 11^{3} 13^{1}$ cycles of size $y_{2}=5^{1} 7^{2} 13^{3}$,
- and $\frac{v n}{y_{3}}=5^{3} 7^{2} 11^{3} 13^{4}$ cycles of size $y_{3}=5^{1}$.

Then we can decompose $K_{v m}$ into $s$ copies of $F_{1}$ and $r$ copies of $F_{2}$ for any $s, r \neq 1$.

### 2.11 Appendix

Lemma 2.11.1 There is a decomposition of $C_{(12: 3)}$ into $7 C_{12}$-factors and $5 C_{6}$ factors.

Proof: Let $G_{0}, G_{1}$, and $G_{2}$ be the partite sets. We will construct each factors by taking a difference between each pair of partite sets. Notice that if the sum of the three differences is congruent to 6 modulo 12 , this gives a $C_{6}$-factor. Likewise, if the sum of the differences is congruent to 4 or 8 modulo 12 we get a $C_{12}$-factor.

The $C_{6}$ factors are obtained by taking differences:

| $G_{0}$ to $G_{1}$ | $G_{1}$ to $G_{2}$ | $G_{2}$ to $G_{3}$ | total $\bmod 12$ |
| :---: | :---: | :---: | :---: |
| 0 | 3 | 3 | 6 |
| 1 | 1 | 4 | 6 |
| 2 | 2 | 2 | 6 |
| 3 | 9 | 6 | 6 |
| 4 | 4 | 10 | 6 |

The $C_{12}$-factors are obtained by taking differences:

| $G_{0}$ to $G_{1}$ | $G_{1}$ to $G_{2}$ | $G_{2}$ to $G_{3}$ | total mod 12 |
| :---: | :---: | :---: | :---: |
| 5 | 10 | 5 | 8 |
| 6 | 6 | 8 | 8 |
| 7 | 0 | 9 | 4 |
| 8 | 5 | 7 | 8 |
| 9 | 7 | 0 | 4 |
| 10 | 11 | 11 | 8 |
| 11 | 8 | 1 | 8 |

Because all possible difference between each pair of partite sets has been taken once, this provides the desired decomposition.

Lemma 2.11.2 There is a decomposition of $C_{(12: n)}$ into $7 C_{3 n}$-factors and $5 C_{2 n}$ factors, for any $n \geq 5$.

Proof: We have to pick differences between $n$ pairs of partite sets. For the first $n-3$ differences sets choose differences that add up to 0 . This can be achieved by taking the difference one would take to decompose $C_{(12: n-3)}$ into $C_{n-3}$-factors. For the last three differences, choose in the same way as we did in the previous lemma.

## Chapter 3

## Finite Abelian Groups And

## Sequenceability ${ }^{1}$

### 3.1 Introduction

In 1961 B. Gordon [19] defined a group $G$ to be sequenceable when there exists a permutation

$$
g_{0}, g_{1}, g_{2}, \ldots, g_{n-1}
$$

[^2]of its elements so that the sequence of partial products
$$
g_{0}, g_{0} g_{1}, g_{0} g_{1} g_{2}, \ldots, g_{0} g_{1} g_{2} \cdots g_{n-1}
$$
are distinct. In that same paper he proved the following theorem.

Theorem 3.1.1 A finite abelian group $G$ is sequenceable if and only if its Sylow 2-subgroup is non-trivial and cyclic.

In 1974 G. Ringel [36] asked when there exists a permutation

$$
g_{1}, g_{2}, \ldots, g_{n-1}
$$

of the non-identity elements of a group such that the sequence

$$
g_{2} g_{1}^{-1}, g_{3} g_{2}^{-1}, \ldots, g_{n-1} g_{n-2}^{-1}, g_{1} g_{n-1}^{-1}
$$

also is a permutation of the non-identity elements. A group $G$ that admits such a permutation is called $R$-sequenceable. As a matter of fact, L. Paige [33] used this concept in 1951, but it was Ringel's problem that motivated the most important paper on this topic (discussed below).

We now provide a context which establishes the close connection between the two
concepts. Given a group $G$ and a subset $S$ of $G$ such that $S$ does not contain the identity element of $G$, we define the Cayley digraph $\overrightarrow{C a y}(G ; S)$ by letting its vertices be the elements of $G$ and having an arc $\left(g_{1}, g_{2}\right)$ if and only if $g_{2}=g_{1} s$ for some $s \in S$. One special such Cayley digraph in which we are particularly interested is when $S=G-\{1\}$, that is, the set $S$ has everything in it other than the identity element. We use the special notation $\vec{K}(G)$ for this Cayley digraph.

It is easy to see that a fixed element $s \in S$ generates a subdigraph consisting of directed cycles whose lengths are all $|s|$, where $|s|$ denotes the order of $s$. Thus, we obtain a factorization of $\overrightarrow{C a y}(G ; S)$ into $|S|$ directed 2-factors. We call this factorization the Cayley factorization of $\overrightarrow{C a y}(G ; S)$ and denote it by $\overrightarrow{\mathcal{F}}(G ; S)$.

If $\vec{D}$ is a subdigraph of $\overrightarrow{C a y}(G ; S)$ with $|S|$ arcs, and $\vec{D}$ has exactly one arc from each directed 2-factor in $\overrightarrow{\mathcal{F}}(G ; S)$, then we say that $\vec{D}$ is orthogonal to $\overrightarrow{\mathcal{F}}(G ; S)$. In this language, the group $G$ is sequenceable when $\vec{K}(G)$ admits an orthogonal Hamilton directed path, and $G$ is $R$-sequenceable when $\vec{K}(G)$ admits an orthogonal directed cycle of length $|G|-1$.

In spite of the similarity between these two concepts, they arose from quite different settings. Gordon was interested in complete Latin squares, whereas, Ringel was considering embeddings of complete graphs into orientable surfaces of positive genus.

We now say a few words about some notational conventions in this paper. We use
$(x, y)$ to denote an arc from $x$ to $y$ in a digraph, and $x y$ to denote an edge joining $x$ and $y$ in a graph. Continuing in this vein, $\left(x_{1}, x_{2}, x_{3}, \ldots, x_{n}\right)$ denotes a directed path of length $n-1,\left(x_{1}, x_{2}, \ldots, x_{n}, x_{1}\right)$ denotes a directed cycle of length $n, x y$ denotes an edge joining $x$ and $y$ in a graph, $x_{1} x_{2} \ldots x_{n}$ denotes a path of length $n-1$ in a graph and $x_{1} x_{2} \ldots x_{n} x_{1}$ denotes a cycle of length $n$ in a graph. We use cyclic notation for permutations and in order to distinguish permutations from directed paths, we are careful with the exposition. Thus, as a permutation, $(1,2,3,4)$ is the cyclic permutation mapping 4 to 1 , and $i$ to $i+1$ for $i=1,2,3$.

For the rest of this paper, we consider only finite abelian groups and use additive notation with one exception. For the direct sum of $a$ copies of the cyclic group $Z_{n}$, we write $Z_{n}^{a}$ rather than $a Z_{n}$.

