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AN ANALYSIS OF MULTIPLICATIVE REGULARIZATION

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AN ANALYSIS OF MULTIPLICATIVE REGULARIZATION

By

Elaheh Gorgin

A DISSERTATION

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Dedicated

To my mother, Sedigheh

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Abstract

Inverse problems arise in many branches of science and engineering. In order to get a good approximation of the solution of this kind of problems, the use of regularization methods is required. Tikhonov regularization is one of the most popular methods for estimating the solutions of inverse problems. This method needs a regularization parameter and the quality of the approximate solution depends on how good the regularization parameter is.

The L-curve method is a convenient parameter choice strategy for selecting the Tikhonov regularization parameter and it works well most of the time. There are some problems in which the L-curve criterion does not perform properly.

Multiplicative regularization is a method for solving inverse problems and does not require any parameter selection strategies. However, it turns out that there is a close connection between multiplicative regularization and Tikhonov regularization; in fact, multiplicative regularization can be regarded as defining a parameter choice rule for Tikhonov regularization.

In this work, we have analyzed multiplicative regularization for finite-dimensional

problems. We also have presented some preliminary theoretical results for infinite-dimensional problems. Furthermore, we have demonstrated with numerical experiments that the multiplicative regularization method produces a solution that is usually very similar to the solution obtained by the L-curve method. This method is guaranteed to define a positive regularization parameter under some conditions. Computationally, this method is not expensive and is easier to analyze compared to the L-curve method.

Chapter 1

Introduction

1.1   Linear inverse problems

Let X and Y be Hilbert spaces (that is, complete inner product spaces) and let $T : X \to Y$ be a linear operator. We will study inverse problems of the following form: given $y \in Y$, find $x \in X$ satisfying $Tx = y$.

An equation $Tx = y$ in which the operator T has certain properties represents an inverse problem. As we will see, an inverse problem is a certain kind of ill-posed problem.

The results in this section are well-known; to learn more, see [12] for further explanations.

1.1.1 Well-posed and ill-posed problems

Hadamard's definition of a well-posed problem says that a mathematical problem is well-posed if and only if the following properties hold:

1. Existence: A solution exists for all admissible data.

2. Uniqueness: The solution is unique for all admissible data.

3. Stability: The solution depends continuously on data.

If the problem is not well-posed, then we call it ill-posed. If one of the first two properties, existence or uniqueness, fails to hold, the problem is ill-posed, but we do not classify it as an inverse problem. The important property makes the equation an inverse problem is that the solution x does not depend continuously on data y (that is, we have an inverse problem if stability fails to hold). The continuity depends on the norms chosen for both spaces. A given problem can be stable under one norm and unstable under a different norm, but normally we have to work with the given norms on X and Y.

For the problem $Tx = y$, existence and uniqueness are equivalent to the existence of T^{-1}, and stability is equivalent to the continuity (boundedness) of T^{-1}. So $Tx = y$ is well-posed if and only if T^{-1} exists and is bounded.

1.1.2 Dealing with lack of existence and lack of uniqueness

If a problem is ill-posed because existence or uniqueness fails, then there are standard methods for defining a related problem that may be well-posed.

Lack of existence: If the existence property fails for $Tx = y$, it just means that $R(T)$ is not all of Y. In this case, for $y \notin R(T)$, we consider the least-squares problem instead: $\min \|Tx - y\|$ for $x \in X$.

Definition 1. *Let $T : X \to Y$ be a linear operator. If $x \in X$ satisfies*

$$\|Tx - y\| = \inf\{\|Tz - y\| \mid z \in X\}, \tag{1.1.1}$$

then x is called a least$-$squares solution of $Tx = y$.

For a given y, the problem $Tx = y$ has a least-squares solution if and only if $y \in R(T) \oplus R(T)^{\perp}$. Also $Tx = y$ has a least-squares solution for all $y \in Y$ if and only if $R(T)$ is closed, because when $R(T)$ is closed, $R(T) \oplus R(T)^{\perp} = Y$.

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Recall that the adjoint operator T^* is defined by $\langle Tx, y \rangle_Y = \langle x, T^*y \rangle_X$ for all $x \in X$ and $y \in Y$.

Theorem 2. *Let $y \in R(T) \oplus R(T)^\perp$. Then $\overline{x}$ is a least-squares solution of $Tx = y$ if and only if $T^*T\overline{x} = T^*y$.*

Lack of uniqueness: If the uniqueness property fails (but existence holds), that is, if the problem $Tx = y$ has more than one solution, we pick the solution of smallest norm and we reformulate the problem as min $\|x\|_X$ such that x is a solution of problem $Tx = y$. Because T is linear, uniqueness is related to $N(T)$. To be precise, if $Tx = y$ has multiple solutions, then $N(T)$ is nontrivial. Conversely, if $N(T)$ is nontrivial and $Tx = y$ has a solution, then this solution is not unique.

If $\hat{x}$ is a solution of $Tx = y$ and $N(T)$ is nontrivial, then the set of all solutions of $Tx = y$ is given by

$$\hat{x} + N(T) = \{\hat{x} + u \mid u \in N(T)\}. \tag{1.1.2}$$

This set has a unique smallest element, namely, the unique element in $N(T)^\perp$. In other words, if $Tx = y$ has a solution, then it has a unique solution belonging to $N(T)^\perp$, and that is the minimum-norm solution.

If the uniqueness and existence both fail, then we pick the minimum-norm least-squares solution of the problem $Tx = y$.

Definition 3. *Let $T : X \to Y$ be a linear operator. If x is a least-squares solution of*

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$Tx = y$ *and*

$$\|x\| = \inf\{\|z\| \mid z \text{ is least-squares solution of } Tx = y\}, \qquad (1.1.3)$$

then x is called the minimum-norm least-squares solution of $Tx = y$.

The set of all least-squares solutions of $Tx = y$ is given by $\hat{x} + N(T^*T)$, which is the same as $\hat{x} + N(T)$ because $N(T^*T) = N(T)$. The set $\hat{x} + N(T)$ has a unique element in $N(T)^\perp$, which is the minimum-norm least-squares solution.

1.1.3 The Moore-Penrose generalized inverse

Recall that $Tx = y$ is well-posed if and only if T^{-1} exists and is bounded. As we discussed, if existence or uniqueness fails, we try to find the minimum-norm least-squares solution instead.

We have seen that for each $y \in R(T) \oplus R(T)^\perp$, there exists a unique minimum-norm least-squares solution of $Tx = y$. We define $T^\dagger : R(T) \oplus R(T)^\perp \to X$ by the condition $T^\dagger y$ is the minimum-norm least-squares solution of $Tx = y$. We write $D(T^\dagger) = R(T) \oplus R(T)^\perp$. The operator $T^\dagger$ is called the Moore-Penrose generalized inverse of T.

If existence or uniqueness fails (i.e., if our problem has no solution or it has more than one solution), then for $y \in D(T^\dagger)$, the best approximate solution is given by $x = T^\dagger y$.

We now see that $\bar{x} = T^\dagger y$ is characterized by the condition $T^*T\bar{x} = T^*y$ and $\bar{x} \in N(T)^\perp$. It follows that $R(T^\dagger) \subset N(T)^\perp$. Also if we define $\hat{T} = T \mid_{N(T^\perp)}: N(T)^\perp \to R(T)$, since $N(\hat{T})$ is trivial and $R(\hat{T}) = R(T)$, then $\hat{T}^{-1}$ exists. Now for all $x \in N(T)^\perp$, $T^\dagger T x = \hat{T}^{-1}\hat{T}x = x$, which implies $N(T)^\perp \subset R(T^\dagger)$. Therefore $R(T^\dagger) = N(T)^\perp$.

Since $R(T)$ is dense in $\overline{R(T)}$, then $D(T^\dagger)$ is dense in $\overline{R(T)} \oplus R(T)^\perp = Y$, and thus $T^\dagger$ is a densely defined operator. Also $T^\dagger$ is a closed operator, i.e., for $y_n \subset D(T^\dagger)$, $y_n \to y$ and $T^\dagger y_n \to x \in X$ imply that $y \in D(T^\dagger)$ and $x = T^\dagger y$. If $R(T)$ is closed, then clearly $D(T^\dagger) = Y$ and $T^\dagger : Y \to X$; therefore by the closed graph theorem, $T^\dagger$ is bounded. Also if $T^\dagger : D(T^\dagger) \to X$ and $T^\dagger$ is bounded, then by Theorem 2.15 of [12], $D(T^\dagger)$ is closed and as a result $R(T)$ is closed. Now we have the following fundamental result.

Corollary 4. *Let $T : X \to Y$ be a linear operator. Then $T^\dagger$ is bounded (i.e., continuous) if and only if $R(T)$ is closed.*

1.1.4 Compact operators

Compact operators are central to the study of the inverse problems. Except for certain degenerate cases, every compact operator K defines an inverse problem $Kx = y$.

Definition 5. *Let $K : X \to Y$ be linear. Then K is a compact operator if and only if it maps bounded sets in X to pre-compact sets in Y. Equivalently K is a compact operator if and only if*

$$\{x_n\} \subset X, \{x_n\} \text{ bounded } \Rightarrow \{Kx_n\} \text{ has a convergent subsequence.}$$

If $K : X \to Y$ is a compact operator, then clearly K is bounded because if it is not, then there exists $\{x_n\} \subset X$ with $\|x_n\|_X = 1$ for all n such that $\|Kx_n\| \to \infty$ as $n \to \infty$. That implies $\{Kx_n\}$ cannot have a convergent subsequence, which is a contradiction.

Here is the degenerate case mentioned above.

Definition 6. *Let $K : X \to Y$ be linear. Then K is a finite-rank operator if and only if $R(K)$ is finite-dimensional.*

A finite-rank operator is a type of compact operator because every bounded subset of a finite-dimensional space is pre-compact.

Theorem 7. *Let $K : X \to Y$ be bounded and finite-rank. Then K is compact.*

Theorem 8. *Let $K : X \to Y$ be compact but not finite-rank. Then $R(K)$ is not closed.*

The above theorem can be proved by way of contradiction. If we assume $R(K)$ is closed, then it is complete, so the new operator $\hat{K} = K \mid_{N(K)^{\perp}}: N(K)^{\perp} \to R(K)$ is continuously invertible. Then $K\hat{K}^{-1} = I_{R(K)}$ is compact and hence $R(K)$ is finite-dimensional.

1.1.5 The singular value expansion (SVE)

A compact operator can always be expressed as a simple form known as the singular value expansion.

Definition 9. *Let X and Y be Hilbert spaces and let $x \in X$ and $y \in Y$. The outer product $y \otimes x : X \to Y$ is the linear operator defined by*

$$(y \otimes x)u = \langle x, u \rangle y \quad \forall u \in X. \tag{1.1.4}$$

Theorem 10. *Let $K : X \to Y$ be compact. Then there exist orthonormal sequences $\{\varphi_n\} \subset X$ and $\{\psi_n\} \subset Y$ and a sequence $\{\sigma_n\}$ of positive numbers such that $\sigma_1 \geq$*

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$\sigma_2 \geq \sigma_3 \geq ... > 0$, *and*

$$K = \sum_n \sigma_n \psi_n \otimes \varphi_n. \tag{1.1.5}$$

The representation $K = \sum_n \sigma_n \psi_n \otimes \varphi_n$ is called the singular value expansion of K. If a compact operator K has infinitely many singular values $\sigma_1, \sigma_2, ...$, then $R(K)$ is non-closed, and also $\lim_{n \to \infty} \sigma_n = 0$.

Theorem 11. *Let $K : X \to Y$ be a compact linear operator and let $\sum_n \sigma_n \psi_n \otimes \varphi_n$ be the SVE of K. Then $\{\sigma_n^2\}_{\{n \in N\}}$ is the set of nonzero eigenvalues of the self-adjoint operator K^*K. Also $\{\varphi_n\}_{\{n \in N\}}$ is a set of eigenvectors of K^*K and the ψ_n is defined by $\psi_n = \frac{K\varphi_n}{\sigma_n}$.*

Each φ_n is an eigenvector of K^*K corresponding to σ_n^2, and $\{\varphi_n\}$ is a complete orthonormal system for $N(K)^\perp$. The following formulas are satisfied:

$$K\varphi_n = \sigma_n \psi_n, \tag{1.1.6}$$

$$K^*\psi_n = \sigma_n \varphi_n, \tag{1.1.7}$$

$$Kx = \sum_n \sigma_n \langle x, \varphi_n \rangle_X \psi_n, \quad x \in X, \tag{1.1.8}$$

$$K^*y = \sum_n \sigma_n \langle y, \psi_n \rangle_Y \varphi_n, \quad y \in Y. \tag{1.1.9}$$

Theorem 12. *Let $\sum_n \sigma_n \psi_n \otimes \varphi_n$ be the SVE of the compact linear operator K. Then*

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we have

1.

$$y \in D(K^\dagger) \Leftrightarrow \sum_n \frac{|\langle y, \psi_n \rangle_Y|^2}{\sigma_n^2} < \infty. \qquad (1.1.10)$$

2. *For* $y \in D(K^\dagger)$,

$$K^\dagger y = \sum_n \frac{\langle y, \psi_n \rangle_Y}{\sigma_n} \varphi_n. \qquad (1.1.11)$$

The first condition is called the Picard condition and it implies that the minimum-norm least-squares solution of $Kx = y$ exists only if the Fourier coefficients $\langle y, \psi_n \rangle$ decay faster than the corresponding singular values σ_n. The second condition gives an explicit formula for the minimum-norm least-squares solution $x^\dagger = K^\dagger y$ of $Kx = y$.

1.1.6 Fredholm integral equations

A Fredholm integral equation of the first kind is an equation of the form

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 \le s \le 1. \qquad (1.1.12)$$

Here the kernel k and the right-hand-side function g are known functions while the function f is unknown and we would like to know it.

We can view this as a linear operator equation $Kf = g$ by defining $K : L^2(a,b) \to$

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$L^2(a, b)$ as

$$(Kf)(s) = \int_a^b k(s, t) f(t) \, dt, \qquad (1.1.13)$$

where $k \in L^2((a, b) \times (a, b))$. Such an operator K is always compact.

Definition 13. *Let K be an integral operator of the first kind with kernel k. We say that k is degenerate if and only if there exists functions $a_1, ..., a_n, b_1, ..., b_n$ such that*

$$k(x, y) = \sum_{i=1}^n a_i(x) b_i(y).$$

It can be shown that K has finite rank if and only if k is degenerate.

1.2 Tikhonov regularization and parameter choice rules

Considering an inverse problem where the data y is perturbed by errors, the solution to the perturbed problem is usually dominated by errors (because of the lack of stability—the solution does not depend continuously on the data). In such a case, it is necessary to use a regularization method to compute a solution that is less sensitive to the perturbation.

Tikhonov Regularization is one of the most popular regularization methods for inverse

problems. It was introduced by Tikhonov [39].

We typically do not know y^* exactly but we rather know an estimate y of y^*. The idea of regularization is to replace the system $Tx = y$ with one that is less sensitive to noise in the data. We will now describe the Tikhonov regularization method. (For more information about Tikhonov regularization, and to learn about other regularization methods, see Engl et al. [12] and Hansen [25]).

Tikhonov regularization defines an approximate solution of $Tx = y$ as the minimizer of the function

$$\|Tx - y\|_Y^2 + \alpha\|x\|_X^2. \tag{1.2.1}$$

Insisting on making $\|Tx - y\|_Y^2$ as small as possible can lead to $\|x\|_X^2 \to \infty$, so we prevent this by adding the penalty term $\alpha\|x\|_X^2$. For each $\alpha > 0$, there exists a unique minimizer of (1.2.1):

$$x_{\alpha,y} = (T^*T + \alpha I)^{-1}T^*y. \tag{1.2.2}$$

For the Tikhonov regularization to work, we would expect that for all $y^* \in D(T^\dagger)$, $x_{\alpha,y^*} \to T^\dagger y^*$ as $\alpha \to 0^+$. More generally, since y^* is not known exactly, we want $x_{\alpha,y} \to T^\dagger y^*$ for $y \notin D(T^\dagger)$ as $(\alpha, y) \to (0^+, y^*)$. We will see that this holds only if α is chosen properly with respect to y, specifically, only if α does not converge to zero

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too quickly with respect to $\|y - y^*\|_Y$.

These convergence properties must hold for the Tikhonov regularization method. But on a specific problem, we do not have a sequence $\{y_n\}$ converging to y^*. We just have one measurement y of y^*, and we need to choose a good value of α. The purpose of this dissertation is to study one method for choosing α.

A variation of Tikhonov regularization is given by

$$\|Tx - y\|_Y^2 + \alpha\|Lx\|_Z^2, \tag{1.2.3}$$

where $L : X \to Z$. In this case the unique minimizer of the Tikhonov functional is given by

$$x_{\alpha,y} = (T^*T + \alpha L^*L)^{-1}T^*y.$$

We will not study this variation here.

1.2.1 Definitions and basic properties

The quality of the approximated solution $x_{\alpha,y}$ depends on choosing a good regularization parameter α. Parameter choice method are described by whether they use a bound δ on the noise level $\|y - y^*\|_Y$. Let α be a regularization parameter choice. Then it is called

- an *a priori* parameter choice if it just depends on the noise level δ: $\alpha = \alpha(\delta)$.

- an *a posteriori* parameter choice if it depends on both the noise level δ and the noisy data y: $\alpha = \alpha(\delta, y)$.

- a purely *a posteriori* (or heuristic) parameter choice if it depends only on the noisy data y: $\alpha = \alpha(y)$.

If we assume $R(T)$ is non-closed, then $T^{\dagger}$ is unbounded. Since in an inverse problem, $T^{\dagger}$ is not bounded and data are not exact, then $T^{\dagger}y$ need not exist and might not be a good approximation for $T^{\dagger}y^*$ even if does exist. That is the reason for using regularization methods for approximating $T^{\dagger}y^*$. We look for an approximation $x_{\alpha, y}$ of $T^{\dagger}y^*$ which depends continuously on the data and has the property that $x_{\alpha, y} \to T^{\dagger}y^*$ as $\delta \to 0$, provided α is chosen properly. However, based on the result of Bakushinskii [3], this is not possible if the parameter choice rule depends only on the noisy data ($\alpha = \alpha(y)$), unless T has finite rank. In other words, no purely *a posteriori* parameter choice strategy can yield a convergent regularization method.

The advantage of a purely *a posteriori* method is that it does not require a knowledge of δ. Since this kind of method can not define a convergent regularization method unless T has finite rank, we consider this kind of method as a heuristic parameter choice rule.

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The following theorem gives the basic convergence results about Tikhonov regularization.

Theorem 14. *[12] Let $x_{\alpha,y}$ be the unique minimizer of the Tikhonov functional, $y^* \in R(T)$, $\|y - y^*\| \leq \delta$ and let $\alpha = \alpha(\delta)$ be such that*

$$\lim_{\delta \to 0} \alpha(\delta) = 0, \tag{1.2.4}$$

and

$$\lim_{\delta \to 0} \frac{\delta^2}{\alpha(\delta)} = 0. \tag{1.2.5}$$

Then

$$\lim_{\delta \to 0} x_{\alpha,y} = T^\dagger y^*. \tag{1.2.6}$$

The theorem implies that sufficient conditions for strong convergence are $\alpha(\delta) \to 0$ and $\frac{\delta^2}{\alpha(\delta)} \to 0$, as $\delta \to 0$.

The condition $\frac{\delta^2}{\alpha(\delta)} \to 0$ implies that α must converge to zero at the slower rate than δ^2. If we have $\alpha = C\delta^p$ (where C is a constant), then in order to obtain convergence we must have $0 < p < 2$.

If $\lim_{\delta \to 0} \alpha(\delta) = 0$ and $\frac{\delta^2}{\alpha(\delta)}$ is only bounded above, then $\|x_{\alpha,y} - T^\dagger y^*\|_X^2 \leq C$ and $x_{\alpha,y} \to T^\dagger y^*$ weakly.

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1.3 Multiplicative regularization

The parameter selection is the most difficult part in the regularization of an inverse problem, and finding a proper Tikhonov regularization parameter is not always straightforward. To avoid the difficulties of choosing an proper regularization parameter, Berg, Broekhoven and Abubaker [41] and [2] proposed a new regularization method.

Let $Tx = y$ be a linear inverse problem, where $T : X \to Y$, $y \in Y$ is given, and x is to be estimated. Multiplicative regularization tries to estimate x by solving

$$\min_{x \in X} J(x; y) \tag{1.3.1}$$

where $J(x; y) = \frac{1}{2} \|Tx - y\|_Y^2 \|x\|_X^2$. The goal is to identify a value of x that makes the residual $\|Tx - y\|_Y^2$ small without letting the regularized solution $\|x\|_X^2$ to be large. Note that we must ignore the global minimizer $x = 0$ and calculate a local minimizer (if one exists) near $T^\dagger y^*$.

To analyze this problem, we notice that

$$\nabla_x J(x, y) = \|Tx - y\|_Y^2 x + \|x\|_X^2 (T^*Tx - T^*y).$$

If x is a minimizer of J, then $\nabla_x J(x, y) = 0$. Assuming $x \neq 0$, $\nabla_x J(x, y) = 0$ is equivalent to

$$\|Tx - y\|_Y^2 x + \|x\|_X^2 (T^*Tx - T^*y) = 0 \Leftrightarrow T^*Tx - T^*y + \frac{\|Tx - y\|_Y^2}{\|x\|_X^2} x = 0.$$

Thus $x \neq 0$ is a minimizer of J only if

$$T^*Tx + \alpha x = T^*y \tag{1.3.2}$$

where

$$\alpha = \frac{\|Tx - y\|_Y^2}{\|x\|_X^2}. \tag{1.3.3}$$

If α satisfies (1.3.2) and (1.3.3), then $x_{\alpha,y} = (T^*T + \alpha I)^{-1} T^* y$ is a stationary point of J. Conversely, if x is a nontrivial stationary point of J, then x must equal $x_{\alpha,y}$ for $\alpha = \frac{\|Tx-y\|_Y^2}{\|x\|_X^2}$.