As mentioned above, R. Friedlander, B. Gordon and M. Miller [18] wrote the most significant paper on Ringel's problem. They conjectured that if $G$ is a finite abelian group whose Sylow 2-subgroup is either trivial or both non-trivial and non-cyclic, then $G$ is $R$-sequenceable. (In other words, the conjecture is saying that if $G$ is not covered by Theorem 3.1.1, then it is $R$-sequenceable.) They established that the conjecture holds in many cases and introduced the following important strengthening of $R$-sequenceability. If $\vec{C}=\left(g_{1}, g_{2}, \ldots, g_{n-1}, g_{1}\right)$ is a directed cycle of length $n-1$ that is orthogonal to $\vec{K}(G)$, where $G$ is an abelian group of order $n$, with the additional properties that 0 is the vertex missed by $\vec{C}$, and there exist three successive elements
$g_{i}, g_{i+1}, g_{i+2}$ on $\vec{C}$ such that $g_{i}+g_{i+2}=g_{i+1}$, then we say that $G$ is $R^{*}$-sequenceable. We sometimes say that $g_{1}, g_{2}, \ldots, g_{n-1}$ is an $R^{*}$-sequence.

Friedlander, Gordon and Miller made considerable progress on the conjecture in [18], but did not solve it completely. Nevertheless, several of their results are important tools for the general conjecture. Some of the missing cases were settled in [21, 31, 37]. The proof of the conjecture is completed in this paper. We express the completion in the form of the following theorem that includes all finite abelian groups.

Theorem 3.1.2 If $G$ is a finite abelian group, then the following hold:
(1) $G$ is sequenceable if the Sylow 2-subgroup is cyclic and non-trivial; and
(2) $G$ is $R$-sequenceable if the Sylow 2-subgroup either is trivial, or the Sylow 2subgroup is non-trivial and non-cyclic.

### 3.2 First Stage of Proof

Part (1) of Theorem 3.1.2 is covered by Theorem 3.1.1. So we move to part (2) which has a natural partition into two subcases. The first subcase is that $G$ has even order with its Sylow 2-subgroup non-trivial and non-cyclic. The second subcase is that $G$ has odd order, that is, the Sylow 2-subgroup is trivial. We consider the first subcase next beginning with some useful results from [18].

Lemma 3.2.1 The cyclic group $Z_{n}$ is $R^{*}$-sequenceable for all odd $n>5$.

Lemma 3.2.2 Let $G$ be an $R^{*}$-sequenceable abelian group and $Z_{n}, n>1$, an odd order cyclic group. Then the following hold:
(1) If $G$ has even order, then $G \oplus Z_{n}$ is $R^{*}$-sequenceable; and
(2) If $G$ has odd order, then $G \oplus Z_{n}$ is $R^{*}$-sequenceable whenever 3 does not divide $n$.

Lemma 3.2.3 Elementary abelian groups are $R$-sequenceable.

The next two results are from [21, 32], respectively.

Lemma 3.2.4 If $G$ is an even order abelian group and its Sylow 2-subgroup is neither $Z_{2}^{3}$ nor $Z_{2} \oplus Z_{4}$, then $G$ is $R$-sequenceable.

Lemma 3.2.5 If $G$ is $R^{*}$-sequenceable, then $Z_{2}^{3} \oplus G$ is $R^{*}$-sequenceable.

We now establish a method for handling the missing even order abelian groups. This is inspired by Häggkvist's Lemma in [20]. Consider the cycle $u_{0} u_{1} u_{2} \ldots u_{r} u_{0}$. The edge $u_{i} u_{j}$ divides the cycle into two subpaths with common end vertices $u_{i}$ and $u_{j}$. The length of the edge $u_{i} u_{j}$ is the length of the shorter of the two paths unless both subpaths have the same length in which case the length of the edge is $(r+1) / 2$.

Lemma 3.2.6 If we label the vertices of $K_{n}$ cyclically as $u_{0}, u_{1}, u_{2}, \ldots, u_{n-1}$, where $n=2 m>4$, then there is a Hamilton path whose first edge has length $m$ and every other edge length is used twice.

Proof: When $m$ is odd, start a path with the edge $u_{0} u_{m}$ which has length $m$. Continue with the edge $u_{m} u_{1}$ and then zig zag back and forth decreasing the length by one with each edge until finishing with the edge $u_{(m-1) / 2} u_{(m+1) / 2}$. We refer to this kind of path as a zig-zag path. At this point we have used one edge of each of the lengths $1,2,3, \ldots, m$.

Next we add the edge $u_{(m+1) / 2} u_{(3 m-1) / 2}$ which has length $m-1$. The unused vertices are $u_{m+1}, u_{m+2}$ through $u_{(3 m-3) / 2}$, of which there are $(m-3) / 2$ such vertices, and $u_{(3 m+1) / 2}, u_{(3 m+3) / 2}$ through $u_{2 m-1}$, of which there are $(m-1) / 2$ such vertices. We now continue with an increasing zig-zag path starting with the edge $u_{(3 m-1) / 2} u_{(3 m+1) / 2}$ and finishing with the edge $u_{m+1} u_{2 m-1}$ of length $m-2$. The resulting path satisfies the conclusions of the lemma. Figure 3.1 shows the path for $m=5$.

The solution when $m$ is even is different in that we describe an iterative procedure for which we show that it results in a path with the desired properties. We require some notation. We denote the current path by $P$ and say the terminal vertex of $P$ is the end vertex distinct from $u_{0}$. The interval $\mathrm{I}\left[u_{i}, u_{j}\right], i \leq j$, denotes the set of vertices $\left\{u_{i}, u_{i+1}, \ldots, u_{j}\right\}$.

Suppose $P$ misses the $\alpha$ vertices I $\left[u_{2 m-\alpha}, u_{2 m-1}\right]$. If, in addition, the remaining vertices missed by $P$ are $\mathrm{I}\left[u_{1}, u_{\alpha-1}\right]$ and $u_{\alpha+1}$, the terminal vertex of $P$ is $u_{\alpha}$, and the edge lengths not used twice by $P$ are $2,3, \ldots, 2 \alpha+1$, then we say the $P$ is R-sided. Note that the interval notation makes no sense when $\alpha=1$. In this case, we treat the interval $\left[u_{1}, u_{0}\right]$ as empty so that $P$ terminates at $u_{1}$ and the vertex $u_{2}$ is not on $P$.

The other possibility is that the remaining vertices missed by $P$ are $\mathrm{I}\left[u_{1}, u_{\alpha+2}\right]$ and $u_{2 m-\alpha-2}$, the terminal vertex of $P$ is $u_{2 m-\alpha-1}$, and the edge lengths not used twice by $P$ are $2,3, \ldots, 2 \alpha+4$. In this case we say the $P$ is L-sided. The interval notation makes no sense here for $\alpha=0$. So we treat the interval $\left[u_{2 m}, u_{2 m-1}\right.$ ] as empty and maintain the remaining conditions.

If $P$ is R-sided with $\alpha \geq 3$, then extend $P$ by adding the 3 -path $u_{\alpha} u_{2 m-\alpha} u_{\alpha+1}$
$u_{2 m-\alpha+2}$. These new edges have lengths $2 \alpha-1,2 \alpha, 2 \alpha+1$ and the terminal vertex of the updated $P$ is now $u_{2 m-\alpha+1}$. Thus, $P$ is now L-sided and $\alpha$ has decreased by 3 .

On the other hand, if $P$ is L-sided with $\alpha \geq 1$, then extend $P$ by adding the 3 -path $u_{2 m-\alpha-1} u_{\alpha+2} u_{2 m-\alpha-2} u_{\alpha}$. These new edges have lengths $2 \alpha+1,2 \alpha+2,2 \alpha+3$ and the terminal vertex of the updated $P$ is now $u_{\alpha}$. Thus, $P$ is now R-sided and $\alpha$ has not changed.


Figure 3.1: A path in $K_{10}$ satisfying the conditions of Lemma 3.2 .6


Figure 3.2: A path in $K_{16}$ satisfying the conditions of Lemma 3.2.6
Construct the initial path $P$ by starting with the edge $u_{0} u_{m}$. Then add an increasing zig-zag path starting with the 2 -path $u_{m} u_{m-1} u_{m+1}$ and continue until finishing with the edge from $u_{(3 m-2) / 2}$ to $u_{m / 2}$ of length $m-1$. Then add the 3 -path $u_{m / 2} u_{(3 m+2) / 2} u_{3 m / 2} u_{(m-4) / 2}$ to complete the initial $P$. Note that $P$ is an R-sided path with $\alpha=(m-4) / 2$.

We now begin iterations of the procedure described above and may continue until
we reach a path $P$ that is L-sided with $\alpha=0$, or R -sided with $\alpha \in\{1,2\}$. If $P$ is L-sided with $\alpha=0$, then $P$ terminates at $u_{2 m-1}$, is missing the vertices $u_{1}, u_{2}, u_{2 m-2}$ and requires edges of lengths 2,3 , and 4 . The completion $u_{2 m-1} u_{1} u_{2 m-2} u_{2}$ does the job.

If $P$ is R-sided with $\alpha=1$, then the terminal vertex is $u_{1}$, the missing vertices are $u_{2 m-1}, u_{2}$, and the unused lengths are 2 and 3 . The completion $u_{1} u_{2 m-1} u_{2}$ works.

If $P$ is R-sided with $\alpha=2$, then the terminal vertex is $u_{2}$, the missing vertices are $u_{2 m-2}, u_{2 m-1}, u_{1}, u_{3}$, and the unused lengths are $2,3,4$ and 5 . There is no completion for this case. If this is the initial $P$, then $m=8$ and Figure 3.2 gives a solution for $m=8$. If this is not the initial $P$, then before the last iteration $P$ was L-sided with $\alpha=2$. So the vertices missed by $P$ are $u_{2 m-4}, u_{2 m-2}, u_{2 m-1}, \mathrm{I}\left[u_{1}, u_{4}\right]$, the terminal vertex is $u_{2 m-3}$, and the missing lengths are 2 through 8 . The completion that works is

$$
u_{2 m-3} u_{3} u_{2 m-4} u_{4} u_{2 m-1} u_{1} u_{2 m-2} u_{2}
$$

This completes the proof.

Lemma 3.2.6 allows us to complete the even order case. Suppose the Sylow 2-subgroup of $G$ is $Z_{2} \oplus Z_{4}$. If this is the entire group $G$, then

$$
(0,2),(1,3),(0,3),(1,1),(1,0),(1,2),(0,1)
$$

is an $R$-sequence.

Write $G$ as the direct sum of its Sylow subgroups. From the preceding paragraph we may assume that there is a summand of the form $Z_{q}$, where $q$ is an odd prime power. So $G$ is a direct sum of $Z_{2} \oplus Z_{4} \oplus Z_{q} \cong Z_{2} \oplus Z_{4 q}$ and an odd order abelian group.

Lemma 3.2.6 tells us that there is a path $P$ (undirected) of length $4 q-1$ in $K_{4 q}$, where we are thinking of this as a Cayley graph on $Z_{4 q}$, such that an initial edge of $P$ has length $2 q$ (that is, joins 0 and $2 q$ ) and all remaining edge lengths occur twice in $P$. Display the vertices of $Z_{4 q} \oplus Z_{2}$ as a $2 \times 4 q$ array with the obvious coordinate system from $Z_{2}$ and $Z_{4 q}$.

Build an undirected cycle $C$ of length $8 q-1$ as follows. Join $(0,1)$ to both $(2 q, 0)$ and $(2 q, 1)$. Given two edges $g_{1} g_{2}$ and $g_{3} g_{4}$ of the same length in $P$, join $\left(g_{1}, 0\right)$ to $\left(g_{2}, 0\right)$ and $\left(g_{1}, 1\right)$ to $\left(g_{2}, 1\right)$, and join $\left(g_{3}, 0\right)$ to $\left(g_{4}, 1\right)$ and $\left(g_{3}, 1\right)$ to $\left(g_{4}, 0\right)$. Finally, if $g$ is the terminal vertex of $P$ distinct from 0 , join $(g, 0)$ to $(g, 1)$.

The preceding construction yields a cycle $C$ (undirected) of length $8 q-1$. Note that the vertex $(0,0)$ is not included in $C$. Also note that three successive vertices are $(2 q, 0),(0,1),(2 q, 1)$ and $(2 q, 1)+(2 q, 0)=(0,1)$. Hence, if we direct $C$ in either direction to obtain a directed cycle, both directed cycles provide an $R^{*}$-sequence for $Z_{2} \oplus Z_{4 q}$. As the remaining summands in the direct sum of $G$ have odd order, we may apply part (1) of Lemma 3.2 .2 as many times as required to obtain that $G$ is

## $R^{*}$-sequenceble.

If the Sylow 2-subgroup of $G$ is $Z_{2}^{3}$, then Lemma 3.2 .3 takes care of the case that $G \cong Z_{2}^{3}$, and Lemmas 3.2.1 and 3.2.5 takes care of the case that there is a cyclic group of odd order bigger than 5 in the direct sum of Sylow p-subgroups. Also, if both $Z_{3}$ and $Z_{5}$ appear in the Sylow subgroups of $G$, then Lemma 3.2.1 tells us that $Z_{15}$ is $R^{*}$-sequenceable. Lemma 3.2 .5 then takes care of this situation.

So we are left with groups of the form $Z_{2}^{3} \oplus Z_{3}^{a}$ and $Z_{2}^{3} \oplus Z_{5}^{b}$, where $a, b>0$. Following are $R^{*}$-sequences for $Z_{2}^{3} \oplus Z_{3} \cong Z_{2}^{2} \oplus Z_{6}$ and $Z_{2}^{3} \oplus Z_{5}$, respectively:

$$
\begin{array}{r}
(0,0,1),(0,1,1),(0,1,0),(0,0,5),(1,0,0),(1,0,1),(0,0,4),(1,1,0) \\
(1,1,4),(1,0,5),(1,1,2),(1,1,5),(0,1,5),(1,0,2),(0,1,3),(1,1,1) \\
(1,0,3),(0,1,2),(0,1,4),(0,0,2),(1,0,4),(0,0,3),(1,1,3)
\end{array}
$$

and

$$
\begin{aligned}
& (0,0,1),(1,1,6),(1,1,5),(0,1,9),(0,1,5),(0,0,3),(1,1,3),(0,1,2), \\
& (1,1,4),(0,1,0),(1,0,3),(0,1,7),(0,0,6),(0,0,9),(1,0,9),(1,0,1), \\
& (0,1,3),(1,1,8),(1,0,4),(1,0,8),(0,1,6),(1,1,9),(1,1,7),(1,0,2), \\
& (0,1,1),(0,0,2),(0,1,4),(1,1,1),(0,0,7),(0,0,8),(0,1,8),(0,0,5), \\
& (1,1,2),(1,0,6),(0,0,4),(1,0,5),(1,0,0),(1,0,7),(1,1,0)
\end{aligned}
$$

We then use part (1) of Lemma 3.2 .2 to obtain that $G$ is $R^{*}$-sequenceable for both forms. This completes the proof of Theorem 3.1 .2 when $G$ has even order.