In this dissertation we study multiplicative regularization as a purely *a posteriori* parameter choice method for Tikhonov regularization. We will evaluate the performance of multiplicative regularization by comparing it with the performance of the L-curve method.

1.4   The L-curve criterion

1.4.1   Definitions of the L-curve

The L-curve criterion is a method for choosing the regularization parameter for Tikhonov regularization method. This method is a purely a-posteriori choice method and it works well most of the time, although there exists some problems in which the L-curve criterion does not perform properly (see [21] and [24]).

The linear L-curve is a set of all points $(\|Tx_{\alpha,y} - y\|_Y, \|x_{\alpha,y}\|_X)$ such that $\alpha > 0$. For the noisy data $y \notin D(T^\dagger)$, the solution norm and residual norm have the following behaviors:

$$\alpha \to 0^+ \quad \Rightarrow \quad \begin{cases} \|x_{\alpha,y}\|_X \to \infty, \\[2mm] \|Tx_{\alpha,y} - y\|_Y \to \|P_{\overline{R(K)}}y - y\|_Y. \end{cases}$$

On the other hand,

$$\alpha \to \infty \quad \Rightarrow \quad \begin{cases} \|x_{\alpha,y}\|_X \to 0, \\[2mm] \|Tx_{\alpha,y} - y\|_Y \to \|y\|_Y. \end{cases}$$

Finding a good regularization parameter balances the norm of the residual $\|Tx_{\alpha,y} - y\|_Y$ and the norm of the regularized solution $\|x_{\alpha,y}\|_X$.

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The log-log L-curve is a set of all points $(\log \|Tx_{\alpha,y} - y\|_Y^2, \log \|x_{\alpha,y}\|_X^2)$ such that $\alpha > 0$.

A graph of $\|x_{\alpha,y}\|_X$ versus $\|Tx_{\alpha,y} - y\|_Y$ tends to have a distinctive shape, especially when plotted in log-log scale. This curve frequently has an L-shaped appearance with an obvious corner separating the vertical and the horizontal parts (see [24]). Using the log-log plot can be useful to increase the sharpness of the corner. The L-curve method chooses the regularization parameter α by picking the value that corresponds to the corner of the L-curve, which is defined as the point of maximum curvature.
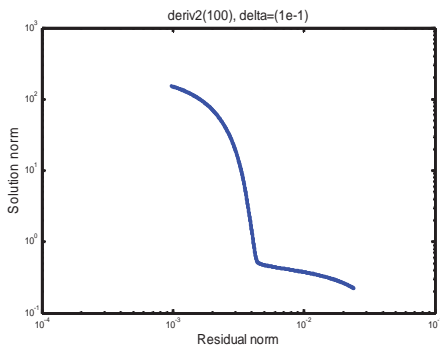
Figure 1.1: The L-curve plot of the test problem "deriv2" (n=100) where the noise level is $\delta = 10^{-1}$. The test problems are described in Section 1.6.

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1.5 Regularization and finite-dimensional problems

Finite dimensional problems can not be unstable. However, if $Ax = b$ is obtained by discretizing an inverse problem $Tx = y$, then A is generally very ill-conditioned. An ill-conditioned system $Ax = b$ has essentially the same character as inverse problem $Tx = y$: the solution is very sensitive to errors in the data.

1.5.1 The singular value decomposition (SVD)

The singular value decomposition is a valuable tool for dealing with problems with ill-conditioned matrices. Let $A \in R^{m \times n}$ be a rectangular matrix where $m \geq n$. Then the singular value decomposition of A can be defined as $A = U \Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^t$, where $U = (u_1, ..., u_m) \in R^{m \times m}$ and $V = (v_1, ..., v_n) \in R^{n \times n}$ are orthogonal matrices and $\Sigma = \text{diag}(\sigma_1, ..., \sigma_n)$ has nonnegative elements such that numbers $\sigma_1, \sigma_2, ..., \sigma_n$ are the singular values of A, arranged in non-increasing order $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$. Every matrix $A \in R^{m \times n}$ has a singular value decomposition.

The minimum-norm least-squares solution of $Ax = b$ is given by

$$A^\dagger b = \sum_{i=1}^{r} \frac{u_i \cdot b}{\sigma_i} v_i, \qquad (1.5.1)$$

where $r = \text{rank}(A)$.

1.5.2 The comparison between SVE and SVD

In the table below, we compare the singular value expansion of an operator K with the singular value decomposition of a matrix A.

Table 1.1
The comparison of the singular value expansion and the singular value
decomposition of a matrix

SVE	SVD
$\sigma_1 \geq \sigma_2 \geq ... > 0$	$\sigma_1 \geq \sigma_2 \geq ... \geq 0$
$\langle \varphi_i, \varphi_j \rangle_X = \delta_{i,j} \quad i,j = 1,2,...$	$v_i \cdot v_j = \delta_{i,j} \quad i,j = 1,...,n$
$\langle \psi_i, \psi_j \rangle_Y = \delta_{i,j} \quad i,j = 1,2,...$	$u_i \cdot u_j = \delta_{i,j} \quad i,j = 1,...,n$
$K\varphi_i = \sigma_i \psi_i$	$Av_i = \sigma_i u_i$
$K^*\psi_i = \sigma_i \varphi_i$	$A^T u_i = \sigma_i v_i$

1.5.3 Tikhonov regularization method

Let $Ax = b$ be an ill-conditioned linear system, where $A \in R^{m \times n}$ and $b \in R^m$ are given and x is supposed to be estimated.

Tikhonov regularization defines an approximate solution of $Ax = b$ as the minimizer of the function

$$\|Ax - b\|^2 + \alpha \|x\|^2.$$

The Tikhonov regularized solution is

$$x_{\alpha,b} = (A^T A + \alpha I)^{-1} A^* b.$$

In terms of the SVD, the regularized solution is given by

$$x_{\alpha,b} = \sum_{i=1}^{n} \frac{\sigma_i^2}{\sigma_i^2 + \alpha} \frac{(u_i \cdot b)}{\sigma_i} v_i. \tag{1.5.2}$$

We call $\frac{\sigma_i^2}{\sigma_i^2 + \alpha}$, $i = 1, 2, ..., n$, the filter factors for Tikhonov regularization method and we use them in order to decrease the effect of small singular values. They filter out the contributions of the smaller singular values to the solution, while leaving the effect of the large singular values almost unchanged. This is true because if $\sigma_i \gg \alpha$ then $\frac{\sigma_i}{\sigma_i^2 + \alpha} \approx \frac{\sigma_i}{\sigma_i^2} = \frac{1}{\sigma_i}$, while if $\sigma_i \ll \alpha$ then $\frac{\sigma_i}{\sigma_i^2 + \alpha} \approx \frac{\sigma_i}{\alpha} = 0$.

The filter factors for Tikhonov regularization filter out the singular values which are smaller than α. Since for the least-squares solution, the largest perturbations are associated with the smallest σ_i, then it is obvious that the regularized solution is less sensitive to the perturbations than the least-squares solution.

1.6 Test problems

We will use a collection of test problems from Hansen's Regularization Tools package [22]. Here are the problems we will use, with a short name for each:

baart: Discretization of the first-kind Fredholm integral equation

$$\int_0^\pi k(s,t)f(t)\,dt = g(s), \quad 0 \le s \le \frac{\pi}{2}, 0 \le t \le \pi,$$

where $k(s,t) = \exp(s\cos(t))$, $g(s) = 2(\frac{\sinh(s)}{s})$ and the solution is $f(t) = \sin(t)$. The equation is discretized by the Galerkin method with orthonormal box functions.

heat: Discretization of the first-kind Volterra integral equation

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 \le t, s \le 1,$$

where $k(s,t) = \frac{t^{\frac{-3}{2}}}{2\sqrt{\pi}}\exp(\frac{-1}{4t})$, The equation is discretized by means of simple quadrature (midpoint rule). An exact solution is constructed, and then the right-hand side is produced as $b = Ax$.

foxgood: Discretization of the first-kind Fredholm integral equation

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 \le t, s \le 1,$$

where $k(s,t) = \sqrt{s^2 + t^2}$, and $g(s) = \frac{(1+s^2)^{\frac{3}{2}} - s^3}{3}$. The exact solution is $f(t) = t$. The equation is discretized by simple quadrature (midpoint rule).

shaw: Discretization of the first-kind Fredholm integral equation

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} k(s,t)f(t)\,dt = g(s), \quad \frac{-\pi}{2} \le t, s \le \frac{\pi}{2},$$

where $k(s,t) = (\cos(s) + \cos(t))^2 \left(\frac{\sin(\pi(\sin(s)+\sin(t)))}{\sin(s)+\sin(t)} \right)^2$. The exact solution is

$$f(t) = 2\exp(-6(t - 0.8)^2) + \exp(-2(t + 0.5)^2),$$

and right-hand side is produced by $b = Ax$. The equation is discretized by simple quadrature.

deriv2: Discretization of the first-kind integral equation

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 \le t, s \le 1,$$

where

$$k(s,t) = \begin{cases} s(t-1), & s < t, \\ \\ t(s-1), & \text{otherwise,} \end{cases}$$

and $g(s) = \frac{(s^3-s)}{6}$. The exact solution is $f(t) = t$. The equation is discretized by the

Galerkin method with orthonormal box functions.

phillips: Discretization of the first-kind integral equation

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 \le t, s \le 1,$$

where $k(s,t) = \phi(s-t)$, and $g(s) = (6-|s|)(1+0.5\cos(\frac{\pi s}{3})) + \frac{9}{(2\pi)\sin(\frac{\pi|s|}{3})}$. The exact

solution is $f(t) = \phi(t)$. The function ϕ is defined as

$$\phi(x) = \begin{cases} 1 + \cos(\frac{\pi x}{3}), & |x| < 3, \\ \\ 0, & |x| \ge 3 \end{cases}$$

The equation is discretized by the Galerkin method with orthonormal box functions.

i_laplace: Discretization of the inverse Laplace transformation

$$\int_0^\infty k(s,t)f(t)\,dt = g(s), \quad 0 \le s, t < \infty,$$

where $k(s,t) = \exp(-st)$, and $g(s) = \frac{1}{(s+0.5)}$. The exact solution is $f(t) = \exp\left(-\frac{t}{2}\right)$.

The equation is discretized by means of Gauss-Laguerre quadrature.

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wing: Discretization of the first-kind Fredholm integral eqaution

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 < t, s < 1,$$

where $k(s,t) = t\exp(-st^2)$, and $g(s) = \frac{\exp(-st_1^2)-\exp(-st_2^2)}{2s}$ for $t_1 = \frac{1}{3}$ and $t_2 = \frac{2}{3}$. The

solution $f(t)$ is given by where

$$f(t) = \begin{cases} 1, & t_1 < t < t_2, \\ \\ 0, & \text{otherwise.} \end{cases}$$

The equation is discretized by the Galerkin method with orthonormal box functions.

Table 1.2
Summary of test problems

Test problems	
deriv2	Computation of the second derivative, a mildly ill-posed problem
phillips	Discretization of a first king Fredholm integral equation
heat	Inverse heat problem, discretization of a first king Volterra integral equation
i_laplace	Inverse Laplace transformation, discretization of a first king Fredholm integral equation
foxgood	Severely ill-conditioned problem, discretized by simple quadrature (midpoint rule)
baart	Discretization of a first king Fredholm integral equation
shaw	One dimensional image resolution model, discretization of a first king Fredholm integral equation
wing	Test problem with a discontinuous solution, discretization of a first king Fredholm integral equation

1.6.1 The degree of ill-posedness

The degree of ill-posedness describes how fast the singular values σ_n converge to zero

as $n \to \infty$. For each test problem, we found the singular values for $n = 100$. If there

exists a positive real number q such that $\sigma_n = cn^{-q}$ (where c is a constant), then the problem is called mildly or moderately ill-posed. If on the other hand we have $\sigma_n = ce^{-qn^t}$, then the problem is called severely ill-posed. The results are given in Table 1.3. It turns out that "deriv2" and "phillips" are mildly ill-posed problems and the rest of the test problems are severely ill-posed.

Table 1.3

The degree of ill-posedness of the test problems

Test problems	$\sigma_n = O(n^{-q})$	$\sigma_n = O(e^{-qn^t})$
deriv2	✓	
phillips	✓	
heat		✓
i_laplace		✓
foxgood		✓
baart		✓
shaw		✓
wing		✓

Next, we tried to find the best fit for each test problem and the results are given in Table 1.4.

Table 1.4

Convergence of the singular values to zero

Test problems	$\sigma_n \approx cn^{-q}$	$\sigma_n \approx ce^{-qn^t}$
deriv2	$(c, q) = (-0.88706, 2.1191)$	
phillips	$(c, q) = (2.8293, 3.6590)$	
heat		$(c, q, t) = (0.45411, 0.85161, 0.57676)$
i_laplace		$(c, q, t) = (4.9452, 0.63472, 1.0893)$
foxgood		$(c, t, q) = (2.0765, 2.5645, 0.76484)$
shaw		$(c, q, t) = (2.9445, 0.37951, 1.2961)$
baart		$(c, q, t) = (5.5648, 0.66672, 1.08300)$
wing		$(c, q, t) = (2.0295, 1.7269, 1.3599)$

1.7 Outline of dissertation

We have seen that for an inverse problems, it is necessary to use a regularization method to compute a solution that is less sensitive to the noise in the data. Most methods require us to choose a good regularization parameter α.

In Chapter 2, we review existing parameter choice rules, especially the L-curve method, which is the most popular purely *a posteriori* rule.

The L-curve parameter choice strategy is a useful method but sometimes it fails to find a proper regularization parameter. Multiplicative regularization was originally introduced to avoid the need to choose a parameter. However, as we saw in Chapter 1, it can be viewed as a parameter choice method for Tikhonov regularization. We analyze multiplicative regularization for finite-dimensional problems in Chapter 3. Numerical results show that both methods, multiplicative regularization and the L-curve method, perform remarkably similarly provided $A \in \mathbb{R}^{m \times n}$ and $\mathrm{col}(A)$ is a proper subspace of $\mathbb{R}^m$.

In Chapter 4, we present some preliminary theoretical and numerical results on the performance of multiplicative regularization on infinite-dimensional problems. The main results and the numerical experiments are found in Chapter 3 and 4.

In Chapter 5, we present our conclusions and discuss the future plans and goal.

Chapter 2

Literature review

2.1 Multiplicative regularization

The parameter selection is the most difficult part in the regularization of an ill-posed problem, and finding a proper Tikhonov regularization parameter is not always straightforward. In order to avoid the difficulties of choosing a proper regularization parameter, Berg, Broekhoven and Abubaker in [41] and [2] proposed a new regularization method. The main advantage of this method is that we do not have to determine the regularization parameter. In this method, no *a priori* knowledge on the data is needed.

Multiplicative regularization has recently been studied in the PhD dissertation of

Orozco Rodríquez [38]. He analyzes the multiplicative regularization as a parameter choice strategy and compares the performance of multiplicative regularization with the L-curve criterion for an image deblurring problem.

Multiplicative regularization has been used in a variety of applications in the past decade (see for example [2] and [1]).

2.2 Popular parameter choice rules

Various methods have been introduced in the literature for selecting the regularization parameter α. Parameter choice strategies are categorized based on their assumptions about the error norm. Here we introduce the most popular ones briefly.

2.2.1 Morozov discrepancy principle

Morozov discrepancy principle is the most popular a-posteriori parameter choice rule for Tikhonov regularization [32]. This method chooses α to satisfy

$$\|Tx_{\alpha,y} - y\| = \delta, \tag{2.2.1}$$

that is, the regularization parameter is chosen via a comparison between the residual and the noise level. The equation (2.2.1) has a solution $\alpha = \alpha(\delta, y) > 0$ as long as $\|y - y^*\| \leq \delta, \|y\| > \delta$, and $(y - y^*) \notin R(T)^{\perp}$. The discrepancy principle defines a convergent regularization method provided $y^* \in R(T)$.

In the real world examples, the information about the noise level is not always available. Therefore it is necessary to consider some other parameter choice rules that avoid knowledge of noise level. Such a heuristic parameter choice rule is called a purely *a posteriori* parameter choice method.

2.2.2 Generalized cross-validation method

This method was introduced by Golub et al. [40] and it is a purely *a posteriori* parameter choice method. This method includes statistical considerations and does not require any prior knowledge about the error. The method selects a regularization parameter in order to minimize the predicted data error of the regularized solution.

Although generalized cross-validation is an accurate purely a-posteriori method for choosing the regularization parameter (see [17]), it has been rarely applied to large scale problems. A major difficulty lies in the evaluation of the cross-validation function which requires the calculation of the trace of an inverse matrix (see [14]).

The next parameter selecting strategy is the L-curve method which is one of the most popular parameter choice methods. We discuss this method in details since we compare the performance of multiplicative regularization with this method.

2.3 The L-curve method

The L-curve criterion is a parameter choice strategy for the Tikhonov regularization method. The use of the L-curve goes back to Miller [31]. In infinite dimensions, no convergence result is possible, as explained in Section 1.2. The L-curve method sometimes fails to find a proper regularization parameter, but it still is a useful method.

2.3.1 The definition of the corner

A difficulty with the L-curve method is that a wide range of regularization parameters corresponding to points on the L-curve near the corner. Therefore the corner of the curve should be selected by some numerical method rather than by visual selection. Hansen and O'Leary [24] proposed a method for choosing α when the errors of the right hand side are white noise[1]. They defined the corner of a continuous L-curve

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[1] $E[b - b^*] = 0$ and $E[(b - b^*) \cdot (b - b^*)] = \sigma^2 I$ where $E[\cdot]$ denotes the expectation.

as the point where the curvature attains a local maximum and they presented an algorithm for locating the corner of the L-curve.

2.3.2 Concavity of the L-curve

In linear scale, independently of the right hand side, the L-curve is always convex. For finite-dimensional problems, Reginska [36] proved the following lemmas that describe the shape of the L-curve in logarithmic scale.

1. The L-curve in logarithmic scale is always concave for α less than or equal the smallest singular value, i.e., $(\alpha \leq \sigma_n)$ and also for α greater than or equal to the largest singular value, i.e., $(\alpha \geq \sigma_1)$.

2. Let $i_1 < i_2$ be such that for $i \in \{i_1, i_2\}$ we have $u_i^T \cdot b \neq 0$ and also for $i \notin \{i_1, i_2\}$ we have $u_i^T \cdot b = 0$. If $\frac{|u_{i_2} \cdot b|}{\sigma_{i_2}^2} < \frac{|u_{i_1} \cdot b|}{\sigma_{i_1}^2}$ then the L-curve in logarithmic scale is strictly concave.

The first result ensures that both ends of a L-curve is always concave but it does not deal with the behavior of the L-curve for $\alpha \in [\sigma_1, \sigma_n]$. The second result constructs a class of specific examples for which the L-curve in logarithmic scale does not have a corner; in particular, it proves that the L-curve does not always have a corner.

Hansen [23] dealt with other conditions under which the L-curve in logarithmic scale
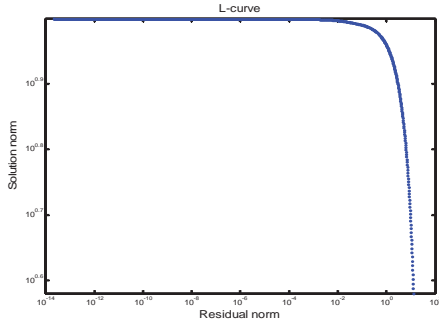
Figure 2.1: The L-curve plot of the problem "shaw" (n=100) where the noise level is zero.
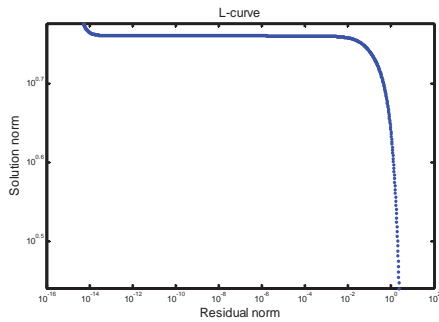
Figure 2.2: The L-curve plot of the problem "foxgood" (n=100) where the noise level is zero. The value of α at corner is 3.0809×10^{-15}.

is concave. He gave the heuristic arguments that an exact unperturbed right hand side b^* satisfying the discrete Picard condition (see, for example, Figure 2.1) or a right hand side consisting of pure noise e, lead typically to a concave L-curve in logarithmic scale. In certain cases, though, we see other behavior. In certain examples, with $b = b^*$, there is a convex corner, though it corresponds to a very small α. This happens especially when the matrix A is so severely ill-conditioned that most of the singular values are zero. For instance, the test problem foxgood indicates a convex corner even when b is in the column space of A (see Figure 2.2).
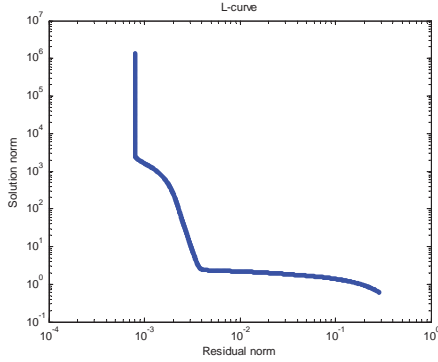
Figure 2.3: The L-curve of the problem "heat" ($n = 100$) for $\delta = 10^{-1}$.