### 3.3 The Gadget

To complete the proof of Theorem 3.1.2 for groups of odd order, we first state the following corollary which is an easy consequence of Lemma 3.2.1 and Lemma 3.2.2.

Corollary 3.3.1 If $G$ is an odd order abelian group whose Sylow 3-subgroup either is trivial, or non-trivial and cyclic, or $R^{*}$-sequenceable, then $G$ itself is $R^{*}$-sequenceable
unless $G \cong Z_{3}$ or $G \cong Z_{5}$ both of which are $R$-sequenceable.

The preceding corollary means that we need only show that abelian groups whose Sylow 3-subgroups are non-trivial and non-cyclic are $R$-sequenceable. The method we employ works, in fact, for all odd order groups and there is no gain in efficiency by restricting ourelves to those groups satisfying the preceding condition on the Sylow 3 -subgroups. Thus, we present the general method.

We work with direct sums. Given the direct sum $G \oplus H$, we shall display the vertices as an $|H| \times|G|$ array, where the columns correspond to the elements of $G$ and the rows correspond to elements of $H$. We develop some lemmas which prove to be very useful, but we need a definition first.

Definition 3.3.2 Let $f$ be a permutation of $H$ and let $g_{1}, g_{2} \in G$. We define the $f$-lift of the arc $\left(g_{1}, g_{2}\right)$ onto $\vec{K}(G \oplus H)$ to be the set of $\operatorname{arcs}\left\{\left(\left(g_{1}, h\right),\left(g_{2}, f(h)\right)\right): h \in H\right\}$. We denote this set of arcs by $\pi_{f}\left(g_{1}, g_{2}\right)$.

In spite of the fact we use functional notation for permutations, we compose permutations from left to right because we move through the arrays from left to right. This gives us the composition rule $(f g)(x)=g(f(x))$.

Lemma 3.3.1 Let $G$ and $H$ be abelian groups. If $\left(g_{1}, g_{2}, \ldots, g_{r+1}\right)$ is a directed path
in $\vec{K}(G)$ of length $r$, and $f_{1}, f_{2}, \ldots, f_{r}$ are permutations of $H$, then the set of arcs

$$
\pi_{f_{1}}\left(g_{1}, g_{2}\right) \cup \pi_{f_{2}}\left(g_{2}, g_{3}\right) \cup \cdots \cup \pi_{f_{r}}\left(g_{r}, g_{r+1}\right)
$$

forms $n=|H|$ vertex-disjoint directed paths of length $r$ in $\vec{K}(G \oplus H)$, where the last vertex of the directed path with initial vertex $\left(g_{1}, h\right)$ is $\left(g_{r+1}, f_{1} f_{2} \cdots f_{r}(h)\right)$.

If $\left(g_{1}, g_{2}, \ldots, g_{r}, g_{1}\right)$ is a directed cycle in $\vec{K}(G)$ of length $r$, and $f_{1}, f_{2}, \ldots, f_{r}$ are permutations of $Z_{n}$, then the set of arcs

$$
\pi_{f_{1}}\left(g_{1}, g_{2}\right) \cup \pi_{f_{2}}\left(g_{2}, g_{3}\right) \cup \cdots \cup \pi_{f_{r}}\left(g_{r}, g_{1}\right)
$$

forms vertex-disjoint directed cycles. The number of directed cycles equals the number of cycles in the disjoint cycle decomposition of $f_{1} f_{2} \cdots f_{r}$.

Proof: It is easy to see that $\pi_{f}\left(g_{1}, g_{2}\right)$ for any permutation $f$ of $H$ generates an orientation of a perfect matching between vertices whose first coordinate is $g_{1}$ and vertices whose first coordinate is $g_{2}$ so that every arc is oriented from $g_{1}$ to $g_{2}$. It then follows directly that we obtain $n$ vertex-disjoint directed paths as claimed.

If we consider the directed path starting at $\left(g_{1}, h\right)$, it is straightforward to see that its terminal vertex is $\left(g_{r+1}, f_{1} f_{2} f_{3} \cdots f_{r}(h)\right)$.

The argument for a directed cycle in $\vec{K}(G)$ is essentially the same except that $\pi_{f_{r}}$ generates an arc from vertices in $G \oplus H$ whose first coordinate is $g_{r}$ to vertices whose first coordinate is $g_{1}$. It is then easy to see that a cycle of length $t$ in the disjoint cycle decomposition of $f_{1} f_{2} f_{3} \cdots f_{r}$ generates a directed cycle of length $r t$ in $G \oplus H$. The rest of the lemma now follows.

Lemma 3.3.1 gives us a way of controlling arcs in $\vec{K}(G \oplus H)$. But we really would like the arcs in the projection of an arc of $\vec{K}(G)$ to be generated by distinct elements of $G \oplus H$. This leads naturally to a known type of permutation. A permutation $f: H \rightarrow H$ is an orthomorphism if the function $g(x)=f(x)-x$ also is a permutation. The next lemma tells us that orthomorphisms are precisely what we need.

Lemma 3.3.2 Let $G$ and $H$ be abelian groups. If $f$ is an orthomorphism of $H$, then the arcs of $\pi_{f}\left(g_{i}, g_{j}\right)$ in $\vec{K}(G \oplus H)$ are generated by the group elements $\left(g_{j}-g_{i}, h\right)$ as $h$ runs through $H$.

Proof: This follows immediately from the definition of an orthomorphism.

There are some special orthomorphisms we use. Let $|H|$ be odd and define the permutation $T_{0}$ on $H$ by $T_{0}(h)=-h$ for $h \in H$. It is easy to see that $T_{0}$ is an orthomorphism because $H$ contains no involutions. We extend this particular orthomorphism to $T_{a}$, $a \in H$, by defining $T_{a}(h)=2 a-h$. It is straightforward to check that $T_{a}$ also is
an orthomorphism. An important feature of these particular orthomorphisms is the following. When $H \cong Z_{n}, n$ odd, then the composition

$$
\begin{equation*}
T_{0} T_{1}=h+2=(0,2, \ldots, n-1,1,3, \ldots, n-2) \tag{3.1}
\end{equation*}
$$

that is, the product is an $n$-cycle.

If $G$ is an $R^{*}$-sequenceable abelian group of order $m$, then we have a directed cycle of length $m-1$ that misses the vertex 0 and has three successive vertices $a, b, c$ for which $a+c=b$. Label the vertices of the directed cycle in succession as $g_{1}, g_{2}, \ldots, g_{m-1}$ so that $a=g_{1}, b=g_{2}, c=g_{3}$. The canonical labelling of the group $G \oplus H$ has the columns labelled so that the leftmost column is labelled $g_{1}$, the next column is labelled 0 , and the remaining columns are labelled $g_{2}$ through $g_{m-1}$ from left to right in that order.

We want to prove that $G \oplus H$ is $R^{*}$-sequenceable whenever possible. It is natural to work with lifts of arcs of the directed cycle in $\vec{K}(G)$, but this directed cycle misses the vertex 0 so that we need to get the vertices of the column labelled 0 involved. We
now describe how to do so.