Furthermore Bazan [5] did a further study of the convexity properties of the L-curve

discussed by Reginska [36] and Hansen [23]. He showed that there exist some situa-

tions in which the L-curve criterion fails since the L-curve has several convex corners.

The heat problem was chosen in his analysis since this problem most of the time

displays more than one convex corner. In this case, Bazan suggested choosing the

regularization parameter corresponding to the sharper corner or else to the right-most

corner.

2.3.3 Other disadvantages of the L-curve

Vogel in [42] studied the behavior of an n-dimensional problem with discrete white

noise as $n \to \infty$, and considered a partially discrete, partially stochastic model[2] for

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[2]Let A_n is an operator from Hilbert space Z into R^n. Then the semi-discrete, semi-stochastic data
model is given by $[b_n]_i = [A_n x^*]_i + \sigma[\eta_n]_i$ for $i = 1, ..., n$ where σ is a fixed operator, x^* is the exact
solution and it is assumed to be deterministic, and η_n are discrete white noise vectors with unit
covariance which means $E([\eta_n(x)]_i) = 0$ for $i = 1, ..., n$, and $E([\eta_n]_i[\eta_n]_j) = \delta_{ij}$ for $i, j = 1, ..., n$, (δ
is the Dirac delta distribution).

analyzing the L-curve in logarithmic scale. He considered more data samples in his analysis rather than assuming that the noise level in data decreases. He showed that the L-curve method is not convergent under certain conditions. The non-convergence of the L-curve method happens when the solution $T^\dagger y^*$ is rough or lacks smoothness, that is, when the Fourier coefficients decay more slowly or at the same rate than the singular values. Vogel showed that the regularization parameters computed by this model stagnate, that is, are bounded away from zero.

Another disadvantage of the L-curve is related to very smooth exact solutions. Hanke [16] proved that for this kind of solutions, the L-curve method would fail. He constructed an infinite-dimensional example and showed the chosen regularization parameter by the L-curve method vanishes too rapidly as the noise to signal ratio in the data level goes to zero. Based on the analysis and numerical examples, he showed that the L-curve method may sometimes fail in the sense that the regularized solutions $x_{\alpha,y}$ do not converge to the true solution x of the equation $Tx = y$ as $\delta \to 0$. This can happen when the solution x is very smooth and the noise level is very small.

2.3.4   Rate of convergence

If we fit the α computed from the L-curve method to a formula of the form $\alpha = C\delta^p$ (where C and p are constant), then we find the following values of p shown in Table

Table 2.1

The rate of convergence for the L-curve method

L-curve	
Test problem	p
deriv2	0.99133
phillips	1.7481
heat	2.578
i_laplace	1.9549
foxgood	1.9582
shaw	2.0207
baart	2.0215
wing	2.0946

2.1 for the test problems.

2.4 Conclusions

The L-curve is a plot of the norm of the regularized solutions versus the norm of the residuals. Under certain conditions, the curve has an L-shaped form with a convex corner; however, this is not always the case. The selected regularization parameter is the point with maximum curvature. The behavior of the L-curve method in some cases is not predictable. The location of the corner depends on the scale of the L-curve and in some scales, the corner may not appear. Unfortunately, the change of scale does not preserves the convexity of the L-curve.

Because of difficulties associated with the L-curve method, we will introduce and

analyze the multiplicative regularization method as a parameter choice strategy.

Chapter 3

Multiplicative regularization for finite-dimensional problems

In order to avoid the difficulties of choosing a proper regularization parameter, Berg, Broekhoven and Abubaker proposed a new regularization method, which we call multiplicative regularization (see [41], [2]). The method is based on minimizing the function

$$J(x, b) = \frac{1}{2} \|Ax - b\|^2 \|x\|^2.$$

Multiplicative regularization tries to identify a value of x that makes the residual small without letting the regularized solution be large. As explained in the Introduction, it turns out that there is a close relationship between multiplicative regularization and

Tikhonov regularization. In fact, multiplicative regularization can be considered as defining a parameter choice method for Tikhonov regularization.

In this chapter, $\|\cdot\|$ denotes the Euclidean norm, and $\langle\cdot,\cdot\rangle$ denotes the Euclidean dot product.

We will study multiplicative regularization for finite-dimensional problems, and we will show by numerical experiments that the performance of this method is comparable to that of the L-curve method.

3.1   Notation and assumption

Let $Ax = b$ be an ill-conditioned problem, where $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$ are given, and x is to be estimated. We always assume that $b \notin N(A^T)$. We assume that b^* is the exact data, x^* is the exact solution ($Ax^* = b^*$, $x^* \in N(A)^{\perp}$), and b is close to b^*. We try to find a local minimizer of J that is near x^*. Notice that $x = 0$ is always a global minimizer of J, but this is not of interest.

To analyze this problem, we notice that

$$\nabla_x J(x, b) = \|Ax - b\|^2 x + \|x\|^2 (A^T A x - A^T b).$$

If x is a minimizer of J, then $\nabla_x J(x, b) = 0$. Assuming $x \neq 0$, $\nabla_x J(x, b) = 0$ is equivalent to

$$\|Ax - b\|^2 x + \|x\|_X^2 (A^T Ax - A^T b) = 0 \Leftrightarrow A^T Ax - A^T b + \frac{\|Ax - b\|^2}{\|x\|^2} x = 0$$

$$\Leftrightarrow A^T Ax + \frac{\|Ax - b\|^2}{\|x\|^2} x = A^T b.$$

Thus $x \neq 0$ is a minimizer of J only if

$$A^T Ax + \alpha x = A^T b, \tag{3.1.1}$$

where

$$\alpha = \frac{\|Ax - b\|^2}{\|x\|^2}. \tag{3.1.2}$$

If α satisfies (3.1.1) and (3.1.2), then $x_{\alpha,b} = (A^T A + \alpha I)^{-1} A^T b$ is a stationary point of J. Conversely, if x is a nontrivial stationary point of J, then x must equal $x_{\alpha,b}$ for

$$\alpha = \frac{\|Ax_{\alpha,b} - b\|^2}{\|x_{\alpha,b}\|^2}. \tag{3.1.3}$$

Thus every nontrivial stationary point of J is a Tikhonov solution $x_{\alpha,b}$, where α satisfies (3.1.3).

Reginska [36] has analyzed a similar regularization method by defining the objective function as

$$\phi(\alpha, b) = \frac{1}{2} \|Ax_{\alpha,y} - b\|^2 \|x_{\alpha,y}\|^2,$$

and selecting α by minimizing this function. We can show that α is a stationary point of $\phi(\cdot, b)$ if and only if α satisfies $\alpha = \frac{\|Ax_{\alpha,y} - b\|^2}{\|x_{\alpha,y}\|^2}$. To prove this, we first derive some results that will be used repeatedly. We can show that $x_{\alpha,b}$ is a smooth function of α for all $\alpha > 0$, and we have the following results:

1. $x'_{\alpha,b} = \frac{\partial}{\partial \alpha}(x_{\alpha,y}) = -N_\alpha^{-1} x_{\alpha,b}$ (where $N_\alpha = (A^T A + \alpha I)$).

2. $\frac{\partial}{\partial \alpha}(\|x_{\alpha,y}\|) = 2\langle x_{\alpha,y}, x'_{\alpha,y} \rangle$.

3.

$$\frac{\partial}{\partial \alpha}(\|Ax_{\alpha,y} - b\|^2) = 2\langle (Ax_{\alpha,b} - b), Ax'_{\alpha,b} \rangle = 2\langle (A^T A x_{\alpha,b} - A^T b), x'_{\alpha,b} \rangle$$

$$= -2\alpha \langle x_{\alpha,y}, x'_{\alpha,y} \rangle = -\alpha \frac{\partial}{\partial \alpha}(\|x_{\alpha,y}\|^2).$$

(Here we use the fact that $A^T A x_{\alpha,b} + \alpha x_{\alpha,b} = A^T b$, that is, $A^T A x_{\alpha,b} - A^T b = \alpha x_{\alpha,b}$.

Using the previous results, we have

$$\phi'(\alpha, b) = -\alpha \langle x_{\alpha,b}, x'_{\alpha,b} \rangle \|x_{\alpha,b}\|^2 + 2\langle x_{\alpha,b}, x'_{\alpha,b} \rangle \|Ax_{\alpha,b} - b\|^2)$$

$$= \langle x_{\alpha,b}, x'_{\alpha,b} \rangle (\|Ax_{\alpha,b} - b\|^2 - \alpha \|x_{\alpha,b}\|^2).$$

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For $b \notin N(A^T)$, $\langle x_{\alpha,y}, x'_{\alpha,y} \rangle$ is never zero. That is true because we have

$$\langle x_{\alpha,y}, x'_{\alpha,y} \rangle = -\langle N_\alpha^{-1} A^T b, N_\alpha^{-2} A^T b \rangle = -\langle A^T b, N_\alpha^{-3} A^T b \rangle,$$

and we know that N_α^{-3} is symmetric positive definite. Therefore, α is a stationary point of $\phi(\cdot, b)$ if and only if $\alpha = \frac{\|Ax_{\alpha,b} - b\|^2}{\|x_{\alpha,b}\|^2}$. So Reginska's approach is closely related to multiplicative regularization.

3.2 Preliminaries

Notice that (3.1.3) is equivalent to $\|Ax_{\alpha,b} - b\|^2 - \alpha\|x_{\alpha,b}\|^2 = 0$. We define

$$F(\alpha, b) = \|Ax_{\alpha,b} - b\|^2 - \alpha\|x_{\alpha,b}\|^2, \tag{3.2.1}$$

for $\alpha > 0$ and all $b \in \mathbb{R}^m$.

Let $r = \text{rank}(A)$. We will now express the Tikhonov solution of $Ax = b$ in terms of the singular value decomposition of A. The reduced SVD of A is

$$A = \sum_{i=1}^{r} \sigma_i \psi_i \otimes \varphi_i,$$

where $\{\varphi_1, \varphi_2, ..., \varphi_r\}$ is an orthonormal set in $\mathbb{R}^n$, $\{\psi_1, \psi_2, ..., \psi_r\}$ is an orthonormal

set in $\mathbb{R}^m$, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. It can be show that $A^T = \sum_{i=1}^{r} \sigma_i \varphi_i \otimes \psi_i$. If we extend $\{\varphi_1, \varphi_2, ..., \varphi_r\}$ to an orthonormal basis $\{\varphi_1, \varphi_2, ..., \varphi_n\}$ of $\mathbb{R}^n$, then $\{\varphi_{r+1}, ..., \varphi_n\}$ is a basis for $N(A)$ and every $x \in \mathbb{R}^n$ is given by $x = \sum_{i=1}^{n} \langle x, \varphi_i \rangle \varphi_i$. We have

$$A^T A x = A^T A \left(\sum_{i=1}^{n} \langle x, \varphi_i \rangle \varphi_i \right)$$
$$= A^T \left(\sum_{i=1}^{r} \sigma_i \langle x, \varphi_i \rangle \psi_i \right)$$
$$= \sum_{i=1}^{r} \sigma_i^2 \langle x, \varphi_i \rangle \varphi_i,$$

and therefore

$$(A^T A + \alpha I)x = \sum_{i=1}^{r} \sigma_i^2 \langle x, \varphi_i \rangle \varphi_i + \alpha \sum_{i=1}^{n} \langle x, \varphi_i \rangle \varphi_i$$
$$= \sum_{i=1}^{r} (\sigma_i^2 + \alpha) \langle x, \varphi_i \rangle \varphi_i + \alpha \sum_{i=r+1}^{n} \langle x, \varphi_i \rangle \varphi_i.$$

If we extend $\{\psi_1, \psi_2, ..., \psi_r\}$ to an orthonormal basis $\{\psi_1, \psi_2, ..., \psi_m\}$ of $\mathbb{R}^m$, then $\{\psi_{r+1}, ..., \psi_m\}$ is a basis for $N(A^T)$ and every $b \in \mathbb{R}^m$ is given by $b = \sum_{i=1}^{m} \langle b, \psi_i \rangle \psi_i$.
Thus we have

$$A^T b = A^T \left(\sum_{i=1}^{m} \langle b, \psi_i \rangle \psi_i \right) = \sum_{i=1}^{r} \sigma_i \langle b, \psi_i \rangle \varphi_i.$$

We know that $(A^T A + \alpha I)x_{\alpha, b} = A^T b$ and it follows that

$$\sum_{i=1}^{r}(\sigma_i^2 + \alpha)\langle x_{\alpha,b}, \varphi_i\rangle\varphi_i + \alpha \sum_{i=r+1}^{n} \langle x, \varphi_i\rangle\varphi_i = \sum_{i=1}^{r} \sigma_i\langle b, \psi_i\rangle\varphi_i$$

$$\Rightarrow \begin{cases} (\sigma_i^2 + \alpha)\langle x_{\alpha,b}, \varphi_i\rangle = \sigma_i\langle b, \psi_i\rangle, & i = 1, 2, ..., r, \\[2ex] \alpha\langle x_{\alpha,b}, \varphi_i\rangle = 0, & i = r+1, ..., n, \end{cases}$$

$$\Rightarrow \begin{cases} \langle x_{\alpha,b}, \varphi_i\rangle = \frac{\sigma_i}{\sigma_i^2 + \alpha}\langle b, \psi_i\rangle, & i = 1, 2, ..., r, \\[2ex] \langle x_{\alpha,b}, \varphi_i\rangle = 0, & i = r+1, ..., n, \end{cases}$$

$$\Rightarrow x_{\alpha,b} = \sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha}\langle b, \psi_i\rangle\varphi_i.$$

We then have

$$\begin{aligned} Ax_{\alpha,b} - b &= A\left(\sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha}\langle b, \psi_i\rangle\varphi_i\right) - \sum_{i=1}^{m}\langle b, \psi_i\rangle\psi_i \\ &= \sum_{i=1}^{r} \frac{\sigma_i^2}{\sigma_i^2 + \alpha}\langle b, \psi_i\rangle\psi_i - \sum_{i=1}^{m}\langle b, \psi_i\rangle\psi_i \\ &= \sum_{i=1}^{r} \frac{-\alpha}{\sigma_i^2 + \alpha}\langle b, \psi_i\rangle\psi_i - \sum_{i=r+1}^{m}\langle b, \psi_i\rangle\psi_i. \end{aligned}$$

For $\alpha > 0$, we have

$$\begin{aligned} F(\alpha, b) &= \|Ax_{\alpha,b} - b\|^2 - \alpha\|x_{\alpha,b}\|^2 \\ &= \sum_{i=1}^{r} \frac{\alpha^2}{(\sigma_i^2 + \alpha)^2}|\langle b, \psi_i\rangle|^2 + \sum_{i=r+1}^{m}|\langle b, \psi_i\rangle|^2 - \alpha\sum_{i=1}^{r} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha)^2}|\langle b, \psi_i\rangle|^2 \\ &= \sum_{i=1}^{r} \frac{\alpha^2 - \alpha\sigma_i^2}{(\sigma_i^2 + \alpha)^2}|\langle b, \psi_i\rangle|^2 + \sum_{i=r+1}^{m}|\langle b, \psi_i\rangle|^2. \end{aligned}$$

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Note that for each $b \in \mathbb{R}^m$, $\lim_{\alpha \to 0^+} x_{\alpha,b}$ exists, and

$$\lim_{\alpha \to 0^+} x_{\alpha,b} = \lim_{\alpha \to 0^+} \sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha} \langle b, \psi_i \rangle \varphi_i = \sum_{i=1}^{r} \frac{\langle b, \psi_i \rangle}{\sigma_i} \varphi_i.$$

We define $x_{0,b} = \sum_{i=1}^{r} \frac{\langle b, \psi_i \rangle}{\sigma_i} \varphi_i$. We will show that $x_{0,b}$ is the minimum-norm least-squares solution of $Ax = b$, that is, that $x_{0,b}$ solves

$$\min \|x\|$$

$$\text{s.t. } A^T A x = A^T b.$$

To see this, let x be any least-squares solution of $Ax = b$. Then we have

$$A^T A x = A^T b \Rightarrow \sum_{i=1}^{r} \sigma_i^2 \langle x, \varphi_i \rangle \varphi_i = \sum_{i=1}^{r} \sigma_i \langle b, \psi_i \rangle \varphi_i$$

$$\Rightarrow \sigma_i^2 \langle x, \varphi_i \rangle = \sigma_i \langle b, \psi_i \rangle, \text{ for } i = 1, 2, ..., r,$$

$$\Rightarrow \langle x, \varphi_i \rangle = \frac{1}{\sigma_i} \langle b, \psi_i \rangle, \text{ for } i = 1, 2, ..., r,$$

$$\Rightarrow x = \sum_{i=1}^{r} \frac{1}{\sigma_i} \langle b, \psi_i \rangle \varphi_i + \sum_{i=r+1}^{n} \beta_i \varphi_i$$

$$\Rightarrow x = x_{0,b} + \sum_{i=r+1}^{n} \beta_i \varphi_i,$$

where $\beta_{r+1}, ..., \beta_n$ can be any real numbers.

Conversely, we can easily show that if $x = \sum_{i=1}^{r} \frac{1}{\sigma_i} \langle b, \psi_i \rangle \varphi_i + \sum_{i=r+1}^{n} \beta_i \varphi_i$, then

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$A^T A x = A^T b$ and x is a least-squares solution. This shows that $x_{0,b}$ is a least-squares solution of $Ax = b$.

So we have $\|x\|^2 = \|x_{0,b}\|^2 + \sum_{i=r+1}^{n} \beta_i^2$, which implies that $\|x_{0,b}\|^2 \leq \|x\|^2$ for every other least-squares solutions.

We define $A^\dagger$ by the condition that $A^\dagger b$ is the unique minimum-norm least-squares solution of $Ax = b$. The previous result shows that $A^\dagger b = \sum_{i=1}^{r} \frac{1}{\sigma_i} \langle b, \psi_i \rangle \varphi_i$ for all $b \in \mathbb{R}^m$, and hence $A^\dagger = \sum_{i=1}^{r} \frac{1}{\sigma_i} \varphi_i \otimes \psi_i$.

We can see that for every $b \in \mathbb{R}^m$, $x_{0,b} = A^\dagger b$, that is, $\lim_{\alpha \to 0^+} x_{\alpha,b} = A^\dagger b$. We also can show that for all $b \in \mathbb{R}^m$ and all $\alpha > 0$, $\|x_{\alpha,b}\| \leq \|x_{0,b}\|$ and as a result $\|N_\alpha^{-1} A^T\| \leq \|A^\dagger\|$. We argue by the way of contradiction. If we assume that there exists $b \in \mathbb{R}^m$ and $\alpha > 0$ such that $\|x_{0,b}\| < \|x_{\alpha,b}\|$, then since for all $x \in \mathbb{R}^n$, $\|Ax_{0,b} - b\|^2 \leq \|Ax - b\|^2$, it follows that

$$\|Ax_{0,b} - b\|^2 + \alpha \|x_{0,b}\|^2 < \|Ax_{\alpha,b} - b\|^2 + \alpha \|x_{\alpha,b}\|^2,$$

which contradicts the definition of $x_{\alpha,b}$ as the unique minimizer of $\|Ax - b\|^2 + \alpha \|x\|^2$.

For $b \in \mathbb{R}^m$, we write $b = \bar{b} + \hat{b}$ where $\bar{b} = Ax_{0,b} = \text{proj}_{\text{col}(A)} b$ and $\hat{b} = b - Ax_{0,b} = \text{proj}_{\text{col}(A)^\perp} b$. Since $x_{\alpha,b} \to x_{0,b}$ as $\alpha \to 0^+$, then $\|Ax_{\alpha,b} - b\|^2 \to \|Ax_{0,b} - b\|^2 = \|\hat{b}\|^2$ as $\alpha \to 0^+$, and $\alpha \|x_{\alpha,b}\|^2 \to 0$ as $\alpha \to 0^+$.

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Thus we extend the definition of F as follows:

$$F(\alpha, b) = \begin{cases} \|Ax_{\alpha,b} - b\|^2 - \alpha\|x_{\alpha,b}\|^2, & \alpha > 0, \\ \|\hat{b}\|^2, & \alpha = 0. \end{cases}$$

In terms of the SVD, we have

$$F(\alpha, b) = \begin{cases} \sum_{i=1}^{r} \frac{\alpha^2 - \alpha\sigma_i^2}{(\sigma_i^2 + \alpha)^2}|\langle b, \psi_i\rangle|^2 + \sum_{i=r+1}^{n} |\langle b, \psi_i\rangle|^2, & \alpha > 0, \\ \sum_{i=r+1}^{n} |\langle b, \psi_i\rangle|^2, & \alpha = 0. \end{cases}$$

For the exact data $b^* \in \mathrm{col}(A)$, it follows that $F(0, b^*) = 0$. We wish to apply the implicit function theorem to show that $F(\alpha, b) = 0$ has a locally unique solution $\alpha = \alpha(b)$ for each b sufficiently close to b^*. To do this, we must extend F from $[0, \infty) \times \mathbb{R}^m$ to $\mathbb{R} \times \mathbb{R}^m$ and prove that F is C^1. First we need to show the following preliminary results.