Definition 3.3.3 Suppose that $G$ is an abelian group with non-zero elements $g_{1}, g_{2}, g_{3}$ satisfying $g_{1}+g_{3}=g_{2}$. Consider $G \oplus H$ with $H$ abelian of odd order $n \geq 3$. The lifts $\pi_{T_{0}}\left(g_{1}, g_{2}\right) \cup \pi_{T_{0}}\left(g_{2}, g_{3}\right)$ consist of $n$ vertex-disjoint directed paths of length 2 using all the vertices of columns $g_{1}, g_{2}, g_{3}$, and whose arcs are generated by $\left(g_{2}-g_{1}, h\right)$ and $\left(g_{3}-g_{2}, h\right)$ as $h$ runs through $H$.

Now for each pair $h,-h$ of additive inverses, replace the pair of directed 2-paths

$$
\left(\left(g_{1}, h\right),\left(g_{2},-h\right),\left(g_{3}, h\right)\right) \text { and }\left(\left(g_{1},-h\right),\left(g_{2}, h\right),\left(g_{3},-h\right)\right), h \neq 0
$$

by the directed 3-paths

$$
\left(\left(g_{1}, h\right),(0,-h),(0, h)\left(g_{3},-h\right)\right) \text { and }\left(\left(g_{1},-h\right),\left(g_{2}, h\right),\left(g_{2},-h\right)\left(g_{3}, h\right)\right.
$$

The directed 2-path $\left(\left(g_{1}, 0\right),\left(g_{2}, 0\right),\left(g_{3}, 0\right)\right)$ is left unaltered. The new collection of directed paths is called the gadget on columns $g_{1}, 0, g_{2}, g_{3}$.

Lemma 3.3.3 The arcs of the gadget on columns $g_{1}, 0, g_{2}, g_{3}$ are generated by the elements $\left(g_{2}-g_{1}, h\right),\left(g_{3}-g_{2}, h\right),\left(0, h^{\prime}\right)$ for all $h \in H$ and all $h^{\prime} \neq 0$ in $H$. Moreover, the terminal vertex of the directed path whose initial vertex is $\left(g_{1}, h\right)$ is $\left(g_{3},-h\right)$.

Proof: The new arc $\left(\left(g_{1}, h\right),(0,-h)\right)$ of the gadget is generated by the group element $\left(g_{3}-g_{2},-2 h\right)$ because $g_{1}+g_{3}=g_{2}$. Similarly, the $\operatorname{arc}\left((0, h),\left(g_{3},-h\right)\right)$ is generated by the group element $\left(g_{2}-g_{1},-2 h\right)$. The two vertical $\operatorname{arcs}((0,-h),(0, h))$ and $\left(\left(g_{2}, h\right),\left(g_{2},-h\right)\right)$ are generated by the group elements $(0,2 h)$ and $(0,-2 h)$. Finally, the $\operatorname{arc}\left(\left(g_{1},-h\right),\left(g_{2}, h\right)\right)$ is generated by $\left(g_{2}-g_{1}, 2 h\right)$, and the $\operatorname{arc}\left(\left(g_{2},-h\right),\left(g_{3}, h\right)\right)$ is generated by $\left(g_{3}-g_{2}, 2 h\right)$. Hence, the claims about which group elements generate the arcs of the gadget follow.

It is easy to see that the directed path beginning at $\left(g_{1}, h\right)$ terminates at $\left(g_{3},-h\right)$ for all $h \in Z_{n}$.

The next lemma is the basis for establishing Theorem 3.1 .2 when $G$ has odd order.

Lemma 3.3.4 Let $G$ be an $R^{*}$-sequenceable abelian group of order $m$. If $H$ is an odd order abelian group for which there are orthomorphisms $f_{1}, f_{2}, \ldots, f_{t}$ of $H$ such that $T_{0} f_{1} f_{2} \cdots f_{t}$ is an $|H|$-cycle and $m-t-3 \geq 0$ is even, then $G \oplus H$ is $R^{*}$-sequenceable.

Proof: We use the canonical labelling of $G \oplus H$. The first four columns of the array correspond to the group elements $g_{1}, 0, g_{2}, g_{3}$ in that order, where $g_{1}+g_{3}=g_{2}$. Employ the gadget on these first four columns. Because of Lemma 3.3.3, it follows that if for each remaining $\left(g_{i}, g_{i+1}\right)$ and $\left(g_{r-1}, g_{1}\right)$, we employ a lift arising from an orthomorphism of $H$, the arcs will have been generated by all elements of $G \oplus H$ other
than $(0,0)$. Moreover, the vertex $(0,0)$ is isolated and the vertices $\left(g_{1}, 0\right),\left(g_{2}, 0\right),\left(g_{3}, 0\right)$ occur in succession. Because $\left(g_{1}, 0\right)+\left(g_{3}, 0\right)=\left(g_{2}, 0\right)$, if the arcs form a single directed cycle, then $G \oplus H$ is $R^{*}$-sequenceable.

From Lemma 3.3.3, the permutation from column $g_{1}$ to column $g_{3}$ is $T_{0}$. We then successively employ the orthomorphisms $f_{1}, f_{2}, \ldots, f_{t}$ for the following lifts. By hypothesis, the product $T_{0} f_{1} f_{2} \cdots f_{t}$ is a cycle of length $|H|$.

There are $m-(t+3)$ further lifts to be employed. If $m-(t+3)=0$, we already have an $|H|$-cycle and we are done. If $m-(t+3)>0$, then it is even and we use $T_{0}$ for each subsequent lift. The product of an even number of $T_{0}$ permutations is the identity as $T_{0}$ is an involution. Thus, the final product is a cycle of length $|H|$ completing the proof.

This method of lifts brings to the fore why the prime 3 is a nagging problem. For $a \in$ $Z_{n}$ satisfying $\operatorname{gcd}(n, a)=1$, let $M_{a}$ denote the permutation of $Z_{n}$ defined by $M_{a}(x)=$ $a x$. When 3 does not divide $n$, it is straightforward to check that both $M_{2}$ and $M_{(n-1) / 2}$ are orthomorphisms. Note that $M_{2} M_{(n-1) / 2}=T_{0}$. Then $T_{0} M_{2} M_{(n-1) / 2} T_{0} T_{1}=T_{0} T_{1}$ is an $n$-cycle and Lemma 3.2 .2 applies for $m \geq 7$. When 3 divides $n$, unfortunately, $M_{(n-1) / 2}$ is not an orthomorphism forcing us to find special arguments for the prime 3. This is what we now examine.

Corollary 3.3.4 If $G$ is an $R^{*}$-sequenceable abelian group of odd order, then $G \oplus Z_{3}$ e
is $R^{*}$-sequenceable for $e \geq 2$.

Proof: It is easy to verify that the permutations $f_{0}=T_{0}, f_{1}=M_{2}$, and $f_{2}=$ $(0,1)(2,6,3,5,8,4)(7)$ satisfy $f_{0} f_{1} f_{2}=(0,1,7,2,8,6,3,5,4)$ for $e=3$. This means that $G \oplus Z_{9}$ is $R^{*}$-sequenceable when $G$ is $R^{*}$ - sequenceable according to Lemma 3.3.4.

For $e=3$, let $f_{0}=T_{0}$. Let $f_{1}$ and $f_{2}$ be the following permutations, respectively:

$$
(0,26,3,8,19,7,10,16,5,24,17,12,20,14,4,22,23,25,11,18,1,13,9,6,15,2)(21)
$$

and

$$
(0,22,21,13,11)(1,6,7)(2,8,15,5,23,10,19,4,24,20,3,16,18,26,14,25)(9,12)(17) .
$$

Again it is easy to verify that the permutation $f_{0} f_{1} f_{2}$ is a 27 -cycle as required. Lemma 3.3.4 then implies that $G \oplus Z_{27}$ is $R^{*}$-sequenceable when $G$ is $R^{*}$-sequenceable.

We now want to show that $G \oplus Z_{3 e}$ is $R^{*}$-sequenceable, when $G$ is $R^{*}$-sequenceable, for all $e \geq 2$ and we proceed by induction on $e$ having established the result for $e=2,3$. Consider $e \geq 4$. Let $N$ be the subgroup of $Z_{3^{e}}$ of order $3^{e-2}$ so that $Z_{3^{e}} / N$ is isomorphic to $Z_{9}$. Use $0,1, \ldots, 8$ as the coset representatives and let $\bar{x}$ correspond
to the element $N+x$ in the quotient group of order 9 .

From above we know there are three orthomorphisms $\overline{f_{0}}, \overline{f_{1}}, \overline{f_{2}}$ of $Z_{3^{e}} / N$ so that $\overline{f_{0} f_{1} f_{2}}=(\overline{0}, \overline{1}, \overline{7}, \overline{2}, \overline{8}, \overline{6}, \overline{3}, \overline{5}, \overline{4})$, and $\overline{f_{0}}(\overline{0})=\overline{f_{1}}(\overline{0})=\overline{0}$ and $\overline{f_{2}}(\overline{0})=\overline{1}$. Suppose that $\overline{f_{i}}(\bar{x})=\bar{y}$. Then let $\alpha$ be any orthomorphism of $N$. Define the $\alpha$-lift action of $f_{i}$ on $N+x$ by letting $f_{i}(n+x)=\alpha(n)+y, n \in N$. It is easy to see that $f_{i}$ acting on the coset $N+x$ picks up all elements of the form $N+(y-x)$ via $f_{i}(n+x)-(n+x)$. Thus, $f_{i}$ is an orthomorphism of $Z_{3^{e}}$ if the action on each coset is defined via the lift of an orthomorphism of $N$ as just described.

We now define $f_{0}, f_{1}, f_{2}$ to ensure that $f_{0} f_{1} f_{2}$ is a cycle of length $3^{e}$. Let $\alpha_{0}, \alpha_{1}, \alpha_{2}$ be orthomorphisms of $N$ such that $\alpha_{0} \alpha_{1} \alpha_{2}$ is a cycle of length $3^{e-2}$ on $N$ by induction. We have that $\overline{f_{0}}$ maps $\overline{0}$ to itself. We use the lift of the orthomorphism $\alpha_{0}$ on $N$ to define $f_{0}$ on $N$. Continuing, we know that $\overline{f_{1}}$ also maps $\overline{0}$ to $\overline{0}$. We use the lift of $\alpha_{1}$ to define $f_{1}$ acting on $N$. Finally, to get the action of $f_{2}$ on $N$, use the lift of $\alpha_{2}$ to define the action of $f_{2}$ mapping $N$ to $N+1$. For all other lifts, use $T_{0}$ on $N$.

We claim that $f_{0} f_{1} f_{2}$ is a cycle of length $3^{e}$. To see this, first note that $f_{0} f_{1} f_{2}$ acts as $[\overline{0}, \overline{1}, \overline{7}, \overline{2}, \overline{8}, \overline{6}, \overline{3}, \overline{5}, \overline{4}]$ on the cosets. Because $\alpha_{0} \alpha_{1} \alpha_{2}$ is a cycle of length $3^{e-2}$ on $N$ and we use the lifts of these three orthomorphisms to give the action of $f_{0}, f_{1}, f_{2}$ on $N$, we see that if $\alpha_{0} \alpha_{1} \alpha_{2}\left(n_{1}\right)=n_{2}$, then $f_{0} f_{1} f_{2}\left(n_{1}\right)=n_{2}+1$. All remaining lifts use $T_{0}$ and there are an even number of them so that $f_{0} f_{1} f_{2}$ is a full cycle of length $9 \cdot 3^{e-2}=3^{e}$ as required.

If every summand in the Sylow 3-subgroup has order at least 9 , then any summand is $R^{*}$-sequenceable by Lemma 3.2.1. Repeated applications of Corollary 3.3.4 yield that the Sylow 3 -subgroup is $R^{*}$-sequenceable. Corollary 3.3.1 then implies that $G$ is $R^{*}$-sequenceable.

When exactly one summand in the Sylow 3 -subgroup is $Z_{3}$, we require a lemma. Two useful items for the proof are given first.

The following are $R^{*}$-sequences for $Z_{3} \oplus Z_{9}$ and $Z_{3} \oplus Z_{27}$, respectively:

$$
\begin{aligned}
& (2,0),(2,3),(0,3),(0,5),(1,2),(2,4),(1,1),(1,8)(0,1),(1,4),(1,0),(2,1),(2,2), \\
& (2,8),(1,6),(0,2),(1,7),(0,8),(1,3),(0,7),(1,5),(0,4),(2,7),(0,6),(2,6),(2,5)
\end{aligned}
$$

and

$$
\begin{aligned}
& (0,1),(0,26),(0,25),(1,24),(2,10),(1,11),(1,25),(0,11),(2,16),(0,8), \\
& (2,26),(1,15),(0,14),(2,4),(1,23),(0,23),(2,20),(1,8),(2,15),(1,0), \\
& (0,10),(0,17),(1,19),(2,14),(0,19),(1,20),(1,13),(1,7),(1,18),(1,3) \text {, } \\
& \quad(2,13),(2,17),(2,7),(0,22),(2,25),(1,6),(0,20),(2,0),(2,8),(2,5), \\
& \quad(0,2),(1,10),(2,1),(2,3),(0,7),(2,18),(2,9),(0,18),(1,14),(0,12) \text {, } \\
& (1,26),(1,2),(0,6),(2,12),(2,22),(1,4),(2,2),(2,21),(0,21),(2,23) \text {, } \\
& \quad(0,16),(1,22),(2,11),(2,24),(2,6),(1,1),(0,24),(1,9),(0,3),(0,4) \text {, } \\
& (0,9),(0,15),(1,5),(1,21),(1,17),(1,12),(0,5),(1,16),(2,19),(0,13) .
\end{aligned}
$$

Lemma 3.3.5 The group $G=Z_{3} \oplus Z_{3^{e}}, e \geq 2$, is $R^{*}$-sequenceable.

Proof: The statement is true for $e=2,3$ because $R^{*}$-sequences are given above. We proceed by induction on $e$ and let $e>3$. Let $N$ be the cyclic subgroup of order $3^{e-2}$. The quotient group $G / N$ is isomorphic to $Z_{3} \oplus Z_{9}$. Let the coset representatives be $\{(i, j): 0 \leq i \leq 2,0 \leq j \leq 8\}$ and let $\overline{(i, j)}$ denote the element $N+(i, j)$ of the quotient group.

Display the elements of $G$ as a $3^{e-2} \times 9$ array where the columns are cosets of the cyclic subgroup of order $3^{e-2}$ and they are written left to right in the order of the $R^{*}$-sequence for $Z_{3} \oplus Z_{9}$ given above, where column $\overline{(0,0)}$ is inserted between $\overline{(2,0)}$ and $\overline{(2,3)}$.