We know that the problem

$$\min \|Ax - b\|^2 + \alpha\|x\|^2$$

has a unique solution for each $\alpha > 0$ and $b \in \mathbb{R}^m$. It is clear that the above problem

is equivalent to

$$\min \|M_\alpha x - (b,0)\|^2,$$

where $M_\alpha x = (Ax, \sqrt{\alpha} x)$. Then the normal equation takes the form $M_\alpha^T M_\alpha x = M_\alpha^T (b,0)$ and $x_{\alpha,b}$ is a least-squares solution of it since

$$M_\alpha^T M_\alpha x_{\alpha,b} = M_\alpha^T (b,0) \Leftrightarrow M_\alpha^T (Ax_{\alpha,b}, \sqrt{\alpha} x_{\alpha,b}) = M_\alpha^T (b,0)$$

$$\Leftrightarrow A^T A x_{\alpha,b} + \alpha x_{\alpha,b} = A^T b$$

$$\Leftrightarrow (A^T A + \alpha I) x_{\alpha,b} = A^T b$$

$$\Leftrightarrow x_{\alpha,b} = N_\alpha^{-1} A^T b.$$

Since $M_\alpha x_{\alpha,b}$ is the projection of $(b,0)$ onto $\mathrm{col}(M_\alpha)$, then by the Pythagorean theorem we have

$$\|M_\alpha x_{\alpha,b}\|^2 + \|M_\alpha x_{\alpha,b} - (b,0)\|^2 = \|(b,0)\|^2,$$

which is equivalent to

$$\|A x_{\alpha,b}\|^2 + 2\alpha \|x_{\alpha,b}\|^2 + \|A x_{\alpha,b} - b\|^2 = \|b\|^2.$$

It follows that

- $\|A x_{\alpha,b}\|^2 \leq \|b\|^2 \Rightarrow \|A N_\alpha^{-1} A^T b\| \leq \|b\|, \ \forall b \in \mathbb{R}^m \Rightarrow \|A N_\alpha^{-1} A^T\| \leq 1.$

- $\|b - A x_{\alpha,b}\|^2 \leq \|b\|^2 \Rightarrow \|b - A N_\alpha^{-1} A^T b\|^2 \leq \|b\|^2, \ \forall b \in \mathbb{R}^m \Rightarrow \|(I - A N_\alpha^{-1} A^T)\| \leq 1.$

• $2\alpha \|x_{\alpha,b}\|^2 \leq \|b\|^2 \Rightarrow \sqrt{2\alpha} \|N_\alpha^{-1} A^T b\| \leq \|b\|, \ \forall b \in \mathbb{R}^m \Rightarrow \|N_\alpha^{-1} A^T\| \leq \frac{1}{\sqrt{2\alpha}}.$

Also we know that $\|N_\alpha^{-1} A^T\| \leq \|A^\dagger\|$, therefore $\|N_\alpha^{-1} A^T\| \leq \min\{\frac{1}{\sqrt{2\alpha}}, \|A^\dagger\|\}$.

We also derive the following important result.

Lemma 15. *For all $\alpha \in (0,1]$, $\|N_\alpha^{-1}\| \leq \frac{1}{\alpha}$.*

Proof. For $\alpha > 0$, $x \cdot N_\alpha x = x \cdot (A^T A + \alpha I)x = \|Ax\|^2 + \alpha \|x\|^2 \geq \alpha \|x\|^2$. Since N_α is a bounded and symmetric operator, it follows that $\|N_\alpha^{-1}\| \leq \frac{1}{\alpha}$, that is, $\|\alpha N_\alpha^{-1}\| \leq 1$ (see [13, Theorem 3.2]). □

Lemma 16. *For each $b \in \mathbb{R}^m$, there exists a unique vector $v_b \in \mathrm{col}(A^\dagger) = N(A)^\perp$ such that $x_{0,b} = A^T A v_b$, namely, $v_b = \sum_{i=1}^r \frac{\langle b, \psi_i \rangle}{\sigma_i^3} \varphi_i$.*

Proof. Since $N(A)^\perp = \mathrm{span}\{\varphi_1, ..., \varphi_r\}$, $v_b \in N(A)^\perp = \mathrm{col}(A^\dagger)$, and we have

$$A^T A v_b = A^T A \left(\sum_{i=1}^r \frac{\langle b, \psi_i \rangle}{\sigma_i^3} \varphi_i \right) = A^T \left(\sum_{i=1}^r \frac{\langle b, \psi_i \rangle}{\sigma_i^2} \psi_i \right) = \sum_{i=1}^r \frac{\langle b, \psi_i \rangle}{\sigma_i} \varphi_i = x_{0,y}.$$

Let $\hat{v}_b$ be another solution in $N(A)^\perp$ such that $x_{0,b} = A^T A \hat{v}_b$. Then $v_b - \hat{v}_b \in N(A)^\perp$ and $v_b - \hat{v}_b \in N(A^T A) = N(A)$, therefore $v_b - \hat{v}_b \in N(A) \cap N(A)^\perp = \{0\}$ and the proof is complete. □

Notice that v_b is the Lagrange multiplier for the problem

$$\min \|x\|$$

$$\text{s.t. } A^T A x = A^T b.$$

This vector will be used later.

3.3   Analysis of multiplicative regularization

To apply the desired form of the implicit function theorem, we must show that F is C^1. First we will show that the F is a continuous function. We start with the following proof.

Lemma 17. *Let $b_0 \in \mathbb{R}^m$ be given. Then $x_{\alpha,b} \to x_{0,b_0}$ as $(\alpha, b) \to (0, b_0)$.*

Proof. We have

$$
\begin{aligned}
\lim_{(\alpha,b)\to(0,b_0)} x_{\alpha,b} &= \lim_{(\alpha,b)\to(0,b_0)} \sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha} \langle b, \psi_i \rangle \varphi_i \\
&= \lim_{(\alpha,b)\to(0,b_0)} \sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha} \langle b - b_0, \psi_i \rangle \varphi_i + \lim_{\alpha\to0} \sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha} \langle b_0, \psi_i \rangle \varphi_i \\
&= \sum_{i=1}^{r} \frac{\langle b_0, \psi_i \rangle}{\sigma_i} \varphi_i = x_{0,b_0}.
\end{aligned}
$$

$\square$

Theorem 18. *The function $F : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ defined by*

$$F(\alpha, b) = \begin{cases} \|Ax_{\alpha,b} - b\|^2 - \alpha \|x_{\alpha,b}\|^2, & \alpha > 0, \\ \\ \|\hat{b}\|^2. & \alpha = 0, \end{cases}$$

is continuous.

Proof. It is obvious that F is continuous on $(0, \infty) \times \mathbb{R}^m$. We must show that, for all $b_0 \in \mathbb{R}^m$,

$$\lim_{(\alpha,b)\to(0,b_0)} F(\alpha, b) = F(0, b_0).$$

We have

$$\lim_{(\alpha,b)\to(0,b_0)} F(\alpha, b) = \lim_{(\alpha,b)\to(0,b_0)} \left(\sum_{i=1}^{r} \frac{\alpha^2 - \alpha\sigma_i^2}{(\sigma_i^2 + \alpha)^2} |\langle b, \psi_i \rangle|^2 + \sum_{i=r+1}^{n} |\langle b, \psi_i \rangle|^2 \right)$$

$$= \lim_{b\to b_0} \sum_{i=r+1}^{n} |\langle b, \psi_i \rangle|^2 = \sum_{i=r+1}^{n} |\langle b_0, \psi_i \rangle|^2 = \|\hat{b}_0\|^2 = F(0, b_0).$$

Notice that

$$\lim_{(\alpha,b)\to(0,b_0)} \sum_{i=1}^{r} \frac{\alpha^2 - \alpha\sigma_i^2}{(\sigma_i^2 + \alpha)^2} |\langle b, \psi_i \rangle|^2 = 0,$$

because $\frac{\alpha^2 - \alpha\sigma_i^2}{(\sigma_i^2 + \alpha)^2} \to 0$ as $\alpha \to 0$, and also $\langle b, \psi_i \rangle \to \langle b_0, \psi_i \rangle$ as $b \to b_0$. $\square$

We can show that $x'_{\alpha,b} \to -v_b$ as $\alpha \to 0$. It is clear that $x'_{\alpha,b}$, in terms of SVD, is

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$$x'_{\alpha,b} = \frac{\partial}{\partial \alpha}(x_{\alpha,b}) = \frac{\partial}{\partial \alpha}\left(\sum_{i=1}^{r} \frac{\sigma_i}{\sigma_i^2 + \alpha}\langle b, \psi_i \rangle \varphi_i\right) = \sum_{i=1}^{r} \frac{-\sigma_i}{(\sigma_i^2 + \alpha)^2}\langle b, \psi_i \rangle \varphi_i.$$

Now we can prove the following lemma.

Theorem 19. *For all $b \in \mathbb{R}^m$, $x'_{\alpha,b} \to -v_b$ as $\alpha \to 0$.*

Proof. We have

$$\lim_{\alpha \to 0} x'_{\alpha,b} = \lim_{\alpha \to 0}\left(\sum_{i=1}^{r} \frac{-\sigma_i}{(\sigma_i^2 + \alpha)^2}\langle b, \psi_i \rangle \varphi_i\right) = -\sum_{i=1}^{r} \frac{\langle b, \psi_i \rangle}{\sigma_i^3}\varphi_i = -v_b,$$

and the proof is complete. $\qquad\qquad\square$

Theorem 20. *The function F defined by (3.2.1) is C^1, with*

$$\frac{\partial F}{\partial \alpha}(\alpha, b) = \begin{cases} -4\alpha x_{\alpha,b} \cdot x'_{\alpha,b} - \|x_{\alpha,b}\|^2, & \alpha > 0, \\[2mm] -\|x_{0,b}\|^2, & \alpha = 0, \end{cases} \qquad (3.3.1)$$

and

$$\nabla_b F(\alpha, b) = \begin{cases} 2(b - Ax_{\alpha,b} + \alpha A x'_{\alpha,b}), & \alpha > 0, \\[2mm] 2\hat{b}, & \alpha = 0. \end{cases} \qquad (3.3.2)$$

Proof. Once again, it is straightforward to show that F is C^1 for $\alpha > 0$. We have

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$$\frac{\partial F}{\partial \alpha}(\alpha, b) = \frac{\partial}{\partial \alpha}(\|Ax_{\alpha,b} - b\|^2 - \alpha\|x_{\alpha,b}\|^2)$$

$$= -2\alpha\langle x_{\alpha,b}, x'_{\alpha,b}\rangle - 2\alpha\langle x_{\alpha,b}, x'_{\alpha,b}\rangle - \|x_{\alpha,b}\|^2$$

$$= -4\alpha\langle x_{\alpha,b}, x'_{\alpha,b}\rangle - \|x_{\alpha,b}\|^2,$$

since $\frac{\partial}{\partial \alpha}(\|Ax_{\alpha,b} - b\|^2) = -\alpha\frac{\partial}{\partial \alpha}(\|x_{\alpha,b}\|^2) = -2\alpha\langle x_{\alpha,b}, x'_{\alpha,b}\rangle$. Also, we have

$$F(\alpha, b) = \langle Ax_{\alpha,b}, Ax_{\alpha,b}\rangle - 2\langle b, Ax_{\alpha,b}\rangle + \langle b, b\rangle - \alpha\langle x_{\alpha,b}, x_{\alpha,b}\rangle$$

$$= \langle AN_\alpha^{-1}A^Tb, AN_\alpha^{-1}A^Tb\rangle - 2\langle b, AN_\alpha^{-1}A^Tb\rangle + \langle b, b\rangle - \alpha\langle N_\alpha^{-1}A^Tb, N_\alpha^{-1}A^Tb\rangle$$

$$= \langle b, AN_\alpha^{-1}A^T AN_\alpha^{-1}A^Tb\rangle - 2\langle b, AN_\alpha^{-1}A^Tb\rangle + \langle b, b\rangle - \alpha\langle b, AN_\alpha^{-2}A^Tb\rangle,$$

and hence

$$\nabla_b F(\alpha, b) = 2AN_\alpha^{-1}A^T Ax_{\alpha,b} - 4Ax_{\alpha,b} + 2b + 2\alpha Ax'_{\alpha,b}$$

$$= 2A(I - \alpha N_\alpha^{-1})x_{\alpha,b} - 4Ax_{\alpha,b} + 2b + 2\alpha Ax'_{\alpha,b}$$

$$= 2Ax_{\alpha,b} + 2\alpha Ax'_{\alpha,b} - 4Ax_{\alpha,b} + 2b + 2\alpha Ax'_{\alpha,b}$$

$$= 2(b - Ax_{\alpha,b} + 2\alpha Ax'_{\alpha,b}).$$

We must show that, for all $b_0 \in \mathbb{R}^m$, $\frac{\partial F}{\partial \alpha}(0, b_0)$ and $\nabla_b F(0, b_0)$ exist and $\frac{\partial F}{\partial \alpha}(\alpha, b) \to$ $\frac{\partial F}{\partial \alpha}(0, b_0)$ and $\nabla_b F(\alpha, b) \to \nabla_b F(0, b_0)$ as $(\alpha, b) \to (0, b_0)$. We have

$$\frac{\partial F}{\partial \alpha}(0,b) = \lim_{\alpha \to 0^+} \frac{F(\alpha,b) - F(0,b)}{\alpha} = \lim_{\alpha \to 0^+} \left(\sum_{i=1}^{r} \frac{\alpha - \sigma_i^2}{(\sigma_i^2 + \alpha)^2} |\langle b, \psi_i \rangle|^2 \right)$$

$$= -\sum_{i=1}^{r} \frac{|\langle b, \psi_i \rangle|^2}{\sigma_i^2} = -\|x_{0,b}\|^2.$$

Also we have

$$\frac{\partial F}{\partial \alpha}(\alpha,b) = \frac{\partial}{\partial \alpha} \left(\sum_{i=1}^{r} \frac{\alpha^2 - \alpha\sigma_i^2}{(\sigma_i^2 + \alpha)^2} |\langle b, \psi_i \rangle|^2 + \sum_{i=r+1}^{n} |\langle b, \psi_i \rangle|^2 \right)$$

$$= \sum_{i=1}^{r} \frac{3\alpha\sigma_i^2 - \sigma_i^4}{(\sigma_i^2 + \alpha)^3} |\langle b, \psi_i \rangle|^2,$$

and we see that

$$\lim_{(\alpha,b) \to (0^+, b_0)} \frac{\partial F}{\partial \alpha}(\alpha,b) = \lim_{(\alpha,b) \to (0^+, b_0)} \left(\sum_{i=1}^{r} \frac{3\alpha\sigma_i^2 - \sigma_i^4}{(\sigma_i^2 + \alpha)^3} |\langle b, \psi_i \rangle|^2 \right)$$

$$= -\sum_{i=1}^{r} \frac{|\langle b_0, \psi_i \rangle|^2}{\sigma_i^2} = -\|x_{0,b_0}\|^2 = \frac{\partial F}{\partial \alpha}(0, b_0).$$

It follows that $\lim_{(\alpha,b) \to (0^+, b_0)} \frac{\partial F}{\partial \alpha}(\alpha,b) = \frac{\partial F}{\partial \alpha}(0,b)$. Also we have $F(0,b) = \|\hat{b}\|^2$, and it follows that $\nabla_b F(0,b) = 2\hat{b}$. Therefore we have

$$\lim_{(\alpha,b) \to (0,b_0)} \nabla_b F(\alpha,b) = \lim_{(\alpha,b) \to (0,b_0)} 2(b - Ax_{\alpha,b} + 2\alpha Ax'_{\alpha,b})$$

$$= 2(b_0 - Ax_{0,b_0}) = 2\hat{b}_0 = \nabla_b F(0,b_0).$$

$\square$

The function F is defined for all $\alpha \geq 0$. To apply the implicit function theorem, we must extend F to all of $\mathbb{R} \times \mathbb{R}^m$. For a fixed b, we define the function $G : [0, \infty) \to \mathbb{R}$ as $G(\alpha) = F(\alpha, b)$, and shift it vertically to get $\hat{G}(\alpha) = F(\alpha, b) - F(0, b)$. Then $\hat{G}$ satisfies $\hat{G}(0) = 0$. Now we form the odd extension of $\hat{G}$ to get

$$
\hat{G}(\alpha) = \begin{cases} \hat{G}(\alpha) = F(\alpha, b) - F(0, b), & \alpha \geq 0, \\ -\hat{G}(-\alpha) = -F(-\alpha, b) + F(0, b), & \alpha < 0. \end{cases}
$$

Finally we shift the function $\hat{G}(\alpha)$ back to get

$$
G(\alpha) = \begin{cases} F(\alpha, b), & \alpha \geq 0, \\ -F(-\alpha, b) + 2F(0, b), & \alpha < 0. \end{cases}
$$

Therefore, we define $F(-\alpha, b) = 2F(0, b) - F(\alpha, b)$ for all $\alpha > 0, b \in \mathbb{R}^m$. It is a standard construction in elementary calculus that the odd extension of a C^1 function (passing through the origin) is C^1. Therefore, it is easy to show that the function F, thus extended to all of $\mathbb{R} \times \mathbb{R}^m$, is C^1. Now we apply the implicit function theorem to show that $F(\alpha, b) = 0$ has a unique solution for each b sufficiently close to b^*. Let $b^* \in \mathrm{col}(A)$ be given. It follows that $F(0, b^*) = 0$ and $\frac{\partial F}{\partial \alpha}(0, b^*) = -\|x_{0, b^*}\|^2 < 0$. (Note that we assume $b^* \notin N(A^T)$, and hence $x_{0, b^*} \neq 0$.) Therefore, the implicit function theorem applies. Then there exist $\delta_1, \delta_2 > 0$ and a C^1 function $r : B_{\delta_2}(b^*) \to (-\delta_1, \delta_1)$ such that for each $b \in B_{\delta_2}(b^*)$, $\alpha = r(b)$ is the unique solution of $F(\alpha, b) = 0$ in $(-\delta_1, \delta_1)$.

We can also show that the solution $\alpha = r(b)$ of $F(\alpha, b) = 0$ must satisfy $\alpha \geq 0$. For each $b \in B_{\delta_2}(b^*)$, we can apply the mean value theorem to find $t \in (0, 1)$ such that

$$F(\alpha, b) = F(0, b) + \frac{\partial F}{\partial \alpha}(t\alpha, b)\alpha.$$

If we take $\alpha = r(b)$, we obtain $F(0, b) + \frac{\partial F}{\partial \alpha}(t\alpha, b)\alpha = 0$, which implies that $\frac{\partial F}{\partial \alpha}(t\alpha, b)\alpha = -F(0, b) \leq 0$. We just need to prove that $\frac{\partial F}{\partial \alpha}(t\ r(b), b) < 0$ for all $b \in B_{\delta_2}(b^*)$. From the continuity of $\frac{\partial F}{\partial \alpha}$ and r, by reducing δ_1 and δ_2 if necessary, it follows that $\frac{\partial F}{\partial \alpha}(t\alpha, b) < 0$, and hence that $\alpha = r(b) \geq 0$.

For $b \in \mathrm{col}(A)$, $\alpha = r(b) = 0$ and for $b \notin \mathrm{col}(A)$, $\|Ax_{\alpha,b} - b\| > 0$ and hence $\|Ax_{\alpha,b} - b\|^2 - \alpha\|x_{\alpha,b}\|^2 = 0$ implies that $\alpha \neq 0$. Thus $\alpha = r(b) > 0$ for $b \notin \mathrm{col}(A)$.

We have seen that $x_{\alpha,b}$ with $\alpha = r(b)$, is a stationary point of $J(\cdot; b)$. We can show that it is a local minimizer of $J(\cdot; b)$ by proving that $\nabla^2_{xx}J(x_{\alpha,b}, b)$ is positive definite. We have

$$\nabla^2_{xx}J(x, b) = \nabla^2_{xx}\left(\frac{1}{2}\|Ax - b\|^2\|x\|^2\right) = \nabla_x(\|x\|^2(A^T Ax - A^T b) + \|Ax - b\|^2 x)$$

$$= \nabla_x(\|x\|^2)(A^T Ax - A^T b) + \|x\|^2 \nabla_x(A^T Ax - A^T b)$$

$$+ \nabla_x(\|Ax - b\|^2)x + \|Ax - b\|^2 \nabla_x(x)$$

$$= \|x\|^2(A^T A) + 2((A^T Ax - A^T b)x^T + 2x(A^T Ax - A^T b)^T) + \|Ax - b\|^2 I,$$

and for $x = x_{\alpha,b}$, since $A^T A x_{\alpha,b} - A^T b = -\alpha x_{\alpha,b}$, then it follows that

$$\nabla^2_{xx} J(x_{\alpha,b}, b) = \|x_{\alpha,b}\|^2 (A^T A) - 4\alpha x_{\alpha,b} x^T_{\alpha,b} + \alpha \|x_{\alpha,b}\|^2 I$$

$$= \|x_{\alpha,b}\|^2 N_\alpha - 4\alpha x_{\alpha,b} x^T_{\alpha,b}.$$

We will now show that for $\alpha = r(b)$, $J(x_{\alpha,b}, b)$ is positive definite by showing that $\langle v, \nabla^2_{xx} J(x_{\alpha,b}, b) v \rangle > 0$ for every $v \in \mathbb{R}^n$, $v \neq 0$. We first show that every vector $v \in \mathbb{R}^n = U \oplus U^\perp$ (where $U = \text{span}\{x_{\alpha,b}\}^\perp$) can be represented uniquely as $v = u + \beta x'_{\alpha,b}$, where $u \in U$ and $\beta \in \mathbb{R}$. We know that $x'_{\alpha,b} \in \mathbb{R}^n$ can be written as $x'_{\alpha,b} = \hat{u} + \gamma x_{\alpha,b}$ for $\hat{u} \in U$, $\gamma \in \mathbb{R}$ and $\gamma \neq 0$ (because if $\gamma = 0$, then that implies $x'_{\alpha,b} \in \text{span}\{x_{\alpha,b}\}^\perp$ and $\langle x_{\alpha,b}, x'_{\alpha,b} \rangle = 0$, which is a contradiction for $b \notin N(A^T)$). It follows that $x_{\alpha,b} = \gamma^{-1} x'_{\alpha,b} - \gamma^{-1} \hat{u}$ and for $v \in \mathbb{R}^n$ we have

$$v = \overline{u} + \lambda x_{\alpha,b} = \overline{u} + \lambda(\gamma^{-1} x'_{\alpha,b} - \gamma^{-1}\hat{u}) = (\overline{u} - \lambda \gamma^{-1}\hat{u}) + \lambda \gamma^{-1} x'_{\alpha,b}.$$

Thus $v = u + \beta x'_{\alpha,b}$, where $u = \overline{u} - \lambda \gamma^{-1}\hat{u} \in U$ and $\beta = \lambda \gamma^{-1} \in \mathbb{R}$. Now suppose there exist $u_1 \in U$ and $\beta_1 \in \mathbb{R}$ such that $v = u + \beta x'_{\alpha,b} = u_1 + \beta_1 x'_{\alpha,b}$. Then we have $u - u_1 = \beta_1 x'_{\alpha,b} - \beta x'_{\alpha,b}$, where $u - u_1 \in U$ and $(\beta_1 - \beta) x'_{\alpha,b} \in \mathbb{R}^n$. Therefore $(\beta_1 - \beta) x'_{\alpha,b} = u - u_1 \in U$, that is, $(\beta_1 - \beta)\langle x_{\alpha,b}, x'_{\alpha,b} \rangle = 0$ and therefore $\beta_1 - \beta = 0$ since $\langle x_{\alpha,b}, x'_{\alpha,b} \rangle \neq 0$. Also it follows that $u - u_1 = 0$ and thus every $v \in \mathbb{R}^n$ has a unique representation.