Even though the columns now correspond to cosets of $Z_{3^{e-2}}$ rather than the group itself-as they did earlier when we defined the lift of an arc onto the array for a direct sum - it should be clear how we define a lift now. Namely, if there is an arc from $\overline{(i, j)}$ to $\overline{\left(i^{\prime}, j^{\prime}\right)}$ in $\vec{K}(G / N)$ and $f$ is a permutation of $N$, then for each $(i, j)+n \in \overline{(i, j)}$, we have an arc to $\left(i^{\prime}, j^{\prime}\right)+f(n)$. We then use the same notation $\pi_{f}$ for the lift.

We then use $\pi_{T_{0}}$ as the lift for the arcs from $\overline{(2,0)}$ to $\overline{(2,3)}$, and from $\overline{(2,3)}$ to $\overline{(0,3)}$. Note that one of the directed paths is $(2,0),(2,3)(0,3)$ and this sequence of three vertices satisfies $(2,0)+(0,3)=(2,3)$. So if we end up with a directed cycle of length $3^{e+1}-1$, we have that $Z_{3} \oplus Z_{3^{e}}$ is $R^{*}$-sequenceable.

It is now clear that if we carry out the obvious gadget operation, we end up with directed paths of length 3, except for the unaltered directed path, whose initial and terminal vertices behave like $\pi_{T_{0}}$ from column $\overline{(2,0)}$ to column $\overline{(0,3)}$. In the proof of Corollary 3.3.4, we show that for all $e>1$ there are two orthomorphisms $f_{1}, f_{2}$ such that $T_{0} f_{1} f_{2}$ is a cycle of length of length $3^{e}$. So we use these two orthomorphisms for the next two lifts of arcs along the $R^{*}$-sequence for $Z_{3} \oplus Z_{9}$. We then use $T_{0}$ for all
subsequent lifts and this leads to a directed cycle of length $3^{e+1}-1$ as required.

We continue now with the subcase that the Sylow 3 -subgroup has exactly one $Z_{3}$ term in the direct sum. The Sylow 3-subgroup is not cyclic so that Lemma 3.3.5 and repeated applications of Corollary 3.3 .4 imply that the Sylow 3 -subgroup is $R^{*}$ sequenceable. Lemma 3.3 .1 then implies that $G$ is $R^{*}$-sequenceable.

If there are two or more $Z_{3}$ terms in the direct sum for the Sylow 3-subgroup, there is a useful fact we exploit. Let

$$
f_{1}=((0,0),(2,0),(0,2),(1,2),(1,0),(0,1))((1,1),(2,2))((2,1))
$$

and

$$
f_{2}=((0,0),(1,0),(1,1),(0,2),(2,2),(0,1))((1,2),(2,0))((2,1))
$$

be two permutations of $Z_{3} \oplus Z_{3}$. It is easy to check that both are orthomorphisms and that $T_{0} f_{1} f_{2}$ is a 9 -cycle.

We then conclude that $Z_{3} \oplus Z_{3} \oplus G$ is $R^{*}$-sequenceable when $G$ is $R^{*}$-sequenceable and has odd order from Lemma 3.3.4. So consider the Sylow 3 -subgroup $H$ itself. If $H$ has a summand $Z$ whose order is at least 9 , then both $Z$ and $Z_{3} \oplus Z$ are $R^{*}$-sequenceable by Lemma 3.2 .1 or Lemma 3.3.5. Then $H$ is $R^{*}$-sequenceable by starting with $Z$ if there are an even number of $Z_{3}$ terms in the direct sum, or starting with $Z_{3} \oplus Z$ if there
are an odd number, and using the preceding fact. Therefore, $H$ is $R^{*}$-sequenceable and Lemma 3.3.1 implies that $G$ is $R^{*}$-sequenceable.

The preceding paragraph means we are left with the subcase that the Sylow 3subgroup is $Z_{3}^{a}$ for some $a \geq 2$. If this is all of $G$, then $G$ is $R$-sequenceable by Lemma 3.2.3. So we may assume that there is a non-trivial Sylow $p$-subgroup for some prime $p>3$. If $p>5$, then we may repeatedly apply Lemmas 3.2.1, 3.2.2, and the above fact to obtain that $G$ is $R^{*}$-sequenceable.

The same process works for $p=5$ except $Z_{3} \oplus Z_{3} \oplus Z_{5}$. Following is an $R^{*}$-sequence for this group which completes the proof of Theorem 3.1.2.

$$
\begin{aligned}
& (0,0,1),(0,2,2),(0,2,1),(1,1,0),(0,2,3),(0,1,1),(0,1,2),(1,2,3),(0,0,2), \\
& (2,1,2),(0,0,4),(0,1,0),(1,0,3),(2,0,0),(2,1,3),(2,0,3),(0,1,3),(1,2,1), \\
& (2,2,1),(1,1,2),(2,1,0),(1,0,2),(1,0,0),(2,0,4),(1,1,1),(2,2,0),(2,2,2), \\
& (2,0,1),(2,2,3),(0,1,4),(2,1,1),(1,2,2),(0,2,0),(2,1,4),(1,1,4),(1,2,4), \\
& (1,1,3),(0,0,3),(1,0,4),(2,2,4),(1,2,0),(2,0,2),(1,0,1),(0,2,4) .
\end{aligned}
$$

## Chapter 4

## Conclusions and Future Work

In this dissertation research related to two problems introduced by the Austrian mathematician Gerhard Ringel is presented. The first was the Hamilton-Waterloo problem. This asks whether complete graphs can be decomposed into combinations of two 2-factors. The second asks whether the non-identity elements of a group can be written in a certain order $g_{1}, g_{2}, \ldots, g_{n-1}$ such that each non-identity element appears exactly once in the sequence $g_{2} g_{1}^{-1}, g_{3} g_{2}^{-1}, \ldots, g_{n-1} g_{n-2}^{-1}, g_{1} g_{n-1}^{-1}$.

### 4.1 The Hamilton-Waterloo Problem

### 4.1.1 Conclusions on the Hamilton-Waterloo Problem

For the Hamilton-Waterloo problem, in Chapter 1 we considered the HamiltonWaterloo problem over the complete graph $K_{v}$, were each connected component of $F_{1}$ is a cycle of size 3 and each connected component of $F_{2}$ is a cycle of size $3 x$. The results obtained are summarized in the following theorem:

Theorem 29 Let $x \geq 2, y \geq 2$, and $r, s \geq 0$ such that $r+s=\left\lfloor\frac{3 x y-1}{2}\right\rfloor$. Then there is a decomposition of $K_{3 x y}$ into $r C_{3}$-factors and s $C_{3 x}$-factors, except possibly when:

- $s=1, y \geq 3$, and $x \in\{3,31,37,41,43,47,51,53,59,61,67,69,71,79,83\}$.
- $s=1, x$ is odd and $y$ is even.
- $s=1, x \geq 6, x \equiv 2(\bmod 12)$.
- $s=1, y \geq 8$ is even and $x \equiv 10(\bmod 12)$.
- $s=1, x \geq 3$ is odd and $y$ is even.
- $1 \leq s \leq \frac{x}{2}-1, x \geq 16, x \equiv 4(\bmod 12), y$ is even.
- $1 \leq s \leq \frac{x}{2}-1, x \geq 10, x \equiv 4(\bmod 6), y$ is odd.
- $(s, x) \in\{(2,12),(4,12)\}$.
- $s=0, x=2, y=2$.
- $x=2$ and $y \in\{4,8\}$.
- $s \in\left\{3,4, \ldots \frac{3(y-1)}{2}\right\}, x=2$ and $y \geq 3$ is odd.
- $x \notin\{2,4\}$ and $y \in\{2,4,6\}$.
- $x=4$ and $y \in\{2,4\}$.
- $x=6$ and $y$ odd.