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Now for all $v \in \mathbb{R}^n$, if $v = u + \beta x'_{\alpha,b}$ and $v \neq 0$, then we have

$$\langle v, \nabla^2_{xx} J(x_{\alpha,b}, b)v \rangle = \langle v, (\|x_{\alpha,b}\|^2 N_\alpha - 4\alpha x_{\alpha,b} x^T_{\alpha,b})v \rangle$$

$$= \|x_{\alpha,b}\|^2 \langle v, N_\alpha v \rangle - 4\alpha |\langle v, x_{\alpha,b} \rangle|^2$$

$$= \|x_{\alpha,b}\|^2 \langle (u + \beta x'_{\alpha,b}), N_\alpha(u + \beta x'_{\alpha,b}) \rangle - 4\alpha |\langle (u + \beta x'_{\alpha,b}), x_{\alpha,b} \rangle|^2$$

$$= \|x_{\alpha,b}\|^2 (\langle u, N_\alpha u \rangle + 2\beta \langle x'_{\alpha,b}, N_\alpha u \rangle + \beta^2 \langle x'_{\alpha,b}, N_\alpha x'_{\alpha,b} \rangle) - 4\alpha |\langle (u + \beta x'_{\alpha,b}), x_{\alpha,b} \rangle|^2$$

$$= \|x_{\alpha,b}\|^2 (\langle u, N_\alpha u \rangle - 2\beta \langle x_{\alpha,b}, u \rangle + \beta^2 \langle x'_{\alpha,b}, N_\alpha x'_{\alpha,b} \rangle) - 4\alpha |\langle u, x_{\alpha,b} \rangle + \beta \langle x'_{\alpha,b}, x_{\alpha,b} \rangle|^2$$

$$= \|x_{\alpha,b}\|^2 \langle u, N_\alpha u \rangle - \beta^2 \|x_{\alpha,b}\|^2 \langle x_{\alpha,b}, x'_{\alpha,b} \rangle - 4\alpha\beta^2 |\langle x_{\alpha,b}, x'_{\alpha,b} \rangle|^2 \text{ (since } \langle u, x_{\alpha,b} \rangle = 0)$$

$$= \|x_{\alpha,b}\|^2 \langle u, N_\alpha u \rangle - \beta^2 \langle x_{\alpha,b}, x'_{\alpha,b} \rangle(\|x_{\alpha,b}\|^2 + 4\alpha \langle x_{\alpha,b}, x'_{\alpha,b} \rangle)$$

$$= \|x_{\alpha,b}\|^2 \langle u, N_\alpha u \rangle + \beta^2 \langle x_{\alpha,b}, N_\alpha x_{\alpha,b} \rangle(\|x_{\alpha,b}\|^2 - 4\alpha \langle x_{\alpha,b}, N_\alpha^{-1} x_{\alpha,b} \rangle).$$

Recall that $\frac{\partial F}{\partial \alpha}(\alpha, b) = 4\alpha \langle x_{\alpha,b}, N_\alpha x_{\alpha,b} \rangle - \|x_{\alpha,b}\|^2 < 0$ for $\alpha = r(b)$. Also if $v \neq 0$, then $u \neq 0$ or $\beta \neq 0$ (or both). If $u \neq 0$, then $\langle u, N_\alpha u \rangle > 0$ and if $\beta \neq 0$, then

$$\beta^2 \langle x_{\alpha,b}, N_\alpha x_{\alpha,b} \rangle(\|x_{\alpha,b}\|^2 - 4\alpha \langle x_{\alpha,b}, N_\alpha^{-1} x_{\alpha,b} \rangle) > 0.$$

Therefore,

$$\langle v, \nabla^2_{xx} J(x_{\alpha,b}, b)v \rangle = \|x_{\alpha,b}\|^2 \langle u, N_\alpha u \rangle + \beta^2 \langle x_{\alpha,b}, N_\alpha x_{\alpha,b} \rangle(\|x_{\alpha,b}\|^2 - 4\alpha \langle x_{\alpha,b}, N_\alpha^{-1} x_{\alpha,b} \rangle) > 0.$$

We now see that there exists $\delta_2 > 0$ such that for all $b \in B_{\delta_2}(b^*) \setminus \operatorname{col}(A)$, $J(\cdot; b)$ has a local minimizer $x_{\alpha,b}$, where $\alpha = r(b)$ satisfies $\alpha = \frac{\|Ax_{\alpha,b} - b\|^2}{\|x_{\alpha,b}\|^2}$. We can also say that this α is a local minimizer of $\varphi(\cdot; b)$.

3.4   Rate of convergence

The standard theory of Tikhonov regularization for an infinite-dimensional problem

shows that if $\alpha = r(b) \to 0$ as $b \to b^*$ and

- $\frac{\|b-b^*\|^2}{\alpha} \to 0$, then $x_{\alpha,b} \to x_{0,b^*}$ strongly;

- $\frac{\|b-b^*\|^2}{\alpha} \leq C$, then $x_{\alpha,b} \to x_{0,b^*}$ weakly;

- $\frac{\|b-b^*\|^2}{\alpha} \to \infty$, $x_{\alpha,b} \nrightarrow x_{0,b^*}$ $(\|x_{\alpha,b}\| \to \infty)$.

In a finite-dimensional problem, $\alpha = r(b) \to 0$ as $b \to b^*$ is sufficient to imply that

$x_{\alpha,b} \to x_{0,b^*}$. This can be proved as follows

$$\|x_{\alpha,b} - x_{0,b^*}\| = \|x_{\alpha,b} - x_{\alpha,b^*} + x_{\alpha,b^*} - x_{0,b^*}\| \leq \|N_\alpha^{-1}T^*(b-b^*)\| + \|x_{\alpha,b^*} - x_{0,b^*}\|$$

$$\leq \|N_\alpha^{-1}T^*\|\|b-b^*\| + \|x_{\alpha,b^*} - x_{0,b^*}\|$$

$$\leq \|T^\dagger\|\|b-b^*\| + \|x_{\alpha,b^*} - x_{0,b^*}\| \to 0, \text{ as } b \to b^*.$$

Nevertheless, since most interesting finite-dimensional problems arise from discretiz-

ing infinite-dimensional problems, we would like to investigate the behavior of $\frac{\|b-b^*\|^2}{\alpha}$

as $b \to b^*$.

Let $b \in \mathbb{R}^m$. We define a new function

$$f(\alpha, b) = \|\hat{b}\|^2 - F(\alpha, b) = \begin{cases} \alpha\|x_{\alpha,b}\|^2 - \|Ax_{\alpha,b} - \bar{b}\|^2, & \alpha > 0, \\ \\ 0, & \alpha = 0. \end{cases}$$

Then $F(\alpha, b) = 0$ is equivalent to $f(\alpha, b) = \|\hat{b}\|^2$. Thus $\alpha = r(b)$ satisfies

$$\alpha = \frac{\|\hat{b}\|^2}{\|x_{\alpha,b}\|^2 - \alpha^{-1}\|Ax_{\alpha,b} - \bar{b}\|^2}.$$

We can show that $\lim_{(\alpha,b)\to(0,b^*)}(\|x_{\alpha,b}\|^2 - \alpha^{-1}\|Ax_{\alpha,b} - \bar{b}\|^2) = \|x_{0,b^*}\|^2$.

That is true because

$$\lim_{(\alpha,b)\to(0,b^*)}(\|x_{\alpha,b}\|^2 - \alpha^{-1}\|Ax_{\alpha,b} - \bar{b}\|^2) = \lim_{(\alpha,b)\to(0,b^*)}\left(\sum_{i=1}^{r}\frac{\alpha - \sigma_i^2}{(\sigma_i^2 + \alpha)^2}|\langle b, \psi_i\rangle|^2 + \sum_{i=r+1}^{m}|\langle b, \psi_i\rangle|^2\right)$$

$$= \lim_{(\alpha,b)\to(0,b^*)}\sum_{i=1}^{r}\frac{\alpha - \sigma_i^2}{(\sigma_i^2 + \alpha)^2}|\langle b, \psi_i\rangle|^2 + \lim_{b\to b^*}\sum_{i=r+1}^{m}|\langle b, \psi_i\rangle|^2)$$

$$= -\sum_{i=1}^{r}\frac{|\langle b^*, \psi_i\rangle|^2}{\sigma_i^2} + \sum_{i=r+1}^{m}|\langle b^*, \psi_i\rangle|^2 = \|x_{0,b^*}\|^2.$$

Notice that $\langle b^*, \psi_i\rangle = 0$ for all $i = r + 1, ..., n$ because $b^* \in \mathrm{col}(A) = \mathrm{span}\{\psi_{r+1}, ..., \psi_n\}^\perp$. Therefore,

$$\frac{\|\hat{b}\|^2}{\alpha} = \|x_{\alpha,b}\|^2 - \alpha^{-1}\|Ax_{\alpha,b} - \bar{b}\|^2 \to \|x_{0,b^*}\|^2 \text{ as } b \to b^*,$$
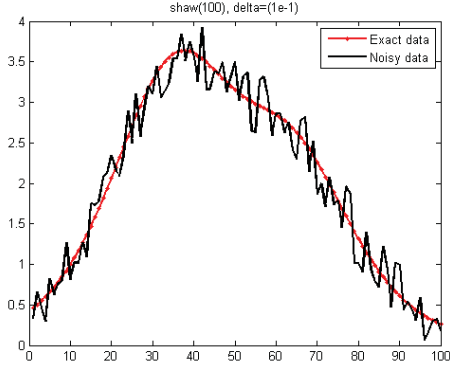
Figure 3.1: The exact data and the noisy data for the test problem "shaw" (n=100), with relative error 10^{-1} in the noisy data.

where $\alpha = r(b)$. We see that α goes to zero at the same rate as $\|\hat{b}\|^2$. This analysis is meaningful if $\hat{b} \neq 0$, which is not possible if $\text{col}(A)$ is all of $\mathbb{R}^m$. If $\text{col}(A) = \mathbb{R}^m$, then multiplicative regularization does not work, that is, it produces $\alpha = r(b) = 0$.

3.4.1 Numerical examples

For our numerical experiments, we use eight test problems from Hansen's Regularization Tools package [22]. See Section 1.6 for a description of these problems. For each experiment, we generate noisy vectors b for several different relative noise levels $\delta = \frac{\|b - b^*\|}{\|b^*\|}$ (see Figure 3.1 for a typical example). The components of the noise vector are uniformly distributed random numbers selected from an interval centered at zero.

Our numerical tests verify that $\alpha \to 0$ as $\delta \to 0$, where α is the regularization parameter selected by multiplicative regularization. If we fit α to a formula of the

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Table 3.1

The rate of convergence for multiplicative regularization

Multiplicative Regularization	
Test problem	p
deriv2	−
phillips	−
heat	2.288
i_laplace	1.9565
foxgood	2.0097
shaw	2.0134
baart	2.0228
wing	2.0692

form $\alpha = C\delta^p$ (where C and p are constant), then we find the values of p shown in Table 3.1 for the test problems.

Multiplicative regularization fails on "deriv2" and "phillips", which will be discussed in Section 3.5.1. The test problem "heat" is a special case and the convergence is slightly faster than predicted. That is probably because most of the components of b are in the col(A) (A is 100×100 and rank(A) = 97). For the other problems, the results are as expected.

3.5   Comparison with the L-curve method

To see the performance of multiplicative regularization, we compare the computed solutions with results obtained by the L-curve method for different noise levels. We
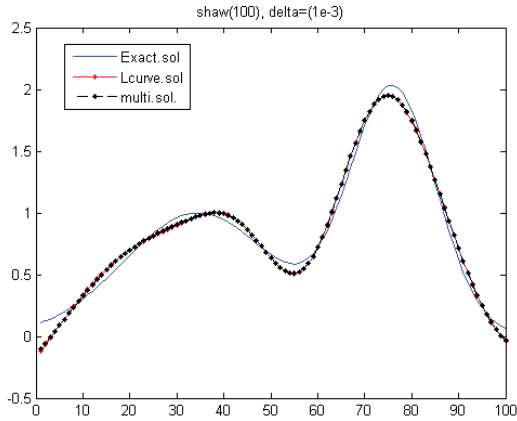
Figure 3.2: Tikhonov regularization: The exact solution for the "shaw" problem, together with the regularized solutions produced by the L-curve method and multiplicative regularization. In this example, the results of the two methods are essentially the same.
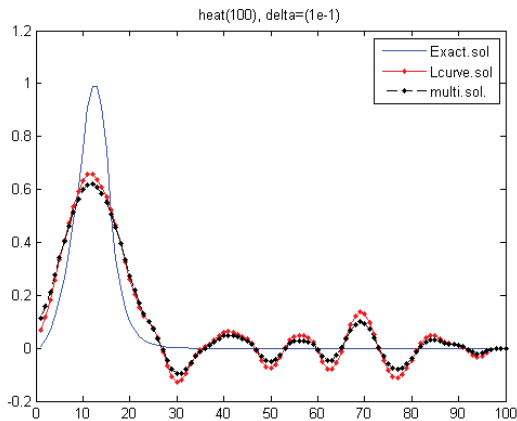
Figure 3.3: Tikhonov regularization: The exact solution for the "heat" problem, together with the regularized solutions produced by the L-curve method and multiplicative regularization. In this example, the two methods produce similar estimates, but the L-curve estimate is slightly better.

compare the solutions produced by both methods on the basis of the error $\frac{\|x-x^*\|}{\|x^*\|}$.

We classify the results of each experiment into one of three categories: we use (MR) if
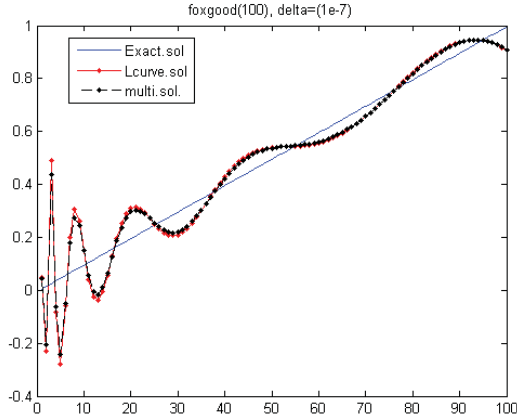
Figure 3.4: The exact solution for the "foxgood" problem, together with the regularized solutions produced by the L-curve method and multiplicative regularization. In this example, the two methods produce similar estimates, but multiplicative regularization estimate is slightly better.

multiplicative regularization produces a better solution, (LC) if the L-curve method performs better and (−) if the results are essentially the same. One solution is classified as the better solution if the error of that solution is at least 10% smaller than the error of the other one. We tested the method on eight problems with seven noise levels for each, a total of 56 cases. On four problems (28 cases), the performance of the two methods is essentially identical (see Figure 3.2 for a typical example). On two problems (14 cases), the performance is similar, but sometimes the L-curve method is slightly better (see Figure 3.2) and sometimes multiplicative regularization is somewhat better (see Figure 3.3). In these cases, L-curve is usually the better method. On two problems (14 cases), multiplicative regularization fails for the reason discussed earlier. The L-curve method also failed in three of these 14 cases.

Table 3.2

Comparison of multiplicative regularization with the L-curve method

	deriv2				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$5.7566 \cdot 10^{-5}$	0.41855	0	$--$	LC
10^{-2}	$7.1802 \cdot 10^{-7}$	0.27427	0	$--$	LC
10^{-3}	$5.7862 \cdot 10^{-9}$	0.30171	0	$--$	LC
10^{-4}	$7.2312 \cdot 10^{-10}$	0.12678	0	$--$	LC
10^{-5}	0	$--$	0	$--$	$-$
10^{-6}	0	$--$	0	$--$	$-$
10^{-7}	0	$--$	0	$--$	$-$
	phillips				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$1.6686 \cdot 10^{-1}$	0.090231	0	$--$	LC
10^{-2}	$3.3077 \cdot 10^{-3}$	0.12628	0	$--$	LC
10^{-3}	$2.2990 \cdot 10^{-5}$	0.21318	0	$--$	LC
10^{-4}	$2.4142 \cdot 10^{-7}$	0.23551	0	$--$	LC
10^{-5}	$1.9277 \cdot 10^{-10}$	0.53181	0	$--$	LC
10^{-6}	$3.0106 \cdot 10^{-10}$	0.055528	0	$--$	LC
10^{-7}	$2.7930 \cdot 10^{-11}$	0.013930	0	$--$	LC
	heat				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$2.1189 \cdot 10^{-4}$	0.30626	$5.4788 \cdot 10^{-4}$	0.34348	LC
10^{-2}	$3.5426 \cdot 10^{-6}$	0.20621	$2.7224 \cdot 10^{-6}$	0.23292	LC
10^{-3}	$2.5138 \cdot 10^{-8}$	0.28105	$1.1999 \cdot 10^{-8}$	0.41573	LC
10^{-4}	$8.5424 \cdot 10^{-11}$	0.36220	$4.3379 \cdot 10^{-11}$	0.47847	LC
10^{-5}	$2.7944 \cdot 10^{-14}$	0.41658	$8.7091 \cdot 10^{-14}$	0.36679	MR
10^{-6}	$1.2293 \cdot 10^{-18}$	0.041476	$2.8514 \cdot 10^{-16}$	0.041328	$-$
10^{-7}	$6.2984 \cdot 10^{-19}$	0.0044045	$1.8570 \cdot 10^{-17}$	0.0044042	$-$
	i_laplace				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$1.5499 \cdot 10^{-2}$	0.24465	$3.2401 \cdot 10^{-2}$	0.23914	$-$
10^{-2}	$1.5208 \cdot 10^{-4}$	0.15536	$2.9277 \cdot 10^{-4}$	0.14565	$-$
10^{-3}	$2.7505 \cdot 10^{-6}$	0.088633	$2.7436 \cdot 10^{-6}$	0.088650	$-$
10^{-4}	$2.8259 \cdot 10^{-8}$	0.061933	$2.7011 \cdot 10^{-8}$	0.062916	$-$
10^{-5}	$1.8743 \cdot 10^{-10}$	0.079925	$2.8186 \cdot 10^{-10}$	0.074567	$-$
10^{-6}	$4.7361 \cdot 10^{-12}$	0.034634	$2.6419 \cdot 10^{-12}$	0.051626	LC
10^{-7}	$3.1414 \cdot 10^{-14}$	0.071051	$2.7174 \cdot 10^{-14}$	0.074970	$-$

Table 3.3

Comparison of multiplicative regularization with the L-curve method

	foxgood				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$2.0157 \cdot 10^{-3}$	0.11777	$6.2084 \cdot 10^{-3}$	0.16537	LC
10^{-2}	$5.0232 \cdot 10^{-5}$	0.039497	$5.8650 \cdot 10^{-5}$	0.038102	–
10^{-3}	$6.9384 \cdot 10^{-7}$	0.020652	$5.6384 \cdot 10^{-7}$	0.022084	–
10^{-4}	$8.9824 \cdot 10^{-9}$	0.059639	$5.7022 \cdot 10^{-9}$	0.072235	LC
10^{-5}	$1.6246 \cdot 10^{-10}$	0.029053	$5.5865 \cdot 10^{-11}$	0.057751	LC
10^{-6}	$1.0776 \cdot 10^{-12}$	0.017196	$4.8795 \cdot 10^{-13}$	0.025274	LC
10^{-7}	$3.6618 \cdot 10^{-15}$	0.10967	$5.3269 \cdot 10^{-15}$	0.095239	MR
	shaw				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$2.4366 \cdot 10^{-2}$	0.18347	$5.9525 \cdot 10^{-2}$	0.18632	–
10^{-2}	$3.8168 \cdot 10^{-4}$	0.086376	$5.1872 \cdot 10^{-4}$	0.080728	–
10^{-3}	$4.8446 \cdot 10^{-6}$	0.084472	$5.0056 \cdot 10^{-6}$	0.083831	–
10^{-4}	$4.4892 \cdot 10^{-8}$	0.043844	$4.9867 \cdot 10^{-8}$	0.044030	–
10^{-5}	$8.1194 \cdot 10^{-10}$	0.049761	$4.9749 \cdot 10^{-10}$	0.053016	–
10^{-6}	$5.3854 \cdot 10^{-12}$	0.029093	$4.9204 \cdot 10^{-12}$	0.029143	–
10^{-7}	$1.8301 \cdot 10^{-14}$	0.066169	$4.4538 \cdot 10^{-14}$	0.061199	–
	baart				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$1.8203 \cdot 10^{-2}$	0.38809	$7.4751 \cdot 10^{-2}$	0.37298	–
10^{-2}	$2.0462 \cdot 10^{-4}$	0.15442	$5.5709 \cdot 10^{-4}$	0.16481	–
10^{-3}	$2.3146 \cdot 10^{-6}$	0.13694	$5.1749 \cdot 10^{-6}$	0.12438	–
10^{-4}	$5.2228 \cdot 10^{-8}$	0.059095	$5.0171 \cdot 10^{-8}$	0.058916	–
10^{-5}	$6.7614 \cdot 10^{-10}$	0.061845	$5.2440 \cdot 10^{-10}$	0.064648	–
10^{-6}	$6.2654 \cdot 10^{-12}$	0.053637	$5.2037 \cdot 10^{-12}$	0.054689	–
10^{-7}	$2.9745 \cdot 10^{-14}$	0.032913	$4.8572 \cdot 10^{-14}$	0.034044	–
	wing				
	L-curve		MR		
δ	α	error	α	error	better
10^{-1}	$3.8860 \cdot 10^{-3}$	0.73238	$2.4572 \cdot 10^{-3}$	0.70125	–
10^{-2}	$3.1947 \cdot 10^{-6}$	0.59493	$1.0035 \cdot 10^{-5}$	0.59475	–
10^{-3}	$5.5346 \cdot 10^{-8}$	0.59242	$9.8831 \cdot 10^{-8}$	0.59341	–
10^{-4}	$1.9538 \cdot 10^{-9}$	0.56667	$9.6922 \cdot 10^{-10}$	0.54962	–
10^{-5}	$3.4017 \cdot 10^{-12}$	0.44772	$7.4998 \cdot 10^{-12}$	0.44230	–
10^{-6}	$3.1521 \cdot 10^{-14}$	0.35107	$7.5627 \cdot 10^{-14}$	0.36499	–
10^{-7}	$4.0807 \cdot 10^{-16}$	0.32416	$6.8258 \cdot 10^{-16}$	0.32399	–

3.5.1 The failure of multiplicative regularization on "deriv2" and "phillips" test problems

The analysis shows that multiplicative regularization chooses a regularization parameter of zero when the noisy data vector lies in the column space of the matrix.

The test problems "deriv2" and "phillips" are both obtained by discretization of first kind Fredholm integral equations. The infinite-dimensional problems are both mildly ill-posed, that is, their singular values decay slowly to zero. In both cases, discretization yields a square, nonsingular matrix A and hence multiplicative regularization chooses $\alpha = 0$ for every b lies in the column space of A (see Figures 3.7, 3.8, 3.9 and 3.10). Also for "deriv2", the L-curve method fails itself for smaller noise levels (see Figure 3.10). In these cases, the L-curve does not have a corner.

We expect multiplicative regularization to work well on "deriv2" and "phillips" if the problem are discretized so that $A \in \mathbb{R}^{m \times n}$, $m > n$. Then $b \notin \text{col}(A)$ is likely to hold if b contains random noise. We discretized these problems to obtain $A \in \mathbb{R}^{m \times n}$, $m = 2n$. This corresponds to collecting more data. As shown in Table 3.4, multiplicative regularization works well with these discretizations and produces better solutions than does the L-curve most of the time (see Figures 3.11, 3.12, 3.13 and 3.14).

Table 3.4

Comparison of multiplicative regularization with the L-curve method

	Re-discretized deriv2				
	L-curve		multi-reg		
δ	α	error	α	error	better
10^{-1}	$2.6408 \cdot 10^{-5}$	0.34582	$9.3739 \cdot 10^{-5}$	0.39561	LC
10^{-2}	$3.0159 \cdot 10^{-7}$	0.24397	$6.7902 \cdot 10^{-7}$	0.23069	–
10^{-3}	$4.3886 \cdot 10^{-9}$	0.21248	$5.7205 \cdot 10^{-9}$	0.19077	MR
10^{-4}	$8.8420 \cdot 10^{-11}$	0.22389	$3.3278 \cdot 10^{-11}$	0.35279	MR
10^{-5}	$2.6816 \cdot 10^{-11}$	0.056869	$3.3157 \cdot 10^{-13}$	0.012736	MR
10^{-6}	0	——	$2.8832 \cdot 10^{-15}$	0.014071	MR
10^{-7}	0	——	$3.1382 \cdot 10^{-17}$	0.0014544	MR
	Re-discretized phillips				
	L-curve		multi-reg		
δ	α	error	α	error	better
10^{-1}	$5.8377 \cdot 10^{-2}$	0.067864	$2.6483 \cdot 10^{-1}$	0.037441	MR
10^{-2}	$1.6060 \cdot 10^{-3}$	0.099886	$2.4835 \cdot 10^{-3}$	0.075813	MR
10^{-3}	$2.0743 \cdot 10^{-5}$	0.096715	$2.3827 \cdot 10^{-5}$	0.089421	–
10^{-4}	$1.0972 \cdot 10^{-7}$	0.28569	$1.9927 \cdot 10^{-7}$	0.20812	MR
10^{-5}	$2.7038 \cdot 10^{-9}$	0.23465	$1.4817 \cdot 10^{-9}$	0.32184	LC
10^{-6}	$4.7798 \cdot 10^{-9}$	0.036692	$6.7599 \cdot 10^{-11}$	0.32768	MR
10^{-7}	$5.5115 \cdot 10^{-9}$	0.029525	$5.3987 \cdot 10^{-11}$	0.31202	LC

3.5.2 More details about the numerical examples

In this section, we will look carefully at some problems for which the two methods perform essentially the same and deal with some other problems for which the two methods differ noticeably.

Figures 3.5 and 3.6 show four typical cases, in which both methods work well. We graph the curvature of the L-curve and the function $\varphi(\cdot; b)$ in each figure. In each case, the curvature has a well-defined maximum and φ has a well-defined minimum. The two values are similar.
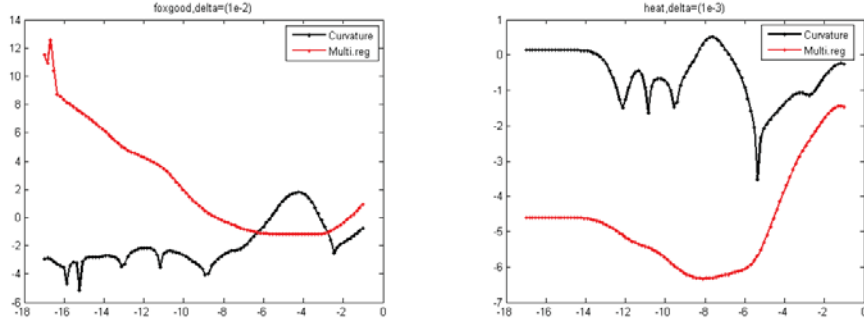
Figure 3.5: The test problem "foxgood", with the noise level 10^{-2} on the left and the test problem "heat", with the noise level 10^{-3} on the right. Both methods perform well.
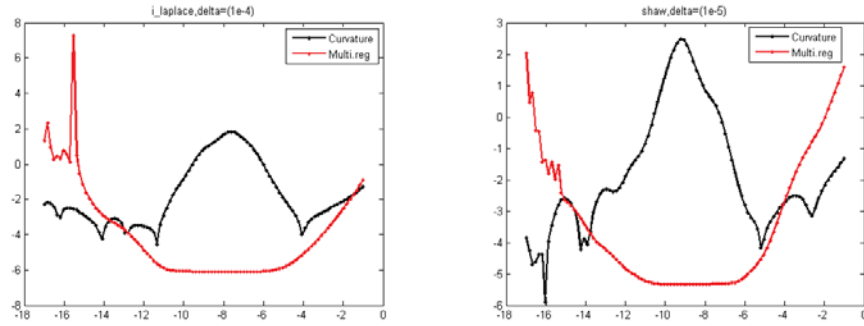
Figure 3.6: The test problem "i_laplace", with the noise level 10^{-4} on the left and the test problem "shaw", with the noise level 10^{-5} on the right. Both methods perform well.

Next, we present some cases in which the two methods differ. Figures 3.7, 3.8 and 3.9 are examples for which the L-curve method succeeds but the multiplicative regularization method fails because $b \in \text{col}(A)$. Notice that for these three cases, the curvature of the L-curve has a well-defined maximum. However, the smallest minimizer of $\varphi(\cdot; b)$, which is the value of α chosen by multiplicative regularization, is $\alpha = 0$ for all three problems.
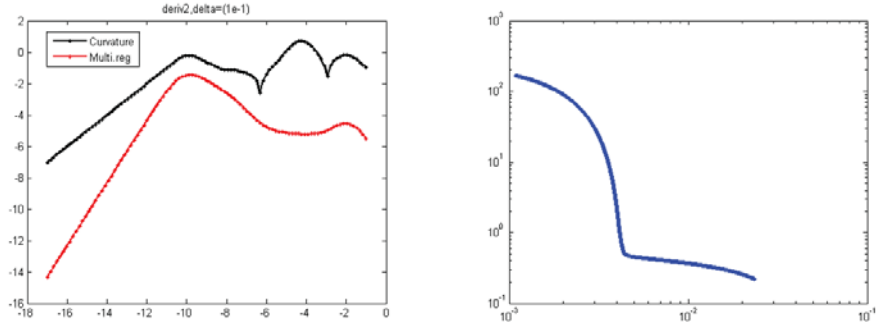
Figure 3.7: The test problem "deriv2" with relative error $\delta = 10^{-1}$ in the noisy data. The curvature has a well-defined corner but multiplicative regularization fails. The graph on the right is the graph of the L-curve.
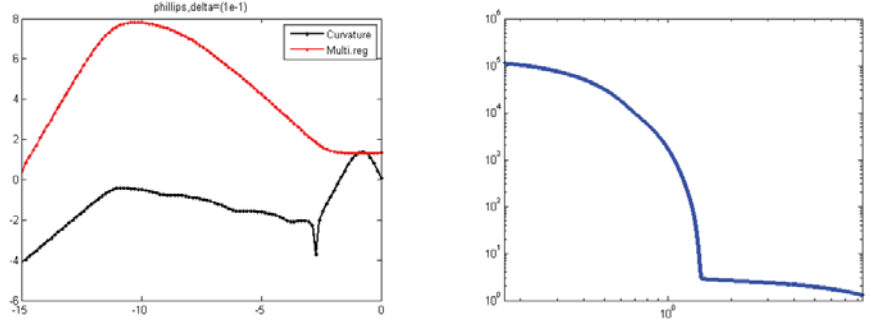
Figure 3.8: The test problem "phillips" with the noise level $\delta = 10^{-1}$ in the noisy data. The curvature has a well-defined corner but multiplicative regularization fails. The graph on the right is the graph of the L-curve.
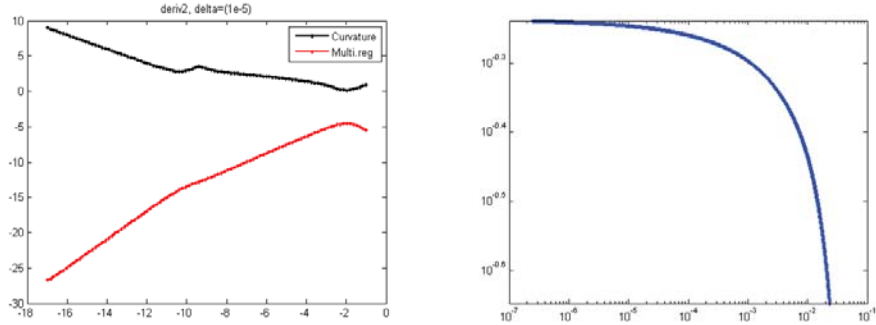
Figure 3.9: The test problem "deriv2" with relative error $\delta = 10^{-5}$ in the noisy data. Both methods fail to identify nonzero regularization parameters. The graph on the right is the graph of the L-curve and it does not have a corner.
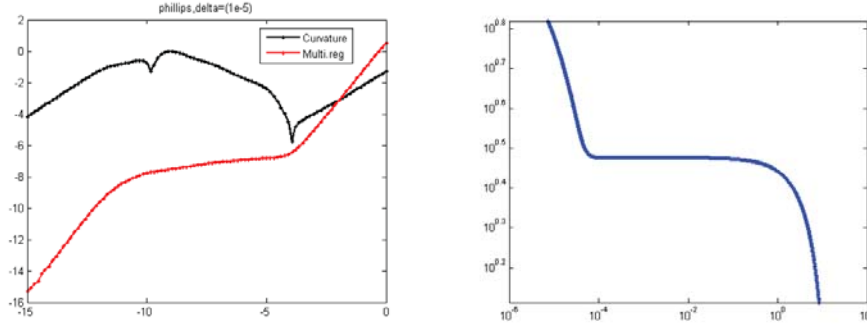
Figure 3.10: The test problem "phillips" with relative error $\delta = 10^{-5}$ in the noisy data. The curvature has a well-defined corner but multiplicative regularization fails. The graph on the right is the graph of the L-curve.

Figure 3.10 is an example for which the L-curve method and the multiplicative regularization method both fail. Notice that the curvature of the L-curve does not have a well-defined maximum and the smallest minimizer of $\varphi(\cdot; b)$ is $\alpha = 0$.

Figures 3.11 and 3.12 are examples of re-discretized test problems "deriv2" and "phillips" with relative error $\delta = 10^{-1}$ in the noisy data. The L-curve method succeeds and the multiplicative regularization method does not fail anymore since $b \notin \mathrm{col}(A)$. Notice that for both cases, the curvature of the L-curve has a well-defined maximum and $\varphi(\cdot; b)$ has a well-defined minimum.

Figure 3.13 is an example of (re-discretized) test problems "phillips" with relative error $\delta = 10^{-5}$ in the noisy data. Notice that the curvature of the L-curve has a well-defined maximum and $\varphi(\cdot; b)$ has a well-defined minimum. Since the smallest minimizer of $\varphi(\cdot; b)$ is not clear, we graph the fixed-point equation $\beta = f_b(\alpha)$ (where $f_b(\alpha) = \frac{\|Ax_{\alpha,b} - b\|^2}{\|x_{\alpha,b}\|^2}$). It is clear that the fixed-point equation finds a nonzero regularization

Figure 3.11: The re-descretized test problem "deriv2" with relative error $\delta = 10^{-1}$ in the noisy data. Both methods identify nonzero regularization parameters. The graph on the right is the graph of the L-curve and it has a well-defined corner.
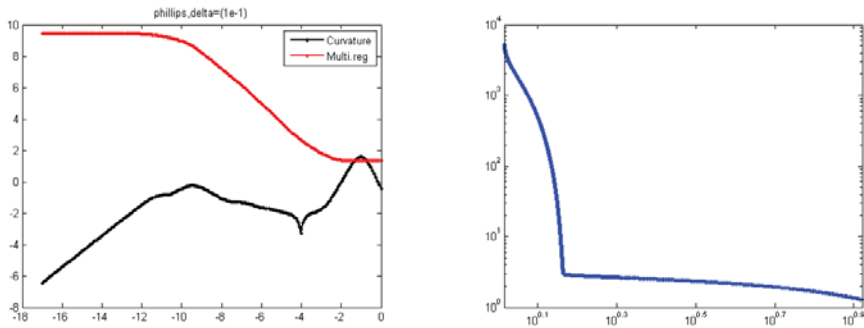
Figure 3.12: The re-discretized test problem "phillips" with relative error $\delta = 10^{-1}$ in the noisy data. Both methods identify nonzero regularization parameters. The graph on the right is the graph of the L-curve and has a well-defined corner.

parameter.
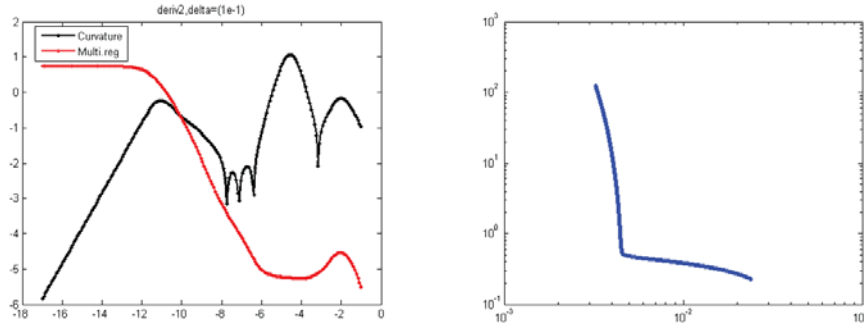
Figures 3.14 is an example of (re-discretized) test problems "deriv2" with relative error $\delta = 10^{-5}$ in the noisy data. The L-curve method fails since the curvature of the L-curve does not have a well-defined corner; however, $\varphi(\cdot; b)$ has a well-defined minimum. Multiplicative regularization performs well since the fixed point equation finds a nonzero regularization parameter.
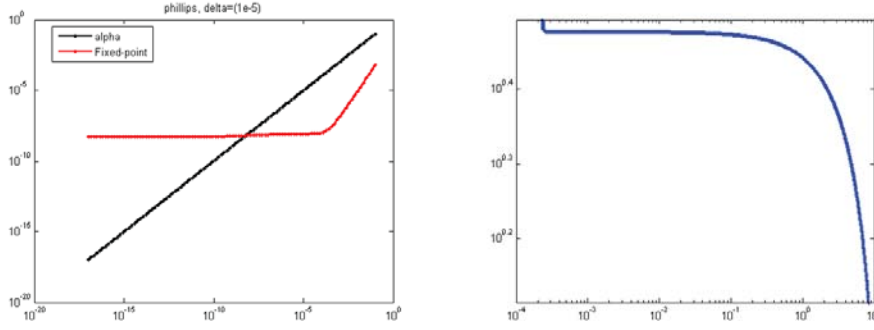
Figure 3.13: The re-discretized test problem "phillips" with relative error $\delta = 10^{-5}$ in the noisy data. Both methods identify nonzero regularization parameters.
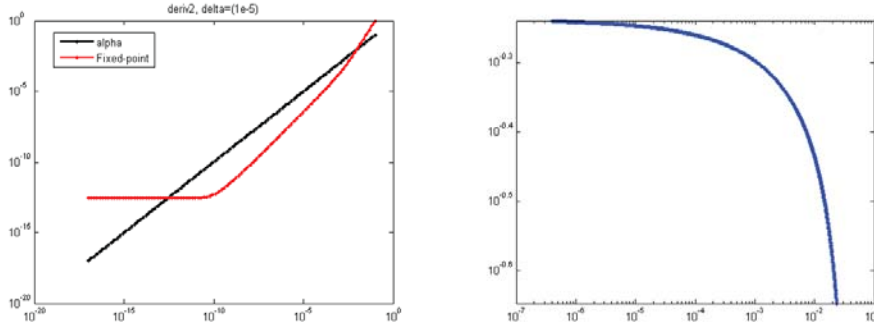
Figure 3.14: The re-discretized test problem "deriv2" with relative error $\delta = 10^{-5}$ in the noisy data. The multiplicative regularization identifies a nonzero regularization parameter and the L-curve method fails to identify a corner.

3.6   Conclusions

Based on the numerical results, both method perform remarkably similarly provided $A \in \mathbb{R}^{m \times n}$ and $\mathrm{col}(A)$ is a proper subspace of $\mathbb{R}^m$. Multiplicative regularization is easy to analyze, while there are no analytic results for L-curve. For instance, we cannot prove that the L-curve always has a corner; indeed, we have just seen examples

in which it does not. On the other hand, provided $b \notin \text{col}(A)$ and b is sufficiently close to b^*, we know that multiplicative regularization defines a positive regularization parameter α. Moreover, we know that $\alpha \to 0$ at the same rate as $\|\hat{b}\|^2$ as $b \to b^*$.

We know that for an infinite-dimensional inverse problem, a purely *a posteriori* parameter choice method can not be convergent. However, if we prove that these results extend to infinite dimensions, namely, that

$$y \to y^*, \|\hat{y}\|_Y \geq \theta \|y - y^*\|_Y \Rightarrow \alpha(y) \to 0 \text{ like } \|\hat{y}\|_Y^2, \qquad (3.6.1)$$

then we will obtain $x_{\alpha,y} \to x^*$ weakly. We know that as $y_k \to y$ and even if $\{y_k\}$ is almost contained in $R(T)$, then multiplicative regularization probably fails. We need to ensure that each y_k contains a significant component in $R(T)^{\perp}$. The condition $\|\hat{y}\|_Y \geq \theta \|y - y^*\|_Y$ is necessary in order to avoid the asymptotic convergence of y to y^* since $\alpha(y)$ should converge to zero at a slower rate than $\|\hat{y}\|_Y^2$. We are not able to prove (3.6.1), but in the next chapter we present some preliminary results.