The methods used in Chapter 1 relied on Hamilton-Waterloo decompositions of the complete equipartite graphs $K_{(x: 3)}$ and $K_{(4: 3)}$. This inspired the work done in Chapter 2.

For the generalization of the Hamilton-Waterloo problem over complete equipartite graphs, in Chapter 2 decompositions of $K_{(v: m)}$ into $s F_{1}$-factors and $r F_{2}$-factors were found. The main result presented was:

Theorem 30 Let $m$ and $n$ be odd, such that $m \equiv 0(\bmod n)$. Let $s$ and $r$ be such that $s, r \neq 1$ and $s+r=v \frac{m-1}{2}$. Let $x_{1}, \ldots x_{m / n}, y_{1}, \ldots y_{m / n}, z_{1}, \ldots, z_{m / n}$ and $w_{1}, \ldots, w_{m / n}$ be such that:

- $\operatorname{gcd}\left(x_{i}, z_{i}\right)=\operatorname{gcd}\left(y_{i}, z_{i}\right)=1$;
- $w_{i} \notin\{2,6\}$;
- 2 divides at most one of $x_{i}, y_{i}$ and $z_{i}$;
- $v=x_{i} y_{i} z_{i} w_{i}$ if 2 divides none of $x_{i}, y_{i}, z_{i}$; and
- $v=2 x_{i} y_{i} z_{i} w_{i}$ if 2 divides one of $x_{i}, y_{i}, z_{i}$.

Furthermore, let $F_{1}$ be $a\left[\left(x_{1} z_{1} n\right)^{\frac{v}{x_{1} z_{1}}}, \ldots,\left(x_{m / n} z_{m / n} n\right)^{\frac{v}{x_{m / n} z_{m / n}}}\right]$-factor, and let $F_{2}$ be a $\left[\left(y_{1} z_{1} n\right)^{\frac{v}{y_{1} z_{1}}}, \ldots,\left(y_{m / n} z_{m / n} n\right)^{\frac{v}{y_{m / n} z_{m / n}}}\right]$-factor. Then there is a decomposition of $K_{(v: m)}$ into $s$ copies of $F_{1}$ and $r$ copies of $F_{2}$.

These results were applied to obtain decompositions of complete graphs, obtaining the following corollaries:

Corollary 4.1.1 Let $m, x$, and $y$ be integers such that:

- $z=s(x, y), w=\frac{\operatorname{gcd}(x, y)}{z} \geq 2$,
- $\frac{x y}{z}$ divides $m$,
- 4 does not divide $x$ nor $y$.
- Neither $x$ nor $y$ is 3 if $\frac{m}{w} \in\{6,12\}$.

Then there exists a decomposition of $K_{m}$ into $s C_{x}$-factors and $r C_{y}$-factors for every $s, r \neq 1$.

Corollary 4.1.2 Let $v, m, n, x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{k}$ be integers such that:

- $m \geq 3$ is odd,
- $n$ divides $m, x_{i}$, and $y_{i}$ for every $i$,
- $k=\frac{m}{n}$,
- $z_{i}=s\left(x_{i}, y_{i}\right)$,
- $\frac{x_{i} y_{i}}{z_{i} n}$ divides $v$ for each $i$,
- $x_{i}$ divides $v$ for each $i$,
- 4 does not divide $x_{i}$ nor $y_{i}$ for any $i$,
- $3 \notin\left\{x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{k}\right\}$ if $k \in\{6,12\}$.

Then there exists a decomposition of $K_{v m}$ into $s\left[x_{1}^{v n / x_{1}}, \ldots, x_{k}^{v n / x_{k}}\right]$-factors and $r$ $\left[y_{1}^{v n / y_{1}}, \ldots, y_{k}^{v n / y_{k}}\right]$-factors for every $s, r \neq 1$.

The importance of Chapter 2 goes beyond the results obtained. The equipartite product we introduced will change the way people study problems on equipartite graphs, and will become one of the main tools to attack such problems. The special greatest common divisor is an extremely interesting concept that can be studied in depth and may yield some extremely useful results.

### 4.1.2 Future Work on the Hamilton-Waterloo Problem

On the Hamilton-Waterloo problem the first goal is to show that $K_{v}$ can be decomposed into $C_{x}$-factors and $C_{y}$-factors if $\frac{x y}{\operatorname{gcd}(x, y)}$ divides $v$ for all $x, y$ odd. To do this I plan to study decompositions of $C_{(x: n)}$ into $C_{x_{1} n}$-factors and $C_{x_{2} n}$-factors, when $x_{1} x_{2}$ is odd and $\frac{x_{1} x_{2}}{s\left(x_{1}, x_{2}\right)}$ does not divide $x$.

Some further research problems on this topic are:

- Find decompositions into cycle sizes divisible by 4.
- Study the Hamilton-Waterloo problem over $K_{(v: m)}$, when $m$ is even.
- Find decompositions of $K_{(v: m)}$ into $1 F_{1}$-factor and $\left\lfloor\frac{(m-1) v}{2}\right\rfloor F_{2}$-factors.


## $4.2 \quad R$-Sequences

### 4.2.1 Conclusions on the problem of $R$-Sequences

On the problem of $R$-sequences, in Chapter 3 the problem over abelian groups was completely solved, as shown in the following theorem:

Theorem 4.2.1 If $G$ is a finite abelian group, then the following hold:
(1) $G$ is sequenceable if the Sylow 2-subgroup is cyclic and non-trivial; and
(2) $G$ is $R$-sequenceable if the Sylow 2-subgroup either is trivial, or the Sylow 2subgroup is non-trivial and non-cyclic.

The problem was studied as two separate cases, abelian groups of even order, and abelian groups of odd order. The even case was proved by finding some special kind of cycles of $\mathbb{Z}_{4 n}$, and using them to find $R^{*}$-sequences of $\mathbb{Z}_{2} \oplus \mathbb{Z}_{4 n}$. The odd case was proved by using orthomorphisms of $\mathbb{Z}_{3^{e}}$ to find $R^{*}$-sequences of $\mathbb{Z}_{n} \oplus \mathbb{Z}_{3^{e}}$.

### 4.2.2 Future Work on $R$-Sequences

In the case of $R$-Sequences, a natural research problem is to study non-abelian groups. One possible starting point is to use the even order construction from Chapter 3 to study when dihedral groups are $R$-sequenceable. Another is to develop a tool using normal subgroups and quotient groups instead of direct products, to aide in the study of $R$-sequenceable non-abelian solvable groups.

I also plan to study the concept of orthogonalizable groups. A group $G$ is called orthogonalizable if for every identity free subset $S \subset G$, the Cayley digraph $\overrightarrow{C a y}(G ; S)$ admits either an orthogonal path or an orthogonal cycle. So far the only results in this problem are over abelian groups of small size [4]. The results obtained in Chapter

3 are a first step towards finding a solution to this problem. The next step would be to work on cyclic groups. I will start by studying groups of prime order.

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