Chapter 4

Multiplicative regularization for infinite-dimensional problems

4.1 Analysis of multiplicative regularization for infinite-dimensional problems

Let X and Y be Hilbert spaces and let $T : X \to Y$ be a nonzero bounded linear operator. We wish to estimate $x \in X$ that satisfies $Tx = y$, which is an inverse problem when $R(T)$ fails to be closed. Multiplicative regularization is based on minimizing the function

$$J(x; y) = \frac{1}{2}\|Tx - y\|_Y^2 \|x\|_X^2.$$

Assuming y is near y^*, we try to find a local minimizer of J near $x^* = T^\dagger y^*$. When $y \notin R(T)$, x is a nontrivial stationary point of $J(\cdot; y)$ if and only if $x = x_{\alpha, y}$, where

$$\alpha = \frac{\|Tx_{\alpha,y} - y\|_Y^2}{\|x_{\alpha,y}\|_X^2}. \tag{4.1.1}$$

We want to study (4.1.1) or equivalently $\|Tx_{\alpha,y} - y\|_Y^2 - \alpha\|x_{\alpha,y}\|_X^2 = 0$.

Let $y \in Y$. Write $y = \bar{y} + \hat{y}$ where $\bar{y} = \mathrm{proj}_{\overline{R(T)}} y$ and $\hat{y} = \mathrm{proj}_{R(T)^\perp} y$. We define the function $F : [0, \infty) \times Y \to R$ by

$$F(\alpha, y) = \begin{cases} \|Tx_{\alpha,y} - \bar{y}\|_Y^2 - \alpha\|x_{\alpha,y}\|_X^2 + \|\hat{y}\|_Y^2, & \alpha > 0, \\ \|\hat{y}\|^2 & \alpha = 0, \end{cases}$$

We want to show that for all y near y^* and $y \notin \overline{R(T)}$, there exists α such that $F(\alpha, y) = 0$. It turns out that we cannot use the implicit function theorem to prove this, because we cannot prove that F is C^1 (though it is continuous).

We define the new function $f(\alpha, y) = \|\hat{y}\|_Y^2 - F(\alpha, y)$, that is,

$$f(\alpha, y) = \begin{cases} \alpha\|x_{\alpha,y}\|_X^2 - \|Tx_{\alpha,y} - \bar{y}\|_Y^2, & \alpha > 0, \\ 0, & \alpha = 0. \end{cases}$$

Then $F(\alpha, y) = 0$ is equivalent to $f(\alpha, y) = \|\hat{y}\|_Y^2$. We need to show that the function

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F is continuous. We start with the following important results.

Lemma 21. *For each $y \in Y$, $Tx_{\alpha,y} \to \overline{y} = proj_{\overline{R(T)}}y$ as $\alpha \to 0^+$.*

Proof. Since $Tx_{\alpha,y} \in R(T)$, then by the Pythagorean theorem we have

$$\|Tx_{\alpha,y} - y\|_Y^2 = \|Tx_{\alpha,y} - \overline{y} + \overline{y} - y\|_Y^2 = \|Tx_{\alpha,y} - \overline{y}\|_Y^2 + \|y - \overline{y}\|_Y^2.$$

It follows that $\|Tx_{\alpha,y} - y\|_Y^2 \geq \|y - \overline{y}\|_Y^2$ and it can be shown that $\|Tx_{\alpha,y} - y\|_Y^2$ is non-increasing as $\alpha \to 0^+$ (see Proposition 5.24 in [12]). Thus

$$\lim_{\alpha \to 0^+} \|Tx_{\alpha,y} - y\|_Y^2 = M \geq \|y - \overline{y}\|_Y^2.$$

We need to show that $M = \|y - \overline{y}\|_Y^2$. By way of contradiction, suppose that $M > \|y - \overline{y}\|_Y^2$. Define ϵ to be $\epsilon = M - \|y - \overline{y}\|_Y^2$. Choose $x \in X$ such that $\|Tx - y\|_Y^2 < M - \frac{\epsilon}{2}$. Then for all $\alpha > 0$ sufficiently small we have

$$\|Tx - y\|_Y^2 + \alpha\|x\|_X^2 < M - \frac{\epsilon}{2} + \alpha\|x\|_X^2 < M < \|y - \overline{y}\|_Y^2 \leq \|Tx_{\alpha,y} - y\|_Y^2 \leq \|Tx_{\alpha,y} - y\|_Y^2 + \alpha\|x_{\alpha,y}\|_X^2,$$

which is a contradiction. This shows that $M = \|y - \overline{y}\|_Y^2$ and

$$\lim_{\alpha \to 0^+} \|Tx_{\alpha,y} - y\|_Y^2 = \|y - \overline{y}\|_Y^2.$$

We have that $\|Tx_{\alpha,y} - y\|_Y^2 = \|Tx_{\alpha,y} - \overline{y}\|_Y^2 + \|y - \overline{y}\|_Y^2$ and as $\alpha \to 0^+$, $\|Tx_{\alpha,y} - y\|_Y^2 \to \|y - \overline{y}\|_Y^2$, thus it follows that $\|Tx_{\alpha,y} - \overline{y}\|_Y^2 \to 0$ as $\alpha \to 0^+$, i.e., $Tx_{\alpha,y} \to \overline{y}$ as

$\alpha \to 0^+$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Lemma 22. *For all* $y \in Y$, $\sqrt{\alpha} N_\alpha^{-1} T^* y \to 0$ *as* $\alpha \to 0^+$.

Proof. We have that $\|Tx_{\alpha,y} - y\|_Y^2 + \|Tx_{\alpha,y}\|_Y^2 + 2\alpha\|x_{\alpha,y}\|_X^2 = \|y\|_Y^2$. Also it follows from the previous lemma that for each $y \in Y$ as $\alpha \to 0^+$, $Tx_{\alpha,y} \to \overline{y}$. Thus as $\alpha \to 0^+$,

$$2\alpha\|x_{\alpha,y}\|_X^2 = \|y\|_Y^2 - \|Tx_{\alpha,y} - y\|_Y^2 - \|Tx_{\alpha,y}\|_Y^2$$

$$\to \|y\|_Y^2 - \|y - \overline{y}\|_Y^2 - \|\overline{y}\|_Y^2$$

$$= \|y\|_Y^2 - \|\hat{y}\|_Y^2 - \|\overline{y}\|_Y^2 = 0.$$

Therefore $\alpha\|x_{\alpha,y}\|_X^2 = \|\sqrt{\alpha} N_\alpha^{-1} T^* y\|_X^2 \to 0$ as $\alpha \to 0^+$ and the proof is complete. □

Now we can prove that the function F is continuous.

Theorem 23. *The function F is continuous.*

Proof. It is straightforward to prove that F is continuous on $(0, \infty) \times Y$. We need to show that for all $y_0 \in Y$, $\lim_{(\alpha,y)\to(0,y_0)} F(\alpha, y) = F(0, y_0)$. We have

$$\lim_{(\alpha,y)\to(0,y_0)} F(\alpha, y) = \lim_{(\alpha,y)\to(0,y_0)} \left(\|Tx_{\alpha,y} - y\|_Y^2 - \alpha\|x_{\alpha,y}\|_X^2 \right).$$

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We can see that $\lim_{(\alpha,y)\to(0,y_0)} \|Tx_{\alpha,y} - y\|_Y^2 = \|\hat{y}_0\|_Y^2$, since

$$y - Tx_{\alpha,y} = y - TN_\alpha^{-1}T^*y = y - TN_\alpha^{-1}T^*(y - y_0) - TN_\alpha^{-1}T^*y_0.$$

Now as $(\alpha, y) \to (0, y_0)$, $TN_\alpha^{-1}T^*(y - y_0) \to 0$ since $\|TN_\alpha^{-1}T^*\| \le 1$ and $TN_\alpha^{-1}T^*y_0 \to \bar{y}_0$ as $\alpha \to 0$. Thus we have

$$y - Tx_{\alpha,y} \to y_0 - \bar{y}_0 = \hat{y}_0, \text{ as } (\alpha, y) \to (0, y_0).$$

Also $\lim_{(\alpha,y)\to(0,y_0)} \alpha\|x_{\alpha,y}\|_X^2 = 0$. That is true because

$$\sqrt{\alpha}x_{\alpha,y} = \sqrt{\alpha}N_\alpha^{-1}T^*(y - y_0) + \sqrt{\alpha}N_\alpha^{-1}T^*y_0 \to 0 \text{ as } (\alpha, y) \to (0, y_0).$$

Therefore,

$$\lim_{(\alpha,y)\to(0,y_0)} F(\alpha, y) = \|\hat{y}_0\|_Y^2 = F(0, y_0).$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Since F is continuous on $[0, \infty) \times Y$ and $f(\alpha, y) = \|\hat{y}\|_Y^2 - F(\alpha, y)$, it follows that f is continuous on $[0, \infty) \times Y$. For $y^* \in R(T)$ we have $f(\alpha, y^*) = \alpha\|x_{\alpha,y^*}\|_X^2 - \|Tx_{\alpha,y^*} - y^*\|_Y^2$. Then clearly

$$\lim_{\alpha\to 0^+} f(\alpha, y^*) = -\|TT^\dagger y^* - y^*\|_Y^2 = 0,$$

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and also

$$\lim_{\alpha \to \infty} f(\alpha, y^*) = -\|y^*\|_Y^2.$$

We will show that $f(\alpha, y^*) > 0$ for all $\alpha > 0$ sufficiently small. We have

$$f(\alpha, y^*) = \alpha \|x_{\alpha, y^*}\|_X^2 - \|Tx_{\alpha, y^*} - y^*\|_Y^2 = \alpha(\|x_{\alpha, y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha, y^*} - y^*\|_Y^2)$$

$$= \alpha(\|x_{\alpha, y^*}\|_X^2 - \alpha^{-1}\|TN_\alpha^{-1}T^*Tx_{0, y^*} - Tx_{0, y^*}\|_Y^2)$$

$$= \alpha(\|x_{\alpha, y^*}\|_X^2 - \alpha^{-1}\|T(I - \alpha N_\alpha^{-1})x_{0, y^*} - Tx_{0, y^*}\|_Y^2)$$

$$= \alpha(\|x_{\alpha, y^*}\|_X^2 - \alpha^{-1}\|\alpha TN_\alpha^{-1}x_{0, y^*}\|_Y^2)$$

$$= \alpha(\|x_{\alpha, y^*}\|_X^2 - \alpha\|TN_\alpha^{-1}x_{0, y^*}\|_Y^2)$$

$$\geq \alpha(\|x_{\alpha, y^*}\|_X^2 - \alpha\|TN_\alpha^{-1}\|^2\|x_{0, y^*}\|_Y^2)$$

$$\geq \alpha(\|x_{\alpha, y^*}\|_X^2 - \frac{1}{2}\|x_{0, y^*}\|_X^2) \quad (\|TN_\alpha^{-1}\|^2 \leq \frac{1}{2\alpha}).$$

Then since $\lim_{\alpha \to 0^+}(\|x_{\alpha, y^*}\|_X^2 - \frac{1}{2}\|x_{0, y^*}\|_X^2) = \frac{1}{2}\|x_{0, y^*}\|_X^2 > 0$, it follows that $f(\alpha, y^*) > 0$ for all $\alpha > 0$ sufficiently small. Therefore there exists $\hat{\alpha} > 0$ such that $f(\alpha, y^*) > 0$ for all $\alpha \in (0, \hat{\alpha})$. We define α^* to be

$$\alpha^* = \arg\max\{f(\alpha, y^*) | \alpha \in (0, \hat{\alpha})\}.$$

Theorem 24. *There exists $\epsilon > 0$ such that for all $y \in B_{\epsilon_1}(y^*) \setminus \overline{R(T)}$, there exists $\alpha \in (0, \alpha^*)$ satisfying $f(\alpha, y) = \|\hat{y}\|_Y^2$.*

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Proof. If ϵ_1 is sufficiently small and $y \in B_{\epsilon_1}(y^*)\backslash\overline{R(T)}$, then $f(0,y) = 0$ and $f(\alpha^*, y) \geq$

$\frac{1}{2}f(\alpha^*, y^*)$. We define $\epsilon_2 = \sqrt{\frac{1}{2}f(\alpha^*, y^*)}$ and consider $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Now if $y \in$

$B_{\epsilon}(y^*) \setminus \overline{R(T)}$, then we have $f(0, y) = 0$, $f(\alpha^*, y) \geq \epsilon_2^2 > 0$, and $\|\hat{y}\|_Y^2 < \epsilon_2^2$. By

intermediate value theorem, since $f(0, y) < \|\hat{y}\|_Y^2 < f(\alpha^*, y)$, there exists $0 < \overline{\alpha} < \alpha^*$

such that $f(\overline{\alpha}, y) = \|\hat{y}\|_Y^2$. $\qquad\square$

By Theorem 24, we can define $\alpha : B_{\epsilon}(y^*) \to [0, \alpha^*]$ such that $\alpha = \alpha(y)$ is the smallest

positive solution of $f(\alpha, y) = \|\hat{y}\|_Y^2$.

Now, we prove the following important result.

Lemma 25. *As $y \to y^*$, $\alpha(y) \to 0$.*

Proof. Let $\{y_k\} \subset Y$, $y_k \to y^*$, $\alpha_k = \alpha(y_k)$ for all $k \in \mathbb{Z}^+$. We have

$$f(\alpha_k, y_k) = \|\hat{y}\|_Y^2 \Leftrightarrow F(\alpha_k, y_k) = 0 \Leftrightarrow \|Tx_{\alpha_k, y_k} - y_k\|_Y^2 = \alpha_k\|x_{\alpha_k, y_k}\|_X^2.$$

If $\{\alpha_k\}$ does not converge to 0, then without loss of generality we can assume that

$\alpha_k \to \overline{\alpha} \in (0, \alpha^*]$ as $y_k \to y^*$. Then as $(\alpha_k, y_k) \to (\overline{\alpha}, y^*)$, we have $F(\alpha_k, y_k) \to$

$F(\overline{\alpha}, y^*)$ and that implies $F(\overline{\alpha}, y^*) = 0$, which contradicts the fact that $F(\alpha, y^*) > 0$

for all $\alpha \in (0, \hat{\alpha})$. So we have $\alpha_k \to 0$ as $y_k \to y^*$. $\qquad\square$

We proved that for each y sufficiently close to y^*, $y \notin \overline{R(T)}$, there exists $\alpha > 0$

satisfying $F(\alpha, y) = 0$.

We would like to study the possibility of determining the rate at which $\alpha = \alpha(y) \to 0$

as $y \to y^*$.

4.2 Rate of convergence

The standard theory of Tikhonov regularization for an infinite-dimensional problem

shows that if $\alpha = \alpha(y) \to 0$ as $y \to y^*$ and $\frac{\|y - y^*\|_Y^2}{\alpha} \leq C$, then $x_{\alpha,y} \to x_{0,y^*}$ weakly.

To see that, we have

$$\|x_{\alpha,y} - x_{0,y^*}\|_X = \|x_{\alpha,y} - x_{\alpha,y^*} + x_{\alpha,y^*} - x_{0,y^*}\|_X$$

$$\leq \|x_{\alpha,y} - x_{\alpha,y^*}\|_X + \|x_{\alpha,y^*} - x_{0,y^*}\|_X.$$

It is clear that $\|x_{\alpha,y^*} - x_{0,y^*}\|_X$ is bounded as $\alpha \to 0$ since

$$\|x_{\alpha,y^*} - x_{0,y^*}\|_X = \|N_\alpha^{-1} T^* y^* - x_{0,y^*}\|_X = \|N_\alpha^{-1} T^* T x_{0,y^*} - x_{0,y^*}\|_X$$

$$= \|(I - \alpha N_\alpha^{-1}) x_{0,y^*} - x_{0,y^*}\|_X = \|\alpha N_\alpha^{-1} x_{0,y^*}\|_X$$

$$\leq \|\alpha N_\alpha^{-1}\| \|x_{0,y^*}\|_X \leq \|x_{0,y^*}\|_X \leq \hat{C}.$$

In fact, since $x_{\alpha,y^*} \to T^\dagger y^*$ strongly as $\alpha \to 0$, it follows that $\|x_{\alpha,y^*} - x_{0,y^*}\|_X \to 0$

as $\alpha \to 0$. Also we have

$$\|x_{\alpha,y} - x_{\alpha,y^*}\|_X \leq \|N_\alpha^{-1} T^*(y - y^*)\|_X \leq \frac{1}{\sqrt{2\alpha}}\|y - y^*\|_Y \leq \frac{1}{\sqrt{2}}C^{\frac{1}{2}}.$$

Thus $\{x_{\alpha,y}\}$ is bounded as $y \to y^*$ and it can be shown that $x_{\alpha,y} \to x_{0,y^*}$ weakly under this condition. To show that, suppose $x_{\alpha,y} \to x \in X$ weakly. We will see that $x = x_{0,y^*}$. We have that $T^*y \to T^*y^*$ strongly (since $y \to y^*$ strongly and the operator T^* is bounded) and also $T^*Tx_{\alpha,y} \to T^*Tx$ weakly (since $x_{\alpha,y} \to x$ weakly and the operator T^*T is bounded). Moreover, since $\{x_{\alpha,y}\}$ is bounded, $\alpha x_{\alpha,y} \to 0$ strongly as $\alpha \to 0$. We have $T^*Tx_{\alpha,y} + \alpha x_{\alpha,y} = T^*y$.

It follows that $T^*Tx = T^*y^*$ and that implies x is a least-square solution of $Tx = y^*$. We have seen that $x_{\alpha,y} \in N(T)^\perp$ and since $N(T)^\perp$ is closed, then $x \in N(T)^\perp$. It follows that x is the minimum-norm least-squares solution of $Tx = y^*$, that is, $x = x_{0,y^*}$.

For an infinite-dimensional inverse problem, a purely *a posteriori* parameter choice method like multiplicative regularization cannot be convergent. We conjecture that if $\|\hat{y}\|_Y \geq \theta\|y - y^*\|_Y$ holds, then $\frac{\|\hat{y}\|_Y^2}{\alpha(y)} \leq C$ holds and $x_{\alpha,y} \to x_{0,y^*}$ weakly. The condition $\|\hat{y}\|_Y \geq \theta\|y - y^*\|_Y$ is necessary to ensure that each y_k contains a significant component in $R(T)^\perp$.

4.2.1 Special cases

We would like to show that $\frac{\|\hat{y}\|^2}{\alpha}$ is bounded above as $y \to y^*$. Unfortunately we are not able to prove that in general, but we can show that it is true for two special cases. First suppose that $\{y_k\} \subset Y$, $y_k = y^* + \hat{y}_k$, $\alpha_k = \alpha(y_k)$ for all $k \in \mathbb{Z}^+$ and $y_k \to y^*$, as $k \to \infty$. Then we have

$$
\begin{aligned}
\frac{\|\hat{y}_k\|_Y^2}{\alpha_k} &= \|x_{\alpha_k, y_k}\|_X^2 - \frac{1}{\alpha_k}\|Tx_{\alpha_k, y_k} - y_k\|_Y^2 \leq \|x_{\alpha_k, y_k}\|_X^2 \\
&= \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y_k, \psi_i\rangle_Y|^2 = \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y^* + \hat{y}_k, \psi_i\rangle_Y|^2 \\
&= \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y^*, \psi_i\rangle_Y|^2 + \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle \hat{y}_k, \psi_i\rangle_Y|^2 \\
&= \|x_{\alpha_k, y^*}\|_X^2 \leq \|x_{0, y^*}\|_X^2 \quad \text{since } \langle \hat{y}_k, \psi_i\rangle_Y = 0 \;\forall i.
\end{aligned}
$$

Thus $\frac{\|\hat{y}_k\|_Y^2}{\alpha_k} \leq C$ as $y_k \to y^*$, where $C = \|x_{0, y^*}\|_X^2$.

For our second special case we suppose $\{y_k\} \subset Y$, $y_k = y^* + \sigma_k \psi_k + \hat{y}_k$, $\alpha_k = \alpha(y_k)$ for all $k \in \mathbb{Z}^+$ and $\hat{y}_k \to 0$, as $k \to \infty$. Then we have

$$
\begin{aligned}
\frac{\|\hat{y}_k\|_Y^2}{\alpha_k} &= \|x_{\alpha_k, y_k}\|_X^2 - \frac{1}{\alpha_k}\|Tx_{\alpha_k, y_k} - y_k\|_Y^2 \leq \|x_{\alpha_k, y_k}\|_X^2 = \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y_k, \psi_i\rangle_Y|^2 \\
&= \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y^* + \sigma_k \psi_k + \hat{y}_k, \psi_i\rangle_Y|^2 \\
&= \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y^* + \sigma_k \psi_k, \psi_i\rangle_Y|^2 \quad \text{since } \langle \hat{y}_k, \psi_i\rangle_Y = 0 \;\forall i \\
&= \sum_{i=1}^{\infty} \frac{\sigma_i^2}{(\sigma_i^2 + \alpha_k)^2}|\langle y^*, \psi_i\rangle_Y|^2 + \frac{\sigma_k^4}{(\sigma_k^2 + \alpha_k)^2} + \frac{2\sigma_k^3}{(\sigma_k^2 + \alpha_k)^2}\langle y^*, \psi_k\rangle_Y \\
&= \|x_{\alpha_k, y^*}\|_X^2 + \frac{\sigma_k^4}{(\sigma_k^2 + \alpha_k)^2} + \frac{2\sigma_k^3}{(\sigma_k^2 + \alpha_k)^2}|\langle y^*, \psi_k\rangle_Y|.
\end{aligned}
$$

The last step follows from the fact $\langle \hat{y}_k, \psi_k \rangle_Y = 0$ since $\hat{y}_k$ is orthogonal to $\text{col}(A)$.

It can be seen that $\frac{\|\hat{y}_k\|_Y^2}{\alpha_k}$ is bounded above since $\|x_{\alpha_k,y^*}\|_X^2$ is bounded by $\|x_{0,y^*}\|_X^2$,

$\frac{\sigma_k^4}{(\sigma_k^2+\alpha_k)^2}$ is bounded by 1 and also

$$\frac{2\sigma_k^3}{(\sigma_k^2+\alpha_k)^2}|\langle y^*, \psi_k\rangle_Y| = \frac{2\sigma_k^4}{(\sigma_k^2+\alpha_k)^2}\frac{|\langle y^*, \psi_k\rangle_Y|}{\sigma_k} \to 0.$$

Therefore $\frac{\|\hat{y}_k\|_Y^2}{\alpha_k}$ is bounded above by a constant C where $C > 0$. Also we can show

that by imposing the condition $\|\bar{y}_k - y_k^*\| < \theta\|\hat{y}_k\|$ (for some $\theta > 0$), $\frac{\alpha}{\|\hat{y}_k\|_Y^2}$ is bounded

above. To see that we have

$$\frac{\|\hat{y}_k\|_Y^2}{\alpha} = \|x_{\alpha_k,y_k^*}\|_X^2 - \frac{1}{\alpha}\|Tx_{\alpha_k,y_k^*} - y_k^*\|_Y^2$$

$$\geq \|x_{\alpha_k,y_k^*}\|_X^2 - \frac{1}{\alpha}\|\bar{y}_k - y_k^*\|_Y^2$$

$$\geq \|x_{\alpha_k,y_k^*}\|_X^2 - \frac{\theta\|\hat{y}_k\|_Y^2}{\alpha}.$$

It follows that $(1+\theta)\frac{\|\hat{y}_k\|_Y^2}{\alpha} \geq \|x_{\alpha_k,y_k^*}\|_X^2$ and thus

$$\frac{\alpha}{\|\hat{y}_k\|_Y^2} \leq \frac{(1+\theta)}{\|x_{\alpha_k,y_k^*}\|_X^2} \leq \hat{C} \text{ as } y_k \to y^*,$$

where $\hat{C} > 0$. Thus $0 < \hat{C} \leq \frac{\|\hat{y}_k\|_Y^2}{\alpha} \leq C$.

4.2.2 Counter-example

Now with a counter-example, we can show that imposing the condition

$$\|\hat{y}\|_Y \geq \theta \|y - y^*\|_Y,$$

is necessary. In the previous section, we proved that for all y sufficiently close to y^* and $y \notin \overline{R(T)}$, there exists $\alpha = \alpha(y) > 0$ satisfying $f(\alpha, y) = \|\hat{y}\|_Y^2$, that is, there exists $\alpha = \alpha(y) > 0$ satisfying $\alpha = \frac{\|\hat{y}\|_Y^2}{\|x_{\alpha,y}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y} - \bar{y}\|_Y^2}$.

We have

$$
\begin{aligned}
\alpha - \frac{\|\hat{y}\|_Y^2}{\|x_{0,y^*}\|_X^2} &= \frac{\|\hat{y}\|_Y^2}{\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2} - \frac{\|\hat{y}\|_Y^2}{\|x_{0,y^*}\|_X^2} \\
&= \frac{\|\hat{y}\|_Y^2 (\|x_{0,y^*}\|_X^2 - (\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y\|_Y^2))}{\|x_{0,y^*}\|_X^2 (\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2)} \\
&= \frac{\|\hat{y}\|_Y^2}{\|x_{0,y^*}\|_X^2} \left(\frac{\|x_{0,y^*}\|_X^2 - (\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2)}{\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2} \right),
\end{aligned}
$$

and thus it follows that

$$
\begin{aligned}
\frac{\|\hat{y}\|_Y^2}{\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2} &= \frac{\|\hat{y}\|_Y^2}{\|x_{0,y^*}\|_X^2} + \frac{\|\hat{y}\|_Y^2}{\|x_{0,y^*}\|_X^2} \left(\frac{\|x_{0,y^*}\|_X^2 - (\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2)}{\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2} \right) \\
&= \frac{\|\hat{y}\|_Y^2}{\|x_{0,y^*}\|_X^2} \left(1 + \frac{\|x_{0,y^*}\|_X^2 - \|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2}{\|x_{\alpha,y^*}\|_X^2 - \alpha^{-1}\|Tx_{\alpha,y^*} - y^*\|_Y^2} \right).
\end{aligned}
$$

We can choose $\{y_k^*\} \subset R(T)$ such that as $k \to \infty$, $y_k^* \to y^*$ and $\|x_{0,y_k^*}\|^2 \to \infty$. To

see that, suppose $y_k^* = y^* + \sqrt{\sigma_k}\psi_k$. Then we have

$$
\begin{aligned}
x_{0,y_k^*} &= T^\dagger y_k^* = T^\dagger(y^* + \sqrt{\sigma_k}\psi_k) = T^\dagger y^* + T^\dagger \sqrt{\sigma_k}\psi_k \\
&= x_{0,y^*} + \frac{\sqrt{\sigma_k}}{\sigma_k}\psi_k = x_{0,y^*} + \frac{1}{\sqrt{\sigma_k}}\psi_k.
\end{aligned}
$$

Clearly as $k \to \infty$, $\sigma_k \to 0$, $y_k^* \to y^*$ and $\|x_{0,y_k^*}\|^2 \to \infty$. Now for each k, choose y_k as

$$
y_k = y_k^* + \hat{y}_k = y^* + \sqrt{\sigma_k}\psi_k + \hat{y}_k,
$$

where $\hat{y}_k$ is sufficiently small such that

$$
\alpha(y_k) = \frac{\|\hat{y}_k\|_Y^2}{\|x_{0,y_k^*}\|_X^2}(1 + \theta_k), \ |\theta_k| < \frac{1}{k}.
$$

To see that, we have

$$
\begin{aligned}
\theta_k &= \frac{\|x_{0,y_k^*}\|_X^2 - \|x_{\alpha_k,y_k^*}\|_X^2 - \alpha_k^{-1}\|Tx_{\alpha_k,y_k^*} - y_k^*\|_Y^2}{\|x_{\alpha_k,y_k^*}\|_X^2 - \alpha_k^{-1}\|Tx_{\alpha_k,y_k^*} - y_k^*\|_Y^2} \\
&= \frac{\sum_{n=1}^{\infty} \frac{3\alpha_k\sigma_n^2 + \alpha_k^2}{(\sigma_n^2 + \alpha_k)^2} \frac{|\langle y_k^*, \psi_n\rangle|^2}{\sigma_n^2}}{\sum_{n=1}^{\infty} \frac{\sigma_n^2 - \alpha_k^2}{(\sigma_n^2 + \alpha_k)^2}|\langle y_k^*, \psi_n\rangle|^2} \\
&\leq \frac{\sum_{n=1}^{\infty} \frac{3\sigma_n^2 + \alpha_k}{(\sigma_n^2 + \alpha_k)} \frac{|\langle y_k^*, \psi_n\rangle|^2}{\sigma_n^2}}{\sum_{n=1}^{\infty} \frac{\sigma_n^2 - \alpha_k^2}{(\sigma_n^2 + \alpha_k)^2}|\langle y_k^*, \psi_n\rangle|^2}.
\end{aligned}
$$

It follows that for each fixed k, $\theta_k \to 0$ as $\hat{y}_k \to 0$ (because $\alpha_k \to 0$ as $\hat{y}_k \to 0$).

We also have that

$$\|y_k - y_k^*\|_Y^2 = \|y_k - \overline{y}_k\|_Y^2 + \|\overline{y}_k - y_k^*\|_Y^2 = \|y_k - \overline{y}_k\|_Y^2 + \|\hat{y}_k\|_Y^2,$$

and clearly $\|\hat{y}_k\|_Y^2 \leq \|y_k - y_k^*\|_Y^2$. We then have

$$\frac{\alpha(y_k)}{\|y_k - y_k^*\|_Y^2} \leq \frac{\alpha(y_k)}{\|\hat{y}_k\|_Y^2} = \frac{1 + \theta_k}{\|x_{0,y_k^*}\|_X^2} \to 0, \text{ as } k \to \infty,$$

since $\|x_{0,y_k^*}\|^2 \to \infty$ and $\theta_k \to 0$ as $k \to \infty$. Hence for this choice of $\{y_k\}$, $\frac{\|y_k - y_k^*\|_Y^2}{\alpha(y_k)}$ is not bounded. In general, we can see that $\frac{\|\hat{y}\|_Y^2}{\alpha(y)}$ need not be bounded if we do not impose the condition $\|\hat{y}\|_Y \geq \theta\|y - y^*\|_Y$.

4.2.3   Numerical experiments

In Chapter 3, we have shown that for a finite-dimensional problem,

$$\lim_{b \to b^*} \frac{\|\hat{b}\|^2}{\alpha} = \lim_{b \to b^*} (\|x_{\alpha,b}\|^2 - \alpha^{-1}\|Ax_{\alpha,b} - \overline{b}\|^2) = \|x_{0,b^*}\|^2, \qquad (4.2.1)$$

which implies as $b \to b^*$, $\frac{\|\hat{b}\|^2}{\alpha} \to \|x_{0,b^*}\|^2$. In this chapter, we will do some numerical experiments to test whether (4.2.1) seems to be true for infinite-dimensional problems.

We construct a noisy data function y as $y = y^* + \delta z$, where y^* is the exact data, δ is

the noise level and z is a discontinuous function. In our example, the function y does not belong to the $D(T^\dagger)$. Now by increasing n (the dimension of the problem) and choosing n sufficiently large, we get to compute the various quantities to 4 correct digits.

We consider some test problems from Hansen's Regularization Tools package [22]. We start with the test problem "baart", which is obtained by discretization of the first-kind Fredholm integral equation

$$\int_0^\pi k(s,t)f(t)\,dt = g(s), \quad 0 \le s \le \frac{\pi}{2}, \tag{4.2.2}$$

where $k(s,t) = \exp(s\cos(t))$ and $f(t) = \sin(t)$. The right hand side is $g(s) = 2(\frac{\sinh(s)}{s}) + \delta z$. The function $z(s)$ is given by

$$z = \begin{cases} s, & 0 \le s \le \frac{\pi}{4}, \\ 2 - 2s, & \frac{\pi}{4} < s \le \frac{\pi}{2}. \end{cases}$$

The equation is discretized by the Galerkin method with orthonormal box functions. The dimension n of the problem should be large enough in order to get accurate results, and we choose n to be 1000. The results for different noise levels are given in Table 4.1. The exact solution of the "baart" test problem satisfies $\|x^*\|^2 = 1.5708$. As we can see, as $y \to y^*$, $\frac{\|\hat{y}\|^2}{\alpha}$ appears to converge to $\|x_{0,y^*}\|^2$.

Table 4.1
Numerical results for the infinite-dimensional test problem "baart"

δ	α	$\|\hat{y}\|$	$\frac{\|\hat{y}\|^2}{\alpha}$	$\|x_{\alpha,y}\|^2$
10^{-2}	$2.3556 \cdot 10^{-6}$	$9.1723 \cdot 10^{-4}$	0.3571	1.8309
10^{-3}	$1.4422 \cdot 10^{-8}$	$9.1723 \cdot 10^{-5}$	0.5833	1.7123
10^{-4}	$1.2021 \cdot 10^{-10}$	$9.1723 \cdot 10^{-6}$	0.6999	1.6731
10^{-5}	$1.1141 \cdot 10^{-12}$	$9.1723 \cdot 10^{-7}$	0.7552	1.5943
10^{-6}	$8.8572 \cdot 10^{-15}$	$9.1723 \cdot 10^{-8}$	0.9499	1.6214
10^{-7}	$7.2416 \cdot 10^{-17}$	$9.1727 \cdot 10^{-9}$	1.1619	1.6763
10^{-8}	$7.0916 \cdot 10^{-19}$	$9.1723 \cdot 10^{-10}$	1.1864	1.5841
10^{-9}	$6.4209 \cdot 10^{-21}$	$9.1725 \cdot 10^{-11}$	1.3104	1.5854
10^{-10}	$5.7308 \cdot 10^{-23}$	$9.1750 \cdot 10^{-12}$	1.4689	1.6303
10^{-11}	$5.4646 \cdot 10^{-25}$	$9.3045 \cdot 10^{-13}$	1.5788	1.5824

Our second example is the test problem "wing" which is obtained from discretization

of the first-kind integral equation

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 < s < 1, \tag{4.2.3}$$

where $k(s,t) = t\exp(-st^2)$, and the solution $f(t)$ is

$$f(t) = \begin{cases} 1, & \frac{1}{3} < t < \frac{2}{3}, \\ \\ 0, & \text{otherwise.} \end{cases}$$

The right hand side is given by $g(s) = \frac{\exp(-\frac{1}{9}s)-\exp(-\frac{4}{9}s)}{2s} + \delta z$ where $z(s)$ is

$$z = \begin{cases} s, & 0 \le s \le \frac{1}{2}, \\ \\ 2 - 2s, & \frac{1}{2} < s \le 1. \end{cases}$$

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Table 4.2

Table 4.2

Numerical results for the infinite-dimensional test problem "wing"

δ	α	$\|\hat{y}\|$	$\frac{\|\hat{y}\|^2}{\alpha}$	$\|x_{\alpha,y}\|^2$
10^{-2}	$1.04525 \cdot 10^{-4}$	$1.5311 \cdot 10^{-3}$	0.0224	0.2720
10^{-3}	$4.3871 \cdot 10^{-7}$	$1.5311 \cdot 10^{-4}$	0.0534	0.2901
10^{-4}	$5.6356 \cdot 10^{-9}$	$1.5311 \cdot 10^{-5}$	0.0416	0.22758
10^{-5}	$1.6054 \cdot 10^{-11}$	$1.5311 \cdot 10^{-6}$	0.1460	0.32125
10^{-6}	$1.7762 \cdot 10^{-13}$	$1.5311 \cdot 10^{-7}$	0.1320	0.27201
10^{-7}	$1.6345 \cdot 10^{-15}$	$\cdot 1.531110^{-8}$	0.1434	0.2933
10^{-8}	$1.0048 \cdot 10^{-17}$	$1.5311 \cdot 10^{-9}$	0.2333	0.3327
10^{-9}	$1.0623 \cdot 10^{-21}$	$1.5311 \cdot 10^{-10}$	0.2206	0.2982
10^{-10}	$1.0555 \cdot 10^{-21}$	$1.5311 \cdot 10^{-11}$	0.2221	0.2976
10^{-11}	$1.0281 \cdot 10^{-23}$	$1.5311 \cdot 10^{-12}$	0.2280	0.3018
10^{-12}	$7.5070 \cdot 10^{-26}$	$1.5311 \cdot 10^{-13}$	0.3123	0.3196

The equation is discretized by the Galerkin method with orthonormal box functions. The dimension of the problem (n) is chosen to be 1000. The numerical results for different noise levels are given in Table 4.3. The exact solution of the "wing" test problem is $\|x^*\|^2 = 0.3340$. One more time as $y \to y^*$, $\frac{\|\hat{y}\|^2}{\alpha}$ appears to converge to $\|x_{0,y^*}\|^2$.

Our last example is "deriv2" which is obtained from discretization of the first-kind integral equation

$$\int_0^1 k(s,t)f(t)\,dt = g(s), \quad 0 \le s \le 1, \tag{4.2.4}$$

where

$$k(s,t) = \begin{cases} s(t-1), & s < t, \\ t(s-1), & \text{otherwise}, \end{cases}$$

and the exact solution is $f(t) = t$. The right hand side is $g(s) = \frac{1}{6}(s^3 - s) + \delta z$. The

Table 4.3

Numerical results for the infinite-dimensional test problem "deriv2"

δ	α	$\|\hat{y}\|$	$\frac{\|\hat{y}\|^2}{\alpha}$	$\|x_{\alpha,y}\|^2$
10^{-2}	$4.5601 \cdot 10^{-6}$	$7.7204 \cdot 10^{-5}$	0.0013	0.3380
10^{-3}	$5.8696 \cdot 10^{-9}$	$7.7204 \cdot 10^{-6}$	0.0102	0.4763
10^{-4}	$1.1966 \cdot 10^{-11}$	$7.7204 \cdot 10^{-7}$	0.0498	0.4957
10^{-5}	$1.5634 \cdot 10^{-14}$	$7.7204 \cdot 10^{-8}$	0.3813	0.6055
10^{-6}	$1.7359 \cdot 10^{-16}$	$7.7206 \cdot 10^{-9}$	0.3434	0.3435
10^{-7}	$1.7885 \cdot 10^{-18}$	$7.7224 \cdot 10^{-10}$	0.3334	0.3334
10^{-8}	$1.8048 \cdot 10^{-20}$	$7.7563 \cdot 10^{-11}$	0.3333	0.3333
10^{-9}	$2.6874 \cdot 10^{-22}$	$9.4647 \cdot 10^{-12}$	0.3333	0.3333
10^{-10}	$8.4016 \cdot 10^{-23}$	$5.2920 \cdot 10^{-12}$	0.3333	0.3333
10^{-11}	$8.1475 \cdot 10^{-23}$	$5.2114 \cdot 10^{-12}$	0.3333	0.3333
10^{-12}	$1.2790 \cdot 10^{-24}$	$6.5295 \cdot 10^{-13}$	0.3333	0.3333

function $z(s)$ is given by

$$
z = \begin{cases} s, & 0 \le s \le \frac{1}{2}, \\ \\ 2 - 2s, & \frac{1}{2} < s \le 1. \end{cases}
$$

We discretize the equation by the Galerkin method with orthonormal box functions to get $T_{mn}x_n = y_m$ where $T_{mn} \in \mathbb{R}^{m \times n}$, $y_m \in \mathbb{R}^m$ and $x_n \in \mathbb{R}^n$ is to be estimated. For this problem n is chosen to be 1000 and $m = 2n$. The numerical results are given in Table 4.2. The exact solution of the "deriv2" test problem is $\|x^*\|^2 = 0.3333$. Clearly as $y \to y^*$, $\frac{\|\hat{y}\|^2}{\alpha} \to \|x_{0,y^*}\|^2$. Notice that we do not expect multiplicative regularization to work for "deriv2" (in finite dimensions) because $\overline{R(T)} = Y$. However, this example suggests that, even for such a problem, multiplicative regularization might work if the problem is discretized with $m > n$.

4.3 Conclusions

Based on our numerical results, we can see that for each test problem, by constructing

a noisy data function y (where $y \notin D(T^{\dagger})$), discretizing the problem $Tx = y$, choosing

n sufficiently large, we get to choose the accurate α and it appears that that as

$y \to y^{*}$, $\frac{\|\hat{y}\|^2}{\alpha}$ converges to $\|x_{0,y^*}\|^2$.

Our special cases and numerical examples suggest that

$$y \to y^{*}, \|\hat{y}\|_Y \geq \theta \|y - y^{*}\|_Y \Rightarrow \alpha(y) \to 0 \text{ like } \|\hat{y}\|_Y^2. \tag{4.3.1}$$

However we are not able to prove the important result (4.3.1) in general.

Chapter 5

Conclusions

5.1   Summary and conclusions

Inverse problems arise in many branches of science and engineering. In order to get a good approximation of the solution of this kind of problems, the use of regularization methods is required. Tikhonov regularization is one of the best methods for estimating the solutions of inverse problems. For a linear inverse problem $Tx = y$, the objective function for Tikhonov regularization method is given by:

$$\|Tx - y\|_Y^2 + \alpha\|x\|_X^2.$$

The quality of the approximate solution computed by Tikhonov regularization depends on choosing a good regularization parameter α.

The L-curve criterion is a purely *a posteriori* parameter choice method for the Tikhonov regularization method. This method works well most of the time, although there exist some problems in which the L-curve criterion does not perform properly. Because of difficulties associated with this method, for instance, we can not prove that the L-curve always has a corner; indeed, we have seen examples in which it does not. Therefore, we have investigated another parameter choice method.

Multiplicative regularization is a strategy for solving inverse problems. This method does not require any parameter selection strategies and, at the same time, can be considered as a parameter choice method for the Tikhonov regularization method. Multiplicative regularization is not an expensive method and is guaranteed to define a positive regularization parameter α provided provided $A \in \mathbb{R}^{m \times n}$, $\text{col}(A)$ is a proper subspace of $\mathbb{R}^m$ and b is sufficiently close to b^*. Under these conditions, multiplicative regularization performs very similarly to the L-curve method. Moreover, $\alpha \to 0$ at the same rate as $\|\hat{b}\|^2$ as $b \to b^*$.

We have presented some preliminary theoretical results for infinite-dimensional problems. For example, for all $y \in Y$ sufficiently close to y^*, there exists a positive regularization parameter $\alpha = \alpha(y)$ and also as $y \to y^*$, $\alpha(y) \to 0$.

5.2 Future work: plans and goals

In general, for an infinite-dimensional inverse problem, a purely *a posteriori* parameter choice method can not be convergent. However, if we conjecture that the results for finite-dimensional problems extend to infinite dimensions, namely, that

$$y \to y^*, \|\hat{y}\|_Y \geq \theta \|y - y^*\|_Y \Rightarrow \alpha(y) \to 0 \text{ like } \|\hat{y}\|_Y^2. \tag{5.2.1}$$

If our conjecture is true, then we obtain the weak convergence of $x_{\alpha,y}$ to x^* under assumption (5.2.1).

We have not been able to prove (5.2.1), but we hope to prove this result in the future.

We also would like to study multiplicative regularization for nonlinear inverse problems.

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